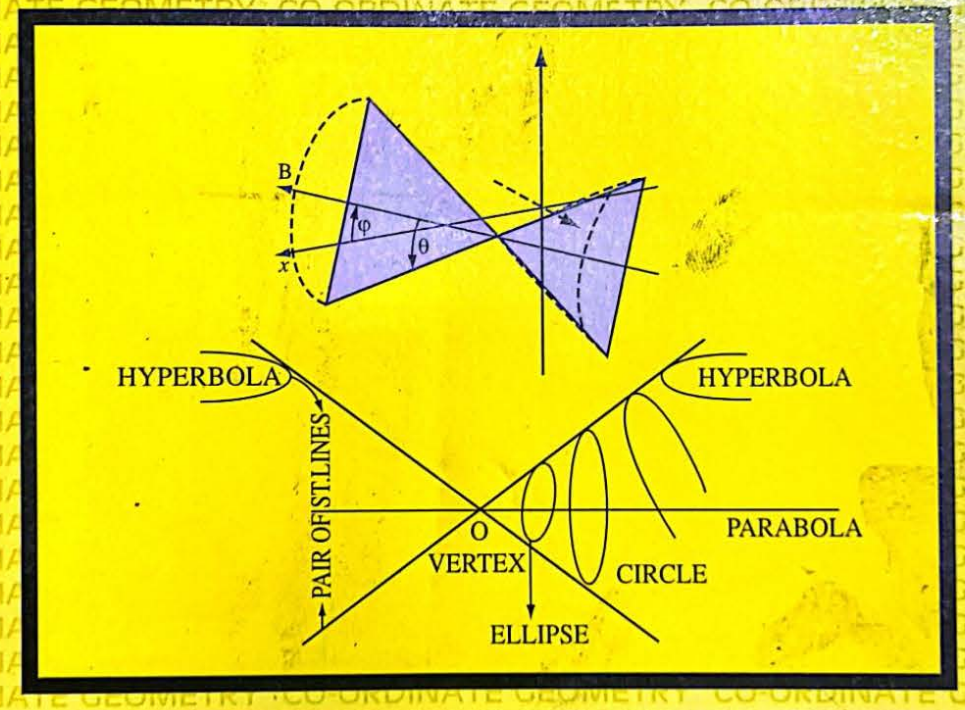
A TEXT BOOK ON CO-ORDINATE GEOMETRY

[TWO AND THREE DIMENSIONS]

WITH VECTOR ANALY



RAHMAN & BHATTACHARJEE

CONTENTS

Chap.	Subject	Page
	Conic Sections in Vector Co-ordieate	1-8
	PLANE CO-ORDINATE CREMETRS	
1.	Co-ordinates	3
2.	Polar Co-ordinates	8
3.	The Straight Lines	11
4.	Change of axes	20
5.	Pairs of Straight Lines	32
6.	General Equation of Second degree	56
7.	Circles	102
8.	System of circles	120
9.	Parabola	133
10.	Ellipse	157
11.	Hyperbola	182
	SOLID GEOMETRY	
1.	Rectangular co-ordinates	3
2.	The Plane	27
3.	The Straight Line	51
4.	Sphere	78
5.	Surfaces-Eillipsoid	98
6.	Other Systems of co-ordinates	119
7.	The cone	125
8.	The Cylinder	147
9.	Hyperboloids and Paraboloids	153
10.	General Equation of Second Degree	162
	(i) Use $\phi-\theta$ method in determining Conics	
11.	Plane Section of Central conicoid	178
12.	Generating lines	192
	Miscellaneous Exercise	203
	Oblique coordinate	208
	VECTOR ANALYSIS	
1.	Introduction of vector Analysis	3
2.	Sclar product or dot product, Solution of Vector Equations	17
3.	Applications of Vectors in Geometry	49
4.	Differentiation of Vectors	79
	Point Functions	
	Formulate involving	
	Physical significance of the	
	gradient of a scalar field	
	divergence of a vector field	
	curl of a vector field	
	Classification of Vector Fields	
	Differential Geometry	

CO-ORDINATES 1

Art. 1. Definition : *Plane Co-ordinate Geometry is that branch of Geometry in which we use two numbers, called co-ordinates to indicate the position of a point in a plane. Co-ordinate Geometry is more powerful than ordinary geometry, because we can use the methods of Algebra in it.*

Art. 2. Rectangular Co-ordinates : In earlier courses in Algebra and Trigonometry, use was made of rectangular co-ordinates. In this system the plane is divided into four quadrants by two perpendicular lines 'XOX' and 'YOY' intersecting at a point O.

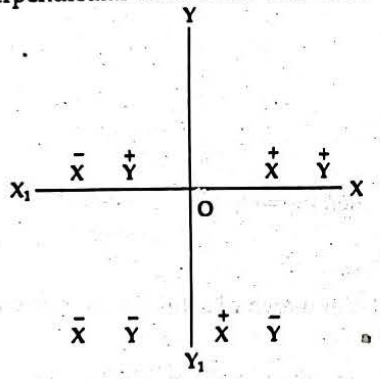


Fig : 1

The straight lines OX and OY with reference to which the co-ordinates are measured are called the axes of reference. OX is called the axis of Y. The point O the intersection of the axes is called the origin.

The distance from the Y-axis is called the X co-ordinate or abscissa of the point. The distance from the X axis is called the Y co-ordinate or ordinate of the point. The two distances taken together are called the co-ordinates of the point and are represented by the symbol. (x, y). Abscissas are positive when measured of the right of the Y axis and negative when measured to left. Ordinates are positive when measured above the X-axis, negative if below.

When a set of points whose co-ordinates are given are to be plotted by a suitable scale is chosen to mark the co-ordinates of points on the plane with reference to the co-ordinate axes already indicated.

Art. 3. Oblique Axes : When the axes of co-ordinate axes not at right angles, they are said to be obliques axes, Sometimes axes inclined to one another at an acute or obtuse angle are more convenient.

Let OX, OY be two straight lines inclined to one another at an angle ω. Let (x, y) be the oblique co-ordinates of any point P referred to OX, OY which are determined as follows. Draw PM parallel to OY to meet OX in. M; PM is called the ordinate and OM, the abscissa of point P repectively. Then OM and PM are together called the co-ordinates of P which is (x, y).

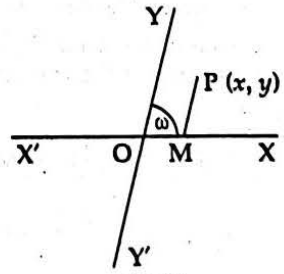


Fig : 2

The convention of signs of co-ordinate according as the point is in one or other of the four quadrants is the same as in the case of rectangular system.

Notes : The co-ordinates x, y are often called cartesian co-ordinates after the name of the discoverer, Des Cartes, a French Mathematician (1595-1695) to distinguish them from other kinds of co-ordinates which are also used to locate the position of a point in the plane.

Art. 4. Distance between two points $P_1(x_1, y_1)$ and $P_2(x_2\ y_2)$ is P_1P_2

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Cor. The distance of the point $P(x, y)$ from the orgin (0, 0)

$$OP = \sqrt{(x^2 + y^2)} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Art. 5. Section Ratio : The co-ordinate of a point $R(x, y)$ which divides the join of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in a given ratio $m_1 : m_2$

(i) Internally are

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

(ii) Externally are

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Co. 1. If $R(x, y)$ is the middle point of the line PQ; then $m_1 = m_2$

$$\therefore\ x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Ex. 1. Determine the co-ordinates of the points $R(x, y)$ which divides the join of points $P(-2, 1)$ and $Q(3, -4)$ externally in the ratio 8 : 3

$R(x, y)$ must be outside segment PQ (external division), then by (2) of Art 5 we have

$$x = \frac{8.3 - 3(-2)}{8 - 3} = 6 \text{ and } y = \frac{8(-4) - 3(1)}{8 - 3} = -7 \qquad \text{Ans. } R(6, -7)$$

Art. 6(a) Area of triangle :

Let ABC be a triangle

whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectivly.

Complete the construction as in Fig (3)

Area of the triangle ABC $= \Delta =$ Area of the trapezium ALMC + Area of the trapezium CMNB—Area of the trapezium ALNB

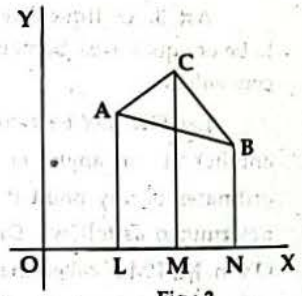


Fig : 3

$$\Delta = \frac{1}{2}(AL + CM)\,LM + (CM + BN)\,MN - \frac{1}{2}(AL + BN)\,LN$$

$$= \frac{1}{2}\{(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) + (y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

In the determinant form $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\text{or, } 2\Delta = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_1 & y_1 \end{vmatrix}$$

Note **1.** Another form which is convenient to use the area of a triangle, is especially useful when the areas of polygons of more than three sides are to be determind, is in the form of an array.

The area of the triangle is

$$\Delta = \text{Area} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1) \quad \ldots \quad \ldots \quad (1)$$

Write down the co-ordinates of three points and again of the first and cross them by arrows as shown in Fig. (4) The downwards directions are regarded positive and upwards direction as negative. Now take the sum of all. The sum is the area of the polygons in absolute value.

If the three points are collinear the, $\Delta = 0$

Note : **2.** The convention of the sign attached to the areas must be duly taken care of. The student will find that the formula (1) gives a positive result when the points (x_1, y_1) (x_2, y_2) and (x_3, y_3) are sitiuated in such a way that in going round the triangle from the first point to the 2nd, form the 2nd to the 3rd etc. the area lies on the left hand side of observer. If the area lies on the right, the result will be negative. In all numerical cases the student should disregard the sign of the area, and give the positive value as final result. But if the area of a quadrilateral found by splitting up the quadrilateral into two triangles and adding up their areas without examining their signs it should be made certain that both triangles are on the left hand side when we travel along the perimeter.

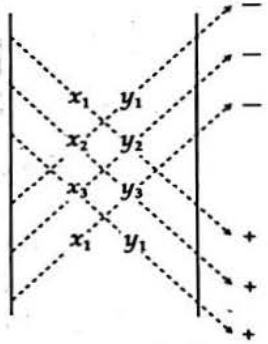


Fig : 4

Art. 6 (b) Find the area of a quadrilateral whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$

The required area

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} \quad \text{By the Art. 6 (a) Note 1}$$

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)] \quad \ldots \quad (2)$$

Art 6. (c) Find the area of a polygon of n sides the co-ordinates of whose angular points taken in order, are

$(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$

The ...

...e required area

$$S = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) \ldots + (x_n y_1 - x_1 y_n)] \quad \ldots \quad \ldots \quad (3)$$

Cor. The co-ordinates of the centroid G (x, y) of a triangle whose vertices are A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) are

$$x = \frac{x_1 + x_2 + x_3}{3} \quad y = \frac{y_1 + y_2 + y_3}{3} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Ex. 2 Find the area of the triangle whose vertices are (2, 3), (5, 7), (−3, 4)

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 \\ 5 & 7 \\ -3 & 4 \\ 2 & 3 \end{vmatrix} = \frac{1}{2}[2.7 + 5.4 + (-3)3 - 2.4 - (-3)(7) - 5.3]$$

$= 11.5$ square units.

Ex. 3 Find the area of a pentagon whose vertices are (−5, −2), (−2, 5), (2, 7), (5, 1), (2, −4)

$$S = \frac{1}{2}\begin{vmatrix} -5 & -2 \\ -2 & 5 \\ 2 & 7 \\ 5 & 1 \\ 2 & -4 \\ -5 & -2 \end{vmatrix} = \frac{1}{2}[(-5)5 + (-2)(7) + (2)(1) + (5)(-4) + (2)(-2) - (-5)(-4) - (2)(1) - (5)(7) - (2)(5) - (-2)(-2)]$$

$$= \frac{1}{2}(-132) = -66$$

Ans. 66 square units

If the vertices are taken in counter clock wise order around the polygon, the sign of the area will be + ve; if taken in a clockwise direction, the sign will be − ve.

EXERCISE I

1. Find the areas of the triangles whose vertices are

(a) (−3, 4,), (6, 2), (4, −3) ... ... Ans. 24.5

(b) $(a, b + c), (b, c + a), (c, a + b)$ ... ... Ans. 0

2. If the three points (−1, 2) (2, −1), (h, 3) are collinear; show that $h = -2$

3. For what value of k are the points (2, 3), (−4, −6) and $(k, 12)$ collinear? Ans. $k = 8$ (D. U. 1957)

4. Find the area of the quadrilateral whose angular points are

(a) (1, 1), (3, 4), (5, −2), (4, −7) Ans. 20.5

(b) (1, 2), (6, 2) (5, 3), (3, 4) Ans. 5.5

(c) (2, 4), (3, 2), (8, 4), (7, 6) Ans. 12

5. If the area of the quadrilateral, whose angular points A, B, C, D taken in order are (1,2), (−5, 6) (7, −4) and $(k, -2)$ be zero, find the value of k Ans. 3

6. Show that the area of the quardilateral, whose vertices taken in order, are (a, 0), (−b, 0), (0, −b) is zero, $(a, b > 0)$ Explain the result with the help of a diagram.

7. Show that the three points (4, 2), (7, 5) (9, 4) lie on a right line.

8. Find the areas of the polygons whose vertices are given

(a) (0, 4), (1, −6), (−2, −3) and (−4, 2) Ans. 25.5

(b) (1, 5) (−2, 4), (−3, −1), (2, −3) (5, 1) Ans. 40

(c) (1, 3), (4, 1), (5, 3), (3, 2), (2, 4) Ans. 35

9. If the area of the quardrilateral formed by the points (1, 3), (2, −5), (6, −2) and $(5, k)$ taken in order be 30, show that $k = 4$.

10. Find the co-ordinates of the orthocentre of the triangle whose angular points are (1, 0), (2, −4), and (−5, −2) R. U 1962

Ans. (11 −13, −7/13)

11. Given the vertical angle and the sum of the sides of a triangle, find the locus of a point where the base is cut in a given ratio. Ans. $2(m+n)^2(1+\cos\theta)(xy - 2k(m+n)(mx+ny)k^2 mn = 0$

ration $= m : n$. $\angle AQB = \theta$

12. If the points $(a, b), (a', b'), ((a - a'), (b - b'))$ are collinear, show that their joint passes through the orgin and that $ab' = ab'$,

??

13. The co-ordinates of A, B, C, D are (6, 3), (−3, 5) (4, −2)

and $(x, 3x)$ respectively and $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$

When Δ is the area of the triangle. Find x. Ans. 11/8

14. In what ratio is the straight line joining the pts. (3, 4) and (8, 1) divided by the x axis? Find the abscissa of this point on the x axis. Ans. 4 : − 1, 29/3 (D. U. 1955)

POLAR CO-ORDINATES (2)

Art. 8. Polar Co-ordinates : In using the cartesian co-ordinates x, y, we indicate the position of a point in a plane in terms of its distances from two lines. It is sometimes preferable to show its location in terms of its distance from a fixed point and its direction from a fixed line through this point. The co-ordinates of points in this system are called **Polar co-ordinates.**

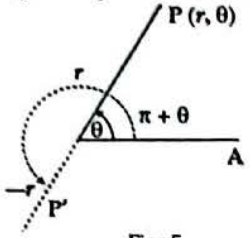


Fig : 5

Let O be a fixed point and OA a fixed straight line. Let P be any point. Then the length of OP usually denoted by r, and the measure of the $\angle AOP$, usually denoted by θ are the polar co-ordiante of P.

O is called the Pole, or the orgin and OA is called the initial line or the directed line OA is called the **Polar axis.** r is called the radius vector and θ is called the vectorial angle.

Thus the polar co-ordinates of P are expressed as (r, θ). The vectorial angle is said to be positive when measured in the anticlockwise direction and negative in the clockwise direction. To find the position of P, first measure the vectorial angles θ and then cut off the radius vector equal to r.

In polar co-ordinates the same point is represented by an indefinitely large number of sets of co-ordinates. Thus

$(r,\theta), (r, \theta + 2\pi), (r, \theta + 4\pi)$ ... ... $(r,\theta - 2\pi) - (r,\theta - 4\pi)$

$(-r, \theta + \pi); (r, \theta + 3\pi)$, ... and $(-r, \theta - \pi), (-r, \theta - 3\pi)$

all represent the same point, viz, P, as the student will easily see on trying to plot the above co-ordinates.

Art. 9. Relation bewteen the polar and cartesian co-ordiantes.

Let the pole and the initial line of the polar system be the same as the origin and the positive direction of the x—axis of the cartesian system. Suppose the co-ordinates of a point be (r, θ) and cartesian co-ordinates (x,y)

Let the axes be rectangular

It is evident from fig (6) that

$x = r\cos\theta$, $y = r\sin\theta$

Also, $x^2 + y^2 = r^2$.

$\frac{y}{x} = \tan\theta$

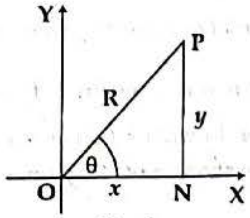


Fig : 6

Thess formula are used for transformations from one system to the other system.

Art. 10. Distance between two points :

To find the distance between two points whose polar co-ordinates are given.

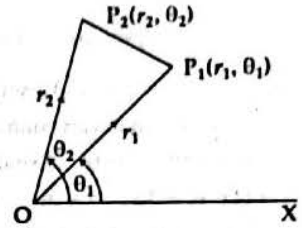


Fig. 7

Let P_1 be the point (r_1, θ_1)

and P_2 the point (r_2, θ_2)

$\angle POP_2 = \angle XOP_2 - \angle POP_1 = \theta_2 - \theta_1$

Then $P_1P_2^2 = OP_1^2 + OP_2^2 - 2OP_1\, OP_2 \cos P_1 OP_2$

$= r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_2 - \theta_1)$

$P_1P_2 = \sqrt{[(r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)]}$

Art. 11. Area of a triangle :

Let P_1, P_2, P_3 be the three vertices whose polar co-ordinates are (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) respectively. The area of the triangle $P_1P_2P_3$.

$\Delta P_1P_2P_3 = \Delta OP_1P_2 + \Delta OP_2P_3 - \Delta\ OP_1P_3$

$= \frac{1}{2} r_1r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) - \frac{1}{2} r_1 r_3 \sin(\theta_3 - \theta_1)$

$= \frac{1}{2} r_1r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) - \frac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3)$

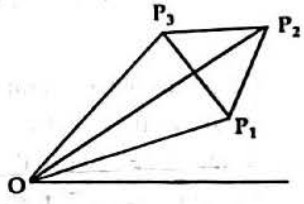


Fig. 7(1)

Continuing on the same way, the area of a polygon $P_1P_2P_3 \ldots P_n$

$= \frac{1}{2} [\, r_1 r_2 \sin(\theta_2 - \theta_1) +$

$r_2 r_3 \sin(\theta_3 - \theta_2) + \ldots$

$\ldots \ldots + r_{n-1} r_n \sin(\theta_n - \theta_{n-1}) + r_n r_1 \sin(1 - \theta_n)]$

Ex. 1. Find the polar co-ordinates of the point whose cartesian co-ordinates are (i) $(-\sqrt{3}, 1)$, (ii) $(5, 12)$

(i) $r = \sqrt{(3 + 1)} = 2$, $\tan\theta = -1/\sqrt{3}$ $\therefore \theta = 5\pi/6$

The polar co-ordinates are $(2, 5\pi/6)$

(ii) $r = \sqrt{(25 + 144)} = 13$, $\tan\theta = 12/5$, $\theta = \tan^{-1} 12/5$

The polar co-ordinates are $(13, \tan^{-1} 12/5)$

Ex. 2. Find the cartesian co-ordinates of the point whose polar co-ordinates are (i) $(5, -\pi - 4)$, (ii) $(2, 330°)$

(i) $x = 5\cos(-\pi/4) = 5/\sqrt{2} = 5\sqrt{2}/2$ and $y = 5\sin(-\pi/4) = -5\sqrt{2}/2$

The cartesian co-ordinates are $(5\sqrt{2}/2, -5\sqrt{2}/2)$

(ii) $x = 2\cos 330° = 2\cos(560° - 30°) = 2\cos 30° = \sqrt{3}$, $y = 2\sin 330° = -2\sin 30° = -1$

The cartesian co-ordinates are $(\sqrt{3}, -1)$

Ex. 3. Find the area of the triangle whose vertices are (0,0), (9, 20°), and (6, 50°)

Area $= \frac{1}{2} r_1 r_3 \sin(\theta_2 - \theta_1) = \frac{1}{2} 6.9 \sin(50° - 20°)$

$= 27 \sin 30° = 13.5$ square units.

Ex. 4. Transform to cartesian co-ordinates the equation $r(\cos 3\theta + \sin 3\theta) = 5k \sin\theta \cos\theta$

or, $r(4\cos^3\theta - 3\cos\theta) + 3\sin\theta - 4\sin^3\theta) = 5k \sin\theta \cos\theta$

or, $r(4x^3 r^3 - 3x - r + 3y - r - 4y^3 r^3) = 5k\, x/r\, y/r + 4y\, r$

or, $4(x^3 - y^3) - 3(x^2 + y^2)(x - y) = 5kxy$

Ex. 5. Transform to polar co-ordinates the equations

(i) $9x^2 + 4y^2 = 36$ (ii) $x^3 = y^2(2a - x)$

(i) Putting $x = r\cos\theta$, $y = r\sin\theta$ the required equation is $9r^2\cos^2\theta + 4r^2 \sin^2\theta = 36$ or, $r^2(4 + 5\cos^2\theta) = 36$

(ii) $r^3\cos^3\theta = r^2\sin^2\theta\,(2a - r\cos\theta)$

or, $r^2\cos^3\theta = 2a\sin^2\theta - r\sin^2\theta\cos\theta$ or, $r\cos\theta = 2a\sin^2\theta$

EXERCISE II

1. Find the polar co-ordinates of the points whose cartesian co-ordiantes are

(i) $(\sqrt{3}, 1)$, (ii) $(-1, 1)$, (iii) $(\sqrt{3}, -1)$ (iv) $(5, \pi^2)$

(v) $(2, -\pi/6)$, Ans (i) $(2, \pi 6)$ (ii) $(\sqrt{2}, 3\pi/4)$

2. Find the polar distance between the points whose polar coordinates are

(i) $(-3, 45°)$ and $(7, 105°)$ Ans (79)

(ii) $(\sqrt{2}, 5\pi/4)$ and $(2, 2\pi/3)$ Ans $1 + \sqrt{3}$

3. Determine the nature of the triangle whose vertices are the points $(0, 0)$, $(3, \pi - 2)$ $(3, \pi - 6)$ and caclulate its area. Ans. $+9(\sqrt{3}) - 4$.

4. Find the areas of the triangle the polar co-ordinates of whose angular points are

(i) $(-3, 30°)$, $(5, 150°)$ and $(7, 210°)$ Ans. $7(\sqrt{3})/2$

(ii) (a, θ) $2a, \theta + \pi/3)$, $(3a, \theta + 2\pi/3)$ Ans. $(5\sqrt{3}/4)a^2$

5. Change the equations to polar Co-ordinates

(i) $(x^2 + y^2)^2 = 2a^2 xy$ ... ... Ans. $r^2 = a^2 \sin 2\theta$

(ii) $x^4 + x^2 y^2 - (x + y)^2 = 0$ ... ... Ans. $r = \pm(1 + \tan\theta)$

6. Transform to co-ordinates the equations

(i) $r^2 - 2r(\cos\theta - \sin\theta) - 7 = 0$, Ans. $x^2 + y^2 - 2x + 2y - 7 = 0$

(ii) $r(1 - e\cos\theta) = ep$ Ans. $x^2 + y^2 = e^2(p + x)^2$

7. A line of length $2a$ has its extremities on two fixed perpenicular lines. Find the locus of the foot of the perpendicular from the intersection of the two fixed lines to a given line. Ans. $r = a\sin 2\theta$

THE STRAIGHT LINE 3

Art. 12. The equation of a straight line

(a) Parallel to the x axis is y = constant

(b) Parallel to the y axis is x = constant

Art. 13. Slope intercept form : The equation of a straight line which cuts of a intercept on the axis of y and is inclined at a given angle θ to the positive direction of x axis is

$y = x\tan\theta + c$

or, $y = mx + c$ Where $m = \tan\theta$ ... ... (1)

Cor. If the straight line passes through the orgin (0, 0) then the equation of the line is $y = mx$ ... ... (2)

Art. 14. Points-Slope Form : The equation of the straight line of slope m passing through the point P (x, y) is

$y - y_1 = m(x - x_1)$ ... ... (1)

Cor. $y - y_1 = m(x - x_1)$ or, $y - y_1 = \tan\theta\,(x - x_1)$

$\frac{y - y_1}{\sin\theta} = \frac{x - x_1}{\cos\theta} = r\,(say)$ or $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r$ ... ... (2)

or, $x = x_1 + rl$ ... ... (3)

$y = y_1 + rm$

Equations in (3) are called Parametric equation of a straight line. It is generally written as $\frac{y - y_1}{l} = \frac{x - x_1}{m} = r$ ... ... (4)

Note : r is the distance between any point (x, y) and the given point (x_1, y_1) on the line

Art. 15. Two point Form : The equation of a straight line passing through the two points P (x_1, y_1) and Q (x_2, y_2) is $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Art. 16. Intercept Form :

The equation of a straight line which cuts off intercepts a and b from the axis is $\frac{x}{a} + \frac{y}{b} = 1$

Since this straight line passes through $(a, 0)$ and $(0, b)$: put $x_1 = a$, $x_2 = 0$, $y = 0$, $y_2 = b$ in *eq* (1) of [Art. 15]

Art. 17. Normal form of the equation of a straight line is $x\cos\theta + y\sin\theta = p$

where. p is the length of the perpendicular form the origin to the line and θ is the angle made by perpendicular with the positive direction *of* x-axis

Art. 18. Equations of the line passing through the intersection of two lines.

The equation of the line passing through the intersection of two lines $P = ax + by + c = 0$ and $Q = a_1x + b_1y + c = 0$

$$ax + by + c + \lambda(a_1x + b_1y + c_1) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where λ is an arbitrary constant.

$$\text{or } P + \lambda Q = 0 \quad \ldots \quad \ldots \quad (2)$$

Art. 19. Every first degree equation in x and y represents a straight line and conversely every straight line can be represented by a first degree equation in x and y.

The general equation of the first degree is of the form $Ax + By + C = 0$

where A, B, C are constants.

Art. 20. Perpendicular distance :

The length of the perpendicular distance d from a point $P(x_1, y_1)$ on the line $ax + by + c = 0$ is

$$d = \pm \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Positive sign is choosen when c is positive and the negative sign when c is negative. This formula gives positive when P lies on same side of the line of the origin, a negative result when P lies on the opposite sides of the origin.

Art. 21. Bisector of angles between two lines :

The equation of the bisectors of the angle between two straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ are

$$\frac{ax + y + c}{\sqrt{(a^2 + b^2)}} = \frac{a_1x + b_1y + c_1}{\pm\sqrt{(a_1^2 + b_1^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Art. 22. Condition for the Concurrency of three lines.

Let the equations to the three lines be

$$a_1x + b_1y + c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$a_3x + b_3 + c_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Let (x_1, y_1) be the common point of intersection. Since all the straight lines are passing through this point.

$$a_1x_1 + b_1y_1 + c_1 = 0,\ a_2x_1 + b_2y_1 + c_2 = 0,\ a_3x_1 + b_3y_1 + c_3 = 0$$

Eliminating x_1 and y_1 from the above equation, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition of the concurency of three straight lines

Art. 23. Condition for Collinearity of three points.

$P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$

If each of them lies on the line $ax + by + c = 0$, then

$$ax_1 + by_1 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$ax_2 + by_2 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$ax_3 + by_3 + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Eliminating a, b and c, From (1) (2) and (3), we have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

which is the required condition for collinearity of the points.

Art. 24. Area of a triangle formed by three given lines.

$$a_1x + b_1y + c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},\ B_1 = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix},\ C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

and similar expression for A_2, B_2, C_2, A_3, B_3, C_3. It is evident that A_1 is the co-factor a_1 in Δ

Solving (2) and (3) the point of intersection (x_1, y_1) is given by $x_1 = A_1/C_1$, $y_1 = B_1/C_1$

Similarly, the point of intersection (x_2, y_2) of (3) and (1) is given by $x_2 = A_2/C_2$, $y_2 = B_2/C_2$ and the point of intersection (x_3, y_3) of (1) and (2) is given by $x_3 = A_3/C_3$ and $y_3 = B_3/C_3$

The area of the triangle with three points as vertices is given by

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2C_1C_2C_3}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \frac{\Delta^2}{2C_1C_2C_3}$$

(See Author's Algebra)

Art. 25. Angle between two lines.

The angle ϕ between the two lines $y = m_1x + c_1$ ... ... (i) and $y = m_2x + c_2$ is ... (ii)

$$\tan\phi = \pm\frac{m_1 - m_2}{1 + m_1m_2} \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

Where m_1 and m_2 are the slopes of the lines.

If the two straight lines are $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$, then

$$\tan\phi = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$$

Art. 26. (a) Condition of parallelism of these (i) and (ii)

is $m_1 = m_2$ ... ... (i)

or, $a_2b_1 - a_1b_2 = 0$ or, $a_1/a_2 = b_1/b_2$

(b) Condition of Perpendicularity of these lines (i) and (ii) is $m_1m_2 = -1$

or, $a_1a_2 + b_1b_2 = 0$ ... ... (3)

(1) The equation to the line parallel to $ax + by + c = 0$ is

$ax + by + k = 0$ ... ... (1)

where k is a constant to be determined by a suitable condition.

(2) The equation to the line Perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$

where k is an arbitrary constant to be determined by a suitable condition

Rule : To obtain the equation of a line perpendicular to a line whose equation is given, interchange the co-efficients of x and y, change the sign of y and suitably change the constant term.

Art. 27. Equation of a straight line in polar co-ordinates.

Let LM be the given straight line, OA the initial line.

ON the perpendicular from the pole O on the line LM. Let ON = p. and the angle which P makes with the initial line viz the $\angle AON = \alpha$. are given.

Let P be any Pt. (r, θ) on LM. Then OP = r, and $\angle AOP = \theta$ so that $\angle NOP = \theta - \alpha$

But from the ΔONP, right angled at N, we have

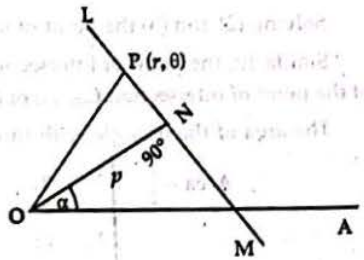


Fig. 8

OP cos NOP = ON, Therefore

$r\cos(\theta - \alpha) = p$

which is the required equation.

Cor. 1. The equation of the line LM in perpendicular form is

$x\cos\alpha + y\sin = p$

Putting $x = r\cos\theta$, $y = r\sin\theta$ (to transform into polar co-ordinates), we get $r\cos(\theta - \alpha) = p$

Cor. 2. The length of the perpendicular from (r_1, θ_1) on this straight line is evidently $= r_1\cos(\theta_1 - \alpha) - p_1$

Cor. 3. The general equation of the straight line in Cartesian co-ordinate is

$Ax + By + C = 0$

Transforming into polar co-ordinates, the equation becomes

$Ar\cos\theta + Br\sin\theta + C = 0$

or, $A\cos\theta + B\sin\theta + C/r = 0$ ... ... (1)

which is the most general form of the equation of a straight line in polar co-ordinates.

Cor. 4. The equation to the line perpendicular to (i) is

$A\cos(\theta + \pi/2) + B\sin(\theta + \pi/2) + C'/r = 0$

or, $B\cos\theta - A\sin\theta + C'/r = 0$

Art. 28. Equation of a straight line through two given points whose co-ordinates are (r_1, θ_1) and (r_2, θ_2)

Let the given points be $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$, P be any point on the line P_1P_2 and has the co-ordinates (r, θ).

Then, since, $\Delta P_1OP_2 = \Delta P_1OP + \Delta POP_2$

$\therefore \frac{1}{2}r_1r_2\sin(\theta_1 - \theta_2) = \frac{1}{2}rr_1\sin(\theta_1 - \theta) + \frac{1}{2}rr_2\sin(\theta - \theta_2)$

or, $r_1r_2\sin(\theta_1 - \theta_2) = rr_1\sin(\theta_1 - \theta) + \theta\, rr_2\sin(\theta - \theta_2)$

or, $\dfrac{\sin(\theta_1 - \theta_2)}{r} = \dfrac{\sin(\theta - \theta_2)}{r_1} + \dfrac{\sin(\theta_1 - \theta)}{r_2}$

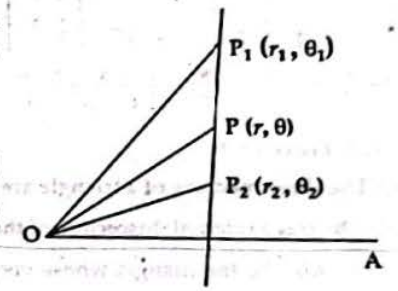


Fig. 9

Art. 29. Equation of the line passing through the intersection of given lines.

Let the equation to the given line be $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha') = p'$. The equation to any straight line through the intersection of the given lines is.

$\{r\cos(\theta - \alpha) - p\} + \lambda\{r\cos(\theta - \alpha') - p'\} = 0$

If this line passes through a given point (r_1, θ_1) then

$\{r_1\cos(\theta_1 - \alpha) - p\} + \lambda\{r_1\cos(\theta_1 - \alpha') - p'\} = 0$

Hence λ is obtained.

Ex. 1. Find the equations to lines passing through (−5, 6)and (a) parallel (b) perpendicular to $7x - 8y = 9$.

(a) Let the equation parallel to $7x - 8y - 9 = 0$ be $7x - 8y = k$ since it passes through (−5, 6) : $7(-5) - 8(6) = k$ or, $-35 - 48 = k$ or, $k = -83$

The required equation is $7x - 8y + 83 = 0$

(b) Let the equation perpendicular to $7x - 8y - 9 = 0$ be $8x + 7y = k$. since it passes through. (−5, 6) of $8(-5) + 7(6) = k$ or, $-40 + 42 = k$ or, $k = 2$

The required equations is $8x + 7y - 2 = 0$

Ex. 2. Determine the equations of the bisectors of the angle between the lines $3x - 4y + 12 = 0$ and $12x + 5y - 30 = 0$

Let P (x, y) be any point on any one of the bisectors,

we have $\frac{3x - 4y + 12}{5} = \pm \frac{12x + 5y - 30}{13}$

or, $21x + 77y = 306$ and $33x - 9y + 2 = 0$

Ex. 3. Find the area of the triangle formed by the lines $2x + y - 3 = 0$, $3x + 2y - 1 = 0$ and $2x + 3y + 4 = 0$

$$\text{Here } \Delta = \begin{vmatrix} 2 & 1 & -3 \\ 3 & 2 & -1 \\ 2 & 3 & 4 \end{vmatrix} = -7$$

By Art. 24

$$C_1 = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5;\ C_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4,\ C_3 = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

Area of the triangle $= \frac{1}{2} \frac{\Delta^2}{C_1C_2C_3} = \frac{49}{40}$

Ex. 4. Prove that

(i) The three medians of a triangle are concurrent.

(ii) The three internal biscectors of the angles of a triangle are concurrent.

(i) Let ABC be the triangle whose vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

The middle point of BC is $\left[\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3)\right]$

The equation to the median through A is

$$\frac{x - x_1}{x_1 - \frac{1}{2}(x_2 + x_3)} = \frac{y - y_1}{y_1 - \frac{1}{2}(y_2 + y_3)}$$

or, $x(2y_1 - y_2 - y_3) - y(2x_1 - x_2 - x_3) = x_1(2y_1 - y_2 - y_3) - y_1(2x_1 - x_2 - x_3)$

Similarly, the equation to the medians through B and C are

$x(2y_2 - y_3 - y_1) - y(2x_2 - x_3 - x_1) = x_2(2y_2 - y_3 - y_1) - y_2(2x_2 - x_3 - x_1)$ and $x(2y_3 - y_1 - y_2) + y(2x_3 - x_1 - x_2)$

$= x_3(2y_3 - y_1 - y_2) - y_3(2x_3 - x_1 - x_2)$

On adding these three equations, their sum identically vanishes. Therefore, the three lines represented by them are concurrent.

(ii) Let the origin be taken inside the triangle. Let equations to the three sides be expressed as (by Art. 17)

$L_1 = x\cos a_1 + y\sin a_1 - p_1 = 0$; $L_2 = x\cos a_2 + y\sin a_2 - p_2 = 0$

$L_3 = x\cos a_3 + y\sin a_3 - p_3 = 0$

where p_2, p_2 and p_3 are all positive.

Since the co-ordinates of the origin when substituted in the equations give results of the same sign, thus the bisectors are given by

$L_1 - L_2 = 0$, $L_2 - L_3 = 0$, $L_3 - L_1 = 0$

On adding these equation, their sum identically vanishes. Therefore the three lines represented by them are concurrent.

EXERCISE III

1. Find the equation of the line which passes through the intersetion of the lines $3x - 5y + 9 = 0$ and $4x + 7y - 28 = 0$ and satisfies the following conditions.

(a) Passes through (4, 2) — Ans. $38x + 87y - 326 = 0$

(b) Is parallel to $2x + 3y - 5 = 0$ — Ans. $82x + 123y - 514 = 0$

(c) Is perpendicular to $4x + 5y - 20 = 0$ — Ans. 205

$x - 164y + 95 = 0$

(d) Whose intercepts are equal — Ans. $41x + 41y - 197 = 0$

2. What are the pts on the axis of x whose perpendicular distance from the straight line. $x/a + y/b = 1$ is a? — Ans. $\pm a/b\sqrt{a^2 \pm b^2}$, 0

3. If p and p_1 be the perpendiculars from the origin upon the st lines whose equation are $x \sec\theta + y \operatorname{cosec}\theta = a$ and

$x\cos\theta - y\sin\theta = a\cos 2\theta$. prove that $4p^2 + p_1^2 = a^2$.

4. Show that the perpendicular let fall from any pts on the st. line $7x - 9y + 10 = 0$ upon the two st. lines $3x + 4y - 5 = 0$ and $12x + 5y - 7 = 0$ are equal to each other.

5. Find the condition that the st. lines $y = m_1x + a_1$. $y = m_2x + a_2$, and $y = m_3x + a_3$ may meet at a pt. — Ans. $m_1(a_2 - a_3) + m_2(a_3 - a_1) + m_3(a_1 - a_2) = 0$

6. Prove that diagonals of the parallelogram formed by the four st. lines.

(i) $\sqrt{(3x)} + y = 0$, $\sqrt{3}y + x = 0$, $\sqrt{3}x + y = 1$ and $\sqrt{3}y + x = 1$

(ii) $x/a + y/b = 1$, $x/b + y/a = 1$. $x/a + y/b = 2$, $x/b + y/a = 2$

(iii) $ax + by + c = 0$, $ax + by + c_1 = 0$, $a_1x + b_1y + c_2 = 0$

$a_1x + b_1y + c_3 = 0$, If $(a^2 + b^2) = (a_1^2 + b_1^2)$

are perpendicular to each other

7. Find the equations to the st. lines passing through the foot of the perpendicular from the pt (h, k) upon the st. line $Ax + By + C = 0$ and bisecting the angle between the perpendicular and the given st. line — Ans. $\pm\{A(y - k) - B(x - h)\} = Ax + By + C$

8. Prove that the area of the parallelogram whose sides are the st. lines. $a_1x + b_1y + c_1 = 0$ $a_1x + b_1y + d_1 = 0$. $a_2x + b_2y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is $\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}$

9. The equation to a pair of opposite sides of a parallelogram are $x^2 - 7x - 6 = 0$ and $y^2 - 11y + 40 = 0$. Find the equations to its diagonals.

Ans. $6x - 6y + 14 = 0, 6x + 5y - 56 = 0$ [R. U. 1982]

10. (a) Show that the area of the triangle formed by the lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$ is $\frac{1}{2}(c_1 - c_2)^2/(m_2 - m_1)$

(b) Find the area of the triangle formed by the sides $x + y + 2 = 0$. $2x - y - 3 = 0$ and $3x + 2y - 5 = 0$ Ans. 338/24

11. Prove that the line $x + \sqrt{3}y = 0$ bisects the Δ whose vertices are $(3, -\sqrt{3})$, $(-1, -\sqrt{3})$ and $(1, \sqrt{3})$ [D. U. 1962]

12. Find the equation to the st. line which passes through the pt. of intersection of $y - 4x - 1 = 0$ and $2x + 5y - 6 = 0$ and is perpendicular to $3y + 4y = 0$ Ans. $66x - 88y + 101 = 0$

13. Find the equation of the bisector of that angle between the lines $4x - 3y + 1 = 0$ and $12x + 5y + 13 = 0$ in which the origin lies.

Ans. $x + 8y + 13/2 = 0$ [D.U. 1964]

14. Find the centre of the inscribed circle of the triangle the equation of whose sides are $4y + 3x = 0$, $2y - 5x = 0$ and $y - 15 = 0$ Ans. (1, 8)

15. Find the equations to the st. lines which are parallel to the st. line $3x + 4y + 5 = 0$ and at distance 2 units from it. Ans. $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$

16. If the sum of the perpendiculars dropped from a variable pt. P on two lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ be always to 10, prove that P must move on a right line.

17. Prove that the three st. lines $2x - 7y + 10 = 0$, $3x - 2y - 1 = 0$ and $x - 12y + 21 = 0$ concur at a point.

18. Find the direction in which a st. lines must be drawn through the pt. (1, 2). so the that its pt. of intersection with the line $x + y = 4$ may be at a distance $\frac{1}{3}(\sqrt{6})$ from this point.

Ans. 15° or 75° to the axis of x

19. Find the equation of the two st. lines passing through the points (1, —1) and inclined at an angle 45° with the st. line [D. U. 1967] Ans. $7x - 3y - 10 = 0$, $3x + 7y + 4 = 0$

20. Show that the length of the perpendicular drawn from the point of intersection of the line $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ on the line $lx + my + n = 0$ is equal to.

$$\begin{vmatrix} l & m & n \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} = (ab_1 - a_1b)\sqrt{(l^2 + m^2)}$$

21. Show that the area of the triangle formed by the st. lines $y - 2x = 0$, $y - 3x = 0$, and $y = 5x + 4$ is 4/3 [D. U. 1964]

22. Show that the area of the triangle formed by the st. line $y = m_1x + c_1$, $y = m_2x + c_2$. and $y = m_3x + c_3$ is $\frac{1}{2}\frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{1}{2}\frac{(c_3 - c_1)^2}{m_3 - m_1} + \frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2}$

23. A straight line moves so that the sum of the receiprocals of its intercepts on the axes is constant, show that it passes through a fixed point.

24. Find the area of the triangle formed by the lines $8x + 7y - 43 = 0$, $2x + 5y = 1$ and $4x - 3y + 11 = 0$ Ans. 26 units.

25. If the variable co-officients λ, μ, υ in the equations of the line $\lambda x + \mu y + \upsilon = 0$ be connected by a relation of the form $\lambda a + \mu b + \upsilon c = 0$ where a. b. c. are constants, then the variable line always passes through a fixed point.

26. Show that the product of the perpendiculars drawn from the two pts $[(\pm\sqrt{(a^2 - b^2)}, 0]$ upon the straight line. $(x/a)\cos\theta + (y/b)\sin\theta = 1$ is b^2

27. Show that the distance of the pt, (x_0, y_0) from the line $ax + b + c = 0$. measured parallel to a line making an angle θ with the x — axis is $-\frac{ax_0 + by_0 + c}{a\cos\theta + b\sin\theta}$

28. Show that the distance from the point (—1, 1) along the straight $3x + 5y = 1$ in the direction of $\tan^{-1} 3/4$ of the positive direction of x-axis is 5/27.

29. Show that perpendicular distance between the two parallel lines $ax + by + c = 0$ and $k(ax + by) + d = 0$ is $\left(\frac{d}{k} - c\right) \div \sqrt{(a^2 + b^2)}$.

30. একটি ত্রিভুজের তিনটি বাহু BC, CA এবং AB দ্বারা নির্ধারিত সরলরেখার সমীকরণ যথাক্রমে $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ দেখাও যে শীর্ষবিন্দু A গামী ও BC বাহুর সমান্তরাল সরলরেখার সমীকরণ,

$$\frac{a_3x + b_3y + c_3}{a_3b_1 - a_1b_3} + \frac{a_2x + b_2y + c_2}{a_1b_1 - a_2b_1} = 0$$

[D. U. 1987]

CHANGE OF AXES 4

Art. 31. Transformation of Co-ordinates.

The co-ordinates of a point or the equation of a curve are alway given with reference to a fixed origin and a set of axes of co-ordinates. The above co-ordinates of the equation of the curve changes when the origin is changed or the direction of axes changed or both. The process of changing the co-ordinate of a point or the equation of a curve is called **transformation of co-ordinates.** Now we have to investigate the mode of the change of the co-ordinates or the equation of the curve according to the transfer from one set to another set of co-ordinate axes.

Art. 32. Change of origin (Translation of axes)

To find the change in the co-ordinates of a point when the origin is shifted to another point O (α, β) where the direction of axes remains unalterad.

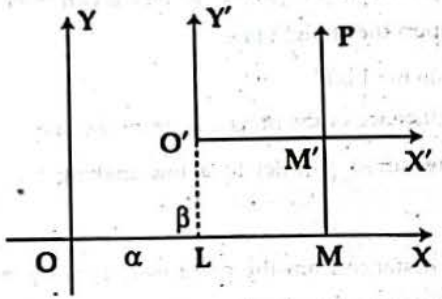


Fig. 10

Let us take a new pair of axes O′ X′ and O′Y′ parallel to the old pair OX and OY; O′ being the new origin whose co-ordinate are (α, β) referred to O′X′ and O′Y′.

let (x′, y′) be the co-ordinates referred to the axes O′X′ and O′Y′ or a point P, whose co-ordinate referred to the old axes are (x, y).

It is required to transform the co-ordinates (x, y) in terms or (x′, y′).

From O′ and P draw O′L and PM perpendiculars to OX. Let PM meet O′X′ in M′.

Then OL = α, LO′ = β, OM = x, MP = y

Also O′M′ = x′ and M′P = y′

Therefore, OM = OL + LM = OL + O′M′.

$\therefore x = \alpha + x'$ ... ... (i)

Similarly, MP = MM′ + M′P = LO′ + M′P

$\therefore y = \beta + y'$ ... ... (ii)

The transformed co-ordinates are

$$\text{and } \left.\begin{aligned} x' &= x - \alpha \\ y' &= y - \beta \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Rule : *In order to shift the origin to (α, β) the transformation is obtained by replacing x by x + a and y by y + β. If from the transformed equation we want to get the old equations then replace x by x − α and y by y − β. This is known as shifting the origin back.*

Art. 33. Rotation of axes (origin fixed)

To find the change in the co-ordinates of a point when the direction of axes is turned through an angle θ where as the origin of co-ordinates remains the same.

Let OX and OY be the old axes and OX′ and OY′ set the new axes. O is the common origin for the two sets of axes. Let the angle X′ OX through which the axes have rotated be represented by θ.

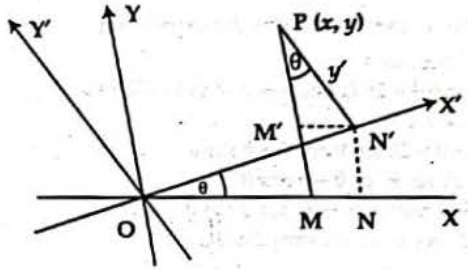


Fig. 11

Let P be any point in the plane and let its co-ordinates referred to the old axes be (x, y), and referred to the new axes be (x′, y′).

Let us try to determine x and y in terms of x′, y′ and θ, Draw PM perpendicular to OX, PN′ perpendicular to OX′, and N′N perpendicular to OX, and N′M′ parallel to OX.

Then $x = OM = ON - MN = ON - M'N'$

$= ON' \cos\theta - PN' \sin\theta = x' \cos\theta - y' \sin\theta$

$y = MP = MM' + M'P = NN' + M'P$

$= ON' \sin\theta + PN' \cos\theta = x' \sin\theta + y' \cos\theta$

Hence the formula for the rotation of the axes through an angle θ are.

$$\text{and } \left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Art. 34. Change of origin with the change of the direction of axes.

It is really the combination of Art. 32 and Art. 33. The best method is to apply Art. 32 first and then Art. 33. Of course the two transformations may also be made simultaneously.

Let us suppose that the system of axes be rectangular. The origin is shifted to the point (α, β) and then the axes are rotated through an angle θ. If the co-ordinates of any point be (x, y) in the old system, and (x′, y′) in the new system, from Art. 32, and Art. 33.

we have.

$$\text{and} \left.\begin{array}{l} x = \alpha + x' \cos\theta - y' \sin\theta \\ y = \beta + x' \sin\theta + y' \cos\theta \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Art. 35. Invariants. [R.U. 1983]

If by the rotation of the rectangular co-ordinate axes about the origin, the expression $ax^2 + 2hxy + by^2$ changes to

$a'x'^2 + 2h'x'y' + b'y'^2$, $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$

Let (x, y) be the co-ordinates of a point P referred to a set of rectangular axes. If the axes are roated through an angle θ about the origin, let the co-ordinates of the same point P be (x', y') referred to the new system of rectangular axes. Then we have by

(3) Art. 33. $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

By using the above transformations the expression

$ax^2 + 2hxy + by^2$ becomes

$a(x' \cos\theta - y' \sin\theta)^2 + 2h(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta) + b(x' \sin\theta + y' \cos\theta)^2$

$= a'x'^2 + 2h'x'y' + b'y'^2$

where $a' = a\cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta$ ... ... (1)

$b' = a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta$ ... ... (2)

$h' = h(\cos^2\theta - \sin^2\theta) - (a - b) \sin\theta \cos\theta$ ... ... (3)

From (1) and (2) we have on simplification

$a' + b' = a + b$

Again $2a' = 2a \cos^2\theta + 4h \sin\theta \text{ cis } \theta + 2b \sin^2\theta$

$= a(1 + \cos 2\theta) + 2h \sin 2\theta + b(1 - \cos 2\theta)$

$= a + b + 2h \sin 2\theta + (a - b) \cos 2\theta$ ... ... (4)

Similarly. $2b' = a + b - 2h \sin 2\theta - (a - b) \cos 2\theta$ ... ... (5)

From (4), (5) and (3) we have

$4(a'b' - h'^2) = (a + b)^2 - \{2h \sin 2\theta + (a - b)\cos 2\theta\}^2$

$- \{2h \cos 2\theta - (a - b) \sin 2\theta\}^2$

$= (a + b) - 4h^2 - (a - b)^2 = 4ab - 4h^2$

or, $a'b' - h'^2 = ab - h^2$

The two quantities $a + b$ and $ab - h^2$ for the expression $ax^2 + 2hxy + by^2$ are called **invariants** of transformation from one system of rectangular axes (Without change of the origin), because their values remain unchanged by the transformation.

Cor. Removal of the-xy-term from the expression $ax^2 + 2hxy + by^2$

If we rotate the rectangular axes through an angle θ about the origin, we have transformations by Eq. 3, Art. 33.

$x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$

Putting the values of x and y in $ax^2 + 2hxy + by^2$

we have, $a'x'^2 + 2h'x'y' + b'y'^2$

an invariant is a property of a mathematical object (or a class of mathematical objects) which remains unchanged after operations or transformations of a certain type are applied to the objects.

where the values of a'. b' and h' are available from

(1), (2) and (3) of Art 35.

The co-efficient of $x'y'$ term in the new expression is $2h'$ where $h' = h(\cos^2\theta - \sin^2\theta) - (a - b) \sin\theta \cos\theta$. In order to remove the $x'y'$ term, the co-efficient h' is zero i.e.

$h(\cos^2\theta - \sin^2\theta) - (a - b) \sin\theta \cos\theta = 0$

$\therefore h \cos 2\theta = \frac{1}{2}(a - b) \sin 2\theta$ or, $\tan 2\theta = \frac{2h}{a - b}$

$$\therefore \theta = \tfrac{1}{2}\tan^{-1}\frac{2h}{a - b} \Rightarrow \tan 2\theta = \frac{2h}{a - b} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Hence if the axes are rotated through an angle $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a - b}$ then xy term in the expression $ax^2 + 2hxy + by^2$ vanishes.

Ex. 1. Determine the equation of the curve $2x^2 + 3y^2 - 8x + 6y - 7 = 0$ when then origin is transferred to the point $(2, -1)$.

when then origin is transferred to the point $(2, -1)$ Put $x = x' + 2$, $y = y' - 1$ in the above equation. Then $2(x' + 2)^2 + 3(y' - 1)^2 - 8(x' + 2) + 6(y' - 1) - 7 = 0$

or, $2x'^2 + 3y'^2 = 18$

Now removing suffixes the equation referred to new axes is $2x^2 + 3y^2 = 18$

Ex. 2. Determine the equation of the parabola

$x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ ... ... (i)

after rotating of axes through 45°

When the axes have been roated through an angle 45°, then

$x = x' \cos 45° - y' \sin 45° = (x' - y')/\sqrt{2}$

$y = x' \sin 45° + y' \cos 45° = (x' + y)/\sqrt{2}$

Put them in (1), then $\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x' - y'}{\sqrt{2}}\right)\frac{x' + y'}{\sqrt{2}} + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x' - y'}{\sqrt{2}}\right) - 4\left(\frac{x' + y'}{\sqrt{2}}\right) + 3 = 0$

On simplification the equation reduces to

$2y'^2 - \sqrt{2}x' - 3\sqrt{2}y' + 3 = 0$

Now dropping the suffixes, the equation is

$2y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$ Ans.

Ex. 3. Remove the first degree terms in

$3x^2 + 4y^2 - 12x + 4y + 13 = 0$

Complete the square,

or, $3(x^2 - 4x) + 4\left(y^2 + y + \frac{1}{4}\right) = 0$

or, $3(x - 2)^2 + 4\left(y + \frac{1}{2}\right)^2 = 0$ ... ... (ii)

Put $x-2=x'$ and $y+\frac{1}{2}=y'$ in (ii)

$\therefore\ 3x'^2+4y'^2=0$

This is satisfied only by $x'=0, y'=0$, which is the new origin.

By shifting origin at $\left(2, \frac{-1}{2}\right)$, the first degree terms can be removed.

Ex. 4. By transforming to parallel axes through a properly chosen point (h, k), prove that the equation.

$$12x^2-10xy+2y^2+11x-5y+2=0$$

can be reduced to one containing only the terms of the 2nd degree.

Transforming to parallel axes through (h, k) we have

$$12(x'+h)^2-10(x'+h)(y'+k)+2(y'+k)^2+11(x'+h)-5(y'+k)+2=0$$

or, $12x'^2-10x'y'+2y'^2+(24h-10k+11)x'+(-10h+4k-5)y'+(12h^2-10hk+2k^2+11h-5k+2)=0$ ... ... (i)

Now equate the co-efficients of x', y' to zero then

$24h-10k+11=0$ also $-10h+4k-5=0$

and $12h^2-10hk+2k^2+11h-5k+2=0$

Solve the first two equations $h=-\frac{3}{2}, k=-\frac{5}{2}$ and it is clear that these values of h, k satisfy the equation (ii). Hence the equation. (i) becomes $12x'^2-10x'y'+2y'^2=0$

5. The clirection cosine of axes remaining the same, choose a new origin such that the new co-ordinates of the pair of points whose old co-ordinate are (5, –13) and (–3,11) may be of the forms (h, k) and (–h, –k). [N.U.H-1997]

Solution : Let the co-ordinate of new origin be (α, β).

Given that the old co-ordinates of the two points are (5, –13) and (–3, 11) whose new co-ordinates are (h, k) and (–h, –k).

So, $(5-\alpha, -13-\beta)=(h, k)$ and

$(-3-\alpha, 11-\beta)=(-h, k)$

Equating corresponding Components,

$5-\alpha=h, \quad -13-\beta=-k$

$-3-\alpha=-h \quad 11-\beta=k$

$\therefore 2-2\alpha=0 \quad -2-2\beta=0$

$\therefore \alpha=1 \quad \therefore \beta=-1$

Thus, The co-ordinate of new origin (1, – 1)

6. Transfer the equation $11x^2-4xy+14y^2-58x-44y+126=0$ to new axes of X and Y whose equation are $x-2y+1=0$ and $2x+y-8=0$ respectively. [N.U.H-2000; 2008]

Solution : Given that

$11x^2-4xy+14y^2-58x-44y+126=0$ ... ... (1)

Let $x-2y+1=0$ and $2x+y-8=0$ be The new axes of X and Y.

New co-ordinate of (x, y) be (x_1, y_1).

So, $x_1=\frac{2x+y-8}{\sqrt{2^2+1^2}}=\frac{2x+y-8}{\sqrt{5}}$

$\Rightarrow 2x+y-8=\sqrt{5}x_1$ ... ... (2)

$y_1=\frac{x-2y+1}{\sqrt{1^2+2^2}}=\frac{x-2y+1}{\sqrt{5}}$ and

$\Rightarrow x-2y+1=\sqrt{5}y_1$ ... ... (3)

From (2) and (3)

$4x+2y-16=2\sqrt{5}x_1$

$x-2y+1=\sqrt{5}y_1$

(+) $5x-15=\sqrt{5}y_1+2\sqrt{5}x_1$

$5x=15+\sqrt{5}y_1+2\sqrt{5}x_1$

$x=\frac{2\sqrt{5}x_1+\sqrt{5}y_1+15}{5}$

$=\frac{2x_1+y_1}{\sqrt{5}}+3$

From (2) ; $y=-2x+\sqrt{5}x_1+8$

$=\frac{-4x_1-2y_1}{\sqrt{5}}-6+\sqrt{5}x_1+8$

$=\frac{-4x_1-2y_1+5x_1}{\sqrt{5}}+2$

$=\frac{x_1-2y_1}{\sqrt{5}}+2$

$\therefore\ x=h+3$

Where $h=\frac{2x_1+y_2}{\sqrt{5}}+3$

$y=k+2k=\frac{x_1-2y_1}{\sqrt{5}}+2$

Putting the values of x and y in (1)

we get

$11(h+3)^2 - 4(h+3)(k+2) + 14(k+2)^2 - 58(h+3) - 44(k+2) + 126 = 0$

$\Rightarrow 11(h^2+6h+9) - 4(hk+2h+3k+6) + 14(k^2+4k+4) - 58h - 174 - 44k - 88 + 126 = 0$

$\Rightarrow 11h^2 - 4hk + 14k^2 - 5 = 0$

$\Rightarrow 11\left(\frac{2x_1+y_1}{\sqrt{5}}\right)^2 - 4\left(\frac{2x_1+y_1}{\sqrt{5}}\right)\left(\frac{x_1-2y_1}{\sqrt{5}}\right) + 14\left(\frac{x_1-2y_1}{\sqrt{5}}\right)^2 - 5 = 0$

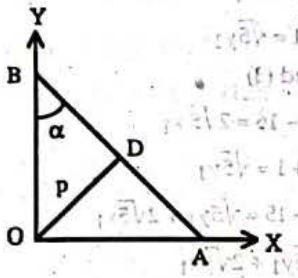
$\Rightarrow 11(4x_1^2+4x_1y_1+y_1^2) - 4(2x_1^2-4x_1y_1-2y_1^2) + 14(x_1^2-4x_1y_1+4y_1^2) - 25 = 0$

$= 50x_1^2 - 75y_1^2 = 25$

$= 2x_1^2 + 3y_1^2 = 1$ which is required equation.

7. Reduce the equation of a straight line in the polar form $r\cos(\theta-\alpha) = p$ [N.U.H. 2001]

Solution :



Let the line AB makes an angle α with Y-axis and OD= p. Form the figure, equation of AB is $\frac{x}{OA} + \frac{y}{OB} = 1$ ... ... (1)

Sinec OD perpendicular to AB and OD = P

From the Δ OBD ; $\sin\alpha = \frac{OD}{OB} = \frac{P}{OB}$

$\therefore OB = \frac{P}{\sin\alpha}$

Again, from the Δ AOD,

$\sin(90° - \alpha) = \frac{OD}{OA} = \frac{P}{OA}$

$\therefore OA = \frac{P}{\cos\alpha}$

Putting the values of OA and OB in (1) we get $\frac{x}{\frac{P}{\cos\alpha}} + \frac{y}{\frac{P}{\sin\alpha}} = 1$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$ ... ... (2)

Relation between cartesian and polar Co-ordinates we can write,

$x = r\cos\theta, \quad y = r\sin\theta$

Equation (2) becomes

$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$

$\Rightarrow r\cos(\theta-\alpha) = p$ which is required polar equation.

??

(1) (2)

8. The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when refered to rectangular axes through the point (2, 3). Find the inclination of the later axes to the former. [N.U.H. 2004]

(2) ??

Solution : Given that

$3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ ... ... (1)

Transfer the origin to a point (2, 3). Then putting $x = x_1 + 2$, $y = y_1 + 3$ in (1), we get (1) ⇒ 3

$(x_1+2)^2 + 2(x_1+2)(y_1+3) + 3(y_1+3)^2 - 18(x_1+2) - 22(y_1+3) + 50 = 0$

$\Rightarrow 3(x_1^2+4x_1+4) + 2(x_1y_1+3x_1+2y_1+6) + 3(y_1^2+6y_1+9) - 18x_1 - 36 - 22y_1 - 66 + 50 = 0$

$= 3x_1^2 + 2x_1y_1 + 3y_1^2 = 1$

Removing suffixes, we get

$3x^2 + 2xy + 3y^2 = 1$ ... ... (2)

(does not change inclination)

Let the axes rotated be an angle θ, Then we know, $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$; where a = 3, b = 3, h = 1

$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2.1}{3-3}$

$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\infty$

$\Rightarrow \theta = \frac{1}{2}\cdot\frac{\pi}{2} = \pi/4 = 45°$

We know that the axes rotated be an angle θ, then

$x = x_1\cos\theta - y_1\sin\theta, \qquad y = x_1\sin\theta + y_1\cos\theta$

$x = x_1\cos45° - y_1\sin45° \qquad y = x_1\sin45° + y_1\cos45°$

$x = \frac{x_1-y_1}{\sqrt{2}} \quad x = \frac{x_1+y_1}{\sqrt{2}}$

Putting the values of x and y in (2) we get

$(2) \Rightarrow 3\left(\frac{x_1-y_1}{\sqrt{2}}\right)^2 + 2\left(\frac{x_1-y_1}{\sqrt{2}}\right)\left(\frac{x_1+y_1}{\sqrt{2}}\right) + 3\left(\frac{x_1+y_1}{\sqrt{2}}\right)^2 = 1$

$\Rightarrow 3(x_1-y_1)^2 + 2(x_1^2-y_1^2) + 3(x_1+y_1)^2 = 2$

$\Rightarrow 3(x_1^2-2x_1y_1+y_1^2) + 2x_1^2 - 2y_1^2 + 3(x_1^2+2x_1y_1+y_1^2) = 2$

$\Rightarrow 8x_1^2 + 4y_1^2 = 2$

$\Rightarrow 4x_1^2 + 2y_1^2 = 1$

Remoing suffixes,

$\therefore 4x^2 + 2y^2 = 1$

Which is the required transformed equation

9. If the direction of axes is turned through an angle 30° and the origin remains unchanged then find the transformation equation of $x^2 + 2\sqrt{3}xy - y^2 - 2a^2 = 0$. Idntify and sketch it. [N.U.H. 2009; 1999]

Solution : given that,

$x^2 + 2\sqrt{3}xy - y^2 - 2a^2 = 0$ ... ... (1)

Since the axes rotated be an angle 30° and origin be unchanged.

So, $x = x_1\cos 30° - y_1\sin 30° = \dfrac{\sqrt{3}x_1 - y_1}{\sqrt{2}}$

$y = x_1\sin 30° + y_1\cos 30° = \dfrac{x_1 + \sqrt{3}y_1}{\sqrt{2}}$

Putting this valus in (1) we get

$(1) = \left(\dfrac{\sqrt{3}x_1 - y_1}{\sqrt{2}}\right)^2 + 2\sqrt{3}\left(\dfrac{\sqrt{3}x_1 - y_1}{\sqrt{2}}\right)$

$\left(\dfrac{x_1 + \sqrt{3}y_1}{2}\right) - \left(\dfrac{x_1 + \sqrt{3}y_1}{2}\right)^2 - 2a^2 = 0$

$\Rightarrow \frac{1}{4}(3x_1^2 - 2\sqrt{3}x_1y_1 + y_1^2) + \frac{2\sqrt{3}}{4}(\sqrt{3}x_1^2 + 3x_1y_1 - x_1y_1 + \sqrt{3}y_1^2) - \frac{1}{4}(x_1^2 + 2\sqrt{3}x_1y_1 + 3y_1^2) - 2a^2 = 0$

$\Rightarrow 3x_1^2 - 2\sqrt{3}x_1y_1 + y_1^2 + 6x_1^2 + 4\sqrt{3}x_1y_1 - 6y_1^2 - x_1^2 - 2\sqrt{3}x_1y_1 - 3y_1^2 - 8a^2 = 0$

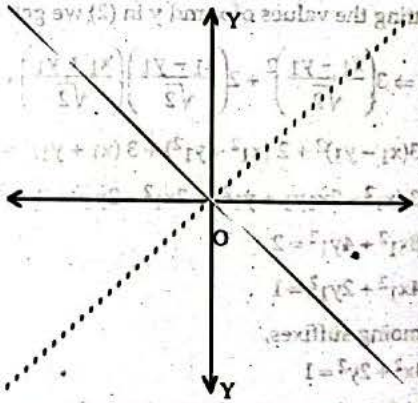
$\Rightarrow 8x_1^2 - 8y_1^2 - 8a^2 = 0$

$= x_1^2 - y_1^2 = a^2$

Removing suffixes, we get

$x^2 - y^2 = a^2$ which is the equation of rectangular hyperbola.

2nd part : we have. $x^2 - y^2 = a^2$. It is rectangular hyperbola whose asymptotes are $y = \pm x$. ie Asymptotes makes an angles with x – axis are ±45°. Its figure below :



10. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involuing x, y and xy both sets of axes being rectangular. [N.U.H. 2003]

Solution : Given equation

$17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$

... ... (1)

Let transfer the origin to a point (4, k)

Then $x = x_1 + h$, $y = y_1 + k$

Putting these values in (1) we get, $17(x_1 + h)^2 + 18(x_1 + h)(y_1 + k) - 7(y_1 + k)^2 - 16(x_1 + 4h - 32(y_1 + k) - 18 = 0$

$\Rightarrow 17(x_1^2 + 2x_1h + h^2) + 18(x_1y_1 + x_1k + y_1h + hk) - 7(y_1^2 + 2y_1k + k^2) - 16x_1 - 16h - 32y_1 - 32k - 18 = 0$

$\Rightarrow 17x_1^2 + 18x_1y_1 - 74y_1^2 + (34h + 18k - 16)x_1 + (18h - 14x - 32)y_1 + 17h^2 + 18hk - 7k^2 - 16h - 32k - 18 = 0$ ... ... (2)

Since there is no terms of x, y. So equating the coefficient of x_1, y_1 from both sides.

$\therefore$ $34h + 18k - 16 = 0$

$18h - 14k - 32 = 0$

By the rule of cross multiplication,

$\therefore \dfrac{h}{-576 - 224} = \dfrac{k}{-288 + 1088} = \dfrac{1}{476 - 324}$

$= \dfrac{h}{-800} = \dfrac{k}{800} = \dfrac{1}{-800}$

$\therefore$ $h = 1$, $k = -1$

Putting the values of h and k in (2) we get $17x_1^2 + 18x_1y_1 - 7y_1^2 + 0 + 0 + 17 + 18(1)(-1) - 7 - 16 - 32(-1) - 18 = 0$

$17x_1^2 + 18x_1y_1 - 7y_1^2 - 10 = 0$

Removing Suffixes,

$\therefore$ $17x^2 + 18xy - 7y^2 - 10 = 0$ ... ... (3)

If we rotated the axes through an angle θ

Then $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2h}{a - b}$

$\therefore$ $2\theta = \tan^{-1}\dfrac{2.9}{17 + 7}$

$\Rightarrow \tan 2\theta = \dfrac{18}{24} = \dfrac{3}{4}$

$\therefore$ $\sin 2\theta = \dfrac{3}{5}$

$\therefore$ $2\sin\theta\cos\theta = \dfrac{3}{5}$ $\qquad \cos 2\theta = \dfrac{4}{5}$

$\therefore$ $\sin\theta = \dfrac{3}{10\cos\theta}$ $\qquad 2\cos^2\theta - 1 = \dfrac{4}{5}$

$= \dfrac{3\sqrt{10}}{10.3} = \dfrac{1}{\sqrt{10}}$ $\qquad \cos^2\theta = \dfrac{9}{10}$ $\therefore \cos\theta = \dfrac{3}{\sqrt{10}}$

We know,

$x = x_1\cos\theta - y_1\sin\theta$

$= \dfrac{3x_1 - y_1}{\sqrt{10}}$

$y = x_1\sin\theta + y_1\cos\theta$

$= \dfrac{x_1 + 3y_1}{\sqrt{10}}$

Putting the values of x, y in (3) we get

$$(3) \Rightarrow 17\left(\frac{3x_1 - y_1}{\sqrt{10}}\right)^2 + 18\left(\frac{3x_1 - y_1}{\sqrt{10}}\right)\left(\frac{x_1 + 3y_1}{\sqrt{10}}\right)$$

$$-7\left(\frac{x_1 + 3y_1}{\sqrt{10}}\right) - 10 = 0$$

$\Rightarrow 17(9x_1^2 - 6x_1y_1 + y_1^2) + 18(3x_1^2 + 9x_1y_1 - 3y_1^2)$

$-7(x_1^2 \Rightarrow 6x_1y_1 + 9y_1^2) - 100 = 0$

$\Rightarrow (153 + 54 - 7)\, x_1^2 = (-102 + 144 - 42)\, x_1y_1 + (17 - 54 - 63)\, y_1^2 \Rightarrow 100$

$\Rightarrow 200x_1^2 - 100y_1^2 = 100$

$\Rightarrow 2x_1^2 - y_1^2 = 1$

Removing suffixes, we get

$2x^2 - y^2 = 1$

Which is the required transformation.

EXERCISE IV

1. Transform to parallel axes through the new origin of the equations.

(a) Origin $(1, -2)$, $2x^2 + y^2 - 4x + 4y = 0$ Ans. $2x^2 + y^2 = 6$ [D. U. 1982]

(b) Origin $(3, 1)$, $x^2 - 6x + 2y^2 + 7 = 0$ Ans. $x^2 + 2y^2 + 4y = 0$.

2. Transform to axes inclined at 45° to the original axes the equations.

(i) $x^2 - y^2 = a^2$ Ans. $2xy = a^2$

(ii) $x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$ Ans. $xy + 6x + 4y = 1$

3. Remove the first degree terms in each of the following equations.

(a) $3x^2 - 4y^2 - 6x - 8y - 10 = 0$ Ans. $5x^2 - 4y^2 = 9$

(b) $2x^2 + 5y^2 - 12x + 10y - 7 = 0$ Ans. $2x^2 + y^2 = 40$

(c) $3x^2 - 4y^2 + 6x + 24y - 135 = 0$ Ans. $3x^2 - 4y^2 = 102$

4. Prove that the value of $g^2 + f^2$ in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ remains uneffected by orthogonal transformation without change of origin.

5. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ to rectangular axes through the point $(2, -1)$ and inclined at an angle $\tan^{-1}\frac{4}{3}$ Ans. $x^2 - 4y^2 + 1 = 0$

6. Determine the angle through which the axes must be rotated to remove the xy term in the equation. $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$ Ans. $\theta = 30°$, $x^2 + 4y^2 = 4$

7. By transforming to parallel axes through a properly chosen point (h, k) prove that the equation $2x^2 + y^2 - xy - 5x - 4y + 11 = 0$ can be reduced to one containing terms of the 2nd degree only. Ans. $(2, 3)$

8. The direction of axes remaining the same, choose a new origin such that the new co-ordinate of the pair of points whose old co-ordinates are $(5, -13)$ and $(-3, 11)$ may be the forms (h, k) and $(-h, -k)$ [N. U. H. 1997] Ans. $(1, -1)$

9. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involving x, y and xy, both sets of axes being rectangular.

[N. U. H. 2003] Ans. $(1, -1)$, $\frac{1}{2}\tan^{-1} 3/4$, $2x^2 - y^2 = 1$

10. Transform the axes inclined at 30° to the original axes the equation, $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ [N. U. H. 2009, 1999] Ans. $x^2 - y^2 = a^2$

11. The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when referred to rectangular axes through the point $(2, 3)$. Find the inclination of the latter axes to the former. (Page 27, Ex 8) [N. U. H. 2004] [R. U. 1960]

PAIR OF STRAIGHT LINES 5

Art. 36. Homogeneous Quadratic Equations.

Let us consider the equation $ax^2 + 2hxy + by^2 = 0$ ... ... (1)

On multiplying it by a, it may written in the form

$a^2x^2 + 2ahxy + aby^2 = 0$, if $a \neq 0$

or, $a^2x^2 + 2ahxy + h^2y^2 - (h^2 - ab)\, y^2 = 0$

or, $\{ax + hy)^2 - \{y\sqrt{(h^2 - ab)}\}^2 = 0$

or, $(ax + hy) + \sqrt{(h^2 - ab)}\, y\}\, \{ax + hy - (h^2 - ab))y\} = 0$

The equation (1) therefore, represents the two straight lines whose equations are

$ax + hy + y\sqrt{(h^2 - ab)} = 0$ ... ... (2)

and $ax + hy - y\sqrt{(h^2 - ab)} = 0$ ... ... (3)

each of which passes through the origin. The two lines (2), (3) form the locus of the equation (1) for (1) is satisfied by the co-ordinates of all points which satisfy (2) and (3).

These two straight lines are real and different if $h^2 > ab$, real and coincident if $h^2 = ab$, and imaginary if $h^2 < ab$.

In the case when $h^2 = ab$, the straight lines though themselves imaginary intersect in a real point. For the origin lies on the locus given by (1) since the equation (1) is always satisfied by the values $x = 0$ and $y = 0$.

Alternative method

If $b \neq 0$ dividing both sides of the equation (1) by $x^2 b$ we have

$$\left(\frac{y}{x}\right)^2 + 2\frac{h}{b}\frac{y}{x} + \frac{a}{b} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

Let m_1, m_2, be the roots of this quadratic equation in $\frac{x}{y}$.

Sum of the roots $= m_2 + m_2 = -2h/b$

and product of the roots $= m_1m_2 = a/b$ ... ... (5)

The Eq. (4) must be equivalent to

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The two lines represented by (4) *i.e.* (1) are given

$\frac{y}{x} - m_1 = 0$, and $\frac{y}{x} - m_2 = 0$,

i.e. $y - m_1x = 0$ and $y - m_2x = 0$ which pass through the origin.

Thus the homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines, real or imaginary, through the origin.

Art. 37. Prove that a homogeneous equation of the n th degree represents n straight lines, real or, imaginary, which all pass through the origin.

Consider the homogeneous equation

$$y^n + a_1xy^{n-1} + a_2x^2y^{n-2} + a_3x^3y^{n-3} + \ldots + a_rx^ry^{n-r} + a_nx^n = 0 \quad \ldots \quad \ldots \quad (1)$$

This can be written as (dividing by x^n)

$$\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + a_3\left(\frac{y}{x}\right)^{n-3} + \ldots + a_r\left(\frac{y}{x}\right)^{n-r} + \ldots + a_n = 0 \quad \ldots \quad (2)$$

Since this is an equation of the n th degree in y/x it must have n roots. Let the roots of this equation be $m_1, m_2, m_3, \ldots m_r, \ldots m_n$.

Then the given equation (1) or (2) must be the same as

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\ldots\left(\frac{y}{x} - m_r\right)\ldots\left(\frac{y}{x} - m_n\right) = 0$$

or, $(y - m_1x)(y - m_2x) \ldots (y - m_rx) \ldots (y - m_nx) = 0$

This equation is satisfied by the points which satisfy the separate equations.

$y - m_1x = 0,\ y - m_2x = 0, \ldots y - m_rx = 0, \ldots y - m_nx = 0$

which all pass through the origin. Conversely, the co-ordinates of all the points which satisfy these n equation satisfy equations (1) also. Hence the theorem.

Note : 1. An equation of the type

$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 \ldots + a_ry^{n-r}x^r \ldots \ldots + a_nx^n = 0$

in which the sum of the powers of x and y in every term is the same (say n) is called a Homogeneous equation (of degree n)

Note. 2. Since the given equation contains real co-efficients, the roots, if imaginary, must occur in pair.

Note. 3. The relation between the roots and the co-efficients can be determined by the symmetric functions of the roots (see author's Higher Algebra. Art. 53 and 54)

Note. 4. If two or more of the roots are equal the corresponding straight lines are coincident.

Art. 38. Angle between the lines represented by the equation. [N. U. H. 1998]

$ax^2 + 2hxy + by^2 = 0$ ... ... (1)

The axes are assumed to be rectangular. Let the lines represented by (1) be $y - m_1x = 0$ and $y - m_2x = 0$

So that (1) and $(y - m_1x)(y - m_2x)$ are the same.

$\therefore m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$ by Art 36 ... ... (2)

If θ be the angle between the straight lines

$y - m_1 x = 0$ and $y - m_2 x = 0$, then by Art. 25.

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{\{(m_1 + m_2)^2 - 4m_1 m_2\}}}{1 + m_1 m_2}$$

$$= \sqrt{\{(-2h/b)^2 - 4a/b\}(1 + a/b)} = 2\sqrt{(h^2 - ab)}/(a + b) \quad \ldots (2)$$

$$\therefore \tan\theta = \frac{2\sqrt{(h^2 - ab)}}{a + b} \quad \ldots \ldots \ldots (3)$$

where θ the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$

Cor. 1. Condition of Perpendicularity.

If the straight lines are perpendicular to each other then $\theta = 90°$ hence $\tan\theta = \infty$ then from (3)

$a + b = 0$

i. e. if the sum of the co-efficients of x^2 and y^2 is zero, then the straight lines will be perpendicular to each other.

Cor. 2. Condition of parallelism or Coincidence.

If the lines of $ax^2 + 2hxy + by^2 = 0$ be coincident, $\theta = 0$

From (3) $\tan\theta = 0$

i. e. $h^2 = ab$

which is the required condition for coincidence.

Here lines cannot be parallel as both of the lines pass through the origin, these are coincident straight lines.

Cor. 3. Two lines represented by $ax^2 + 2hxy + by^2 = 0$ will be real if $h^2 > ab$

Cor. 4. Two lines of above equation will be imaginary if $h^2 < ab$.

Art. 39. The bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 = 0$

[N. U. H. 2008, 1998]

Let the lines $y - m_1 x = 0$ and $y - m_2 x = 0$ represented by

$ax^2 + 2hxy + by^2 = 0$ ... ... (1) have slopes m_1 and m_2 and make angles θ_1 and θ_2 with OX. Let one of the bisectors make an angle θ with OX. from the diagram we have

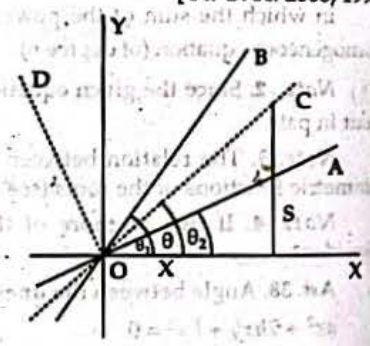


Fig. 12

$$\theta = \theta_1 + \frac{1}{2}(\theta_2 - \theta) \text{ or, } \frac{1}{2}\pi + \theta_1 + \frac{1}{2}(\theta_2 - \theta_1) = \frac{1}{2}(\theta_1 + \theta_2)$$

$$\text{or, } \frac{1}{2}\pi + \frac{1}{2}(\theta_1 + \theta_2)$$

$$\therefore 2\theta = \theta_1 + \theta_2$$

$$\text{or, } \pi + (\theta_1 + \theta_2) \quad \ldots \ldots \ldots (2)$$

In either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

$$\text{or, } \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{m_1 + m_2}{1 - m_1 m_2} \quad \ldots \ldots \ldots (3)$$

From Eq. (1), we have

$b(y - m_1 x)(y - m_2 x) = by^2 + 2hxy + ax^2$

$$\therefore m_1 + m_2 = -2\frac{a}{b},\ m_1 m_2 = \frac{a}{b} \quad \ldots \ldots \ldots (4)$$

Also if C (x, y) be any point on one of the bisectors, (see fig)

$$\tan\theta = y/x \quad \ldots \ldots \ldots (5)$$

From (3) by (4) and (5), we have

$$\frac{2xy}{x^2 - y^2} = \frac{2h}{a - b}$$

i. e. the required equation of the bisectors is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad \ldots \ldots \ldots (6)$$

Note. 1. In (2), first value θ is the angle made by the internal bisector OC with OX and the second value of θ is the angle made by the external bisector OD with OX.

Alternative Method :

Let the given equation $ax^2 + 2hxy + by^2 = 0$ represent the line

$y - m_1 x = 0$ and $y - m_2 x = 0$,

$$\therefore m_1 + m_2 = -2\frac{h}{b},\ m_1 m_2 = \frac{a}{b} \quad \ldots \ldots \ldots (1)$$

The equations of the required bisectors are

$$\frac{y - m_1 x}{\sqrt{(1 + m_1^2)}} = \pm\frac{y - m_2 x}{\sqrt{(1 + m_2^2)}} \quad \ldots \ldots \ldots (2)$$

Squaring, we have

$(y - m_1 x)^2 (1 + m_2^2) = (y - m_2 x)^2 (1 + m_1^2)$

or, $(m_1^2 - m_2^2)(x^2 - y^2) = 2xy\ \{(m_1 - m_2) - m_1 m_2 (m_1 - m_2)\}$

or, $(m_1 + m_2)(m_1 - m_2)(x^2 - y^2) = 2xy\ (m_1 - m_2)(1 - m_1 m_2)$

or, $(m_1 + m_2)(x^2 - y^2) = 2xy\ (1 - m_1 m_2)$

or, $-2\frac{h}{b}(x^2 - y^2) = 2xy\frac{(b - a)}{b}$

or, $h(x^2 - y^2) = xy(a - b)$

or, $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

GENERAL EQUATION OF SECOND DEGREE

Art. 40. Find the condition that the general equation of the second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

may represent a pair of straight lines. [N. U. H. 1997, 2002, 2007, 2009]

If we transfer the origin to a point (α, β), the point of intersection of two straight lines and keep the direction of the axes unaltered, the eq. (1) reduces to

$a(x+\alpha)^2 + 2h(x+\alpha)(y+\beta) + b(y+\beta)^2 + 2g(x+\alpha) + 2f(y+\beta) + c = 0$ by Art. 32 (Rule)

or, $ax^2 + 2hxy + by^2 + 2(a\alpha + h\beta + g)x + 2(h\alpha + b\beta + f)y + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

... ... (2)

The equation (2) may represent a pair of straight lines, if it is reduced to a homogeneous equation in x and y. This is possible if the co-effcients of x and y and the constant terms are saperately zero. *i. e.*

$a\alpha + h\beta + g = 0$ ... ... (3)

$h\alpha + b\beta + f = 0$ ... ... (4)

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$ ... (5)

The relation (5) may be written as

$\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$ ... ... (6)

By (3) and (4), the relation (6) becomes $g\alpha + f\beta + c = 0$ ... (7)

Now if we eliminate α and β from (3), (4) and (7), the required condition is

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad (8)$$

or, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ... ... (9)

Alternative method — By Calculus

Suppose $F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$\equiv (lx + my + n)(l'x + m'y + n')$ ... ... (1)

This relation being true for all values of x and y, it must admit of partial differentiation w. r. to each of the variables x and y. So we have, by taking partial differentaiation of (1)

$\frac{\delta F}{\delta x} = 2(ax + hy + g) = l'(lx + my + n) + l(l'x + m'y + n')$ ... (2)

and $\frac{\delta F}{\delta y} = 2(hx + by + f) = m'(lx + my + n) + m(l'x + m'y + n')$ ... (3)

Let (α, β) be the point of intersection of the lines represented by the general equation (1)

$\therefore l\alpha + m\beta + n = 0, \quad l'\alpha + m'\beta + n' = 0$

Putting α, β respectively for $x : y$ in (2) and (3) we have

$a\alpha + h\beta + n = 0$ ... ... (4)

$h\alpha + b\beta + f = 0$ ... ... (5)

Again puttin (α, β) in the left hand side of (1) we have

$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

or, $\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + (g\alpha + f\beta + c) = 0$

By (4) and (5)

$g\alpha + f\beta + c = 0$ ... ... (6)

Eliminating α, β from (4), (5) and (6) we have

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the required condition. It is both necessary and sufficient.

Cor. 1. Solving (4) and (5), we have

$\alpha = \frac{hf - bg}{ab - h^2} = \frac{G}{C}$ and $\beta = \frac{gh - af}{ab - h^2} = \frac{F}{C}$

where G, F, C are the co-factors of g, f, c, in

Note : 1. From the above discussions we see that if $F(x, y) = 0$ be the equation of two lines, the point of intersection (α, β) would be obtained by solving the equations.

$\frac{\delta F(\alpha, \beta)}{\delta \alpha} = 0, \frac{\delta F(\alpha, \beta)}{\delta \beta} = 0$

Cor. 2. If $a = 0$ and $b \neq 0$, we solve it for y and proceed as before. We thus find the condition $2fgh - bg^2 - ch^2 = 0$

Cor. 3. If $a = 0$, $b = 0$ and $h \neq 0$, then the general equations become $2hxy + 2gx + 2fy + c = 0$

or, $xy + (\frac{g}{h})x + (\frac{c}{h})y + \frac{c}{2h} = 0$ $\therefore$ Dividing by $2h$

or, $(x + \frac{f}{h})(y + \frac{g}{h}) = (\frac{fg}{h^2}) - \frac{c}{2h}$

or, $(hx + f)(hy + g) = fg - \frac{1}{2}ch$

and the condition that it may represent a pair of lines is

$fg - \frac{1}{2}ch = 0$ or, $2fg - ch = 0$

Note : 2 There is another method of finding the condition that the general second degree equation may represent a pair of straight lines. This process will be demonstrated by a worked out example. See worked sum No. 1.

Art. 41. (A) Angle between the lines given by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$ ... (1)

$ll' = a, mm' = b, nn' = c$

$lm' + l'm = 2h, mn' + m'n = 2f, nl' + n'l = 2g$ ... ... (2)

$lx + my + n = 0$ and $l'x + m'y + n' = 0$

or, $y = -\frac{l}{m}x - \frac{n}{m}$ and $y = -\frac{l'}{m'}x - \frac{n'}{m'}$

If θ be the angle between the lines then

$$\tan\theta = \frac{-l/m - (-l'/m')}{1 + (-\frac{l}{m})(-\frac{l'}{m'})}$$

$$= \frac{l'm - lm'}{l'l + mm'}$$

$$\frac{\sqrt{(lm' + l'm)^2 - 4\,ll'mm'}}{ll' + mm'} = \frac{\sqrt{2h^2 - ab}}{a + b} \quad \ldots \quad \ldots \quad \text{by (2)}$$

On account of relation (2), we have $(lx + my)(l'x + m'y) = ll'x^2 + 2(lm' + l'm)\,xy + mm'\,y^2 = ax^2 + 2hxy + by^2$

hence it follows that the lines represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

are parallel to the lines given by $ax^2 + 2hxy + by^2 = 0$

(B) Condition for perpendicularity of lines [N. U. H. 2007]

The lines will be perpendicular of $\theta = 90°$ *i. e.* $\tan\theta = \infty$

From (3), we have $ll' + mm' = 0$. *i. e.* if $a + b = 0$

This also follows from (4). Hence the given equation represents a pair of perpendicular lines, if $\Delta = 0$ and $a + b = 0$

(C) Condition for parallelism of the lines.

The lines in (4) will be parallel if $\frac{l}{l'} = \frac{m}{m'}$ or, $lm' = l'm$

or, $lm' - l'm = 0$

or, $(lm' + l'm)^2 - 4ll'\,mm' = 0$ or $h^2 - ab = 0$ by (4).

This also follows from (4). Hence the given equation represents a pair of parallel lines is $\Delta = 0$ and $h^2 = ab$

(D) Condition for coincidence of the lines. [N. U. H. 2007]

The line in (1) will be concident if $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$

i. e. $lm' - l'm = 0,\ mn' - m'n = 0,\ nl' - nl' = 0$

or, $(lm' + l'm)^2 - 4ll'\,mm' = 0,\ (mn' + m'n)^2 - 4mm'\,nn' = 0,\ (nl' + n'l)^2 - 4nn'll' = 0$

or, $h^2 - ab = 0,\ f^2 - bc = 0,\ g^2 - ac = 0$ by (2)

In such a case, the given equation becomes a perfect square. Hence the given equation will represent a pair of coincident lines if

$\Delta = 0,\ f^2 - bc = 0,\ g^2 - ac = 0,\ h^2 - ab = 0$

which are equavalent to $gh = af,\ hf = bg,\ fg = ch$

(E) Bisectors of the angles between the lines.

Let (α, β) be the points of intersection of the two lines represented by the equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

If we transfer the origin to (α, β) without changing the direction of axes, the first degree terms and the term independent of x and y will vanish. Then the transformed equation will be a homogeneous equation and will be a form $ax^2 + 2hxy + by^2 = 0$

The equations of the bisectors of the angle between the lines with (α, β) as origin by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

If we go back to our old axes, the above equations, of the bisectors are given by

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{(a-b)} = \frac{(x-\alpha)(y-\beta)}{h}$$

which is the required (See Art. 39 Rule) equation of the bisectors of the angles between the given lines.

It should be noted that with reference to the new origin the co-ordinates of the old origin are (−α, −β)

Art. 42. Lines Joining the origin to the intersection of a curve and a line.

Let the second degree equation representing the curve be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

and the first degree equation representing the line be

$lx + my = n \ldots \ldots$ or $\frac{lx}{n} + \frac{my}{n} = 1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$

Let the straight line meet the curve (1) at P and Q. We are to find the equation of the pair of lines OP and OQ.

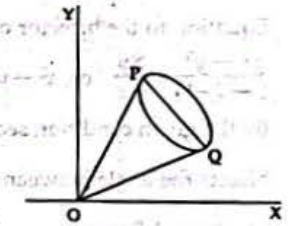


Fig. 13

Now making (1) homogeneous with the help of (3) we get

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0$$

Which, on simplification, reduces to an equation of the form

$Ax^2 + 2Hxy + By^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$

This is a Homogeneous equation of 2nd degree. It represents a pair of straight lines through the origin. Moreover the co-ordinates of P and Q satisfy (4), because the co-ordinates of P and Q satisfy (1) and (3). Therefore, (4) must represent OP and OQ and is our required equation.

Ex. 1. Prove that the equation $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents a pair of parallel lines. [R. U. 1997; D. U. 1986]

$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$

or, $x^2 + x(6y + 4) + (9y^2 + 12y - 5) = 0$ quadratic in x

$$x = \frac{-(6y + 4) \pm \sqrt{(6y + 4)^2 - 4\cdot 1\,(9y^2 + 12y - 5)}}{2}$$

$= (-6y - 4 \pm 6)/2 = -3y - 2 \pm 3$

$x + 3y + 5 = 0$ and $x + 3y - 1 = 0$

Hence the given equation represents two straight line $x+3y+5=0$ and $x+3y-1=0$. As these two lines differ only in constant terms, so these form a pair of parallel straight lines. See Art. 26 (1).

Ex 2. Show that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines.

$ax^2+2xy+by^2+\lambda(x^2+y^2)=0$ is equal to the angles between the other two lines of the system.

The equation of the straight lines bisecting the angles between

$ax^2+2hxy+by^2=0$ is $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$

Again, the equation of the straight lines bisecting the angles between $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$

or, $(a+\lambda)x^2+2hxy+(a+\lambda)y^2=0$ is

$$\frac{x^2-y^2}{(a+\lambda)-(b+\lambda)}=\frac{xy}{h} \text{ or, } \frac{(x^2-y^2)}{a-b}=\frac{xy}{h}$$

which is the same as the equation of bisectors for the first pair. Hence the result.

Ex. 3. If the pair of straight lines $x^2-2axy-y^2=0$ and $x^2-2bxy-y^2=0$ be such that each pair bisects the angle between the other pair, prove that $ab=-1$.

[D. U. 1953, 57, 61, R. U. 1964, 79]

Equation to the bisector of first pair is

$$\frac{x^2-y^2}{1-(-1)}=\frac{xy}{-a} \text{ or, } x^2-y^2=\frac{2xy}{-a} \quad \ldots \quad (1)$$

By the given condition second pair $x^2-y^2=2bxy$ ... (2)

bisects the angle between the first pair. Hence comparing we have $\dfrac{1}{-a}=b$

or, $ab=-1$ Proved.

Ex. 4 Prove that two of the lines represented by the equation $ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0$ will be at right angles if

$(b+d)(ad+be)+(a-e)^2(a+e+c)=0$

Let $ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=(x^2+kxy-y^2)(ax^2+k'xy-ey^2)=0$

$=ax^4+(k'+ak)x^3y+(-ek-k')xy^3+(kk'-a-e)x^2y^2+ey^4$

Now compare the co-efficients of x^4, x^3y, xy^3, x^2y^2 and y^4 from both sides, then

$k'+ak=b$, ... (1)

$-ek-k'=d$, ... (2)

$-a-e+kk'=c$ ... (3)

Find the value of k and k' from 1st and 3rd equations.

$k=(b+d)/(a-e)$, $k'=-(ad+bc)/(a-e)$

Therefore from (3), we have $kk'=a+e+c$

$$\frac{b+d}{a-e}\times\frac{ad+be}{a-e}=a+e+c$$

or, $(b+d)(ad+be)+(a-e)^2(a+e+c)$ proved.

Ex. 5. Prove the equation $y^3-x^3+3xy(y-x)=0$. represents three straight lines equally inclined to one another. [C. U. (Hons) 1987]

Changing the equation $y^3-x^3+3xy(y-x)=0$

in polar co-ordinates ; $x=r\cos\theta$, $y=r\sin\theta$, we have $r^3\sin^3\theta-r^3\cos^3\theta+3r^3\sin\theta.\cos\theta(\sin\theta-\cos\theta)=0$

or, $\sin^3\theta-\cos^3\theta+3\sin\theta\cos\theta(\sin\theta-\cos\theta)=0$

or, $(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+4\sin\theta\cos\theta)=0$

or, $(\tan\theta-1)(\tan^2\theta+4\tan\theta+1)=0$

Either $(\tan\theta-1=0$ *i. e.* $\theta=45°$. say $\theta_1=45°$ ... ... (1)

Again $\tan^2\theta+4\tan\theta+1=0$

or, $\tan\theta=[-4\pm\sqrt{(16-4)}]/2=-2\pm\sqrt{3}$

Let $\tan\theta_2=-2+(\sqrt{3})=\tan 165°$

$\therefore \theta_2=165°$

$\tan\theta_3=-2-\sqrt{3}=-\tan 75°=\tan(-75°)=\tan(360°-75°)=\tan 285°$

$\therefore \theta_3=285°$

Now $\theta_2-\theta_1=165°-45°=120°$

$\theta_3-\theta_2=285°-165°=120°$

$\theta_3-\theta_1=285°-45°=360°-75°-45°=-120°$

Hence the straight lines are inclined to each other at an angle of 120°. here RO is produced to OR'.

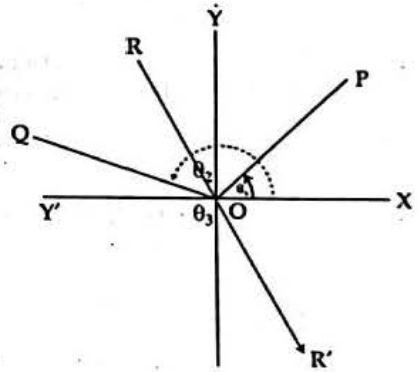


Fig : 14

Ex. 6. Show that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines if $a:h=h:b=g:f$. Also show that the distance between them is $2\sqrt{\left\{\dfrac{(g^2-ac)}{a(a+b)}\right\}}$

[N. U. H. 2006]

We know the condition that the general equation represents two straight lines is $abc+2fgh-af^2-bg^2-ch^2=0$ and it will represent two parallel straight lines if in addition, $ab-h^2=0$

Now $abc+2fgh-af^2-bg^2-ch^2=0$

or, $h^2c+2fgh-af^2-bg^2-ch^2=0$, Since $ab=h^2$

or, $bg^2-2fgh+af^2=0$

or, $bg^2 - 2fg\sqrt{(ab)} + af^2 = 0 \quad (\therefore h^2 = ab)$

or, $(\sqrt{bg} - \sqrt{af})^2 = 0$ or, $\sqrt{bg} = \sqrt{af}$

or, $\frac{g}{f} = \sqrt{(\frac{a}{b})} = \sqrt{(\frac{ab}{b^2})} = h/b \therefore h^2 = ab.$ or, $\frac{a}{h} = \frac{h}{b}$

$\therefore \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ which are the equired conditions

Let AB and CD be the lines $lx + my + n = 0$, $lx + my + n_1 = 0$ respectively. Take any point P (x_1, y_1) on AB. Draw PM perpendicular to CD.

$l^2 = a, m^2 = bn_1 = c, l(n + n_1) = 2g.$

$\therefore lx_1 + my_1 + n_1 = 0$ ... ... (1)

$\therefore$ PM $= \frac{lx_1 + my_1 + n_1}{\sqrt{(l^2 + m^2)}} = \frac{n_1 - n}{\sqrt{(l^2 + m^2)}}$ from (1)

$= \frac{\sqrt{\{(n + n_1)^2 - 4nn_1\}}}{\sqrt{(l^2 + m^2)}} = \frac{\sqrt{(4g^2/l^2 - 4c)}}{\sqrt{(a + b)}} = \frac{\sqrt{(4g^2/a - 4c)}}{\sqrt{(a + b)}}$

$= \frac{2\sqrt{\{(g^2 - ac)}}{\sqrt{\{a(a + b)}} = 2\sqrt{\left(\frac{g^2 - ac}{a(a + b)}\right)}$

Ex. 7. Prove that the straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = (bf^2 - ag^2)$

[D. U. 1956, C. U. (Hons) 1977, C. U. 1981]

Let the lines represented by the given equation be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

Since they are equadistant from the origin.

$\frac{n}{\sqrt{(l^2 + m^2)}} = \frac{n'}{\sqrt{(l'^2 + m'^2)}}$

or, $n^2(l'^2 + m'^2) = n'^2(l^2 + m^2)$

or, $n^2l'^2 - n'^2l^2 = n'^2m^2 - n^2m'^2$

or, $(nl' + n'l)(nl' - n'l) = (nm' + n'm)(n'm - nm')$

or, $(nl' + n'l)^2 \{(nl' + n'l)^2 - 4nn'll'\} = (nm' + n'm))^2$

$\{(n'm + nm')^2 - 4nn'mm'\}$ ... ... (2)

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$ comparing the co-efficients we have

$ll' = a, mm' = b, nn' = c, mn' + m'n = 2f, nl' + n'l = 2g, lm' + l'm = 2h$

Putting the values in (2) we have

$g^2(g^2 - ac) = f^2(f^2 - bc)$ or, $f^4 - g^4 = c(bf^2 - ag^2)$

Ex. 8. Show that the lines joining the origin to the intersection of $7x^2 + 8x^2 - 7x^2 + 6x - 12y = 0$ and $2x + y - 1 = 0$ are at right angles.

Making the first equation homogeneus with the help of

$2x + y = 1$, we have

$7x^2 + 8xy - 7x^2 + (6x - 12y)(2x + y) = 0$ or, $19x^2 - 10xy - 19y^2 = 0$ which is the equation of the required straight lines.

Since the sum of the co-efficients of x^2 and y^2 is equal to zero, the straight lines are at right angles.

Ex. 9. The axes being rectangular, find the equation to the pair of sraight lines meeting at the origin which are perpendicular to the pair given by the equation $ax^2 + 2hxy + by^2 = 0$

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then we have $m_1 + m_2 = -2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$ ... ... (1)

Now the equations of the lines perpendicular to the above lines and passing through the origin are $m_1y + x = 0$ and $m_2y + x = 0$

Their joint equation is

$(m_1y + x)(m_2y + x) = 0$, or $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

or, $(\frac{a}{b})y^2 - (\frac{2h}{b})xy + x^2 = 0$ by (1)

or, $bx^2 - 2hxy + ay^2 = 0$ Ans.

Ex. 10. The line joining the origin to the point P (2,3) is the diameter of a circle which cuts the lines $5x^2 - 12xy + 3y^2 = 0$ at Q and R. Find the combined equation of the lines PQ, PR.

P is a point (2, 3) and O is the origin. The circle on OP as diameter cuts at Q and R, the lines OQ & OR and whose combined equations are $5x^2 - 12xy + 3y^2 = 0$. PQ and PR are perpendicular to the chord OQ, OR respectively.

Let $5x^2 - 12xy + 3y^2$

$= 3(y - m_1x)(y - m_2x)$

$\therefore m_1 + m_2 = \frac{12}{3}, m_1m_2 = \frac{5}{3}$

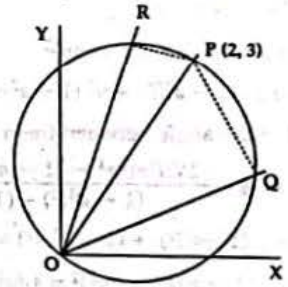


Fig. 15

The combined equation of the perpendiculars through O to OQ, OR is

$(m_1y + x)(m_2y + x) = 0$

or, $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

by (1) ... ... (2)

or, $5y^2 + 12xy + 4x^2 = 0$

Now the combined equation of the lines PR, PQ parallel to the lines represented by (2) is obtained by changing the origin O to

P (2, 3) or by putting $x - 2$ for x and $y - 3$ for y in (2)

Therefore, $5(y - 3)^2 + 12(x - 2)(y - 3) + 3(x - 2)^2 = 0$

or, $3x^2 + 12xy + 5y^2 - 48x - 54y + 129 = 0$

Ex. 11. Find the area of the triangle formed by the lines. [N.U.H. 2005]

$ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$

[N.U.H. R. U. Hons. 1978; D. U. '60; C. U. Hons. 1991]

Let the lines represented by the given pair be $y = m_1x$ and $y = m_2x$, so that $m_1 + m_2 = -2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$ ... ... (1)

These lines will meet the, third line at points.

$$\left(\frac{-n}{1+nm_1}, \frac{-nm_1}{1+n_1m_2}\right) \text{and} \left(\frac{-n}{1+mm_2}, \frac{-nm_2}{1+mm_2}\right)$$

which will be two vertices of the triangle and the third vertex is clearly (0, 0). Hence the area of the triangle,

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-n}{l+mm_1} & \frac{-nm_1}{1+mm_1} & 1 \\ \frac{-n}{l+mm_2} & \frac{-nm_2}{1+mm_2} & 1 \end{vmatrix} = \frac{n^2(m_2-m_1)}{2(l+mm_1)(l+mm_2)} = \frac{\frac{n^2}{2}\sqrt{[(m_1+m_2)^2-4m_1m_2]}}{l^2+lm(m_1+m_2)+m^2m_2m_2}$$

$$= \frac{n^2\sqrt{(h^2-2ab)}}{(am^2+bl^2-2hlm)}$$ by (1) Ans.

Ex. 12. The circle $x^2+y^2=a^2$ cuts off an intercept on the straight line $lx+my=1$, which subtends an angle of 45° at the origin, show that $4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$

The equation of the straight lines joining the origing to the points of intersection of

$x^2+y^2=a^2$ ... ... (1)

and $lx+my=1$ ... ... (2)

is $x^2+y^2=a^2(lx+my)^2$

or $x^2(1-a^2l^2)+y^2(1-a^2m^2)-2a^2lmxy=0$

Let the angle between them be $\phi = 45°$ (given)

$$\tan\phi = \frac{2\sqrt{[l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)]}}{(1-a^2l^2)+(1-a^2m^2)}$$ by Art. 41

or, $(1-a^2l^2)+(1-a^2m^2) = 2\sqrt{[l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)]}$

or, $(2-a^2l^2-a^2m^2)^2 = 4(a^2l^2+a^2m^2-1)$

or, $4[a^2l^2+m^2)-1] = [a^2(l^2+m^2)-2]^2$ Proved.

13. Prove that the equation $m(x^3-3xy^2)+y^3-3x^2y=0$ represent three straight lines equally inclined to one another. [N.U.H 1999]

Solution : Given that

$m(x^3-3xy^2)+y^3-3x^2y=0$ ... ... (1)

Since the equation (1) is a homogeneous is of degree 3. So it represents straight lines through origin.

Let $y = x\tan\theta$ represented the equation (1) whene θ is the slope of the line represented by (1)

$\therefore$ (1) $\Rightarrow m(x^3-3x.x^2\,\text{ten}^2\theta)+x^3\tan^3\theta-3x^2.x\tan\theta=0$

$\Rightarrow mx^3-3x^3m\tan^2\theta+x^3\tan^3\theta-3x^3\tan\theta=0$

$\Rightarrow m-3m\tan^2+\tan^3\theta-3\tan\theta=0$

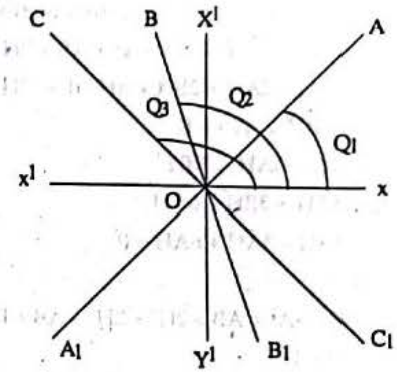
$\therefore\ m = \dfrac{3\tan\theta-\tan^3\theta}{1-3\text{ten}^2\theta}$

$\Rightarrow m = \tan3\theta$

$\therefore\ \tan3\theta = \tan\alpha$ let $m = \tan\alpha$

$\therefore\ 3\theta = n\pi+\alpha$

$\therefore\ \theta = \dfrac{n\pi}{3}+\dfrac{\alpha}{3}$, $n = 0, 1, 2$



Let A A_1 B B_1 and CC_1 makes an angles θ_1, θ_2 and θ_3 with x – axis.

$\theta_1 = \alpha/3, n=0$

$\theta_2 = \dfrac{\pi}{3}+\dfrac{\alpha}{3}, n=1$

$\theta_3 = \dfrac{2\pi}{3}+\dfrac{\alpha}{3}; n=2$

Angle between the lines AA_1 and BB_1 is $\theta_2-\theta_1 = \dfrac{\pi}{3}$

Angle between lines BB_1 and CC_1 is $\theta_3-\theta_2 = \dfrac{2\pi}{3}+\dfrac{\alpha}{3}-\dfrac{\pi}{3}\dfrac{d}{3} = \pi/3$

Angle between lines CC_1 and AA_1 is $\pi-(\theta_3-\theta_1)$

$= \pi-\theta_1+\theta_1$

$= \pi-\dfrac{2\pi}{3}-\dfrac{\alpha}{3}+\dfrac{\alpha}{3}$

$= \pi/3$

So, three lines equally inclined to one another.

14. Prove that two of the lines represented by equation $ax^4+bx^3y+cx^2y+dxy^3+ay^4=0$ will bisect angle between the other tow if $c+6a=0, b+d=0$ [N.U.H . 2000,'02,'09]

Solution : Given that

$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4=0$ ... ... (1)

Since equation (1) is a homogeneous equation of degree 4. So, it represent 4 straight lines through origin.

Let $Ax^2+2Hxy+By^2=0$ ... ... (2)

be the two of the lines represented by (1).

According to the problem, other two lines represented by (1) is the bisectors of the lines represented by (2)

Equation of the bisectors of the two lines represented by (2)

$$\frac{x^2-y^2}{A-B} = \frac{xy}{H}$$

$\Rightarrow Hx^2-Hy^2-Axy+Bxy=0$ ... ... (3)

$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4 = (Ax^2+2Hxy+By^2)(Hx^2-Hy^2-Axy+Bxy)$

$\Rightarrow ax^4+bx^3y+cx^2y^2+xy^4 = AHx^4-AHx^2y^2-A^2x^3y+ABx^3y$

$+2H^2x^3y-2H^2xy^3-2AHx^2y^2+2BHx^2y^2+BHx^2y^2-B_1+y^4-BAxy^3+Bxy^3$

Equating similar Co-efficient from both sides

$\therefore$ $a = AH, a = -BH, b = -A^2 + AB + 2H^2,$

$c = -AH - 2AH + 2BH + BH, \ d = -2H^2 - AB + B^2$

$\therefore$ $AH = BH \ \therefore A = -B$

$\therefore$ $c + 6a = -3AH + 3BH$

$= -3AH + 3BH + 6AH$

$= -3AH - 3AH + 6AH = 0$

$= 0$

$b + d = -A^2 + AB + 2H^2 - 2H^2 - AB + B^2$

$= -A^2 + B^2$

$= -A^2 + A^2 \ [\therefore B = A]$

$= 0$ Proved.

15. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$ represents two straight lines. Prove that sauare of the distance of their point of intersection from the origin is $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$.

[N.U.H -2001]

Solution : Given that

$ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$ $\therefore$ ... (1)

Since equation (1) reprsents pair of st-lines

Let $l_1x + m_1y + n_1 = 0$ ... ... (2)

and $l_2x + m_2y + n_2 = 0$ ... ... (3)

by the two lines represented by (1)

$\therefore$ $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (l_1x + m_1y + n_1).\quad (l_2x + m_2y + n_2)$

$\Rightarrow$ $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2$

Equating the similar Co-efficients from both sides

$\therefore$ $l_1l_2 = a, m_1m_2 = b, l_1m_2 + l_2n_1 = 2h, l_1n_2 + l_2n_1 = 2g$

$m_1n_2 + m_2n_1 = 2f, n_1n_2 = c.$

Now, $l_1x + m_1y + n_1 = 0$

$l_2x + m_2y + n_2 = 0$

$$\therefore \frac{x}{m_1n_2 - m_2n_1} = \frac{y}{n_1l_2 - n_2l_1} = \frac{1}{l_1m_2 - l_2m_1}$$

$$\therefore x = \frac{m_1n_2 - m_2n_1}{l_1m_2 - l_1m_1}, y = \frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1}$$

$\therefore$ Point of intersection of the lines (2) and (3) say A,

$$A\left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1}, \frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1}\right)$$

$$\therefore OA^2 = \left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1} - 0\right)^2 + \left(\frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1} - 0\right)^2$$

$$= \frac{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}{(l_1m_2 - l_2m_1)^2}$$

$$= \frac{(m_1n_2 + m_2n_1)^2 - 4m_1m_2n_1n_2 + (n_1l_2 + n_2l_1)^2 - 4l_1l_1n_1n_2}{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}$$

$$= \frac{4f^2 - 4bc + 4g^2 - 4ca}{4h^2 - 4ab}$$

$$= \frac{f^2 + g^2 - bc - ca}{h^2 - ab}$$

$$= \frac{f^2 + g^2 - c(a+b)}{h^2 - ab} \text{ (Proved)}$$

16 If two straight lines represented by the equation $x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan m\varphi + y^2\sin^2\varphi = 0$ makes an angle α and β with x – axis respectively, then show that $\tan\alpha - \tan\beta = 2$.

[N.U.H-2003, 2008]

Solution : Given equation

$x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan\varphi + y^2\sin^2\varphi = 0$ ... ... (1)

since the two lines represented by the equation (1)

makes an angle α and β with x – axis.

So their equation are

$y = x\tan\alpha$ and $y = x\tan\beta$

$$\therefore \tan\alpha + \tan\beta = \frac{2\tan\varphi}{\sin^2\varphi}$$

$$= \frac{2\sin\varphi}{\cos\varphi} \times \frac{1}{\sin^2\varphi}$$

$$= 2\sec\varphi \operatorname{cosec}\varphi$$

$$\text{and } \tan\alpha.\tan\beta = \frac{\tan^2\varphi + \cos^2\varphi}{\sin^2\varphi}$$

$$= \frac{\sin^2\varphi}{\cos^2\varphi} \cdot \frac{1}{\sin^2\varphi} + \frac{\cos^2\varphi}{\sin^2\varphi}$$

$$= \sin^2\varphi + \cot^2\varphi$$

Now, $(\tan\alpha - \tan\beta)^2 = (\tan\alpha + \tan\beta)^2 - 4\tan\alpha.\tan\beta$

$= (2\sec\varphi \operatorname{cosec}\varphi)^2 - 4.(\sec^2\varphi + \cot^2\varphi)$

$= 4\sec^2\varphi\operatorname{cosec}^2\varphi - 4\sec^2\varphi - 4\cot^2\varphi$

$= 4\{\sec^2\varphi\,(\operatorname{cosec}^2\varphi - 1) - \cot^2\varphi\}$

$= 4\{\sec^2\varphi.\,\cot^2\varphi - \cot^2\varphi\}$

$= 4\{\cot^2\varphi\,(\sec^2 - 1)\}$

$= 4\{\cot^2\varphi.\,\tan^2\varphi\}$

$\therefore (\tan\alpha - \tan\beta)^2 = 4$

$\therefore \tan\alpha - \tan\beta = 2$ (Proved)

17. If one of the straigh line given the equation $ax^2 + 2hxy + by^2 = 0$ concides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented them be perpendicular. Then show that $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{-aa_1bb_1}$ [N.U.H 1998, 2004]

Solution : Given equations

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots \quad (1)$$

$$a_1x^2 + 2h_1xy + b_1y^2 = 0 \quad \ldots \quad (2)$$

Let $y = m_1x$ and $y = m_2x$ be the two lines represented by (1)

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \quad \ldots \quad (3)$$

$$m_1m_2 = \frac{a}{b} \quad \ldots \quad (4)$$

According to the problem, $y = m_1x$ and $y = -\frac{1}{m_2}x$ be the two lines represented by (2)

$$\therefore \quad m_1 - \frac{1}{m_2} = -\frac{2h_1}{b_1} \quad \ldots \quad (5)$$

$$m_1.-\frac{1}{m_2} = \frac{a_1}{b_1} \quad \ldots \quad (6)$$

From (3) and (6) we get

$$m_1 + m_2 = -\frac{2h}{b}$$

and $-m_1 - \frac{a_1}{b_1}m_2 = 0$

$$(+) \quad m_2 - \frac{a_1}{b_1}m_2 = -\frac{2h}{b}$$

$$\Rightarrow \quad m_2(1 - \frac{a_1}{b_1}) = -\frac{2h}{b}$$

$$\Rightarrow \quad m_2.\frac{b_1 - a_1}{b_1} = -\frac{2h}{b}$$

$$\therefore \quad m_2 = -\frac{2hb_1}{b(b_1 - a_1)}$$

From (3); $m_1 + m_2 = -\frac{2h}{b}$

$$m_1 = -\frac{2h}{b} - m_2$$

$$= -\frac{2h}{b} + \frac{2hb_1}{b(b_1 - a_1)}$$

$$= \frac{2h}{b}\{\frac{b_1}{b_1 - a_1} - 1\}$$

$$= \frac{2h}{b}.\frac{b_1 - b_1 + a_1}{b_1 - a_1}$$

$$m_1 = \frac{2ha_1}{b(b_1 - a_1)}$$

From (4); $m_1m_2 = \frac{a}{b}$

$$\Rightarrow \frac{2ha_1}{b(b_1 - a_1)} \times \frac{-2hb_1}{b(b_1 - a_1)} = \frac{a}{b}$$

$$\Rightarrow \frac{4h^2a_1b_1}{b^2(b_1 - a_1)} = -\frac{a}{b}$$

$$\Rightarrow \frac{h^2a_1b_1}{b_1 - a_1} = -\frac{1}{4}ab$$

$$\Rightarrow \frac{h^2a_1^2b_1^2}{b_1 - a_1} = -\frac{1}{4}aa_1bb_1$$

$$\therefore \quad \frac{ha_1b_1}{b_1 - a_1} = \frac{1}{2}\sqrt{-aa_1bb_1} \quad \ldots \quad (7)$$

Again From (4); $m_1m_2 = \frac{a}{b} \therefore m_1 = \frac{a}{bm_2}$

From (5); $m_1 - \frac{1}{m_2} = -\frac{2h_1}{b_1}$

$$\frac{1}{m_2} - m_1 = \frac{2h_1}{b_1}$$

$$\Rightarrow \frac{1}{m_2} - \frac{a}{bm_2} = \frac{2h_1}{b_1}$$

$$\Rightarrow \frac{b - a}{bm_2} = \frac{2h_1}{b_1}$$

$$\Rightarrow m_2 = \frac{b_1(b - a)}{2h_1b}$$

$$m_1 = \frac{a}{bm_2}$$

$$= \frac{a}{b}\cdot\frac{2h_1b}{b_1(b-a)} = \frac{2h_1a}{b_1(b-a)}$$

From (6): $m_1 \cdot \frac{-1}{m_2} = \frac{a_1}{b_1}$

$$\Rightarrow -\frac{2h_1a}{b_1(b-a)} \times \frac{2h_1b}{b_1(b-a)} = \frac{a_1}{b_1}$$

$$\Rightarrow -\frac{4h_1^2ab}{b_1^2(b-a)^2} = \frac{a_1}{b_1}$$

$$\Rightarrow \frac{4h_1^2a^2b^2}{(b-a)^2} = -aa_1bb_1 \quad \ldots \quad (8)$$

$$\therefore \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{-aa_1bb_1} \quad \ldots$$

From (7) and (8) we get

$$\frac{ha_1b_1}{b_1-a_1} = \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{-aa_1bb_1} \text{ (Proved)}$$

EXERCISE V

[তারকা চিহ্নের অঙ্কগুলি অনার্স ছাত্রদের জন্য]

Find the lines represented by (নিম্নে সমীকরণের দ্বারা প্রকাশিত সরলরেখাগুলো নির্ণয় কর।)

1. $x^2\cos 2\theta - 4xy\cos\theta + 2y^2 + x^2 = 0.$ Ans. $y = x\cos\theta,\ y = x\cos\theta$
2. $3x^2 - 16xy + 5y^2 = 0$ Ans. $x - 5y = 0;\ 3x - y = 0$
3. $y^3 - xy^2 - 14x^2y + 24x^3 = 0$ Ans. $y = -4x,\ y = 2x,\ y = 3x$

Prove that the following equations represent two straight lines; find also their point of intersection and the angles between them.

4. $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$ Ans. $(\mathfrak{G}/2, -5/2))$; $90°$
5. $2y^2 - xy - x^2 + y + 2x - 1 = 0$ Ans. $(1, 0)$, $\tan^{-1}13$ [R. U. 1960]
6. $x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ Ans. $(\frac{-11}{13}, \frac{10}{13}; 90°)$
7. $2y^2 + 3xy - 5y - 6x + 2 = 0$ Ans. $(-1, 2, \tan^{-1}\frac{3}{2})$ [D. U. 1952]

Find the value of λ or k so that the following equations may represent pairs of straight lines.

8. $\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$ Ans. 4 [D. U. 1960]
9. $2x^2 - y^2 + xy - 2x - 5y + k = 0$ Ans. -4
10. $6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$ Ans. $+12$
11. $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ Ans. $3, \frac{9}{2}$
12. $kxy - 8x + 9y - 12 = 0$ Ans. 6

13. $6x^2 + 2kxy + 12y^2 + 22x + 31y + 20 = 0$ Ans. $\frac{171}{20}, \frac{17}{2}$ [D. U. 1962]
14. $6x^2 - 7xy + 16x - 3y^2 - 2y + k = 0$ Ans. 8 [D.U. (Hons) 1962]
15. $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ Ans. $\lambda = 2$ [D. U. 1962]

16. Find the angle between the straight lines.

$(x^2 + y^2)(\cos^2\theta\sin^2\alpha + \sin^2\theta) = (x\tan\theta - y\sin\alpha)^2$ Ans. 2θ [R. U. 1961]

16. Show that $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ equation representing two straight lines whose included angle is 45° (দেখাও যে $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ দুটি সরলরেখা প্রকাশ করে এবং এদের অন্তর্ভুক্ত কোণ 45°)

17. If the two straight line represented by $x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ make angles α and β with the axis of x, then show that $\tan\alpha - \tan\beta = 2$. ($x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ এর দুটি সরলরেখা x অক্ষরেখার সাথে αও β কোণ করলে দেখাও যে, $\tan\alpha - \tan\beta = 2$.) [U. N. H. 2003, 2008]

18. Show that the straight lines represented by $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ are perpendicular to each other. (দেখাও যে, $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ এর দুটি সরলরেখা পরস্পরের উপর লম্ব হবে।)

19. If the origin is joined by the two points ofintersection of the line $lx + my + n = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ there will be two straight lines show that they will be coincident. (প্রমাণ কর যে, $lx + my + n = 0$ এবং $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ বক্ররেখার ছেদ বিন্দুদ্বয়কে মূল বিন্দুতে সংযোগ করলে যে সরলরেখা দুটি উৎপন্ন হবে তারা সমপতিত হবে যদি $a^2l^2 + b^2m^2 = n^2$) [R. U. 1980]

20. The straight line $y = mx$ bisects the angle between two straight lines represented by $ax^2 - 2hxy + by^2 = 0$ if $h(1 - m^2) + m(a - b) = 0$. (দেখাও যে, $y = mx$ সরলরেখা $ax^2 - 2hxy + by^2 = 0$ দ্বারা প্রকাশিত সরলরেখাদ্বয়ের অন্তর্ভুক্ত কোণকে দ্বিখণ্ডিত করবে যদি $h(1 - m^2) + m(a - b) = 0$) [D. U. 1977]

21. Show that the two straight lines represented by $ax^2 + 2\lambda xy - ay^2 = 0$ will be perpendicular to each other. (দেখাও যে, $ax^2 + 2\lambda xy - ay^2 = 0$ পরস্পরের উপর লম্ব দুটি সরলরেখা প্রকাশ করে।)

22. Show that the four straight lines represented by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ will be on the sides of a square. (দেখাও যে, $12x^2 + 7xy - 12y^2 = 0$. এবং $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$. এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি বর্গক্ষেত্রের বাহুর উপর থাকবে।)

23. Show that the four straight lines represented by $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ will make equal angles with each other. (দেখাও যে, $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ এর দ্বারা প্রকাশিত চারটি সরলরেখা একে অন্যের সাথে সমান কোণে থাকবে।)

24. Show that the four straight lines represented by $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ (দেখাও যে, $y^2 - 4y + 3 = 0$ এবং $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ will form a parallelogram. এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক উৎপন্ন করবে।)

25. Show the four straight lines represented by $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ and $x^2 + 4xy - 2y^2 = 0$ will form a rhombus through the origin. (দেখাও যে, $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ এবং $x^2 + 4xy - 2y^2 = 0$ এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি রম্বস প্রকাশ করবে।)

26. If the two straight lines through the origin represented by $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ are perpendicular to each other then find the combined equation of straight lines passing through the origin. (মূলবিন্দুগামী সরলরেখা দুটি $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ দ্বারা প্রকাশিত সরলরেখা দুটির উপর পরস্পর লম্ব হলে মূলবিন্দুগামী সরলরেখাদ্বয়ের যুগ্ম সমীকরণ নির্ণয় কর।) Ans. $2y^2 - 5xy + 2x^2 = 0$ [C. U. 1990]

27. Prove that the angles between the straight lines. $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ have the same pair of bisectors. Interpret the case when $\lambda = -(a + b)$ (দেখাও যে $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ দ্বারা প্রকাশিত সরলরেখা দুইটির কোণের দ্বিখণ্ডকদ্বয় একই হবে। যখন $\lambda = -(a + b)$ তখন এটা কি প্রকাশ করে?) [R. U. Hons. 1977; '80]

28. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin. show that $h(g^2 - f^2) = fg(a - b)$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে, $h(g^2 - f^2) = fg(a - b)$) [D. U.(Hons.) 1959; C. U. Hons.. 1989]

29. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, prove that the square of the distance of their point of intersection from the origin is $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে সরলরেখা দুইটির ছেদবিন্দু হতে মূলবিন্দুর দূরত্ব $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$ হবে।) [U. N. H. 2001]

30. Show that the pair of straight lines joining the origin O to the intersections A and B of the line $lx + my = 1$ with the conic. $ax^2 + by^2 = 1$ has the equation. $(a - l^2)x^2 - 2lmxy + (b - m^2)y^2 = 0$ ($lx + my = 1$ এবং $ax^2 + by^2 = 1$ কণিকের ছেদবিন্দু দুইটি A এবং B হলে, দেখাও যে সরলরেখা দুইটির সংযুক্ত সমীকরণ হবে $(a - l^2)x^2 - 2lmxy + (b - m^2)y^2 = 0$) [R. U. (Pass) 1985; R. U. 1962, '82]

31. দেখাও যে, $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ এবং $4x^2 - 4xy + y^2 + 4x - 2y - 3 = 0$ দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক প্রকাশ করে।

32. Prove that two of the lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect angle between the other two if $c + 6a = 0$, $b + d = 0$. [N. U. H. 2000, '02, '09][C. H. 1979, 78]

33. Find the equations of the bisectors of the angles between the lines represented by

(a) $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$ [D. U. 1962]
Ans. $7(23x + 25)^2 - 7(23y - 43)^2 = 44(23x + 25)(23y - 43)$

(b) $2x^2 + 7xy + 6y^2 + 13x + 22y + 20 = 0$ Ans. $7x^2 - 7y^2 + 8xy - 4x - 58y - 83 = 0$ [R. U. 1962]

34. Prove that the angles between the lines joining the origin to the intersections of the line $y = 3x + 2$ with the curve. $x^2 + 3y^2 + 2xy + 4x + 8y - 11 = 0$ is $\tan^{-1}(2\sqrt{2})/3$

35. Prove that one of the line represented by the equations $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ will bisect the angle between the other two if $(3a + c)^2(bc + 2ad - 3ad) = (b + 3d)^2(bc + 2ab - 3ad)$

36. Show that the lines joining the origin to the intersection of $x^2 + y^2 + 19x + 4y - 3 = 0$ and $3x + 4y = 1$ are at right angles. [R. H. 1988; D. U. 1961]

37. দেখাও যে $3x^2 + 3xy - 3y^2 + 2x + 5y = 0$ এবং $3x - 2y = 1$ সরলরেখার ছেদবিন্দু দুটির সাথে মূলবিন্দুর সংযোগ সরলরেখা দুটি পরস্পর লম্ব হবে। [C. U. 1987]

37. (a) Find the angle between the lines joining the origin to the pt of intersection of $y = x + 1$ with $x^2 - 3y^2 + 2xy - 3x + 3y + 1 = 0$ [R. U. 1981] Ans. $\tan^{-1} 2\sqrt{5}/5$.

38. দেখাও যে $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$ দ্বারা প্রকাশিত দু' জোড়া পরস্পর লম্ব সরলরেখা প্রকাশ করে। যদি দু' জোড় সরলরেখা সমাপতিত হয় তা হবে যদি $2b^2 = a^2 + 3ac$

38. (a) Prove that $x^4 + y^4 - 4xy(x^2 - y^2) + 2cx^2y^2 = 0$ represents a pair of pair of perpendicular straight lines, also show that if $c = 1$ then those of straight lines will be coincident. [C. U(P) 1989]

39. Prove that two of the straight lines represented by the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$. will be at right angles if. $a^2 + ac + bd + d^2 = 0$. [C. H. 1980]

40. Show that lines joining the origin to the point of intersections of $y^2 - 4ax = 0$ and $y = mx + c$ will be coincident of $c = \frac{a}{m}$

41. Prove that the straight lines represented by the equation $\cos 3\alpha(x^3 - 3xy^2) + \sin 3\alpha(y^3 - 3x^2y) + 3a(x^2 + y^2) - 4a^3 = 0$ form an equailateral triangle whose area is $3\sqrt{3}a^2$. [C.H.1989]

42. Prove that the equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to one another. [N. U. H. 1999] [R. U. 1987]

[প্রমাণ কর যে $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ দ্বারা তিনটি সরলরেখা প্রকাশ করে এবং এরা একে অন্যের সাথে একই কোণে থাকবে।]

43. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve

$(x - h)^2 + (y - k)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

(প্রমাণ কর যে $kx + hy = 2hk$ এবং বক্ররেখা $(x - h)^2 + (y - k)^2 = c^2$ ছেদবিন্দুর সাথে মূলবিন্দুর সংযোগ সরলরেখাদ্বয় পরস্পরের উপর লম্ব হবে যদি $h^2 + k^2 = c^2$)

44. If one of the straight line given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented by them be perpendicular, prove that $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$. [N. U. H. 1998, 2004]

(যদি $ax^2 + 2hxy + by^2 = 0$ রেখাদ্বয়ের একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখাদ্বয়ের একটি রেখার সাথে সমাপতিত হয় এবং অপর দুটি রেখা পরস্পরের সাথে লম্ব হয়, তবে দেখাও যে $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$.

45. Find the equation of the line through the origin and perpendicular to the lines $5x^2 - 7xy - 3y^2 = 0$

($5x^2 - 7xy - 3y^2 = 0$ রেখাদ্বয়ের উপর লম্ব এবং মূলবিন্দুগামী সরলরেখাদ্বয়ের সমীকরণ নির্ণয় কর।

Ans. $3x^2 + 7xy + 5y^2 = 0$) [D. U. H. 1979]

46. Show that $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ represents pair of straight lines.

(দেখাও যে, $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ সমীকরণ দুটি সরলরেখা প্রকাশ করে।) [D. U. 1979]

47. Show that the st. lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ will be at right angles. if $g(a_1 + b_1) = g(a + b)$ [D. U. '76]

48. Show that the equation $bx^2 - 2hxy \quad ay^2 = 0$ represents a pair of st. lines which are at right angle to the pair given by equation $ax^2 + 2hxy + by^2 = 0$ [R. U. 1985]

(দেখাও যে, $bx^2 - 2hxy + ay^2 = 0$ এর সরলরেখা দুটি $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ের সাথে পরস্পর লম্ব হবে।) [D. U. 1976, R. U. 1978, '85]

49. If one of the lines $ax^2 + 2hxy + by^2 = 0$ be perpendicular to one of the lines $a'x^2 + 2h'xy + b'y^2 = 0$, prove that $(aa' - bb')^2 + 4(a'h + bh')(ah' + b'h) = 0$. [R.H. 1977]

(যদি $ax^2 + 2hxy + by^2 = 0$ সরলরেখার একটি $a'x^2 + 2h'xy + b'y^2 = 0$ এই সরলরেখার একটির উপর পরস্পর লম্ব হয়, তা হলে প্রমাণ কর যে $(aa' - bb')^2 + 4(a'h + bh')(ah' + b'h) = 0$.) [R.H. 1997]

50. Prove that $(ax + by)(\alpha x + \beta y) + kxy - (a + \alpha)x - (b + \beta)y + 1 = 0$ represents a pair of straight lines if $k = (a - \alpha)(b - \beta)$. Find the co-ordinates of their point of intersection.

(প্রমাণ কর যে $(ax + by)(\alpha x + \beta y) + kxy - (a + \alpha)x - (b + \beta)y + 1 = 0$ দুটি সরলরেখা প্রকাশ করবে, যদি $k = (a - \alpha)(b - \beta)$. হয় সরলরেখা দুটির ছেদবিন্দু নির্ণয় কর।) [D. U. 1975]

51. Prove that product of the perpendiculars from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{\{(a - b)^2 + 4h^2\}}}$ [C. U. 1986; R. U. 1979]

* 52. দেখাও যে $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ে একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখা দুটির একটি রেখার সাথে সমাপতিত হবে যদি

$(ab_1 - a_1b)^2 = 4(a_1h - ah_1)(bh_1 - b_1h)$. [. খ. ৩. ১৯৯৭] [R. U. 1988; D.U. 1989]

* 53. If $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, the area of the triangle formed by their bisectors and the axis of x is

$$\Delta = \frac{\sqrt{\{(a - b)^2 + 4h^2\}}}{2h} \frac{ca - g^2}{ab - h^2}$$ [C.H. 1989; R.U. 1982]

(S দ্বারা দুটি সরলরেখা সূচিত হলে, দেখাও যে তাদের অন্তর্গত কোণের দ্বিখণ্ডকদ্বয় x অক্ষের সাথে ত্রিভুজ উৎপন্ন করে, তার ক্ষেত্রফল Δ হবে)

* 54. Show that four lines given by the equations $(y - mx)^2 = c^2(1 + m^2)$, $(y - nx)^2 = c^2(1 + n^2)$ form a rombus.

* 55. The vertices of a triangle lie on the lines $y = m_1x$, $y = m_2x$, $y = m_3x$ the circumcentre being at the origin, prove that the locus of the other centre is the line $L \equiv x(\sin\theta_1 + \sin\theta_2 + \sin\theta_3) - y(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) = 0$ (একটি ত্রিভুজের শীর্ষবিন্দুগুলো যথাক্রমে $y = m_1x$, $y = m_2x$, $y = m_3x$ সরলরেখার উপর থাকবে। এদের পরিলিখিত বৃত্তের কেন্দ্র মূল বিন্দুতে হলে দেখাও যে অন্য কেন্দ্রগুলোর সঞ্চারপথ একটি সরল রেখা L হবে।)

* 56. The distance from the origin to orthocentre of the triangle formed by the lines $x/\alpha + y/\beta = 1$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{(a + b)\alpha\beta\sqrt{(\alpha^2 + \beta^2)}}{a\alpha^2 - 2h\alpha\beta + b\beta^2} = L$$

($x/\alpha + y/\beta = 1$ এবং $ax^2 + 2hxy + by^2 = 0$ সরলরেখার দ্বারা সৃষ্ট ত্রিভুজের লম্ববিন্দু হতে মূল বিন্দুর দূরত্ব L হবে।)

* 57. Show that the centroids of the triangles of which the three perpendiculars lie along the lines. $y - m_1x = 0$, $y - m_2x = 0$, $y - m_3x = 0$ lie on

$y(3 + m_2m_3 + m_3m_1 + m_1m_2) = x(m_1 + m_2 + m_3 + 3m_1m_2m_3)$

(দেখাও যে কোন ত্রিভুজের লম্বত্রয় $y - m_1x = 0$, $y - m_2x = 0$, $y - m_3x = 0$ সরলরেখা বরাবর থাকলে ত্রিভুজের ভরকেন্দ্রে $y(3 + m_2m_3 + m_3m_1 + m_1m_2) = x(m_1 + m_2 + m_3 + 3m_1m_2m_3)$ এর উপর থাকবে।)

* 58. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles of by the lines $ax^2 + 2hxy + by^2 = 0$. Prove that the locus of its vertex is.

$S = (a + b)(x^2 + y^2) + 2h(fy + gx) + (a - b)(fx - gy) = 0$

(কোন ত্রিভুজের ভূমি (f, g) একটি নির্দিষ্ট বিন্দুগামী এবং এর বাহুগুলো $ax^2 + 2hxy + by^2 = 0$ সরলরেখার দ্বারা লম্বভাবে দ্বিখণ্ডিত হয়। দেখাও যে ত্রিভুজের শীর্ষবিন্দুর সঞ্চারপথ S হবে।)

* 59. Show that the equation $E = ax^2 + 2hxy + by^2 = [(a + b)\sin^2\theta + \sqrt{(h^2 - ab)}\sin 2\theta](x^2 + y^2)$ represents two straight lines having the same bisectors as $ax^2 + 2hxy + by^2 = 0$ and making equal angles θ with them respectively. (দেখাও যে সমীকরণ E দুটি সরলরেখা প্রকাশ করে যাদের দ্বিখণ্ডকদ্বয় $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ের দ্বিখণ্ডকদ্বয় একই হয় এবং যথাক্রমে তাদের সাথে θ সমান কোণ উৎপন্ন করে।)

* 60. Show that the equation $S = (ab - h^2)(ax^2 + 2hxy + by^2 + 2gx + 2fy) + af^2 + bg^2 - 2fgh = 0$ represents a pair of straight lines, and that these straight lines form a rhombus with the lines $ax^2 + 2hxy + by^2 = 0$ provided that $(a - b)fg + h(f^2 - g^2) = 0$ (দেখাও যে S সমীকরণ দুটি সরলরেখা প্রকাশ করে এবং দেখাও যে S সমীকরণ এর দুটি সরলরেখা এবং $ax^2 + 2hxy + by^2 = 0$ সরলরেখা দুটি একত্রে একটি রম্বস উৎপন্ন করবে যদি $(a - b)fg + h(f^2 - g^2) = 0$ হয়।)

61. $ax^2 + 2hxy + by^2 = 0$ দ্বারা নির্ধারিত সরলরেখাযুগল এবং $x + y = 1$ সরলরেখা দ্বারা উৎপন্ন ত্রিভুজের ক্ষেত্রফল নির্ণয় কর। Ans. $\sqrt{(h^2 - ab)}/(a + b - 2h)$ [D. U. 1987]

62. Find the condition that two of the lines given by $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ may be at right angles (কোন শর্তে $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ সমীকরণের তিনটি সরলরেখার মধ্যে দুটি সরলরেখা পরস্পর লম্ব হবে) Ans. $a^2 + 3bd + 3ac + d^2 = 0$

63. Show that the or the centre of the triangle formed by the lines $ax^2 + 2hxy + by^2 =$ and $lx + my = 1$ is a point (x_1, y_1) such that $\dfrac{x_1}{l} = \dfrac{y}{m} = \dfrac{a + b}{am^2 - 2hlm + bl^2}$

GENERAL EQUATION OF SECOND DEGREE 6

Art. 43. Conic Section regarded as locus.

Definition : If a point P moves in a plane such a way that the ratio of its distance PS from a fixed point S in the plane to its perpendicular distance PM from a fixed straight line XM in it, is always a constant, the locus of the point P is called a Conic Section or briefly a Conic.

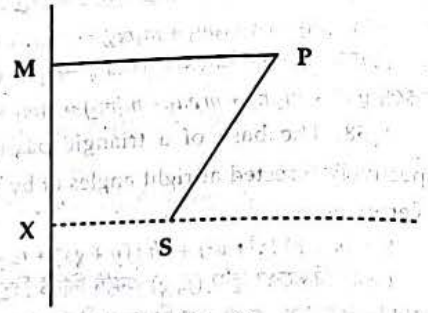


Fig. 16

The constant ratio is called the **eccentricity** of the conic and is generally represented by the letter 'e'. The fixed point S is called the **focus** and the fixed straight line XM is called **directrix** of the conic.

The conic is called a **Parabola, and Ellipse or a Hyperbola according** as the eccentricity $e = , < \text{or} > . 1$.

That is for a **Parabola,** SP = PM

for an **Ellipse,** SP = ePM $e < 1$

for **Hyperbola,** SP = ePM $e > 1$

The eccentricity is a positive number.

Art. 44. Conic section regarded as a section of a double right circular Cone.

Literally the term conic section means the section of a conic but the term is used in a restricted sense to mean the section of a particular kind of cone known as **Circular** cone. This is a cone generated by the revolution of an infinite straight line about a fixed point in such a way that it is always inclined at a constant angle to a fixed straight line passing through the fixed point.

The fixed point is called the **Vertex**, the fixed straight line is called the **Axis**, the revolving line is called **Generator** and the constant angle of inclination of the revolving line to the fixed line is called the **Semivertical** Angle of the Cone.

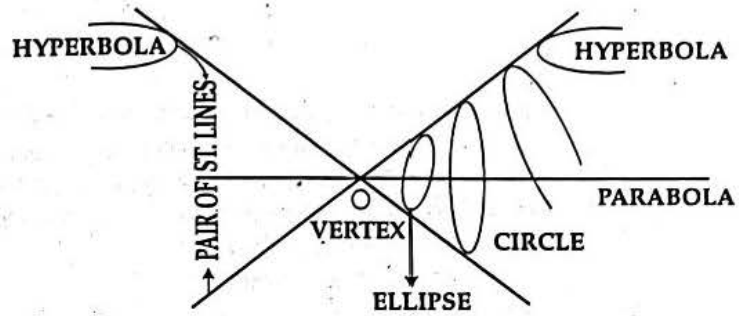


Fig : 17

If the cutting plane while passing through the vertex cuts the cone again, the curve section is a pair of straight lines intersecting at the vertex.

If the cutting plane without passing through the vertex is perpendicular to the axis the curve of the section is a Circle.

If the cutting plane neither passes through the vertex nor is perpendicular to the axis, the curve of the section is a **parabola** ; an ellipse or a **hyperbola** according as the inclination of the cutting plane to the axis is equal, greater or less than the inclination of the generator to the same line.

The different positions of the cutting plane and the curves of sections obtained are shown in the **above fig.**

Let us explain the conics above in more details. The above figure is a double right circular cone and a plane cuts it in different positions giving different conic sections such as —

(1) *A pair of straignt lines as obtained from the double cone by a section through the common vertex cutting the double cone in two generators.*

(2) *A circle is obtained by a plane section perpendicular to the common axis.*

(3) *A parabola is obtained by a plane section parallel to a generator of the double cone.*

(4) *An ellipse is obtained by a section cutting only one half of the double cone, and not perpendicular to the axis or parallel to the generator.*

(5) *A hyperbola is obtained by a section cutting both halves of the double cone and not passing through the common verte.*

Note : The circle is a degenerate form of ellipse and a pair of straight lines is a degenerate form of hyperbola.

Art. 44. (a) একটি সরল বৃত্তীয় কোণকে (Right Circular cone) এর কোন সমতল (Plane) দ্বারা বিভিন্ন ছেদিতাংশকে কোণকচ্ছেদ (Conic Section) **বা সংক্ষেপে কণিক বলা হয়।**

একটি সরলরেখা একটি নির্দিষ্ট বিন্দু দিয়ে এবং একটি নির্দিষ্ট সরলরেখার সাথে একটি ধ্রুব কোণে ঘুরে যে কোণক উৎপন্ন করে একে সরল বৃত্তীয় কোণক বলা হয়। নির্দিষ্ট বিন্দুটিকে বলা হয় শীর্ষক বিন্দু (Vertex), নির্দিষ্ট সরল রেখাকে বলা হয় অক্ষরেখা (Axis)। ঘূর্ণায়মান সরলরেখাকে বলা হয় কারিকা রেখা (Generating line) এবং অক্ষরেখার সাথে কারিকারেখার ধ্রুব কোণকে বলা হয় অর্ধশীর্ষ কোণ (Semivertical angle)।

মনে করি COD একটি সরলরেখা O বিন্দুতে বিয়ারিং এর মধ্যে স্থির আছে। এখন সরলরেখাটি O বিন্দুর চারদিকে ঘুরলে, COD সরলরেখার মাথা C এবং D দুটি বৃত্ত CB ও AD অঙ্কন করবে এবং চিত্রটি একটি দ্বিকোণক (Double Cone) হবে। O এর শীর্ষবিন্দু।

দ্বি-কোণককে একটি সমতল (plane)-এর সাহায্যে বিভিন্নভাবে O-এর মধ্যে দিয়ে বিভক্ত করতে পারা যায়। O-এর মধ্য দিয়ে সমতলটি কোণটিকে বিভক্ত করলে কোণকচ্ছেদ COD ও AOB দুটি সরলরেখা হবে। [Fig. 18 (a)]।

যদি কারিকাদ্বয়ের একটি শীর্ষবিন্দুগামী সমতলের উপর থাকে, তাহলে সমতলের দ্বারা কোণকটির ছেদিতাংশের সঞ্চার পথ একটি সরল রেখা হবে। আবার যদি সমতলটি কেবল শীর্ষ বিন্দু দিয়ে যায়, কিন্তু ডবল কোণকের কোন একটি কোণককে ছেদ না করে তাহলে সমতলটির কোণকচ্ছেদের সঞ্চার পথ একটি বিন্দু হবে।

সমতলটির দ্বারা কোণকচ্ছেদের সঞ্চার পথ একজোড়া সমান্তরাল সরলরেখা হবে না।

(The degenerate locus of two parallel lines cannot be obtained as plane section of a cone)

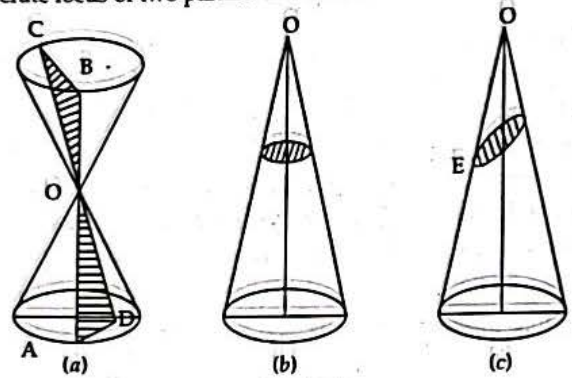


Fig : 18 (a)

সমতলটি যদি O দিয়ে না যায় এবং কোণকের ভূমির সাথে সমান্তরাল অথবা অক্ষরেখার সাথে লম্ব হয়ে কোণককে বিভক্ত করে, তখন কোণকচ্ছেদটির বৃত্ত হবে। [Fig. 19 (b)]

ছেদকারী সমতলটি যদি যে কোন কারিকারেখার সমান্তরাল হয়ে কোণককে ছেদ করে; তখন ছেদিতাংশটি একটি O পরাবৃত্ত হবে। [Fig. 18 (c)]

ছেদকারী তলটি যদি সমান্তরাল অবস্থা হতে যে কোন কোণে অথবা ছেদকারী সমতল ভূমির সাথে সমান্তরাল না হয়ে কোণককে সম্পূর্ণরূপে ছেদ করে, সে কোণকচ্ছেদ E-কে উপবৃত্ত (Ellipse) বলা হবে। [Fig. 18 (c)]।

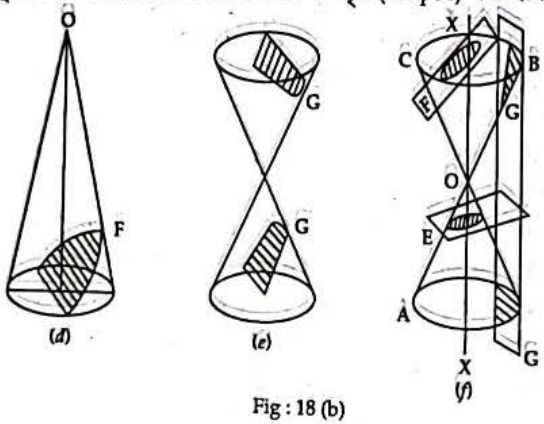


Fig : 18 (b)

ছেদকারী সমতলটি ঘুরে যদি উভয় কোণকে ছেদ করে অথবা ছেদকারী সমতলটি কোণকের অক্ষের সাথে সমান্তরাল হয়ে কোণককে ছেদ করে, তখন যে G বক্ররেখা দুটি উৎপন্ন হয়, তাকে অধিবৃত্ত (Hyperbola) বলা হয়। [Fig. 18 (c)]

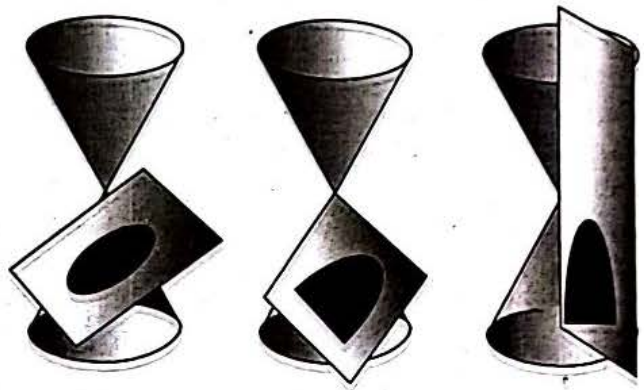


Fig : 18 (c)

টীকা : বৃত্তকে উপবৃত্তের এবং সরলরেখাদ্বয়কে অধিবৃত্তের অপজাত আকার বলা হয়।

উল্লিখিত পর্যালোচনা হতে দেখা যায় যে, সরলরেখাদ্বয়, বৃত্ত, পরাবৃত্ত, উপবৃত্ত, অধিবৃত্তকে সাধারণত **কণিক** বলা হয়।

উল্লিখিত কোণচ্ছেদগুলো একটি দ্বিমাত্রিক সমীকরণ হতে পাওয়া যায়।

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

এ সমীকরণ হতে (i) $h^2 - 4ab < 0$, হলে বক্ররেখাটি উপবৃত্ত, বৃত্ত, একটি বিন্দু অথবা কোন বক্ররেখাই হবে না।

(ii) যদি $h^2 - 4ab > 0$ বক্ররেখাটি অধিবৃত্ত দুটি ছেদকারী সরলরেখা হবে।

(iii) যদি $h^2 - 4ab = 0$ হলে বক্ররেখাটি পরাবৃত্ত, দুটি সমান্তরাল সরলরেখা অথবা কোন বক্ররেখাই হবে না।

Art. 45. Cartesian equation of a conic given its focus, directrix and eccentricity.

Let S (α, β) be the given focus and $lx + my + n = 0$ the given directrix of a conic whose eccentricity is e. Then if P (x, y) be any position of the moving point tracing the conic, wd have from fig. 18.

$$SP^2 = (x - \alpha)^2 + (y - \beta)^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{and} \quad PM = \frac{lx + my + n}{(\sqrt{l^2 + m^2})} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$\therefore$ $SP^2 = ePM$ *by* Art. 43.

$SP = e^2 PM^2$

$$\text{or}, (x - \alpha)^2 + (y - \beta)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\text{or}, x^2 \{(l^2 + m^2) - e^2 l^2\} - 2e^2 lmxy + y^2 \{(l^2 + m^2) - e^2 m\}$$
$$- 2x \{\alpha (l^2 + m^2) - e^2 nl^2\} - 2 \{\beta(l^2 + m^2) - e^2 mn\}$$
$$+ (l^2 + m^2)(\alpha^2 + \beta^2) - e^2 n^2 = 0$$

which is a relation connecting the co-ordinates (x, y) of their moving point P in any position. This equation of the conic is traced out by the point P.

When $e < 1$, the equation represents an ellipse, when $e > 1$, it represents a hyperbola and a parabola when $e = 1$.

The equation (4) is of the form.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

which is called the General Equation of the Second degree in a plane. If the focus is chosen for the origin, $\alpha = 0$; $\beta = 0$, so that the equation (3) is

$(l^2 + m^2)(x^2 + y^2) = e^2 (lx + my + n)^2$ ... ... (5)

With rectangular axis, it represents a conic; whose focus is at the origin whose eccentricity is $\sqrt{(l^2 + m^2)}$

Note : For further and detailed information, see Art 43 and Art. 44.

Art. 46. Polar Equation of a Conic.

To find the polar equation of a conic, the focus being the pole.

Let S be the focus SX the initial line and ZM the directrix. Take any point P on the conic and let its co-ordinates be (r, θ).

Then $SP = r$, $\angle XSP = \theta$

Let the length of the semilatus rectum and the eccentricity be l and e respectively. Draw PN perpendicular to the initial line and PM perpendicular to the directrix.

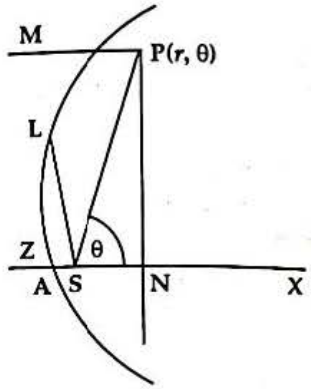


Fig : 19 .

Then

$r = SP = ePM = e(NZ + SN)$

$eZS + eSP\cos\theta = e.\dfrac{l}{e} + e.r.\cos\theta$; $SL = e$,

$SZ = l + er\cos\theta$ or, $r = \dfrac{l}{1 - e\cos\theta}$ ... ... (1)

If the initial line be SZ, then the equation (1) becomes

$r = \dfrac{l}{1 - e\cos(\pi - \theta)} = \dfrac{l}{1 + e\cos\theta}$

$r = \dfrac{l}{1 + e\cos\theta}$ ... ... (2)

If $e = 0$, the equation reduces to $r = l$ or $r =$ constant.

The curve is a circle

If $e = 1$, The curve is a parabola

If $e < 1$, The curve is an ellipse

If $e > 1$, The curve is a hyperbola.

Art. 47. To show that the general equation of second degree in x and y represents a conic.

Let the general equation be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

Let the axis be turned through an angle θ, then Eq. (1) takes the form by putting.

$x\cos\theta - y\sin\theta$ for x

and $x\sin\theta + y\cos\theta$ for y, (See Art. 33)

$x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2xy[h(\cos^2\theta + \sin^2\theta)$

$-(a - b)\sin\theta\cos\theta] + y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta + 2x(g\cos\theta + f\sin\theta) + 2y(f\cos\theta$

$-g\sin\theta) + c = 0$ ... ... (2)

Now if θ be chosen that

$h(\cos^2\theta - \sin^2\theta) - (a - b)\sin\theta\cos\theta = 0$

or, $\tan 2\theta = 2h/(a - b)$

then in *eq.* (2) term containing xy vanishes and it takes the form, *by* putting the value of θ.

$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$ ... ... (3)

where *by* the Principle of Invariant (*see* Art. 33)

$A + B = a + b$ and $AB = ab - h^2$ ... ... (4)

Case. I. If $A \neq 0$ and $B \neq 0$ then eq. (3) can be written as $A(x + G/A)^2 + B(y + F/B)^2 = G^2/A + F^2/B - C$ ... ... (5)

Now shift the origin to the point $(-G/A, -F/B)$

Eq. (5) takes the form

$Ax^2 + By^2 = G^2/A + F^2/B - C = K$ (say)

or, $Ax^2 + By^2 = K$ or, $\dfrac{x^2}{K/A} + \dfrac{y^2}{K/A} = 1$ ... ... (6)

(i) If $K = 0$ *eq.* (6) will represent a pair of straight lines (real or imaginary)

(ii) If K/A and K/B in eq. (6) are both positive, then it represents an imaginary ellipse.

If K/A and K/B in eq. (6) are both negative, then it represents an **Imaginary ellipse.**

Hence K/A and K/B have the same sign i. e. $\dfrac{K^2}{AB} > 0$, Since K^2 is always positive.

So, $AB > 0$ i.e. $ab - h^2 > 0$ by (4)

Hence eq. (6) represents an ellipse.

(iii) If K/A and K/B are of opposite sign, *eq.* (6) will represent a **hyperbola** i. e; $\dfrac{K^2}{AB} < 0$ or $AB < 0$ is K^2 as always positive or, $ab - h^2 < 0$ by (4)

Hence *eq.* (6) will represent a hyperbola

In case $ab - h^2 < 0$ and $A + B = 0$

i.e. $ab - h^2 < 0$ and $a + b = 0$ then *by* eq. (4)

eq. (1) will represent a **rectangular hyperbola.**

Case II. Let either A or B be zero ; then eq. (3) takes the form if $A = 0$

or, $By^2 + 2Gx + 2Fy + C = 0$

or, $B(y + F/B)^2 = -2Gx - C + F^2/B = -2G\left(x + \frac{BC - F^2}{BG}\right)$ ... ... (7)

(i) If $G = 0$, then eq. (7) will represent a pair of parallel straight lines.

(ii) If $G = 0$ and $F^2/B - C = 0$, (7) will represent a pair of coincident lines.

(iii) $G \neq 0$, shift the origin to $\left(\frac{BC - F^2}{BG}, \frac{-F}{B}\right)$,

then eq. (7) takes the form $y^2 = -\frac{2G}{4}x$ ... ... (8)

which represents a parabola.

Since $AB = ab - h^2$ from (4)

or, $ab - h^2 = 0$, as A is zero

Hence when $ab - h^2 = 0$, conic (1) will represent a parabola.

Case III. When $A = B$ in eq. (3) i. e. when $a = b$ and $h = 0$ from (1), the eq. (1) will represent a circle.

Hence Eq. (1) represents an ellipse, parabola or hyperbola according as $ab - h^2 >, =$ or < 0

Art. 48. From the above discussions the general equation of the second degree.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent

(1) a pair of straight lines if the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(i) two parallel lines if $\Delta = 0, ab = h^2$

(ii) two perpendicular lines if $\Delta = 0, a + b = 0$

(2) a circle if $a = b, h = 0$,

(3) a parabola if $ab = h^2, \Delta \neq 0$

(4) an ellipse if $ab - h^2 > 0, \Delta \neq 0$

(5) a hyperbola if $ab - h^2 < 0, \Delta \neq 0$

(6) a rectangular hyperbola if. $a + b = 0, ab - h^2 < 0, \Delta \neq 0$

Note : $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 =$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Art. 49. Centre of a conic.

Definition : The centre of a conic is the point in the plane of the conic such that every chord of the conic passing through it is bisected at it.

Let the equation to the conic be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (2)

and let $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r$ ... ... (3)

be the equation of any line through (x_1, y_1)

If the live (2) meets the conic (1) at a distance r from (x_1, y_1) then $(x = x_1 + lr, y = y_1 + mr)$ is a point of intersection of (1) and (2) The from (1)

$a(x_1 + lr)^2 + 2h(x_1 + lr)(y_1 + mr) + b(y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0$

or, $(al^2 + 2hlm + bm^2)r^2 + 2r\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\} + (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) = 0$ ... ... (4)

From Eq. (4) we see that r has two roots say r_1 and r_2 which are the distances from (x_1, y_1) i.e. the line (2) inersects the conic (1) at two points.

If (x_1, y_1) be the middle point of the chord, then r_1 and r_2 have equal but opposite sign i. e. $r_1 = -r_2$, or $r_1 + r_2 = 0$

The sum of the roots of r in (3) $= 0$

or $(ax_1 + by_1 + g)l + (hx_1 + by_1 + f)m = 0$ ... ... (5)

If (x_1, y_1) is the centre of the conic, then the relation (4) holds good for all values of l and m

$\therefore$ $ax_1 + hy_1 + g = 0$ ... ... (6)

$hx_1 + hy_1 + f = 0$ ... ... (7)

The above two equations given the centre of the conic (1). Solve (6) and (7) for x_1 and y_1

$\therefore$ $x_1 = \frac{hf - bg}{ab - h^2}$, $y_1 = \frac{gh - af}{ab - h^2}$

Hence the co-ordinates of centre (1) is $\left(\frac{hf - bg}{ab - h^2}\right), \left(\frac{gh - af}{ab - h^2}\right)$

Cor. 1. When $ab - h^2 = 0$, the co-ordinates of the centre are at infinity and the curve is a parabola which has no centre.

Cor. 2. If $hf - bg = 0$ and $ab - h^2 = 0$

i. e. $a/h = h/b = g/f$

The equations (3) and (4) represent the same line. Is this case any point on the straight line (3) or (4) may be regarded as the centre, for every such point satisfies the definition of the centre.

The locus in this case is a pair of parallel straight lines.

Note : The equations (3) and (4) can be obtained by, diferentiating (1) partially with respect to x and y respectively.

If $F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Then, $\frac{\delta F}{\delta x} \equiv 2(ax + hy + g) = 0$, $\frac{\delta F}{\delta y} \equiv 2(hx + by + f) = 0$,

Solve $\frac{\delta F}{\delta x} = 0$ and $\frac{\delta F}{\delta y} = 0$ for x and y. Thus the co-ordinates for the centre (x, y) are obtained.

Art. 50. Equation referred to centre of the conic. (Rectangular axex)

Let the equation to the conic be

$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

$\frac{\delta F}{\delta x} \equiv 2(ax + hy + g) = 0,\ \frac{\delta F}{\delta y} \equiv 2(hx + by + f) = 0$ ... ... (2)

If (x_1, y_1) be the co-ordinates of the centre of the conic, then these co-odinates are obtained by solving these equations,

$ax_1 + hy_1 + g = 0$ and $hx_1 + by_1 + f = 0$ ... ... (3)

If the origin is transferred to (x_1, y_1) the Eq. (1) becomes $ax^2 + 2hxy + by^2 + c_1 = 0$ ... (4)

where $c_1 = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2y_1 + c$ ... ... (5)

by (2) Art. 49

$= x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + gx_1 + fy_1 + c$

$= gx_1 + fy_1 + c$ by (3) ... ... (6)

Put the values of x_1, and y_1 from (5), Art. 49. in (6)

then $c_1 = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} \times \frac{\Delta}{ab - h^2}$ ... ... (7)

Where Δ is called the discriminant of the general equation (1) The equation referred to the new axes through the centre is

$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0$ ... ... (8)

Cor. If $\Delta = 0$, then $c_1 = 0$ then Eq. (8) will represent two straight lines

Working Rule : For the reduction of the equation of a conic referred to centre as origin.

Let the equation of the conic be.

$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

then write $\frac{\delta F}{\delta x} \equiv (ax + hy + g) = 0$ or $ax + hy + g = 0$ ... ... (2)

and $\frac{\delta F}{\delta y} = 2(hx + by + f) = 0$ or, $hx + by + f = 0$ ... ... (3)

The equations (2) and (3) *i. e.* $\frac{\delta F}{\delta x} = 0$, and $\frac{\delta F}{\delta y} = 0$

are the same as these giving the co-ordinates of the centre.

Now the constant $= \frac{\Delta}{ab - h^2} = gx + fy + c$ which is represented by another constant c_1.

$c_1 = gx + fy + c$ ... ... (4)

where x and y are found from the equations $\frac{\delta F}{\delta x} = 0$, and $\frac{\delta F}{\delta y} = 0$

and g, f and c are found from the given equation (1)

Hence the equation reduces to $ax^2 + 2hxy + by^2 + c_1 = 0$

In other words "Same terms of second degree as in the given equation + new constant = 0.

which we can always reduce to the FORM

$Ax^2 + 2H\,xy + By^2 = 1$ ... ... (5)

If the axes of the equation (5) are rotated about its new (centre) origin in such a way that xy-term vanishes, the new equation of the conic becomes $a_1x^2 + b_1y^2 + c_1 = 0$ ... (6)

which is called the **Standard Form** of the equation of the conic. Then by Invariants. (See Art. 35) we have from (5) and (6)

$a_1 + b_1 = a + b$

$a_1b_1 = ab - h^2$

where a_1 and b_1 can be easily found.

Thus the equation (1) reduces to the standard form

$a_1x^2 + b_1y^2 + c_1 = 0$

Ex. Reduce the equation $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$ to the standard form.

Let $f(x, y) \equiv 32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$

$\delta f/\delta x = 0$ or, $64x + 52y - 64 = 0$ ... ... (1)

and $\delta f/\delta y = 0$ or, $52x - 14y - 52 = 0$ ... ... (2)

Solve (1) and (2) for x and y $\therefore$ $x = 1, y = 0$ Centre is at $(1, 0)$

New constant, $c_1 = gx + fy + c = -32(1) - 26(0) - 148 = -180$

Therefore, the equation of the conic referred as origin is

$32x^2 + 52xy - 7y^2 - 180 = 0$ by (5) working Rule (xi)

When the xy — term is removed by rotation of the axes, let the reduced equation be

$a_1x^2 + b_1y^2 = 180$ ... ... (3) by (6) (cor. Art 50)

Then $a_1 + b_1 = 32 - 7 = 25$.

$\therefore$ here $a = 32$, $b = -7$, $h = 26$

and $a_1b_1 = -32.7 - (26)^2 = -900$

Solve for a_1 and b_1 $\therefore$ $a_1 = 45, b_1 = -20$

Therefore, the equation (3) is

$45x^2 - 20y^2 = 180$ or, $x^2/_4 - y^2/\frac{9}{2} = 1$

The given equation represents a hyperbola.

Art. 51. Axes : To find the lengths and position of the axes of the conic. [C. U. 1986]

$ax^2 + 2hxy + by^2 = 1$ ... ... (1)

Let the axes be rectangular.

Let the conic be cut by a concentric circle of radius r. The diameters through the points of intersection will be equally inclined to the axes of the conic, and will be coincident if the radius of the circle be equal to either of the semi-axes of the conic.

Now suppose that equation of the circle as $\frac{x^2 + y^2}{r^2} = 1$ ... (2)

Making (1) homogeneous with the help of (2), we have

$\left(a - \frac{1}{r^2}\right)x^2 + 2hxy + \left(b - \frac{1}{r^2}\right)y^2 = 0$ ... ... (3)

which represents a pair of straight lines passing through the origin and the points of intersection of (1) and (2). When these lines (diameters) coincide they become axes of conic (1).

They will be coincident (by condition of equal roots) ... ... (4)

$(a - 1/r^2)(b - 1/r^2) = h^2$ ... ... (5)

or, $\frac{1}{r^4} - (a+b)\frac{1}{r^2} + ab - h^2 = 0$ ...

This is a quadratic equation in r^2. Hence the square of lengths of the semi-axes of the conic. (4) are the roots of (5)

Let r_1^2 and r_2^2 be the two roots. The semi-axes of the conic are of lengths r_1 and r_2.

Now multiply the equation (3) by $(a - 1/r^2)$ then it becomes

$$(a - 1/r^2)^2 x^2 + 2h(a - 1/r^2)xy + (a - 1/r^2)(b - 1/r^2)y^2 = 0$$

or, $(a - 1/r^2)^2 x^2 + 2h(a - 1/r^2)xy + h^2y^2 = 0$ by (4)

or, $[(a - 1/r^2)x + hy]^2 = 0$ or, $(a - 1/r^2)x + hy = 0$

The equations of the axes are therefore

$$\left(a - \frac{1}{r_1^2}\right)x + hy = 0 \text{ and } \left(a - \frac{1}{r_2^2}\right)x + hy = 0 \quad \ldots \quad \ldots \quad (6)$$

and length of the corres ponding semi-axes are r_1 and r_2 where r_1^2 and r_2^2 are the roots of (5)

Cor. For the general equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (7)

which reduces to the equation

$ax^2 + 2hxy + by^2 = k$ by Eq 8 Art. 50.

where $k = -c_1$ from (7) Art. 50

Making the equation homogeneous by (2) we have

$$ax^2 + 2hxy + by^2 = k\,\frac{x^2 + y^2}{r^2}$$

$$\text{or, } \left(a - \frac{k}{r^2}\right)x^2 + 2hxy + \left(b - \frac{k}{r^2}\right)y^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

In this case the square of the Semi-axes are given by (sec relation 4)

$(a - k/r^2)(b - k/r^2) = h^2$ ... ... (9)

and the equations of the axes are given by

$(a - k/r_1^2)x + hy = 0$ and $(a - k/r_2^2)x + hy = 0$ ... ... (10)

Note : As the minor or conjugate axis is perpendicular to the major or transverse axis, the equation of minor or conjugate axis is easily obtained if equation of the major or transverse axis is known. say if $5x + 3y + 6 = 0$ is the equation of major axis the equation of the minor axis is $3x - 5y + k = 0$, Since it pass through the centre, then k is known.

Sometimes the minor or conjugate axis obtained from the fromula $(b - 1/r_2^2)x + hy = 0$ seems to be not perpendicular to the major or transverse axis. For example see worked out Example No. 10.

Art. 51. (a) Working Rule for finding the Co-ordinates of foci, feet of directrices, vertices and equations of the latus rectum, directrices and tangents at the vertices.

On the line PQ, suppose A is a point whose coordinates (h, k) are known. The line PQ makes an angle θ with X axis. Then the co-ordinates of any ohter point (say) P (x, y) can be determined if the distance AP (= d) is known. From the figure.

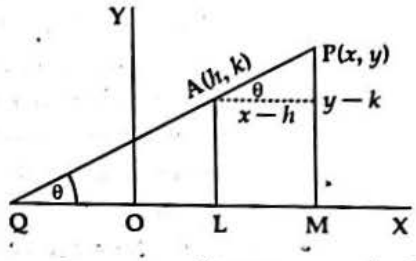


Fig : 20

AP (= d), $\sin\theta = \frac{y-k}{AP} = \frac{y-k}{d}$ or, $y = k + d\sin\theta$

$\cos\theta = \frac{x-h}{AP} = \frac{x-h}{d}$ or, $x = h + d\cos\theta$

Thus $x = h + d\cos\theta$

$y = k + d\sin\theta$ ... ... (XI)

Now let us try to apply in conic (Ellipse and Hyperbola)

* The important points associated with them are the foci, vertices and feet of the directrices. These points lie either on the major axis or on the transverse axis. But centre of the conic is common to them. Therefore, we may take (h, k) as the co-ordinates of the centre of the conic and θ is the inclination of the major or transverse axis to the x axis. d stands for the distance between the centre and the required point on the conics. Now we are giving the distances of the points.

Distance for foci S and S′ ; $d = \pm ae = \pm r_1 e$

Distance of vertices A and A′; $d = \pm r_1$

Distance of the feet of the directrices Z and Z′ ; $d = \pm a/e = \pm r_1/e$ where r_1 is the length of semi major or semi transverse axis.

Thus d, h and θ are known, so any point S or S′ ; A or A′, Z or Z′ can be known.

Extremities of minor axis are B and B′. Here the distance d between the centre and B or B′ ; $d = \pm b = \pm r^2$; and θ slope of minor axis is known.

Length of the latus rectum LSL′ $= \frac{2b^2}{a} = \frac{2r_2^2}{r_1}$

For the determination of the extremities of latus rectum, we are to consider the known point as focus and (h, k) are the co-ordinates of the focus here. The distance between S and L or L′ is given by

$d = \pm b^2/a = \pm r_2^2/r_1$

The slope of the latus rectum is the same as the slope of minor axis. Therefore θ is known.

Thus we use the **Working Rule** in this book as **(XI)**

* Feet of directrices mean the points of intersection of major axis and directrices i. e. Z and Z′

Lat us rectum, is parallel to minor or conjugate axis and passes through the focus S or S′

Directrices are parallel to minor or conjugate axis and passes through Z or Z′

Tangents at the vertices A and A′ are parallel to minor or conjugate axis and pass through A or A′. So their equations can be easily determined.

Notes : Slope of the axis as given by tan θ. From it we get sin θ and cos θ. If tan θ is negative, then θ is obtuse. In this case we keep perpendicular positive and base negative. Thus sin θ is positive and cos θ is negative in this case.

Let us explain the above method in the following Example.

Ex. ***Reduce the equation***

$f(x, y) \equiv 8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ ... ... (1)

to the standard form and find its all properties.

In the Eq. (i) $a = 8, h = 2, b = 5, g = -12, f = -12, c = 0$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -$ ve and $ab - h^2 = 36$.

Therefore, $\Delta \neq 0, ab - h^2 > 0$

(i) The equation (i) will represent an ellipse.

$\frac{\delta f(x, y)}{\delta x} \equiv 16x + 4y - 24 = 0$, and $\frac{\delta f(x, y)}{\delta y} \equiv 10y + 4x - 24 = 0$

or, $4x + y - 6 = 0$ and $2x + 5y - 12 = 0$, Centre of the conic is the intersection of these two lines. Solve $x_1 = 1, y_1 = 2$.

(ii) Centre (1, 2)

$C_1 = gx_1 + fy_1 + c = -12.1 - 12.2 + 0 = -36$

The eq. (1) referred to the centre (1, 2) is

$8x^2 + 4xy + 5y^2 - 36 = 0$, or, $\frac{2}{9}x^2 + \frac{1}{9}xy + \frac{5}{36}y^2 = 1$ ... ... (2)

Here $A = 2/9, H = 1/18, B = 5/36$ ... ... (3)

∴ The lengths of the semi-axes are given by

$\frac{1}{r^4} - (A + B)\frac{1}{r^2} + AB - H^2 = 0$ by 5, Art. 51.

Put the values of A, B and H from (3) then

$r^4 - 13r^2 + 36 = 0$ or, $r^2 = 9$ or, 4 i. e.; $r_1 = 3, r_2 = 2$ ... ... (4)

(iii) The lengths of the axes are 6 and 4.

The equations of the major and minor axes are

$(A - 1/r_1^2)/x + Hy = 0$ and $(A - 1/r_2^2)x + Hy = 0$

or, $2x + y = 0$ and $x - 2y = 0$ by (3) and (4), referred to the centre (1, 2),

or, $2(x - 1) + (y - 2) = 0$ and $(x - 1) - 2(y - 2) = 0$; referred to the old orgin:

or, $2x + y - 4 = 0$ and $x - 2y + 3 = 0$

(iv) Equation of the major axis is $2x + y - 4 = 0$ ... ... (5)

(v) Equation of the minor axis is $x - 5y + 3 = 0$ ... ... (6)

(Eccentricity. $e^2 = 1 - r_2^2/r_1^2 = 1 - 4/9 \therefore e = \sqrt{5}/3$ ... ... (7)

Now d for foci S and S′ $= \pm ae = \pm r_1 e = \pm \sqrt{5}$ by (4) and (7)

(vi) Solpe of major axis, eq. (5)

$\tan\theta = -2$ and $\sin\theta = 2/\sqrt{5}, \cos\theta = -1/\sqrt{5}$ ... ... (8)

(h, k) *i. e.* : (1, 2) are the centre of the conic (1) by (II) of

Art. 51. (a) $d = \sqrt{(1^2 + 2^2)} = \sqrt{5}$

(vii) Focus S ; $(h + d\cos\theta, k + d\sin\theta) = (1 + \sqrt{5}.-1/\sqrt{5}, 2 + \sqrt{5}.2/\sqrt{5} = (0, 4)$,

Focus S′, $(1 - \sqrt{5}.-1/\sqrt{5}, 2 - \sqrt{5}.2/\sqrt{5}) = (2, 0)$

d for vertices A and A′, $= \pm a = \pm r_1 = \pm 3$

d for feet of directrices Z and Z′ $= \pm a/e = \pm r_1/e = \pm 9/\sqrt{5}$.

(viii) Vertex A, $(1 + 3.-1/\sqrt{5}, 2 + 3.2/\sqrt{5}) = (1 - 3/\sqrt{5}, 2 + 6/\sqrt{5})$

Vertex A′ $(1 - 3.-1/\sqrt{5}, 2 - 3.2/\sqrt{5}) = (1 + 3/\sqrt{5}, 2 - 6/\sqrt{5})$

(ix) Point Z $(1 + 9/\sqrt{5}.-1/\sqrt{5}, 2 + 9/\sqrt{5}.2-\sqrt{5}) = (-44/5, 28/5)$

Point Z′ $(1 - 9/\sqrt{5}.-1\sqrt{5}, 2 - 9/\sqrt{5}.2-\sqrt{5}) = (14/5, -8/5)$

(x) d for the points B and B′ $= \pm b = \pm r^2 = \pm 2$

Slope of minor axis : $\tan\theta = \frac{1}{2}, \sin\theta = 1/\sqrt{5}., \cos = 2/\sqrt{5}$

(h, k) is the centre (1, 2)

Points B and B′ $= (h \pm d\cos\theta, k \pm d\sin\theta)$

$= (1 \pm 2.2/\sqrt{5}, 2 \pm 2.1/1/\sqrt{5}) = (1 \pm 4 - \sqrt{5}, 2 \pm 2\sqrt{5})$

(xi) Length of the latus rectum $= \frac{2b^2}{a} = 2r_2^2/r_1 = 8/3$

(xii) Distance between S and L or, L′ is given by

$d = \pm b^2/a = \pm r_2^2/r_1 = \pm 4/3$

and (h, k) stands for S (0, 4) or, S′ (2, 0) by (vii)

Solpe of latus rectum is the same as the solpe of minor axis

Points : L and L′ $(0 \pm 4/3.2\sqrt{5}, 4 \pm 4/3.1/\sqrt{5}) = (+ 8/3\sqrt{5}, 4 + 4./3\sqrt{5})$

Similarly another pair of points with respect to S′(2, 0) can be determined.

(xiii) Equation of the latus rectum which is parallel to minor axis $x - 2y + 3 = 0$ is $x - 2y + k = 0$. Since it passes through the foci S (0, 4) or S′ (2, 0), hence $k = 8, -2$, Hence equations of the latar recta are $x - 2y + 8 = 0$ and $x - 2y - 2 = 0$

(xiv) Directrix through Z $(-4/5, 28/5)$ or, Z′ $(14/5, -18/5)$ is also parallel to minor axis $x - 2y + 3 = 0$. Therefore the equation is $x - 2y + 3 = 0$, where $\lambda = 12, -10$.

Hence the equations of the directrices are $x - 2y + 12 = 0$ and $x - 2y - 10 = 0$

Cor. Eccentricity of the conic, e is given by $e^2 = 1 - \frac{r_1^2}{r_2^2}$

If r_1^2 and r_2^2 are positive, the conic is an ellipse. If r_2^2 is negative the conic is a hyperbola. If $r_1^2 = r_2^2$ and r_2^2 is negative, the conic is a rectangular hyperbola.

Art 52. Tangents : Def : Let P be a given point on a curve and Q be any other point on it. If Q tends to P, the straight line PQ tends, in general, to a definite straight line to which the chord PQ of a curve tends as Q tends to P. This straight line is called the tangent to the curve at P.

Popularly tangent is defined as 'Tangent is the limiting position of a secant'.

Art. 52. (a) To find the equation of the tangent to the conic

$F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

Let (x_1, y_1) be any point on the conic

Let a line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r$ ... ... (2)

Pass through the point (x_1, y_1). If the line (5) meets the conic $F(x, y) = 0$, the general co-ordinates $(x_1 + lr, y_1 + mr)$ will satisfy $F(x, y) = 0$,

$a(lr + x_1)^2 + 2h(lr + x_1)(mr + y_1) + b(mr + y_1)^2 + 2g(lr + x_1) + 2f(mr + y_1) + c = 0.$

or, $r^2(a^2l^2 + 2hlm + bm^2) + 2r[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + (ax_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c) = 0.$

or, $r^2(a^2l^2 + 2hlm + bm^2) + 2r(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + F(x_1, y_1) = 0$ ... ... (3)

If the point (x_1, y_1) is on the conic $F(x_1, y_1) = 0$ then

$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$ ... ... (4)

or, $F(x_1, y_1) = 0$

Then from (3). one value of r is zero. If the line (2) is an tangent to the conic, then the other value of r is also zero from (3)

$(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0$ ... ... (5)

Since the tangent can be drawn in any direction and hence its equation is obtained by elimination l, m between (2) and (5)

Therefore, the equation of the tangent at (x_1, y_1) is

$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0$

or, $x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$

or, $x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c - (gx_1 + fy_1 + c)$

or, $2x(ax_1 + hy_1 + g) + 2y(hx_1 + by_1 + f) + 2(gx_1 + fy_1 + c) = 0$ by (4)

or, $x\frac{\delta F}{\delta x_1} + y\frac{\delta F}{\delta y_1} + 2c_1 = 0$, where $c_1 = gx_1 + fy_1 + c$

or $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... ... (6)

Hence the equation of the tangent at (x_1, y_1) to the conic $F(x, y) = 0$ is

$T = x\frac{\delta F}{\delta x_1} + y\frac{\delta F}{\delta y_1} + 2C_1 = 0$ ... ... (7)

where $C_1 = (gx_1 + fy_1 + c)$

or, $T = axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Note : T is obtained from $F(x, y) = 0$ by writing

(1) xx_1 and yy_1 for x^2 and y^2

(2) $xy_1 + x_1y$ for $2xy$

(3) $x + x_1$, and $y + y_1$ for $2x$ and $2y$ respectively and

(4) retaining the constant term.

Art. 53. Equations of tangents to different curves.

The equation to the tangent at (x_1, y_1) to

(i) the circle, $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(iii) the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(iv) the hyperbola, $x^2/a^2 - y^2/b^2 = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(v) the rectangular hyperbola $x^2 - y^2 = a^2$ is $xx_1 - yy_1 = a^2$

(vi) the rectangular hyperbola asymptotes as axes of co-ordinates $xy = c^2$ is $(xy_1 + x_1y) = c^2$

Art. 54. Normal : The normal at any point P of a conic is the straight line which passes through P and is perpendicular to the tangent at P.

Art. 54. (a) Normal to the general conic.

$F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

Let $P(x_1, y_1)$ be any point on the conic $F(x, y) = 0$, it is required to find a normal at this point.

The equation to any line through $P(x_1, y_1)$ is

$y - y_1 = m(x - x_1)$ ... ... (2)

The equation of the tangent at P (x_1, y_1) to the conic $F(x, y) = 0$ is by (7) of Art. 52.

$x\frac{\delta F}{\delta x_1} + y\frac{\delta F}{\delta y_1} + 2c_1 = 0$, or, $x(F_x)_1 + y(F_y)_1 + 2c_1 = 0$

$y = -\left(\frac{F_x}{F_y}\right)x + 2c_1/(F_y)$ ... ... (3)

or, $y = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}x - \frac{gx_1 + fy_1 + c}{hx_1 + by_1 + f}$ by (8) Art. 52 ... ... (4)

If the straight line (2) is perpendicular to (3) or (4), then $mx\frac{F(x_1)}{(Fy_1)} = -1$.

or, $m = \left(\frac{F_y}{F_x}\right)_1 = \frac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}$ ... ... (5)

The equation of the normal at (x_1, y_1) to the conic $F(x, y) = 0$ is from (2) and (5)

$$y - y_1 = \frac{hx_1 + by_1 + f}{ax_1 + hy_1 + f_1}(x - x_1)$$

$$\text{or, } \frac{y - y_1}{hx_1 + by_1 + f_1} = \frac{x - x_1}{ax_1 + by_1 + g} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\text{or, } \frac{y - y_1}{(F_y)_1} = \frac{x - x_1}{(F_x)_1} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $(F_x)_1$ is the differentiation of the given equation $F(x, y) = 0$. with respect to x_1 treating y_1 as constant, and $(F_y)_1$ is the differentiation with respect to y_1 treating x_1 as constant.

Art. 55. Normals to different curves.

The equation to the normal at (x_1, y_1). to the curve, by (6) of Art. 54.

1. Circle $x^2 + y^2 = a^2$ is $\frac{y - y_1}{y_1} = \frac{x - x_1}{x_1}$ or, $xy_1 = x_1y$

2. Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\frac{y - y_1}{y_1 + f} = \frac{x - x_1}{x_1 + g}$

3. Parabola, $y^2 = 4ax$ is $\frac{y - y_1}{y_1} = \frac{x - x_1}{-2a}$ or, $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

4. Ellipse. $\frac{x^2}{a^2} + \frac{y^2}{y^2} = 1$ is $\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}$

5. Hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $\frac{y - y_1}{-y_1/b^2} = \frac{x - x_1}{x_1/a^2}$

Art. 56. Find the condition that a line may touch a conic.

Let the line be $lx + my + n = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

and the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad (2)$

If (1) touches (2) at (x_1, y_1) it must be of the form $T = 0$ the tangent at (x_1, y_1)

or, $(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c) = 0$ By 8, Art. 52.

This equation is the same as (1) if

$$\frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n} = k \text{ (say)}$$

Then $ax_1 + hy_1 + g - kl = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$

$hx_1 + by_1 + f - mk = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$

$gx_1 + fy_1 + c - nk = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$

Since the point (x_1, y_1) is on $lx + my + n = 0$

hence $lx_1 + my_1 + n = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$

The eliminant of x_1, y_1 and k from (3), (4), (5) and (6) is the required condition. From the above equations we have the eliminant—

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

or, $Al^2 + Bm^2 + Cn^2 + 2F\,mn + 2Glm + 2Hlm = 0$

where A, B, C etc. are the cofactors of a, b, c etc:

in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

The equation (7) is called the TANGENTIAL EQUATION of the conic.

Art. 57. Cord of Contact. Pole and Polar.

The tangents are drawn to a conic from any point (x_1, y_1). To find the equation of the straight line (chord of contact) joining the points of contact (real, coincident or imaginary).

Let a pair of tangents be drawn at point $T(x_1, y_1)$ to the conic, (See Fig. Art. 58)

$ax^2 + 2hxy + by^2 + 2gx + 2fy = c = 0 \quad \ldots \quad \ldots \quad (1)$

The equation of the tangents at $P(x_2, y_2)$ and $Q(x_3, y_3)$ of the conic (1) are respectively.

$axx_2 + h(x_2y + xy_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad \ldots \quad (2)$

and $axx_3 + h(x_3y + xy_3) + byy_3 + g(x + x_3) + f(y + y_3) + c = 0 \quad \ldots \quad (3)$

Since (2) and (3) pass through $T(x_1, y_1)$

Hence $ax_1x_2 + h(x_2y_1 + x_1y_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \quad \ldots \quad (4)$

and $ax_1x_3 + h(x_3y_1 + x_1y_3) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0 \quad \ldots \quad (5)$

But (4) and (5) are the conditions that the points (x_2, y_2) and (x_3, y_3) shall lie on

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \quad \ldots \quad \ldots \quad (6)$

Hence the equation (6) is the equation of the line joining the point of the contact of the tangent from the point (x_1, y_1) to given conic.

The eq. of the chord of contact is (6)

Art. 58. If through a point P (within or without a conic) there be drawn any straight line to meet the conic in Q and R, the locus of the intersection of the tangents at Q and R is called the polar of P.

Let a straight line QR be drawn through P to meet the conic in Q and R. Tangents are drawn at Q and R to meet at T. The locus of T is called the polar of P with respepct to the conic.

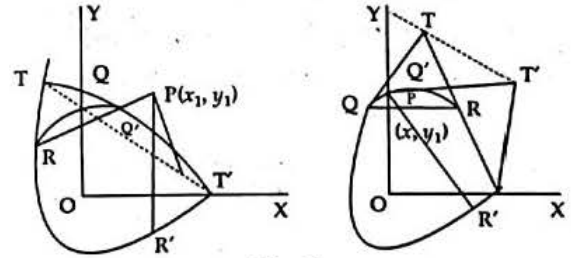


Fig : 21

TT' is the required polar of P with respect to the conic. The given point P is called the pole of the polar line. QR is the chord of contact of tangents drawn from the point (x', y') and therefore its equation by Art. 57 is

$$axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y') + c = 0 \quad \ldots \quad \ldots \quad (1)$$

Since the line QR passes through $P(x_1, y_1)$, then (1) becomes

$$ax_1x' + h(x_1y' + y_1x') + by_1y' + g(x_1 + x') + f(y_1 + y') + c = 0 \quad \ldots \quad (2)$$

As the relation (2) is true so the variable point (x', y') always lies on the straight line (2) whose equation is

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ which is the equation of polar of the point (x_1, y_1).

It is to be noted that polar of (x_1, y_1) is real, whether the point (x_1, y_1) lies outside, on or inside the given conic, while the tangents from (x_1, y_1) to the conic are only real if the point (x_1, y_1) lies outside or on the conic.

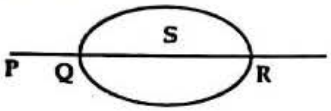


Fig : 22

বিকল্প পদ্ধতি :

Solid geometry Art. 45 আলোচনা করা যেতে পারে।

মনে করি, $P(x_1, y_1)$ বিন্দুতে PQR সরলরেখা কণিক (1) কে Q এবং R বিন্দুতে ছেদ করে। QR এর উপর S এমন একটি বিন্দু PQ, PS, PR যে Harmonic Progression (সমজাস প্রগমণ অথবা) $\frac{1}{PQ}, \frac{1}{PS}, \frac{1}{PR}$ সমান্ত প্রগমণ (Arithmetic Progression) থাকে। S বিন্দুর সঞ্চার পথকে প্রদত্ত কণিকের P বিন্দুর পোলার বলা হয় এবং P বিন্দুকে পোলার পোল বলা হয়।

$$\frac{1}{PQ} + \frac{1}{PR} = \frac{2}{PS} \text{ or, } \frac{PQ + PR}{PQ.PR} = \frac{2}{PS} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{or, } PS = \frac{2PQ.PR}{PQ + PR} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

যদি $S(x', y')$ এবং $PS = r$ হয় PQR-এর নতি x অক্ষের সাথে θ উৎপন্ন করে তখন PS-এর সমীকরণ

$$\frac{x' - x_1}{\cos\theta} = \frac{y' - y_1}{\sin\theta} = r' \text{ or, } r'\cos\theta = x' - x_1, r'\sin\theta = y' - y_1$$

$$\text{or, } r'l = x' - x_1, r'm = y' - y_1 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

এদের মান (1) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ বসালে

$$(al^2 + 2hlm + bm^2)r'^2 + 2[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m]r'$$
$$+ F(x_1, y_1) = 0, F(x_1, y_1) = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \quad \ldots \quad (4)$$

r' এর দুটি বীজ, $PQ = r_1, PR = r_2$ মনে করি।

$$r_1 + r_2 = PQ + PR = \frac{-2[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m]}{al^2 + 2hlm + bm^2} \quad \ldots \quad (5)$$

$$r_1r_2 = PQ.PR = \frac{F(x_1, y_1)}{al^2 + 2hlm + bm^2} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

(5) এবং (6) হতে মান বসিয়ে $\ldots \; PS = \frac{2PQ.PR}{PQ + PR}$

$$\text{or, } r' = \frac{2F(x_1, y_1)}{-[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m]}$$

or, $(ax_1 + hy_1 + g)lr' + (hx_1 + by_1 + f)mr' + F(x_1, y_1) = 0$; (3) হতে

or, $(ax_1 + hy_1 + g)(x' - x) + (hx_1 + by_1 + f)(y' - y) + F(x_1, y_1) = 0$

এখন $F(x_1, y_1)$ এর মান (4) হতে বসিয়ে সরল করলে

$$ax'x_1 + h(x'y_1 + x_1y') + by'y_1 + g(x' + x_1) + f(y' + y_1) + c = 0$$

অতএব $S(x', y')$ এর সঞ্চার পথ

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

এটি একটি সরলরেখা। একে $P(x_1, y_1)$ বিন্দুর পরিপ্রেক্ষিতে (1) কণিকের পোলারের সমীকরণ বলা হয়।

Note : The form of the equation of the polar is the same as that of the tangent at a point on the conic. See Eq. 8 Art 52. The polar of a point P w. r to the conic is a straight line. It coincides with the chord of contact of tangents from P, if P lies outside the conic. If P lies on the conic, it coincide with tangent at P.

Note : More about Poles and Polars and their properties such as conjugate points and lines etc, See Art, 75, 76, 77, 105 The eq. of the polar of a point $P(x_1, y_1)$ w. r. to a conic, can be had by the method shown on Art. 48 in Solid Geometry.

Art. 59. Chord in terms of middle points.

Let the equation of the conic be.

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let the equation of the chord which has (x_1, y_1) as its middle point be —

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\text{or, } x = x_1 + lr, y = y_1 + mr \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

putting the values of x and y form (3) in (1) we have

$$(al^2 + 2hlm + bm^2)r^2 + 2(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m]r + (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

As (x_1, y_1) are the co-ordinates of the middle point of the chord, the roots of this equation must be equal in magnitude but opposite in sign i.e. the sum of the roots is zero. The Eq. (4) reduces to the form

$$\therefore \; Ar^2 - B = 0$$

Hence the co-efficient of r is zero i. e.

$$(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Now eliminating l and m from (2) and (5), we have

$$(x - x_1)(ax_1 + by_1 + g) + (hx_1 + by_1 + f)(y - y_1) + 0$$

$$\text{or, } axx_1 + h(xy_1 + x_1y) + byy_1 + gx + fy =$$
$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + fy_1 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Now adding $gx_1^2 + fy_1 + c$ to both sides of (6), we have

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c \quad \ldots \quad \ldots \quad (7)$$

$$= ax_1 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \quad \ldots$$

which is the required equation of the chord in terms of the middle point.

The equation (7) can be written as $T = S_1$

where $T = axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(x + y_1) + c$

and $S_1 = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

Note : T is used here as a symbol. It is not to be confused with the tangent to the conic.

Art. 60. The equation of the chord in terms of the middle Point : (x_1, y_1) for different curves.

The equation of the chord in terms of the middle point w. r. to the—

(1) **Circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

(2) **Parabola** $y^2 = 4ax$ is $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

(3) **Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

(4) **Hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

Art. 61. Pair of tangents.

Let $P(x_1, y_1)$ be a point outside the conic

$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The equation of any line through $P(x_1, y_1)$

$$\text{is } \frac{x - x_1}{l} = \frac{y - y_1}{m} = r \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$x = x_1 + lr,\ y = y_1 + mr \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Putting the values of x and y from (2) in (1), we have $(al^2 + 2hlm + bm^2)\, r^2 + 2r\, \{ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\} + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

$$\text{or, } (al^2 + 2hlm + bm^2)\, r^2 + 2r\, \{ax_1 + by_1 + g)l + (hx_1 + by_1 + f)\, m\} + S_1 = 0 \quad \ldots \quad (3)$$

If Eq. (2) be the tangent to the conic (1) then roots of (3) are equal [*c.f.* $B^2 = 4AC$]

$$\text{Hence } (al^2 + 2hlm + bm^2)\, S_1 = \{ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}^2 \quad \ldots \quad \ldots \quad (4)$$

Now eliminating l and m from (2) and (4) we have

$$\{a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2\}\, S_1$$

$$= \{ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1)\}^2$$

$$\text{or, } (S + S_1 - 2T)\, S = (T - S_1)^2$$

where T is the tangent to the conic (1) at (x_1, y_1)

or, $SS_1 = T^2$ or, $T^2 = SS_1$

which is the equation of the pair of tangnts from (x_1, y_1) to the conic S,

Art. 61. (a) Director Circle. [C.H. 1984]

Def. *Director circle is the locus of the point from which perpendicular tangents can be drawn to the conic.*

The equation of pair of tangents drawn from (x_1, y_1) to the conic S is $SS_1 = T^2$, see eq. (5) Art 61 ... (1) If the tangents given by (1) are perpendicular, then sum of the coefficients of x^2 and y^2 is zore.

$$\therefore\ (a + b)S_1 - (ax_1 + hy_1 + g)^2 - (hx_1 + by_1 + f)^2 = 0$$

Then locus of (x_1, y_1) is the Director circle given by

$$(a + b)S - (ax + hy + g)^2 - (hx + by + f)^2 = 0$$

$$\text{or, } (ab - h^2)(x^2 + y^2) + 2x(bg - hf) + 2y(af - gh) + c(a + b) - g^2 - f^2 = 0 \quad \ldots \quad \ldots \quad (2)$$

$$\text{or, } C(x^2 - y^2) - 2Gx - 2Fy + A + B = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad [\text{See Art. 49}]$$

Particular Cases for.

(a) (i) **Circle** $x^2 + y^2 = a^2$ Here $a = b = 1, h = 0, g = 0$

$f = 0, c = -a^2$ then the locus (2) is $x^2 + y^2 = 2a^2$

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$, here $a = b = 1, h = 0$

then the locus (2) is $x^2 + y^2 + 2gx + 2fy + 2c = 0$

(b) **Parabola** $y^2 = 4ax$, here $a = f = h = c = 0, g = 2a,\ b = 1$ then the locus (2) is $-4ax - 4a^2 = 0$ ro, $x = -a$ which is the directrix of the parabola.

(c) **Ellipcse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, here $a = \frac{1}{a^2}, b = \frac{1}{b^2}, c = -1, h = b = g = f = 0.$

The locus is $x^2 + y^2 = a^2 + b^2$

(d) **Hyperbola** $\frac{x}{a^2} - \frac{y^2}{b^2} = 1$, here $a = \frac{1}{a^2},\ b = \frac{-1}{b^2},\ c = 1,\ h = g = f = 0.$

The locus is $x^2 + y^2 = a^2 - b^2$

Art. 62. Diameter.

Def : *The locus of the middle points of a system of parallel chord of a conic is called diameter which always passes through the centre of the conic.*

Let (x_1, y_1) be the middle point of a chord so that its equation with respect to a conic.

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (x)$$

$$\text{is } axx_1 + h(yx_1 + xy_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$$

$$= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

$$\text{or, } x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = F(x_1, y_1)$$

$$\text{or, } x\frac{\delta F}{\delta x_1} + \frac{\delta F}{\delta y_1} + 2c_1 = 2F(x_1, y_1) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The slope of the chord from (2) is

$$\text{or}, -x\frac{\delta F}{\delta x_1} \div \frac{\delta F}{\delta y_1} = \frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Since the chords are parallel, their slope is constant. Let us assume it to be 'm'

$$m = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} \text{ or, } (a + mh)x_1 + (h + mb)y_1 + (g + mf) = 0$$

Therefore the locus of middle point (x_1, y_1) of the parallel chords is ... ... (4)

$$(a + mh)x + (h + mb)y + (g + mf) = 0$$

which is the straight line called diameter.

If the Eq. (4) is parallel to $y = m_1x$, i.e. if the slope of the diameter is equal to m_1 then

$$-\frac{a + mh}{h + mb} = m_1 \text{ or}, a + h(m + m_1) + bmm_1 = 0 \quad \ldots \quad \ldots \quad (5)$$

Since the relation (5) is symmetrical in m and m_1 the slope of the diameter bisecting chords parallel to $y = m_1x$ is m.

Thus we come to the conclusion that if a diameter bisects chords parallel to a second diameter, then the second diameter bisects chords parallel to the first.

Such two diameters are called **Conjugate diameters** of a conic and their slopes are connected by the relation.

$$a + h(m + m_1) + bmm_1 = 0$$

Conjugate diameters. *Two diameters are said to be conjugate if each is the locus of mid points of chord parallel to the other.*

The results obtained for the ellipse and the hyperbola are the particular case of (5) $mm_1 = -a^2/b^2$ and $mm_1 = a^2/b^2$. See Art. 134 and Art. 143.

Art. 62. (a) Conjugate Points : *Two points are said to be conjugate with respect to a conic if the polar of one passes through the other.*

The condition that P (x_1, y_1) and Q(x_2, y_2) are conjugate points with respect to the conic given by the general eq. is.

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$$

Cor. If the pole of a line l' lies on a line l, then the pole of the second line l' lies on l.

Two lines are said to be conjugate lines with respect to a conic the pole of one lies on the other.

Find the condition when two diameters will be conjugate.

(দুটি ব্যাসের অনুবন্ধী ব্যাস হওয়ার শর্ত নির্ণয় কর)

মনে করি, একটি কনিক

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots \quad (1)$$

AB যদি এর ব্যাস হয় এর সমীকরণ

$\frac{\delta S}{\delta x} + m\frac{\delta S}{\delta y} = 0$, m; PQ ব্যাসের নতি x অক্ষের সাথে

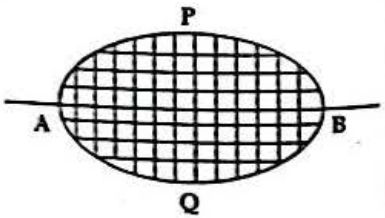


Fig : 23

$$\text{or}, (2ax + 2hy + 2g) + m(2hx + 2by + 2f) = 0$$

$$\text{or}, (a + hm)x + (h + bm)y + (g + mf) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$y = -\frac{a + hm}{h + bm}x - \frac{g + mf}{h + bm} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

AB ব্যাসে, PQ ব্যাসের সমান্তরাল জ্যাগুলোর মধ্যবিন্দুতে ছেদ করে এবং অর্থাৎ মধ্যবিন্দুর সঞ্চার পথ।

PQ এর নতি m_1; x অক্ষের সাথে হলে, (3) হতে

$$m_1 = -\frac{a + hm}{h + hm}$$

$$\text{or}, \ a + hm + hm_1 + bmm_1 = 0$$

$$\text{or}, a + h(m + m_1) + bmm_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

এখন AB জ্যা PQ এর সমান্তরাল জ্যাগুলোর মধ্যবিন্দু এবং PQ জ্যা AB জ্যার সমান্তরাল জ্যাগুলোর মধ্যবিন্দুর সঞ্চার পথ হবে। AB এবং PQ-কে অনুবন্ধী ব্যাস বলে।

(4) হতে m ও m_1–কে পরিবর্তন করলে (4) সমীকরণের পরিবর্তন হয় না। দুটি ব্যাসের x অক্ষের সাথে নতি যথাক্রমে m ও m_1 হল এদের অনুবন্ধী হবার শর্ত

$$a + h(m + m_1) + bmm_1 = 0$$

অনু: কণিক বৃত্ত হলে, $x^2 + y^2 = c^2$, $y = mx$, $y = m_1x$ ব্যাস দুটি অনুবন্ধী হওয়ার শর্তে $[a = 1, h = 0, b = 1]$ $1 + mm_1 = 0$, or, $mm_1 = -1$

(2) পরাবৃত্ত $y^2 = 4ax$ হলে, $y = mx$, $y = m_1x$ ব্যাস দুটি অনুবন্ধী হওয়ার শর্ত, $a = 0$. $b = 1$. $h = 0$. $g = 2a$. $f = 0$, $c = 0$. $0 + 0(m + m_1) + 1mm_1 = 0$ or, $mm_1 = 0$

(3) উপবৃত্ত $x^2/a^2 + y^2/b^2 = 1$ হলে, $y = mx$, $y = m_1x$ ব্যাস দুটি অনুবন্ধী হওয়ার শর্ত, $a = 1/a^2$, $b = 1/b^2$, $h = 0$, $\frac{1}{a^2} + 0(m + m_1) - \frac{1}{b^2}mm_1 = 0$ or, $mm_1 = -\frac{b^2}{a^2}$

(4) অধিবৃত্ত, $x^2/a^2 - y^2/b^2 = 1$ এর জন্য $y = mx$, $y = m_1x$ ব্যাস দুটি অনুবন্ধী হওয়ার শর্ত $a = 1/a^2$, $b = -1/b^2$, $h = 0$. $\frac{1}{a^2} + 0(m + m_1) - \frac{mm_1}{b^2} = 0$ or, $mm_1 = \frac{a^2}{b^2}$

(5) কণিকটি $Ax^2 + 2Hxy + By^2 = 0$ হলে, এর সরলরেখা দুটির $y = mx$, $y = m_1x$, x-অক্ষের সাথে নতি যথাক্রমে m ও m_1 হলে $m_1 + m_1 = -\frac{2H}{B}$, $m_1m = A/B$

$y = mx$, $y = m_1x$ ব্যাস দুটি অনুবন্ধী হওয়ার শর্ত $a + h(m + m_1) + bmm_1 = 0$

$$\text{or}, a - 2\frac{hH}{B} + bA/B = \text{or}, aB - 2hH + bA = 0.$$

Ex. Find the condition that the two lines given by the equation $Ax^2 + 2Hxy + By^2 = 0$ may be conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$ [D.U. 1975]

The conic $ax^2 + 2hxy + by^2 = 1$ does not contain the first degree term so its centre is at (0, 0). If $y = m_1x$ and $y = m_2x$ be its two conjugate diameters, then by (5) Art. 62

$$a + h(m_1 + m_2) + bm_1m_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

If the lines $y = m_1x$ and $y = m_2x$ are also given by

$Ax^2 + 2Hxy + By^2 = 0$, then

$m_1 + m_2 = -2H/B$ and $m_1m_2 = A/B$, see Art. 36.

Put these values in (1), then (1) becomes

$a + h(-2H/B) + b.\ A/B = 0$ or, $aB - 2Hh + bA = 0$

or $aB + bA = 2hH$ is the required condition.

Asymptotes

Art. 62. (b) For definition see Art. 144. Chapter XI From the conclusion of Art. 144. we know that the equations of the asympotes differ from the equation of the conic by a constant.

Let the given conic be

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

The equation of the asymptotes are given by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0$ ... ... (2)

or, $S + \lambda = 0$ ... ... (3)

Choose the value of λ such that eq. (3) may represent a pair of straight lines. Then

$ab(c + \lambda) + 2fgh - af^2 - bg^2 - (c + \lambda)h^2 = 0$. see (9), Art. 40

or, $\lambda = -(abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2) = -\dfrac{\Delta}{ab - h^2}$

... ... (4)

Therefore eq. (2) becomes

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \dfrac{\Delta}{ab - h^2} = 0$

or, $S - \dfrac{\Delta}{ab - h^2} = 0$ ... ... (5)

1. *Note :* if (α, β) be the center of the conic (1), then

$g\alpha + f\beta + c = \dfrac{\Delta}{ab - h^2}$ = or, $c - \dfrac{\Delta}{ab - h^2} = -g\alpha - f\beta$

Hence the eq. of the asymptotes is also given by

$ax^2 + 2hxy + by^2 + 2gx + 2fy - g\alpha - f\beta = 0$

2. *Note* : From (2), Art. 151, Chapter XI, we know that asymptotes of the conic and it conjugate are same. So we may write down the equation in the form.

(Conic) + (Conjugate conic) = 2(asymptotes)

Hence the equation of the conjugate to the conic (1) whose asymptotes are (5) is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \dfrac{2\Delta}{ab - h^2} = 0$ ... ... (6)

3. *Note* : The asymptotes pass through the center of the conic

Ex. Find the asymptotes of the $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$ and find also the equation of its conjugate conic.

The equation of the asymptotes of the conic are given by

$3x^2 - 5xy - 2y^2 + 17x + y + 14 + \lambda = 0$ ... ... (1)

This will give a pair of straight lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $3.(-2).(14 + \lambda) + 2\left(\frac{1}{2}\right)(17/2) - (5/2)\,3\left(\frac{1}{2}\right)^2 - (-2)(17/2)^2 - (14 + \lambda) - 5/2)^2 = 0$

or, $-84 - 6\lambda - 85/4 - 3/4 + 289/2 - 350/4 - 25\lambda/2 = 0$

or, $-49\lambda = 196$

or, $\lambda = -4$.

Hence the equations of the asymptotes of the conic are given by

$3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$ ... ... (2)

Let the equation to the conjugate conic be

$3x^2 - 5xy - 2y^2 + 17x + y + k = 0$

But we know (conic) + (conjugate conic) = 2(asymptotes)

or, $S + (S - 14 + k) = 2(S - 14 + 10)$, or, $k = 6$

The equation of the conjugate conic is

$3x^2 - 5xy - 2y^2 + 17x + y + 6 = 0$

Note : Eqs. (2) and (3) can be obtaind by using the formula Find Δ which is $-196/4$ and $ab - h^2 = -6 - 25/4 = -49/4$

Now use the formula (5) and (6) of Art. 62(c)

Art. 62. (b) Reduction of the equation of a non -central conic.

To find the axis, latus rectum, focus, foot of the dcirectrix and the equations of the direcitrix and latus rectum of the parabola.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

(i) Eq. (1) will represent a parabola if the second degree terms form a perfect square, i. e, $\Delta \neq 0$ and $h^2 = ab$.

Let us put $a = \alpha^2$, and $b = \beta^2$ so that $h = \alpha\beta$;

Hence the eq. (1) becomes $(\alpha x + \beta y)^2 = -(2gx + 2fy + c)$ ... ... (2)

If $\alpha x + \beta y = 0$ is the axis and $2gx + 2fy + c = 0$ is tangent at the vertex, then these two straight lines should be at right angles.

(ii) In case if they are not perpendicular to each other, let us introduce a constant in (2). Then eq. (2) becomes

$(\alpha x + \beta y + \lambda)^2 = -2gx - 2fy - c + 2\lambda(\alpha x + \beta y) + \lambda^2$

or, $(ax + \beta y + \lambda)^2 = 2(\alpha\lambda - g)x - 2(\beta\lambda - f)y + (\lambda^2 - c)$ ... ... (3)

(iii) Select λ so that lines $\alpha x + \beta y + \lambda = 0$ ... ... (4)

and $2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + (\lambda^2 - c) = 0$ ... ... (5)

may be perpendicular to each other. Eq. (4) is the axis and eq. (5) is the tangent at the vertex of the parabola. By the condition of perpendicularity (See Art. 35*b*) we have.

(iv) Let us consider these lines (3) and (4) as new co-ordinate axes and the co-ordinates of a point P referred to new system be (x', y'), but the co-ordinates of this points $P(x, y)$ referred to the old system of co-ordinate axes. Then,

PM = x' = the perp. distance of the point P (x, y) from the new y axis i.e. from the straight line (5)

$$= \frac{2(\alpha\lambda - g)\,x + 2\,(\beta\lambda - f)\,y + (\lambda^2 - c)}{2\sqrt{[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2]}} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

PN = y' = the prep. distance of P (x, y) from the x — axis i. e. from the straight line

$$(4) = \frac{\alpha\lambda + \beta y + x}{\sqrt{(\alpha^2 - \beta^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Let us try to form the relation $PN^2 = 4p$ PM (Where the 4p is the latus rectum of the parabola) from the relations (7) and (8).

Therefore, $\left[\frac{\alpha x + \beta y + \lambda}{\alpha^2 + \beta^2}\right]^2 (\alpha^2 + \beta^2)$

$$= \sqrt{2}\,\frac{[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2][2x(\alpha\lambda - g) + 2y(\beta\lambda - f) + (\lambda^2 - c)]}{\sqrt{2}[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2]}$$

or, $PN^2(\alpha^2 + \beta^2) = 2\sqrt{[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2]}$. PM. by (7) and (8)

$$\text{or, } (y')^2 = \frac{2\sqrt{[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2]}}{\alpha^2 + \beta^2}\,x' \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Now removing x' by X, y' by Y and

$$4p = 2\sqrt{[(\alpha\lambda - g)^2 + (\beta\lambda - f)^2]}/(\alpha^2 + \beta^2) \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

the standard equation of the conic is $Y^2 = 4p.\,X$. ... ... (11)

which is the equation of the parabola.

A few properties of (11) are given below.

Axis of the parabola is Y = 0, from (3), $\alpha x + \beta y + \lambda = 0$

λ is given by (6)

Vertex. The co-ordinates of the vertex will be obtained by solving the equations (4) and (5), i. e., X= 0 and Y = 0.

Latus rectum is given by relation (10)

Focus X = p, Y = 0 Use relations (7), (8) and (10), intersection of X = p and Y = 0 give the co-ordinates of the focus S.

Foot of the directrix, X = — p, Y = 0. Use relations (7) and (10), Z is the foot of the directrix.

Equation of the directrix X = — p. Use relations (7) and (10)

Equations of latus rectum. X = p. Use relations (7) and (10)

Ex. Reduce the equation —

$$16x^2 - 24xy + 9x^2 - 104x - 172y - 44 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

to the standard form. Find all its properties.

As the second degree terms of (i) form a perfect square the eq.

(i) will represent a parabola since $\Delta \neq 0$ and $ab = h^2$

$$\therefore\ (4x - 3y)^2 = 104x + 172y - 44 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The lines $4x - 3y = 0$ and $104x + 172y - 44 = 0$ are not at right angles, hence let us introduce a constant λ in (2)

$$(4x - 3y + \lambda)^2 = (104x + 172y - 44) + 2\lambda(4x - 3y) + \lambda^2$$

$$\text{or, } (4x - 3y + \lambda)^2 = x\,(104 + 8\lambda) + y\,(172 - 6\lambda) + (\lambda^2 - 44)$$

The lines $4x - 3y + \lambda = 0$ ... ... (4)

and $x\,(104 + 8\lambda) + y\,(172 - 6\lambda) + (\lambda^2 - 44) = 0$ ... ... (5)

are at right angles if (apply $a_1a_2 + b_1b_2 = 0$, sec Art. (25b)

$$4(104 + 8\lambda) + (-3)\,(172 - 6\lambda) = 0 \text{ or, } \lambda = 2 \quad \ldots \quad \ldots \quad (6)$$

Now eq. (3) becomes by (6)

$$(4x - 3y + 2)^2 = 40\,(3x + 4y - 1)$$

which can be written as

$$\left(\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}}\right)^2 . (25) = 40{\cdot}5\left(\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}}\right)$$

$$\text{or, } PN^2 = \frac{40{\cdot}5}{25}\,PM \text{ or, } Y^2 = 8X$$

Which is the stantard form of eq. (1)

$$\text{or, } Y^2 = 4P\,X. \text{ Where } P = \frac{1}{4}\,.\,8 = 2$$

For other properties of $Y^2 = 8x = 4pX$ ... ... (7)

$$\text{Here } Y = \frac{4x - 3x + 2}{\sqrt{(4^2 + 3^2)}} \text{ and } X = \frac{3x + 4y - 1}{\sqrt{(3^2 + 4^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

(i) Axis of (7) is Y = 0 or, $4x - 3y + 2 = 0$, by (8) ... ... (9)

(ii) Tangent at vertex is X = 0, or, $3x + 4y - 1 = 0$ by (8) ... ... (10)

(iii) Vertex is the intersection of eq. (9) and (10) by (8)

i. e. $4x - 3y + 2 = 0$ and $3x + 4y - 1 = 0$, Solve.

$$x = -\frac{1}{5}, y = \frac{2}{5} \quad \text{Vertex}\left(\frac{-1}{5}, \frac{2}{5}\right)$$

(iv) Latus rectum is 4p = 8

(v) Focux $X = p$, $Y = 0$, by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} = 2$ and $\frac{4x-3y+2}{\sqrt{(4^2+3^2)}} = 0$

or, $3x+4y-11=0$ and $4x-3y+2=0$, Solve

$\therefore x=1, y=2$. The focus is $(1, 2)$

(vi) Foot of the directrix. $X = -p$ and $Y = 0$ by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} = -2$ and $\frac{4x-3y+2}{\sqrt{4^2+3^2}}$

or, $3x+4y+9=0$ and $4x-3x+2=0$. Now solve. The foot of the directrix is $(-7/5, -6/5)$

(vii) Equation of the directrix. $X = -p$ by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} = -2$ ro, $3x+4y+9=0$

(viii) Equation of the latus rectum $X = p$ by (8)

or, $\frac{3x+4y-1}{\sqrt{(3^2+4^2)}} =$ or, $3x+4y-11=0$

Note : The properties of a parabola can be determined by other method also. The demonstration of such a method is givne below.

Ex. Find the focus, foot of the directrix and equations of latus rectum, axis and directrix of the conic.

$x^2+2xy+y^2-2x-1=0$ ... ... (1)

As the 2nd degree terms of eq. (1) form a perfect square i. e. $\Delta \neq 0$ and $h^2 = ab$, the eq (1) is a parabola. The eq. (1) may be written as $(x+y)^2-2x-1=0$ ... ... (2)

The axis of the parabola is parallel to the line

$x+y=0$ and is $x+y+k=0$ ... ... (3)

The equation of the directrix is $x-y+\lambda=0$ ... ... (4)

as the disectrix is perpendicular to the axis (3)

Let S (x_1, y_1) be the focus. If P (x, y) be any point on eq. (5) then by the definition of parabola.

$SP^2 = PM^2$, (PM is perpendicular to the directrix)

or, $(x-x_1)^2+(y-y_1)^2 = \left(\frac{x-y+\lambda}{\sqrt{(1^2+1^2)}}\right)^2$

or, $(x+y^2)-x(2\lambda+4x_1)+y(2\lambda-4y_1)+$

$(-\lambda^2+2x_1^2+2y_1^2)=0$ ... ... (4)

Identify the equations (4) and (1), the

$2\lambda+4x_1=2$ ... ... (5)

$2\lambda-4y_1=0$ ... ... (6)

$-\lambda^2+2x_1^2+2y_1^2=-1$ ... ... (7)

From (5), $x_1 = \frac{1}{2}(1-\lambda)$ and from (6). $y_1 = \frac{1}{2}\lambda$. Now from (7)

$-\lambda^2+2.\left(\frac{1}{2}\right)^2(1-\lambda)^2+(\lambda/2)^2=-1$ or, $\lambda=3/2$, $x_1=-1/4$, $y_1=3/5$

(i) Hence the focus is $\left(-\frac{1}{4}, \times \frac{3}{4}\right)$

(ii) The equation of the directrix is eq. (4) i. e ;

$x-y+3/2=0$ or, $2x-2y+3=0$ ... ... (8)

(iii) The eq. of the axis eq (3) $x+y+k=0$. since it passes through the focus S $(-1/4, 3/4)$, hence $k=-\frac{1}{2}$

$\therefore x+y-\frac{1}{2}=0$ or, $2x+2y-1=0$ ... ... (9)

is the eq. of the axis.

(iv) The foot of the directrix is obtained by solving eqs (8) and (9) i. e. $2x-2y+3=0$ and $2x+2y-1=0$ $x=-\frac{1}{2}, y=1$ i. e; $Z\left(\frac{1}{-2}, 1\right)$

(v) Vertex A. of the parabola is the middle point of S $(-1/4, 3/4)$ and $Z\left(-\frac{1}{2}, 1\right)$

$\therefore$ A $\left(\frac{1}{2}-1/4-\frac{1}{2}), \frac{1}{2}(3/4+1)\right)$ i.e. A $(-3/8, 7/8)$

(vi) Eq. of the latus rectum. The latus rectum is perp. to the axis whose eq is (9) at S$(-1/4, 3/4)$. Hence the eq. of latus rectum is $2x-2y+c=0$, since it passes through $(-1/4, 3/4)$, then $c=2$. Hence $2x-2y+2=0$

(vii) Length of the latus rectum. It is four times the distance between the vertex A $(-3/8, 7/8)$

and the focus S$(-1/4, 3/4)$

Latus rectum $= 4\sqrt{((-1/4+3/8)^2+(3/4-7/8)^2)} = 1/\sqrt{2}$

(viii) Equation of the tangent at vertex.

It is parallel to the directrix whose equation is $2x-2y+3=0$ by (8). Thus the equation to the tangent at the vertex is $2x-2y+d=0$; as it passes through A$(-3/8, 7/8)$ then $d=5/2$

Hence the eq. is $2x-2y+5/2=0$

EXAMPLES

Ex. 1. What conic does the equation $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent.

In this equation, we have by comparing with the general equation.

$h = -9$, $a = 13$ and $b = 37$ etc.

Now $h^2 - ab = (-9)^2 - (13)(37) = -$ ve, and $\Delta \neq 0$; $h^2 < ab$ or $ab - h^2 > 0$ See Art. 48.

Hence the conic is an ellipse.

Ex. 2. Test the nature of the conic given by the equation $3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$

If we compare this equation with the general equation of second degree, then we have.

$h = -4$, $a = 3$ and $b = -3$, $c = 8$, $g = 5$ and $f = -13/2$ Then by Art. 48. we have

$a + b = 3 - 3 = 0$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & -4 & -5 \\ -4 & -3 & -13/2 \\ 4 & -13/2 & 8 \end{vmatrix}$$

Hence the equation represents a rectangular hyperbola.

Ex. 3. Test the nature of the conic given by $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

Here we have after comparison with the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 9$, $b = 16$, $c = 19$, $g = -9$, $f = -101/2$, $h = -12$

Then by Art. 48, we have

$h^2 - ab = (-12)^2 - 9.16 = 144 - 144 = 0$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 9 & -12 & -9 \\ -12 & -16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 9 \end{vmatrix} \neq 0$$

$\therefore h^2 - ab = 0$ and $\Delta \neq 0$. Hence the conic is a parabola.

Ex. 4. Find the centre of the conic represented by $2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

Let $f(x, y) \equiv 2x^2 + y^2 - 3xy - 5x + 4y + 6 = 0$

$\frac{\delta f(x, y)}{\delta x} \equiv 4x - 3x - 4 = 0, \frac{\delta f(x, y)}{\delta y} \equiv 2y - 3x + 4 = 0$

The centre of the conic is the intersection of two lines.

$4x - 3y - 5 = 0$ and $2y - 3x + 4 = 0$

Solve $x = 2$, $y = 1$

Hence centre of the conic is at $(2, 1)$

Ex. 5. Find the lengths and equation of the axis of the conic $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$.

Let $f(x, y) \equiv 36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$

The centre is given by

$\frac{\delta f}{\delta x} \equiv 72x + 24y - 72 = 0$ and $\frac{\delta f}{\delta x} \equiv 24x + 58y + 126 = 0$

Solving the centre is $(2, -3)$

$C_1 = gx_1 + fy_1 + c = -36.2 + 63.-3 + 81 = -180$

Therefore equation of the conic referred to centre $(2, -3)$ as origin is

$36x^2 + 24xy + 29x^2 - 180 = 0$ by (5) working Rule Art. 50 or, $x^2/5 + \frac{2}{15}xy + \frac{29}{180}y^2 = 1$

Now semi-axis of the conic are given by (See Eq. 4. Art. 51)

$(1/5 - 1/r^2)(29/180 - 1/r^2) = (1/15)^2$

or, $1/r^4 - \frac{13}{36}\frac{1}{r^2} + \frac{1}{36} = 0$ or, $r^4 - 13r^2 + 36 = 0$

or, $(r^2 - 9)(r^2 - 4) = 0$, $\therefore r = 3$, or, 2 i. e., $r_1 = 3$, $r_2 = 2$

The curve is an ellipse.

Major axis $= 6$, Minor axis $= 4$

Now equation to major axis referred to the centre $(2, -3)$ as origin is $(a - 1/r_1^2)x + hy = 0$

or, $(1/5 - 1/9)x + y/15 = 0$ by (6) Art. 51

or, $4x + 3y = 0$

And referred to the old axis (see Art. 32) is

$4(x - 2) + 3(y + 3) = 0$ or, $4x + 3y + 1 = 0$

Similarly the equation to minor axis referred to the centre

$(2, -3)$ is $(a - 1/r^2)x + hy = 0$ or, $(1/5 - 1/4)x + y/15 = 0$

or, $3x = 4y$ and referred to the old axes is (see Art. 32)

$3(x - 2) = 4(y + 3)$ or, $3x - 4y - 18 = 0$

Lengths of the axes are 6 and 4.

And equations of major and minor axes respectively are $4x + 3y + 1 = 0$ and $3x - 4y - 18 = 0$.

Ex. 6. Reduce the equation $6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ to the standard form. Find also its, lengths, equations and directions of the axes. [D.U. 1977 ; C.U. 1982]

Let $f(x, y) \equiv 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ ... ... (1)

The centre is given by

$\frac{\delta f}{\delta x} \equiv 12x + 5y - 4 = 0, \frac{\delta f}{\delta x} \equiv 5x - 12y + 7 = 0$

Solving, centre is $(1/13, 8/13)$

$$C = -C_1 = -(gx_1 + fy_1 + c) = -\left(-2 \times \frac{1}{13} + \frac{7}{2} \times \frac{8}{13} + 11\right) = -13$$

Transferring the origin to (1/13, 8/13) the equation (1) reduces to $6x^2 + 5xy - 5y^2 + 13 = 0$ by (5), Working Rule Art. 50

The squares of the semiaxes are given by

$(a - k/r^2)(b - k/r^2) = h^2$ see (9) Art. 5

or, $(6 - {}^{-13}/r^2)(-6 - {}^{-13}/r^2) = (5/2)^2$

or, $169r^4 = 169 \times 4$ or, $r^2 = \pm 2$

i. e. $r_1^2 = 2$, and $r_2^2 = -2$

They are of opposite signs and equal, hence the conic (1) forms a rectangular hyperbola.

The lengths of the axes are $2\sqrt{2}$ and $2\sqrt{2}\,i$

The equations of axes are given (by eq. 10, Art. 51) with respect to the centre.

$(a - k/r_1^2)x + hy = 0$ and $(b - k/r_2^2)x + hy = 0$

or, $(6 + 13/2)x + 5/2.y = 0$ and $(6 - 13/2)x + 5y/2 = 0$

or, $y = -5x$ and $x - 5y = 0$

The equations with respect to the axes (See Art. 22)

are $(y - 8/13) = -5(x - 1/13) = 0$ and $x - 1/3 - 2(y - 8/13) = 0$

or, $y + 5x - 1 = 0$ and $x - 5y + 3 = 0$

Slopes are $\tan^{-1}(-5)$ and $\tan^{-1}(1/5)$ with the x — axis.

Ex. 7. Show that $25x^2 + 2xy + 25y^2 - 130x - 130y + 169 = 0$ represent an ellipse referred to rectangular cartesian co-ordinates (x, y). Find the centre and the lengths of the principal axes and also the directions of the axes. [D.U. 1965]

Let $f(x, y) = 25x^2 + 2xy + 25y^2 - 130x - 130y + 169 = 0$ ... (1)

The centre is given by

$$\frac{\delta f(x,y)}{\delta x} \equiv 50x + 2y - 130 = 0\,;\ \frac{\delta f(x,y)}{\delta y} \equiv 50y + 2x - 130 = 0$$

Solve, centre is (5/2, 5/2)

$k_1 = gx_1 + fy_1 + c = -65.5/2,\ -65.5/2 + 169 = -156$

Transfer the origin to (5/2, 5/2). the eq. (1)

Reduces to $25x^2 + 2xy + 25y^2 - 156 = 0$ by (5) Art. 50

$$\text{or, } \frac{25}{156}x^2 + \frac{25}{156}xy + \frac{25}{156}y^2 = 1 \quad \ldots \quad (2)$$

or, $ax^2 + 2hxy + by^2 = 1$

The squares of the semi-axes are given by

$(a - 1/r^2)(b - 1/r^2) = h^2$, by (9) Art. 51

$$\text{or, } \left(\frac{25}{156} - 1/r^2\right)\left(\frac{25}{156} - 1/r^2\right) = \left(\frac{1}{156}\right)^2$$

$$\text{or, } \frac{25}{156} - 1/r^2 = \pm\frac{1}{156}$$

or, $r^2 = 6$ or, 13/2

$\therefore r_1 = (\sqrt{6}),\ r_2 = \sqrt{(13/2)}$

The eq. (1) represents an ellipse.

Lenght of minor axis $= 2r_1 = 2(\sqrt{6})$

Lenght of major axis $= 2r_2 = 2(\sqrt{13/2})$

The equation of major axis is

$$(a - 1/r_1^2)x + hy = 0 \text{ or, } \left(\frac{25}{165} - \frac{1}{6}\right)x + \frac{1}{156}y = 0$$

or, $x - y = 0$ referred to the centre (5/2. 5/2)

or, $(x - 5/2) - (y - 5/2) = 0$ referred to old axes,

or, $y = x$. Slope is 45° with the x — axis

The eq of the minor axis is—

$(a - 1/r_2^2)x + hy = 0$ or, $(25/156 - 2/13)x + 1/156y = 0$

or, $x + y = 0$ referred to the centre (5/2, 5/2)

or, $(x - 5/2) + (y - 5/2) = 0$ referred to old axes.

or, $x + y - 5 = 0$. or, $y = -x + 5$

Slope of the line is $\tan\theta = -1$ i. e. $\theta = 135°$

Hence eq. (1) represents an ellipse, centre (5/2, 5/2), major axis $= 2\sqrt{6}$, minor axis $= 2(\sqrt{13/2})$. Direction of major axis is 45° and direction of minor axis is 135° with the x-axis.

Ex. 8. Find the equation of the asymptotes of the conic

$$7x^2 + 52xy - 32y^2 - 170x + 140y = 0 \quad \ldots \quad (1)$$

The equations of the a symptotes of a conic differ from the equation of conic only by a constant. Hence the asymptotes are given by $7x^2 + 52xy - 32y^2 - 70x^2 + 140y + \lambda = 0$ ... (2)

The centre of the conic (1) is given by

$$\frac{\delta f(x,y)}{\delta x} \equiv 14x + 52y - 170 = 0\,;\ \frac{\delta f(x,y)}{\delta x} \equiv 52x - 64y + 140 = 0$$

Solve the equations $7x + 26y - 85 = 0$ and $13x - 16y + 35 = 0$

The centre is (1, 3)

The asymptoptes pass through the centre (1, 3) then from (2)

$7.1 + 52.3 - 32.9 - 170.1 + 140.3 + \lambda = 0$ or, $\lambda = -125$

Hence the asymptotes are—

$$7x^2 + 52xy - 32y^2 - 170x + 140y - 125 = 0 \quad \ldots \quad (3)$$

The possible asymptotes are parallel to the factors of—

$7x^2 + 52xy - 32y^2$ i. e. $(x + 8y)$ or, $7x - 4y$ Factors of eq. (3) are $(x + 8y + A)(7x - 4y + B)$

Now compare the co-effcients of (3) and (4)

$7A + B = -170, -4A + 8B = 140, A = -25, B = 5$

Aymptotes from (4) are

$x + 8y - 25 = 0$ and $7x - 4y + 5 = 0$

[D.U.(P) 1986 (C. H. 1977)]

Ex. 9. Reduce the equation

$4x^2 - 4xy + y^2 - 8x - y + 6 = 0$ ... ... (1)

to the standard from. Find all its properties.

In the eq. (1), $a = 4, b = 1, h = -2, g = -4, f = -3, c = 5$.

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -64, ab - h^2 = 4 - 4 = 0$

$\Delta \neq 0$ and $ab = h^2$. Hence eq. (1) represents a parabola. The eq. (1) may be written as

$(2x - y)^2 = 8x + 6y - 5$ ... ... (2)

or, $(2x - y + \lambda)^2 = 8x + 6y - 5 + \lambda^2 + 2\lambda(2x - y)$

or, $(2x - y + \lambda)^2 = x(4\lambda + 8) + y(-2\lambda + 6) + (\lambda^2 - 5)$ ... ... (3)

The lines $2x - y + \lambda = 0$ ... ... (4)

and $x(4\lambda + 8) + y(-2\lambda + 6) + (\lambda^2 - 5) = 0$ ... ... (5)

may be perpendicular if $a_1a_2 + b_1b_2 = 0$

i. e. $2(4\lambda + 8)(-1) + (-2\lambda + 6) = 0$ or, $\lambda = -1$ ... ... (6)

The eq. (3) becomes

$(2x - y - 1)^2 = 4(x + 2y - 1)$ which can be written as

$$\left(\frac{2x - y - 1}{\sqrt{(2^2 + 1^2)}}\right)^2 5 = 4\sqrt{5}\left(\frac{x + 2y - 1}{\sqrt{(1^2 + 2^2)}}\right)$$

It is of the form $PN^2 = 4(\sqrt{5})/5.\ PM$.

or, $Y^2 = 4/\sqrt{5}.\ X$. which is the standard equation of a parabola.

(i) Latus rectum $= 4p = 4/\sqrt{5}$ or, $p = 1/\sqrt{5}$

$X = (x + 2y - 1)/\sqrt{5}$ and $Y = (2x - y - 1)/\sqrt{5}$ (7)

$\therefore\ Y^2 = 4pX$.

(ii) Axis is $Y = 0$ or, $2x - y - 1 = 0$ from (7) ... ... (8)

(iii) Tangent at the vertex $X = 0$ i. e. $x + 2y - 1 = 0$ from (7) .... (9)

(iv) Vertex A is the intersection of eqs. (8) and (9).

Solve $x = 3/5, y = 1/5$; A $(3/5, 1/5)$

(v) Focus is $S(p, 0)$ i.e. ; $X = p$ and $Y = 0$

or, $\dfrac{x + 2y - 1}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$ and $\dfrac{2x - y - 1}{\sqrt{5}} = 0$ from (7)

or, $x + 2y - 2 = 0$ and $2x - y - 1 = 0$

Solve, $x = 4/5,\ y = 3/5$; $S(4/5, 3/5)$

(vi) Equation of latus rectum is $X = p$

or, $(x + 2y - 1)/\sqrt{5} = 1/\sqrt{5}$ or, $x + 2y - 2 = 0$

(vii) Foot of the directrix $Z(-p, 0)$ i. e. $X = -p, Y = 0$

or, $(x + 2y - 1)/\sqrt{5} = -1/\sqrt{5}$ and $(2x - y - 1)/\sqrt{5} = 0$

Solve, $x = 4/5, y = -1/5 \therefore Z(4/5, -1/5)$.

(viii) Equation of the directrix is $X = -p$

or, $(x + 2y - 1)/\sqrt{5} = -1/\sqrt{5}$ or, $x + 2y = 0$

Ex. 10. Reduce the equation $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ to the standard form. Find also the equations of latus rectum, directrices and axes. [N. U. H. 1997]

Let $f(x, y) \equiv x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ ... ... (1)

(i) its centre is at $\left(0, \frac{1}{2}\right)$ [See Ex. 4. for method.]

Now transfer the origin to the point $\left(0, \frac{1}{2}\right)$, the eq. (1) reduces to $x^2 + 12xy - 4y^2 + c_1 = 0$

where $c_1 = gx_1 + fy_1 + c = -3 \cdot 0 + 2.\frac{1}{2} + 9 = 10$

(ii) The reduced equation is $x^2 + 12xy - 4y^2 + 10 = 0$

or, $-x^2/10 - 12xy/10 + 4y^2/10 = 1$

(iii) The lengths of the semi axes are given by

$(a - 1/r^2)(b - 1/r^2) = h^2$ or, $(-1/10 - 1/r^2)(4/10 - 1/r^2)$
$= (-6/10)^2$

or, $r^2 = 5/4$ or, -2, $\therefore\ r_1^2 = 5/4 : r_2^2 = -2$ ... ... (2)

The conic is a hyperbola.

(iv) Eq. of the transverse axis is

$(a - 1/r_1^2)x + hy = 0$ or, $(-1/10 - 4/5)x - 3/5y = 0$ or, $3x + 2y = 0$

referred to the centre $\left(0, \frac{1}{2}\right)$ or, $3(x - 0) + 2(y - 1/2) = 0$

referred to the origin, or, $3y + 2y - 1 = 0$ ... ... (3)

Slope of it, $\tan\theta = -3/2$. $\sin\theta = 3/\sqrt{13}, \cos\theta = -2/\sqrt{13}$ ... ... (4)

through $\left(0, \frac{1}{2}\right)$, then $k = 3/2$, The equation is $4x - 6y + 3 = 0$

(v) Eccentricity. $e^2 = 1 - r_2^2/r_1^2 = 1 + 2.4/5$

or, $e = \sqrt{(13/5)}$ ... ... (5)

(vi) d for foci S and S′ $= \pm ae = \pm r_1e = \pm \frac{1}{2}\sqrt{13}$

d for feet of the directriecs Z and Z′ $= \pm a/e = \pm 5/2\sqrt{13}$

(vii) Here (h, k) are the centre $\left(0, \frac{1}{2}\right)$ of conic (1)

Points S and S′ = $(h + d\cos\theta, k + d\sin\theta)$

$$= (0 \pm \frac{1}{2}\sqrt{13}, -2\sqrt{\frac{1}{13}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{13}\, 3\sqrt{\frac{1}{13}}) = (-1, 2) \text{ and } (1, -1)$$

(viii) points Z and Z′ = $(0 \pm \frac{5}{2}\sqrt{\frac{1}{13}}, -2\sqrt{\frac{1}{13}}, \frac{1}{2} \pm \frac{5}{2\sqrt{13}} \times 3\sqrt{\frac{1}{13}})$

$$= (-5/13, 14/13) \text{ and } (5/13, -1/13)$$

(ix) Latus rectum and directrix are parallel to the conjugate axis $2x - 3y + 3/2 = 0$ Therefore their slopes are the same as conjugate axis.

(x) The equation of the latus rectum is $2x - 3y + k = 0$. Since it passes through $S(-1, 2)$ or $S'(1, -1)$: hence $k = 8$ or, -5, The equations are $2x - 3y + 8 = 0$ and $2x - 3y - 5 = 0$

(xi) Similarly the equation of directrix is $2x - 3y + \lambda = 0$

Since it passes through $Z(-5/13, 14/13)$ or, $(5/13, -1/13)$

hence $\lambda = 4, -1$.

The equation of directrices $2x - 3y + 4 = 0$ and $2x - 3y - 1 = 0$

For reference Art. 15 (a) and Art. 51

Ex. 11. $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$ সমীকরণ দ্বারা প্রকাশিত বক্ররেখাটি অঙ্কন কর।

এ সমীকরণে $a = 4, b = 1, h = -2, g = -6, f = 3, c = 9$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 4 \cdot 19 + 2(3)(-6)(-2) - 4(3)^2$

$- 1.(6)^2 - 9(-2)^2 = 36 + 72 - 36 - 36 = 0$

$ab - h^2 = 4 \cdot 1 - (2)^2 = 0$

যেহেতু $\Delta = 0, ab - h^2 = 0$ সুতরাং প্রদত্ত সমীকরণটি যুগল সমান্তরাল অথবা সমপতিত সরলরেখা প্রকাশ করবে।

$\therefore 4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$

or, $(2x - y)^2 - 12x + 6y + 9 = 0$

or, $(2x - y + \lambda)^2 = (12 + 4\lambda)x - (6 + 2\lambda)y + \lambda^2 - 9$

λ-এর মান এমনভাবে স্থির করতে হবে যেন

$2x - y + \lambda = 0, (12 + 4\lambda)x - (6 + 2\lambda)y + \lambda - 9 = 0$

পরস্পরের উপর লম্ব হয়।

অতএব $3(12 + 4\lambda) + (6 + 2y) = 0$ or, $\lambda = -3 \therefore [aa_1 + bb_1 = 0]$

$\therefore \quad (2x - y - 3)^2 = 0$ যা এক যুগল সমপতিত সরলরেখা প্রকাশ করে।

$\therefore \quad 2x - y - 3 = 0$ or, $\frac{y}{-3} + \frac{x}{3/2} = 1$

এর লেখচিত্র একটি সরলরেখায় যা y অক্ষরেখাকে $(0, -3)$ এবং x — অক্ষরেখাকে $(3/2, 0)$ বিন্দুতে ছেদ করবে।

Ex. 12. $16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$ সমীকরণের দ্বারা প্রকাশিত বক্ররেখার লেখচিত্র অঙ্কন কর।

প্রদত্ত সমীকরণের দ্বিমাত্রিক পদগুলোর সমষ্টি একটি পূর্ণ বর্গ হলে সমীকরণটি একটি পরাবৃত্ত (parabola) হতে পারে।

$16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

or, $(4x - 3y + \lambda)^2 + (77 - 8\lambda)x + 2(3\lambda - 32)y + 95 - \lambda^2 = 0 \quad \ldots \quad \ldots \quad (2)$

λ এর মান এমনভাবে স্থির করা যায় যেন

$4x - 3y + \lambda = 0, (77 - 8\lambda)x + 2(3\lambda - 32)y + 95 - \lambda^2 = 0$

$\therefore$ পরস্পর লম্ব হয়।

$\therefore 4(77 - 8\lambda) - 3 \cdot 2(3\lambda - 32) = 0, 50\lambda = 308 + 192$ or, $\lambda = 10$

অতএব (2) হতে—

$(4x - 3y + 10)^2 - 3x - 4y - 5 = 0$

or, $\frac{(4x - 3y + 10)^2}{5^2} = \frac{1}{5}\frac{(3x + 4y + 5)}{5} \quad \ldots \quad \ldots \quad \ldots \quad (3)$

$\therefore Y^2 = 4pX$ একটি পরাবৃত্তের সমীকরণ $\quad \ldots \quad \ldots \quad \ldots \quad (4)$

পরাবৃত্তের অক্ষ, $4x - 3y + 10 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$

A শীর্ষ বিন্দুতে স্পর্শকের সমীকরণ $3x + 4y + 5 = 0 \quad \ldots \quad \ldots \quad (6)$

যখন $x = 0$, (1) হতে —

$9y^2 - 64y + 95 = 0$, or, $(9y - 19)(y - 5) = 0 \therefore y = 5, 19/9$

$\therefore (0, 5), (0, 19/9)$

আবার $y = 0$; $16x^2 + 77x + 95 = 0$, x এর মান কাল্পনিক অর্থাৎ বক্ররেখাটি অক্ষরেখাকে ছেদ করবে না।

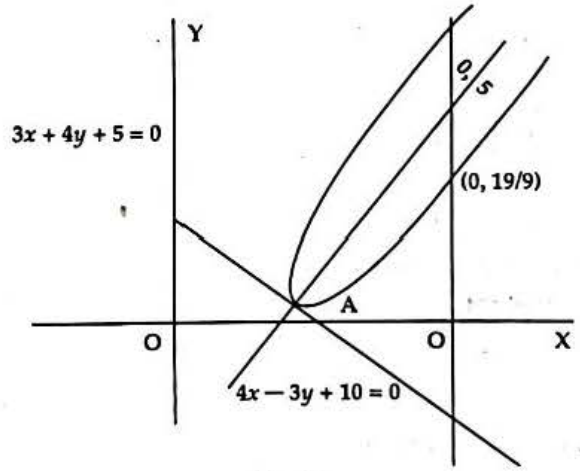


Fig. 24

নাভিলম্ব (3), (4) হতে $4p = \frac{1}{5}$

নাভিলম্ব সমীকরণ, $\frac{3x+4y+5}{\sqrt{(3^2+4^2)}} = \frac{1}{4}\,\frac{1}{5}$

or, $3x+4y+5 = \frac{1}{4}$

S, (focus) নাভিবিন্দু ঃ পরাবৃত্তের অক্ষরেখা

$4x-3y+10=0$ এবং নাভিলম্বের ছেদক

$3x+4y+5=1/4 \quad \therefore S(-217/100.\ 11/25)$

পরাবৃত্তের অক্ষরেখাকে লিখা যায়—

$4x-3y+10=0$ or, $\frac{x}{-10/4}+\frac{y}{10/3}=1$

A বিন্দুতে স্পর্শকের সমীকরণকে লিখা যায়—

$3x+4y+5=0$ or, $\frac{x}{-5/3}+\frac{y}{-5/4}=1$

সমীকরণ (3) $X=0, Y=0$ বসালে (3) ধনাত্মক হয়। সুতরাং পরাবৃত্তটি A বিন্দুর ডানদিকে থাকবে। চিত্রটি আকারের রূপরেখা দেয়া হল।

Ex. 13. $17x^2-12xy+8y^2+46x-28y+17=0$ সমীকরণটির প্রমাণরূপ কি হবে এবং এর লেখচিত্র অঙ্কন কর।

$f(x,y) \equiv 17x^2-12xy+8y^2+46x-28y+17=0 \quad \ldots \quad \ldots \quad (1)$

$\frac{\delta f}{\delta x} \equiv 17x-6y+23=0, \frac{\delta f}{\delta y} \equiv -6x+8y-14=0 \quad \ldots \quad \ldots \quad (2)$

এদের সমাধান করলে, $x=-1, y=1$

বক্ররেখাটির কেন্দ্র $(-1,1)$

$-c_1 = -(gx+fy+c) = 23+14-17 = 20$

নতুন বক্ররেখাটি, $x = \frac{17}{20}x^2 - \frac{12}{20}xy + \frac{8}{20}y^2 = 1 \ldots \quad \ldots \quad \ldots \quad (3)$

বক্ররেখাটির অক্ষদ্বয়ের দৈর্ঘ্য

$\left(\frac{17}{20}-\frac{1}{r^2}\right)\left(\frac{8}{20}-\frac{1}{r^2}\right) = \left(\frac{-6}{20}\right)^2$

বা, $(17r^2-20)(2r^2-5) = 9r^4$

বা, $25r^4-125r^2+100=0$ বা, $r^4-5r^2+4=0$

$\therefore r^2 = 4, 1$ যেহেতু r^2 এর দুটি মান ধনাত্মক

অতএব, বক্ররেখাটি উপবৃত্ত হবে,

(1) এ $y=0, 17x^2+46x+17=0$

বা, $x = -\frac{23}{17} \pm \frac{4}{17}\sqrt{15} = -0{\cdot}44, -2{\cdot}26$ (প্রায়)

যখন $x=0$, (1) হতে $8y^2-28y+17=0$.

বা, $y = \frac{1}{8} \pm \sqrt{(14^2-8.17)} = \frac{7}{4} \pm \frac{1}{4}\sqrt{15} = 2.72 : 0.78$

পরাক্ষের (Major axis) সমীকরণ—

$(A-1/r_1^2)x+Hy=0$

বা, $A=17/20, B=8/20, H=-6/20$

বা, $(17/20-1/4)x-6y/20=0$

বা, $12x-6y=0$, বা, $2x-y=0$

বা, $2(x+1)-(y-1)=0$ পূর্বের মূলবিন্দু অনুসারে,

বা, $2x-y+3=0$ বা, $x/(-3/2)+y/3=1$

উপাক্ষের (Minor axis) সমীকরণ

$(A-1/r_2^2)x+Hy=0$

বা, $(17/20-1)x-6y/20=0$

বা, $-3x/20-6y/20=0$ বা, $x+2y=0$

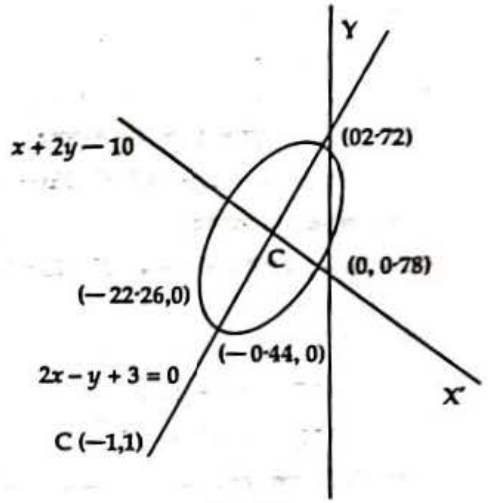


Fig : 25

বা, $(x+1)+2(y-1)=0$ বা, $x+2y-1=0$

বা, $x/1+y/\frac{1}{2}=1$

এখন উল্লিখিত বিষয়গুলো অক্ষতলে সন্নিবেশ করলে বক্ররেখার আকারের প্রকৃতি পাওয়া যায়।

Ex. 14. $3x^2-8xy-3y^2+10x-13y+8=0$ সমীকরণটির লেখচিত্র অঙ্কন কর। [D.U. 1982]

$f(x,y) \equiv 3x^2-8xy-3y^2+10x-13y+8=0 \ldots \quad \ldots \quad \ldots \quad (1)$

$\frac{\delta f}{\delta x} \equiv 6x-8y+10=0, \frac{\delta f}{\delta y} \equiv -8x-6y-13=0 \quad \ldots \quad \ldots \quad (2)$

(1) এর কেন্দ্র থাকলে তা (2) হতে সমীকরণদ্বয়ের ছেদ বিন্দু হবে।

কেন্দ্র $(-41/25, 1/50)$

$c_1 = gx+fy+c = -5(41/25)-(13/2)(1/50)+8 = -33/100$

(1) এ প্রমাণ রূপ $3x^2-8xy-3y^2 = 33/100$

$Ax^2+2Hxy+By^2=1$

বা, $\frac{100}{11}x^2 - \frac{800}{33}xy - \frac{100}{11}y^2 = 1$

অক্ষরেখাদ্বয়ের দৈর্ঘ্য

$1/r^4-(A+B)/r^2+(AB-H^2)=0$

বা, $r^4 = \frac{33^2}{100^2 \times 5^2}$

বা, $r^2 = \pm\frac{33}{500}$

$\therefore r_1^2 = 33/500, r_2^2 = -33/500$

অতএব (1) সমীকরণ একটি অধিবৃত্ত (Hyperbola) প্রকাশ করবে।

এর আড় অক্ষের সমীকরণ—

$(A - 1/r_1^2)x + Hy = 0$

বা, $(100/11 - 500/33)x - 400/33y = 0$

বা, $x - 2y = 0$ একে পূর্বে মূল বিন্দু অনুসারে

$(x + 41/25) - 2(y - 1/50) = 0$ বা, $25x - 50y + 42 = 0$

এর অনুবন্ধী অক্ষের সমীকরণ—

$50x + 25y + c = 0$ বা, $50 \times -41/25 + 25 \times 1/50 + c = 0$ বা, $c = \frac{163}{2}$

$50x + 25y + \frac{163}{2} = 0$

(1) এ $x = 0$ হলে, $y = 0, -13/3$

(1) এ $y = 0$ হলে, $x = -2, -4/3$

বক্ররেখাটি এর অক্ষদলকে $(-2, 0), (-4/3, -13/3)$ বিন্দুতে ছেদ করবে। এর অসীমতট রেখাগুলো

$(3x + y)(x - 3y) + 10x - 13y + 8 = 0$

$\therefore 3x + y - 49/6 = 0$ এবং $x - 3y + 43/10 = 0$

নাভি লম্বের দৈর্ঘ্য $= \frac{2b^2}{a} = \frac{2r_2^2}{r_1} = 0.23$

নাভিবিন্দু (Foci) $= \pm r_1 e = \pm 0.343.$

$e^2 = 1 - r_2^2/r_1^2 = 6/5.$

উল্লিখিত বিষয়াদি অক্ষতলে সন্নিবেশ করলে বক্ররেখার আকৃতির রূপ দেয়া যেতে পারে।

Which can be written as

$$\left(\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}}\right)^2 (25) = 40.5 \left(\frac{3x - 4y + 2}{\sqrt{3^2 + 4^2}}\right)$$

or, $PN^2 = \frac{40.5}{25} PM$ or, $Y^2 = 8X$

which is the standard form of eq. (1)

or, $Y^2 = 4PX$. where $P = \frac{1}{4}.8 = 2$

For other properties of $y^2 = 8x = 4pX$ ... (7)

Here $Y = \frac{4x - 3y + 2}{\sqrt{(4^2 + 3^2)}}$ and $X = \frac{3x + 4y - 1}{\sqrt{(3^2 + 4^2)}}$ ... (8)

(i) Axis of (7) is $Y = 0$ or, $4x - 3y + 2 = 0$, by (8) ...

(ii) Tangent at vertex is $X = 0$, or, $3x + 4y - 1 = 0$ by (8) ... (10)

(iii) Vertex is the intersection of eq (9) and (10 by (8) i.e.

$4x - 3y + 2 = 0$ and $3x + y - 1 = 0$,

Solve. $x = -\frac{1}{5}$, $y = \frac{2}{5}$ vertex $(-\frac{1}{5}, \frac{2}{5})$

(iv) Latus rectum is $4p = 8$

(v) Focus $X = p$, $Y = 0$, by (8)

or, $\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}} = 2$ and $\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}} = 0$

or, $3x + 4y - 11 = 0$ and $4x - 3y + 2 = 0$. solve

$\therefore$ $x = 1$, $y = 2$. The focus is $(1, 2)$

(vi) Foot of the directrix, $X = -p$ and $Y = 0$ by (8)

or, $\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}} = 2$ and $\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}} = 0$

or, $3x + 4y + 9 = 0$ and $4x - 3y + 2 = 0$.Now solve. The foot of the directix is $(-7/5, -6/5)$

(vii) Equation of the directrix, $X = -p$, by (8)

or, $\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}} = -2$ or, $3x + 4y + 9 = 0$

(viii) Equation of the latus rectum $X = p$ by (8)

or, $\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}} = 2$ or, $3x + 4y - 11 = 0$

Note : "The properties of a parabola can be determined by other method also. The demonstration of such a method is given below.

Ex. 14(a) What conic does the equation $5x^2 - 2xy + 5y^2 - 8x - 8y - 8 = 0$ represent? Reduec the equation to standard form. [N.U.H. 1996]

Ans. The given equation

$5x^2 - 2xy + 5x^2 - 8x - 8y - 8 = 0$ ... ... (i)

Comparing (i) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get, $a = 5, h = -1, b = 5, g = -4, f = -4, c = -8.$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 5.5.-8 + 2(-4)(-4)(-1) - 5(-4)^2 - 5(-4)^2 - (-8)(-1)^2$

$= -200 - 32 - 80 - 80 + 8 = -384$

$\therefore \Delta \neq 0$

$ab - h^2 = 5.5 - (-1)^2 = 25 - 1 = 24$

$\therefore ab - h^2 > 0$

Thus, the given conic represents an ellipse,

Lef (α, β) be the centre of the conic (i)

$\therefore \alpha = \frac{hf - bg}{ab - h^2}$, $\beta = \frac{hg - af}{ab - h^2}$

$= \frac{(-1)(-4)-5(-4)}{5.5-(-1)^2}$ $\quad = \frac{(-1)(-4)-5(-4)}{5.5-(-1)^2}$

$= \frac{24}{24}$ $\quad = \frac{24}{24}$

$= 1$ $\quad = 1$

$\therefore$ Centre is at (1, 1)

Transfer the origin to a point (1,1) When direction of axes are unchanged, then (1) becomes

$ax^2 + 2hxy + by^2 = c_1$

ie $5x^2 - 2xy + 5y^2 = c_1$

where $c_1 = -(\alpha g + \beta f + c)$

$c_1 = -\{1.(-4) + 1.(-4) - 8\} = 16$

$\therefore 5x^2 - 2hxy + 5y^2 = 16$

$\Rightarrow \frac{5}{16}x^2 - 2\frac{1}{16}xy + \frac{5}{16}y^2 = 1$

$\Rightarrow Ax^2 + 2Hxy + By^2 = 1$ .... (2)

Where $A = \frac{5}{16}, H = \frac{-1}{16}, B = \frac{5}{16}$

Removing xy term when the axes being rotated.

$\therefore a_1x^2 + b_1y^2 = 16$ .... (3)

$a_1 + b_1 = a + b$

$a_1b_1 = ab - h^2$

$\therefore a_1 + b_1 = 5 + 5 = 10$

$a_1b_1 = 5.5 - (-1)^2$

$a_1b_1 = 24$

Now, $(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1b_1$

$= 10^2 - 4.24 = 100 - 96 = 4$

$\therefore a_1 - b_1 = 2$ take + ve sign

$\therefore \quad a_1 + b_1 = 10$

$\quad a_1 - b_1 = 2$

$(+) \quad 2a_1 = 12 \qquad \therefore a_1 = 6$

$b_1 = 10 - a_1 = 10 - 6 = 4$

$\therefore (3) \Rightarrow 6x^2 + 4y^2 = 16$

$\Rightarrow \frac{x^2}{\frac{8}{3}} + \frac{y^2}{4} = 1$

$\Rightarrow \frac{x^2}{\left(\sqrt{\frac{8}{3}}\right)^2} + \frac{y^2}{y^2} = 1$

which is standard form of an ellipse.

Ex. 14(b) Find the asymptotes the hyperbola $x^2 + 2xy - y^2 + 2x + 4y = 0$ and the equation of the conjugate hyperbola. [N.U.H. 1996]

Ans. The given hyperbola

$x^2 + 2xy - y^2 + 2x + 4y = 0$ ..... (1)

Let equation of asymptote of the hyperbola (1) is

$x^2 + 2xy - y^2 + 2x + 4y + \lambda = 0$ ..... (2)

Since asymptotes is a pair of st. lines

So $\Delta = 0$ is $\boxed{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$

$\therefore 1.(-1).\lambda + 2.(+2)(+1). - 1.(+2)^2 - (-1)(+1)^2 - \lambda.1^2 = 0$

$\Rightarrow -\lambda + 4 - 4 + 1 - \lambda = 0$

$\Rightarrow 2\lambda + 1 = 0$

$\therefore \lambda = \frac{1}{2}$

$\therefore (2) \Rightarrow x^2 + 2xy - y^2 + 2x + 4y + \frac{1}{2} = 0$

Which is the required asymptotes.

Let equation of the conjugate hyperbola is

$x^2 + 2xy - y^2 + 2x + 4y + k = 0$ ..... (3)

We know, equation of the conjugate hyperbola

$= 2 \times$ (equation of asymptotes) $-$ equation of the hyperbola

$\Rightarrow x^2 + 2xy - y^2 + 2x + 4y + k = 2 \times (x^2 + 2xy - y^2 + 2x + 4y + \frac{1}{2}) - (x^2 + 2xy - y^2 + 2x + 4y)$

$\Rightarrow k = 1$

$(3) \Rightarrow x^2 + 2xy - y^2 + 2x + 4y + 1 = 0$

Which is the required conjugate hyperbola.

Ex. 14(c) Reduce the equation $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ to the standard from and identify it. [N.U.H. 1997]

Ans. The given equation

$x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$ ..... (1)

Compare (–1) with the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ We get

$a = 1, b = -4, h = 6, g = -3, f = 2, c = 9$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1.(-4).9 + 2.2.(-3).6 - 1.2^2 + 4.(-3)^2 - 9.6^2$

$= -36 - 72 - 4 + 36 - 324$

$= -400 \neq 0$

$ab - h^2 = 1.-4 - 6^2 = -40 < 0$

Thus, the given conic represents a hyperbola.

Let (α, β) be the centre of the conic (1)

$\therefore \alpha = \dfrac{hf - bg}{ab - h^2}$ $\quad \beta = \dfrac{hg - af}{ab - h^2}$

$= \dfrac{6.2 + 4(-3)}{1 - 4 - 6^2}$ $\quad = \dfrac{6.-3 - 1.2}{1.-4 - 6^2}$

$= \dfrac{0}{-40}$ $\quad = \dfrac{-20}{-40}$

$= 0$ $\quad = \dfrac{1}{2}$

$\therefore$ Centre is at $(0, \frac{1}{2})$

Now, transfer the origine to a point $(0, \frac{1}{2})$

Then (1) becomes

$x^2 + 12xy - 4y^2 = c_1$ ..... (2)

Where $C_1 = -(g\alpha + f\beta + C)$

$= -(-3.0 + 2.\frac{1}{2} + 9)$

$= -10$

$\Rightarrow x^2 + 12xy - 4y^2 = -10$ .....(3)

Now, transform the axes such that remove the xy term, then

$a_1 x^2 + b_1 y^2 = -10$ .....(4)

Where $a_1 b_1 = a + b = 1 - 4 = -3$

and $a_1 b_1 = ab - h^2 = -4 - 6^2 = -40$

Now, $(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4 a_1 b_1$

$= (-3)^2 - 4.-40$

$= 9 + 160 = 169$

$\therefore a_1 - b_1 = 13$ Take + ve sign

Again, $a_1 + b_1 = 3$

$\dfrac{a_1 - b_1 = 13}{(+)\ 2a_1 = 10}$ $\quad \therefore a1 = 5$

$b_1 = -3 - a_1 = -3 - 5 = -8$

$(4) \Rightarrow 5x^2 - 8y^2 = -10$

$\Rightarrow \dfrac{x^2}{-2} + \dfrac{y^2}{\frac{5}{4}} = 1$

Which is required standard form.

Ex. 14(d) Find the nature of the Conic $x^2+2xy+y^2-6x-2y+4 = 0$. Reduce it to standard form. Find vertex, Focus, equation and length of latus rectum of the Conic. [N.U.H - 1999]

Ans. The given equation

$x^2+2xy+by^2-6x-2y+4 = 0$ .....(1)

Compare (1) with the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 1, h = 1, b = 1, g = -3, f = -1, c = 4$

$\therefore \Delta = abc + 2fgh - af^2 - by^2 - ch^2$

$= 1.1.4 + 2.(-1)(-3).1 - 1.(-1)^2 - 1(-3)^2 - 4.1^2$

$= 4 + 6 - 1 - 9 - 4 = -4$

$\therefore \Delta \neq 0$

$ab - h^2 = 1.1 - 1^2 = 0$

Thus, the given conic represents a porabola

(1) can be written as

$(x + y)^2 = 6x + 2y - 4$

$\Rightarrow (x + 4 + \lambda)^2 = 6x + 2y - 4 + 2\lambda x + 2\lambda y + \lambda^2$

$\Rightarrow (x + y + \lambda)^2 = (6 + 2\lambda)x + (2 + 2\lambda)y + \lambda^2 - 4$ ..... (2)

Find the value of λ, let the lines

$x + y + \lambda = 0$ and $(6 + 2\lambda)x + (2 + 2\lambda)y + \lambda^2 - 4 = 0$

are perpendicular $[a_1 a_2 + b_1 b_2 = 0]$

$\therefore 1.(6 + 2\lambda) + 1.(2 + 2\lambda) = 0$

$\Rightarrow 6 + 2\lambda + 2 + 2\lambda = 0$

$\Rightarrow 4\lambda = -8 \therefore \lambda = -2$

(2) $\Rightarrow (x + y - 2)^2 = (6-4)x + (2-4)y + (-2)^2 - 4$

$\Rightarrow (x + y - 2)^2 = 2x - 2y$

$\Rightarrow \left(\dfrac{x+y-2}{\sqrt{1^2+1^2}}\right)^2 2 = \dfrac{2(x-y)}{\sqrt{1^2+1^2}}.\sqrt{2}$

$\Rightarrow \left(\dfrac{x+y-2}{\sqrt{2}}\right)^2 = 4.\dfrac{\sqrt{2}}{4}.\left(\dfrac{x-y}{\sqrt{2}}\right)$

$\therefore Y^2 = 4AX$. Where $Y = \dfrac{x+y-2}{\sqrt{2}}, X = \dfrac{x-y}{\sqrt{2}}$

$A = \dfrac{\sqrt{2}}{4}$

Which is required standard form.

For Vertex $X = 0, \quad Y = 0$

$\Rightarrow \dfrac{x-y}{\sqrt{2}} = 0$ $\quad \Rightarrow \dfrac{x+y-2}{\sqrt{2}} = 0$

$\Rightarrow x - y = 0$ $\quad \Rightarrow x + y - 2 = 0$

$x = y$

Putting this in $x + y - 2 = 0$

$\therefore x + x - 2 = 0$

$\therefore x = 1 \qquad \therefore x = y$

$\therefore y = 1$

$\therefore$ Co-ordinate of vertex A (1, 1).

For Focus $X = A$, $Y = 0$

$\Rightarrow \frac{x-y}{\sqrt{2}} = \frac{\sqrt{2}}{4} \qquad \Rightarrow \frac{x+y-2}{\sqrt{2}} = 0$

$\Rightarrow x - y = \frac{2}{4} \qquad \Rightarrow x + y - 2 = 0$

$\Rightarrow x - y - \frac{1}{2} = 0$

Adding these we get

$2x - \frac{5}{2} = 0 \quad \therefore x = \frac{5}{4}$

$\therefore y = 2 - x = 2 - \frac{5}{4} = \frac{3}{4}$

$\therefore$ Co-ordinate of Focus $\left(\frac{5}{4}, \frac{3}{4}\right)$

Equation of latus rectum $X = A$

$\Rightarrow \frac{x-y}{\sqrt{2}} = \frac{\sqrt{2}}{4}$

$\Rightarrow x - y - \frac{1}{2} = 0$

$\Rightarrow 2x - 2y = 1 = 0$

Length of latus rectum $= 4A = 4.\frac{\sqrt{2}}{4} = \sqrt{2}$.

Ex. 14(e) Find the nature of the conic represented by $5x^2 + 2xy + 5y^2 + 10x - 22y + 29 = 0$ convert it into the standard form and find the equation of the axes and its length if possible.

[N.U.H-2000]

Ans. The given equation

$5x^2 + 2xy + 5y^2 + 10x - 22y + 29 = 0$ .....(1)

Comparing (1) with the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 5, h = 1, b = 5, g = 5, f = -11, c = 29$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 5.5.29 + 2.(-11).5.1 - 5.(-11)^2 - 5(5)^2 - 29.1^2$

$= 5.145 - 110 - 5.121 - 5.25 - 29$

$= 725 - 110 - 605 - 125 - 29$

$= 725 - 869$

$= -144 \neq 0$

$ab - h^2 = 5.5 - 1^2$

$= 25 - 1$

$= 24 > 0$

Thus, the given conic represents an ellipse.

let (α, β) be the centre of the conic (1)

$\alpha = \frac{hf - bg}{ab - h^2} \qquad \beta = \frac{hg - af}{ab - h^2}$

$= \frac{1.-11 - 5.5}{5.5 - 1^2} \qquad = \frac{1.5 - 5.-11}{5.5 - 1^2}$

$= \frac{-36}{24} = \frac{-3}{2} \qquad = \frac{60}{24} = \frac{5}{2}$

center is at $\left(\frac{-3}{2}, \frac{5}{2}\right)$

Transfer the origin to a point $\left(\frac{-3}{2}, \frac{5}{2}\right)$, then

$(1) \Rightarrow 5x^2 + 2xy + 5y^2 + c_1 = 0$ ... ... (2)

Where $c_1 = \alpha g + \beta f + C$

$= -\frac{3}{2}.5 + \frac{5}{2}.-11 + 29$

$= -\frac{15}{2} - \frac{55}{2} + 29$

$= \frac{-15 - 55 + 58}{2}$

$= \frac{-70 + 58}{2} = -6$

$\Rightarrow 5x^2 + 2xy + 5y^2 = 6$

$\Rightarrow \frac{5}{6}x^2 + 2\frac{1}{6}xy + \frac{5}{6}y^2 = 1$

$\Rightarrow Ax^2 + 2Hxy + By^2 = 1$ ... ... (3)

where $A = \frac{5}{6}, H = \frac{1}{6}, B = \frac{5}{6}$

Removing xy term, Rotating the axes then

$(2) \Rightarrow a_1x^2 + b_1y^2 = 6$ ... ... (4)

Where $a_1 + b_1 = a + b$

$a_1b_1 = ab - h^2$

$\therefore \quad a_1 + b_1 = 5 + 5 = 10$

$a_1b_1 = 5.5 - 1^2 = 24$

Now, $(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1b_1$

$= 10^2 - 4.24$

$= 100 - 96 = 4$

$\therefore a_1 - b_1 = 2\ (+\text{ve})$

$\therefore a_1 + b_1 = 10$

$(+)\ 2a = 12 \qquad \therefore a_1 = 6$

$b_1 = 10 - a_1 = 10 - 6 = 4$

$(4) \Rightarrow 6x^2 + 4y^2 = 6$

$= \dfrac{x^2}{1^2} + \dfrac{y^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = 1$

Which is required standard form of an ellipse.

If 2r be the length of axes, then

$\dfrac{1}{r^4} - (A + B)\dfrac{1}{r^2} + (AB - H^2) = 0$

$\Rightarrow \dfrac{1}{r^4} - \left(\dfrac{5}{6} + \dfrac{5}{6}\right)\dfrac{1}{r^2} + \left(\dfrac{5}{6} \cdot \dfrac{5}{6} - \dfrac{1}{6^2}\right) = 0$

$\Rightarrow \dfrac{1}{r^4} - \dfrac{5}{3r^2} - \dfrac{24}{36} = 0$

$\Rightarrow 1 - \dfrac{5}{3}r^2 - \dfrac{2}{3}r^4 = 0$

$\Rightarrow 2r^4 + 5r^2 - 3 = 0$

$\Rightarrow 2r^4 + 6r^2 - r^2 - 3 = 0$

$\Rightarrow 2r^2(r^2 + 3) - 1(r^2 + 3) = 0$

$\Rightarrow (2r^2 - 1)(r^2 + 3) = 0 \qquad \therefore r_1^2 = \dfrac{1}{2}, r_2^2 = -3$

$\therefore$ Length of major axis $= 2r_1 = 2.\dfrac{1}{\sqrt{2}} = \sqrt{2}$

Length of minor axis $= -2r_2 = 2\sqrt{3}$

Eq^n of major axis $(A - \dfrac{1}{r_1^2}(x - \alpha) + H(y - \beta) = 0$

$\Rightarrow \left(\dfrac{5}{6} - \dfrac{1}{\frac{1}{2}}\right)(x + \dfrac{3}{2}) + \dfrac{1}{6}\left(y - \dfrac{5}{2}\right) = 0$

$\Rightarrow \left(\dfrac{5}{6} - 2\right)\left(\dfrac{2x + 3}{2}\right) + \dfrac{1}{6}\left(\dfrac{2y - 5}{2}\right) = 0$

$\Rightarrow -\dfrac{7}{6}(2x + 3) + \dfrac{1}{6}(2y - 5) = 0$

$\Rightarrow -7(2x + 3) + 1(2y - 5) = 0$

$\Rightarrow -14x - 21 + 2y - 5 = 0$

$\Rightarrow 14x - 2y + 26 = 0$

$\Rightarrow 7x - y + 13 = 0$

Eq^n, of minor axis $\left(A - \dfrac{1}{r_2^2}\right)(x - \alpha) + H(y - \beta) = 0$

$\Rightarrow \left(\dfrac{5}{6} - \dfrac{1}{-3}\right)\left(x + \dfrac{3}{2}\right) + \dfrac{1}{6}\left(y - \dfrac{5}{2}\right) = 0$

$\Rightarrow \dfrac{7}{6}\left(\dfrac{2x + 3}{2}\right) + \dfrac{1}{6}\left(\dfrac{2y - 5}{2}\right) = 0$

$\Rightarrow 7(2x + 3) + 1(2y - 5) = 0$

$\Rightarrow 14x + 21 + 2y - 5 = 0$

$\Rightarrow 14x + 2y + 16 = 0$

$\Rightarrow 7x + y + 8 = 0$

Ex. 14(f) Reduce the equation $x^2 + 2xy + y^2 - 2x - 1 = 0$ to its standard form and then indentify it. Find the Co-ordinates of its focus, the foot of direction, the equation of axes and directrix, the equations of axes and directrix. [N.U.H. 2001]

Ans. The given equation

$x^2 + 2xy + y^2 - 2x - 1 = 0 \ldots \quad \ldots \quad (1)$

Comparing (1) with the help of $ax^2 + 2hxy + by^2 + 2gx + 2f + c = 0$

We get $a = 1, b = 1, h = 1, g = -1, f = 0, c = 1$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1.1.1 + 2.0.-1.1 - 1.0 - 1.1 - 1.1$

$= 1 - 1 - 1 = -1 \neq 0$

$ab - h^2 = 1.1 - 1^2 = 0$

Thus the conic (1) represents a parabola.

$(1) \Rightarrow (x + y)^2 = 2x + 1$

$\Rightarrow (x + y + \lambda)^2 = 2x + 2\lambda x + 2\lambda y + \lambda^2 + 1$

$\Rightarrow (x + y + \lambda)^2 = (2 + 2\lambda)x + 2\lambda y + \lambda^2 + 1 \quad \ldots \quad \ldots \quad (2)$

The lines $x + y + \lambda = 0$ and $(2 + 2\lambda)x + 2\lambda y + \lambda^2 + 1 = 0$

are perpendicular if $(a_1a_2 + b_1b_2 = 0)$

$\therefore 1.(2 + 2\lambda) + 1.2\lambda = 0$

$4\lambda + 2 = 0 \therefore \lambda = -\dfrac{1}{2}$

$(2) \Rightarrow \left(x + y - \dfrac{1}{2}\right)^2 = x - y + \dfrac{5}{4}$

$$\Rightarrow \left(\frac{x+y-\frac{1}{2}}{\sqrt{1^2+1^2}}\right)^2 . 2 = \frac{x-y+\frac{5}{4}}{\sqrt{1^2+1^2}} . \sqrt{2}$$

$$\Rightarrow \left(\frac{x+y-\frac{1}{2}}{\sqrt{2}}\right)^2 = 4.\frac{1}{4\sqrt{2}} . \left(\frac{x-y+\frac{5}{4}}{\sqrt{2}}\right)$$

$\Rightarrow Y^2 = 4AX$

Where $Y = \dfrac{x+y-\frac{1}{2}}{\sqrt{2}}$, $X = \dfrac{x-y+\frac{5}{4}}{\sqrt{2}}$, $A = \dfrac{1}{4\sqrt{2}}$

Which is standard form of the parabola.

It focus $X = A$, $Y = 0$

$$\Rightarrow \frac{x-y+\frac{5}{4}}{\sqrt{2}} = \frac{1}{4}\sqrt{2} \text{ and } \frac{x+y-\frac{1}{2}}{\sqrt{2}} = 0$$

$\Rightarrow x - y + 1 = 0$ $\qquad$ $x + y - \frac{1}{2} = 0$

$\therefore x + y - \frac{1}{2} = 0$

$(+)\ 2x + \frac{1}{2} = 0$

$\therefore x = -\frac{1}{4}$ $\quad$ $\therefore y = x + 1 = -\frac{1}{4} + 1 = \frac{3}{4}$

$\therefore$ Focus $\left(\frac{-1}{4}, \frac{3}{4}\right)$

At foot of directrix : $X = -A$, $\quad Y = 0$

$$\therefore \frac{x-y+\frac{5}{4}}{\sqrt{2}} = -\frac{1}{4\sqrt{2}} \qquad \frac{x+y-\frac{1}{2}}{\sqrt{2}} = 0$$

$\therefore x - y + \frac{3}{2} = 0$ $\qquad$ $y = x + \frac{3}{2} = \frac{-1}{2} + \frac{3}{2} = 1$

and $x + y - \frac{1}{2} = 0$

$(+)\ 2x + 1 = 0 \therefore x = \frac{-1}{2}$

$\therefore$ Foot of directrix $\left(-\frac{1}{2}, 1\right)$

Equation of axis $Y = 0$

$\therefore x + y\frac{-1}{2} = 0$ is $2x + 2y - 1 = 0$

Equation of directrix $X = -A$

$$= \frac{x-y+\frac{5}{4}}{\sqrt{2}} = -\frac{1}{4\sqrt{2}}$$

$$= x - y + \frac{3}{2} = 0$$

$$= 2x - 2y + 3 = 0$$

Ex. 14(g) Reduce the equation $x^2 + 2xy + y^2 + 2x - 1 = 0$ to the standard form and identify it.

[N.U.H. 2002]

Ans. The given equation

$x^2 + 2xy + y^2 + 2x - 1 = 0$ .... (1)

Comparing (1) with the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 1, b = 1, h = 1, g = 1, f = 0, c = -1$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1.1.-1 + 2.0 - 1.0 - 1.1^2 + 1.1^2$

$= -1 - 1 + 1 = -1 \neq 0$

$ab - h^2 = 1.1 - 1^2 = 0$

Thus the given conic represents a parabola.

$(1) \Rightarrow (x + y)^2 = -2x + 1$

$\Rightarrow (x + y + \lambda)^2 = -2x + 2\lambda x + 2\lambda y + \lambda^2 + 1$

The lines $x + y + \lambda = 0$ and $(-2 + 2\lambda)x + 2\lambda x + \lambda^2 + 1 = 0$

are perpendicular if $a_1a_2 + b_1b_2 = 0$

$\Rightarrow (x + y + \lambda)^2 = (-2 + 2\lambda)x + 2\lambda y + \lambda^2 + 1$ ... ... (2)

$\therefore 1.(-2+2\lambda) + 1.2\lambda = 0 \therefore \lambda = \frac{1}{2}$

Putting the values of λ in (2) we get

$$\left(x + y + \frac{1}{2}\right)^2 = -x + y + \frac{1}{4} + 1$$

$$\Rightarrow \left(x + y + \frac{1}{2}\right)^2 = -x + y + \frac{5}{4}$$

$$\Rightarrow \left(\frac{x+y+\frac{1}{2}}{\sqrt{1+1}}\right)^2 . 2 = \frac{-x+y+\frac{5}{4}}{\sqrt{1^2+1^2}} . \sqrt{2}$$

$$\Rightarrow \left(\frac{x+y+\frac{1}{2}}{\sqrt{2}}\right)^2 = 4.\frac{1}{4\sqrt{2}}\left(\frac{-x+y+\frac{5}{4}}{\sqrt{2}}\right)$$

$\therefore Y^2 = 4AX$ Where $Y = \dfrac{x+y+\frac{1}{2}}{\sqrt{2}}, X = \dfrac{-x+y+\frac{5}{4}}{\sqrt{2}}, A = \frac{1}{4}\sqrt{2}$

Which is the standard form of the parabala.

Ex. 14(h) Find the equation of the axes and its latus rectum of the conic $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$. [N.U.H. 2002]

Ans. The given Conic

$x^2 - 5xy + y^2 + 8x - 2oy + 15 = 0$ .... (1)

Compare (1) with the equation $ax^2 + 2hxy + hy^2 + 2gx + 2fy + C = 0$

We get

$a = 1, b = 1, h = \frac{-5}{2}, g = 4, f = -10, C = 15$

$\Delta\ abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1.1.15 + 2.-10.4.\frac{-5}{2} - 1.100 - 1.16 - 15.\frac{25}{4}$

$= 15 + 200 - 100 - 16 - \frac{375}{4}$

$= 215 - 116 - \frac{375}{4}$

$= 99 - \frac{375}{4}$

$= \frac{396-375}{4}$

$= \frac{21}{4} \neq 0$

$\therefore ab - h^2 = 1.1 - \frac{25}{4} = \frac{-21}{4} < 0$

Hence the given conic represents a hyperbola.

Let (α, β) be the centre of the conic (1)

$\therefore \alpha = \dfrac{hf - bg}{ab - h^2}$ $\quad \beta = \dfrac{hg - af}{ab - h^2}$

$= \dfrac{\frac{-5}{2}.-10 - 1.4}{\frac{-21}{4}} = -4$ $\quad = \dfrac{\frac{-5}{2}.4 - 1.-10}{\frac{-21}{4}} = 0$

$\therefore$ Centre is at $(-4, 0)$

Transfer the origin to a point $(-4, 0)$, direction of axes are unchanged, then

$(1) \Rightarrow x^2 - 5xy + y^2 + C_1 = 0$ .... (2)

Where $c_1 = g\alpha + f\beta + c$

$= 4.-4 + 0 + 15$

$= -1$

$(2) \Rightarrow x^2 - 5xy + y^2 = 1$ .... (3)

$\Rightarrow Ax^2 + 2Hxy + By^2 = 1$

Where $A = 1, B = 1, H = -\frac{5}{2}$

When rotation of axes, then

$a_1x^2 + b_1y^2 = 1$ ... . (4)

Where $a_1 + b_1 = a + b = 1 + 1 = 2$

$a_1b_1 - h_1^2 = ab - h^2$

$\Rightarrow a_1b_1 = 1.1 - \frac{25}{4} = -\frac{21}{4}$

$\therefore (a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1b_1$

$= 2^2 - 4.-\frac{21}{4}$

$= 4 + 21 = 25$

$\therefore a_1 - b_1 = 5$ (+ve sign)

and $a_1 + b_1 = 2$

$(+)\ 2a_1 = 7$

$b_1 = 2 - a_1 = 2 - \frac{7}{2}$

$b_1 = -\frac{-3}{2}$

$\therefore (4) \Rightarrow \frac{7}{2}x^2 - \frac{3}{2}y^2 = 1$

$= \dfrac{x^2}{\left(\sqrt{\frac{2}{7}}\right)^2} - \dfrac{y^2}{\left(\sqrt{\frac{2}{3}}\right)^2} = 1$

Which is standard form of a hyperbola.

If 2r be the length of axes then

$(AB - H^2)\, r^4 - (A + B) r^2 + 1 = 0$

$\Rightarrow (1.1 - \frac{25}{4})\, r^4 - (1+1)\, r^2 + 1 = 0$

$\Rightarrow \frac{-21}{4} r^4 - 2r^2 + 1 = 0$

$\Rightarrow 21r^4 - 8r^2 + 4 = 0$

$\Rightarrow 21r^4 + 14r^2 - 6r^2 - 4 = 0$

$\Rightarrow 7r^2 (3r^2 + 2) - 2 (3r^2 + 2) = 0$

$\Rightarrow (3r^2 + 2)(7r^2 - 2) = 0$

$\therefore 3r^2 + 2 = 0 \qquad 7r^2 - 2 = 0$

$\therefore r_1^2 = \frac{2}{7}, \quad r_2^2 = -\frac{2}{3}$

$r_1 = \sqrt{\frac{2}{7}}, \qquad r_2 = i\sqrt{\frac{2}{3}}$

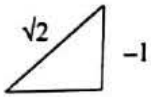
Eq^n. of transverse axis $\left(A - \frac{1}{r_1^2}\right)(x-\alpha) + H(y-\beta) = 0$

$\Rightarrow \left(1 - \frac{7}{2}\right)(x+4) - \frac{5}{2}(y-0) = 0$

$\Rightarrow -5(x+4) - 5y = 0$

$\Rightarrow x+y+4 = 0$

Slope $\tan\theta = -1$



$\sin\theta = \frac{-1}{\sqrt{2}} \qquad \cos\theta = \frac{-1}{\sqrt{2}}$

Eq^n. of conjugate axis

$(A - \frac{1}{r_1^2})(x-\alpha) + H(y-\beta) = 0$

$\Rightarrow (1 + \frac{3}{2})(x+4) - \frac{5}{2}(y-0) = 0$

$\Rightarrow -5(x+4) - 5y = 0$

$\Rightarrow x - y + 4 = 0$

Ecentricity $e = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 + \frac{2/3}{2/7}} = \sqrt{\frac{10}{3}}$

Foci S $(\alpha \pm r_1 e\cos\theta, \beta \pm r_1 e \sin\theta)$

$S\left(-4 \pm \sqrt{\frac{2}{7}} \cdot \sqrt{\frac{10}{3}} \cdot \frac{1}{\sqrt{2}} \cdot 0 \pm \sqrt{\frac{2}{7}} \cdot \sqrt{\frac{10}{3}} \cdot \frac{-1}{\sqrt{2}}\right)$

$S\left(-4 \pm \sqrt{\frac{10}{21}}, \pm\sqrt{\frac{10}{21}}\right)$

$\therefore S\left(-4 + \sqrt{\frac{10}{21}}, \sqrt{\frac{10}{21}}\right)$

and $S'\left(-4 - \sqrt{\frac{10}{21}}, -\sqrt{\frac{10}{21}}\right)$

Latus rectum parallel to conjugate axis passes through S and S'.

Any line parallel to conjugate axis is

$x - y + \lambda = 0 \qquad \ldots \qquad \ldots (5)$

It passes through S

$\therefore -4 + \sqrt{\frac{10}{21}} + \sqrt{\frac{10}{21}} + \lambda = 0 \qquad \therefore \lambda = 4$

$\therefore (5) \Rightarrow x - y + 4 = 0$

Again it passes through S^1

$\therefore -4 - \sqrt{\frac{10}{21}} + \sqrt{\frac{10}{2}} + \lambda = 0 \qquad \therefore \lambda = 4$

$\therefore x - y + 4 = 0$ Ans.

Ex. 14(i) Reduece the equation $8x^2 + 4xy + 5y^2 - 24x + 24y = 0$ to the standard form and identify the conic. Find the equation of axes and eccentricity of the conice.

Ans: Given conic

$8x^2 + 4xy + 5y^2 - 24x - 24y = 0 \ldots\ldots\ldots (1)$

Compare (1) with the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 8, b = 5, h = 2, g = -12, f = -12, c = 0$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 8.5.0 + 2. - 12. - 12.2 - 8.\ 144 - 5.144 - 0$

$= 576 - 1240 - 720$

$= 576 - 1960$

$= -1394 \neq 0$

$ab - h^2 = 8.5 - 2^2 = 36 > 0$

Thus the conic (1) represents an ellipse

Let (α, β)be the centre of the conic

$\therefore \alpha = \frac{hf - bg}{ab - h^2} \qquad \beta = \frac{hg - af}{ab - h^2}$

$\alpha = \frac{2. -12 - 5.-12}{36} \qquad \beta = \frac{2. -12 - 8.-12}{36}$

$= \frac{36}{36} \qquad = \frac{62}{36}$

$= 1 \qquad = \frac{31}{18}$

$\therefore$ centre $(1, \frac{31}{18})$

Transfer the origin to a point $(1, \frac{31}{18})$

Then (1) becomes

$8x^2 + 4xy + 5y^2 + C_1 = 0 \ldots\ldots (2)$

$c1 = g\alpha + f\beta + C$

$= -12.1 - 12.\frac{3}{18} + 0$

$= -12 - \frac{62}{3}$

$= \frac{-36-62}{3} \quad = -\frac{98}{3}$

$(2) \Rightarrow 8x^2 + 4xy + 8y^2 = \frac{98}{3}$ ..... (4)

Removing xy term when axes be being rotated

$a_1 x^2 + b_1 y^2 = \frac{98}{3}$ ..... (4)

where $a_1b_1 = a + b = 8 + 5 = 13$

$a_1b_1 = ab - h^2 = 36$

$(a_1 - b_1)^2 = (a_1 - b_1)^2 - 4\, a_1b_1$

$= (13)^2 - 4.36 = 169 - 144 = 25$

$\therefore a_1 - b_1 = 5$

$a_1 + b_1 = 13$ $\qquad b_1 = 13 - a_1 = 13 - 9 = 4$

$(+)2a_1 = 18$

$a_1 = 9$

$(4) \Rightarrow 9x^2 + 4y^2 = \frac{98}{3}$

$= \frac{x^2}{\frac{98}{27}} + \frac{y^2}{\frac{98}{12}} = 1$

Which is required standard form. conjugate hyphorbola

Ex. 14(j) Find the equation of the asymptotes of the hyperbola $6x^2 - 7xy - 3y^2 - 2x - 8y - 6 = 0$ and also find the equation of its conjugate hyperbola. [N.U. 2003]

Sol. Given hyperbala

$6x^2 - 7xy - 3y^2 - 2x - 8y - 6 = 0$ ... ... (1)

Equation of the asymptotes of the hyperbola (1) is

$6x^2 - 7xy - 3y^2 - 2x - 8y + k = 0$ ... ... (2)

Since asymptotes is a pair of st. lines

$\therefore 6(-3)k + 2(-4)(-1)(-7/2) - 6(-4)^2 + 3(-1)^2 - \left(\frac{-7}{2}\right)^2 = 0$

$\Rightarrow -18k - 28 - 96 + 3 - \frac{49k}{4} = 0$

$\Rightarrow -\left(18 + \frac{49}{4}\right)k = 121$

$\Rightarrow \frac{-121}{4}k = 121 \therefore k = 4$

$(2) \Rightarrow 6x^2 - 7xy - 3y^2 - 2x - 8y - 4 = 0$ ... ... (3)

Which is the required equation of the asymptotes.

Let equation of the conjugate hyperbola be $6x^2 - 7xy - 3y^2 - 2x - 8y + \lambda = 0$ ... ... (4)

we know

Equation of the conjugate hyperbola = 2 × (eq^n of the asymptotes) – eq f the hyperbola.

$\Rightarrow 6x^2 - 7xy - 3y^2 - 2x - 8y + \lambda = 2(x^2 - 7xy - 3y^2 - 2x - 8y - 4) - (x^2 - 7xy - 3y^2 - 2x - 8y - 6)$

$\lambda = -8 + 6 = -2$

$6x^2 - 7xy - 3y^2 - 2x - 8y - 2 = 0$

which is required conjugate hyperbola.

Ex. 14(k) Reduece the equation $5x^2 - 2xy + 5y^2 - 8x - 8y - 8 = 0$ to the standard from identify the conic and trace the conic. [N.U.H. 2004]

Ans: The given equation

$5x^2 - 2xy + 5y^2 - 8x - 8y - 8 = 0$ ..... (1)

Here $a = 5, b = 5, h = -1, g = -4, f = -4, c = -8$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 5.5 - 8. + 2.-4. -4. - 1.-5.(-4)^2 - 5(-4)^2 + 8(-1)^2$

$= -384$

$\therefore \Delta \neq 0$

$ab - h^2 = 5.5 - (-1)^2 = 24 > 0$

Hence the given conic represents an ellipse

Let (α, β) be the centre of the conic

$\therefore \alpha = \frac{hf - bg}{ab - h^2}$ $\qquad \beta = \frac{hg - af}{ab - h^2}$

$= \frac{(-1)(-4) - 5(-4)}{24}$ $\qquad = \frac{(-1)(-4) - 5(-4)}{24}$

$= \frac{24}{24}$ $\qquad = \frac{24}{24}$

$= 1$ $\qquad = 1$

$\therefore$ Centre (1,1)

Transfer the origin to a point(1,1)direction of axes unchanged

$(1) \Rightarrow 5x^2 - 2xy + 5y^2 + C_1 = 0$ .....(2)

Where $C_1 = g\alpha + f\beta + c = -4.1 + 4.1 - 8 = -16$

$(2) \Rightarrow 5x^2 - 2xy + 5y^2 - 16 = 0$

$\Rightarrow 5x^2 - 2xy + 5y^2 - 16$ ..... (3)

$\Rightarrow \frac{5}{16}x^2 - \frac{2}{16}xy + \frac{5}{16}y^2 = 1$

$\Rightarrow Ax^2 + 2Hxy + By^2 = 1$

where $A = \frac{5}{16}, H = \frac{-1}{16}, B = \frac{5}{16}$.

Removing xy term, when axes related a_1b_1

$\therefore a_1x^2 + b_1y^2 = 16$ ..... (4)

where $a_1b_1 = a + b = 5 + 5 = 10$

$a_1b_1 = ab - b^2 = 5.5 - 1^2 = 24$

$(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4\, a_1b_1$

$= (10)^2 - 4.24 = 4$

$\therefore a_1 - b_1 = 2$ (+ve)

$= \frac{a_1 + b_1 = 10}{(+)2a_1 = 12}$ $\quad \therefore -a_1 = 6$

$b_1 = 10 - a_1 - 10 - 6 = 4$

$(4) \Rightarrow 6x_2 + 4y_2 = 16$

$\Rightarrow \frac{6}{16}x^2 + \frac{4}{16}y^2 = 1$

$\Rightarrow \frac{x^2}{\frac{8}{3}} + \frac{y^2}{4} = 1$

$\Rightarrow \frac{x^2}{\left(\sqrt{\frac{8}{3}}\right)^2} + \frac{y^2}{2^2} = 1$

which is required standard form length of semi axes are given by

$(AB - H^2)r^4 - (A+B)r^2 + 1 = 0$

$\Rightarrow \left\{\frac{5}{16} \cdot \frac{5}{16} - \left(\frac{-1}{16}\right)^2\right\}r^4 - \left(\frac{5}{10} + \frac{5}{16}\right)^2 + 1 = 0$

$\Rightarrow \left(\frac{25}{16.16} - \frac{1}{16.16}\right)r^4 - \frac{10}{16}r^2 + 1 = 0$

$\Rightarrow 24r^4 - 10.16r^2 + 16.16 = 0$

$\Rightarrow 3.8r^4 - 10.2.8r^2 + 2.8.16 = 0$

$= 3r^4 - 20r^2 + 32 = 0$

$= 3r^4 - 12r^2 - 8r^2 + 32 = 0$

$= 3r^2 - (r^2 - 4) - 8(r^2 - 4) = 0$

$(3r^2 - 8)(r^2 - 4) = 0$

$\therefore r^2 = 4, \quad r^2 = \frac{8}{3}$

$\therefore r_1 = 2, \quad r_2 = \frac{2\sqrt{2}}{\sqrt{3}}$

$\therefore$ Length of major axis $2r = 2.2 = 4$

Length of minor axis $2r_2 = 2.2\sqrt{\frac{2}{3}} = 4\sqrt{\frac{2}{3}}$

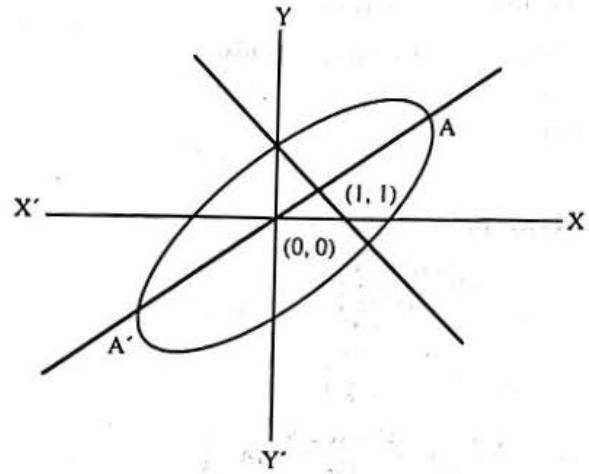
Eq^n, of major axis $(A - \frac{1}{r_1^2})(x - \alpha) + H(y - \beta) = 0$

$\Rightarrow (\frac{5}{16} - \frac{1}{4})(x-1) - \frac{1}{16}(y-1) = 0$

$\Rightarrow \frac{1}{16}(x-1) - \frac{1}{16}(y-1) = 0$

$\Rightarrow (x-1) - (y-1) = 0$

$\Rightarrow x - y = 0$



Eq^n, of minor axis $(A - \frac{1}{r_1^2})(x - \alpha) + H(y - \beta) = 0$

$\Rightarrow \left(\frac{5}{16} - \frac{3}{8}\right)(x-1) - \frac{1}{16}(y-1) = 0$

$\Rightarrow -\frac{1}{16}(x-1) - \frac{1}{16}(y-1) = 0$

$\Rightarrow x - 1 + y - 1 = 0$

$\Rightarrow x + y - 2 = 0$

Ex. 14(I) Identify the conic $9x^2 + 24xy + 16y^2 + 22x + 46y + 9 = 0$ and reduce it to echelon form. [N.U.H. 2005]

Sq^n: The given equation

$9x^2 + 24xy + 16y^2 + 22x + 46y + 9 = 0$ ..... (1)

Comparing (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get $a = 9, h = 12, b = 16 g = 11, f = 23\ c = 9$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9.16.9 + 2.23.11.12 - 9.(23)^2 - 16.(11)^2 - 9.(12)^2$

$= 1296 + 6072 - 4761 - 1936 - 1296$

$= -625 \neq 0$

$ab - h^2 = 9.16 - (12)^2$

$= 144 - 144 = 0$

Hence the given conic represents a porabola.

$(1) \Rightarrow (3x + 4y)^2 = -22x - 46y - 9$

$\Rightarrow (3x + 4y + k)^2 = -22x - 46y - 9 + 6kx + 8ky + k^2$

$\Rightarrow (3x + 4y + k)^2 = (6k - 22)x + (8k - 46)y + k^2 - 9$..... (2)

To find the values of k, The lines $3x + 4y+k=0$

and $(6k - 22)x+ (8k- 46)y+k^2- 9 =0$ are perpendicular

$3.(6k- 22)+ 4.(8k-46)=0$

$\Rightarrow 18k- 66+ 32x-184=0$

$\Rightarrow 50k =250 \therefore k=5$

$(2) \Rightarrow (3x+4y+5)^2= 8x- 6y+16$

$$\left(\frac{3x+4y+5}{\sqrt{3^2+4^2}}\right)^2.25 \qquad = 2\left(\frac{4x-3y+8}{\sqrt{4^2+3^2}}\right).5$$

$$\left(\frac{3x+4y+5}{\sqrt{3^2+4^2}}\right)^2 \qquad = \frac{2}{5}\left(\frac{4x-3y+8}{\sqrt{4^2+3^2}}\right)$$

$Y^2=4Ax$ where $Y=\frac{3x-4y+5}{\sqrt{25}}, X=\frac{4x-3y+8}{\sqrt{25}}, 4A=\frac{2}{5}$

Which is required standard form.

Ex. 14(m) Reduce the conic $8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$ to the standard form and find its all properties. Also draw the graph of the Conic. [N.U.H. 2007]

Solutions : The given conic

$8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0 \quad \ldots \quad \ldots \quad (1)$

Comparing (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get $a = 8, h = 2, b = 5, g = -8, f = -7, c = 13$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 8.5.13 + 2.(-7)(-8).2 - 8(-7)^2 - 5(-8)^2 - 13(2)^2$

$= -20 \neq 0$

$ab - h^2 = 8.5 - 2^2 = 36 > 0$

Hence the given Conic represents an ellipse.

Let (α, β) be the centre of the conic.

$$\therefore \alpha = \frac{hf - bg}{ab-h^2} \qquad \beta = \frac{hg - af}{ab - h^2}$$

$$= \frac{2.-7 - 5.-8}{8.5 - 2^2} \qquad = \frac{2.-8 - 8.-7}{8.5 - 2^2}$$

$$= \frac{26}{36} \qquad = \frac{40}{36} = \frac{10}{9}$$

$$= \frac{13}{18}$$

$$\therefore \text{Centre}\left(\frac{13}{18}, \frac{10}{9}\right)$$

Transfer the to a point $\left(\frac{13}{18}, \frac{10}{9}\right)$ When axes being unchanged

$(1) \Rightarrow 8x^2 + 4xy + 5y^2 + C = 0 \quad \ldots \quad \ldots \quad (2)$

Where $C_1 = g\alpha + f\beta + C$

$= -8\alpha - 6.\beta + C = 0$

$= 8.\frac{13}{18} - 6.\frac{10}{9} + \beta = 0$

$C_1 = -\frac{5}{9}$

$(2) \Rightarrow 8x^2 + 4xy + 5y^2 - \frac{5}{9} = 0$

$\Rightarrow 8x^2 + 4xy + 5y^2 = \frac{5}{9} \quad \ldots \quad \ldots \quad (3)$

$\Rightarrow \frac{72}{5}x^2 + \frac{36}{5}xy + 9y^2 = 1$

$\therefore Ax^2 + 2Hxy + By^2 = 1 \quad \ldots \quad \ldots \quad (4)$

Where $A = \frac{72}{5}, H = \frac{18}{5}, \beta = 9$

Now, we remove xy term, then (3) becom

$a_1x^2 + b_1y^2 = \frac{5}{9} \quad \ldots \quad \ldots \quad (5)$

where $a_1 + b_1 = a + b = 8 + 5 = 13$

$a_1b_1 = ab - h^2 = 8.5 - 2^2 = 36$

$\therefore (a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1b_1$

$= (13)^2 - 4.36 = 25$

$\therefore \quad a_1 - b_1 = 5 \text{ (+ve)}$

and $a_1 + b_1 = 13$

$(+)\ 2a_1 = 18 \qquad \therefore b_1 = 13 - a_1 = 13 - 9 = 4$

$\therefore a_1 = 9$

$\therefore (5) \Rightarrow 9x^2 + 4y^2 = \frac{5}{9}$

$$\Rightarrow \frac{x^2}{\frac{5}{81}} + \frac{y^2}{\frac{5}{36}} = 1$$

$$\Rightarrow \frac{x^2}{\left(\frac{\sqrt{5}}{9}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{5}}{6}\right)^2} = 1$$

Which is required standard form.

Length of seme axes is given by

$(AB - H^2)r^4 - (A + B)\, r^2 + 1 = 0$

$$\Rightarrow \left(\frac{72}{5} \cdot 9 - \frac{18.18}{5.5}\right) r^4 - \left(\frac{72}{5+9}\right) r^2 + 1 = 0$$

$$\Rightarrow \frac{2916}{25} r^4 - \frac{117}{5} r^2 + 1 = 0$$

$$\Rightarrow 2916r^4 - 585r^2 + 25 = 0$$

$$r^2 = \frac{585 \pm \sqrt{(585)^2 - 4(2916)(25)}}{2.2916} = \frac{585 \pm 225}{5832}$$

$$r^2 = \frac{585 + 225}{5832} \text{ and } \frac{585 - 225}{5832}$$

$$r^2 = \frac{810}{5832} \text{ and } \frac{360}{5832}$$

$$r^2 = \frac{5}{36} \text{ and } \frac{5}{81}$$

$$r = \frac{\sqrt{5}}{6} \text{ and } \frac{\sqrt{5}}{9}$$

$$\therefore r_1 = \frac{\sqrt{5}}{6} \text{ and } r_2 = \frac{\sqrt{5}}{9}$$

Length of Major axis $2r_1 = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$

Length of minor axis $2r_2 = \frac{2\sqrt{5}}{9}$

Eq^n. of major axis $(A - \frac{1}{r_1^2})(x - \alpha) + H(y - \beta) = 0$

$$\Rightarrow \left(\frac{72}{5} - \frac{36}{5}\right)\left(x - \frac{13}{18}\right) + \frac{18}{5}\left(y - \frac{10}{9}\right) = 0$$

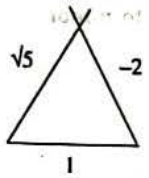
$$\Rightarrow 2x + y - \frac{23}{9} = 0$$

Equations of minor axis $\left(A - \frac{1}{r_2^2}\right)(x - \alpha) + H(y - \beta) = 0$

$$\Rightarrow \left(\frac{72}{5} - \frac{81}{5}\right)\left(x - \frac{13}{18}\right) + \frac{18}{3}\left(y - \frac{10}{9}\right) = 0$$

$$\Rightarrow x - 2y + \frac{3}{2} = 0$$

Slope of Major axis $\tan\theta = \frac{-2}{1}$



$$\therefore \sin\theta = \frac{-2}{\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

Ecentricity $e = \sqrt{1 - \frac{r_2^2}{r_1^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

veritices A $(\alpha \pm r_1\cos\theta, \beta \pm r_1\sin\theta)$

$$A\left(\frac{\beta}{18} \pm \frac{\sqrt{5}}{6} \cdot \frac{1}{\sqrt{5}}, \frac{10}{9} \pm \frac{\sqrt{5}}{6} \cdot \frac{-2}{\sqrt{5}}\right)$$

$$A\left(\frac{13}{18} \pm \frac{1}{6}, \frac{10}{9} \pm \frac{1}{3}\right)$$

$$\therefore A\left(\frac{8}{9}, \frac{13}{9}\right), A'\left(\frac{5}{9}, \frac{7}{9}\right).$$

Focus S $(\alpha \pm r_1\, e\cos\theta)$ $(\beta \pm r_1\, e\sin\theta)$

$$S\left(\frac{13}{18} \pm \frac{\sqrt{5}}{6} \cdot \frac{\sqrt{5}}{3} \cdot \frac{1}{\sqrt{5}}, \frac{10}{9} \pm \frac{\sqrt{5}}{6}, \frac{\sqrt{5}}{3}, \frac{-2}{\sqrt{5}}\right)$$

$$S\left(\frac{13}{18} \pm \frac{\sqrt{5}}{18}, \frac{10}{9} \pm \frac{\sqrt{5}}{9}\right)$$

$$S\left(\frac{13 + \sqrt{5}}{18}, \frac{10 + \sqrt{5}}{9}\right), S'\left(\frac{13 - \sqrt{5}}{18}, \frac{10 - \sqrt{5}}{9}\right)$$

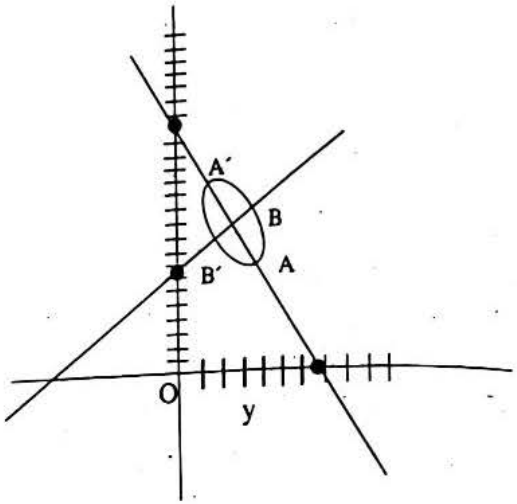
Foot of directrices $Z\left((\alpha \pm \frac{r_1}{e}\cos\theta, \beta \pm \frac{r_1}{e}\sin\theta)\right)$

$$Z\left(\frac{13}{18} \pm \frac{\sqrt{5}}{6} \times \frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{10}{9} \pm \frac{\sqrt{5}}{6} \cdot \frac{3}{\sqrt{5}} \frac{-2}{\sqrt{5}}\right)$$

$$Z\left(\frac{13}{18} \pm \frac{1}{2\sqrt{5}}, \frac{10}{9} \pm \frac{1}{\sqrt{5}}\right)$$

$$\therefore Z\left(\frac{13}{18} + \frac{1}{2\sqrt{5}}, \frac{10}{9} + \frac{1}{\sqrt{5}}\right), Z'\left(\frac{13}{18} - \frac{1}{2\sqrt{5}}, \frac{10}{9} - \frac{1}{\sqrt{5}}\right)$$

Latus rectum parallel to minor axis passes through S, S′



$\therefore$ Eq^n, of the line parallel to minor axis $x - xy + k = 0$

It passes Through S, S′

$x - 2y + k = 0$

$$\frac{13+\sqrt{5}}{18} - 2.\frac{10+\sqrt{5}}{9} + K = 0$$

$$\therefore k = -\frac{13+\sqrt{5}-40-4\sqrt{5}}{18}$$

$$= -\frac{-27-3\sqrt{5}}{18} = \frac{+(9+\sqrt{5})}{6}$$

Again, $\frac{13-\sqrt{5}}{18} - 2.\frac{10-\sqrt{5}}{9} + k = 0$

$$k = \frac{10-2\sqrt{5}}{9} - \frac{13-\sqrt{5}}{18}$$

$$= \frac{40-4\sqrt{5}-13+\sqrt{5}}{18} = \frac{27-3\sqrt{5}}{18} = \frac{9-\sqrt{5}}{6}$$

$\therefore$ Eq^2 of Latus rectum are

$$x - 2y + \frac{9+\sqrt{5}}{6} = 0$$

and $x - 2y + \frac{9-\sqrt{5}}{6} = 0$

Again, directrices parallel to minor axis pass through Z, Z′

$$\therefore \frac{13}{18} + \frac{1}{2\sqrt{5}} - 2.\left(\frac{10}{9} + \frac{1}{\sqrt{5}}\right) + k = 0$$

$$k = \frac{3}{2}\left(1 + \frac{1}{\sqrt{5}}\right)$$

Again, $\frac{13}{18} - \frac{1}{2\sqrt{5}} - 2\left(\frac{10}{9} - \frac{1}{\sqrt{5}}\right) + k = 0$

$$\therefore k = \frac{1}{6}(9+\sqrt{5})$$

$\therefore$ Eq^n of directries

$$x - xy + \frac{1}{6}(9+\sqrt{5}) = 0 \text{ and } x - 2y + \frac{1}{6}(9-\sqrt{5}) = 0$$

(iv) Latus rectum is $4p = 8$

(v) Focus $X = p$, $Y = 0$, by (8)

$$\text{or, } \frac{3x+4y-1}{\sqrt{3^2+4^2}} = 2 \text{ and } \frac{4x-3y+2}{\sqrt{4^2+3^2}} = 0$$

or, $3x + 4y - 11 = 0$ and $4x - 3y + 2 = 0$. solve

$\therefore$ $x = 1$, $y = 2$, The focus is (1, 2)

(vi) Foot of the directrix, $X = -p$ and $Y = 0$ by (8)

$$\text{or, } \frac{3x+4y-1}{\sqrt{3^2+4^2}} = 2 \text{ and } \frac{4x-3y+2}{\sqrt{4^2+3^2}} = 0$$

or, $3x + 4y + 9 = 0$ and $4x - 3y + 2 = 0$.Now solve. The foot of the directix is $(-7/5, -6/5)$

(vii) Equation of the directrix, $X = -p$, by (8)

$$\text{or, } \frac{3x+4y-1}{\sqrt{3^2+4^2}} = -2 \text{ or, } 3x + 4y + 9 = 0$$

(viii) Equation of the latus rectum $X = p$ by (8)

$$\text{or, } \frac{3x+4y-1}{\sqrt{3^2+4^2}} = 2 \text{ or, } 3x + 4y - 11 = 0$$

Note : "The properties of a parabola can be determined by other method also. The demonstration of such a method is given below.

Ex 15. Find the focus, foot of the directrix and equations of latus rectum, axis and diretrix of the conic $x^2 + 2xy + y^2 - 2x - 1 = 0$ [N.U.H. 2001] ... ... (1)

As the 2nd degree terms of eq. (1) form a perect Square i. e. $\Delta = 0$ and $h^2 = ab$, the eq. (1) is a parabola. The eq. (1) may be written as

$(x+y)^2 - 2x - 1 = 0$ ... ... (2)

The axis of the parabola is parallel to the line

$x + y = 0$ and is $x + y + k = 0$ ... ... (3)

The equation of the directrix is $x - y + \lambda = 0$ ... ... (4)

as the directrix is perpendicular to the axis (3)

Let $S(x_1, y_1)$ be the focus. If $P(x, y)$ be any point of eq. (5) then by the definition of parabola.

$SP^2 = PM^2$, (PM is perpendicular to the directrix)

$$\text{or, } (x-x_1)^2 + (y-y_1)^2 = \left(\frac{x-y+\lambda}{\sqrt{1^2+1^2}}\right)^2$$

$$\text{or, } (x+y)^2 - x(2\lambda + 4x_1) + y(2\lambda - 4y_1) + (-\lambda^2 + 2x_1^2 + 2y_1^2) = 0 \quad \ldots \quad (5)$$

Identify the equations (5) and (1) then

$2\lambda + 4x_1 = 2$ ... ... (5)

$2\lambda - 4y_1 = 0$ ... ... (7)

$-\lambda^2 + 2x_1^2 + 2y_1^2 = -1$ ... ... (8)

Form (6), $x_1 = \frac{1}{2}(1-\lambda)$ and from (7), $y_1 = \frac{1}{2}\lambda$, Now from (7)

$-\lambda^2 + 2\left(\frac{1}{2}\right)^2(1-\lambda)^2 + 2(\lambda/2)^2 = -1$ or, $\lambda = 3/2$

$\therefore x_1 = -1/4,\ y_1 = 3/4$

(i) Hence the focus is $(-\frac{1}{4}, \frac{3}{4})$

(ii) The equation of the directrix is eq. (4) i.e.

$x - y + 3/2 = 0$ or, $2x - 2y + 3 = 0$ ... ... (9)

EXERCISE VI

Test the nature of the conics given by the following equations (in rectangular axe
(নিম্নলিখিত সমীকরণগুলির প্রমাণরূপ নির্ণয় কর।)

1. $4x^2 + 9y^2 - 8x + 36y - 31 = 0$ Ans. ellipse [D.U. 197
2. $2x^2 - 3y^2 + 8x + 30y - 27 = 0$ Ans. hyperbol
3. $x^2 - 2y^2 - 6xy - 4y + 6 = 0$ Ans. Hyperbo
4. $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ Ans. Two straight lin
5. $(y-a)^2 + (x-a)^2 = 0$ Ans. a Circl
6. $xy + a^2 = a(x+y)$ Ans. Pair of straight lin
7. $x^2 - xy - 2y^2 - x - 4y - 2 = 0$ Ans. Pair of straight lin
8. $x^2 + 2xy + y^2 + 2x - 1 = 0$ Ans. Parabola [R. U. (PS) 198
9. $x^2 + 2xy + y^2 - 6x + 10y + 25 = 0$ Ans. Parabol
10. $34x^2 + 24xy + 41y^2 + 48x + 14y - 108 = 0$ Ans. Ellips
11. Reduce the equation $8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$ to standard form [N.U.H. 2007] Ans. Ellips
12. Reduce the equation $5x^2 - 24xy - 5y^2 + 4x + 58y - 59 = 0$ to the standard form. Ans. $x^2 - y^2 = 2$ [R.U. 198
13. Find the position and magnitudes of the conic whose equations is $x^2 - 5xy + y^2 + 8x$ $20y + 15 = 0$ (C.U.1990, C.H. 1980) Ans. $x + y + 4 = 0,\ x - y + 4 = 0, 3, 2\sqrt{\frac{2}{7}},\ 2\sqrt{\frac{2}{3}}, 135°, 4$
14. Find the lengths of the axes and centre of the conic $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ Ans. (2,3); $2\sqrt{6}$.

15. Find the entre of the following coincs
(i) $x^2 + y^2 + xy + x + y - 1 = 0$ Ans. $(-1/3, -1/3)$
(ii) $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ Ans. $(-1, 1)$
(iii) $3x^2 - 5xy + 6y^2 + 11x - 17y + 13 = 0$ Ans. $(-1, 1)$

16. What conics do the equations represent?
(1) $2x^2 + 3xy - 2y^2 - 5x + 5y = 0$ [E.U. 1961] Ans. Rect hyperbola.
(ii) $7x^2 - 6xy - y^2 + 4x - 4y = 0$ Ans. Pair of staight lines

17. Find the lengths, equations and directions of axes of the following conics.
(i) $17x^2 + 12xy + 8y^2 - 46x - 28y + 33 = 0$ Ans. $4/\sqrt{5}, 2/\sqrt{5}$
$2x + y - 3 = 0,\ x - 2y + 1 = 0,\ \tan^{-1}(-2),\ \tan^{-1}\frac{1}{2}$ [C.U.(S) 1992]
(ii) $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ Ans. $4/\sqrt{3},\ x - y + 2 = 0,\ x + y = 0,\ 45°,\ 135°$

18. Find the eccentricity of the conics (কণিকের বিকেন্দ্রিকতা)
(a) $34x^2 + 24xy + 41y^2 + 48x + 14y - 108 = 0$; $(-\frac{16}{26}, \frac{1}{25})$ Ans. $1/\sqrt{2}$
(b) $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ Ans. 2

19. Find the focus (নাভি) of the conics
(a) $9x^2 + 24xy + 16y^2 - 2x + 14y + 1 = 0$ Ans. $[-1/25, -11/50]$
(b) $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ Ans. $[-1 \pm 2\sqrt{2}/3,\ 1 \pm 2\sqrt{2}/3]$
(c) $4x^2 + y^2 - 4xy - 10y - 19 = 0$ Ans. $[-6/5, -7/5]$
(d) 20 $8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$ Ans. [2,3,] [1,1]

20. Find the directrix (নিয়ামক) of the conics
(a) $8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$ Ans. $x + 2y - 10 = 0,\ x + 2y - 1 = 0$
(b) $9x^2 - 6xy + y^2 - 14x - 2y - 12 = 0$ Ans. $2x + 6y - 7 = 0$ [C.U. 1979, D.U. 1979]

21. Calculate the latus rectum (নাভিলম্ব) of the conic
$25x^2 + 120xy + 144y^2 - 82x - 163y + 32 = 0$ Ans. 1/13

22. Find the co-ordinates of vertex, focus and equation of the latus rectum of the conic $x^2 + 2xy + y^2 - 6x - 2y + 4 = 0$ $\lambda = -2$. [N.U.H. 99] Ans. (1,1); (5/4, 3/4); $2x - 2y - 1 = 0$

23. (a) Find the equations of latus rectum and directrix of the conic $55x^2 - 30xy + 39y^2 - 40x - 24y - 464 = 0$ Ans. $3x + 5y - 21 = 0,\ 3x + 5y + 13 = 0$
Ans. (1/2, 1/2), $c_1 = -480$ $5x + 5y = 36,\ 3x + 5y + 28 = 0$

24. Find the vertex, directrix and focus of the conics
(a) $9x^2 - 6xy + y^2 - 14x - 2y + 12 = 0$ Ans. (1,1); $2x + 6y - 7 = 0$; (21/20, 23/20)
(b) $x^2 - 4xy + 4y^2 + 10x - 8y + 13 = 0$ Ans. $(-39/25, 13/25)$, $10x + 5y - 2 = 0$, $(-9/5, 2/5)$

25. Find the focus and directrix of the conic
$17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ Ans. $(-1 \pm \sqrt{(3/5)}, 1 \pm \sqrt{(3/5)}$; $x + 2y = 1 \pm 120/\sqrt{15}$.

26. Find the tangents at the vertex (শীর্ষ সম্পর্ক) of the conic
$11x^2 + 16xy - y^2 - 70x - 40y + 82 = 0$ Ans. $x - y = -1 \pm 2\sqrt{5}/6$.

27. $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$ কণিককে প্রমাণ রূপে প্রকাশ করা। এর অক্ষদ্বয়ের ও নাভিলম্বের সমীকরণ নির্ণয় কর। Ans. $x + y + 4 = 0, x - y + 4 = 0, x - y = -4 \pm 2\sqrt{(10/21)}$

28. $3x - 4y + 7 = 0,\ 4x + 3y + 1 = 0$ দুইটি অধিবৃত্তের অসীমতটরেখা মূলবিন্দুগামী হলে এর সমীকরণ নির্ণয় কর। অক্ষদ্বয়ের দৈর্ঘ্য এবং সমীকরণ নির্ণয় কর। Ans. $\frac{2}{5\sqrt{14}}, \frac{2}{5\sqrt{14}}, x + 7y - 6 = 0\ 7x - y + 8 = 0$

29. Find the latus rectum (নাভিলম্ব) and the co-ordinates of vertex (শীর্ষ) of the parabola. $(12x + 5y)^2 - 52x - 78y + 9 = 0$ Ans. 4/13, (36/169, 15/169)

30. Find the vertex, focus, and latus rectum of the parabola $x^2 - 4x + 3y = 1$ Ans. (2,5/3), (2, 11/12), 3 [R.U. 1987, D.U. 1956]

31. Find the vertex, focus, axis, directrix and latus rectum of the parabola $4y^2 - 20x - 8y + 39 = 0$ [D. U. 1952] Ans. (9/4,1), (3,1), $y = 1$, $2x - 1 = 0$,5.

32. Find the co-ordinates of the vertex and the length of the latus rectum of the parabola $x^2 - 4x + 2y = 0$ [D.U.1958] Ans. (2,2),2

33. For the parabola $4(y - 1)^2 = -7(x - 3)$ find (1) the latus rectum (2) the co-ordinates of the focus and directrix. Ans. 7/4, (41/16,1), (3,1)

34. Find the latus rectum of the hyperbola $6x^2 - 7xy - 3y^2 = 1$ Ans. $\frac{260}{3 + \sqrt{130}}$

35. Find the latus rectum of parabola $9x^2 + 24xy + 16y^2 + 22x + 46y + 9 = 0$. Find also it focus (নাভি।) and directrix (নিয়ামক) [N.U.H. 05] Ans. 2/5, (−9/5, 1/10), $8x - 6y + 17 = 0$

36. Reduce the equation $3x^2 - 2xy - y^2 + 2x + y - 1 = 0$ to the standard form and sow that $x - 3y + 2 = 0$ is the dianeter of the conic. Find its conjugate axis and centre of the conic.

($3x^2 - 2xy - y^2 + 2x + y - 1 = 0$ কে প্রমাণরূপে প্রকাশ কর এবং দেখাও যে, $x - 3y + 2 = 0$ কণিকের ব্যাস। এর অনুবন্ধী ব্যাস ও কেন্দ্র নির্ণয় কর।) Ans. $\left(\frac{-1}{8}, \frac{5}{8}\right)$ সমঅধিবৃত্ত, $16x - 8y + 7 = 0$

[N.U.H. 98] [C.H. 1984; D.U. 1967]

37. If $y = 3x$ is a diameter of the conic $4x^2 + 6xy - 5y^2 = 1$, then show that $12y - 13x = 0$ is the conjugate diameter. (দেখাও যে, $4x^2 + 6xy - 5y^2 = 1$ এর যে ব্যাস $y = 3x$ এর সাথে অনুবন্ধী হয় তাহার সমীকরণ $12y - 13x = 0$)

38. Show that conjugate diameter of the diameter $3x^2 - 8xy + 7y^2 - 4x + 2y - 7 = 0$ is $9x - 17y - 1 = 0$. (দেখাও যে $3x^2 - 8xy + 7y^2 - 4x + 2y - 7 = 0$ কণিকের যে ব্যাস $3x - y - 5 = 0$ এর সাথে অনুবন্ধী হয় তার সমীকরণ $9x - 17y - 1 = 0$) Ans. উপবৃত্ত

39. Reduce the equation $8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ to the standard form. Find also its axes, ecentricity and the latus rectum. [N.U.H 2003]

($8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ কণিকের প্রমাণরূপ দাও। ইহার অক্ষদ্বয়, বিকেন্দ্রীকতা এবং নাভিলম্ব নির্ণয় কর।) Ans. উপবৃত্ত 6, 4; $\pm\sqrt{5}$, 8/3 [C.U. 1980]

40. Reduce the eq. $3x^2 + 8xy - 3y^2 + 10x - 13y + 8 = 0$ into standard form. What is the shape of the conic ? Find its centre, axes and latera recta. ($3x^2 + 8xy - 3y^2 + 10x - 13y + 8 = 0$-কে ইহার প্রমাণ রূপে প্রকাশ কর। কণিকাটি কি? ইহার কেন্দ্র, অক্ষদ্বয় এবং নাভিলম্বদ্বয় নির্ণয় কর।)

[D. H. 1987; C.H. 1982] Ans. সমপরাবৃত্ত, (−11/25,−75/50), 1537/250, − 1537/250, $2\sqrt{1537}/\sqrt{250}$

41. Reduce the equation $x^2 - 6xy + 9y^2 - 2x - 3y + 1 = 0$ to standard form. What is the shape ? Find also its axes, focic, directrics. ($x^2 - 6xy + 9y^2 - 2x - 3y + 1 = 0$ কে প্রমাণরূপে প্রকাশ কর এবং কণিকাটি কি? ইহার অক্ষরেখা, নাভিবিন্দু এবং নিয়ামক নির্ণয় কর।)

[C. H. 1978] $y^2 = \frac{9x}{10\sqrt{10}}$, $20x - 60y + 7 = 0$, (13/40, 9/40), $12x + 4y - 3/4 = 0$

42. Reduce the eq. $2x^2 + 3xy - 2y^2 - 10x + 5y = 0$ to standard formand draw the graph of the figure. ($2x^2 + 3xy - 2y^2 - 10x + 5y = 0$ কণিকাটির প্রমাণরূপে প্রকাশ কর এবং চিত্রটি অঙ্কন কর।)

Ans. Two perpendicular lines. [D.U. 1979, '80]

43. Find the shape of the conics (নিম্নলিখিত কণিকগুলির আকার কি হবে) ?

(i) $3x^2 + 4xy + 5y^2 - 2x + 2y + 5 = 0$ Ans. ellipse

(ii) $4x^2 + 2xy - y^2 + 2x - 2y + 5 = 0$ Ans. hyperbola

(iii) $3x^2 + 4xy - 3y^2 - 2x + 2y - 7 = 0$ Ans. Rec. hyperbola

(iv) $3x^2 + 6xy + 3y^2 + 3x + y - 4 = 0$ Ans. Parabola

(v) $x^2 + 4xy + 6y^2 + 2x + 4y + 1 = 0$ Ans. Intersecting st. lines

(vi) $5x^2 + 5y^2 + 26xy - 82x - 98y + 149 = 0$ Ans. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

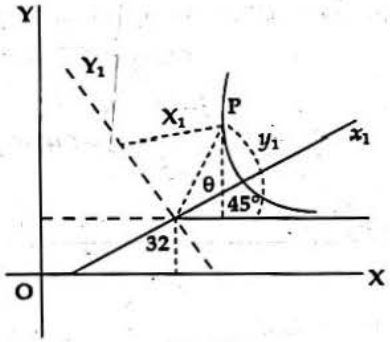


Fig : 26

(vii) $12x^2 + 24xy + 19y^2 - 12x - 40y + 31 = 0$ Ans. locus is the single pl. (−2/3,2)

(viii) $4x^2 - 12xy + 9y^2 - 52x + 26y + 81 = 0$ Ans. $y^2 = \frac{8}{\sqrt{13}}x$

(ix) $3x^2 + 10xy + 3y^2 - 2x - 14y - 5 = 0$ Ans. pair of st. lines

(x) $x^2 + 2xy + y^2 - 6x - 2y + 4 = 0$ Ans. Parabola C.H. 1989

(xi) $13x^2 + 10xy + 13y^2 - 4x - 68y + 28 = 0$ Ans. Ellipse

(xii) $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$, Find eqs. of axes. Ans. $2x - 2y + 7 = 0$ [C. H. 1986]
Ans. $4x^2/5 - y^2/2 = 1$, (0, 1/2), $3x + 2y - 1 = 0$, $4x - 6y + 3 = 0$

(xiii) $x^2 - 6xy + y^2 + 8x - 20y + 15 = 0$, অক্ষরেখার সমীকরণ নির্ণয় কর।
Ans. $y = x + 3$, $y + x + 3 = 0$ [C.H. 1987]

(xiv) $5x^2 - 24xy - 5y^2 + 4x + 58y - 59 = 0$
Ans. $2x + 3y - 7 = 0$, $3x + 2y - 8 = 0$ Rect. hyperbola.

(xv) $x^2 - 2xy + y^2 + 6x + 2y - 19 = 0$ [D.U. 1989] Ans. Parabola

THE CIRCLE 7

Art. 63. Definition :

A circle is the locus of a point in a plane which moves so that its distance from a given point is always equal to a given distance. The given point is called the centre and the given distance is the radius of the circle.

Art. 63. (a) The equation of a circle with centre at (h, k) and radius 'a' is

$(x-h)^2 + (y-k)^2 = a^2$ ... ... (1)

Cor. *If the centre of the circle is the origin, the equation becomes* $x^2 + y^2 = a^2$ (2)

Art. 64. Equation of the circle in Polar Co-ordinates.

Let the centre C of the circle be at (r_1, θ_1) and a be the radius of the circle. Let P be any point (r, θ) on the circle.Then the angle $\angle COP = \theta - \theta_1$ and from the Δ OCP

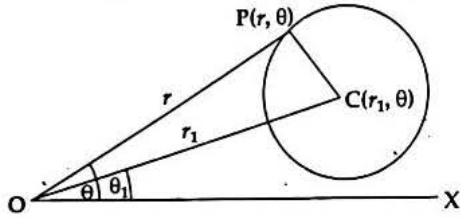


Fig : 27

$CP^2 = OC^2 + OP^2 - 2OC.\ OP \cos COP$

or, $a^2 = r_1^2 + r^2 - 2rr_1 \cos(\theta - \theta_1)$. The equation of the circle in terms of polar co-ordinates is

$r^2 + r_1^2 - 2r_1 r \cos(\theta - \theta_1) = a^2$ ... ... (1)

where (r_1, θ_1) is the contric of the circle.

Cor. 1. If the pole lies on the circle, then $r_1 = a$. The equation of the circle now is

$r = 2a\cos(\theta_1 - \theta)$ ... ... (2)

Cor. 2. If θ_1 = zero, we see that initial line is a diameter and the pole is an extremity of the diameter. The equation of the circle of radius a is $r = 2a\cos\theta$ ... (3)

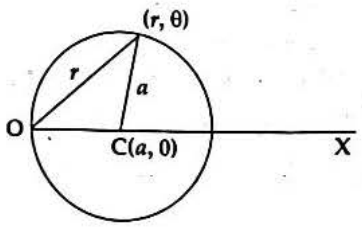


Fig : 28

Cor. 3. If the centre is the origin (i.e. the pole), the equation of the circle becomes

$r = a$ ... ... (4)

Art. 65. (a) General equation to a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (1)

or, $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

The centre of the circle is at $(-g, -f)$ ... ... (2)

The radius of the circle is positive

$a^2 = g^2 + f^2 - c$ or, $a = \sqrt{(g^2 + f^2 - c)}$ ... ... (3)

Note : **1.** If $g^2 + f^2 - c = 0$, the rdius of the circle is zero and the equation (1) represents a real circle.

Note. **2.** If $g^2 + f^2 - c = 0$, the radius of the circle is zero and the equation (1) represents a circle of zero radius. This is known as point circle.

Note. **3.** If $g^2 + f^2 - c < 0$. i.e. $g^2 + f^2 - c$ is negative. The radius $\sqrt{(g^2 + f^2 - c)}$ becomes imaginary. The equation ... (1) represents an imaginary circle.

(b) General equation of a circle in Polar Co -ordinates.

The equation, (1) Art. 64 is

$r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1) = a^2$

or, $r^2 - 2rr_1 \cos\theta_1 \cos\theta - 2r_1 r \sin\theta_1 \sin\theta = a^2 - r_1^2$

or, $r^2 + 2r(g\cos\theta + f\sin\theta) + c = 0$ ... ... (1)

which is the general equation of a circle

where $g = -r_1 \cos\theta$

$f = r_1 \sin\theta$

$c = r_1^2 - a^2$ ... ... (2)

The co-ordinates of centre = $(\sqrt{(g^2 + f^2)}, \tan^{-1}(f/g))$ ... ... (3)

The radius of the circle = $\sqrt{(g^2 + f^2 - c)}$ ... ... (4)

Art. 66. The equation of the circle whose diameter is the line joining the points (x_1, y_1) and (x_2, y_2) is ...

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Art. 67. Position of a points with respect to a circle.

Any point (x_1, y_1) will be outside, on or inside the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (1)

according as its distance from the centre $(-g, -f)$ is greater, equal or less than the radius $\sqrt{(g^2 + f^2 - c)}$ of the circle (1) i.e. according as $(x_1 + g)^2 + (y_1 + f)^2 >= < g^2 + f^2 - c$

or, $x_1^2 + y_1^2 + 2gx_1 + 2f_1 + c >= < 0$.

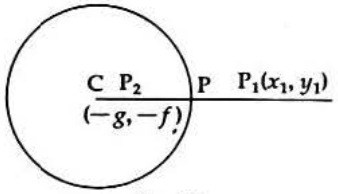


Fig : 29

Rule. *Substitue the given point in the equation of the circle and the point will be outside, on or inside the circle according as the results is + ve, zero or — ve*

Art. 68. The equation of the tangent to the circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (See Art. 52 *a*)

(Alternative Method).

Let P (x_1, y_1) and Q $(x_2 y_2)$ be the two points on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (1)

The equation of the line PQ is

$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$ or, $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$ ... ... (2)

Since the points (x_1, y_1) and (x_2, y_2) are on the circle (1) they will satisfy Eq. (1).

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ ... ... (3)

$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$ ... ... (4)

subtracting (4) from (3), we have

$x_1^2 - x_2^2 + y_1^2 - y_2^2 + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0$

or, $\frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$

or, $\frac{y - y_1}{x - x_1} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f}$ by (2) ... ... (5)

When Q coincides with P, the line PQ becomes the tangent at P of the circle (1). Then in limit $x_1 = x_2$ and $y_1 = y_2$ in eq. (5) we have,

or, $\frac{y - y_1}{x - x_1} = -\frac{2x_1 + 2g}{2y_1 + 2f} = -\frac{x_1 + g}{y_1 + f}$

or, $xx_1 + yy_1 + xg + yf = x_1^2 + y_1^2 + x_1 g + y_1 f$ ... ... (6)

Adding $x_1 g + y_1 f$ to both sides of Eq. (6) we have

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

or, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = -c$ ... ... by (3)

or, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Hence the equation of the tangent to the circle (1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Cor. The equation of the tangent at (x, y) to the circle $x^2 + y^2 = a^2$ is

$xx_1 + yy_1 = a^2$

Art. 69. The equation of the normal (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$y(x_1 + g) - x(y_1 + f) + fx_1 - gy_1 = 0$ (See Art. 53 (a))

Cor. The equation of the nomal

at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - x_1 y = 0$

Note : *In the case of a circle the equation of normal at any point on it can be easily found since it is a straight line joining the point and the centre of circle.*

Art. 70. Length and mid point of the chord intercepted by the circle $x^2 + y^2 = a^2$ on the line $y = mx + c$.

The line joining centre O to the middle point C of a chord of a circle is perpendicular to the chord whose equation is

$y = mx + c$ ... ... (1)

Hence the equation of OC is $y - 0 = -1/m\,x$ or, $my + x = 0$ ... ... (2)

Solve (1) and (2), the co-ordinates of the middle point C of the chord AB is $-mc/(1 + m^2), c/(1 + m^2)$ ... ... (3)

Again OC = perpendicular distance of C (0,0) from AB

$OC = \frac{c}{\sqrt{(1 + m^2)}}$ and $CA^2 = a^2 - c^2/(1 + m^2)$

Length of chord AB = 2AC.

$= 2\sqrt{\left\{\frac{a^2(1 + m^2) - c^2}{(1 + m^2)}\right\}}$

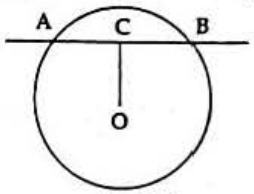


Fig : 30

Art. 70. (a) Length of the tangent T from a external point (x_1, y_1) to the circle. $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$T^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Cor. The length of the tangent T from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is given by $T^2 = x_1^2 + y_1^2 - a^2$

Note : The equation of the circle is to be arranged in such a way that right hand side is zero and the co-efficients of x^2 and y^2 must be unity.

Condition of Tangency

Art. 71. Show that the straight lines $y = mx \pm a\sqrt{(1 + m^2)}$ are always tangents to the circle $x^2 + y^2 = a^2$

Let $y = mx + c$ ... (1) meet the circle $x^2 + y^2 = a^2$... ... ... (2)

$x^2 + (mx + c)^2 = a^2$ [put value of y from (1) in (2)]

or, $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ ... ... (3)

If the line (1) always touches the circle (2) then x has equal roots in (2)

$\therefore\ 4m^2c^2 = 4(1 + m^2)(c^2 - a^2)$ or $c^2 = a^2(1 + m^2)$

or, $c = \pm a\sqrt{(1 + m^2)}$ ... ... (4)

which is, therefore, the condition of tangency of the lines (1) to the circle (2)

Put the values of c in (1), then the straight lines

$y = mx \pm a\sqrt{(1 + m^2)}$

are always tangents to the circle $x^2 + y^2 = a^2$ whatever be the value of m.

The general equation of the tangent to the circle $x^2 + y^2 = a^2$ is represented by the form

$y = mx \pm a\sqrt{(1 + m^2)}$ ... ... (5)

where m is a variable parameter.

Cor. $y = mx \pm a\sqrt{(1 + m^2)}$ are the equation of the tangents to the circle which are parallel to $y = mx$.

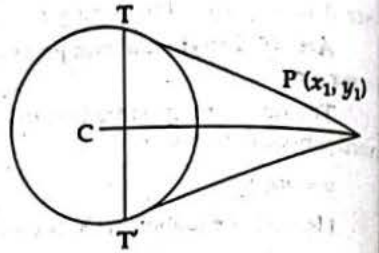


Fig : 31

Art. 72. Chord of contact : If two tangents be drawn to a circle from an external point the straight line joning the points of contact is called the Chord of contact of tangent drawn from it. From the fig (31) T and T' are the points of contact of the tangents PT and PT' and TT' is the chord of contact.

Art. 73. The equation of the chord of contact of the tangents drawn to a circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ from a point (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... ... (1)

Note : 1. The Eq. (1) Art. 73. indicates that it is of the same form as that of equation of the tangent (Art. 68) to the circle at (x_1, y_1). The reason is obvious. If P is on the circle. T and T' coincide with P and in this limiting position of chord of contact becomes the tangent to the circle at the point P (x_1, y_1)

Hence the identical form.

Note : 2. If the point P (x_1, y_1) be within the circle, the two tangents will be imaginary; yet the equation of the chord of contact obtained by above method is the same as above and is real. We thus see, since this line is alway real that you may have a real straight line joining the imaginary points of contact of two imaginary tangents. There is nothing surprising in this for real curves can pass throjugh imaginary points.

Ex. Find the points of intersection of the circle $x^2 + y^2 = 1$ with the straight line $x = 2$.

The figure shows that the straight line does not meet the circle at all ... (1)

Put the value of x in the circle $x^2 + y^2 = 1$

$y^2 = -3$

or, $y = \pm\sqrt{(-3)}$

$= \pm\sqrt{3}i$

The straight line $x = 2$, cuts the circle $x^2 + y^2 = 1$ in the two imaginary points, $(2, i\sqrt{3})$ and $(2, -i\sqrt{3})$ ... ... (2)

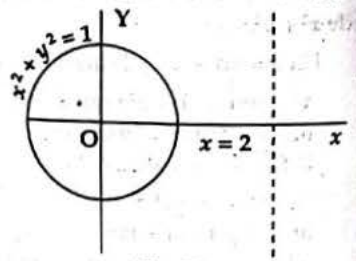


Fig : 32

Of the two statements (1) and (2) the mathematician prefers statement (2). One great advantage in this that he can give his proposition in a form which has no exceptions. Thus he can say. "Every straight line meets a circle in two points."

Note : 3. The equation (7) can be written in the form

$y(y_1 + f) = -(x_1 + g)x - (gx_1 + fy_1 + c) = y = \frac{(x_1 + g)}{y_1 + f}x - \frac{gx_1 + fy_1 + c}{y_1 + f}$

'm' of the chord is $\frac{-(x_1 + g)}{y_1 + f}$ ... ... (i)

Again the equation to the line passing through P (x_1, y_1) and C $(-g, -f)$ is $\frac{(y - y_1)}{y_1 + f}$

$= \frac{x - x_1}{x_1 + g}$ or, $y = \frac{y_1 + f}{x_1 + g}(x - x_1) + y_1$

'm' of the chord is $\frac{-(y_1 + f)}{x_1 + g}$ ... ... (ii)

From (i) and (ii), it is seen that PC is at right angles to TT'

Art. 74. Find the polar of the point (x_1, y_1) w.r., to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

If the points of contact $T(x_2, y_2)$; $T'(x_3, y_3)$; the tangents at these points are

$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0$ ... ... (1)

and $xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0$ (see fig 31) ... ... (2)

Both of these tangents (1) and (2) will pass through P (x_1, y_1); we have, $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$ ... ... (3)

$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0$ ... ... (4)

relations (3) and (4) show that the two points

$T(x_2, y_2)$; and $T'(x_3, y_3)$; are on the line

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ∴ ... ... (5)

which is, therefore, the required equation of the polar of the point (x_1, y_1); (see Art. 57)

Note : 1. It will be noticed that the equation of the polar is the same as that of the chord of contact of the tangents real and imaginary from that point to the conic.

Note : 2. The pole is different from the pole of polar co-ordinates.

Cor. If $g = 0$, $f = 0$ in (5); then $xx_1 + yy_1 = a^2$ which is the polar of the point $P(x_1, y_1)$ w.r. to the circle $x^2 + y^2 = a^2$

Art. 75. Pole of line.

Find the pole of a line with respect to any circle $x^2 + y^2 = a^2$

Let the equation to the line and circle respectively be

$Ax + By + C = 0$ ..... (1) and $x^2 + y^2 = a^2$ ... ... (2)

If (x_1, y_1) be the pole then the equations to the polar is $xx_1 + yy_1 = a^2$

Compare equations (1) and (3), then $\frac{x_1}{A} = \frac{y_1}{B} = -\frac{a^2}{C}$ or $x_1 = -\frac{A}{C}a^2$, $y_1 = \frac{-B}{C}a^2$

The required pole $\left(-\frac{A}{C}a^2, \frac{-B}{C}a^2\right)$

We can proceed similarly if the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$\left\{\frac{A(g^2 + f^2 - c)}{Ag + Bf - C} - g : \frac{B(g^2 + f^2 - c)}{Ag + Bf - C} - f\right\} \quad \ldots \quad \ldots \quad (3)$$

which are the co-ordinates of the required pole.

Art. 76. conjugate points.

If the polar of a point P passes through T, then the polar of T passes through A ; and two such points which lie on the polar of the other are called conjugate points.

Let the co-ordinates of P and T be (x_1, y_1) and (x_2, y_2) respectively.

The polar of (x_1, y_1) w.r. to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

This line passes through T (x_2, y_2), if $x_2x_1 + y_2y_1 = a^2$

This shows that (x_2, y_2) lies on $xx_2 + yy_2 = a^2$ which is the polar of T (x_2, y_2). Hence the proposition

Art. 77. Conjugate Lines.

Two lines the pole of either of which with respect to a circle lies on the other are called Conjugate lines with respect to the circle.

If the pole of the line $Ax + By + C = 0$, $C \neq 0$

with respect to a circle lies on $A_1x + B_1y + C_1 = 0$, $C_1 \neq 0$;

then the pole of $A_1x + B_1y + C_1 = 0$;

will lie on the first line $Ax + By + C = 0$.

Let the circle be $x^2 + y^2 = a^2$ ... ... (1)

The pole of the line $Ax + By + C = 0$, w. r. to the circle (1) is

$(-Aa^2/C, -Ba^2/C)$ ... ... (2)

It lies on the line $A_1x + B_1y + C_1 = 0$;

then $-AA_1a^2 - BB_1a^2 + CC_1 = 0$, Or, $(AA_1 + BB_1)a^2 = CC_1$

Now the pole of the line $A_1x + B_1y + C_1 = 0$;

w.r. to the circle i. e. $(-A_1a^2/C_1, -a^2. B_1/C_1)$

will lie on the line $Ax + By + C = 0$

if $-AA_1a^2 - BB_1a^2 + CC_1 = 0$, which is true as proved in (3)

Art. 78. The equation to the pair of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is

$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$

or, $SS_1 = T^2$ (see Art 61)

Art. 79. Number of tangents from a point.

To show that from any point there can be drawn two tangents real, coincident or imaginary to a circle.

Let the given point be (x_1, y_1). The equation to any tangent to the circle $x^2 + y^2 = a^2$ be

$y = mx + a\sqrt{(1 + m^2)}$ ... ... (1)

If this line (1) passes through (x_1, y_1), we have

$y_1 = mx_1 + a\sqrt{(1 + m^2)}$ or, $(y_1 - mx_1)^2 = a^2(1 + m^2)$ ... ... (2)

or, $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$ ... ... (3)

The equation (3) is quadratic in m gives two values of m corresponding to any given values of x_1 and y_1.

Hence two tangents from (x_1, y_1) can be drawn. They may be real, coincident or imaginary according as

$4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2)$ is + ve. zero or − ve,

i. e. $a^2(x_1^2 + y_1^2 - a^2) \gtreqless 0$ i. e. $x_1^2 + y_1^2 \gtreqless a^4$

according as the point (x_1, y_1) is outside on or inside the circle. If the point is inside the circle, tangents are imaginary.

Art. 80. Parametric Representation.

If we put $x = a\cos\theta$, $y = a\sin\theta$ in the equation of the circle $x^2 + y^2 = a^2$ the values of x and y will stisfy the equation of the circle, whatever be the value 'θ'. Hence the co-ordinates of any point on the above circle can be written as $x = a\cos\theta$, $y = a\sin\theta$ which are called parameter.

$(a\cos\theta, a\sin\theta)$ is symbolically written as point 'θ'

Art. 81. Common tangents to two circles.

Let O_1 and O_2 be the centres of two circles whose radii are r_1 and r_2 Then P_1P_2 and Q_1Q_2 are called the direct common tangents and p_1p_2 and q_1q_2 are called the transverse common tangents.

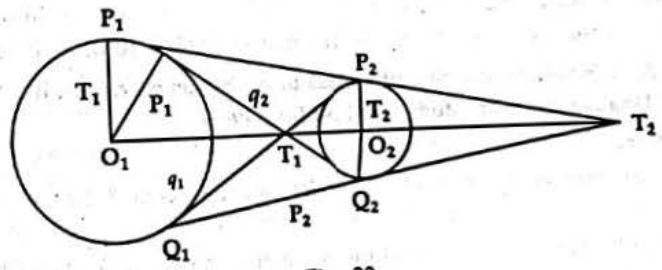


Fig : 33

Let them intersect in T_1 and T_2 respectively. Then O_1, T_1, O_2 and T_2 lie on a straight line.

Since $\Delta T_1O_1P_1$ and $\Delta T_2O_2P_2$ are similar.

Then $\dfrac{T_1O_1}{T_2O_2} = \dfrac{r_1}{r_2} = \dfrac{O_1P_1}{O_2P_1}$ ... ... (1)

Similarly from the $\Delta T_1 O_1 P_1$ and $\Delta T_2 O_2 P_2$

$$\frac{T_1 O_2}{T_2 O_2} = \frac{O_1 P_1}{O_2 P_2} = \frac{r_1}{r_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Therefore, the points T_1 and T_2 divide $O_1 O_2$ internally and externally in the radii of the circles.

These points are known as the centre of similitude and the circle on $T_2 T_1$ as diameter is called the circle of similitude of the two circles.

If the co-ordinates of T_2 be (x_1, y_1) then any line through it is

$$y - y_1 = m(x - x_1) \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Now the condition of tangency is applied. The length of perpendicular from O_1 on this line (3) is equal to radius of the circle r_1. This will give the value of m, hence the equations of tangents are obtained. So for the other pair which passes through T_1.

Art. 82. The power of a point.

If any line drawn through a fixed point P meets the circle at Q and R. the product PQ. PR is constant and is called the **Power of P. w. r. to the circle.**

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let P (x_1, y_1) be any point. The equation PR is of the form (See Art. 14).

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now putting the value of x and y from (2) in (1) we have $(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$

The roots r_1, r_2 of the equation (3) are the measure PQ and PR Hence

$r_1 . r_2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

or, PQ. PR $= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c =$ (length of tangent)^2

The power of P is (+ ve) or (− ve) according as PQ and P R are drawn in the same or in opposite senses i. e. according as PQ and PR are drawn in the same or in opposite senses i. e. according as P is outside, or inside the circle. The power of P is zero when P lies on the circle.

Art. 83. A. R. Khalifas Solution of a class of problems on circles in Analytic Geometry (Published by Dhaka University studies Vol. XI June 1963).

Proposition.

To find the general equation of all circles passing through two given points (x_1, y_1) and (x_2, y_2)

In the consideration of the fact that the equation of a circle must be of the second degree, in which the co-efficients of x^2 and y^2 must be equal and the term containing xy must be absent (for rectangular co-ordinates) and in order that the circle may pass through the two points, the co-ordinates of the points must satisfy it, the equation of a circle passing through the given points (x_1, y_1) and (x_2, y_2) may be written as

$$S \equiv (x - x_1)(x - y_2) + (y - y_1)(y - y_2) = 0 \quad \ldots \quad \ldots \quad (1)$$

It may be seen in passing that this equation is obtained in Analytical conics in another context, namely as the equation of the circle drawn upon the straight line joining the two given points of its diameter. We do not require that properly. This may only be noted how easily the equation can be written down at once from above considerations.

The equation of a st. line through the two points (x_1, y_1) and (x_2, y_2) is written very easily from similar considerations in the same way as

$$L = (x - x_1)(y - y_2) - (x - x_2)(y - y_1) = 0 \ldots \quad \ldots \quad \ldots \quad (2)$$

Therefore, the general equation of all circles passing through the two points (x_1, y_1) and (x_2, y_2) is S = AL.

Or, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) =$

$$A\{(x - x_1)(y - y_2) - (x - x_2) + (y - y_1)\} = 0 \quad \ldots \quad \ldots \quad (3)$$

where A is a constant

Now we work out some problems with the help of this result.

EXAMPLES

Ex. 1. Find the equation of the circle passing through the three points (−3, 2), (1, 7) and (5, −3)

The general equation of all circles passing through the first two points (− 3, 2), (1, 7) is

$$(x + 3)(x - 1) + (y - 2)(y - 7) = A\{(x + 3)(y - 7) - (x - 1)(y - 2)\} \quad \ldots \quad (1)$$

Since the circle passes through the third point (5, − 3), we have 8.4 + (−5) (− 10) $= A\{(8(-10) - 4(-5)\}$ or, $A = -30/41$

Thus from (4), we have

$30(x^2 + y^2 + 2x - 9x + 11) = -41(-5x + 4y - 23)$

or, $30(x^2 + y^2) - 145x - 106y - 613 = 0$

which is the rquired equation.

Ex. 2. Find the pole of the straight line $2x - y = 6$ w. r. to the circle $5x^2 + 5y^2 = 9$

Let P (x_1, y_1) be the co-ordinates of the pole.

The equation to the polar of P (x_1, y_1) with respect to the circle

$5x^2 + 5y^2 = 9$ is $5xx_1 + 5yy_1 = 9 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

This polar is identical with the line $2x - y = 6 \quad \ldots \quad \ldots \quad (2)$

Now comparing the co-efficients of x, y from (1) and (2) we have. $5x_1/2 = -5y_1/1 = 9/6$, $x_1 = 3/5, y_1 = -3/10 \quad \therefore \quad (3/5, -3/10)$ Ans.

Ex. 3. ABCD is a square whose side is a, taking AB and AD as axes, prove that the equation to the circle circumscribing the square is $x^2 + y^2 = a(x + y)$ [D. U. 1975]

If AB and AD are axes, clearly the centre of the circle is at $(a/2, a/2)$ and radius is $\frac{1}{2}\sqrt{(a^2 + a^2)} = a/\sqrt{2}$

The required equation of the circle is $(x - a/2)^2 + (y - a/2)^2 = (a/\sqrt{2})^2$

or, $x^2 + y^2 - ax - ay = 0$ or $x^2 + y^2 = a(x + y)$ proved.

Ex. 4. Find the locus of the middle points of a system of parallel chords of a circle.

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ... (i)

The equation to the chord in terms of its middle points (x_1, y_1) is $T = S_1$

or, $(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$ ... ... (ii)

It is evident that if the chords be parallel to the given line $lx + my = 0$ and (ii) be one of such chord, then comparing the equation $lx + my = 0$ with the equation (ii) we have

$(x_1 + g) : l = (y_1 + f) : m$

Thus the locus of (x_1, y_1) is the line $x + g : l = y + f : m$ which is satisfied by the equations $x + g = 0$ and $y + f = 0$ i. e ; the co-ordinates of the centre $(-g, -f)$ of circle.

Hence the locus of the middle points of a system of parallel chord is a straight line passing through the centre, which is the diameter of the system.

Ex. 5. Show that the locus of the poles of tangents to the circle $x^2 + y^2 = a^2$ w. r. to the circle $x^2 + y^2 = 2bx$ is the conic.

$(a^2 - b^2)x + a^2y^2 - 2a^2bx + a^2b^2 = 0$

The polar of any point (x_1, y_1) w. r. to the circle

$x^2 + y^2 - 2bx = 0$ is

$xx_1 + yy_1 - b(x + x_1) = 0$ or, $x(x_1 - b) + yy_1 - bx_1 = 0$ ... ... (i)

(i) is tangent to the circle $x^2 + y^2 = a^2$

$$\frac{-bx_1}{\sqrt{[(x_1 - b)^2 + y_1^2]}} = \pm a$$

or. $b^2x_1^2 = a^2(x_1 - b)^2 + y_1^2a^2$ or $b^2x_1^2 = a^2x^2_1 - 2a^2x_1^2b + a^2b^2 + y_1^2a^2$

or, $x_1^2(a^2 - b^2) - 2a^2bx_1 + a^2b^2 + y_1^2a^2 = 0$

Hence the locus of (x_1, y_1) is $x^2(a^2 - b^2) + a^2y^2 - 2a^2bx + a^2b^2 = 0$

Ex. 6. Show that the equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

is $(gx + fy)^2 + c(x^2 + y^2)$

The equation to the pair tangents is $S\,S_1 = T^2$

or, $(x^2 + y^2 + 2gx + 2fy + c)c = x\ [x.0 + y.0 + g(x + 0) + f(y + 0) + c]^2$

or, $c(x^2 + y^2 + 2gx + 2fy + c) = (gx + fy + c)^2$

or, $c(x^2 + y^2) + c(2gx + 2fy + c) = (gx + fy)^2 + 2(gx + fy)c + c^2$

or, $c(x^2 + y^2) = (gx + fy)^2$

Ex. 7. prove that, if the polar of a point P w. r. to the circle $x^2 + y^2 = 37$ touches the circle. $(x - 3)^2 + (y + 2)^2 = 25$, the locus of P must be a conic, whose equation you are required to obtain.

Let the co-ordinates of P be (x_1, y_1)

Polar of (x_1, y_1) w. r. to $x^2 + y^2 = 37$ is $xx_1 + yy_1 = 37$

If (1) touches $(x - 3)^2 + (y + 2)^2 = 25$

$$\frac{3x_1 - 2y_1 - 37}{\sqrt{(x_1^2 + y_1^2)}} = 5$$

Length of the perpendicular from the centre $(3, -2)$ to the line(1) is equal to the radius of the circle. 5

or, $(3x_1 - 2y_1 - 37)^2 = 25(x_1^2 + y_1^2)$

or, $16x_1^2 + 21y_1^2 + 12x_1y_1 + 222x_1 - 148y_1 - 1369 = 0$

The locus of (x_1, y_1) is $16x^2 + 21y^2 + 12xy + 222x - 148y - 1369 = 0$

Ex. 8. Find the equation of the circle inscribed in the triangle determined by the lines.

$2x - 3y + 21 = 0$ ... ... (i)

$3x - 2y + 20 = 0$ ... ... (ii)

$2x + 3y + 9 = 0$ ... ... (iii)

The centre of the circles lies at the point of intersection of the bisectors of the interior angles of the triangle. Let (h, k) be the co-ordinates of the centre of the incircle. Bisector of angle (i) and (ii) is

$$\frac{2h - 3k + 21}{\sqrt{(4 + 9)}} = +\frac{3h - 2k - 6}{\sqrt{(9 + 4)}} \text{ or, } h + k + 3 = 0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Biector of angle (i) and (iii) is

$$\frac{2h - 3k + 21}{\sqrt{(4 + 9)}} = +\frac{2h + 3k + 9}{\sqrt{(4 + 9)}} \text{ or, } 6k - 12 = 0 \text{ or, } k = 2 \quad \ldots \quad \ldots \quad \text{(v)}$$

From (iv), $h = -1$. Co. ordinates of centre $(-1, 2)$

Radius of the circle, $= \dfrac{2(-1) + 3(2) + 9}{\sqrt{13}} = \sqrt{13}$

Hence equation to circle is $(x + 1)^2 + (y - 2)^2 = (\sqrt{13})^2$, or, $x^2 + y^2 + 2x - 4y - 8 = 0$

Ex. 9. Find the equation of the direct common tangents to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 + 6x - 8y = 0$

The centres of the two circles are (0 0) and $(-3, 4)$ and their radii are 4 and 5 respectively. So the point dividing externally the line of centres in the ratio 4 : 5 is given by

$$x_1 = \frac{5.0 - 4.(-3)}{5 - 4} = 12,\ y_1 = \frac{5.0 - 4.4}{5 - 4} = -16$$

The direct tangents pass through $(12, -16)$

Hence their equation are given by $y + 16 = m(x - 12)$ ... ... (1)

In order that this line may touch the circle if the perpendicular from the centre (0, 0) of the circle on the line (i) is equal to the radius 4.

$$\frac{12m + 16}{\sqrt{(1 + m^2)}} = \pm 4 \text{ or, } 8m^2 + 24m + 15 = 0 \qquad m = \frac{-24 \pm \sqrt{(576 - 480)}}{16} = \frac{-6 \pm \sqrt{6}}{4}$$

The equation (i) becomes $y + 16 = \pm\dfrac{-6 \pm \sqrt{6}}{4}(x - 12)$ or, $6x + 4y - 8 = \pm 16(x - 12)$

which are the required equations.

Ex. 10. Find the equation of the transverse common tangents of the circle $x^2+y^2-x-2y+1=0$ and $2x^2+2y^2-3x+4y=0$

The centres of the circles are

$x^2+y^2-x-2y+1=0$ ... ... (i)

$x^2+y^2-\frac{3}{2}x+2y=0$ ... ... (ii)

are $(\frac{1}{2}, 1)$ and $(\frac{3}{4}, -1)$ and their radii are $\frac{1}{2}$ and $\frac{5}{4}$ (See fig 34) respectively.

The point dividing O_1 and O_2 internally in the ratio of $\frac{1}{2}:\frac{5}{4}$, i.e.,

co-ordinates of T_1 are $x_1=\dfrac{5/4\,\frac{1}{2}+\frac{1}{2}.3/4}{\frac{5}{4}+\frac{1}{2}}=\frac{4}{7}$, $y_1=\dfrac{5/4.1+\frac{1}{2}(-1)}{\frac{5}{4}+\frac{1}{2}}=\frac{3}{7}$

The transverce tangents are the lines through T_1 and distance from the centre O_1 (1/2, 1) so they are given by

$y-3/7=m(x-4/7)$ ... ... (iii)

$\dfrac{1-3/7-m\ (1/2-4/7)}{\sqrt{(1+m^2)}}=\pm\frac{1}{2}$

or, $\left(\frac{4}{7}+m/14\right)^2=\frac{1}{4}(1+m^2)$ or, $(8+m)^2=\frac{196}{4}(1+m^2)$

or, $64+16m+m^2=49+49m^2$

or, $48m^2-16m-15=0$ or, $m=\frac{3}{4}, -\frac{5}{12}$

Thus the transverse tangents are $y-3/7=\frac{3}{4}\left(x-\frac{4}{7}\right)$ and $y-\frac{3}{7}=-\left(\frac{5}{12}\right)\left(x-\frac{4}{7}\right)$

or, $4y=3x$ and $12y+5x=8$

Ex. 11. The circle $x^2+y^2-2x+4y-20=0$ meets the straight line $2x-y=2$ at A and B. Find the equation of the straight line joining A and B with the origin and obtain the angle between them.

The equation of the straight lines OA and OB is obtained by making the equation of the circle $x^2+y^2-2x+4y-20=0$

homogeneous with the line $2x-y=2$ or, $\frac{1}{2}(2x-y)=1$

The joint equation is $x^2+y^2-2x\frac{1}{2}(2x-y)+4y\frac{1}{2}(2x-y)-20\,[\frac{1}{2}(2x-y)]^2=0$

or, $x^2+y^2-2x^2+xy+4xy-2y^2-5(4x^2+y^2-4xy)=0$

or, $21x^2+6y^2-25xy=0$

And the angle between the straight lines is $\tan 2\varphi=\dfrac{2\sqrt{(h^2-ab)}}{a+b}$

or. $\tan 2\varphi=\dfrac{2\sqrt{(625/4-21.6)}}{21+6}$ $\therefore$ $2\varphi=\tan^{-1}\frac{11}{27}$

EXERCISE VII

1. Find the equation to the circle whose radius is 8 and the centre is (−4, 2).
Ans. $x^2+y^2+8x+4y+4=0$

2. Find the equation to the circle which is tangent to both axes, its centre being in the first quadrant and radius is 8.
Ans. $x^2+y^2-16x-16y+64=0$

2 (a) Find the equation of a circle which touches both the axes and passes through the pt. (−2, −1). [D. U. 1984] Ans. $x^2+y^2+10x+10y+25=0$, $x^2+y^2+2x+2y+1=0$

3. Find the equation of the circle which touches the axes of co-ordinates and passes through (3, 4). Ans. $x^2+y^2-2ax-2ay+a^2=0$. where $a=7\pm\sqrt{24}$

4. Find the co-ordinates and centre of the circle.
$x^2+y^2-6x+14y+33=0$. [R. U. 1958]
Ans. (3, −7), 5

5. Find the equation of the circle passing through the points (use Khalifa's method)

(a) (1, 3), (2, −1), (−1, 1) Ans: $5x^2+5y^2-11x-9y-12=0$

(b) (−4, −3), (−1, −7), (0, 0) Ans: $x^2+y^2+x+7y=0$

6. Show that following points are concyclic

(a) (3, 5), (3, −5), (2, 4), (2, −4)

(b) $(am_1, a/m_1)$, $(am_2, a/m_2)$, $(am_3, a/m_3)$, $(am_4, a/m_4)$ if $m_1m_2m_3m_4=1$.
[R. U. 1963, D. U. 1964]

7. Find the equation of the circle which passes through (3, 5) and (5, −3) and has its centre on the line $2x+y=27$ (use Khalifa's method)
Ans. $x^2+y^2-24x-6y-68=0$ [D. U. 1965]

8. Determine the centres and radii of the circles $x^2+y^2-2x+2y-7=0$, $x^2+y^2-6x-2y-6=0$ and $x^2+y^2-8x-4y-5=0$ and show that their centres are collinear. [D. U. 1962]

9. Find the locus of the middle points of the chords of the circles $x^2+y^2=a^2$ which subtends a right angle at the origin. Ans. $x^2+y^2=a^2/2$ [D. U. 1959, '77, '84; R. U. 1980]

10. Find the equation of the circle circumscribing the triangle formed by the lines.

(a) $2x-y+7=0$, $3x+5y-9=0$, $x-7y-13=0$

Ans. $169x^2+169y^2-8x+498y-3707=0$

(b) $x+2y-5=0$, $2x+y-7=0$, $x-y+1=0$ Ans. $3x^2+3y^2-13x-11y+20=0$

11. Find the equation of the circle inscribed in the triangle formed by the lines $2x-3y+21=0$, $3x-2y-6=0$, $2x+3y+9=0$

Ans. $x^2+y^2+2x-4y-8=0$

12. Find the area of an equilateral triangle inscribed in the circle
$x^2+y^2+2gx+2fy+c=0$ Ans. $(g^2+f^2-c)\,3\sqrt{3}/4$

13. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x+6y+14=0$ touch at the points (3, −1)

14. Find the equation of the circle through the points of intersection of the circles. $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 8x - 2y - 1 = 0$ and through the point (1, 2). Ans $x^2 + y^2 + 4x - 7y - 5 = 0$ [R. U. 1963]

15. Show that the equation $(3x - 3y + 1)^2 + (6x - 1)(3y + 2) = 0$ represents a circle and that the straight lines $x = \frac{1}{6}$ and $y = -2/3$ touch it.

16. Find the equation to the circle which is concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passes through (5, 4) Ans. $x^2 + y^2 - 8x + 12y - 49 = 0$

17. Show that the equation $(y - x + 3)^2 + 2(x - 2)(y + 2) = 0$ represents a circle of which $x = 2$ and $y + 2 = 0$ are two tangents. What is represented by $x - y = 3$.

Ans. $x - y = 3$ is the equation of the polar of the point $(2, -2)$ w.r. to the given circle.

18. Find the equation to the circle through the point (1, 2) (3, 4) and tangents to the lines $3x + y - 3 = 0$ [একটি বৃত্ত (1, 2) (3, 4) বিন্দুগামী এবং $3x + y - 3 = 0$ সরলরেখাকে স্পর্শ করিলে বৃত্তটির সমীকরণ নির্ণয় কর।] [C. U. 1982] Ans. $x^2 + y^2 - 8x - 2y + 7 = 0$, $x^2 + y^2 - 3x - 7y + 9/2 = 0$.

19. Find the equation of the circle through the points of intersection of $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ and touching the line $x + 2y = 0$ [একটি বৃত্ত $x^2 + y^2 - 4 = 0$ এবং $x^2 + y^2 - 2x - 4y + 4 = 0$ বৃত্তদ্বয়ের ছেদবিন্দুগামী এবং $x + 2y = 0$ সরলরেখাকে স্পর্শ করলে, বৃত্তটির সমীকরণ নির্ণয় কর।]

Ans. $x^2 + y^2 - x - 2y = 0$ [D. U. 1964 ; C. U. 1980]

20. Prove that the locus of a point whose distance from a fixed point is in a constant ratio to the tangent drawn from it to a given circle is a circle.

21. Find the polar of the point $(-16/3, -1/3)$ w. r. to the circle $x^2 + y^2 + 4x - 6y + 3 = 0$ [$x^2 + y^2 + 4x - 6y + 3 = 0$ বৃত্তটি অনুসারে $(-16/3, -1/3)$ বিন্দুটির পোলার নির্ণয় কর।]

Ans. $x + y + + 2 = 0$ [D. U. 1980]

22. Prove that the polar of the point (p, q) with respect to the circle $x^2 + y^2 = a^2$ touches $(x - c)^2 + (y - d)^2 = b^2$ if $b^2(p^2 + q^2) = (a^2 - cp - dq)^2$

23. Find the equation of the circle which has for its diameter the chord cut off on the $3x + y + 5 = 0$ by the circle $x^2 + y^2 = 16$ Ans. $x^2 + y^2 + 3x + y - 11 = 0$

24. Find the pole of the lines $2x - y + 10 = 0$ w. r. to $x^2 + y^2 - 7x - 5y - 1 = 0$

Ans. $(3/2, -3/2)$ [D. U. 1961]

25. Show that the polar of the point (h, k) w. r. to the circle $x^2 + y^2 - 2\lambda x + c = 0$ where λ is a variable parameter, always passes through a fixed point.

26. Find the equations of the tangents to the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ which are parallel to the straight line $3x - 4y + 1 = 0$

Ans. $3x - 4y + 2 = 0$, $3x - 4y - 38 = 0$ [D. U. 1952, '57, R. U. 1985]

27. Show that the equation of the circle which has for its diameter the chord cut off on straight line $ax + by + c = 0$ by the circle $(a^2 + b^2)(x^2 + y^2) = 2c^2$ is $(a^2 + b^2)(x^2 + y^2) + 2c(ax + by) = 0$

28. Prove that the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6) is a tangent to the parabola $5y^2 + 448x = 0$

29. Find the condition that the chord of contact of tangents from the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ should subtend a right angle at the centre. Ans. $x_1^2 + y_1^2 = 2a^2$

[R. U. 1958, 1979]

[(x_1, y_1) বিন্দু হতে $x^2 + y^2 = a^2$ বৃত্তের পরিপ্রেক্ষিতে স্পর্শ জ্যা বৃত্তের কেন্দ্রের সমকোণ করার শর্ত নির্ণয় কর।]

30. Find the equation of the pair of tangents from the point (5, 5) to the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ Ans. $x = 5$, $y = 5$.

31. Find the equations of the pair of lines joining the origin to the points of intersection of the line $x/a + y/b = 1$ with the circle $x^2 + y^2 = c^2$ and deduce that the line is a tangent to the circle. then $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

32. Find the equation the circle whose diameter is the common chord of circles $x^2 + y^2 + 6x + 2y + 6 = 0$ and $x^2 + y^2 + 8x + y + 10 = 0$ Ans. $5(x^2 + y^2) + 26x + 12y + 22 = 0$ [C. U. 1979]

[$x^2 + y^2 + 6x + 2y + 6 = 0$ এবং $x^2 + y^2 + 8x + y + 10 = 0$ বৃত্ত দুইটির সাধারণ জ্যাকে ব্যাস ধরে বৃত্তের সমীকরণ নির্ণয় কর।]

33. A point moves so that the sum of the square of its distances from the four sides of a square is constant. Prove that it always lies on a circle.

34. If the tangent from a point P to the circle $x^2 + y^2 - a^2 = 0$ be equal to the perpendicular from P to the straight line $lx + my + n = 0$ find the locus of. P.

Ans. $(l^2 + m^2)(x^2 + y^2 - a^2) = (lx + my + n)^2$

[$lx + my + n = 0$ সরল রেখার উপর P হতে লম্বের দূরত্ব এবং P হতে $(x^2 + y^2 - a^2) = 0$ বৃত্তের স্পর্শকের সমান হয়। তাহলে বিন্দুর সঞ্চারপথ নির্ণয় কর।]

35. Find the equations to the common tangents of the circles

(a) $x^2 + y^2 - 22x + 4y + 100 = 0$, $x^2 + y^2 + 22x - 4y - 100 = 0$.

Ans. $24x + 7y = 125$, $7x - 24y = 250$; $4x - 3y = 35$, $3x + 4y = 50$

(b) $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 + 4x + 2y - 4 = 0$

Ans. $x = 1$, $2y = 1$, $y = 2$, $3x + 4y = 5$, $4x - 3y - 10 = 0$

36. Find the direct common tangents of circles.

$x^2 + y^2 - 2x - 4y - 4 = 0$, $x^2 + y^2 + 6x - 8y + 9 = 0$ Ans. $15y - 24x + 612 = \sqrt{51}(x - 13)$

$24x + 15y + 612 = -\sqrt{51}(x - 13)$

37. Prove that the four points at which the two curves $ax^2 + 2hxy + by^2 = c$ and $a_1x^2 + 2h_1xy + b_1y^2 = c_1$ intersect, are concyclic if

$a - b \therefore a_1 - b_1 = h : h_1$

[প্রমাণ কর যে, $ax^2 + 2hxy + by^2 = c$ এবং $a_1x^2 + 2h_1xy + b_1y^2 = c_1$ এর ছেদবিন্দু চারটি একই বৃত্তের উপর থাকলে $a - b : a_1 - b_1 = h : h_1$]

38. Proved that polar of the point (1, — 2) with respect to the circle $x^2 + y^2 + 6y + 5 = 0$ and $x^2 + y^2 + 2x + 8y + 5 = 0$ are the same. Prove also that there is another point the polar of which w. r. to these circles are the same, and find its co-ordinates. Ans. (2, 1) [D. U. 1960]

39. Prove that the locus of the poles of chord of the circle $x^2 + y^2 = a^2$ which subtends right angles at the point (h, k) is the circle.

$(x^2 + y^2)(h^2 + k^2 - a^2) - 2a^2hx - 2a^2ky + 2a^4 = 0$ [C. H. 1989]

Also if R be the radius of this circle and δ is the distance of its centre from the centre of the given circles, prove that $\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{a^2}$

[$x^2 + y^2 = a^2$ বৃত্তের জ্যা এর পোল (h, k) বিন্দুতে সমকোণ করলে প্রমাণ কর যে, জ্যা এর পোলগুলোর সঞ্চারপথ $(x^2 + y^2)(h^2 + k^2 - a^2) - 2a^2hx - 2a^2ky + 2a^4 = 0$ এই বৃত্ত হবে।

40. Prove that the length of the common chord of the two circles whose equations are $(x - a)^2 + (y - b)^2 = c^2$ and

$(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{(4c^2 - 2(a - b)^2)}$ [C. H. 1977]

প্রমাণ কর যে, $(x - a)^2 + (y - b)^2 = c^2$ এবং $(x - b)^2 + (y - a)^2 = c^2$ বৃত্ত দুটির সাধারণ জ্যা এর দৈর্ঘ্য $\sqrt{(4c^2 - 2(a - b)^2)}$ হবে।

40. (a). Show that the length of the common chord of the two circles, $x^2 + y^2 + 2\lambda x + c = 0$ and

$x^2 + y^2 + 2\mu y - c = 0$ is $2\sqrt{((\lambda^2 - c)(\mu^2 + c)/(\lambda^2 + \mu^2))}$

[দেখাও যে, $x^2 + y^2 + 2\lambda x + c = 0$ এবং $x^2 + y^2 - 2\mu y - c = 0$ বৃত্তদ্বয়ের সাধারণ জ্যা এর দৈর্ঘ্য $2\sqrt{(\lambda^2 - c)(\mu^2 + c)/(\lambda^2 + \mu^2))}$] [D. U. 1987]

41. Show that poles of the lines $3x - 11y - 13 = 0$, $8x + y - 2 = 0$ and $3x + 2y + 1 = 0$ with respect to the circle $x^2 + y^2 - 4x + 3 = 0$ are collinear.

[দেখাও যে $x^2 + y^2 - 4x + 3 = 0$ বৃত্তের পরিপ্রেক্ষিতে $3x - 11y - 13 = 0$, $8x + y - 2 = 0$ এবং $3x + 2y + 1 = 0$ সরলরেখাগুলো সমরেখ হবে।] [D. U. 1979]

41. (a) Find the condition that the lines $y = mx + c$, $y = m_1x + c_1$ may be conjugate lines w. r to the circle $x^2 + y^2 = a^2$ Ans. $a^2(mm_1 + 1) = cc_1$ [R. U. 1981]

[$x^2 + y^2 = a^2$ বৃত্তের পরিপ্রেক্ষিতে $y = mx + c$, $y = m_1x + c_1$ সরলরেখা দুটি অনুবন্ধী হলে এর শর্ত নির্ণয় কর।

42. The circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ intersect. at A and B. Find the equation of a circle whose diameter is AB Ans. $2x^2 + 2y^2 + 2x + 6y + 1 = 0$ [C. H. 1978]

42. (a) show that the eq. of the circle on the chord $x\cos\alpha + y\sin\alpha = p$ of the circles $x^2 + y^2 = a^2$ as diameter is $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0$ [C. H. 1989 [R. U. 1981, '88]

[$x^2 + y^2 = a^2$ বৃত্তের $x\cos\alpha + y\sin\alpha = p$ জ্যাকে অপর একটি বৃত্তের ব্যাস ধরে বৃত্ত অঙ্কন করলে দেখাও যে বৃত্তটি $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0$ হবে।]

43. Find the equation of the circle through the intersection of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 2x + 4y + 1 = 0$ which touches the straight line $x + 2y + 5 = 0$

Ans. $5(x^2 + y^2) + 2x + 4y - 15 = 0$ [D. U. 1975]

[একটি বৃত্ত $x + 2y + 5 = 0$ সরল রেখাকে স্পর্শ করে $x^2 + y^2 = 4$ এবং $x^2 + y^2 + 2x + 4y + 1 = 0$ ছেদ বিন্দুগামী হলে, বৃত্তটির সমীকরণ নির্ণয় কর।]

44. Find the equation of the circle through the points of intersection of the circle $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having its centre on the line $\frac{x}{a} - \frac{y}{b} = 2$

Ans. $x^2 + y^2 - 3ax + by = 0$ [R. U. 1978]

45. এমন একটি বৃত্ত অঙ্কন কর যার কেন্দ্র $2x - y = 4$ সরলরেখার উপর এবং $x^2 + y^2 = 2x$, $x^2 + y^2 = 4y$ বৃত্তদ্বয়ের ছেদবিন্দু দিয়ে যায়। [D. H. 1983] Ans. $x^2 + y^2 - 3x + 2y = 0$

46. Circles are drawn with the centres on the axis of x an touching the stright line $y = x\tan\theta$. Show that the points of contract of tangents from a fixed point (ξ, η) will lie on the curve given by

$S = (x - \xi)^2(y^2 - x^2\sin^2\theta) - 2xt(x - \xi)(y - \eta)\sin^2\theta + y^2\cos^2\theta(y - \eta)^2 = 0$. (C. H. 1989)

47. If he given two circles can always be put in the form $s_1 = x^2 + y^2 + ax + by + c = 0$, $s_2 = x^2 + y^2 + a_1x + b_1y + c_1 = 0$ and that one of the circles is within the other, then prove that aa1 and b are always positive.

(যদি দুইটি বৃত্ত s_1 এবং s_2 সর্বদা প্রদত্ত আকারে হয় এবং একটির ভিতর অন্যটি হয় তা হলে প্রমাণ কর যে, aa_1 এবং b সর্বদা ধনাত্মক হবে।)

SYSTEM OF CIRCLES 8

Art. 84. Angle of intersection of two circles.

The angle of intersection of two curves is measured by the angle between the tangents to the curves at a point of their intersection.

Since in the case of a circle the radius to the point of contact of tangent is perpendicular to the tangent, the angle between the two tangents to the two circles at a common point equal to the angle between the radii of the circles drawn to the same point.

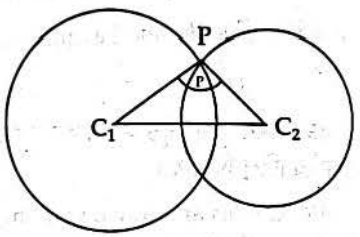


Fig : 34

Hence if ϕ be the angle of intersection of two circles whose radii are r_1 and r_2 and d be the distance between their centers then

$$\cos\phi = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \ldots \quad \ldots \quad (1)$$

(See Properties of Triangle in Author's Trigonometry]

If the equation of two circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ then $r_1^2 = g_1^2 + f_1^2 - c_1$

$r_2^2 = g_2^2 + f_2^2 - c_2$

and $d^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

from Eq.(1) by relation (2)

$$\cos\phi = \frac{g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 - \{(g_1 - g_2)^2 + (f_1 - f_2)^2\}}{2r_1r_2}$$

$$\text{or, } \cos\varnothing = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2r_1r_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Cor. 1. If $\phi = 0$, the two circles touch each other internally. From (3), we have $2r_1r_2 = 2g_1g_2 + 2f_1f_2 - c_1 - c_2$

Cor. 2. If $\varphi = 180°$, the two circles touch each other externally. From (3), we have $-2r_1r_2 = 2g_1g_2 + 2f_1f_2 - c_1 - c_2$

Cor. 3. If $\phi = 90°$, the two circles said to be orthogonal when the tangents at their points of intersection are right angles. From (3), we have

$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0$

or, $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

which is the condition for orthogonality of two intersecting circles.

Alternative Method for condition of orthogonality.

Let the equation to the intersecting circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$(-g_1, -f_1)$, $\sqrt{(g_1^2 + f_1^2 - c_1)}$, centre and radius of the first circle

$(-g_2, -f_2)$, $\sqrt{(g_2^2 + f_2^2 - c_2)}$, centre and radius of the 2nd circle

As the two circles interesect at P

at right angles i. e. $\angle C_1PC_2 = 90°$

Therefore, from the figure, we have

$C_1P^2 + C_2P^2 = C_1C_2^2$

or, $g_1^2 + g_2^2 + f_1^2 - c_1 + f_2^2 - c_2$

$= (-g_1 + g_2)^2 + (-f_1 + f_2)^2$

or, $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

which is the required condition.

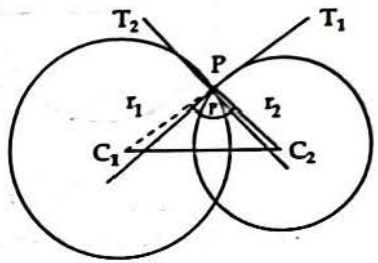


Fig : 35

Art. 85. Radical Axis.

The radical axis of two circles is the locus of a point which moves so that the lengths of tangents drawn from it to the two circles are equal. Otherwise, the locus of points whose powers with respect to two circles are equal is a straight line called the **radical axis of the circles.**

See Art. 82.

Art. 86. Radical axis of two circles

$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

and $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$

Let (x_1, y_1) be the point from which the lenghts of the two tangents to (1) and (2) are equal, see Art. 70 (a).

Then, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$

or, $2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$

Hence the locus of the point (x_1, y_1) is a straight line i. e. the radical axis of the two circles is.

or, $2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$

which is a straight line.

Notes. **1.** If $S_1 = 0$ and $S_2 = 0$ be the equations of two circles then the radical axis of the two circles is given by $S_1 - S_2 = 0$

provided that co-efficients of x^2 and y^2 terms are unity in both cases or the same.

Note 2. Since $S_1 - S_2 = 0$ represents are curve passing through the intersection of $S_1 = 0$ and $S_2 = 0$; therefore the radical axis of two circle is their common chord.

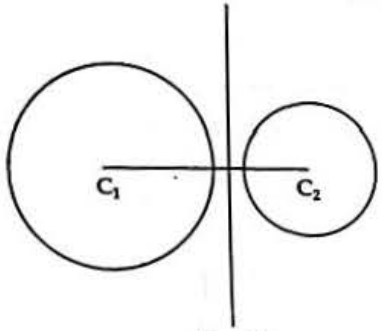


Fig : 36

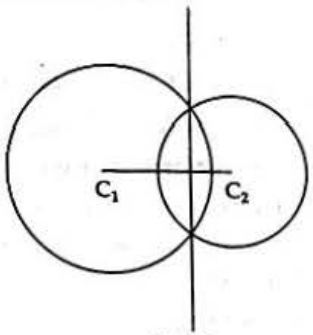


Fig : 37

Note. 3. It should be observed from the figures that the points of intersection may be real Fig. (36) or imaginary Fig. (37) but in both cases the radical axis is a real line.

Art. 87. Properties of Radical Axis.

(a) The radical axis of two circles is perpendicular to the line joining the centres.

Let the equations of the two circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ... ... (1)

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ... ... (2)

$(-g_1, -f_1)$ and $(-g_2, -f_2)$ are the centres of circles (1) and (2) respectively.

The equations of the line joing the centres is

$y + f_1 = \frac{f_1 - f_2}{g_1 - g_2}(x + g_1)$ ... ... (3)

The slope of line (3) $\frac{f_1 - f_2}{g_1 - g_2} = m_1$ (say) ... ... (4)

The equation of the radical axis of (1) and (2) is

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ ... ... (5)

Slope of the radical axis (5), $-\frac{g_1 - g_2}{f_1 - f_2} = m_2$ (say) ... ... (6)

From (4) and (5), $m_1m_2 = -1$

Hence they are at right angles.

(b) ***The radical axes of three circles, taken in pair meet in a point.***

Let the equations of the three circles be $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ the co-efficients of x^2 and y^2 being the same in all cases.

The radical axis of the circles $S_1 = 0$, and $S_2 = 0$

$S_1 - S_2 = 0$ ... ... (1)

The radical axis of $S_2 = 0$, $S_3 = 0$ and $S_3 = 0$, $S_1 = 0$ are respectively

$S_2 - S_3 = 0$ ... ... (2)

and $S_3 - S_1 = 0$ ... ... (3)

Since the sum of the left sides of those Eqs. (1), (2) and (3) vanishes the lines represented by them meet at a point.

The point in which radical axes of the three circles meet is called the **radical centre** of the three circles. The lenghts of the tangents from the radical centre to the three circles are equal.

Art. 88. Co-axial circles. Def : If a system of circles is such that the radical axis of one pair is the same as that of any other pair, the circles are said to form a **Co-axial system.**

Art. 89. Various forms of Co-axial circles.

(a) $S_1 - \lambda S_2 = 0$

where $S_1 = 0$ and $S_2 = 0$ are any two circles of the system. The general equation of all circles through the intersection of two circles $S_1 = 0$ and $S_2 = 0$ can be put in the form $S_1 - \lambda S_2 = 0$ except $\lambda = 1$.

Hence $S - \lambda S_1 = 0$ represents a series of circles all of which will pass through two fixed points. i.e the intersection (real or imaginary) of the given circles and thus will have a common radical axis and form a family of co-axial circles. Their **common radical axis or common Chord is**

$S_1 - S_2 = 0$ when $\lambda = 1$ in (a)

(b) In the same way we may show that the equation $S - \lambda L = 0$ represents for all values of λ a circle of a co-axial system of which $L = 0$ is the radical axis and the circle $S = 0$ is a member. If $L = 0$ intersects the circle $S = 0$ in real point, then $S - \lambda L = 0$ represents a circle through these points.

Art. 90. Centre locus of the system $S_1 - \lambda S_2 = 0$

Where $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ... ... (1)

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ... ... (2)

Radical axis of $S_1 = 0$ and $S_2 = 0$ is

$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ ... ... (3)

The equation of the system of circles is $S_1 - \lambda S_2 = 0$

or, $x^2(1 - \lambda) + y^2(1 - \lambda) + 2x(g_1 - \lambda g_2) + 2y(f_1 - \lambda f_2) + (c_1 - \lambda c_2) = 0$ ... ... (4) by (1) and (2)

If (x_1, y_1) be the co-ordinates of the centre of (4), then

$x_1 = -\frac{g_1 - \lambda g_2}{1 - \lambda}$ or $\lambda = \frac{x_1 + g_1}{x_1 + g_2}$ ... ... (5)

$y_1 = -\frac{f_1 - \lambda f_2}{1 - \lambda}$ or, $\lambda = \frac{y_1 + f_1}{y_1 + f_2}$ ... ... (6)

From (5) and (6) we have,

$\frac{x_1 + g_1}{x_1 + g_2} = \frac{y_1 + f_1}{y_1 + f_2}$ ... ... (7)

which shown that the locus of (x_1, y_1) is the line

$x(f_1 - f_2) - y(g_1 - g_2) + f_1g_2 - f_2g_1 = 0$ ... ... (8)

which is evidently perpendicular to the radical axis (3) i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ ... ... (9)

It also passes through $(-g_1, -f_1)$ and $(-g_2, -f_2)$, the centre of the given circles forming the system as is expected.

Art. 91. Reduction of the general equation $S_1 - \lambda S_1 = 0$ of a system of co-axial circles.

Take the line of centres of the circles as the axis x and the radical axis of the system as the axis of y.

Since the centres of the circles $S_1 = 0$, $S_2 = 0$ lie on the x — axis their equation are of the form $(x - h)^2 + (y - 0)^2 = a^2$

Hence $f_1 = 0 = f_2$ so the equations reduce to

$x^2 + y^2 + 2g_1x + c_1 = 0$ ... ... (1)

$x^2 + y^2 + 2g_2x + c_2 = 0$ ... ... (2)

The radical axis of (1) and (2) is $2(g_1 - g_2)x + c_1 - c_2 = 0$ ... ... (3)

But the radical axis is the y axis i.e. $x = 0$.

$\therefore$ Eq. (3) becomes $c_1 - c_2 = 0$ or, $c_1 = c_2 = c$ (say)

Now the equations (1) and (2) becomes respectively

$x^2 + y^2 + 2g_1x + c = 0$ and $x^2 + y^2 + 2g_2x + c = 0$

which differ only in the co-efficient of x

$\therefore S_1 - \lambda S_2 = 0$

or, $x^2 + y^2 + \dfrac{2(g_1 - \lambda g_2)}{1 - \lambda}x + c = 0$ ... Put, $m = \dfrac{g_1 - \lambda g_2}{1 - \lambda}$

or, $x^2 + y^2 + 2mx + c = 0$ ... ... (4)

where m is a variable parameter, the different circles of the system will be obtained.

Art. 92. Point Circles, Limiting Points.

Point Circles : Circles whose radii are zero, are called Points Circles.

Limiting Points : Limiting point of co-axial circle, the Centre of the Points Circles belonging to the family.

Art. 93. Find limiting points of the system of circles represented by

$x^2 + y^2 + 2\lambda x + c = 0$, or $(x + \lambda)^2 + y^2 = \lambda^2 - c$ ... ... (1)

When the radius of the circle is zero, the circles reduce to points circles i.e.; $\lambda^2 - c = 0$, or, $\lambda = \pm(\sqrt{c})$

Hence the point circles or limiting points of the co-axial system are given by. $(x - \sqrt{c})^2 + y^2 = 0$, $(x + \sqrt{c})^2 + y^2 = 0$

The limiting points are $(\sqrt{c}, o)$ and $(-\sqrt{c}, 0)$

The limiting points are real, co-incident or imaginary according as c is + ve, zero or-ve i.e. $c >, =$ or, < 0

Cor. The radical axis $x = 0$ meet every circle of the system $x^2 + y^2 + 2\lambda x + c = 0$.

i. e. $(y^2 + c = 0)$ in the points $(0, \pm\sqrt{-c})$

These points are real if $c \le 0$ and imaginary $c > 0$, Hence eq. (1) represents a co-axial system intersecting in real points if $c < 0$ and a non-intersecting system if $c > 0$.

If $c = 0$ every circle of the system touches the y-axis at the origin and system is said to be tangential.

Art. 94. Properties of limiting point.

(a) *The polar of either limiting point with respect to any circle the system passes through the other.*

The polar of $(\sqrt{c}, 0)$ which is a limiting point *w.r.* to the circle

$x^2 + y^2 + 2\mu x + c = 0$ ... ... (1)

is $x(\sqrt{c}) + y \cdot 0 + \mu(x + \sqrt{c}) + c = 0$ Art. 74 (5)

or, $x + \sqrt{c} = 0$ as $\sqrt{c} + \mu \neq 0$

which passes through the point $(-\sqrt{c}, 0)$ the other limiting point of the system and hence they are **Conjugate points.**

(b) *Every circle through the limiting points of a co-axial system is orthogonal to all circles of the system.*

The general equation to a circle passing through the limiting points $(-\sqrt{c}, 0)$ is $x^2 + y^2 + 2\lambda x + c = 0$

where λ is a variable parameter. Now the circles

$x^2 + y^2 + 2\mu x + c = 0$ ... (1) and $x^2 + y^2 + 2\lambda x + c = 0$ ... ... (2)

will be orthogonal if $2\mu . 0 + 2 . 0 . \lambda - c - (-c) = 0$

which is evidently satisfied.

Hence any circle of the system (2) is orthogonal to every circle of the system (1).

These two systems of circles (1) and (2) are such that the centres of the other system lie on the radical axis of the other and the common point of one system are limiting points of the other. They are, therefore, called the conjugate system of co-axial circles.

EXAMPLES

Ex. 1. Prove that the circles $x^2 + y^2 - 3x + 8y - 2 = 0$ ... ... (i)

and $x^2 + y^2 + 4x - 5y - 24 = 0$ cut orthogonally ... ... (ii)

O_1 $(3/2, -4)$, $r_1 = [\sqrt{(9/4 + 16 + 2)}] = 9/2$, are the co-ordinates of centre and radius respectively of (i)

$O_2(-2, 5/2)$, $r_2 = \sqrt{[(4 + 25/4 + 24)]} = \sqrt{(137/4)}$, be the co-ordinates of the centre and radius respectively of ... ... (iii)

$O_1(3/2, -4)$: $r_1 = [\sqrt{(9/4 + 16 + 2)}] = 9/2$,

$O_1O_2^2 = (3/2 + 2)^2 + (-4 - 5/2)^2 = 218/4$.

$r_1^2 + r_2^2 = 81/4 + 137/4 = 218/4$

$O_1O_2^2 = r_1^2 + r_2^2$, Hence they are orthogonal

Ex. 2. Find the equation of the circle which cuts the circle

$x^2 + y^2 - 6x + 8 = 0$ ... (i) and $x^2 + y^2 - 2x - 2y - 7 = 0$ ... ... (ii)

Orthogonally.

Let the equation to the circle passing through the origin be $x^2 + y^2 + 2gx + 2fy = 0$ ... ... (iii)

Since (iii) cuts (i) and (ii) orthogonally, then

$-6g + 2f.0 - 8 = 0$ and $-2g - 2f + 7 = 0$

or, $6g + 8 = 0$. or, $g = -4/3$ and $2g + 2f - 7 = 0$, or, $2f = 29/3$

The required equation is $x^2 + y^2 - 2(4/3)x + 29y/3 = 0$

or, $3x^2 + 3y^2 - 8x + 29y = 0$

Ex. 3. Find radical axis of the pairs of circles

$x^2 + y^2 = 144$ ... (i) $x^2 + y^2 - 15x + 11y = 0$ ... ... (ii)

The equation to the radical axis of (i) and (ii)

is $x^2 + y^2 - 144 - (x^2 + y^2 - 15x - 11y) = 0$ or, $15x + 11y - 144 = 0$

Ex. 4. Find the radical centre of the three circles

$x^2 + y^2 + x + 2y + 3 = 0$, $x^2 + y^2 + 2x + 4y + 5 = 0$

and $x^2 + y^2 - 7x - 8y - 9 = 0$

The radical axis of the circles is $x^2 + y^2 + x + 2y + 3 - (x^2 + y^2 + 2x + 4y + 5) = 0$

or, $x + 2y + 2 = 0$ ... ... (1)

$x^2 + y^2 + x + 2y + 3 = 0$ and $x^2 + y^2 - 7x - 8y - 9 = 0$

is $x^2 + y^2 - 7x - 8y - 9 - (x^2 + y^2 + x + 2y + 3) = 0$

or, $4x + 5y + 6 = 0$ ... ... (1)

Solve (1) and (2) for x and y.

$x = -2/3$ and $y = -2/3$. The centre $(-2/3, -2/3)$

Ex. 5. Find the equation of the circle which cuts orthogonally each of the three circles.

$S_1 \equiv x^2 + y^2 + x + 2y + 3 = 0$ ... ... (i)

$S_2 \equiv x^2 + y^2 + 2x + 4y + 5 = 0$ ... ... (ii)

$S_3 \equiv x^2 + y^2 - 7x - 8y + 9 = 0$ ... ... (iii)

The raidcal axes of the pairs of circles (i), (ii) and (iii) are respectively given by the equations

$S_2 - S_1 \equiv x + 2y + 2 = 0$ ... ... (iv)

$S_1 - S_3 \equiv 8x + 10y - 6 = 0$ or, $4x + 5y - 3 = 0$ ... ... (v)

The intersecting of the lines (iv) and (v) is $(16/3, -11/3)$, which is the radical centre of the circles.

The square of the tangent from $(16/3, -11/3)$ to the circle $(1) = (16/3)^2 + (11/3)^2 + (16/3 - 22/3 + 3) = (16/3)^2 + (11/3)^2 + 1$

Hence the equation of the radical circles is

$(x - 16/3)^2 + (y + 11/3)^2 = (16/3)^2 + (11/3)^2 + 1$

or, $x^2 + y^2 - \frac{32}{3}x + \frac{22}{3}y = 1$ or, $3(x^2 + y^2) - 32x + 22y - 3 = 0$

Ex. 6. Find the co-ordinates of the limiting point of the co-axial system determined by circles $x^2 + y^2 + 4x + 2y + 5 = 0$ and $x^2 + y^2 + 2x + 4y + 7 = 0$. [C. U. 1986]

The radical axis of two given circles is

$(x^2 + y^2 + 4x + 2y + 5) - (x^2 + y^2 + 2x + 4y + 7) = 0$

or, $2x - 2y - 2 = 0$ or, $x - y - 1 = 0$

Hence the equation of any circle co-axial with the given circles

is $x^2 + y^2 + 4x + 2y + 5 + \lambda(x - y - 1) = 0$

or, $x^2 + y^2 + x(4 + \lambda) + y(2 - \lambda) + (5 - \lambda) = 0$ ... (1) (By Art. 89)

Radius of this circle

$= \sqrt{[\{(4 + \lambda)/2\}^2 + \{(2 - \lambda)/2\}^2 - (5 - \lambda)\}]}$

Thus the radius is zero *i.e.*

$16 + \lambda^2 + 8\lambda + 4 - 4\lambda + \lambda^2 - 20 + 4\lambda = 0$ or, $2\lambda^2 + 8\lambda = 0$

$\therefore \lambda = 0, -4$

Co-ordinates of centre of (i) $(-(4 + \lambda)/2, -(2 - \lambda)/2)$

Putting the value λ the limiting points are $(-2, -1)$, $(0, -3)$

Ex. 7. Find the equations of the circle through the intersection of the circles.

$x^2 + y^2 + 5x + 6y + 7 = 0$ ... ... (i)

$x^2 + y^2 + 4x + 3y + 2 = 0$ ... ... (ii)

whose centre lies on the line $x + 3y + 5 = 0$

Any circle through common points of the given circles is

$x^2 + y^2 + 5x + 6y + 7 - \lambda(x^2 + y^2 + 4x + 3y + 2) = 0$

or, $(1 - \lambda)x^2 + (1 - \lambda)y^2 + x(5 - 4\lambda) + y(6 - 3\lambda) + (7 - 2\lambda) = 0$ (iii) Centre of (iii) is

$\frac{4\lambda - 5}{2(1 - 7)}, \frac{3\lambda - 6}{2(1 - \lambda)}$

Since the centre lies on $x + 3y + 5 = 0$

or, $\frac{4\lambda - 5}{2(1 - \lambda)} + 3\frac{(3\lambda - 6)}{2(1 - \lambda)} + 5 = 0$ or, $\lambda = 13/3$

From (iii), putting the value of λ we have

$10(x^2 + y^2) + 37x + 21y + 5 = 0$

Ex. 8. Find the equation of the circle whose diameter is the common chord of the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ [R. U. 1976]

The circle whose diameter is the common chord of the circle

$S_1 \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ ... ... (i)

and $S_2 \equiv x^2 + y^2 + 4x + 3y + 2 = 0$ ... ... (ii)

passes through their points of intersection and has its centre on the radical axis of (i) and (ii). The radical axis of (i) and (ii) is $x^2 + y^2 + 2x + 3y + 1 - (x^2 + y^2 + 4x + 3y + 2) = 0$

or, $2x + 1 = 0$ ... ... (iii)

Any circle passing through the point of intersection of the two circles is $S_1 + \lambda S_2 = 0$

or, $x^2 + y^2 + \frac{2(1+2\lambda)}{1+\lambda}x + \frac{3(1+\lambda)}{1+\lambda}y + \frac{1+2\lambda}{1+\lambda} = 0$

Its centre is $\left(-\frac{1+2\lambda}{1-\lambda}, -\frac{3}{2}\right)$

As the centre lies on $2x + 1 = 0$

$\therefore\ 2\left(-\frac{1+2\lambda}{1+\lambda}\right) + 1 = 0$, or $\lambda = -\frac{1}{3}$

Hence the required circle is $S_1 - \frac{1}{3}S_2 = 0$

or, $2(x^2 + y^2) + 2x + 6y + 1 = 0$

Ex. 9. Prove that the two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

The radical axis of the two circles is

$x^2 + y^2 + 2ax + c^2 - (x^2 + y^2 + 2by + c^2) = 0$ or, $ax - by = 0$ (1)

The circles will touch each other if this line touches them i.e. if its distance from th centre of the circles is equal to the radius of that circle. Now the centre of the circle.

$x^2 + y^2 + 2ax + c^2 = 0$ is $(-a, o)$, and radius $= \sqrt{(a^2 - c^2)}$

$\therefore$ The required condition is

$\frac{a(-a) - b(0)}{\sqrt{(a^2 + b^2)}} = \sqrt{(a^2 - c^2)}$ or, $\frac{a^4}{a^2 + b^2} = a^2 - c^2$

or, $a^4 - a^2c^2 + b^2a^2 - b^2c^2 = a^4$ or, $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

Ex. 10. The circle $x^2 + y^2 + 2x - 4y - 11 = 0$ and the line $x - y + 1 = 0$ intersect at A and B Find the equation of the circle on AB as diameter and the equation of the circle through A, I orthogonal to the given circles.

The equation of the circle through the point of intersection

or $x^2 + y^2 + 2x - 4y - 11 = 0$ ... ... (1)

and $x - y + 1 = 0$ ... ... (2)

is $x^2 + y^2 + 2x - 4y - 11 + \lambda(x - y + 1) = 0$

Represents a circle through A and B. If its centre

$(-1 - \lambda/2, 2 + \lambda/2)$ lies on the line $x - y + 1 = 0$, $\lambda = -2$

Hence the equation of the circle on AB as diameter from (3) is

$x^2 + y^2 - 2y - 13 = 0$

The circle (3) is orthogonal to (1) if by cor. Art. 84

$(2 - \lambda) + 2(4 - \lambda) = +\lambda - 22$ or, $\lambda = -16$

Hence the equation of the orthogonal circle by putting the value of λ in (3) is $x^2 + y^2 - 14x + 12y - 27 = 0$

EXERCISE VIII

1. Find the angle between the circles.

(a) $x^2 + y^2 - 8x - 12y = 0$ and $x^2 + y^2 - 2x + 4y = 0$ Ans. $\cos^{-1} 2/\sqrt{65}$

(b) $x^2 + y^2 + 4x - 6y + 12 = 0$ and $x^2 + y^2 - 2x + 2y - 23 = 0$ Ans. $60°$.

2. Showing that the following pair of circles intersect orthogonally.

(a) $x^2 + y^2 - 2ax + 2by = 0$ and $x^2 + y^2 + 2bx + 2ay = 0$

(b) $x^2 + y^2 - 2px + 2py + 2pq = 0$ and $x^2 + y^2 + 2px + 2py - 2pq = 0$

3. Find the equation to the circle which cuts the circles $x^2 + y^2 + 2x + 4y + 6 = 0$ and $x^2 + y^2 + 4x + 6y + 2 = 0$ orthogonally and passes through the point (1, 1). Ans. $x^2 + y^2 - 16x + 12y + 2 = 0$.

4. Find the value of k for which the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$ are orthogonal. Ans. -18.

5. Find the radical axis of the following pairs of circles.

(a) $x^2 + y^2 - 3x - 4y + 5 = 0$ and $3x^2 + 3y^2 - 7x + 8y + 11 = 0$.
Ans. $x + 10y = 2$.

(b) $x^2 + y^2 + 4x + 8y + 19 = 0$ and $x^2 + y^2 + 8x + 4y + 19 = 0$.
Ans. $x - y = 0$

6. Find the radical centre of the circles.

(a) $x^2 + y^2 + 3x - 2y - 4 = 0$, $x^2 + y^2 - 2x - y - 6 = 0$
$x^2 + y^2 - 1 = 0$ Ans. $(-1, -3)$

(b) $x^2 + y^2 + 12x + 11 = 0$, $x^2 + y^2 - 4x - 21 = 0$ and
$x^2 + y^2 - 4x + 16y + 43 = 0$ Ans. $(-2, -4)$

7. Find the equation to the circle which has for is diameters chord cut off on the line $3x + 2y = 6$ by the circle $x^2 + y^2 = 16$
Ans. $13(x^2 + y^2) - 36x - 24y - 136 = 0$ [R. U. 1960]

8. Find the equation to the circle which cuts orthogonally the three circles.

(a) $x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$
$x^2 + y^2 + 7x - 9y + 29 = 0$ Ans. $x^2 + y^2 - 16x - 18y - 4 = 0$ [C. U. '80]

(b) $x^2 + y^2 = a^2$, $(x - c)^2 + y^2 = a^2$, $x^2 + (y - b)^2 = a^2$
Ans. $x^2 + y^2 - cx - by + a^2 = 0$. [C. U. 1981 ; R. U. 1954]

(c) $x^2 + y^2 + 2x + 4y + 3 = 0$, $x^2 + y^2 + 3x + 5y = 0$ and $x^2 + y^2 + 4x + 5y - 1 = 0$. Ans. $x^2 + y^2 - 2x - 4y - 13 = 0$

9. Find the equation of the circle through the intersection of the circle $x^2 + y^2 + 2x + 3y + 4 = 0$ and the line $2x + 3y + 4 = 0$ and which passes through the point (2, 3) Ans. $17(x^2 + y^2) - 26x - 29y - 52 = 0$

10. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g_1x + 2f_1y = 0$ touches each others show that $f_1g = fg$.

11. Show that the circles $x^2 + y^2 - 4x - 7y + 6 = 0$, $x^2 + y^2 + 3x - 14y - 1 = 0$ and $3(x^2 + y^2) + 2x - 35y + 4 = 0$ have a common radical axis.

12. Find the general equation of all circles any pair of which have the same radical axis as the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 2x + 4y = 6$ Ans. $(1 + \lambda)(x^2 + y^2) + 2\lambda(x + 2y) = 4 + 6\lambda$

13. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having it centre on the line $x/a - y/b = 2$ Ans. $x^2 + y^2 - 3ax + by = 0$

14. Find the equation of the circle passing through the points of intersection of the line $2x + 3y + 4 = 0$ and the circle $x^2 + y^2 + 2x + 3y + 4 = 0$ and cutting the circle $x^2 + y^2 + 5x + 6y + 7 = 0$ orthogonally. Ans. $10(x^2 + y^2) + 14x + 21y + 28 = 0$

15. Find the co-ordinates of the limiting points of the co-axial system determined by the circles.

$x^2 + y^2 - 2x + 8y + 11 = 0$, [R. U. 1980]

and $x^2 + y^2 + 4x + 2y + 5 = 0$ Ans. $(-2, -1), (0, -3)$

16. Find the length of the common chord of the circles.

(a) $x^2 + y^2 - 12x + 16y - 69 = 0$,

and $x^2 + y^2 - 9x + 12y - 59 = 0$ Ans. 10.

17. Circles are drawn through the point $(c, 0)$ touching the circles $x^2 + y^2 = a^2$, Show that the locus of the poles of the axis of x, w, r, to these circles is the curve $4a^2(x - c)^2 = (a^2 - c^2)\{a^2 - (c - 2x)^2\} y^2$

18. Find the limiting points of the co-axial system determined by the circles.

(a) $x^2 + y^2 + 2x - 6y = 0$, and $2x^2 + 2y^2 - 10y + 5 = 0$ Ans. $(1, 2), (3, 1)$

(b) $(x - 1)^2 + (y - 2)^2 + \lambda(x^2 + y^2 + 2x + 5) = 0$ Ans. $(1, 2), (-2, -1)$

19. If the origin be at one of the limiting points of a system of coaxial circles of which $x^2 + y^2 + 2gx + 2fy + c = 0$ is a member, prove that the equation of the system of circles cutting them all orthogonally is $(x^2 + y^2)(g + \mu f) + c(x + \mu y) = 0$ [R. U. 1977]

20. Prove that the limiting points of the system $x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$ subtend a right angle at the origin if $\frac{c}{g^2} - \frac{k}{f^2} = 2$

21. Prove that two circles will pass through the two pts. (o, a) and $(0, -a)$ and touch the straight ine $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$. [C. H. 1989 ; C. U. 1981, '86]

22. The straight line $lx + my - 1 = 0$ meets the lines $ax^2 + 2hxy + by^2 = 0$ in the points P and Q : show that the equation of the circle described on PQ as diameter is $S = (x^2 + y^2)(am^2 - 2hlm + bl^2) - 2x(bl - hm) - 2y(am - hl) + a + b = 0$. [R. H. 1988]

[$lx + my - l = 0$ সরলরেখা $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়কে যথাক্রমে P ও Q বিন্দুতে ছেদ করে। দেখাও যে PQ-কে ব্যাস ধরে বৃত্ত অঙ্কন করলে বৃত্তটি S হবে]

23. From fixed point (x_1, y_1) perpendiculars are drawn to the straight lines $L = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Show that the equation of the circles circumscribing the quadrilateral so formed is

$S = (ab - h^2)\{x(x - x_1) + y(y - y_1) - (hf - bg)(x - x_1) - (gh - af)(y - y_1) = 0$

[(x_1, y_1) একটি নির্দিষ্ট বিন্দু হতে L সরলরেখাগুলোর উপর লম্ব টানা হল। দেখাও যে চতুর্ভুজের পরিলিখিত বৃত্তের সমীকরণ ও হবে।]

*24. Show that the general equation of all circles cutting at right angles the circles represented by

$x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$, $x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0$ is

$$S = \begin{vmatrix} x^2 + y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} + k \begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

[দেখাও যে $x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$, $x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0$ বৃত্তদ্বয়কে লম্বভাবে ছেদকারী বৃত্তের সাধারণ সমীকরণ S হবে।

*25. Circles are drawn through the point $(a, 0)$ touching the circle $x^2 + y^2 = c^2$. Show that the locus of the pole of the x-axis with respect to these circles is the curve $4c^2(x - a)^4 = (c^2 - a^2)\{c^2 - (a - 2x)^2\} y^2$. [C. H. 1988]

(a, 0) বিন্দুগামী এবং $x^2 + y^2 = c^2$ বৃত্তকে স্পর্শকারী বৃত্তসমূহ অঙ্কিত হল। প্রমাণ কর যে ঐ সমস্ত বৃত্তসমূহের প্রেক্ষিতে x অক্ষের পোল (Pole)-সমূহের সঞ্চারপথ হবে $4c^2(x - a)^4 = (c^2 - a^2)\{c^2 - (a - 2x)^2\}y^2$

*26. Show that if $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ be the equations of three circles of which each two cut orthogonally, the equation.

$l_1S_1 + l_2S_3 + l_3S_3 = 0$ represents real circle except in certain cases where it represents a straight line.

(দেখাও যে যদি $S_1 = 0, S_2 = 0, S_3 = 0$ বৃত্তগুলোর মধ্যে দুটি করে লম্বভাবে পরস্পর ছেদ করলে, $l_1S_1 + l_2S_2 + l_3S_3 = 0$ একটি বাস্তব বৃত্ত প্রকাশ করবে এবং বিশেষ ক্ষেত্রে এর একটি সরলরেখা হবে।)

*27. Show that circle on the chord $x \cos \alpha + y \sin \alpha - p = 0$ of the circle $x^2 + y^2 - a^2 = 0$ as diameter if

$S = x^2 + y^2 - a^2 - 2p(x \cos \alpha + y \sin \alpha - p) = 0$

$x \cos \alpha + y \sin \alpha - p = 0$ জ্যাকে ব্যাস ধরে দেখাও যে $x^2 + y^2 - a^2 = 0$ বৃত্তে বৃত্ত অঙ্কন করলে বৃত্তটি ও হবে]

*28. Find the limiting points of the system of circles

$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + c_1) = 0$ and that the square of the distance between them is $\{(c - c_1)^2 - 4f^2g^2 + 4f^2c + 4g^2c_1\}/(f^2 + g^2)$ [C. H. 1988 (1986)]

[$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + c_1) = 0$

উক্ত পদ্ধতির বৃত্তসমূহের সীমায়ীত বিন্দুসমূহ নির্ণয় কর এবং দেখাও যে তাদের দূরত্বের বর্গ $\{(c - c_1)^2 - 4f^2g^2 + 4fg^2c + 4g^2c_1\}/(f^2 + g^2)$

29. The straight line $x \cos \alpha + y \sin \alpha = p$ being denoted by (α, p) find the equation of the circle circumscribing the triangle formed by the lines $(\alpha\, p)$, $(\beta\, q)$, $(\gamma\, r)$ and show that if it passes through the origin then

$S \equiv qr \sin(\beta - \gamma) + rp \sin(\gamma - \alpha) + pq \sin(\alpha - \beta) = 0$ [C. H. 1989]

[$x \cos \alpha + y \sin \alpha = p$ সরলরেখাকে (αp) দ্বারা প্রকাশ করলে দেখাও যে (αp), (βq), (γr) দ্বারা সৃষ্ট ত্রিভুজের পরিলিখিত সমীকরণ S হবে]

*30. Show that the equations of the circle circumscribing (পরিলিখিত) the triangle formed by the lines $ax + by + c = 0$, $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ is

$$\begin{vmatrix} \dfrac{a^2 + b^2}{ax + by + c} & a & b \\ \dfrac{a_1^2 + b_1^2}{a_1x + b_1y + c_1} & a_1 & b_1 \\ \dfrac{a_2^2 + b_2^2}{a_2x + b_2y + c_2} & a_2 & b_2 \end{vmatrix} = 0$$

*31. Circles are drawn with the centres of the axis of x and touching the straight line $y = x \tan \theta$. show that the points of contact of tangents from a fixed point (ξ, η) will lie on the curve given by

$$S = (x - \xi)^2 (y^2 - x^2 \sin^2 \theta) - 2xy(x - \xi)(y - \eta) \sin^2\theta + y^2 \cos^2\theta\, (y - \eta)^2 = 0$$

[x-অক্ষের উপর কেন্দ্র এবং $y = x \tan \theta$ সরলরেখাকে স্পর্শ করে এরূপ বৃত্ত আঁকা হল। দেখাও যে (ξ, η) একটি নির্দিষ্ট বিন্দু হতে বৃত্তগুলোর উপর স্পর্শকের ছেদবিন্দুগুলো একটি বক্ররেখার উপর থাকবে যার সমীকরণটি S হবে।]

[C. H. 1989]