KORELACIJSKA TEORIJA STOHASTIČKIH PROCESA

$$\hat{X}(u) := \int_{-\infty}^{\infty} X(t) e^{-iut} \, dt$$

$$X \mapsto \hat{X}$$

$$F(s) = \int_{\Omega} f(t) \underbrace{K(s,t)}_{\text{jezgra}} dt$$ – INTEGRALNE TRANSFORMACIJE
– osnovno svojstvo – linearnost

$$f \mapsto F$$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(u) e^{iut} \, du$$

$$\int_{-\infty}^{\infty} |X(t)| \, dt < \infty$$ – uvjet

$X(t)$: proces

$T > 0$

$$X_T(t) = \begin{cases} X(t), & -T < t < T \\ 0, & \text{inače} \end{cases}$$

– $X_T(t)$ je integrabilno

$$\int_{-\infty}^{\infty} |X(t)| \, dt = \int_{-T}^{T} |X(t)| \, dt < \infty$$

$$\hat{X}_T(u) = \int_{-\infty}^{\infty} X_T(t) e^{-iut} \, dt = \int_{-T}^{T} X(t) e^{-iut} \, dt$$

ENERGIJA

$$E(t) = \int_{-\infty}^{\infty} X_T(t)^2 \, dt = \int_{-T}^{T} X(t)^2 \, dt < \infty =$$

$$\hookrightarrow = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{X}_T(u)|^2 \, du$$

SNAGA

$$P(T) = \frac{1}{2T} \int_{-T}^{T} X(t)^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\hat{X}(u)|^2}{2T} \, du$$

SLUČAJNE VARIJABLE

VREMENSKO USREDNJENJE

$$R_{XX} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E}(X^2(t)) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T \to \infty} \frac{\mathbb{E}|\hat{X}_T(u)|^2}{2T}}_{S_{XX}(u)} \, du$$

$$= A[E(X^2(t))]$$

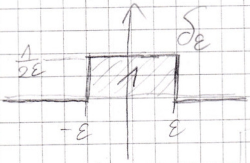
OPERATOR

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(u)\,du$$

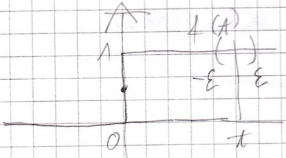
SPEKTRALNA GUSTOĆA
SNAGE PROCESA

$$A[f(t)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\,dt$$

$$\frac{1}{2T}\int_{-T}^{T} f(t)\,dt$$

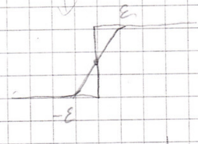


$$f * \delta_\varepsilon = \int_{-\infty}^{\infty} f(\tau)\,\delta_\varepsilon(t-\tau)\,d\tau = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} f(\tau)\,d\tau = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t+\varepsilon)\,d\tau$$

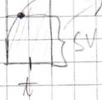


$$(f * \delta_\varepsilon)(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(\tau)\,d\tau = \frac{1}{2}$$

postala neprekinuta f-ja



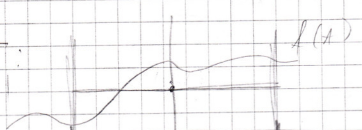
$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t+\varepsilon)\,d\tau \Rightarrow$$



izrazom se dobiva srednja vrijednost

- integral "popravlja" fju (zaglađuje je, postaje prekinuta)

za T :



PRIMJER

$$X(t) = A\cos(w_0 t + \vartheta)$$

$$\vartheta \sim U\left[0, \frac{\pi}{2}\right]$$

$$\mathbb{E}[X(t)^2] = \mathbb{E}[A^2\cos^2(w_0 t + \vartheta)] =$$

$$= \mathbb{E}\left[\frac{A^2}{2} + \frac{A^2}{2}\cos(2w_0 t + 2\vartheta)\right] =$$

$$= \frac{A^2}{2} + \frac{A^2}{2}\int_0^{\frac{\pi}{2}} \cos(2w_0 t + 2h)\cdot\frac{2}{\pi}\,dh =$$

$$= \frac{A^2}{2} + \frac{A^2}{\pi}\sin(2w_0 t)$$

$$P_{XX} = \underset{\substack{\nearrow \\ \text{vrem.} \\ \text{usred.}}}{A}\left[\mathbb{E}(X(t)^2)\right] = \lim_{T\to\infty}\frac{1}{2T}\underbrace{\int_{-T}^{T}\frac{A^2}{2} + \frac{A^2}{\pi}\sin(2w_0 t)\,dt}_{} =$$

$$2T\cdot\frac{A^2}{2} - \frac{A^2}{\pi}\Big[\underbrace{-\cos(2w_0 T) + \cos(2w_0 T)}_{0}\Big]\cdot\frac{1}{2\ldots}$$

$$\Rightarrow P_{XX} = \frac{A^2}{2}$$

$$\hat{X}_T(w) = \int_{-T}^{T} A\cos(w_0 t + \vartheta)e^{-iwt}\,dt =$$

$$= \frac{A}{2}e^{i\vartheta}\int_{-T}^{T} e^{i(w-w_0)t}\,dw + \frac{A}{2}e^{-i\vartheta}\int_{-T}^{T} e^{i(w+w_0)t}\,dt =$$

$$= AT e^{i\vartheta}\frac{\sin(w_0-w)T}{(w_0-w)T} + AT e^{-i\vartheta}\frac{\sin(w_0+w)T}{(w_0+w)T}$$

$$\mathbb{E}|\hat{X}_T(w)|^2 = \int_0^{\frac{\pi}{2}} \hat{X}_T(w)\,\overline{\hat{X}_T(w)}\,\frac{2}{\pi}\,dh$$

$$S_{XX}(u) = \lim_{T\to\infty} \frac{\mathbb{E}|\hat{X}_T(u)|^2}{2T} = \frac{A^2\pi}{2}\left[\delta(u-u_0)+\delta(u+u_0)\right]$$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(u)\,du = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\frac{A^2\cancel{\pi}}{2}\left[\delta(u-u_0)+\delta(u+u_0)\right]du =$$

$$= \frac{A^2}{4}\left[1+1\right] = \frac{A^2}{2}$$

SVOJSTVA:

1) $S_{XX} \geq 0$ - pozitivnost

2) $S_{XX}(-u) = S_{XX}(u)$ - simetričnost

3) $S_{XX}(u)$ - realna f-ja

4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(u)\,du = A\left[\mathbb{E}[X(t)^2]\right]$

$S_{XX}(u),\ A\left[R_{XX}(t, t+\tau)\right]$

AUTOKORELACIJSKA F-JA

$$A\left[R_{XX}(t, t+\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(u)\,e^{iu\tau}\,du$$

$$S_{XX}(u) = \int_{-\infty}^{\infty} A\left[R_{XX}(t, t+\tau)\right] e^{-iu\tau}\,d\tau$$

$$S_{XX}(u) = \lim_{T\to\infty}\frac{1}{2T}\mathbb{E}\left(|\hat{X}_T(u)|^2\right) =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\mathbb{E}\left(\hat{X}_T(u)\,\overline{\hat{X}_T(u)}\right) =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\underbrace{\mathbb{E}\left[X(t_1)X(t_2)\right]}_{R_{XX}(t_1,t_2)} e^{-iut_1}e^{iut_2}\,dt_1\,dt_2 =$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\,e^{-iu(t_1-t_2)}\,dt_1\,dt_2 = \begin{bmatrix} t = t_2 \\ t_1 = t+\tau = t_2+\tau \\ dt_1\,dt_2 = dt\,d\tau \end{bmatrix}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\int_{-T-t}^{T-t} R_{xx}(t+\tau, t) e^{-i\omega\tau} d\tau \right) dt =$$

$$= \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau) dt \right) e^{-i\omega\tau} d\tau =$$

$$= \int_{-\infty}^{\infty} A\left[R_{xx}(t, t+\tau) \right] e^{-i\omega\tau} d\tau$$

Ako je $\overbrace{R_{xx}(t, t+\tau) = R_{xx}(\tau)}^{\text{svojstvo stacionarnosti}}$, onda je i usrednjenje stac.

$$\boxed{\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau \\ R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega \end{aligned}}$$

PRIMJER 2:

$$X(t) = A\cos(\omega_0 t + \vartheta)$$

$$\vartheta \sim U[0, 2\pi]$$

$$R_{xx}(t, t+\tau) = \mathbb{E}\left[X(t) X(t+\tau) \right] =$$

$$= A^2 \int_0^{2\pi} \cos(\omega_0 t + h) \cos(\omega_0 (t+\tau) + h) \frac{1}{2\pi} dh =$$

$$= \frac{A^2}{2} \cos \omega_0 \tau \rightarrow \text{ovisi samo o } \tau \rightarrow \text{stacionaran proces}$$

$$= \frac{A^2}{4} \left[e^{i\omega_0\tau} + e^{-i\omega_0\tau} \right]$$

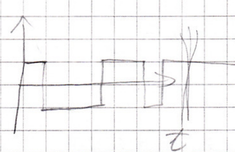
$$R_{xx} \circ\!\!-\!\!\bullet S_{xx}$$

$$e^{i\omega_0\tau} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{-i\omega t} d\omega$$

$$S_{xx} = \frac{A^2 \pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$$

PRIMJER 3 (Telegrafski signal)

$$X(t) = (-1)^{N(t)}, \quad N \sim \mathcal{P}(\lambda),$$



$$X(t) \sim \begin{pmatrix} -1 & 1 \\ e^{-\lambda t} sh\lambda t & e^{-\lambda t} ch \lambda t \end{pmatrix}$$

$$P(X(t)=1) = P(N(t)=0) + P(N(t)=2) + \cdots =$$
$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!}\right] =$$
$$= e^{-\lambda t} ch \lambda t = \frac{1}{2}\left[1 + e^{-2\lambda t}\right]$$

$$P(X(t)=-1) = \frac{1}{2}\left[1 - e^{-2\lambda t}\right]$$

$$\mathbb{E}[X(t)] = e^{-2\lambda t}$$

$$P(X(t_2)=1 \mid X(t_1)=1) = e^{-\lambda\tau} ch \lambda\tau, \quad \tau = t_2 - t_1$$

$$P(X(t_1)=1, X(t_2)=1) = P(X(t_1)=1) \cdot P(X(t_2)=1 \mid X(t_1)=1) =$$
$$= e^{-\lambda t_1} ch\lambda t_1 \cdot e^{-\lambda\tau} ch\lambda\tau$$

$$R_{XX}(t_1,t_2) = \mathbb{E}(X(t_1)X(t_2)) = 1\cdot 1 \cdot P(X(t_1)=1, X(t_2)=1) + 1\cdot(-1)\cdot P(\ldots) +$$
$$+ (-1)\cdot 1 \cdot P(\ldots) + (-1)(-1) P(\ldots) =$$
$$= e^{-2\lambda(t_2-t_1)}$$

nije stac. zbog očekivanja

$R_{XX}(t, t+\tau) = e^{-2\lambda\tau}$ → da bi bio stacionaran, mora ga se simetrizirati:

$$Y(t) = A\, X(t), \quad A \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$R_{YY}(t,t+\tau) = \mathbb{E}\left[\underbrace{A^2}_{=1} X(t)X(t+\tau)\right] = R_{XX}(\tau)$$

$$\mathbb{E}[Y(t)] = \underset{\substack{\| \\ 0}}{\mathbb{E}(A)} \cdot \mathbb{E}(X(t)) = 0$$

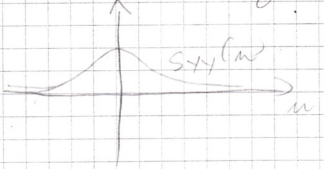
$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau =$$

ako je proces realan (R_{XX} realno) $\quad S_{XX} = \int_{-\infty}^{\infty} R_{XX}(\tau)\left[\cos(\omega\tau) - i\sin(\omega\tau)\right] d\tau$

$R_{XX}(-\tau) = R_{XX}(\tau) \Rightarrow$ dio sa sin iščezava

$$= 2\int_0^{\infty} e^{-2\lambda\tau} \cdot \cos\omega\tau \, d\tau = \int_0^{\infty}\left[e^{-(2\lambda - i\omega)\tau} + e^{-(2\lambda + i\omega)\tau}\right.$$

$$= \frac{4\lambda}{4\lambda^2 + \omega^2}$$



PRIMJER

$$Y(t) = X(t) \cdot C \cdot \cos(\omega_0 t)$$

$X(t) \rightarrow \boxed{} \rightarrow Y(t)$

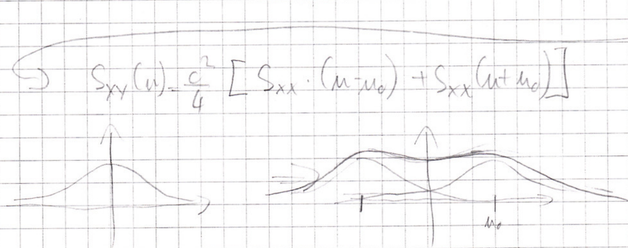
$$R_{YY}(t, t+\tau) = E\left[Y(t)Y(t+\tau)\right] =$$

$$= C^2 E\left[X(t)X(t+\tau)\right] \cdot \cos(\omega_0 t) \cdot \cos(\omega_0(t+\tau)) =$$

$$= \frac{C^2}{2} R_{XX}(\tau)\left[\cos\omega_0\tau + \cos(2\omega_0 t + \omega_0\tau)\right]$$

$$A\left[R_{YY}(t, t+\tau)\right] = \frac{C^2}{2} R_{XX}(\tau)\cos\omega_0\tau = \frac{C^2}{2} R_{XX}(\tau) \cdot \frac{e^{i\omega_0\tau} + e^{-i\omega_0\tau}}{2}$$

$$\Rightarrow S_{XY}(\omega) = \frac{C^2}{4}\left[S_{XX}(\omega - \omega_0) + S_{XX}(\omega + \omega_0)\right]$$



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$t \mapsto R_{XX}(t, t+\tau)$

$A(R_{XX}(t,t+\tau)) = \underbrace{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t,t+\tau)\,dt}_{f(\tau)}$ = (ako stacionaran) R_{XX}

$\int_{-\infty}^{\infty} A(R_{XX}(t,t+\tau)\, e^{-iu\tau} d\tau) = S_{XX}(u)$

$\hat{1} = 2\pi\delta(u)$ → isto kao, samo $f(u) = \delta(u)$

$\widehat{e^{i\tau a}} = 2\pi\delta(u-a) \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut} \underbrace{\delta(u-a)}_{f(u)} = \delta(u-a)$ IFT

$\frac{1}{2\pi} e^{i\tau a} = \delta(u-a)$ / $\mathcal{F}$

$\frac{1}{2\pi}\widehat{e^{i\tau a}} = \delta(u-a)$

ŠUM (BIJELI ŠUM)

$X(t)$ → ŠUM

$Z(t) = X(t) + Y(t)$

$R_{ZZ}(t,t+\tau) = E[\overset{X(t)+Y(t)}{Z(t)}\ \overset{X(t+\tau)+Y(t+\tau)}{Z(t+\tau)}] = \dots = E[X(t)X(t+\tau)] +$
$+ E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)]$
$+ E[Y(t)Y(t+\tau)] =$

$= R_{XX}(t,t+\tau) + \underbrace{R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau)}_{\text{KROS-KORELACIJSKE FUNKCIJE}} + R_{YY}(t,t+\tau)$

$R_{XY} \neq R_{YX}$ u općenitom slučaju

$A_t \,/\, \mathcal{F}_x$

$$S_{ZZ}(u) = S_{XX}(u) + \underbrace{S_{XY}(u) + S_{YX}(u)}_{?} + S_{YY}(u)$$

$$S_{XY}(u) = \lim_{T\to\infty} \frac{1}{2T} E\left[\overline{\hat{X}_T(u)} \cdot Y_T(u)\right]$$

$$S_{YX}(u) = \lim_{T\to\infty} \frac{1}{2T} E\left[\overline{\hat{Y}_T(u)} \cdot X_T(u)\right]$$

UZAJAMNE SPEKTRALNE GUSTOĆE SNAGE

UZAJAMNA PROSJEČNA SNAGA

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[X(t)\, Y(t)\right] dt$$

Parsevalova jednakost

$$2\pi \int f(t)\, g(t)\, dt = \int \hat{f}(u)\, \overline{\hat{g}(u)}\, du$$

$$= \int_{-\infty}^{\infty} \hat{X}_T(u) \cdot \overline{\hat{Y}_T(u)}$$

$P_{XY} = P_{YX}$ (za realne procese)

ŠUM:

- stacionaran, Gaussov, centriran ($\mu = 0$)
- $R_{XX}(\tau)$ vrlo brzo pada u 0 kod τ raste

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]\, E[X(t+\tau)]$$

$$\approx 0 \text{ za } \tau \gg 0$$

$\Rightarrow$ $X(t)$ i $X(t+\tau)$ su slabo korelirane

BIJELI ŠUM:

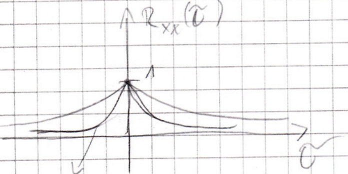
$$\boxed{R_{XX}(\tau) = \delta(\tau)}$$

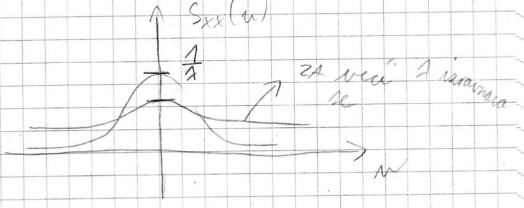
Što autokor. f-ja brže pada, spektar se širi (zastupljena su sve veće frekvencije).

PRIMJER: centrirani telegrafski signal

$$R_{XX}(\tau) = e^{-2\lambda|\tau|}$$

$$S_{XX}(u) = \frac{4\lambda}{4\lambda^2+u^2} \qquad \lambda>0$$





za veći λ brži je pad

- bijeli šum: $R_{XX}(\tau)=N_0\delta(\tau)$, $N_0\in\mathbb{R}$

$$S_{XX}(u) = \int_{-\infty}^{\infty} N_0\delta(\tau)e^{-iu\tau}\,d\tau = N_0 \quad \text{– spektar konstantan}$$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(u)\,du = \infty$$ → TAKAV PROCES ZAPRAVO NE POSTOJI

→ bijeli šum je aproksimacija za OBOJENI ŠUM

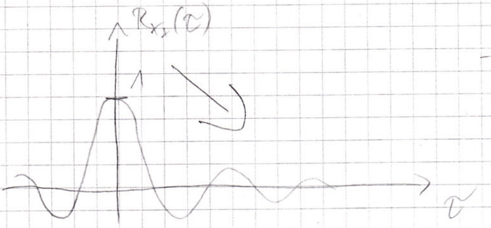
↳ zastupljen samo dio spektra (spektralna gustoća je konstanta u zanimljivom području)

$$S_{XX}(u) = \begin{cases} \frac{A}{v}, & |u|<v \\ 0, & \text{inače} \end{cases}$$

$$R_{XX}(\tau) = \check{S}_{XX}(u) = \frac{1}{2\pi}\int_{-v}^{v} \frac{A}{v} e^{iu\tau}\,du = \frac{A}{2\pi v}\cdot \frac{e^{iu\tau}}{i\tau}\Big|_{-v}^{v} =$$

$$= \frac{A}{2\pi v}\cdot\frac{e^{i\tau v}-e^{-i\tau v}}{i\tau} \overset{2\text{ – RASPIŠI SAM}}{=} \cos(-\tau v)+i\sin(-\tau v) = \cos\tau v - i\sin\tau v =$$

$$= \frac{A}{\pi}\cdot\frac{\sin(\tau v)}{\tau v}$$



- što je ν veći, brže je pad amplitude

$$R_{xx}(\tau) = \frac{\sin(\tau\nu)}{\tau\nu}$$

BIJELI ŠUM

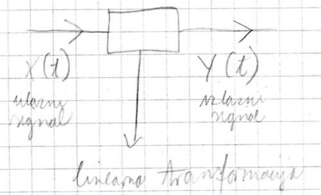
$\nu \to \infty$

$\tau = 0 \to R_{xx}(\tau) = 1$

$\tau \neq 0 \to R_{xx}(\tau) = 0, \ \tau > 0$

$\Longrightarrow R_{xx}(\tau) = \delta(\tau)$

LINEARNI SUSTAVI



$Y(t) = L(x(t))$

lin. op.:

$L(\alpha_1 x_1 + \dots + \alpha_n x_n) = \alpha_1 L(x_1) + \dots + \alpha_n L(x_n)$

$L(x(t)) = x'(t)$ - linearni operator

- integralna reprezentacija L

$$x(t) = \int_{-\infty}^{\infty} x(s)\,\delta(t-s)\,ds \quad / L$$

$$L(x(t)) = \int_{-\infty}^{\infty} x(s)\,L(\delta(t-s))\,ds$$

$h(t) = L(\delta(t))$ - impulsni odziv sustava

$$= \int_{-\infty}^{\infty} x(s)\,h(t-s)\,ds$$

$\Rightarrow$ $y(t) = x(t) * h(t)$; s.v.: $Y(t) = X(t) * h(t)$

$$f * g = g * f$$

Dokaz:

$$X(t), \quad Y(t) = L(X(t))$$

$X(t)$ stacionaran

$$\to m_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\xi) h(\xi) d\xi\right] =$$

$$= \int_{-\infty}^{\infty} m_X \cdot h(\xi) d\xi = m_X \cdot \int_{-\infty}^{\infty} h(\xi) d\xi = \text{konstanta}$$

$$R_{YY}(t, t+\tau) = E[Y(t) Y(t+\tau)] = E\left[\int_{-\infty}^{\infty} X(t-\xi_1) h(\xi_1) d\xi_1 \cdot \right.$$

$$\left. \cdot \int_{-\infty}^{\infty} X(t+\tau-\xi_2) h(\xi_2) d\xi_2\right] =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{E[X(t-\xi_1) X(t+\tau-\xi_2)]}_{R_{XX}(\tau+\xi_1-\xi_2)} h(\xi_1) h(\xi_2) d\xi_1 d\xi_2 =$$

$$= \int_{-\infty}^{\infty} h(\xi_1) \underbrace{\left[\int_{-\infty}^{\infty} R_{XX} \cdot h(\xi_2) d\xi_2\right]}_{(R_{XX} * h)(\tau+\xi_1)} d\xi_1 = \int_{-\infty}^{\infty} (R_{XX} * h)(\underbrace{\tau + \xi_1}_{\tau - (-\xi_1)}) \cdot h(\xi_1) d\xi_1 =$$

$$= \left| \begin{matrix} s = -\xi_1 \\ ds = -d\xi_1 \end{matrix} \right| = \int_{-\infty}^{\infty} (R_{XX} * h)(\tau - s) h(-s) ds =$$

$$= (R_{XX} * h)(\tau) * h(-\tau) = (R_{XX} * h * \tilde{h})(\tau), \quad \tilde{h}(t) = h(-t)$$

$\hookrightarrow$ ovisnost samo o τ $\Rightarrow$ Y stacionaran

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = \dots = R_{XX}(\tau) * h(\tau)$$

$$R_{YX}(t,t+\tau) = R_{XX}(\tau) * \tilde{h}(\tau)$$

$$\widehat{f * g}(\omega) = \hat{f}(\omega)\,\hat{g}(\omega)$$

$$e^{-i\omega(\tau-s)} \cdot e^{-i\omega s}$$

$$\widehat{f*g}(\omega) \int (f*g)(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$\Rightarrow R_{YY}(t) = R_{XX}(\tau) * h(\tau) * \tilde{h}(\tau) \quad /\ \hat{}$$

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot H(\omega) \cdot \tilde{H}(\omega)$$

$$H(\omega) = \hat{h}(\omega)$$

FUNKCIJA SUSTAVA

$$\tilde{H}(\omega) = \widehat{h(-t)} = \int_{-\infty}^{\infty} h(-\tau)\, e^{-i\omega\tau} = /\ s = -\tau\ / =$$

2x konjugacija, jedna ide pod integral

$$= \overline{\overline{\int_{-\infty}^{\infty} \underbrace{h(s)}_{\in \mathbb{R}} e^{i\omega s}\, ds}}$$

$$= \overline{\int_{-\infty}^{\infty} h(s)\, e^{-i\omega s}\, ds} = \overline{H(\omega)}$$

$$S_{YY}(\omega) = S_{XX}(\omega)\, H(\omega)\, \overline{H(\omega)} = S_{XX}(\omega)\, |H(\omega)|^2 \Big/ H(\omega)$$

$$R_{XY} = \dots \quad /\ \hat{}$$

$$R_{YX} = \dots \quad /\ \hat{}$$

$$S_{XY}(\omega) = S_{XX}(\omega)\, H(\omega)$$

$$S_{YX}(\omega) = S_{XX}(\omega) \cdot \overline{H(\omega)}$$

$\Rightarrow$ Kako odrediti $H(\omega)$ za $y(t) = L(x(t))$

– podmetnemo $x(t) = e^{i\omega t}$ od $H(\omega)$

$$y(t) = \int_{-\infty}^{\infty} x(t-\xi)\, h(\xi)\, d\xi = \int_{-\infty}^{\infty} e^{i\omega(t-\xi)} h(\xi)\, d\xi =$$

$$= \underbrace{e^{i\omega t}}_{x(t)} H(\omega)$$

$$\Rightarrow \boxed{H(t) = \frac{y(t)}{x(t)} = e^{i\omega t}}$$

$\Rightarrow$ samo ako podmetnemo taj signal

PRIMJER 10

Dan je sustav $y(t) = x'(t)$. Neka je $x(t)$ stacionaran.

R_{YY}, R_{XY}, R_{YX}

$L(x(t)) = x'(t)$ – linearan operator $\Rightarrow$ linearan sustav

1°) $H(\omega) = i\omega$ $\Leftarrow$ $x(t) = e^{i\omega t}$, $y(t) = x'(t) = i\omega e^{i\omega t}$

2°) $y(t) = x(t) * h(t) \;/\hat{}$

$\hat{y}(\omega) = \hat{x}(\omega) H(\omega)$

$y(t) = x'(t)$ $\Rightarrow$ $H(\omega) = i\omega$

$\hat{y}(\omega) = (i\omega)\hat{x}(\omega)$

$$S_{YY}(\omega) = S_{XX}(\omega) \overbrace{H(\omega)\overline{H(\omega)}}^{|H(\omega)|^2} = S_{XX}(\omega) \cdot \omega^2$$

$$S_{XY}(\omega) = S_{XX}(\omega)(i\omega)$$

$$S_{YX}(\omega) = S_{XX}(\omega)(-i\omega)$$

$$R_{YY}(\tau) = -R_{XX}'''(\tau) = R_{X'X'}$$

$$R_{XY}(t, t+\tau) = R_{XX}'(\tau) = R_{XX'}$$

$$R_{YX}(t, t+\tau) = -R_{XX}'(\tau) = R_{X'X}$$

$f(\tau) \xrightarrow{\mathcal{F}} \hat{f}(\omega)$

$f'(\tau) \xrightarrow{\mathcal{F}} -i\omega \hat{f}(\omega)$

$f''(\tau) \to (i\omega)^2 \hat{f}(\omega)$

$-\omega^2$

PRIMJER 11:

$x(t)$

$$y(t) = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} x(\tau)\, d\tau$$

IZGLAĐIVANJE ULAZNOG SIGNALA

$$S_{YY}(\omega) = ? = S_{XX}(\omega)\, |H(\omega)|^2$$

- linearan
- svođenje na konvoluciju

$$= \begin{cases} 1, & \tau \in (t-\varepsilon, t+\varepsilon) \\ 0, & \text{inače} \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{2\varepsilon} \chi_{(t-\varepsilon, t+\varepsilon)}(\tau)\, x(\tau)\, d\tau =$$

$$\chi_{(t-\varepsilon, t+\varepsilon)}(\tau) = \chi_{(-\varepsilon,\varepsilon)}(t-\tau)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\varepsilon} \chi_{(-\varepsilon,\varepsilon)}(t-\tau)\, x(\tau)\, d\tau = \left(\frac{1}{2\varepsilon} \chi_{(-\varepsilon,\varepsilon)} * x\right)(\tau)$$

$$y(t) = h(t) * x(t)$$

$$h(t) = \frac{1}{2\varepsilon} \chi_{(-\varepsilon,\varepsilon)}(t)$$

$$H(\omega) = \frac{1}{2\varepsilon} \int_{-\infty}^{\infty} \chi_{(-\varepsilon,\varepsilon)}(\tau)\, e^{-i\omega\tau}\, d\tau =$$

$$= \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} e^{-i\omega\tau}\, d\tau = \frac{1}{2\varepsilon} \left. \frac{e^{-i\omega\tau}}{-i\omega} \right|_{-\varepsilon}^{\varepsilon} = \frac{\sin(\varepsilon\omega)}{\varepsilon\omega}$$

$$S_{YY}(u) = S_{XX}(u) \cdot \frac{\sin^2(\varepsilon u)}{(\varepsilon u)^2}$$

PRIMJER dif. jdba kao lin. sustav

- izlazni signal je rješenje dif. jdbe kojoj je ulazni signal desna strana

izlaz

$$Y(t) \rightsquigarrow a_n Y^{(n)}(t) + \dots + a_1 y'(t) + a_0 y(t) = x(t)$$

ulaz

- funkcija sustava - FT

$$a_n (iu)^n \hat{y}(u) + \dots + a_0 \hat{y}(u) = \hat{x}(u)$$

$$\hat{y}(u) \cdot \left(a_n (iu)^n + \dots + a_1 (iu) + a_0\right) = \hat{x}(u)$$

$$\hat{y}(u) = \frac{1}{\quad} \cdot \hat{x}(u)$$

$H(u)$

BITNO:

- $a_n, \dots, a_0$ → KONSTANTNI KOEFICIJENTI
- nema slobodnih članova