Week 4, MAT 110

Differential Calculus & Coordinate Geometry

EXPONENTIAL GROWTH & DECAY

Standard exponential equation: $X(t) = Ae^{kt}$

Example (1) A culture initially has P_0 number of bacteria. At $t=1h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple. $\rightarrow 3P_0$

$t = ?$

Initial time $t_0 = 0$

Initial number of bacteria $P(0) = P_0$. {We are using $P(t)$ instead of $X(t)$}.

$\therefore P(t) = Ae^{kt}$

$P(0) = P_0 = Ae^{k(0)}$ [at $t=0$]

$P_0 = Ae^0 = A$

$\therefore$ if $A = P_0$, we can say $P(t) = Ae^{kt}$

$\Rightarrow P(t) = P_0 e^{kt}$ —— (1)

Given at $t=1$, the no. of bacteria is measured to be $\frac{3}{2}P_0$

$\therefore$ sub $t=1$ into (1) we have:

$P(1) = P_0 e^{k(1)}$

$\frac{3}{2}P_0 = P_0 e^k$

$\rightarrow e^k = \frac{3}{2}$

$\ln e^k = \ln\frac{3}{2}$

$k = \ln\frac{3}{2}$

$= 0.4055$

$\ln e^x = x$

$\therefore P(t) = P_0 e^{0.4055t}$ —— (1)

$P(t) = P_0 e^{kt}$ $\quad K = 0.4055, \; P(t) = 3P_0$

$t = ?$

To find the time at which the number of bacteria has tripled, we solve:

$$3P_0 = P_0 e^{0.4055t}$$

$$e^{0.4055t} = 3$$

$$\ln e^{0.4055t} = \ln 3$$

$$0.4055t = \ln 3$$

$$t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ h}$$

Example ② How many days (t?) will it take for an insect population to double if it's growth rate is 5%?

$r = 5\% = 0.05$ $\quad P(t) = P_0 e^{rt}$ $\quad \underbrace{[A = P_0, \; k = r]}_{\text{In this case we are using the above notations}}$

Given $2P_0 = P_0 e^{0.05t}$ $\quad \boxed{\ln e^{x} = x}$

$$2 = e^{0.05t}$$

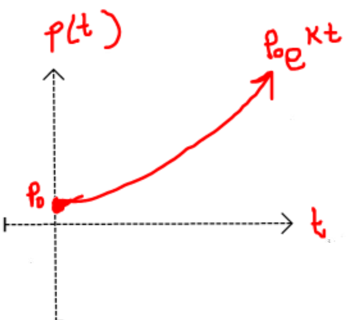
$$\ln 2 = \ln e^{0.05t}$$

$$\ln 2 = 0.05t$$

$$t = \frac{\ln 2}{0.05}$$

$$t = 13.86$$

or $t \approx 14$ days



$x(t) = Ae^{kt} \Rightarrow P(t) = P_0 e^{kt}$

$t = 0$ initial time

$P(0) = Ae^{k(0)} = Ae^{0} = A$

$P_0 = A$

$P(0) = P_0$ = initial population

Example 3 In 1950, the population of Augusta was 6,000. By 1990, the population grew to 12,000. In what year will the population reach 30,000?

40 yrs

time

$t = 40,\ P_0 = 6000$

i

$P(t) = P_0 e^{kt}$

$P(40) = 6000e^{k(40)}$

$12{,}000 = 6000e^{40k}$

ii

$12{,}000 = 6{,}000e^{40k}$

$2 = e^{40k}$

$\ln e^{40k} = \ln 2$

$40k = \ln 2$

$\boxed{k = \frac{\ln 2}{40}}$

iii

$P(t) = P_0 e^{kt}$

$30000 = 6{,}000e^{\frac{\ln 2}{40}t}$

$5 = e^{\frac{\ln 2}{40}t}$

$\ln e^{\frac{\ln 2}{40}t} = \ln 5$

$\frac{\ln 2}{40}t = \ln 5$

$t = \frac{\ln 5}{\ln 2}(40) = \frac{40\ln 5}{\ln 2}$

$t = 92.8 \approx 93$ years

$\therefore$ 1950 + 93 years = 2043

$\therefore$ In 2043, the population will reach 30,000.

$P(t) = \frac{1}{2} P_0$

Example (4) The half life of lead is 22 years. How long will it take for a lead object to decay to 80% of its original amount? $\rightarrow P_0$

$t = ?$ 20%

(i)

$$P(t) = P_0 e^{kt} \quad (A = P_0)$$

$$P(22) = P_0 e^{k(22)}$$

$(100 - \frac{20}{100}) P_0$

$$\frac{1}{2} P_0 = P_0 e^{22k}$$

$$e^{22k} = \frac{1}{2}$$

$$\ln e^{22k} = \ln\left(\frac{1}{2}\right)$$

$$22k = \ln(0.5)$$

$$\boxed{k = \frac{\ln(0.5)}{22}}$$

(ii) $t = ?$ while $P(t)$ is 80% ↓ to original amount

$$P(t) = P_0 e^{kt}$$

$$0.80 P_0 = P_0 e^{\frac{\ln(0.5)}{22} t}$$

$$e^{\frac{\ln(0.5)}{22} t} = 0.80$$

$$\ln e^{\frac{\ln(0.5)}{22} t} = \ln(0.80)$$

$$\frac{\ln(0.5)}{22} t = \ln(0.80)$$

$$t = \frac{(\ln 0.80)(22)}{\ln(0.5)}$$

$$t \approx 7.08 \text{ years}$$