

MAT 110 Differential Calculus & Coordinate Geometry

Week 2

- Topics:
- ① Limit Definition \rightarrow Differentiability Test
 - ② Basic Rules of Differentiation
 - ③ Chain Rule
 - ④ Higher Derivatives
 - ⑤ Techniques of Differentiation
 \hookrightarrow Implicit Differentiation

Definition of Limit

Any function $y = f(x)$ is differentiable at $x=a$ if the following limit exists.

$$\frac{dy}{dx} = f'(x) = \lim_{x \rightarrow a}$$

$$\frac{f(x) - f(a)}{x-a} \quad \text{OR} \quad \frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

1st Principle of Derivative.

This is also known as

Example: Use the limit definition to find the derivative of

$$f(x) = x^2 - 1 \quad \text{at } x=2.$$

In general

given $f(x) = x^2 - 1$
$f(2) = 2^2 - 1$
$= 3$

Using limit definition

$$\frac{dy}{dx} = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \rightarrow 2} \frac{x^2 - 1 - 3}{x-2}$$

$$= \lim_{x \rightarrow 2} \frac{x^2 - 4}{x-2}$$

$$= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x \rightarrow 2} x+2 = 2+2 = 4$$

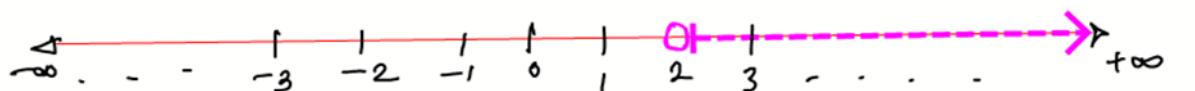
$\therefore f(x)$ is differentiable
at $x=2$.

$> < () \rightarrow$ open interval
 $\geq \leq [] \rightarrow$ closed interval $x = 2, \dots, 3, \dots$

i) $x \geq 2 \Rightarrow x = 2 \text{ or } x > 2$



ii) $x > 2 \Rightarrow x \text{ is greater than } 2$



b) $x = 2, 0, 0, 0, 0, 0, 1, 2, 3, \dots$ ✓

Differentiability & Continuity:

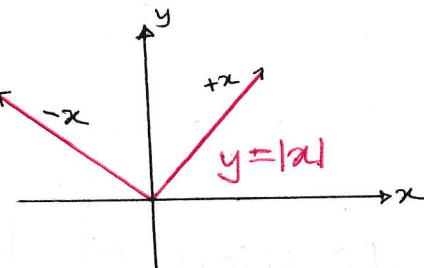
If f is differentiable at $x=a$, then f must be continuous at a .

Example:

To show that $f(x) = |x|$ is continuous at 0, show that

$$\lim_{x \rightarrow 0} |x| = |0| = 0.$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) \\ &= \lim_{x \rightarrow 0^-} (-x) \\ &= \lim_{x \rightarrow 0^+} (x) \\ &= (-0) \\ &= 0 \end{aligned}$$

$> < ()$ open interval
 $\geq \leq []$ closed interval

$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$, hence $f(x) = |x|$ is continuous at $x=0$

for $f(0)$ we should consider $f(x)=x$ since it is a closed interval

Test the Differentiability by 1st Principal of

Derivative: (Performing two sided limit test)

$$\begin{aligned} f'(x) &= f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0^-} \frac{-x - 0}{x - 0} \\ &= \lim_{x \rightarrow 0^-} -\frac{x}{x} = \lim_{x \rightarrow 0^-} -1 = -1 \end{aligned}$$

$$\begin{aligned} f'_+(x) &= f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\ &\leftarrow R.H.L = \lim_{x \rightarrow 0^+} \frac{x - 0}{x - 0} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1 \end{aligned}$$

$\therefore L.H.L \neq R.H.L$
 So two sided limit does not exist

\therefore The derivative does not exist at $x=0$

continuity $\not\Rightarrow$ Differentiability.

Differentiability \Rightarrow Continuity

Test the Differentiability of the following functions:

1) $f(x) = \begin{cases} \cos x, & x > 0 \\ -\cos x, & x < 0 \end{cases}$ at $x=0$

R.H. function
L.H. function

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{-\cos x - (\cos 0^\circ)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{-\cos x + 1}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{\frac{d}{dx}(-\cos x + 1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x \rightarrow 0^-} \frac{-(-\sin x) + 0}{1}$$

$$= \lim_{x \rightarrow 0^-} \frac{\sin x}{1} = \sin 0^\circ = 0$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{\cos x - \cos 0^\circ}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x - 0}{1}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\sin 0^\circ}{1} = 0$$

\therefore L.H.L = R.H.L

\therefore $f(x)$ is differentiable at $x=0$

$\therefore f(x)$ is continuous at $x=0$

12 $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$

$\lim_{x \rightarrow 0^-} x^2 \sin \frac{1}{x} = 0$

$\lim_{x \rightarrow 0^+} x^2 \sin \frac{1}{x} = 0$

$\lim_{x \rightarrow \infty} x^2 \sin \frac{1}{x} = \infty$

$\lim_{x \rightarrow -\infty} x^2 \sin \frac{1}{x} = \infty$

11 $\because -1 \leq \sin x \leq 1$
 $\Rightarrow -1 \leq \sin(\frac{1}{x}) \leq 1$
 $\Rightarrow -x \leq x \sin \frac{1}{x} \leq x$ [g ≤ f ≤ h]

$f(0) = 0$

$$\begin{aligned} \lim_{x \rightarrow 0} g(x) &= \lim_{x \rightarrow 0} (-x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} h(x) &= \lim_{x \rightarrow 0} (x) \\ &= 0 \end{aligned}$$

by Squeeze Theorem

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= 0 \\ \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} &= 0 \end{aligned}$$

①

$f'_-(x) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \rightarrow 0^-} \frac{x^2 \sin \frac{1}{x} - 0}{x}$

$= \lim_{x \rightarrow 0^-} x \sin \frac{1}{x}$

$= 0$ by squeeze Thm

$f'_+(x) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \rightarrow 0^+} \frac{x^2 \sin \frac{1}{x} - 0}{x}$

$= \lim_{x \rightarrow 0^+} x \sin \frac{1}{x}$

$= 0 \rightarrow$ by squeeze Thm

$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^+} f'(x)$

$f(x)$ is differentiable at $x=0$

Hence $f(x)$ is continuous at $x=0$

3

$$f(x) = \begin{cases} x \cos \frac{1}{x} & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$$

$x < 0 \text{ or } x > 0$

at $x=0$

$\begin{array}{c} x < 0 \\ x > 0 \end{array}$

$f(x) = x \cos \frac{1}{x} \quad \begin{cases} x=0 \\ f(0)=0 \end{cases} \quad f(x) = x \cos \frac{1}{x}$

11

$\therefore -1 \leq \cos x \leq 1$

$\Rightarrow -1 \leq \cos(\frac{1}{x}) \leq 1 \quad [g \leq f \leq h]$

$$\begin{aligned} \lim_{x \rightarrow 0} g(x) \\ = \lim_{x \rightarrow 0} -1 \\ = -1 \end{aligned}$$

$$\lim_{x \rightarrow 0} f(x)$$

$$\begin{aligned} \lim_{x \rightarrow 0} h(x) \\ = \lim_{x \rightarrow 0} 1 \\ = 1 \end{aligned}$$

$\lim_{x \rightarrow 0} \cos(\frac{1}{x})$ does not exist.

$$f'_+(x) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{x \cos \frac{1}{x} - 0}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{x \cos \frac{1}{x} - 0}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{x \cancel{\cos \frac{1}{x}}}{\cancel{x}}$$

$$= \lim_{x \rightarrow 0^-} \cos \frac{1}{x}$$

= any number between
between -1 & $+1$

use
squeeze
thm

= any number between
 -1 & $+1$

L.H.L \neq R.H.L

$\therefore f(x)$ is not differentiable at $x=0$



4 Let $f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ 12\sqrt{x}, & x \geq 9 \end{cases}$ i) Is $f(x)$ continuous at $x=9$?

R.H.L $\rightarrow 12\sqrt{9}$

$$\therefore f(9) = 12\sqrt{9} = 12(3) = 36$$

ii) Determine whether $f(x)$ is differentiable at $x=9$.

$x=9$ or $x>9$

i) $\lim_{x \rightarrow 9^-} x^2 - 16x$

$$\lim_{x \rightarrow 9^+} 12\sqrt{x}$$

$$= 9^2 - 16(9) = -63$$

$$\text{L.H.L} \neq \text{R.H.L}$$

$f(x)$ is not continuous at $x=9$

ii) $\lim_{x \rightarrow 9^-} \frac{f(x) - f(9)}{x - 9}$

$$= \lim_{x \rightarrow 9^-} \frac{x^2 - 16x - 12\sqrt{9}}{x - 9}$$

$$= \lim_{x \rightarrow 9^-} \frac{x^2 - 16x - 36}{x - 9}$$

$$= \lim_{x \rightarrow 9^-} \frac{x^2 - 18x + 2x - 36}{x - 9}$$

$$= \lim_{x \rightarrow 9^-} \frac{(x-18)(x+2)}{(x-9)}$$

$$= \frac{(-9)(11)}{0}$$

$$= \infty$$

$\because x < 9$
open interval

hence for the input value we need to consider $f(x) = 12\sqrt{x}$ as $x > 9$
which mean $x = 9$ or > 9 .

$x < 9$ means x is less than 9, NOT EQUAL to 9. \therefore we cannot substitute $x = 9$ into the function

$$f(x) = x^2 - 16.$$

$$\lim_{x \rightarrow 9^+} \frac{f(x) - f(9)}{x - 9}$$

$$= \lim_{x \rightarrow 9^+} \frac{12\sqrt{x} - 12\sqrt{9}}{x - 9}$$

$$= \lim_{x \rightarrow 9^+} \frac{12(\sqrt{x} - 3)}{x - 9}$$

$$= \lim_{x \rightarrow 9^+} \frac{12(\sqrt{x} - 3)}{(\sqrt{x} + 3)(\sqrt{x} - 3)}$$

$$= \lim_{x \rightarrow 9^+} \frac{12}{\sqrt{x} + 3}$$

$$= \frac{12}{6}$$

$$= 2$$

$f(x)$ is not differentiable at $x=9$

5 Let $f(x) = \begin{cases} x^2 & ; x \leq 1 \\ \sqrt{x} & ; x > 1 \end{cases}$

① Is $f(x)$ continuous at $x=1$?
 ② Determine whether $f(x)$ is differentiable at $x=1$.

① $\lim_{x \rightarrow 1^-} f(x)$
 $= \lim_{x \rightarrow 1^-} x^2$
 $= 1^2 = 1$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \sqrt{x} \\ &= \lim_{x \rightarrow 1^+} \sqrt{1} = 1 \end{aligned}$$

$$L.H.L = R.H.L$$

$f(x)$ is continuous at $x=1$

② $\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \rightarrow 1^-} \frac{x^2 - 1^2}{x - 1}$$

$$= \lim_{x \rightarrow 1^-} \frac{(x+1)(x-1)}{(x-1)}$$

$$= \lim_{x \rightarrow 1^-} x + 1$$

$$= 1 + 1$$

$$= 2$$

for $f(1)$

we will consider

$$f(x) = x^2 ; x \leq 1$$

{ if $f(x) = \sqrt{x} \rightarrow x > 1$
 contains open interval}

$$\lim_{x \rightarrow 1^+}$$

$$= \lim_{x \rightarrow 1^+} \frac{\sqrt{x} - 1^2}{x - 1}$$

$$= \lim_{x \rightarrow 1^+} \frac{(\sqrt{x} - 1)}{(\sqrt{x} + 1)(\sqrt{x} - 1)}$$

$$= \lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x} + 1}$$

$$= \frac{1}{\sqrt{1} + 1}$$

$$= \frac{1}{2}$$

$$L.H.O.L \neq R.H.O.L$$

$f(x)$ is not differentiable at $x=1$

It proves that continuity does not imply differentiability.

6 Show that

$$f(x) = \begin{cases} x^2 + 1; & x \leq 1 \\ x; & x > 1 \end{cases}$$

① is not continuous at $x = 1$
② is not differentiable at $x = 1$

TRY

Basic Rules of Derivatives

Definition: Differentiation means the rate of change of one quantity with respect to another. The speed is calculated as the rate of change of distance with respect to time. This speed at each instant is not the same as the average calculated. Speed is the same as the slope, which is nothing but the instant rate of change of the distance over a period of time.

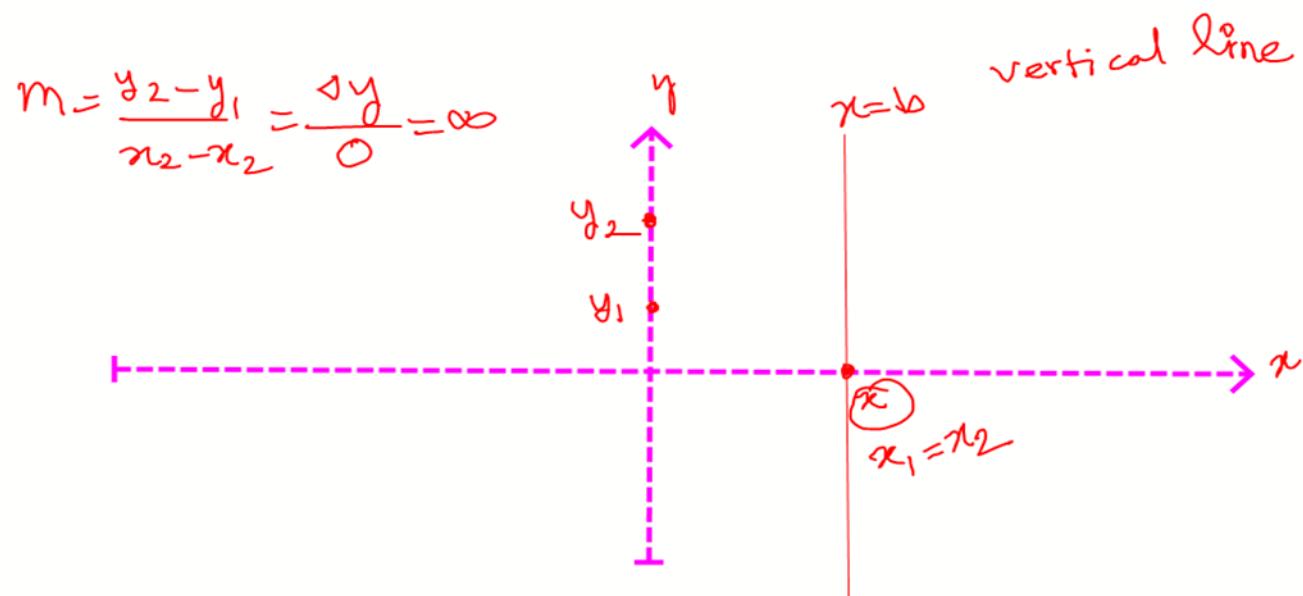
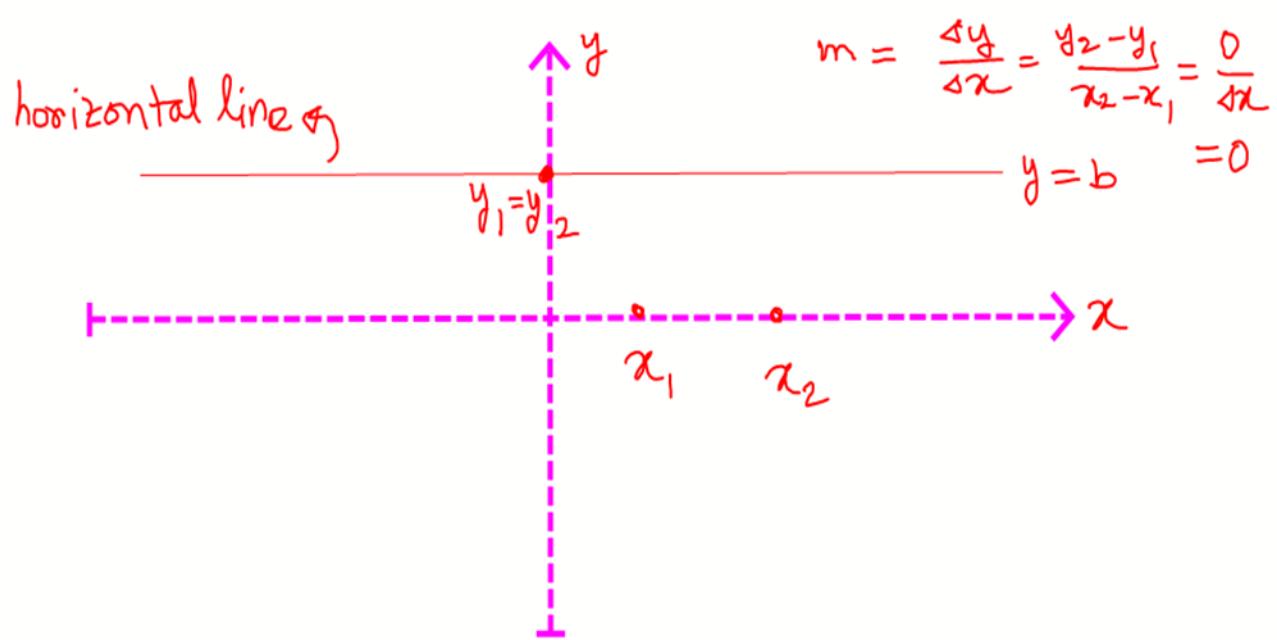
The ratio of a small change in one quantity with a small change in another quantity which is dependent on the first quantity is called differentiation.

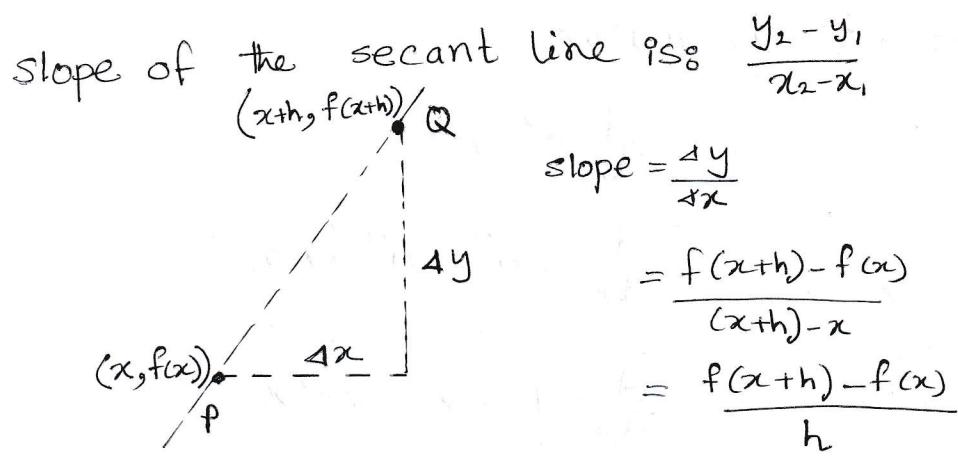
One of the important concepts in calculus is mainly focused on the differentiation of a function.

The useful tools that are determined by differentiation are:

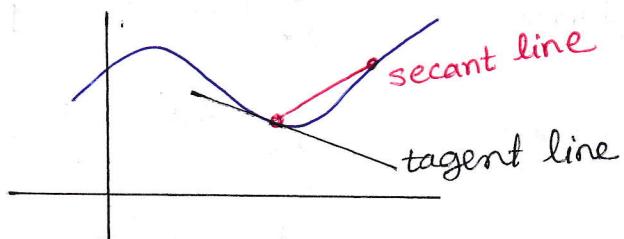
- The maximum or minimum value of a function
- The velocity and acceleration of moving objects
- The tangent (slope) of a curve

If $y = f(x)$ is differentiable, then the differentiation is represented as $f'(x)$ or $\frac{dy}{dx}$.





A secant line is a line that intersects a curve at a minimum of two distinct points.



tangent line: slope of a curve

Rules:

① Addition: $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

② Product: $\frac{d}{dx}(uv) = u \frac{d}{dx}(v) + v \frac{d}{dx}(u)$

③ Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}u - u \frac{d}{dx}v}{v^2}; v \neq 0$

④ Constant: $\frac{d}{dx}[c] = 0 ; \frac{d}{dx}[cu] = c \frac{d}{dx}(u)$

⑤ Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Important Formulae

$$\left\{ \begin{array}{l} \frac{d}{dx} c = 0 \\ \frac{d}{dx} x^n = nx^{n-1} \\ \frac{d}{dx} \sin x = \cos x \\ \frac{d}{dx} \cos x = -\sin x \end{array} \quad \begin{array}{l} \frac{d}{dx} e^x = e^x \\ \frac{d}{dx} a^x = a^x \ln a \\ \frac{d}{dx} \tan x = \sec^2 x \\ \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x = -\csc^2 x \end{array} \quad \begin{array}{l} \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{d}{dx} \sec x = \sec x \tan x \\ \frac{d}{dx} \csc x = -\csc x \cot x \\ \frac{d}{dx} \ln |x| = \frac{1}{x} \\ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \end{array} \quad \begin{array}{l} \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \\ \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2} \\ \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}} \end{array} \right.$$

Chain Rule

Let $y = f(g(x))$ composite function
 ↳ a function that is written inside another function.

$$y' = f'(g(x))g'(x)$$

Consider $y = 4 \cos(\underline{\underline{x^3}})$

$$\begin{aligned} \frac{dy}{dx} &= 4 (-\sin \underline{\underline{x^3}}) (\underline{\underline{3x^2}}) \\ &= -12x^2 \sin x^3. \end{aligned}$$

Another way to approach chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

while $y = f(g(x))$
if $y = f(u)$ and $u = g(x)$

introducing
 $\xrightarrow{g} f(g(x)) = f(u)$

Consider $y = 4 \cos x^3$

$$\text{Let } u = x^3$$

$$y = 4 \cos x^3 \\ = 4 \cos u$$

$$\boxed{\frac{du}{dx}} = 3x^2$$

$$\boxed{\frac{dy}{du}} = -4 \sin u$$

$$\begin{aligned}\frac{dy}{dx} &= \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dx}} \\ &= (-4 \sin u)(3x^2) \\ &= (-4 \sin x^3)(3x^2) \\ &= -12x^2 \sin x^3.\end{aligned}$$

Note:
 $\cos^3 x = (\cos x)^3$
 $\cos x^3 \neq (\cos x)^3$

Examples on Chain Rule:

$$\boxed{1} \quad y = \csc^3 x \\ = (\csc x)^3$$

$$\begin{aligned}\frac{dy}{dx} &= 3 \csc^2 x (-\csc x \cot x) \\ &= -3 \csc^3 x \cot x\end{aligned}$$

$$x^n = n x^{n-1}$$

$$\frac{d}{dx} [\csc x]$$

$$[2] \quad y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \times \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}}} \times \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$$

$$\sqrt[3]{a} = a^{\frac{1}{3}}$$

$$\sqrt{a} = a^{\frac{1}{2}}$$

$$\frac{a}{b} = \frac{a}{b} \times \frac{c}{c}$$

$$\sqrt{a^2} = a$$

$$= \frac{-\sqrt{(1+x^2)^2}}{\sqrt{1+2x^2+x^4-1+2x^2-x^4}} \times \frac{-4x}{(1+x^2)^2}$$

$$= \frac{-(-1+x^2)}{\sqrt{4x^2}} \times \frac{-4x}{(1+x^2)^2} = \frac{-4x}{2x(1+x^2)}$$

$$= \frac{2}{1+x^2}$$

$$= \cancel{-\frac{4x}{1+x^2}} = \cancel{-\frac{4x}{1+x^2}}$$

$$[3] \quad y = \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\}$$

$$= \left[\sin 2 \left[\tan^{-1} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \right] \right]$$

$$\frac{dy}{dx} = \cos \left\{ 2 \tan^{-1} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \right\} \cdot \frac{1}{1 + \left[\left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \right]^2} \cdot \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}-1} \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$$\frac{d(u)}{dx} v$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$= \cos \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\} \cdot \frac{1}{1 + \frac{1-x}{1+x}} \quad \left(\frac{1-x}{1+x} \right)^{-\frac{1}{2}} \cdot \frac{-1-x - 1+x}{(1+x)^2}$$

$$= \cos \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\} \cdot \frac{1}{1+x+1-x} \cdot \sqrt{\frac{1+x}{1-x}} \cdot \frac{-2}{(1+x)^2}$$

$$= \cos \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\} \cdot \frac{1+x}{2} \cdot \frac{\cancel{\sqrt{1+x}}}{\sqrt{1-x}} \cdot \frac{\cancel{(-2)}}{\cancel{(1+x)}^2 + 1} \cdot \frac{1}{\sqrt{1+x}}$$

$$= -\cos \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\} \frac{1}{\sqrt{1-x^2}}.$$

[4] $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (\sin x)^{\cos x}$ i $v = (\cos x)^{\sin x}$ ii

Then $y = u + v$ & $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$u = (\sin x)^{\cos x} \quad \text{--- } \textcircled{i}$$

$\ln u = \ln(\sin x)^{\cos x} \rightarrow$ introduce \ln on both sides

$$= \cos x \ln(\sin x)$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \cos x \left[\frac{1}{\sin x} (\cos x) \right] + (-\sin x) \ln(\sin x)$$

$$\frac{du}{dx} = u \left[\cos x \cot x - \sin x \ln(\sin x) \right]$$

$$\frac{du}{dx} = (\sin x)^{\cos x} \left[\cos x \cot x - \sin x \ln(\sin x) \right] \quad \text{--- (a)}$$

$$v = (\cos x)^{\sin x} \quad \text{--- (ii)}$$

$$[uv]' = u'v + v'u$$

$$\ln v = \ln(\cos x)^{\sin x} = \sin x \ln(\cos x)$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \cos x (\ln \cos x) + \sin x \frac{1}{\cos x} \cdot (-\sin x)$$

$$\frac{dv}{dx} = v \left[\cos x \ln(\cos x) + \tan x (-\sin x) \right]$$

$$\frac{dv}{dx} = (\cos x)^{\sin x} (\cos x \ln(\cos x) - \sin x \tan x) \quad \text{(b)}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = \text{(a)} + \text{(b)}$$

[5] $x = a \cos^3 \theta ; y = a \sin^3 \theta$; Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \quad x = f(\theta) \Rightarrow y = f(\theta)$$

$$x = a \cos^3 \theta = a (\cos \theta)^3$$

$$\frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta)$$

$$= -3a \sin \theta \cos^2 \theta$$

$$y = a \sin^3 \theta = a (\sin \theta)^3$$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta \cdot \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 3a \sin^2 \theta \cdot \cos \theta \left(\frac{1}{-3a \sin \theta \cos^2 \theta} \right)$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{-3a \sin \theta \cos^2 \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

6. $x = \sin^2 \theta$, $y = \tan \theta$, Find $\frac{dy}{dx}$

$$\frac{dx}{d\theta} = 2 \sin \theta \cdot \cos \theta$$

$$\frac{dy}{d\theta} = \sec^2 \theta$$

$$x = f(\theta) \\ y = g(\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \sec^2 \theta \cdot \frac{1}{2 \sin \theta \cdot \cos \theta}$$

$$= \frac{1}{2} \sec^2 \theta \cdot \frac{1}{\sin \theta} \cdot \frac{1}{\cos \theta}$$

$$= \frac{1}{2} \sec^2 \theta \cdot \cosec \theta \cdot \sec \theta$$

$$= \frac{1}{2} \sec^3 \theta \cosec \theta$$

$$\frac{d\theta}{dx} = \frac{1}{2 \sin \theta \cos \theta}$$

7. $x = a \sec^2 \theta$, $y = a \tan^2 \theta$; find $\frac{dy}{dx}$

$$\frac{dx}{d\theta} = 2a \sec \theta \cdot \sec \theta \tan \theta \\ = 2a \sec^2 \theta \tan \theta$$

$$\frac{dy}{d\theta} = 2a \tan \theta \cdot \sec^2 \theta \\ = 2a \sec^2 \theta \tan \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 2a \sec^2 \theta \tan \theta \cdot \frac{1}{2a \sec^2 \theta \tan \theta} \\ = 1$$

Higher Derivatives

Alternative forms of higher derivatives are shown:

First derivative: $\frac{dy}{dx}$, y' , y_1 , Dy , $f'(x)$

Second derivative: $\frac{d^2y}{dx^2}$, y'' , y_2 , D^2y , $f''(x)$

n th derivative: $\frac{d^n y}{dx^n}$, $y^{(n)}$, y_n , $D^n y$, $f^{(n)}(x)$

Let $y = x^3 + 3\sin x \rightarrow f(x)$

$$\frac{dy}{dx} = 3x^2 + 3\cos x \rightarrow 1^{\text{st}} \text{ derivative}$$

$$\frac{d^2y}{dx^2} = 6x - 3\sin x \rightarrow 2^{\text{nd}} \text{ derivative}$$

$$\frac{d^3y}{dx^3} = 6 - 3\cos x \rightarrow 3^{\text{rd}} \text{ derivative}$$

Let $y = e^{ax}$

$$y_1 = ae^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

⋮

$$y_n = a^n e^{ax} \rightarrow n^{\text{th}} \text{ derivative}$$

Techniques of Differentiation

✓ Implicit Differentiation

Implicit functions

Any function $y = f(x)$ can be presented as $f(x, y) = 0$.
 $f(x, y) = 0$ is known as Implicit function.

The technique we apply to evaluate $\frac{dy}{dx}$ from
 $f(x, y) = 0$ is known as Implicit Differentiation.

Examples on Implicit Differentiation:

1 $xy = 1$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$1 \cdot y + x \cdot \frac{dy}{dx} = 0$$

$$x \cdot \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x^2}$$

2 Find $\frac{dy}{dx}$

$$5y^2 + \sin y = x^2$$

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5 \frac{d}{dx} y^2 + \frac{d}{dx} \sin y = \frac{d}{dx}[x^2]$$

$$5[2y \frac{dy}{dx}] + \cos y \cdot \frac{dy}{dx} = 2x \frac{dx}{dm}$$

$$10y \frac{dy}{dm} + \cos y \frac{dy}{dm} = 2x$$

$$\frac{dy}{dm} (10y + \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

3 $3x^4 - \underline{x^2 y} + 2y^3 = 0$

$$12x^3 - 2x y - \underline{x^2 \frac{dy}{dx}} + 6y^2 \cdot \frac{dy}{dx} = 0$$

$$\begin{aligned} \frac{dy}{dx} (-x^2 + 6y^2) &= 2xy - 12x^3 \\ &= 2x(y - 6x^2) \end{aligned}$$

$$\frac{dy}{dx} = \frac{2x(y - 6x^2)}{6y^2 - x^2}$$

$$\boxed{4} \quad x^3 + y^3 + 4x^2y - 25 = 0$$

$$3x^2 + 3y^2 \frac{dy}{dx} + 4x^2 \frac{dy}{dx} + 8x \cdot y - 0 = 0$$

$$\frac{dy}{dx} (3y^2 + 4x^2) = -8xy - 3x^2$$

$$\frac{dy}{dx} = \frac{-(8xy + 3x^2)}{3y^2 + 4x^2}$$

$$\boxed{5} \quad x^y = y^x$$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$\frac{dy}{dx} \cdot \ln x + y \cdot \frac{1}{x} = 1 \cdot \ln y + x \cdot \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} \ln x - \frac{x}{y} \frac{dy}{dx} = \ln y - \frac{1}{x}$$

$$\frac{dy}{dx} (\ln x - \frac{x}{y}) = \ln y - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\ln y - \frac{1}{x}}{\ln x - \frac{x}{y}}$$

Examples on Higher order Derivatives:

1 $y = x^n$

$$y' = n x^{n-1}$$

$$y'' = n(n-1)x^{n-2}$$

$$y''' = n(n-1)(n-2)x^{n-3}$$

$$y^{(4)} = n(n-1)(n-2)(n-3)x^{n-4}$$

$$\vdots \\ y^{(n)} = n(n-1)(n-2)(n-3) \dots (n-(n-1)) x^{n-n}$$

$$= n(n-1)(n-2)(n-3) \dots (1)x^0$$

$$= n(n-1)(n-2)(n-3) \dots (1)$$

$$= n!$$

2 $y = (ax+b)^n$

$$y' = n(ax+b)^{n-1} a = an(ax+b)^{n-1}$$

$$y'' = an(n-1)(ax+b)^{n-2} (a) = a^2 n(n-1)(ax+b)^{n-2}$$

$$y''' = a^3 n(n-1)(n-2)(ax+b)^{n-3}$$

$$y^{(4)} = a^4 n(n-1)(n-2)(n-3)(ax+b)^{n-4}$$

$$\vdots \\ y^{(n)} = a^n n(n-1)(n-2)(n-3) \dots (n-(n-1))(ax+b)^{n-n}$$

$$= a^n n(n-1)(n-2)(n-3) \dots (1)(ax+b)^0$$

$$= a^n n(n-1)(n-2)(n-3) \dots 1$$

$$= a^n n!$$

$$3 \quad y = \log_e(ax+b)$$

$$\Rightarrow y = \ln(ax+b)$$

$$y' = \frac{a}{ax+b} = a(ax+b)^{-1} = 0! a^1 (ax+b)^{-1}$$

$$y'' = a(-1)(ax+b)^{-2}(a) = -a^2(ax+b)^{-2} = -1! a^2 (ax+b)^{-2}$$

$$y''' = 2a^2(ax+b)^{-3}(a) = 2a^3(ax+b)^{-3} = 2! a^3 (ax+b)^{-3} = (-1)^2 2! a^3 (ax+b)^{-3}$$

$$y^{(4)} = -6a^4(ax+b)^{-4} = -3! a^4 (ax+b)^{-4} = (-1)^3 3! a^4 (ax+b)^{-4}$$

$$y^{(5)} = 24a^5(ax+b)^{-5} = 4! a^5 (ax+b)^{-5} = (-1)^4 4! a^5 (ax+b)^{-5}$$

NOTE

$$(-1)^{\text{odd no.}} = -1$$

$$(-1)^{\text{even no.}} = +1$$

$$(-1)^{n+1} \quad n=1, 2, 3, 4, 5, \dots \text{ natural no.}$$

$$= 1, -1, 1, -1, 1, -1, 1, -1, 1, \dots$$

$$(-1)^{n-1} = 1, -1, 1, -1, 1, -1, 1, -1, 1, \dots$$

$$y^{(n)} = (-1)^{n-1} (n-1)! a^n (ax+b)^{-n}$$

$$4 \quad y = \frac{1}{x+a} = (x+a)^{-1}$$

$$y' = -(x+a)^{-2} = (-1)^1 1 \cdot (x+a)^{-2} = (-1)^1 1!_0 (x+a)^{-2}$$

$$y'' = +2(x+a)^{-3} = (-1)^2 1 \cdot 2 \cdot (x+a)^{-3} = (-1)^2 2!_0 (x+a)^{-3}$$

$$y''' = -6(x+a)^{-4} = (-1)^3 1 \cdot 2 \cdot 3 \cdot (x+a)^{-4} = (-1)^3 3!_0 (x+a)^{-4}$$

$$y^{(n)} = (-1)^n 1 \cdot 2 \cdot 3 \cdots n (x+a)^{-(n+1)} = (-1)^n n!_0 (x+a)^{-(n+1)}$$

$$[5] \quad y = \sin(ax+b)$$

$$y' = a \cos(ax+b) = a \sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y'' = a^2 \cos\left(ax+b+\frac{\pi}{2}\right) = a^2 \sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y''' = a^3 \cos\left(ax+b+\frac{\pi}{2}\right) = a^3 \sin\left(ax+b+\frac{3\pi}{2}\right)$$

$$y^{(n)} = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

$$\boxed{\cos x = \sin\left(x+\frac{\pi}{2}\right)}$$

$$[6] \quad y = \cos(ax+b)$$

$$y' = -a \sin(ax+b) = a \cos\left(ax+b+\frac{\pi}{2}\right)$$

$$\boxed{-\sin x = \cos\left(x+\frac{\pi}{2}\right)}$$

$$y'' = -a^2 \sin(ax+b+\frac{\pi}{2}) = a^2 \cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y''' = -a^3 \sin(ax+b+\frac{2\pi}{2}) = a^3 \cos\left(ax+b+\frac{3\pi}{2}\right)$$

$$\vdots \\ y^{(n)} = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

$$[7] \quad y = \sin^3 x$$

$$\Rightarrow y = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

$$\sin^3 x = \sin^2 x \sin x$$

$$= \frac{1}{2} (1 - \cos 2x) \sin x$$

$$= \frac{1}{2} (\sin x - \cos \cancel{2x} \sin \cancel{x})$$

$$\cos a \sin b = \frac{1}{2} [\sin(a+b) - \sin(a-b)]$$

$$y^{(n)} = \frac{3}{4} \sin\left(x+\frac{n\pi}{2}\right) - \frac{1}{4} (3^n) \sin\left(3x+\frac{n\pi}{2}\right)$$

$$= \frac{1}{2} \left[\sin x - \left\{ \frac{1}{2} (\sin 3x - \sin x) \right\} \right]$$

$$= \frac{1}{2} \left[\sin x - \frac{1}{2} \sin 3x + \frac{1}{2} \sin x \right]$$

$$= \frac{1}{2} \left[\frac{3}{2} \sin x - \frac{1}{2} \sin 3x \right]$$

$$= \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

Refer to $\sin 3x$