MAT 110 Differential Calculus & Coordinate Geometry

$\boxed{\text{Week 1}} \rightarrow$ Continued

Computing Limits (Examples)

Algebric Manipulation:

$\boxed{1}$ $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{x+1}-1} = \dfrac{0}{0}$

$a^2-b^2 = (a+b)(a-b)$

$$= \lim_{x\to 0} \frac{x}{(\sqrt{x+1}-1)}\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{(x+1)-1}$$

$$= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x}$$

$$= \lim_{x\to 0} \sqrt{x+1}+1$$

$$= \sqrt{0+1}+1 = 2$$

$\boxed{2}$ $\lim\limits_{x\to 2} \dfrac{2x^2-5x+2}{5x^2-7x-6}$

$$= \lim_{x\to 2} \frac{2x^2-4x-x+2}{5x^2-10x+3x-6}$$

$$= \lim_{x\to 2} \frac{2x(x-2)-1(x-2)}{5x(x-2)+3(x-2)}$$

$$= \lim_{x\to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

$$= \lim_{x\to 2} \frac{2x-1}{5x+3}$$

$$= \frac{2(2)-1}{5(2)+3} = \frac{3}{13}$$

$\boxed{3}$ $\lim\limits_{x\to\infty} (\sqrt{x^6+5} - x^3)$ $\qquad a^2-b^2$

$$= \lim_{x\to\infty} (\sqrt{x^6+5}-x^3)\left(\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3}\right)$$

$$= \lim_{x\to\infty} \frac{x^6+5-x^6}{\sqrt{x^6+5}+x^3}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{x^6+5}+x^3}$$

$\sqrt{x^6} = x^3$

Note $\boxed{\dfrac{n}{\infty} \approx 0}$

$$\rightarrow \lim_{x\to\infty} \frac{\frac{5}{x^3}}{\frac{\sqrt{x^6+5}+x^3}{x^3}} \qquad \frac{5}{3} = \frac{5/2}{3/2}$$

$$= \lim_{x\to\infty} \frac{5/x^3}{\frac{\sqrt{x^6+5}}{x^3}+\frac{x^3}{x^3}}$$

$$= \lim_{x\to\infty} \frac{5/x^3}{\sqrt{\frac{x^6+5}{x^6}}+1}$$

$$= \lim_{x\to\infty} \frac{5/x^3}{\sqrt{1+\frac{5}{x^6}}+1} = \frac{0}{\sqrt{1+0}+1}$$

$$= \frac{0}{2} = 0$$

$\boxed{4}$ $\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-x^3\right)$

$$=\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-x^3\right)\left(\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}\right)$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^6+5x^3}\right)^2-\left(x^3\right)^2}{\sqrt{x^6+5x^3}+x^3}$$

While solving Rational function: $f(x)=\frac{P(x)}{Q(x)}$

divide both numerator & denominator by the maximum power (degree) of x that occurs in the denominator.

$$=\lim_{x\to\infty}\frac{x^6+5x^3-x^6}{\sqrt{x^6+5x^3}+x^3}$$

$$=\lim_{x\to\infty}\frac{5x^3}{\sqrt{x^6+5x^3}+x^3}$$

$$=\lim_{x\to\infty}\frac{\frac{5x^3}{x^3}}{\frac{\sqrt{x^6+5x^3}+x^3}{x^3}}$$

$$=\lim_{x\to\infty}\frac{5}{\frac{\sqrt{x^6+5x^3}}{x^3}+\frac{x^3}{x^3}}$$

$$=\lim_{x\to\infty}\frac{5}{\sqrt{\frac{x^6+5x^3}{x^6}}+1}$$

$$=\lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{5}{x^3}}+1}$$

$$=\frac{5}{\sqrt{1+0}+1}$$

$$=\frac{5}{2}$$

$$=2\frac{1}{2}$$

Find the Limit of the following functions:

$\boxed{1}$ $f(x) = \begin{cases} x^2+1, & x>0 \\ 1, & x=0 \\ 1+x, & x<0 \end{cases}$ Find $\lim_{x\to 0} f(x)$

$R\in(-\infty, +\infty)$

$x<0$ $\boxed{x=0}$ $x>0$

$-\infty$ $f(x)=1+x$ $f(x)=x^2+1$ $+\infty$

$\lim_{x\to 0^-} 1+x$

$= 1+0$

$= 1$

$\boxed{f(0)=1}$

$\lim_{x\to 0^+} x^2+1$

$= 0^2+1$

$= 1$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} = f(0)$

$\therefore$ limit exists and $f(x)$ is continuous at $x=0$.

$\boxed{2}$ $f(x) = \begin{cases} 3x-1; & x<1 \\ 3-x; & x>1 \end{cases}$ Find $\lim_{x\to 1} f(x)$

$x<1$ $\boxed{x=1}$ $x>1$

$f(x)=3x-1$ $f(x)=3-x$

$\lim_{x\to 1^-} 3x-1$

$= 3(1)-1$

$= 2$

$\lim_{x\to 1^+} 3-x$

$= 3-1$

$= 2$

$\therefore$ L.H.L = R.H.L

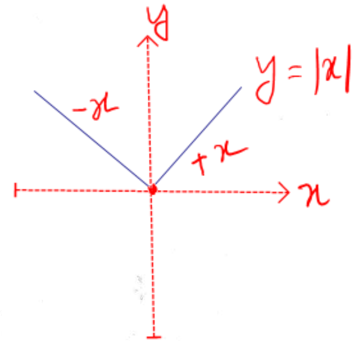
$\therefore$ Limit of this function exists

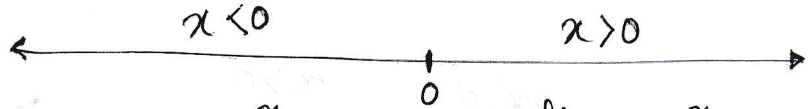
The function is continuous at $x=1$.

$\boxed{3}$ $\lim\limits_{x \to 0} \dfrac{x}{|x|} = ?$

$$f(x) = \frac{x}{|x|} = \begin{cases} \dfrac{x}{+x} \ ; & x > 0 \\ \dfrac{x}{-x} \ ; & x < 0 \end{cases} = \begin{cases} 1 \ ; & x > 0 \\ -1 \ ; & x < 0 \end{cases}$$

$$f(x) = \frac{x}{|x|} \ ; \ x \neq 0$$





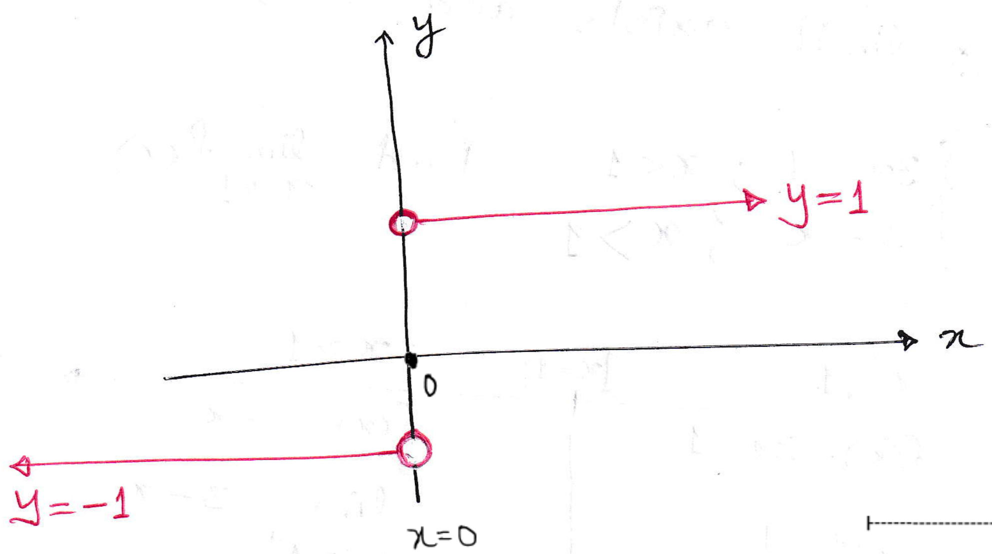
$$\lim_{x \to 0^-} \frac{x}{-x} \qquad\qquad \lim_{x \to 0^+} \frac{x}{+x}$$

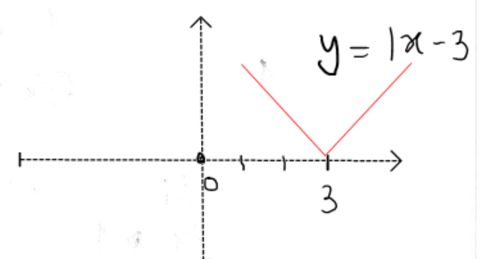
$$= \lim_{x \to 0^-} -1 \qquad\qquad = \lim_{x \to 0^+} 1$$

$$= -1 \qquad\qquad = 1$$

The function is not continuous at $x = 0$

$L.H.L \neq R.H.L$



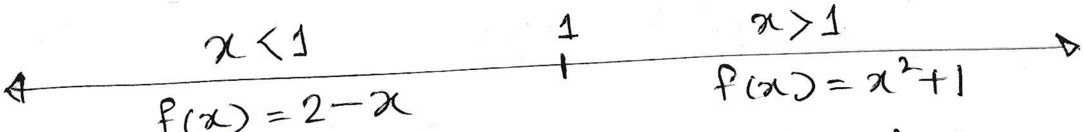


<u>TRY</u> $f(x) = \begin{cases} \dfrac{|x-3|}{x-3} \ ; & x \neq 3 \\ 0 \ ; & x = 3 \end{cases}$

Is $f(x)$ continuous at $x = 3$?

Does limit exists for $\lim\limits_{x \to 3} f(x)$?

$\boxed{4}$ $f(x) = \begin{cases} 2-x & , x<1 \\ x^2+1 & , x>1 \end{cases}$ Find $\lim\limits_{x \to 1} f(x)$



$\lim\limits_{x \to 1^-} 2-x$

$= 2-1$

$= 1$

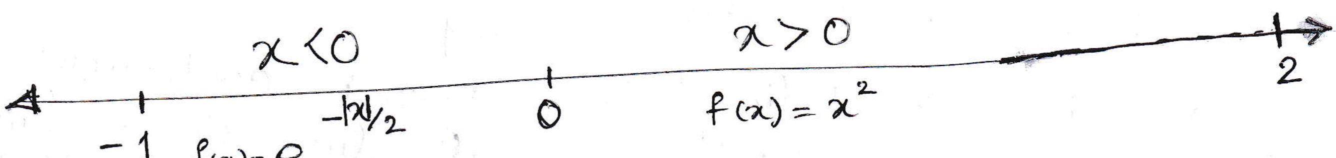
$\lim\limits_{x \to 1^+} x^2+1$

$= 1^2+1$

$= 2$

$\therefore$ L.H.L $\neq$ R.H.L $\therefore f(x)$ is not continuous at $x=1$.

$\boxed{5}$ $f(x) = \begin{cases} e^{-|x|/2} & ; -1<x<0 \\ x^2 & ; 0<x<2 \end{cases}$ Find $\lim\limits_{x \to 0} f(x)$



$\lim\limits_{x \to 0^-} e^{-|x|/2}$

$= e^{-|0|/2}$

$= e^0$

$= 1$

$\lim\limits_{x \to 0^+} x^2$

$= 0^2$

$= 0$

L.H.L $\neq$ R.H.L

$f(x)$ is not continuous at $x=0$

$\boxed{6}$ $f(x) \begin{cases} \dfrac{1}{x+2} \,; & x<-2 \\ x^2-5 \,; & -2\leq x\leq 3 \\ \sqrt{x+13} \,; & x>3 \end{cases}$

$\lim_{x\to -2}$ $\lim_{x\to 3}$

Find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 3} f(x)$

$-\infty$ $x<-2$ $\boxed{x=-2}$ $-2<x<3$ $\boxed{x=3}$ $x>3$ $+\infty$

$f(x)=\dfrac{1}{x+2}$

$\lim_{x\to -2^-} \dfrac{1}{x+2}$

$= \dfrac{1}{-2+2}$

$= -\infty$

Limit does not exist at $x=-2$ for $f(x)$ as $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

$f(x)=x^2-5$

$\lim_{x\to -2^-} x^2-5$

$=(-2)^2-5$

$=-1$

$f(x)=x^2-5$

$\lim_{x\to 3^-} x^2-5$

$=3^2-5$

$=4$

$f(x)$ is continuous at $x=3$ as $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$f(x)=\sqrt{x+13}$

$\lim_{x\to 3^+} \sqrt{x+13}$

$=\sqrt{3+13}$

$=\sqrt{16}$

$=4$

$\boxed{7}$ $f(x)= \begin{cases} x^2 \,; & x<1 \\ 2.4 \,; & x=1 \\ x^2+1 \,; & x>1 \end{cases}$ Does $\lim_{x\to 1} f(x)$ exist ?

$x<1$ $x=1$ $x>1$

$f(x)=x^2$

$\lim_{x\to 1^-} x^2$

$=1^2$

$=1$

$f(1)=2.4$

$f(x)=x^2+1$

$\lim_{x\to 1^+} x^2+1$

$=1^2+1=2$

$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) \neq f(1)$

Limit does not exist

$\boxed{8}$ $f(x) = \begin{cases} 2x+1 & ; x<1 \\ 3-x & ; x \geq 1 \end{cases}$ Find $\lim\limits_{x \to 1} f(x)$.

TRY

$\boxed{9}$ $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$ $x \neq 0 \to x<0$ or $x>0$

Is $f(x)$ continuous at $x=0$

$x<0$ | $\boxed{x=0}$ $f(0)=1$ | $x>0$

$x<0$: $f(x) = \dfrac{e^{1/x^2}}{e^{1/x^2}-1}$

$x>0$: $f(x) = \dfrac{e^{1/x^2}}{e^{1/x^2}-1}$

$$\lim_{x \to 0^-} \frac{e^{1/x^2}}{e^{1/x^2}-1}$$

$$= \lim_{x \to 0^-} \frac{e^{1/x^2}}{e^{1/x^2}\left[1 - \frac{1}{e^{1/x^2}}\right]}$$

$$= \lim_{x \to 0^-} \frac{1}{1 - \frac{1}{e^{1/x^2}}}$$

$$= \frac{1}{1 - \frac{1}{e^{1/0}}}$$

$$= \frac{1}{1 - \frac{1}{e^{\infty}}}$$

$$= \frac{1}{1 - \frac{1}{\infty}} = \frac{1}{1-0} = \frac{1}{1}$$

$$= 1$$

$$\lim_{x \to 0^+} \frac{e^{1/x^2}}{e^{1/x^2}-1}$$

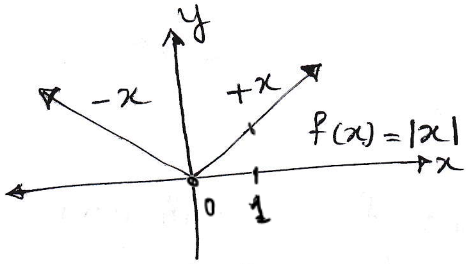
Similarly

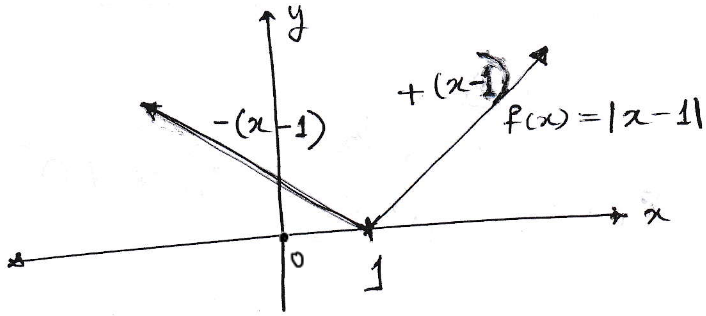
$= 1$

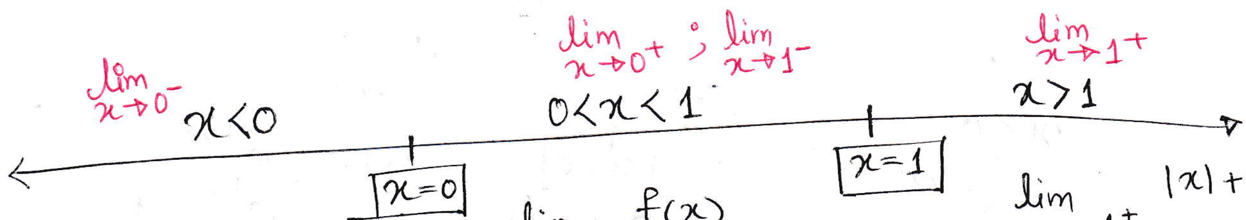
$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x=0$

$\boxed{10}$ $f(x) = |x| + |x-1|$. Find $\lim\limits_{x \to 0} f(x)$ & $\lim\limits_{x \to 1} f(x)$







$\lim\limits_{x \to 0^-} f(x)$

$= \lim\limits_{x \to 0^-} |x| + |x-1|$

$= \lim\limits_{x \to 0^-} (-x) + [-(x-1)]$

$= \lim\limits_{x \to 0^-} -x - x + 1$

$= \lim\limits_{x \to 0^-} -2x + 1$

$= -2(0) + 1$

$= +1$

$\lim\limits_{\substack{x \to 0^+ \\ x \to 1^-}} f(x)$

$$\boxed{\begin{aligned} &= \lim_{0^+, 1^-} |x| + |x-1| \\ &= +x + [-(x-1)] \\ &= +x - x + 1 \\ &= 1 \end{aligned}}$$

$\lim\limits_{x \to 0^+} 1 = 1$

$\lim\limits_{x \to 1^-} 1 = 1$

$\lim\limits_{x \to 1^+} |x| + |x-1|$

$= \lim\limits_{x \to 1^+} (+x) + (x-1)$

$= \lim\limits_{x \to 1^+} (x + x - 1)$

$= \lim\limits_{x \to 1^+} 2x - 1$

$= 2(1) - 1$

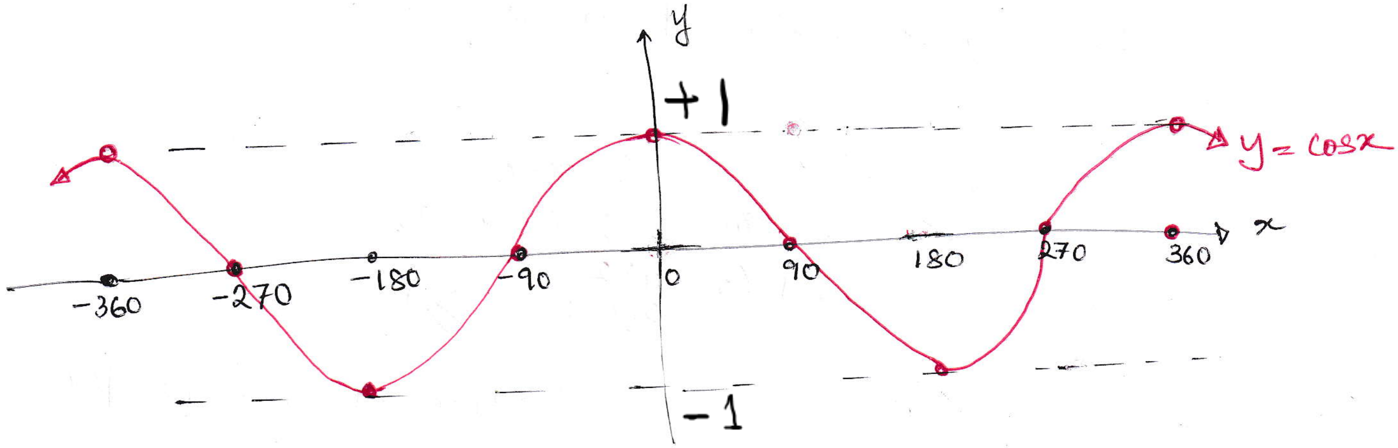
$= 1$

$\therefore f(x)$ is continuous at $x = 0$ & at $x = 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$ also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

Squeeze Theorem

$\boxed{11}$ $f(x) = \begin{cases} x\cos\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $\longrightarrow x<0 \; ; \; x>0$ Does limit exist at $x=0$?



we know ; $-1 \leq \cos x \leq 1$

$\therefore -1 \leq \cos\frac{1}{x} \leq 1$

$\therefore -1(x) \leq (x)\cos\frac{1}{x} \leq (x)\,1$

multiply (x) on all sides to form the given function

$\underbrace{-x}_{g(x)} \leq \underbrace{x\cos\frac{1}{x}}_{f(x)} \leq \underbrace{x}_{h(x)}$

$\lim_{x\to 0}(-x) = 0$ $\quad$ $\lim_{x\to 0} x = 0$

$\therefore \lim_{x\to 0} f(x) = 0$ by squeeze Theorem

$x<0$		$x>0$
$f(x) = x(\cos\frac{1}{x})$	$x=0$, $f(0)=0$	$f(x) = x(\cos\frac{1}{x})$

$\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} x(\cos\frac{1}{x})$

$= 0$

$\lim_{x\to 0^+} x(\cos\frac{1}{x})$

$= 0$

f(x) is continuous at $x=0$ as L.H.L = R.H.L

Yes the limit exists at $x=0$

TRY (i) $f(x)=\begin{cases}(x-a)\sin\frac{1}{x-a}; & x\neq a\\ 0 & ; x=a\end{cases}$

Does limit exist at $x=a$?

(ii) $f(x)=\begin{cases}1 & ; x<0\\ 1+\sin x & ; 0\leq x\leq \pi/2\\ 2+(x-\frac{\pi}{2})^2, & x\geq \pi/2\end{cases}$

↳ Refer to example $\boxed{6}$ on page $\boxed{3}$.

L' Hôspital's Rule

[Indeterminate form: $\frac{0}{0}$ or $\frac{\infty}{\infty}$]

$\boxed{1}$ $\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$ $\quad \frac{\sin 2(0)}{\sin 5(0)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$

$= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin 2x)}{\frac{d}{dx}(\sin 5x)}$ $\quad [f(g)]' = f'(g)g'$

$= \lim_{x\to 0} \frac{2\cos 2x}{5\cos 5x}$

$= \frac{2(1)}{5(1)} = \frac{2}{5}$

$\boxed{2}$ $\lim_{x\to 0} x\ln x$ $\quad 0.\ln 0 = 0.\infty = \infty$

$= \lim_{x\to 0} \frac{\ln x}{1/x}$ $\quad \to \frac{\ln 0}{\frac{1}{0}} = \frac{\infty}{\infty}$

$= \lim_{x\to 0} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\frac{1}{x})}$ $\quad \to \left[\frac{u}{v}\right]' = \frac{vu'-v'u}{v^2}$

$= \lim_{x\to 0} \frac{1/x}{\frac{1.1-0.x}{x^2}}$

$= \lim_{x\to 0} \frac{1}{x}\cdot\frac{x^2}{1}$

$= \lim_{x\to 0} x = 0.$

$\boxed{3}$ $\lim_{x \to \pi} \dfrac{\sin x}{x-\pi}$ · $\dfrac{\sin \pi}{\pi - \pi} = \dfrac{0}{0}$

$= \lim_{x \to \pi} \dfrac{\frac{d}{dx} \sin x}{\frac{d}{dx}(x-\pi)}$

$= \lim_{x \to \pi} \dfrac{\cos x}{1}$

$= \lim_{x \to \pi} \cos x$

$= \cos \pi = -1$

$\boxed{4}$ $\lim_{x \to \infty} x e^{-x}$

$= \lim_{x \to \infty} \dfrac{x}{e^x}$ $\quad \dfrac{\infty}{e^\infty} = \dfrac{\infty}{\infty}$

$= \lim_{x \to \infty} \dfrac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)}$

$= \lim_{x \to \infty} \dfrac{1}{e^x}$

$= \dfrac{1}{e^\infty} = \dfrac{1}{\infty} \approx 0$

$\left[f(g(x))\right]' = f'(g(x))g'$

$\boxed{5}$ $\lim_{x \to 0} \dfrac{\ln(\sin x)}{\ln(\tan x)}$ $\quad \dfrac{\ln(\sin 0^\circ)}{\ln(\tan 0^\circ)} = \dfrac{\ln 0}{\ln 0} = \dfrac{\infty}{\infty}$

$= \lim_{x \to 0} \dfrac{\frac{d}{dx} \ln(\sin x)}{\frac{d}{dx} \ln(\tan x)}$

$= \lim_{x \to 0} \dfrac{\frac{1}{\sin x} \cdot \cos x}{\frac{1}{\tan x} \cdot \sec^2 x}$

$= \lim_{x \to 0} \dfrac{\cos x}{\sin x} \times \dfrac{\tan x}{\sec^2 x}$

$= \lim_{x \to 0} \dfrac{\cos x}{\sin x} \times \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}}$

$= \lim_{x \to 0} \dfrac{\cos x}{\sin x} \times \dfrac{\sin x}{\cos x} \times \dfrac{\cos^2 x}{1}$

$= \lim_{x \to 0} \cos^2 x = (\cos^2 0^\circ)$

$= 1$

$\boxed{6}$ $\lim_{x \to 0} \dfrac{x - \tan^{-1} x}{x^3}$ $\quad \dfrac{0-0}{0} = \dfrac{0}{0}$

$= \lim_{x \to 0} \dfrac{\frac{d}{dx}(x - \tan^{-1} x)}{\frac{d}{dx}(x^3)}$

$= \lim_{x \to 0} \dfrac{1 - \frac{1}{1+x^2}}{3x^2}$

$= \lim_{x \to 0} \dfrac{\frac{1+x^2-1}{1+x^2}}{3x^2}$

$= \lim_{x \to 0} \dfrac{x^2}{1+x^2} \times \dfrac{1}{3x^2}$

$= \lim_{x \to 0} \dfrac{1}{3(1+x^2)}$

$= \dfrac{1}{3}$

$\boxed{7}$ $\lim_{x \to 0} (e^x + x)^{1/x}$ $(e^0+0)^{1/0} = 1^\infty = \infty$

Let $y = (e^x + x)^{1/x}$ $\quad \because f(x) = (e^x+x)^{1/x}$ [change of variable]

$\ln y = \ln (e^x + x)^{1/x}$ (take ln on both sides)

$\ln y = \frac{1}{x} \ln (e^x + x)$

$\ln y = \frac{\ln (e^x + x)}{x}$ → (take $\lim_{x \to 0}$ on both sides)

$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln (e^x + x)}{x}$ $\quad \frac{\ln (e^0+0)}{0} = \frac{\ln 1}{0} = \frac{0}{0}$

$\ln y = \lim_{x \to 0} \frac{\frac{d}{dx} \ln (e^x + x)}{\frac{d}{dx}(x)}$ apply L'Hôspital's rule

$= \lim_{x \to 0} \frac{\frac{1}{e^x + x} [e^x + 1]}{1}$

$= \frac{1}{e^0 + 0} [e^0 + 1]$

$= \frac{1}{1} [1 + 1]$

$\therefore \ln y = 2$

$\log_e y = 2$

$y = e^2$

$\lim_{x \to 0} (e^x + x)^{1/x} = \lim_{x \to 0} (y)$ $\quad \because (e^x + x)^{1/x} = y$

$= \lim_{x \to 0} e^2$ $\quad \because y = e^2$

$= e^2$

TRY

i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

ii) $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$

iii) $\lim\limits_{x \to \pi} (x-\pi)\cot x \longrightarrow \dfrac{\cos x}{\sin x}$

iv) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

v) $\lim\limits_{x \to \infty} \dfrac{3x+5}{6x-8}$

vi) $\lim\limits_{x \to \infty} \sqrt[3]{\dfrac{3x+5}{6x-8}}$

vii) $\lim\limits_{x \to \infty} \dfrac{4x^2-x}{2x^3-5}$