0 | lets find a general formula

The problem is to find the slope of f in the θ direction at the point $P_0 \in \mathbb{R}^2$.

We want to turn this 2d problem into a 1d problem, so that we can find the derivative generally, then just apply it. For each version of the problem, we are given a function $f : \mathbb{R}^2 \to \mathbb{R}$ (with a 2d input), but we would rather have a function $h : \mathbb{R} \to \mathbb{R}$ that encodes the planar slice that we care about.

We can do this by composing f with some other $g : \mathbb{R} \to \mathbb{R}^2$ s.t. $f \circ g(t)$ is that planar slice. This way, g just has to spit out xy-coordinates in the domain of f.

For convienence, lets also ensure that $g(0) = P_0$, In summary, we want a function $g(t)$ that parameterizes a line that passes through P_0 and has slope θ, ie. goes in the direction $\langle \cos\theta, \sin\theta \rangle$.

$$g(t) = \vec{P_0} + t\langle \cos\theta, \sin\theta \rangle$$

Now, to solve any of the following problems, we can take

$$\frac{d}{dt}\Big|_0 f \circ g$$

1 | $f(x, y) = x^2 + y^2$

Before we do anything, lets find $\frac{d}{dt} f \circ g$ generally for any P_0 and θ:

$$\begin{aligned}
\frac{d}{dt} f \circ g &= \frac{d}{dt}\left((P_{0x} + t\cos\theta)^2 + (P_{0y} + t\sin\theta)^2 \right) \\
&= \frac{d}{dt}\left(P_{0x}^2 + t^2\cos^2\theta + 2P_{0x}t\cos\theta + P_{0y}^2 t^2 \sin^2\theta + 2tP_{0y}\sin\theta\right) \\
&= \frac{d}{dt}\left(P_{0x}^2 + P_{0y}^2 + t^2\left(\cos^2\theta + \sin^2\theta\right) + 2t\left(P_{0x}\cos\theta + P_{0y}\sin\theta\right)\right) \\
&= 2\cancelto{0}{t}\left(\cos^2\theta + \sin^2\theta\right) + 2\left(P_{0x}\cos\theta + P_{0y}\sin\theta\right) \\
&= 2\left(P_{0x}\cos\theta + P_{0y}\sin\theta\right)
\end{aligned}$$

Because this tells us the slope, we can find the steepest slope through optimization.

$$\frac{d}{d\theta} 2\left(P_{0x}\cos\theta + P_{0y}\sin\theta\right) = 2\left(-P_{0x}\sin\theta + P_{0y}\cos\theta\right) = 0$$

Likewise, the slope is zero for all \$$\theta$\$s s.t.

$$P_{0x}\cos\theta + P_{0y}\sin\theta = 0$$

(I don't actually know how to solve these equations, but Wolfram Alpha does! Is there a trig identity for $\sin + \cos$?)

Now, we just plug values in.

(a) 0	(b) 24	(c) 6
(d) $14\sqrt{2}$	(e) 20	(f) 19.32
(g) (above)	(h) $\frac{4\pi}{3}, \frac{\pi}{3}$	(i) (above)

2 | $f(x,y) = x^2 - y^2$

We can follow the same process as above

$$\begin{aligned} f \circ g &= (P_{0x} + t\cos\theta)^2 - (P_{0y} + t\sin\theta)^2 \\ &= P_{0x}^2 + t^2\cos^2\theta + 2tP_{0x}\cos\theta - P_{0y}^2 - t^2\sin^2\theta - 2tP_{0y}\sin\theta \\ &= P_{0x}^2 - P_{0y}^2 + t^2\left(\cos^2\theta - \sin^2\theta\right) + 2t\left(P_{0x}\cos\theta - P_{0y}\sin\theta\right) \end{aligned}$$

$$h(t) = \frac{d}{dt}\Big|_0 f \circ g = 2\left(P_{0x}\cos\theta - P_{0y}\sin\theta\right)$$

$$\frac{d}{d\theta}h = -2\left(P_{0x}\sin\theta + P_{0y}\cos\theta\right) = 0$$

$$\theta = n\pi - \tan^{-}\left(\frac{P_{0y}}{P_{0x}}\right)$$

(a) 0 (b) -24 (c) 6

(d) 0 (e) -10 (f) -5.17

(g) (above) (h) $\frac{2\pi}{3}$ steepest, add ninties for others