

1 | Position of m_i

In a rigid body consisting of N point masses, the vector to the position of m_i is defined as $\vec{r}_i(t)$, which is defined as follows:

$$\vec{r}_i(t) = \vec{R}_{CM}(t) + \vec{r}_i'(t) \quad (1)$$

whereas, $\vec{R}_{CM}(t)$ is the position vector of the center of mass of the rigid body as a whole, and $\vec{r}_i'(t)$ the vector from the center of mass to m_i .

2 | Velocity of m_i

The velocity of m_i is simply determined by the first derivative of the position equation as per above. Namely, that:

$$\vec{v}_i(t) = \vec{V}_{CM}(t) + \vec{v}_i'(t) \quad (2)$$

where, $\vec{v}_i(t)$ is the velocity vector of m_i , and $\vec{V}_{CM}(t)$ is the velocity vector of the center of mass of the rigid body, and $\vec{v}_i'(t)$ is the velocity vector from center of mass to m_i .

3 | Deriving KE_{total}

3.1 | Setting up

From definition of KE_{total} itself, KE_{total} is the sum of all energies of each point mass in the rigid body.

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 \quad (3)$$

3.2 | Derivation, part 1

Expanding this equation and substituting the value of v_i , and additionally setting $M = \sum m_i$ (namely, that M represents the total mass of the rigid body) we could derive:

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \sum_{i=1}^N \frac{1}{2} m_i (v_i \cdot v_i) \quad (4)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i ((V_{CM}^{\rightarrow} + \vec{v}_i') \cdot (V_{CM}^{\rightarrow} + \vec{v}_i')) \quad (5)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i (V_{CM}^{\rightarrow 2} + 2 \times (\vec{v}_i' \cdot V_{CM}^{\rightarrow}) + \vec{v}_i'^2) \quad (6)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i V_{CM}^{\rightarrow 2} + \sum_{i=1}^N m_i \times (\vec{v}_i' \cdot V_{CM}^{\rightarrow}) + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (7)$$

$$= \frac{1}{2} V_{CM}^{\rightarrow 2} \sum_{i=1}^N m_i + V_{CM}^{\rightarrow} \sum_{i=1}^N m_i \vec{v}_i' + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (8)$$

3.3 | Dealing with the Middle Term

At this point, we must note that $\sum_{i=1}^N m_i \vec{v}_i' = 0$. Per the definition of the center of mass, the following holds:

$$r_{CM}^{\rightarrow} = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i \quad (9)$$

Changing reference frame to that of the center of mass itself, this equation therefore becomes:

$$r_{CM}'^{\rightarrow} = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (10)$$

It is important to realize here that $r_{CM}'^{\rightarrow} = 0$ because of the fact that — at the reference point of the center of mass, the center of mass is at a zero-vector distance away from itself.

In order to figure a statement with respect to the *velocity* of r_i' , we take the derivative of the previous equation with respect to time.

$$0 = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (11)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (12)$$

$$= \left(\frac{1}{M}\right) \sum_i m_i \vec{v}_i' \quad (13)$$

Given that $\frac{1}{M}$ could not be zero for an object with non-zero mass, it is concluded therefore that $\sum_i m_i \vec{v}_i' = 0$.

3.4 | Derivation, part 2

As $\sum_i m_i \vec{v}_i' = 0$, the KE_{total} work-in-progress equation's middle term (which contains the statement $\sum_i m_i \vec{v}_i'$) is therefore zero. Substituting that in and removing the term, we therefore result in:

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \frac{1}{2} V_{CM}^2 \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (14)$$

Replacing the definition of $M = \sum m_i$, we result in

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \frac{1}{2} M V_{CM}^2 + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (15)$$

$$KE_{total} = \frac{1}{2} M V_{CM}^2 + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (16)$$

The left term of this equation ($\frac{1}{2} M V_{CM}^2$) is the clear original statement for $KE_{translational}$. As component masses of a rigid body cannot experience translational motion about its center of origin, the second term is therefore rotational only and so $KE_{rotational}$.

Therefore:

$$KE_{total} = KE_{translational} + KE_{rotational} \quad (17)$$