

Week 9

→ Lagrange Multiplier

→ Optimization

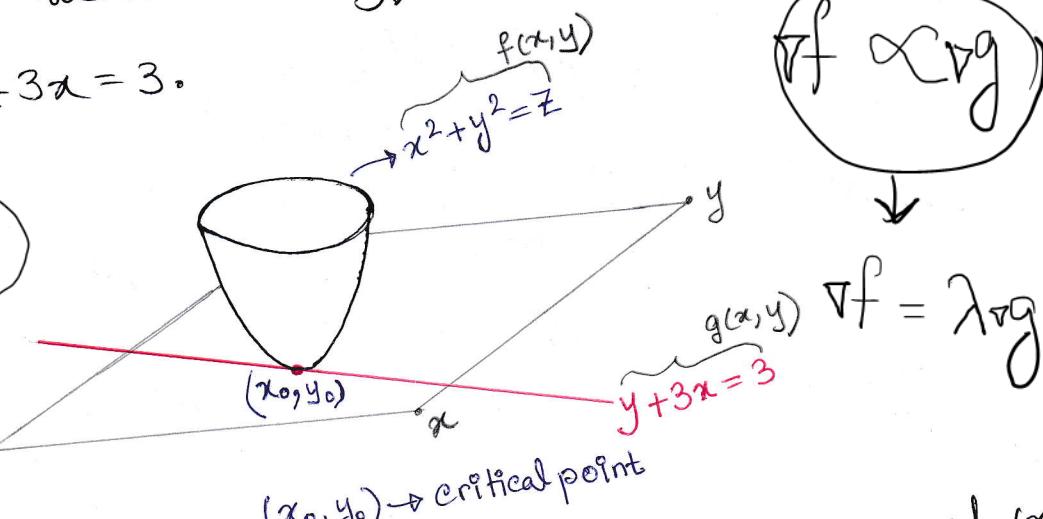
Lagrange Multiplier:Find maxima/minima of $f(x,y)$ subject to

constraint $g(x,y) = c$
 restriction/condition

For example we have $f(x,y) = x^2 + y^2$ and constraint

$$g(x,y) = y + 3x = 3.$$

$$\begin{aligned} \checkmark \rightarrow y & \quad x \propto y \\ \checkmark \quad \rightarrow x & \\ y = \left(\frac{1}{2}\right)x & \\ x = 2y & \end{aligned}$$



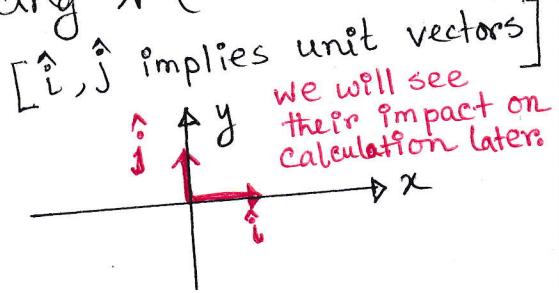
We will decide whether extremum occurs at (x_0, y_0) or not.
maximum/minimum
 specifically

$\nabla g \neq 0$ $\nabla \rightarrow$ slope/gradient
 on xy plane

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \quad \text{introducing } \lambda \text{ (a scalar multiple)}$$

$$\text{while } \nabla f(x, y) = \frac{df}{dx} \hat{i} + \frac{df}{dy} \hat{j}$$

$$\nabla g(x, y) = \frac{dg}{dx} \hat{i} + \frac{dg}{dy} \hat{j}$$



Chapter 13.9 pg 996 Anton's Calculus 10th Ed.

Exercise:

Use Lagrange multipliers to find the maximum & minimum values of f subject to the given constraint. Also, find the pts at which these extreme values occur

$$(5) f(x, y) = xy; \quad \underbrace{4x^2 + 8y^2 = 16}_{g(x, y)}$$

$$\nabla f = y\hat{i} + x\hat{j} \quad \nabla g = 8x\hat{i} + 16y\hat{j}$$

At a constraint, for relative extremum we must have

$$\nabla f = \lambda \nabla g \\ \Rightarrow y\hat{i} + x\hat{j} = \lambda (8x\hat{i} + 16y\hat{j})$$

Equating factors of like terms such as
(coefficients of \hat{i} & coefficients of \hat{j})

$$x = \lambda 16y \quad \textcircled{1}$$

$$y = \lambda 8x \quad \textcircled{i}$$

$$\lambda = \frac{y}{8x}$$

$$\lambda = \frac{x}{16y}$$

Compare

$$\frac{y}{8x} = \frac{x}{16y}$$

$$\frac{y}{x} = \frac{x}{2y}$$

$$x^2 = 2y^2 \Rightarrow y^2 = \frac{1}{2}x^2 \quad \textcircled{1}$$

2

Substitute y^2 into $g(x, y)$

$$4x^2 + 8\left(\frac{1}{2}x^2\right) = 16$$

$$4x^2 + 4x^2 = 16$$

$$8x^2 = 16$$

$$x^2 = 2$$

$$x = \pm \sqrt{2}$$

Substitute $x = \pm \sqrt{2}$ into ①

$$y^2 = \frac{1}{2} (\pm \sqrt{2})^2 = \frac{1}{2} (2) = 1$$

$$\therefore y = \pm \sqrt{1} = \pm 1$$

All possible combination of (x, y) :

| | | | | |
|----------------|------------------------------|----------------------------|------------------------------|------------------------------|
| (x, y) | $(-\sqrt{2}, -1)$ | $(\sqrt{2}, 1)$ | $(-\sqrt{2}, 1)$ | $(\sqrt{2}, -1)$ |
| $f(x, y) = xy$ | $(-\sqrt{2})(-1) = \sqrt{2}$ | $(\sqrt{2})(1) = \sqrt{2}$ | $(-\sqrt{2})(1) = -\sqrt{2}$ | $(\sqrt{2})(-1) = -\sqrt{2}$ |
| max/min | max | max | min | min |

→ $f(x, y)$ has maxima at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, 1)$

The maximum = $\sqrt{2}$

→ $f(x)$ has minima at $(-\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$

The minimum = $-\sqrt{2}$

$$\textcircled{6} \quad f(x,y) = x^2 - y^2; \quad g(x,y) = x^2 + y^2 - 25 = 0$$

$$\nabla f = 2x\hat{i} - 2y\hat{j} \quad \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$2x\hat{i} - 2y\hat{j} = \lambda (2x\hat{i} + 2y\hat{j})$$

Equating factors of like terms:

$$2x = \lambda 2x \quad \textcircled{1}$$

$$-2y = \lambda 2y \rightarrow \textcircled{2}$$

$$2x - \lambda 2x = 0$$

$$0 = 2y\lambda + 2y$$

$$2x(1-\lambda) = 0$$

$$2y(\lambda+1) = 0$$

$$2x = 0$$

$$2y = 0$$

$$x = 0$$

$$y = 0$$

Substitute either $x=0$ or $y=0$ into $g(x,y)$

Substituting $y=0$ into $g(x,y)$

$$x^2 + 0^2 - 25 = 0$$

$$x^2 = 25$$

$$x = \pm 5$$

combination of (x,y) :

All possible

| (x,y) | $(5,0)$ | $(-5,0)$ |
|----------------------|------------------|---------------------|
| $f(x,y) = x^2 - y^2$ | $5^2 - 0^2 = 25$ | $(-5)^2 - 0^2 = 25$ |
| max/min | max | max |

$f(x,y)$ has maxima at $(5,0), (-5,0)$

The maximum = 25

$$(7) \quad f(x,y) = 4x^3 + y^2 ; \quad \underbrace{2x^2 + y^2 - 1 = 0}_{g(x,y)}$$

$$\nabla f = 12x^2 \hat{i} + 2y \hat{j} \quad \nabla g = 4x \hat{i} + 2y \hat{j}$$

$$12x^2 \hat{i} + 2y \hat{j} = \lambda (4x \hat{i} + 2y \hat{j})$$

Equating factors of like terms:

$$12x^2 = 4x\lambda \quad \textcircled{1}$$

$$3x^2 = x\lambda$$

$$3x^2 - x\lambda = 0$$

$$2y = \lambda 2y \quad \textcircled{2}$$

$$2y - \lambda 2y = 0$$

$$2y(1-\lambda) = 0$$

$$2y = 0$$

$$y = 0$$

Substitute $y = 0$ into $g(x, y)$

$$2x^2 + 0^2 - 1 = 0$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

All possible combination of (x, y) :

| | | |
|-----------------------------|---|---|
| (x, y) | $(\frac{1}{\sqrt{2}}, 0)$ | $(-\frac{1}{\sqrt{2}}, 0)$ |
| $f(x, y)$ $= 4x^3 + y^2$ | $4(\frac{1}{\sqrt{2}})^3 + 0^2$ $= \sqrt{2}$ | $4(-\frac{1}{\sqrt{2}})^3 + 0^2$ $= -\sqrt{2}$ |
| max/min | max | min |

$f(x, y)$ has maximum at $(\frac{1}{\sqrt{2}}, 0)$

The maximum = $\sqrt{2}$

$f(x, y)$ has minimum at $(-\frac{1}{\sqrt{2}}, 0)$.

The minimum = $-\sqrt{2}$

$$\textcircled{8} \quad f(x,y) = x - 3y ; \quad \underbrace{x^2 + 3y^2 - 16 = 0}_{g(x,y)}$$

$$\nabla f = \hat{i} - 3\hat{j} \quad \nabla g = 2x\hat{i} + 6y\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$\hat{i} - 3\hat{j} = \lambda(2x\hat{i} + 6y\hat{j})$$

Equating Factor of Like terms:

$$1 = \lambda 2x \quad \textcircled{1}$$

$$-3 = \lambda 6y \rightarrow \textcircled{2}$$

$$\frac{1}{2x} = \lambda$$

$$-\frac{1}{2y} = \lambda$$

Compare

$$\frac{1}{2x} = -\frac{1}{2y}$$

$$x = -y$$

$$\text{or } y = -x$$

substitute either $x = -y$ or $y = -x$ into $g(x,y)$

Substituting $y = -x$ into $g(x,y) \Rightarrow x^2 + 3(-x)^2 = 16$

$$4x^2 = 16$$

$$x^2 = 4$$

$$x = \pm 2$$

Sub $x = +2$ into $y = -x \Rightarrow y = -(2) = -2$

∴ the critical pt is $(2, -2)$

Sub $x = -2$ into $y = -x \Rightarrow y = -(-2) = 2$

so the critical pt is $(-2, 2)$

| (x, y) | $(-2, 2)$ | $(2, -2)$ |
|-------------------|-------------------------|------------------------|
| $f(x,y) = x - 3y$ | $= -2 - 3(2)$ $= -8$ | $= 2 - 3(-2)$ $= 8$ |
| max/min | min | max |

$f(x,y)$ has maximum 8
at $(2, -2)$

$f(x,y)$ has minimum -8
at $(-2, 2)$

$$\textcircled{9} \quad f(x, y, z) = 2x + y - 2z ; \quad \underbrace{x^2 + y^2 + z^2 = 4}_{g(x, y, z)}$$

$$\nabla f = 2\hat{i} + \hat{j} - \hat{k} \quad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$2\hat{i} + \hat{j} - \hat{k} = \lambda (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Equating Factors of Like terms:

$$2 = \lambda 2x \rightarrow \textcircled{i} \quad 1 = \lambda 2y \rightarrow \textcircled{j} \quad -2 = \lambda 2z \rightarrow \textcircled{k}$$

$$\lambda = \frac{1}{x} \quad \lambda = \frac{1}{2y} \quad \lambda = \frac{-2}{2z} = -\frac{1}{z}$$

compare \textcircled{i}, \textcircled{j}

$$\frac{1}{x} = \frac{1}{2y}$$

$$x = 2y$$

$$y = \frac{1}{2}x$$

compare \textcircled{i}, \textcircled{k}

$$\frac{1}{x} = -\frac{1}{z}$$

$$x = -z$$

$$z = -x$$

Substitute "y" and "z" into $g(x, y)$

$$x^2 + \left(\frac{1}{2}x\right)^2 + (-x)^2 = 4$$

$$x^2 + \frac{x^2}{4} + x^2 = 4$$

$$4x^2 + x^2 + 4x^2 = 16$$

$$9x^2 = 16$$

$$x^2 = \frac{16}{9} \Rightarrow x = \pm \frac{4}{3}$$

$$x = \frac{4}{3} \Rightarrow y = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{2}{3} \quad \& \quad z = -\left(\frac{4}{3}\right) = -\frac{4}{3}$$

$$\therefore (x, y, z) = \left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right)$$

$$x = -\frac{4}{3} \Rightarrow y = \frac{1}{2}\left(-\frac{4}{3}\right) = -\frac{2}{3} \quad \& \quad z = -\left(-\frac{4}{3}\right) = \frac{4}{3}$$

$$\therefore (x, y, z) = \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$$

| | | |
|---------------------------------|---|---|
| (x, y, z) | $(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3})$ | $(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})$ |
| $f(x, y, z)$ $= 2x + y - 2z$ | $2(\frac{4}{3}) + \frac{2}{3} - 2(-\frac{4}{3})$ $= \frac{8}{3} + \frac{2}{3} + \frac{8}{3}$ $= \frac{18}{3} = 6$ | $2(-\frac{4}{3}) + (-\frac{2}{3}) - 2(\frac{4}{3})$ $= -\frac{8}{3} - \frac{2}{3} - \frac{8}{3}$ $= -\frac{18}{3} = -6$ |
| max/min | max | min |

$f(x, y)$ has maximum at $(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3})$

The maximum = 6

$f(x, y)$ has minimum at $(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})$

The minimum = -6

⑩ $f(x, y, z) = 3x + 6y + 2z$; $g(x, y, z) = 2x^2 + 4y^2 + z^2 = 70$

$$\nabla f = 3\hat{i} + 6\hat{j} + 2\hat{k} \quad \nabla g = 4x\hat{i} + 8y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$3\hat{i} + 6\hat{j} + 2\hat{k} = \lambda(4x\hat{i} + 8y\hat{j} + 2z\hat{k})$$

Equating factors of Like terms:

$$3 = 4x\lambda \quad \textcircled{i}$$

$$6 = 8y\lambda \quad \textcircled{j}$$

$$2 = 2z\lambda \quad \textcircled{k}$$

$$\lambda = \frac{3}{4x}$$

$$\lambda = \frac{6}{8y} = \frac{3}{4y}$$

$$\lambda = \frac{1}{2z}$$

compare $\textcircled{i}, \textcircled{j}$

compare $\textcircled{i}, \textcircled{k}$

$$\frac{3}{4x} = \frac{1}{2z}$$

$$3z = 4x$$

$$\boxed{z = \frac{4x}{3}}$$

$$\frac{3}{4x} = \frac{3}{4y}$$

$$\frac{1}{x} = \frac{1}{y}$$

$$\boxed{y = x}$$

Substitute "y" and "z" into $g(x, y)$

$$2x^2 + 4(x)^2 + \left(\frac{4x}{3}\right)^2 = 70$$

$$2x^2 + 4x^2 + \frac{16x^2}{9} = 70$$

$$18x^2 + 36x^2 + 16x^2 = 630$$

$$70x^2 = 630$$

$$x^2 = 9$$

$$x = \pm 3$$

$$\text{Sub } x=3 \Rightarrow y=x=3 \quad \& \quad z = \frac{4x}{3} = \frac{4(3)}{3} = 4$$

$$\therefore (x, y, z) = (3, 3, 4)$$

$$\text{Sub } x=-3 \Rightarrow y=x=-3 \quad \& \quad z = \frac{4x}{3} = \frac{4(-3)}{3} = -4$$

$$\therefore (x, y, z) = (-3, -3, -4)$$

| (x, y, z) | $(3, 3, 4)$ | $(-3, -3, -4)$ |
|----------------------------------|---------------------------|-------------------------------|
| $f(x, y, z)$ $= 3x + 6y + 2z$ | $3(3) + 6(3) + 2(4) = 35$ | $3(-3) + 6(-3) + 2(-4) = -35$ |
| max/min | max | min |

$f(x, y, z)$ has maximum at $(3, 3, 4)$

The maximum = 35

$f(x, y, z)$ has minimum at $(-3, -3, -4)$

The minimum = -35

$$(11) f(x,y,z) = xyz, \quad g(x,y,z) = x^2 + y^2 + z^2 - 1$$

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \quad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Equating factors of Like terms

$$yz = \lambda 2x \quad (i)$$

$$xz = 2y\lambda \quad (j)$$

$$xy = 2z\lambda \quad (k)$$

$$\lambda = \frac{yz}{2x}$$

$$\lambda = \frac{xz}{2y}$$

$$\lambda = \frac{xy}{2z}$$

compare (i), (j), (k)



$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z} \Rightarrow \frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z} \quad (i), (k)$$

(i), (j)

$$\frac{yz}{x} = \frac{xz}{y} \quad \left\{ \begin{array}{l} \frac{yz}{x} = \frac{xy}{z} \\ \frac{z}{x} = \frac{x}{z} \\ x^2 = z^2 \\ z^2 = x^2 \end{array} \right.$$

substitute y^2, z^2

into $g(x, y) =$

$$x^2 + (x^2) + (x^2) = 1$$

$$3x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$$2^3 = 8 \text{ out come}$$

$$y^2 = x^2 = \left(\pm \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow y = \pm \frac{1}{\sqrt{3}}$$

All possible combination of (x, y, z)

| | | | | | |
|--------------------|--|--|--|--|--------------------------------------|
| (x, y, z) | $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | Also $z = \pm \frac{1}{\sqrt{3}}$ |
| $f(x, y, z) = xyz$ | $\frac{1}{3\sqrt{3}}$ | $-\frac{1}{3\sqrt{3}}$ | $-\frac{1}{3\sqrt{3}}$ | $-\frac{1}{3\sqrt{3}}$ | |
| max/min | maximum | minimum | minimum | minimum | |
| (x, y, z) | $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | |
| $f(x, y, z) = xyz$ | $-\frac{1}{3\sqrt{3}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{1}{3\sqrt{3}}$ | $-\frac{1}{3\sqrt{3}}$ | |
| max/min | minimum | maximum | maximum | maximum | |

The maximum is $\frac{1}{3\sqrt{3}}$ and it occurs at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

The minimum is $-\frac{1}{3\sqrt{3}}$ and it occurs at

$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

$$(12) f(x, y, z) = x^4 + y^4 + z^4 ; g(x, y, z) = x^2 + y^2 + z^2 = 1$$

$$\nabla f = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k} \quad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k} = \lambda (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$4x^3 = \lambda 2x \quad (i)$$

$$4x^3 - \lambda 2x = 0$$

$$2x(2x^2 - \lambda) = 0$$

$$2x = 0 \quad 2x^2 = \lambda$$

$$(x, y, z) = (0, 0, 0)$$

$$4y^3 = \lambda 2y \quad (j)$$

$$4y^3 - \lambda 2y = 0$$

$$2y(2y^2 - \lambda) = 0$$

$$2y = 0$$

$$2y^2 - \lambda = 0$$

$$y = 0$$

$$2y^2 = \lambda \quad (k)$$

$$4z^3 = \lambda 2z \quad (k)$$

$$z = 0 \quad \text{and} \quad 2z^2 = \lambda$$

comparing (i), (j), (k)

$$\Rightarrow 2x^2 = 2y^2 = 2z^2$$

$$\Rightarrow x^2 = y^2 = z^2$$

substitute $y^2 = x^2$, $z^2 = x^2$ into $g(x, y, z)$:

$$x^2 + (x^2) + (x^2) = 1$$

$$3x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$$y^2 = x^2 = \left(\pm \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow y = \pm \frac{1}{\sqrt{3}}$$

$$z^2 = x^2 = \left(\pm \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \Rightarrow z = \pm \frac{1}{\sqrt{3}}$$

$$2^n = 2^3 = 8 \text{ possible outcome}$$

$$n = 3$$

↑
3 variables

| (x, y, z) | $f(x, y, z) = x^4 + y^4 + z^4$ | max/min |
|---|--|---------|
| $(0, 0, 0)$ | 0 | — |
| $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 \\ = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$ | maximum |
| $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |
| $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ | $\frac{1}{3}$ | " |

Optimization

Q1 At what point ∞ points on the circle $x^2+y^2=1$ does $f(x,y)=xy$ have an absolute maximum, and what is that maximum?

Given $f(x,y)=xy$

$$\nabla f = y\hat{i} + x\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\hat{i} + x\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$$

Equating factors of like terms:

$$y = \lambda 2x \quad \textcircled{1}$$

$$\lambda = \frac{y}{2x}$$

compare

$$\frac{y}{2x} = \frac{x}{2y}$$

$$y^2 = x^2$$

$$x = \pm \frac{1}{\sqrt{2}} \Rightarrow y^2 = x^2 \Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

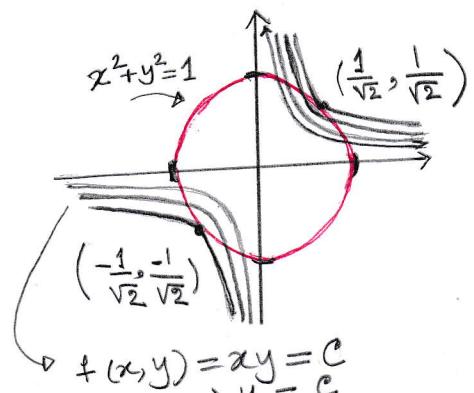
All possible outcomes of (x,y)

| (x,y) | $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ | $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ | $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ | $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ |
|---------|--|---|---|--|
| xy | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |

The function $f(x,y)=xy$ has an absolute maximum of $\frac{1}{2}$ occurring at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$$g(x,y) = x^2 + y^2 - 1 = 0$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$



substitute $y^2 = x^2$ into "g"

$$x^2 + (x^2) - 1 = 0$$

$$2x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$y^2 = x^2 \Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

$$(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

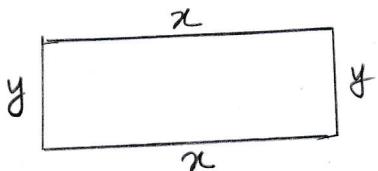
2 Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

let x = length of the rectangle
 y = width of the rectangle
 A = area of the rectangle

$$\therefore A = xy$$

$$\text{maximize } A = xy$$

$$\text{perimeter of rectangle } p = x + x + y + y \\ = 2x + 2y$$



$$x, y > 0$$

$$\therefore f(x, y) = xy$$

constraint (perimeter)

$$g(x, y) = 2x + 2y - p = 0$$

$$\nabla g = 2\hat{i} + 2\hat{j}$$

$$\nabla f = y\hat{i} + x\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\hat{i} + x\hat{j} = \lambda (2\hat{i} + 2\hat{j})$$

Equating factors of like terms:

$$y = \lambda 2 \quad \textcircled{i}$$

$$x = \lambda 2 \quad \textcircled{j}$$

$$\lambda = \frac{y}{2}$$

$$\text{compared } \frac{y}{2} = \frac{x}{2} \Rightarrow x = y$$

Substitute $x = y$ into "g"

$$2(y) + 2y = p$$

$$y = \frac{p}{4}$$

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\therefore The dimensions of the rectangle are:

$$x = \frac{p}{4}, y = \frac{p}{4}$$

3 Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.

We will find points on the sphere that minimize and maximize the square of the distance to $(1, 2, 2)$

\therefore We want to find the relative extrema of

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

subject to the constraint $x^2 + y^2 + z^2 = 36$

$$\nabla f(x, y, z) = 2(x-1)\hat{i} + 2(y-2)\hat{j} + 2(z-2)\hat{k}$$

$$\nabla g(x, y, z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$2(x-1)\hat{i} + 2(y-2)\hat{j} + 2(z-2)\hat{k} = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Equating Factor of Like terms:

$$2(x-1) = 2\lambda x \quad \textcircled{i} \quad \begin{cases} 2(y-2) = 2\lambda y \\ \lambda = \frac{y-2}{y} \end{cases} \quad \textcircled{j}$$

$$\lambda = \frac{x-1}{x}$$

$$2(z-2) = 2\lambda z \quad \textcircled{k} \quad \begin{cases} 2(z-2) = 2\lambda z \\ \lambda = \frac{z-2}{z} \end{cases}$$

Compare $\textcircled{i}, \textcircled{j}$

$$\frac{x-1}{x} = \frac{y-2}{y}$$

$$xy - 2x = xy - y$$

$$xy - xy = 2x - y$$

$$0 = 2x - y$$

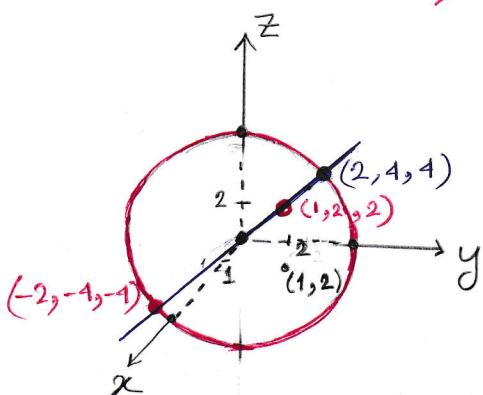
$$y = 2x$$

Compare $\textcircled{i}, \textcircled{k}$

$$\frac{x-1}{x} = \frac{z-2}{z}$$

$$xz - z = xz - 2x$$

$$z = 2x$$



Substitute $y = 2x$, $z = 2x$ into "g"

$$x^2 + (2x)^2 + (2x)^2 = 36$$

$$x^2 + 4x^2 + 4x^2 = 36 \Rightarrow 9x^2 = 36$$

$$x^2 = 4 \Rightarrow x = \pm 2$$

$$x=2 \Rightarrow y = 2x = 2(2) = 4$$

$$z = 2x = 2(2) = 4$$

$$\therefore (x, y, z) = (2, 4, 4)$$

$$x = -2 \Rightarrow y = 2(-2) = -4$$

$$z = 2(-2) = -4$$

$$\therefore (x, y, z) = (-2, -2, -4)$$

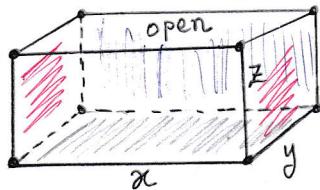
$$f(2, 4, 4) = (2-1)^2 + (4-2)^2 + (4-2)^2 = 9$$

$$f(-2, -4, -4) = (-2-1)^2 + (-4-2)^2 + (-4-2)^2 = 81$$

$\rightarrow (2, 4, 4)$ is the pt on the sphere closest to $(1, 2, 2)$

$\rightarrow (-2, -4, -4)$ is the pt on the sphere that is farthest to $(1, 2, 2)$.

④ Consider a rectangular box having volume of 32 ft^3 . Minimize the surface area of the box. Consider the box is open at the top



$$V = xyz = 32$$

$$\text{Surface Area } S = xy + 2xz + 2yz$$

Area of $xy \rightarrow$ bottom

$2yz + 2xz \rightarrow$ sides

$xz + xz \rightarrow$ front back

Minimize "f" subject to "g"

$$f = xy + 2xz + 2yz \quad g = xyz = 32$$

$$\nabla f = (y+2z)\hat{i} + (x+2z)\hat{j} + (2x+2y)\hat{k} \quad \left\{ \begin{array}{l} \nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k} \end{array} \right.$$

$$\nabla f = \lambda \nabla g$$

$$(y+2z)\hat{i} + (x+2z)\hat{j} + (2x+2y)\hat{k} = \lambda (yz\hat{i} + xz\hat{j} + xy\hat{k})$$

Equating factors of like terms:

$$y+2z = \lambda yz - \textcircled{i}$$

$$\lambda = \frac{y+2z}{yz}$$

$$\lambda = \frac{1}{2} + \frac{2}{y}$$

$$x+2z = \lambda xz - \textcircled{j}$$

$$\lambda = \frac{x+2z}{xz}$$

$$\lambda = \frac{1}{2} + \frac{2}{x}$$

$$2x+2y = \lambda xy - \textcircled{k}$$

$$\lambda = \frac{2x+2y}{xy}$$

$$\lambda = \frac{2}{y} + \frac{2}{x}$$

compare \textcircled{i}, \textcircled{j}

$$\frac{1}{2} + \frac{2}{y} = \frac{1}{2} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$\boxed{y=x}$$

compare \textcircled{i}, \textcircled{k}

$$\frac{1}{2} + \frac{2}{y} = \frac{2}{y} + \frac{2}{x}$$

$$\frac{1}{2} = \frac{2}{x}$$

$$\boxed{z = \frac{1}{2}x}$$

Substitute $y=x$, $z=\frac{1}{2}x$ into "g"

$$xyz = 32$$

$$x(x)(\frac{1}{2}x) = 32$$

$$x^3 = 64$$

$$x = \sqrt[3]{64} = 4$$

$$\therefore y=x \quad \therefore y=4$$

$$\therefore z=\frac{1}{2}x \quad \therefore z=\frac{1}{2}(4)=2$$

To minimize the surface area of the box,
the dimension should be
 $x=4$, $y=4$, $z=2$.