**1. međuispit iz Stohastičkih procesa**
**26.11.2014.**

1. **(6 bodova)**
Razdioba vektora $(X, Y)$ dana je u tablici

| $X \backslash Y$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| 1 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ |
| 2 | $\frac{2}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ |
| 3 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |

Odredi **1.** Marginalne razdiobe varijabli $X$ i $Y$; **2.** Uvjetna očekivanja $E(X \mid Y)$ i $E(Y \mid X)$.

2. **(8 bodova)** Ponavljamo pokus. Neka je $0 < p < 1$ vjerojatnost uspjeha u pokusu. Neka slučajna varijabla $X$ broji neuspješne pokuse do prve pojave uspjeha. Neka je $Y$ broj uspjeha u prvih $n$ ponavljanja pokusa, $n \geq 1$. Precizno izvedite funkcije izvodnice za slučajne varijable $X$ i $Y$ te, na temelju funkcija izvodnica, odredite njihove disperzije.

3. **(6 bodova)** Student rješava ispit koji se sastoji od 20 zadataka na zaokruživanje, točno-netočno. Student nasumično zaokružuje jedan od odgovora, pri čemu pogriješi s vjerojatnošću $\frac{2}{3}$. Za svaki točno riješeni zadatak student dobiva 3 boda, a za svaki netočan gubi 2 boda. Odredite funkciju izvodnicu za ukupan zbroj bodova osvojenih na ispitu, te, pomoću nje, vjerojatnost da student na ispitu osvoji ukupno 0 bodova.

4. **(8 bodova)**
**(a)** Iskažite (po komponentama i matrično) Chapman-Kolmogorovljeve jednadžbe za homogeni Markovljev lanac, te ih dokažite.
**(b)** Izvedite sustav pomoću kojeg se računaju stacionarne vjerojatnosti ergodičkog homogenog Markovljevog lanca.
**(c)** Koristeći sustav iz $(b)$, odredite stacionarne vjerojatnosti općenitog Markovljevog lanca sa 2 stanja, takvog da je prijelaz is svakog stanja u svako stanje moguć u jednom koraku.

5. **(6 bodova)** Kockar počinje igru sa $n$ žetona, $1 \leq n \leq M$. U svakoj igri dobiva ili gubi žeton s istom vjerojatnošću. Igra prestaje kad igrač osvoji $M$ žetona ili kad izgubi sve žetone.
**(a)** Izvedite formulu za očekivanu duljinu igre.
**(b)** Za gore opisano slučajno pomicanje odredite matricu prijelaznih vjerojatnosti.

6. **(6 bodova)**
**a)** Napiši definiciju bitnog skupa stanja Markovljevog lanca.
**b)** Napravi klasifikaciju stanja za Markovljeve lance zadane sljedećim matricama prijelaznih vjerojatnosti (na mjestu zvjezdica su pozitivni brojevi).

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & * & 0 & * & 0 \\ * & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} 0 & * & * & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & * & 0 & * \end{bmatrix}$$

**Dozvoljena je upotreba kalkulatora.**

MI 2014.    STOHA

1.

| $X \backslash Y$ | -1 | 0 | 1 | |
|---|---|---|---|---|
| 1 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ | $\frac{1}{3}$ $p_1$ |
| 2 | $\frac{2}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ | $\frac{5}{12}$ $p_2$ |
| 3 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{4}$ $p_3$ |
| | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{5}{12}$ | 1 |
| | $q_1$ | $q_2$ | $q_3$ | |

a) $X \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{5}{12} & \frac{1}{4} \end{pmatrix}$ $Y \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{4} & \frac{5}{12} \end{pmatrix}$

UVJETNA RAZDIOBA : $p_{i|j} = \frac{r_{ij}}{q_j}$ $q_{j|i} = \frac{r_{ij}}{p_i}$

b) $E(X|Y)$, $E(Y|X) = ?$

$E(X) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{5}{12} + 3 \frac{1}{4} = \frac{23}{12}$

$E(Y) = \frac{1}{12}$

$\frac{r_{ij}}{q_j}$

$E(X|Y=-1) = 1 \cdot \frac{\frac{1}{12}}{\frac{1}{3}} + 2 \cdot \frac{\frac{2}{12}}{\frac{1}{3}} + 3 \cdot \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4} + 1 + \frac{3}{4} = 2$

$E(X|Y=0) = 1 \cdot \frac{\frac{1}{12}}{\frac{1}{4}} + 2 \frac{\frac{1}{12}}{\frac{1}{4}} + 3 \frac{\frac{1}{12}}{\frac{1}{4}} = 6 \cdot \frac{\frac{1}{12}}{\frac{1}{4}} = 2$

$E(X|Y=1) = 1 \cdot \frac{\frac{2}{12}}{\frac{5}{12}} + 2 \cdot \frac{\frac{2}{12}}{\frac{5}{12}} + 3 \cdot \frac{\frac{1}{12}}{\frac{5}{12}} = \frac{6}{5} + \frac{3}{5} = \frac{9}{5}$

$E_{(X|Y)} \sim \overbrace{\begin{pmatrix} E(X|Y=-1) & E(X|Y=0) & E(X|Y=1) \\ q_1 & q_2 & q_3 \end{pmatrix}}^{\text{zajedno}}$

$E(X|Y) \sim \begin{pmatrix} \frac{9}{5} & 2 \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}$

$\frac{r_{ij}}{p_i}$

$E(Y|X=1) = -1 \cdot \frac{\frac{1}{12}}{\frac{1}{3}} + 0 \cdot \frac{\frac{1}{12}}{\frac{1}{3}} + 1 \cdot \frac{\frac{2}{12}}{\frac{1}{3}} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

$E(Y|X=2) = -1 \cdot \frac{\frac{2}{12}}{\frac{5}{12}} + 0 \cdot \frac{\frac{1}{12}}{\frac{5}{12}} + 1 \frac{\frac{2}{12}}{\frac{5}{12}} = 0$

$E(Y|X) \sim \begin{pmatrix} E(Y|X=1) & E(Y|X=2) & E(Y|X=3) \\ p_1 & p_1 & p_3 \end{pmatrix}$

$E(Y|X=3) = -1 \cdot \frac{\frac{1}{12}}{\frac{1}{4}} + 0 \cdot \frac{\frac{1}{12}}{\frac{1}{4}} + 1 \cdot \frac{\frac{1}{12}}{\frac{1}{4}} = 0$

$E(Y|X) \sim \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

② Ponavlja se pokus $p \in (0,1)$ uspjeh

$X$ = neuspješni pokus do prve pojave uspjeha $p$ uspjelo je

$Y$ - broj uspjeha u prvih $n$ ponavljanja

funkcije izvodnice za $X$ i $Y$ te disperzije pomoću njih

$$\psi(z) = c_0 + c_1 z + c_2 z^2 + \dots = \sum_{n=0}^{\infty} c_n z^n \qquad \boxed{\psi(z) = \sum_{k=0}^{\infty} p_k z^k}$$

$$\psi(z) = E(z^X) = \sum_{n=0}^{\infty} p_n z^n$$

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n & \dots \\ p & pq & pq^2 & pq^3 & & pq^n & \end{pmatrix}$$

$$X: \quad \psi(z) = \underline{\sum_{k=0}^{\infty} p_k z^k} = \underline{\sum_{k=0}^{\infty} p q^k z^k} = \underline{p \sum_{k=0}^{\infty} (qz)^k = \frac{p}{1-qz}}$$

GEOMETRIJSKA RAZDIOBA

$X \sim \mathcal{G}(p)$

broj ponavljanja pokusa dok se ne dogodi $A$

BINOMNA RAZDIOBA: broj pojavljivanja $A$ u $n$ događaja

$$Y \sim B(n,p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$\psi(z) = \underline{\sum_{k=0}^{\infty} p_k z^k} = \underbrace{\sum_{k=0}^{\infty} \binom{n}{k} p^k z^k q^{n-k}}_{(pz)^k} = \underline{(pz+q)^n}$$

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$D(X), D(Y) = ?$

$$\text{OČEKIVANJE} \quad \psi'(z) = \sum_{n=1}^{\infty} n p_n z^{n-1} \quad \Rightarrow \quad E(X) = \sum_{n=1}^{\infty} n p_n = \psi'(1)$$

$$\text{DISPERZIJA} \quad \psi''(z) = \sum_{n=2}^{\infty} n(n-1) p_n z^{n-2} \qquad \psi''(1) = \sum_{n=0}^{\infty} n(n-1) p_n = E(X(X-1)) = E(X^2 - X)$$

$$= E(X^2) - E(X)$$

$$= E(X^2) - \psi'(1)$$

$$E(X^2) = \psi''(1) + \psi'(1)$$

$$D(X) = E(X^2) - E(X)^2 = \psi''(1) + \psi'(1) - (\psi'(1))^2$$

$$\frac{pq}{(q-1)^2} = \frac{pq}{(-p)^2} = \frac{q}{p}$$

2. nastavak

$$D(X) = ? \quad \Psi(z) = \frac{p}{1-qz} \quad \Psi'(z) = \frac{-qp}{(qz-1)^2} \quad \Psi'(1) = \frac{-qp}{(q-1)^2} = \frac{-qp}{p^2} = -\frac{q}{p} =$$

ili: se: $\frac{-p}{qz-1}$

$$\Psi''(z) = -\frac{qp \cdot 2(1-qz) \cdot (-q)}{(1-qz)^{4\,3}} \quad \Psi''(1) = \frac{-2q^2 p}{(1-q)^3} = \frac{-2q\,p}{(-p)p^2} = \frac{2q}{p^2}$$

$$D(X) = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{2q^2 + qp - q^2}{p^2} = \frac{q(p+q)}{p^2} = \underline{\underline{\frac{q}{p^2}}}$$

ČUDNE OPERACIJE

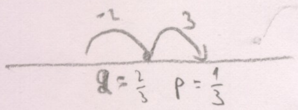
$$D(Y) = \quad \Psi(z) = (pz+q)^n \quad \Psi'(z) = n(pz+q)^{n-1} \cdot p \quad \Psi'(1) = np$$

$$\underline{\Psi''(1) + \Psi'(1) - (\Psi'(1))^2} \quad \Psi''(z) = n(n-1)p^2(pz+q)^{n-2} \quad \Psi''(1) = n(n-1)p^2$$

$$D(Y) = n(n-1)p^2 + np - n^2p^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = -np(p-1) = \underline{npq}$$

(3) 20 zadataka točno-netočno pogreške $\frac{2}{3}$ točno rješenje 3, netočno -1 -2

funkcije izvodnica, $P = ?$ 0 bodova



$n = 20$

$$X \sim \begin{pmatrix} -2 & 3 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad \Psi_{S_n}(z) = \Psi_{X_1}(z) \cdots \Psi_{X_n}(z) = \Psi_X(z)^n$$

$$\underline{\Psi(z) = \left(\frac{2}{3} z^{-2} + \frac{1}{3} z^{3}\right)^{20}}$$

$$= \left(\frac{2}{3}\right)^{20} z^{-40} \sum_{k=0}^{20} \binom{20}{k} \left(\frac{z^5}{2}\right)^k$$

za 0 bodova traži se član uz $z^0$

$$k = 8 \quad \binom{20}{8} \cdot \frac{1}{2^8} \cdot \frac{2^{20}}{3^{20}} = \frac{2^{12}}{3^{20}} \cdot \binom{20}{8}$$

$z^{5 \cdot 8} = z^{40}$, $z^{40} \cdot z^{-40} = z^0$

označava se

$$P(X=0) = [z^0]\,\Psi(z) = \ldots$$

4 a) Chapman - Kolmogorovljeve jednadžbe za homogeni Markovljev lanac

$$\underline{p_{ij}(m) = \sum p_{ik}(r)\, p_{kj}(m-r)}$$ $r = 1, 2, \ldots m-1$

Prijelazne vjerojatnosti

$$\underline{P(X_{n+1}=j \mid X_n=i)}$$

matrično $$\underline{\Pi(m) = \Pi(r)\,\Pi(m-r)}$$

$$\underline{\Pi(m) = \Pi^m}$$

$$\underline{p_{ij} := P(X_{n+1}=j \mid X_n=i)}$$
$$\underline{= P(X_1=j \mid X_0=i)}$$

Dokaz

$$p_{ij}(m) = P(X_m=j \mid X_0=i) = \frac{P(X_m=j, X_0=i)}{P(X_0=i)}$$

$$= \sum_k \frac{P(X_m=j, X_{m-1}=k, X_0=i)}{P(X_0=i)} \quad / \cdot P(X_{m-1}=k, X_0=i) \quad / \; P(X_{m-1}=k, X_0=i)$$

$$= \sum_k P(X_{m-1}=k \mid X_0=i) \cdot P(X_m=j \mid X_0=i, X_{m-1}=k)$$

Zbog markovljevog svojstva (odsustvo pamćenja)

prijelazne vjerojatnost $\rightarrow$ $$p_{ij}(m) = \sum_k (X_{m-1}=k \mid X_0=i) \cdot P(X_m=j \mid X_{m-1}=k)$$ [nema $X_0=i$, "prošlost" od $X_{m-1}$]

$$= \sum_k p_{ik}(m-1)\, p_{kj}$$

$$\Pi(m) = \Pi(m-1)\,\Pi$$

$$\Pi(m) = \Pi(m-2)\,\Pi^2 = \ldots \Pi^m$$

$$\underline{\Pi(m) = \Pi^m = \Pi^r \cdot \Pi^{m-r} = \Pi(r) \cdot \Pi(m-r)}$$ ■

4 b) Ergodički homogeni Markovljev lanac - računanje stac. vjerojatnosti

Igra p. $\Pi_j = \lim_{n\to\infty} p_{ij}(n)$ $\quad \Pi_j$ - stacionarna vrijednost

$$p(n) = p(0)\Pi^n = p(0)\Pi(n)$$

$$p_j(n) = \sum_i p_i(0)\, p_{ij}(n)$$

$$\lim_{n\to\infty} p_j(n) = \lim_{n\to\infty} \sum_i p_i(0)\, p_{ij}(n) = \underbrace{\sum_i p_i(0)}_{1} \cdot \Pi_j = \Pi_j$$

S više stanja:

$$p(n) = p(n-1)\Pi$$

$$p_j(n) = \sum_k p_k(n-1)\, p_{kj}$$

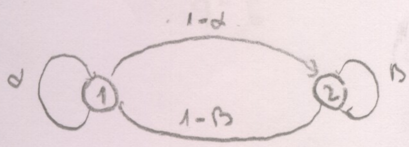
$$\Pi_j = \sum_k \Pi_k\, p_{kj} \quad \forall j$$

$$\sum_k \Pi_k = 1$$

matrično: $\Pi^T \Pi = \Pi$

$$\sum_j \Pi_j = 1$$

c) Stac. vjerojatnosti općenitog Markovljevog lanca s 2 stanja, prijelaz iz svakog stanja u svako stanje u jednom koraku




$\alpha = P(X_1 = 1 \mid X_0 = 1)$ $\quad 1-\alpha = P(X_1 = 2 \mid X_0 = 1)$ $\quad 0<\alpha,\beta<1$

$\beta = P(X_1 = 2 \mid X_0 = 2)$ $\quad 1-\beta = P(X_1 = 1 \mid X_0 = 2)$

$$\Pi = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \end{array}$$

SVOJSTVENE VRIJEDNOSTI

$$\det|\lambda I - \Pi| = \begin{vmatrix} \lambda-\alpha & -1+\alpha \\ -1+\beta & \lambda-\beta \end{vmatrix}$$

$$= \lambda^2 - (\alpha+\beta)\lambda - 1 + \alpha + \beta$$

$\lambda_1 = 1$

$\lambda_2 = \alpha + \beta - 1$

$\Downarrow$

svojstveni vektori

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha-1 \\ 1-\beta \end{pmatrix}$

S - matrica svojstvenih vrijednosti

$$\Pi = SDS^{-1}$$

$$\Pi^n = SD^nS^{-1}$$

D - dijag. matrica svoj. vrij.

$$\Pi^n = \begin{bmatrix} 1 & \alpha-1 \\ 1 & 1-\beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\alpha+\beta-1)^n \end{bmatrix} \begin{bmatrix} 1 & \alpha-1 \\ 1 & 1-\beta \end{bmatrix}^{-1}$$

⋮

str 8. (u knjižici)

⋮

$$\Pi_1 = \frac{1-\beta}{2-\alpha-\beta} \qquad \Pi_2 = \frac{1-\alpha}{2-\alpha-\beta}$$

⑤ $n$ žetona $1 \leq n \leq M$ gubitak žetona s istom vj kee i dobivanje, igra prestaje kad igrač ima $M$ ili $0$ žetona

a) formula za očekivanu duljinu igre

$$Y_n = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$d_n = p(1+d_{n+1}) + q(1+d_{n-1})$$

$$p d_{n+1} = d_n - q d_{n-1} - 1$$

$p = q = \frac{1}{2}$

$$\frac{1}{2} d_{n+1} = d_n - \frac{1}{2} d_{n-1} - 1$$

hom. $\rightarrow d_n = A + Bn$

part. $\rightarrow d_{n_p} = Cn^2$ (dvostruka nultočka karak. jedn.)

$$d_0 = d_M = 0$$

$$d_n = Mn - n^2 \qquad d_n = n(s-n)$$

$p \neq q$

$$d_n = A + B\left(\frac{q}{p}\right)^n + d_p$$

partikularno rješenje $\rightarrow d_n = \alpha n$

$$p\alpha(n+1) = \alpha n - q\alpha(n-1) - 1$$

$$\alpha = \frac{1}{q-p}$$

$$d_n = A + B\left(\frac{q}{p}\right)^n + \frac{n}{q-p}$$

$$d_0 = 0$$

$$d_s = 0$$

$$d_n = \frac{n}{q-p} - \frac{s}{q-p} \cdot \frac{1-\left(\frac{q}{p}\right)^n}{1-\left(\frac{q}{p}\right)^s}$$

$p$ - dobitak

$q$ - gubitak

55. b) Matrica prijelaznih vjerojatnosti $S=\{0,1,\dots,M\}$ $\underbrace{\phantom{0,1,\dots,M}}_{M+1 \text{ vrijednost}}$

$$p_{00}=P\{X_n=0 \mid X_{n-1}=0\}=1$$

$$p_{mm}=P\{X_n=m \mid X_{n-1}=m\}=1$$

$$p_{i,i+1}=P\{X_n=i+1 \mid X_{n-1}=i\}=p$$

$$p_{i,i-1}=P\{X_n=i-1 \mid X_{n-1}=i\}\ q$$

$p_{ij}=0$ ostalo

$$\Pi=\begin{bmatrix} 1 & 0 & \dots & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} \in M_{(M+1)\times(M+1)}$$

⑥ a) Def. bitnog skupa stanja Markovljevog lanca

bitno stanje $i \to j \Rightarrow j \to i \quad \forall j$ ; $i \to j \quad j \not\to i \quad \exists j$ - NEBITNO STANJE

$H \subset S$ bitni skup : - svako stanje $i \in H$ bitno

- $i,j \in H \quad i \leftrightarrow j$ (međusobno dostižna)
- $i \in H \quad j \notin H \Rightarrow i \not\to j$

b) KLASIFIKACIJA STANJA ZA MARKOVLJEVE LANCE $(* > 0)$

povezanost u jednom koraku

$$\begin{bmatrix} x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & x & 0 & x \\ x & x & 0 & x & 0 \\ x & 0 & 0 & 0 & x \end{bmatrix}$$

$1\to1 \quad 1\to2$

$2\to4$

$3\to3 \quad 3\to5$

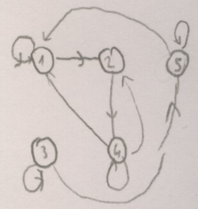
$4\to1 \quad 4\to2 \quad 4\to4$

$5\to1 \quad 5\to5$

$1\to2\to4\to1$

$\{1,2,4\} \to$ bitni skup

$3,5 \to$ nebitna stanja



$$\begin{bmatrix} 0 & x & x & 0 & 0 & x \\ 0 & x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & 0 & 0 \\ 0 & x & 0 & x & 0 & x \end{bmatrix}$$

$1\to2 \quad 1\to3 \quad 1\to6$

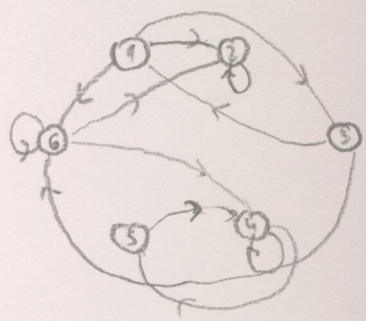
$2\to2$

$3\to1 \quad 3\to6$

$4\to4 \quad 4\to5$

$5\to4$

$6\to2 \quad 6\to4 \quad 6\to6$



bitni skupovi: $\{2\}$, $\{4,5\}$

nebitna stanja $1,3,6$