Increasing and Decreasing Function

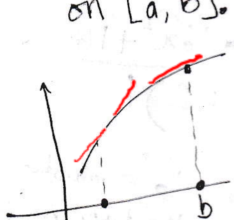
Theorem of increasing and decreasing function

Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b).

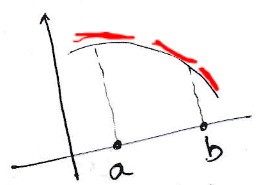
(i) If $f'(x) > 0$ for every value of x in (a,b), then f is ↑ on $[a,b]$.

(ii) If $f'(x) < 0$ for every value of x in (a,b), then f is ↓ on $[a,b]$.

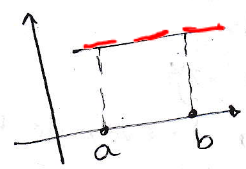
(iii) If $f'(x) = 0$ for every value of x in (a,b), then f is constant on $[a,b]$.



Graph has +ve slope
f is ↑



Graph has -ve slope
f is ↓



Graph has '0' slope
f is constant

Theorem of Concavity

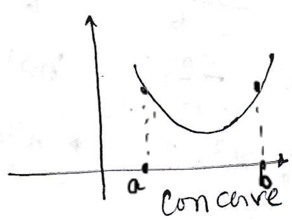
Let f be twice differentiable on an open interval I.

(a) If $f''(x) > 0$ on I, then f is concave up on I

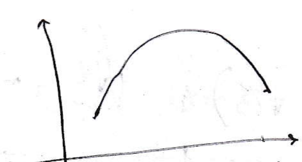
(b) If $f''(x) < 0$ on I, then f is concave down on I

$y = mx + c$
$y' = m$
$y'' = 0$

$I \in (a,b)$



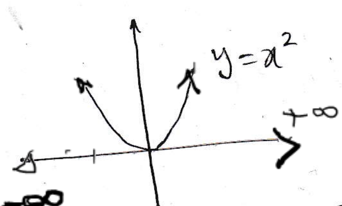
concave up



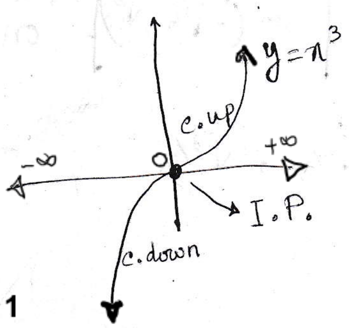
concave down

Inflection point

An inflection pt is a pt on a curve at which concavity changes the direction.



always c.up ∴ no I.P.



I.P. is $x = 0$

Exercise

1. Find the open interval on which $f(x)$ is
a) increasing b) decreasing c) c. up d) c. down
e) the x-coordinate of all inflection points.

(i) $f(x) = \dfrac{x^2}{x^2+2}$

$y = (m)x + c$

$y' = m$

$f'(x) = \dfrac{2x(x^2+2) - x^2(2x)}{(x^2+2)^4}$

$f'(x) = \dfrac{4x}{(x^2+2)^4}$

Let $f'(x) = 0$ to evaluate the turning points

$f'(x) = 0 \Rightarrow \dfrac{4x}{(x^2+2)^4} = 0$

$\Rightarrow 4x = 0$; $x^2+2 \neq 0$
$x^2 \neq -2$

$\Rightarrow x = 0$

$\mathbb{R} \in (-\infty, +\infty)$

$-\infty$ — $x<0$ $(-\infty, 0)$, $x = -1$ — $\boxed{x=0}$ — $x>0$ $(0, +\infty)$ — $+\infty$

$f'(-1) = \dfrac{4(-1)}{((-1)^2+2)^4} = -0.44$ ↓

$\therefore f'(x) < 0$

(b) f is ↓ on $(-\infty, 0)$

$f'(1) = 0.44$

$\therefore f'(x) > 0$ ↑

(a) $\therefore f$ is ↑ on $(0, +\infty)$

$f''(x) = \dfrac{4(x^2+2)^2 - 4x\{2x(x^2+2)(2x)\}}{(x^2+2)^4}$

$= \dfrac{4x^4 + 16x^2 + 16 - 16x^4 - 32x^2}{(x^2+2)^4}$

$$\boxed{\therefore f''(x) = \frac{-12x^4 - 16x^2 + 16}{(x^2+2)^4}}$$

Let $f''(x) = 0$ to evaluate the turning pt

$\dfrac{-12x^4 - 16x^2 + 16}{(x^2+2)^4} = 0$

$-12x^4 - 16x^2 + 16 = 0$ $x^2 + 2 \neq 0 \Rightarrow x^2 \neq -2$

$3x^4 + 4x^2 - 4 = 0$ $ax^2 + bx + c = 0$

$x^2 = \dfrac{-4 \pm \sqrt{16+48}}{6}$ $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-4 \pm 8}{6} = \dfrac{-12}{6}, \dfrac{4}{6}$

$x^2 = -2, \dfrac{2}{3}$ but $x^2 \neq -2$

$\therefore x^2 = 2/3 \rightarrow x = \pm\sqrt{2/3}$

$x = \pm 0.816$

$(-\infty, -\sqrt{2/3})$ | $(-\sqrt{2/3}, \sqrt{2/3})$ | $(\sqrt{2/3}, +\infty)$

$-\infty$ — $x = -1$ — $\boxed{x = -\sqrt{2/3} = -0.816}$ — $x = 0$ — $\boxed{x = \sqrt{2/3} = 0.816}$ — $x = 1$ — $+\infty$

$f''(-1) = -0.14$ $f''(x) < 0$ ↓

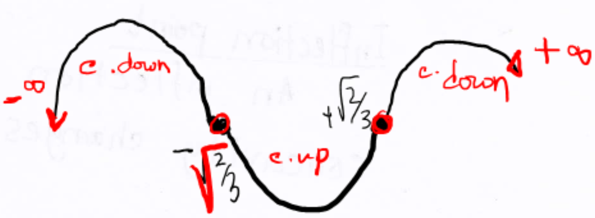
d) f is c. down on $(-\infty, -\sqrt{2/3})$

$f''(0) = 1$ ↑ $f''(x) > 0$

(c) f is c. up on $(-\sqrt{2/3}, \sqrt{2/3})$

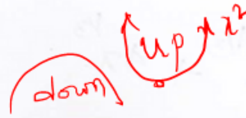
$f''(1) = -0.14$ ↓ $f''(x) < 0$

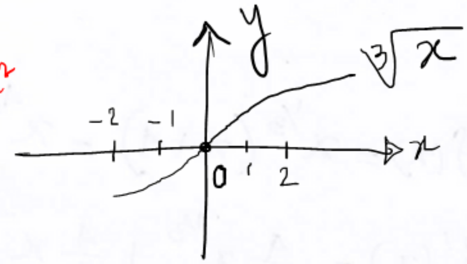
d) f is c. down on $(\sqrt{2/3}, +\infty)$

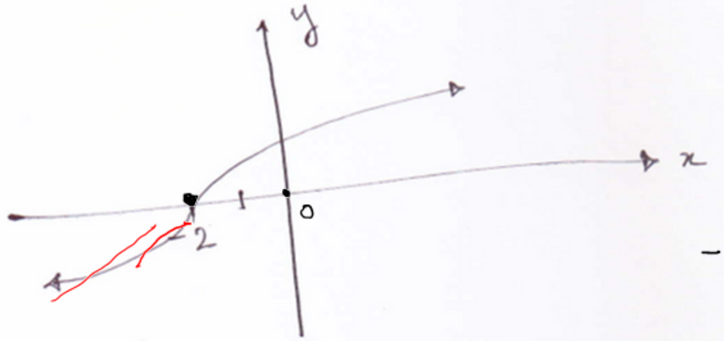


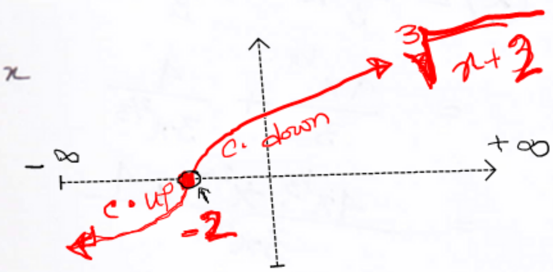
e) $x = -\sqrt{2/3}, \sqrt{2/3}$.

1(ii) $f(x)=\sqrt[3]{x+2}$









a) f is always increasing.

$f'(x) > 0$. Graph has +ve slope

b) f is never decreasing

c) f is c. up on $(-\infty, -2)$

d) f is c. down on $(-2, +\infty)$

e) x-coordinate inflection point $x = -2$.

maximum minimum

Maxima and Minima

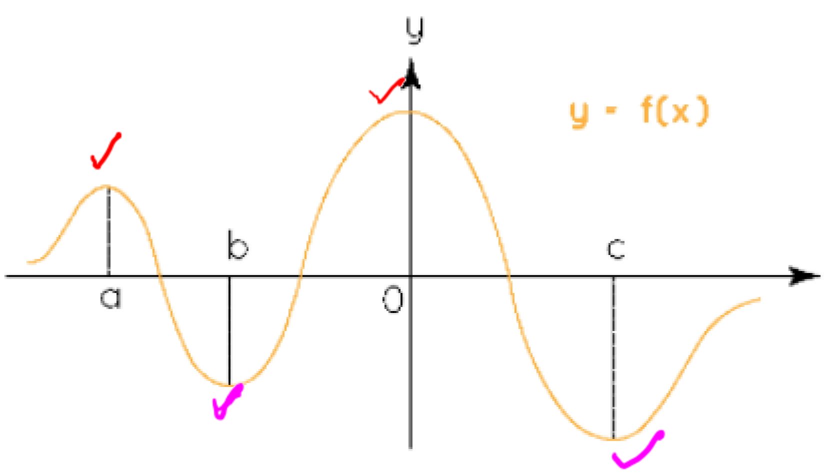
Maxima and minima are known as the extrema of a function. Maxima and minima are the maximum or the minimum value of a function within the given set of ranges. For the function, under the entire range, the maximum value of the function is known as the absolute maxima and the minimum value is known as the absolute minima.

There are other maxima and minima of a function, which are not the absolute maxima and minima of the function and are known as local maxima and local minima. Let us learn more about local maxima and minima, absolute maxima and minima, and how to find the maxima and minima of the function.

What are Maxima and Minima of a Function?

Maxima will be the highest point on the curve within the given range and minima would be the lowest point on the curve.

The combination of maxima and minima is extrema. In the image given below, we can see various peaks and valleys in the graph. At x = a and at x = 0, we get maximum values of the function, and at x = b and x = c, we get minimum values of the function. All the peaks are the maxima and the valleys are the minima.



There are two types of maxima and minima that exist in a function, which are:

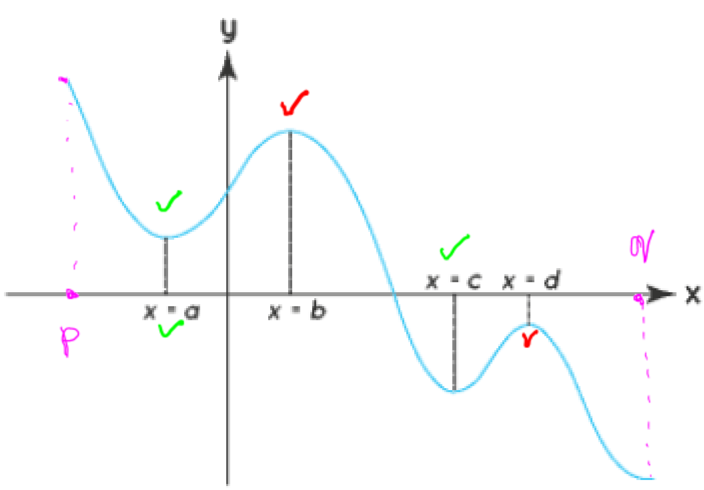
- Local Maxima and Minima
- Absolute or Global Maxima and Minima

Local Maxima and Minima (Relative Maxima and Minima)

Local maxima and minima are the maxima and minima of the function which arise in a particular interval. Local maxima would be the value of a function at a point in a particular interval for which the values of the function near that point are always less than the value of the function at that point. Whereas local minima would be the value of the function at a point where the values of the function near that point are greater than the value of the function at that point.

In the image given below, we can see that x = b and x = d, are the local maxima, and x = a and x = c, are the local minima.

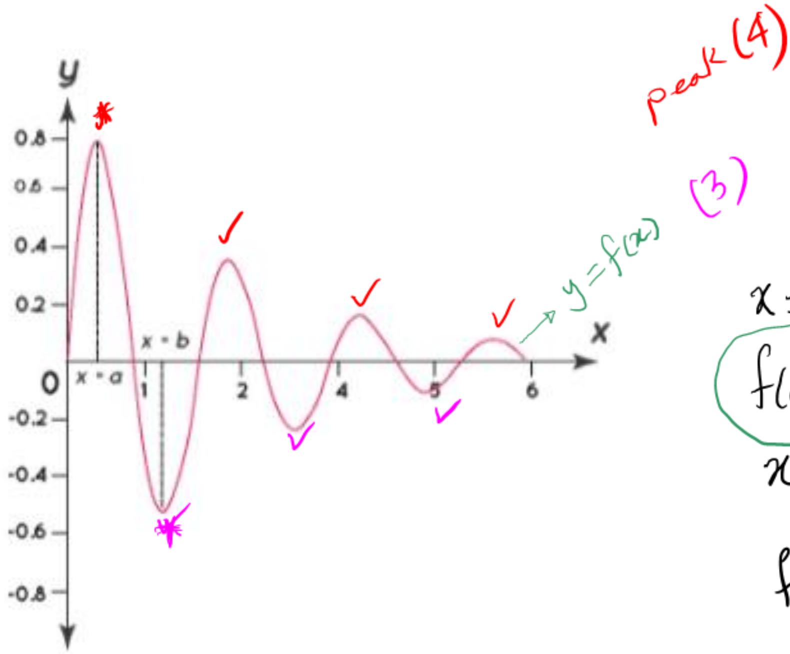
$I \in (P, Q)$



Absolute Maxima and Minima

The highest point of a function within the entire domain is known as the absolute maxima of the function whereas the lowest point of the function within the entire domain of the function, is known as the absolute minima of the function. There can only be one absolute maximum of a function and one absolute minimum of the function over the entire domain. The absolute maxima and minima of the function can also be called the global maxima and global minima of the function.

In the image given below, point x = a is the absolute maxima of the function and at x = b is the absolute minima of the function.



$x = a$

$f(a) = ?$

$x = 2 \Rightarrow f(2) = ?$

$f = b \Rightarrow f(b) = ?$

e) x ...

Critical Number (C.N.)

C.N.'s are values in the domain of f at which $f'(x) = 0$ or $f'(x) = \infty$

All C.N. are not S.P.

Stationary Point (S.P.)

S.P. are values in ~~which~~ the domain of f at which $f'(x) = 0$.

All S.P. are C.N.

Exercise $\boxed{2}$ Locate the critical numbers and identify which critical numbers correspond to stationary points.

(i) $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2) - 2x(x)}{(x^2+2)^2}$$

$$f'(x) = \frac{-x^2+2}{(x^2+2)^2}$$

$$f'(x) = 0$$

$$\frac{-x^2+2}{(x^2+2)^2} = 0 \qquad x^2+2 \neq 0$$

$$-x^2+2 = 0$$

$$x^2 = 2$$

$x = \pm\sqrt{2} \longrightarrow$ C.N. & S.P.

$x^2+2 \neq 0$ otherwise $f'(x) = \infty$

$x^2 = -2$

$x = \pm i\sqrt{2} \longrightarrow$ C.N.

(ii) $f(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3}$

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$= \frac{4x^{1/3}}{3} + \frac{4}{3x^{2/3}}$

$= \frac{4x^{1/3}\cdot x^{2/3} + 4}{3x^{2/3}}$

$= \frac{4x+4}{3x^{2/3}}$

$= \frac{4(x+1)}{3x^{2/3}}$

$f'(x) = 0$

$\frac{4(x+1)}{3x^{2/3}} = 0$

$x+1=0$; $3x^{2/3} \neq 0$ otherwise $f'(x) = \infty$

$\boxed{x=-1}$ → C.N. & S.P. $\quad$ → $\boxed{x=0}$ → C.N.

(iii) $f(x) = x^{2/3}$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f'(x) = 0$ otherwise $f'(x) = \infty$

$\frac{2}{3x^{1/3}} = 0$; $3x^{1/3} \neq 0$

$\boxed{x=0}$ — C.N.

~~otherwise~~ no. S.P.

$f'(x)$

$f'(x) > 0$ C.N. $f'(x) < 0$

$-\infty$ $(-\infty, x_0)$ ↑ $x = x_0$ $(x_0, +\infty)$ ↓ $+\infty$

x_0

$(-\infty, x_0)$ C.N. $(x_0, +\infty)$

$-\infty$ $x = x_0$ $+\infty$

$f'(x) < 0$ ↓ $f'(x) > 0$ ↑

4.2.3 THEOREM (*First Derivative Test*) *Suppose that f is continuous at a critical point x_0.*

(a) *If $f'(x) > 0$ on an open interval extending left from $\boxed{x_0}$ and $f'(x) < 0$ on an open interval extending right from $\boxed{x_0}$, then f has a relative maximum at x_0.* $x = x_0$

x_0

(b) *If $f'(x) < 0$ on an open interval extending left from $\boxed{x_0}$ and $f'(x) > 0$ on an open interval extending right from $\boxed{x_0}$, then f has a relative minimum at x_0.*

(c) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

$-\infty$ $(-\infty, x_0)$ C.N. $(x_0, +\infty)$ $+\infty$

$f'(x) > 0$ ↑ $x = x_0$ $f'(x) > 0$ ↑

OR $f'(x) < 0$ ↓ $f'(x) < 0$ ↓

4.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.* → C.N.

(a) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(b) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(c) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.* → do 1st derivative test

$f \Rightarrow f' \Rightarrow f''$

Find C.N.

do not perform 1st derivative Test

Find $f''(x)$.

Maxima Minima Examples

(1) $f(x) = 2x^3 - 9x^2 + 12x$

$f'(x) = 6x^2 - 18x + 12$ → $f''(x) = 12x - 18$

Slove $f'(x) = 0$

$6x^2 - 18x + 12 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2, 1$ → critical numbers

Solving $f'(x) = 0$ yields

$\boxed{x = 1,\ x = 2}$ → C.N.

$f''(1) = -6 < 0$

∴ f has relative maximum at $\boxed{x = 1}$

$f''(2) = 6 > 0$

∴ f has relative minimum at $\boxed{x = 2}$

$(-\infty, 1)$		$(1, 2)$		$(2, +\infty)$
$x < 1$		$1 < x < 2$		$x > 2$
$f'(0) = 0 - 0 + 12 = 12$ ↑	$\boxed{x=1}$	$f'(1.5) = -1.5$ ↓	$\boxed{x=2}$	$x = 3$, $f'(3) = 12$ ↑

$-\infty$ ... $+\infty$

∴ $f(x)$ has relative maximum at $\boxed{x = 1}$

$f(x)$ has relative minimum at $\boxed{x = 2}$

(2) $f(x) = \frac{x}{2} - \sin x, \quad 0 < x < 2\pi$

$f'(x) = \frac{1}{2} - \cos x$

Solve $f'(x) = 0$

$\frac{1}{2} - \cos x = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}\frac{1}{2}$

$= 60^\circ, 300^\circ$ over $[0, 2\pi]$

$f''(x) = \sin x$

$f''(60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2} > 0$

$f(x)$ has relative min at $x = 60^\circ$

$f''(300^\circ) = \sin 300^\circ = -\frac{\sqrt{3}}{2} < 0$

∴ $f(x)$ has relative max at $x = 300^\circ$

$x < 60^\circ$		$60^\circ < x < 300^\circ$		$x > 300^\circ$
$f'(30^\circ) = \frac{1}{2} - \cos 30^\circ = -0.37$ ↓	$\boxed{x = 60^\circ}$	$f'(120^\circ) = 1$ ↑	$\boxed{x = 300^\circ}$	$f'(310^\circ) = -0.14$ ↓

∴ $f(x)$ has relative maximum at $x = 300^\circ$ & relative minimum at $x = 60^\circ$

(3) Given $f'(x) = x^3(x^2-5) = x^5 - 5x^3$

solve $f'(x) = 0$ for C.N.

$x^3(x^2-5)=0$

$x^3=0$ $\quad$ $x^2-5=0$

$x=0$ $\quad$ $x = \pm\sqrt{5}$ $\quad$ C.N. are $x = -\sqrt{5}, 0, +\sqrt{5}$

$f'(x) = x^5 - 5x^3$

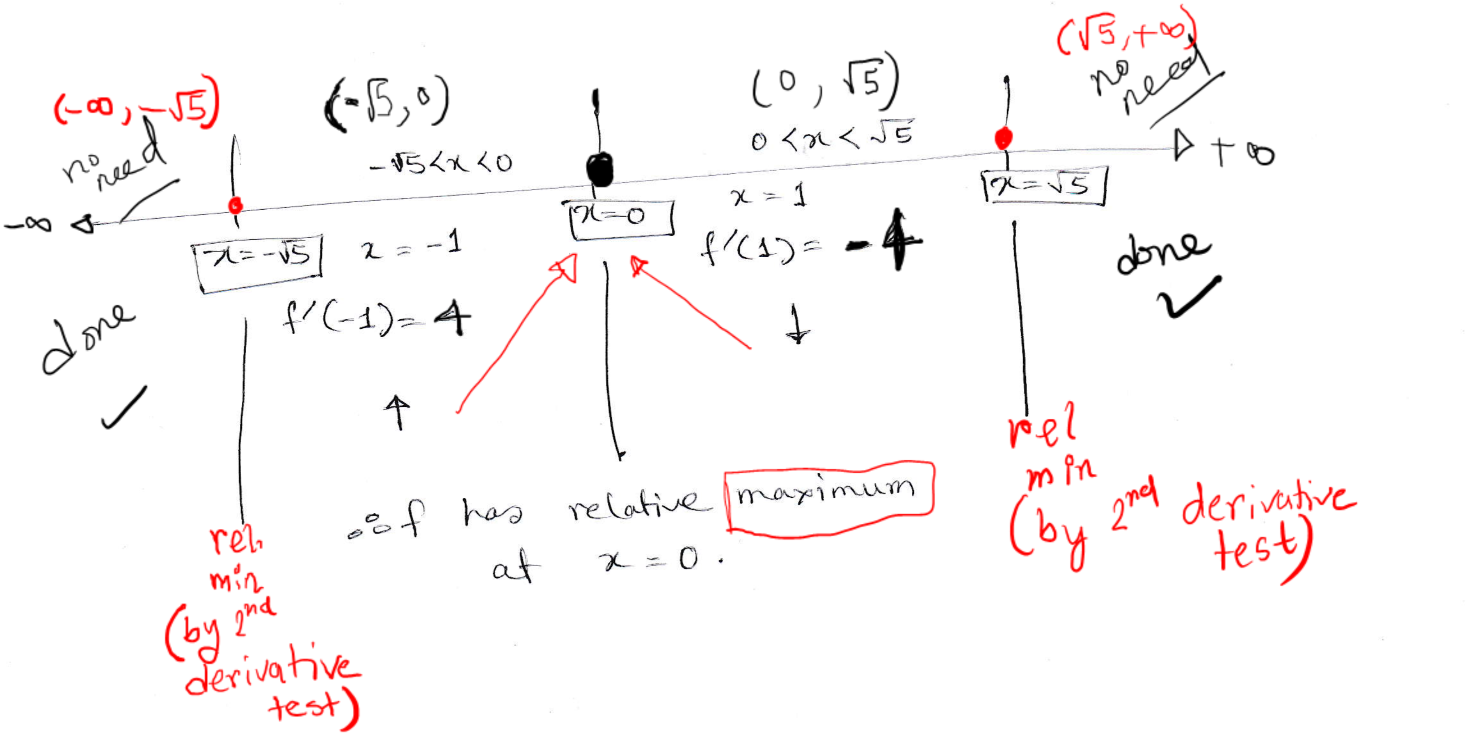
$f''(x) = 5x^4 - 15x^2$

$f''(0) = 0$ the test is inconclusive $\rightarrow$ do 1st derivative test

$f''(-\sqrt{5}) = 50 > 0$ f has relative minimum at $x = -\sqrt{5}$

$f''(\sqrt{5}) = 50 > 0$ f has relative minimum at $x = \sqrt{5}$

Do 1st derivative test for $x = 0$



$\therefore$ f has relative $\boxed{\text{maximum}}$ at $x = 0$.