Week 8 Differential Calculus & Coordinate Geometry

-> Introduction to Vector Calculus
-> Fields, Gradient, Curl & Divergence

ARITHMETIC OPERATIONS ON VECTORS

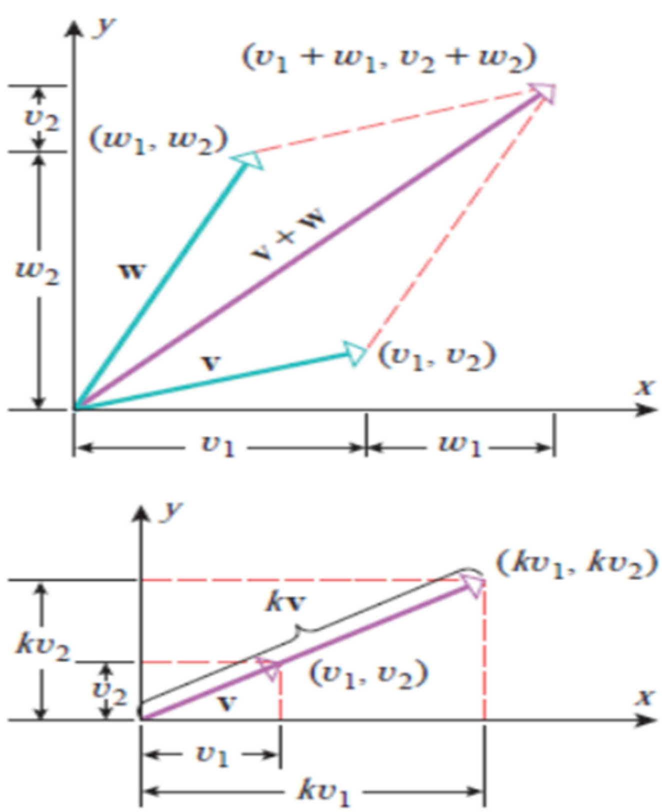
A vector has a magnitude and direction. Examples: Velocity, acceleration etc.

11.2.4 THEOREM *If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are vectors in 2-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \tag{1}$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \tag{2}$$
$$k\mathbf{v} = \langle kv_1, kv_2 \rangle \tag{3}$$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space and k is any scalar, then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \tag{4}$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle \tag{5}$$
$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \tag{6}$$



▲ Figure 11.2.8

RULES OF VECTOR ARITHMETIC

The following theorem shows that many of the familiar rules of ordinary arithmetic also hold for vector arithmetic.

11.2.6 THEOREM *For any vectors* **u**, **v**, *and* **w** *and any scalars* k *and* l, *the following relationships hold:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(l\mathbf{u}) = (kl)\mathbf{u}$

(*f*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*g*) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

DEFINITION OF THE DOT PRODUCT

11.3.1 DEFINITION If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the ***dot product*** of **u** and **v** is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$u = \langle 1, 2, 2 \rangle$, $v = \langle 0, 1, 2 \rangle$ $\quad$ $u \cdot v = (1)(0) + (2)(1) + (2)(2)$

$= 6 \rightarrow$ scalar number

ALGEBRAIC PROPERTIES OF THE DOT PRODUCT

The following theorem provides some of the basic algebraic properties of the dot product.

11.3.2 THEOREM *If* **u**, **v**, *and* **w** *are vectors in 2- or 3-space and* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ $\rightarrow$ Norm ~ square

(*e*) $\mathbf{0} \cdot \mathbf{v} = 0$

CROSS PRODUCT

We now turn to the main concept in this section.

11.4.2 DEFINITION If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \tag{3}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \tag{4}$$

~~Observe that the right side of Formula (3) has the same form as the right side of Formula (2), the difference being notation and the order of the factors in the three terms. Thus, we~~ We can rewrite (3) as :

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{5}$$

However, this is just a mnemonic device and not a true determinant since the entries in a determinant are numbers, not vectors.

Example 1 Let $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find

(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Solution (a).

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

Similarly try (b)

- The cross product is defined only for vectors in 3-space, whereas the dot product is defined for vectors in 2-space and 3-space.
- The cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar.

ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Vector Valued Function:

A vector valued function is also called a vector function. It is a function with the following two properties:

1. The domain is a set of real numbers
2. The range is a set of vectors

Vector functions are, therefore, simply an extension of scalar functions, where both the domain and the range are the set of real numbers.

In this tutorial we'll consider vector functions whose range is the set of two or three dimensional vectors. Hence, such functions can be used to define a set of points in space.

Given the unit vectors i,j,k parallel to the x,y,z-axis respectively, we can write a three dimensional vector valued function as:

$$r(t) = x(t)i + y(t)j + z(t)k$$

It can also be written as:

$$r(t) = \langle x(t), y(t), z(t) \rangle$$

DIRECTIONAL DERIVATIVES

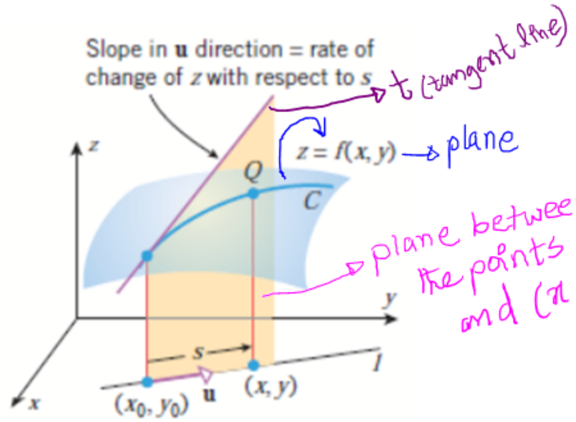
13.6.1 DEFINITION If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of* u** at (x_0, y_0) is denoted by $D_{\mathbf{u}} f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0} \tag{2}$$

provided this derivative exists.

here u is a unit vector. magnitude = 1

$s = d((x, y), (x_0, y_0))$



▲ **Figure 13.6.2**

The curve "c" intersects the blue & the pink plane.

The tangent line "t" to the curve "c" is the directional derivative of $z = f(x, y)$

Directional Derivative:(geometric interpretation)
rate of change of "z=f(x,y)" in the direction of "u" with respect to the distance "s"

directional derivative represents the ***instantaneous rate of change of f(x, y) with respect to distance in the direction of* u** at the point (x_0, y_0).

consider:

If the point changes, the directional derivative will change as well. → if (x_0, y_0) changes

The instantaneous rate of change is **the change in the rate at a particular instant**, and it is same as the change in the derivative value at a specific point. For a graph, the instantaneous rate of change at a specific point is the same as the tangent line slope. That is, it is a curve slope.

$(x_0, y_0) = (1, 2)$

▶ **Example 1** Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

using definition 13.6.1

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} = u_1\hat{i} + u_2\hat{j}$$

$u_1 = \frac{\sqrt{3}}{2}$, $u_2 = \frac{1}{2}$

Solution. It follows from Equation (2) that

$x_0 + su_1 , y_0 + su_2$

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2$$

we have

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2\right]_{s=0}$$
$$= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2} + \sqrt{3}$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point (1, 2) in the direction of $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved. ◀

13.6.3 THEOREM

(a) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(b) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

Redoing

Example 1: (using Thm 13.6.3) $f_x = y$; $f_y = x$ ∵ $f(x, y) = xy$

$f_x(1,2) = 2$; $f_y(1,2) = 1$

$D_u(1,2) = f_x(1,2)\,u_1 + f_y(1,2)\,u_2$

$= (2)\left(\frac{\sqrt{3}}{2}\right) + (1)\left(\frac{1}{2}\right) = \sqrt{3} + \frac{1}{2}$

$u = u_1\hat{i} + u_2\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

▶ **Example 3** Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution. The partial derivatives of f are

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 - z^3, \quad f_z(x, y, z) = -3yz^2 + 1$$
$$f_x(1, -2, 0) = -4, \quad f_y(1, -2, 0) = 1, \quad f_z(1, -2, 0) = 1$$

$\therefore a = 2i + j - 2k$

$|a| = \sqrt{2^2 + 1^2 + (-2)^2}$

$= \sqrt{9}$

Since $\mathbf{a}$ is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \underbrace{\frac{2}{3}}_{u_1}\mathbf{i} + \underbrace{\frac{1}{3}}_{u_2}\mathbf{j} - \underbrace{\frac{2}{3}}_{u_3}\mathbf{k}$$

Formula (5) then yields

$$D_{\mathbf{u}}f(1, -2, 0) = \underbrace{(-4)}_{f_x}\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3 \blacktriangleleft$$

$f_y = 1$ $\quad f_z = 1$

THE GRADIENT

13.6.4 DEFINITION

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

The symbol ∇ (read "del") is an inverted delta.

Formulas (4) and (5) can now be written as

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \tag{10}$$

and

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u} \tag{11}$$

respectively. For example, using Formula (11) our solution to Example 3 would take the form

$\hat{i} \cdot \hat{i} = 1$

$$D_{\mathbf{u}}f(1, -2, 0) = \nabla f(1, -2, 0) \cdot \mathbf{u} = (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right)$$

$\hat{i}^2 = 1$

$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k} = -4\hat{i} + 1\hat{j} + 1\hat{k}$

$$= (-4)\left(\tfrac{2}{3}\right)\hat{i}^2 + \tfrac{1}{3}\hat{j}^2 - \tfrac{2}{3}\hat{k}^2 = -3$$

$\hat{j}^2 = 1$

$k^2 = 1$

PROPERTIES OF THE GRADIENT

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\| \|\mathbf{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta \qquad (12)$$

where θ is the angle between $\nabla f(x, y)$ and $\mathbf{u}$.

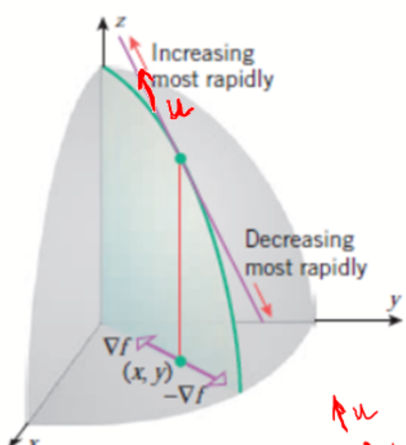
as $\cos\theta \in [-1, 1]$ & $\underbrace{\|\nabla f(x,y)\|}_{\text{Norm}} \geq 0$ hence D_u will observe the maximum value while $\cos\theta = 1$.

$\cos\theta = 1 \Rightarrow \theta = 0^\circ$

At (x, y), the surface $z = f(x, y)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x, y)\|$.

D_u will observe the minimum value while $\cos\theta = -1$ or $\theta = \pi$.

At (x, y), the surface $z = f(x, y)$ has its minimum slope in the direction that is opposite to the gradient, and the minimum slope is $-\|\nabla f(x, y)\|$.



▲ **Figure 13.6.5**

maximum occurs while $\vec{u}$ & ∇f are in the same direction

minimum occurs while $\vec{u}$ & ∇f are in the opposite direction

$\uparrow u$ — min

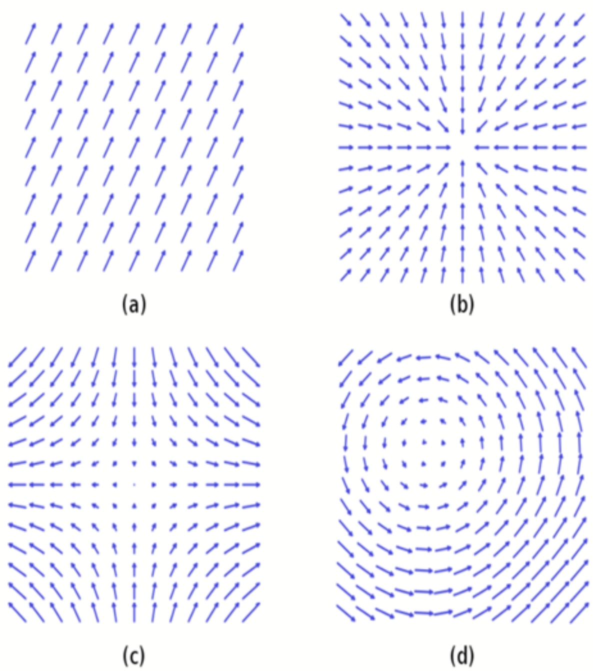
$\rightarrow -\nabla f$

Vector Field

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three dimensional) vector given by $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$).

The standard notation for the function $\vec{F}$ is,

$$\vec{F}(x, y) = f(x, y)\,\vec{i} + g(x, y)\,\vec{j}$$
$$\vec{F}(x, y, z) = f(x, y, z)\,\vec{i} + g(x, y, z)\,\vec{j} + h(x, y, z)\,\vec{k}$$



A vector field in the plane can be visualised as a collection of arrows with a given magnitude and direction each attached to a point in the plane.

Divergence of a Vector Field

- The divergence of a vector field $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ is computed as

$$\operatorname{div}(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

  In three dimensions, the divergence of the vector field $\mathbf{G}(x, y, z) = \langle G_1(x, y, z), G_2(x, y, z), G_3(x, y, z) \rangle$ is computed as

$$\operatorname{div}(\mathbf{G}) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}$$

- The divergence of a vector field measures the density of change in the strength of the vector field. In other words, the divergence measures the instantaneous rate of change in the strength of the vector field along the direction of flow.

In other words divergence is an operation on a vector field that tells us how the field behaves towards or away from a point.

Calculating Divergence at a Point

If $\mathrm{F}\,(x, y, z) = e^x \mathrm{i} + yz \mathrm{j} - yz^2 \mathrm{k}$, then find the divergence of $\mathbf{F}$ at $(0, 2, -1)$.

The divergence of $\mathbf{F}$ is

$\frac{\partial}{\partial x}\left(e^x\right) + \frac{\partial}{\partial y}\left(yz\right) - \frac{\partial}{\partial z}\left(yz^2\right) = e^x + z - 2yz.$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4$. If $\mathbf{F}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

The divergence and curl can now be defined in terms of this same odd vector ∇ by using the cross product and dot product. The divergence of a vector field $\mathbf{F} = \langle f, g, h \rangle$ is

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle.$$

$$\vec F = x^2y\,\vec i + xyz\,\vec j - x^2y^2\,\vec k$$

The curl measures the amount of rotation of the vector field at a point.
Imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf does not rotate as it moves through the fluid.
Curl F=0 => Irrotational Vector Field.

$$\operatorname{curl}\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$$

$$= -2x^2y\,\vec i + yz\,\vec k - \left(-2xy^2\,\vec j\right) - xy\,\vec i - x^2\vec k$$

$$= -\left(2x^2y + xy\right)\vec i + 2xy^2\,\vec j + \left(yz - x^2\right)\vec k$$

$$= +\vec i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -x^2y^2 \end{vmatrix} - \vec j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2y & -x^2y^2 \end{vmatrix} + \vec k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2y & xyz \end{vmatrix}$$

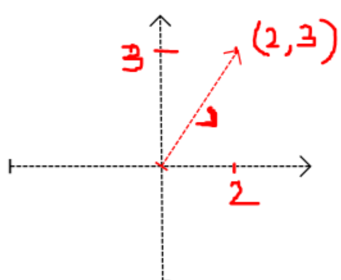
$$= \vec i\left(-2x^2y - xy\right) - \vec j\left(-2xy^2 - 0\right) + \vec k\left(yz + x^2\right)$$

$$= -\left(2x^2y + xy\right)\vec i + 2xy^2\,\vec j + \left(yz + x^2\right)\vec k$$

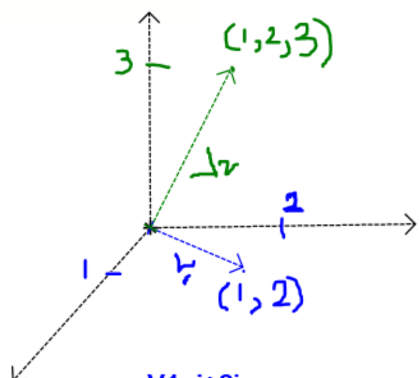
Compute $\operatorname{div}\vec F$ for $\vec F = x^2y\,\vec i + xyz\,\vec j - x^2y^2\,\vec k$

$$\operatorname{div}\vec F = \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}\left(-x^2y^2\right) = 2xy + xz$$

v=2i+3j



V2=i+2j+3k



V1=i+2j

Conservative vector fields

Definition 1.1. Let $\vec F\colon D \to R^n$ be a vector field with domain $D \subseteq \mathbb{R}^n$. The vector field $\vec F$ is said to be **conservative** if it is the gradient of a function. In other words, there is a differentiable function $f\colon D \to \mathbb{R}$ satisfying $\vec F = \nabla f$. Such a function f is called a **potential function** for $\vec F$.

Example 1.2. $\vec F(x,y,z) = (y^2z^3, 2xyz^3, 3xy^2z^2)$ is conservative, since it is $\vec F = \nabla f$ for the function $f(x,y,z) = xy^2z^3$.

Example 1.3. $\vec F(x,y,z) = (3x^2z, z^2, x^3+2yz)$ is conservative, since it is $\vec F = \nabla f$ for the function $f(x,y,z) = x^3z + yz^2$.

Ex 1.2 $\nabla f = f_x i + f_y j + f_z k = y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle = \vec F$

$\therefore \nabla f = \vec F$