1 | Elastic collision

We are given that the object m_1 collides with the rod with velocity v_0, and the rod is floating in free space. Given m_1, v_0, m_2, I_0, and r, we are to figure to the final velocity of m_1 after collision v_f, the velocity of m_2 after collision v_{CM}, and of course the rotation of the rod after collision ω.

We are assuming that this collision elastic.

We have, then, for conservation of linear momentum:

$$m_1v_0 = m_1v_f + m_2v_{CM} \tag{1}$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2}m_1{v_0}^2 = \left(\frac{1}{2}m_1{v_f}^2\right) + \left(\frac{1}{2}m_2{v_{CM}}^2\right) + \left(\frac{1}{2}I_0\omega^2\right) \tag{2}$$

$$\Rightarrow m_1{v_0}^2 = \left(m_1{v_f}^2\right) + \left(m_2{v_{CM}}^2\right) + \left(I_0\omega^2\right) \tag{3}$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1r^2\left(\frac{v_0}{r}\right) = m_1r^2\left(\frac{v_f}{r}\right) + I_0\omega \tag{4}$$

$$\Rightarrow m_1rv_0 = m_1rv_f + I_0\omega \tag{5}$$

We now have a system of three equations that can be combined to solve for three unknowns v_f, v_{CM}, and ω.

We will begin with the kinetic energy expression:

$$m_1{v_0}^2 = \left(m_1{v_f}^2\right) + \left(m_2{v_{CM}}^2\right) + \left(I_0\omega^2\right) \tag{6}$$

Now, this last term seems a little concerning to deal with. Therefore, we will use the final expression to simplify it:

$$m_1rv_0 = m_1rv_f + I_0\omega \tag{7}$$

$$\Rightarrow m_1rv_0 - m_1rv_f = I_0\omega \tag{8}$$

and also:

$$\frac{m_1rv_0 - m_1rv_f}{I_0} = \omega \tag{9}$$

Therefore, $I_0\omega^2$ would be:

$$\frac{(m_1rv_0 - m_1rv_f)^2}{I_0} \tag{10}$$

Supplying that back:

$$
\begin{aligned}
& m_1{v_0}^2 = m_1{v_f}^2 + m_2{v_{CM}}^2 + \left(I_0\omega^2\right) && (11)\\
\Rightarrow & m_1{v_0}^2 = m_1{v_f}^2 + m_2{v_{CM}}^2 + \frac{(m_1rv_0 - m_1rv_f)^2}{I_0} && (12)\\
\Rightarrow & m_1{v_0}^2 = m_1{v_f}^2 + m_2{v_{CM}}^2 + \frac{(r(m_1v_0 - m_1v_f))^2}{I_0} && (13)\\
\Rightarrow & m_1{v_0}^2 = m_1{v_f}^2 + m_2{v_{CM}}^2 + \frac{r^2(m_1v_0 - m_1v_f)^2}{I_0} && (14)\\
\Rightarrow & m_1{v_0}^2 - m_1{v_f}^2 = m_2{v_{CM}}^2 + \frac{r^2(m_1v_0 - m_1v_f)^2}{I_0} && (15)\\
\Rightarrow & m_1I_0{v_0}^2 - m_1I_0{v_f}^2 = m_2I_0{v_{CM}}^2 + r^2(m_1v_0 - m_1v_f)^2 && (16)
\end{aligned}
$$

At this point, we realize that we haven't use the first expression yet. Therefore, we will endeavor to use it to rid of the one v_{CM} we have here.

$$
\begin{aligned}
& m_1v_0 = m_1v_f + m_2v_{CM} && (17)\\
\Rightarrow & m_1v_0 - m_1v_f = m_2v_{CM} && (18)\\
\Rightarrow & \frac{m_1v_0 - m_1v_f}{m_2} = v_{CM} && (19)
\end{aligned}
$$

```
var("m1 m2 I vf v0 r w vcm")
# eqn = m1*I*v0^2 - m1*I*vf^2 == m2*I*((m1*v0-m1*vf)/m2)^2 + r^2*(m1*v0-m1*vf)^2
# eqn = m1*m2*I*v0^2-m1*m2*I*vf^2 ==m2*I*(m1*v0-m1*vf)^2 + m2*r^2*(m1*v0-m1*vf)^2
# eqn = m1*m2*I*v0^2 - m1*m2*I*vf^2 == m2*I*((m1*v0)^2-2*((m1)^2*vf*v0)+(m1*vf)^2) + m2*r^2*((m1*v0)^2-2
# eqn = I*v0^2 - I*vf^2 == m1*I*(v0^2-2*vf*v0+vf^2) + m1*r^2 *(v0^2-2*vf*v0+vf^2)
# eqn = -2*I*vf^2 == -2*m1*I*vf*v0 + r^2*v0^2-2*r^2*m1*vf*v0+r^2*vf^2
# eqn = -(2*I-r^2)*vf^2 == -2*m1*vf*v0*(I+r^2)+r^2*v0^2
eqn = -(I+m1*I+m1*r^2)*vf^2 + 2*m1*v0*(I+r^2)*vf-(m1*I+m1*r^2-I)*v0^2==0
simplify(solve(eqn, vf))
```

Ok, so now, supplying all of this back:

$$
\begin{aligned}
& m_1I_0{v_0}^2 - m_1I_0{v_f}^2 = m_2I_0{v_{CM}}^2 + r^2(m_1v_0 - m_1v_f)^2 && (20)\\
\Rightarrow & m_1I_0{v_0}^2 - m_1I_0{v_f}^2 = m_2I_0\left(\frac{m_1v_0 - m_1v_f}{m_2}\right)^2 + r^2(m_1v_0 - m_1v_f)^2 && (21)\\
\Rightarrow & m_1I_0{v_0}^2 - m_1I_0{v_f}^2 = m_2I_0\frac{(m_1v_0 - m_1v_f)^2}{{m_2}^2} + r^2(m_1v_0 - m_1v_f)^2 && (22)\\
\Rightarrow & m_1m_2I_0{v_0}^2 - m_1m_2I_0{v_f}^2 = m_2I_0(m_1v_0 - m_1v_f)^2 + m_2r^2(m_1v_0 - m_1v_f)^2 && (23)\\
\Rightarrow & m_1m_2I_0{v_0}^2 - m_1m_2I_0{v_f}^2 = m_2I_0((m_1v_0)^2 - 2((m_1)^2v_fv_0) && (24)\\
& + (m_1v_f)^2) + m_2r^2((m_1v_0)^2 - 2((m_1)^2v_fv_0) + (m_1v_f)^2) && (25)\\
\Rightarrow & I_0{v_0}^2 - I_0{v_f}^2 = m_1I_0({v_0}^2 - 2v_fv_0 + {v_f}^2) + m_1r^2({v_0}^2 - 2v_fv_0 + {v_f}^2) && (26)\\
\Rightarrow & I_0{v_0}^2 - I_0{v_f}^2 = (m_1I_0{v_0}^2 - 2m_1I_0v_fv_0 + m_1I_0{v_f}^2) + (m_1r^2{v_0}^2 - 2m_1r^2v_fv_0 + m_1r^2{v_f}^2) && (27)\\
\Rightarrow & -I_0{v_f}^2 + 2m_1I_0v_fv_0 - m_1I_0{v_f}^2 + 2m_1r^2v_fv_0 - m_1r^2{v_f}^2 = (m_1I_0{v_0}^2) + (m_1r^2{v_0}^2) - I_0{v_0}^2 && (28)\\
\Rightarrow & -(I_0 + m_1I_0 + m_1r^2){v_f}^2 + 2m_1v_0(I_0 + r^2)v_f = (m_1I_0{v_0}^2) + (m_1r^2{v_0}^2) - I_0{v_0}^2 && (29)\\
\Rightarrow & -(I_0 + m_1I_0 + m_1r^2){v_f}^2 + 2m_1v_0(I_0 + r^2)v_f = (m_1I_0 + m_1r^2 - I_0){v_0}^2 && (30)\\
\Rightarrow & -(I_0 + m_1I_0 + m_1r^2){v_f}^2 + 2m_1v_0(I_0 + r^2)v_f - (m_1I_0 + m_1r^2 - I_0){v_0}^2 = 0 && (31)
\end{aligned}
$$

There are probably more elegant ways to do this, but now, its quadratic equation time!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{32}$$

$$\begin{cases} a = -(I_0 + m_1 I_0 + m_1 r^2) \\ b = 2m_1 v_0 (I_0 + r^2) \\ c = -(m_1 I_0 + m_1 r^2 - I_0){v_0}^2 \\ x = v_f \end{cases} \tag{33}$$

Let's tackle that middle term first.

$$\begin{aligned} b^2 &= (2m_1 v_0 (I_0 + r^2))^2 && (34) \\ &= 4{m_1}^2{v_0}^2(I_0 + r^2)^2 && (35) \\ &= 4{m_1}^2{v_0}^2({I_0}^2 + 2I_0 r^2 + r^4) && (36) \\ &= (4{m_1}^2{v_0}^2{I_0}^2 + 8{m_1}^2{v_0}^2 I_0 r^2 + 4{m_1}^2{v_0}^2 r^4) && (37) \end{aligned}$$

and,

$$\begin{aligned} 4ac &= 4(I_0 + m_1 I_0 + m_1 r^2)(m_1 I_0 + m_1 r^2 - I_0){v_0}^2 && (38) \\ &= 4{v_0}^2((m_1{I_0}^2 + m_1 I_0 r^2 - {I_0}^2) + ((m_1 I_0)^2 + {m_1}^2 I_0 r^2 - m_1{I_0}^2) + ({m_1}^2 r^2 I_0 + (m_1 r^2)^2 - m_1 I_0 r^2)) && (39) \\ &= 4{m_1}^2{v_0}^2{I_0}^2 + 8{m_1}^2{v_0}^2 I_0 r^2 + 4{m_1}^2{v_0}^2 r^4 - 4{I_0}^2{v_0}^2 && (40) \end{aligned}$$

Therefore, subtracting the two terms, we will note that they are only different by:

$$b^2 - 4ac = 4{I_0}^2{v_0}^2 \tag{41}$$

Finally, then; performing the square root:

$$\sqrt{b^2 - 4ac} = 2I_0 v_0 \tag{42}$$

Adding negative b to this expression:

$$\begin{aligned} &2m_1 v_0 (I_0 + r^2) + 2I_0 v_0 && (43) \\ \Rightarrow &2m_1 v_0 I_0 + 2I_0 v_0 + 2m_1 v_0 r^2 && (44) \end{aligned}$$

```
var("m1 m2 I vf v0 r w vcm")
a = -(I+m1*I+m1*r^2)
b = 2*m1*v0*(I+r^2)
c = -(m1*I+m1*r^2-I)*v0^2
expand(b^2)
expand(4*(I+m1*I+m1*r^2)*(m1*I+m1*r^2-I)*v0^2)
# expand(sqrt(expand(b^2-4*a*c)))
```

Performing the actual solution digitally:

$$v_{cm} = \frac{2I_0 m_1 v_0}{m_1 m_2 r^2 + I_0 m_1 + I_0 m_2} \tag{45}$$

$$v_f = \frac{(m_1 m_2 r^2 + I_0 m_1 - I_0 m_2) v_0}{m_1 m_2 r^2 + I_0 m_1 + I_0 m_2} \tag{46}$$

and finally, we have

$$\omega = \frac{2 m_1 m_2 r v_0}{m_1 m_2 r^2 + I_0 m_1 + I m_2} \tag{47}$$

2 | **Rigid Body Kinetic Energy**

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2} m_i {v_i}^2 \tag{48}$$

Because of the fact a point v_i can be defined as a sum of the velocity from the origin plus the displace from from origin ($v_i = v_{CM} + v_i'$), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM} + v_i')(V_{CM} + v_i') \tag{49}$$

Now, we shall foil the above expression:

$$\begin{aligned}
KE &= \sum_i \frac{1}{2} m_i ({V_{CM}}^2 + 2 v_{CM} v_i' + {v_i'}^2) && (50)\\
&= \sum_i \frac{1}{2} m_i {V_{CM}}^2 + \sum_i m_i V_{CM} v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 && (51)\\
&= \frac{1}{2} M {V_{CM}}^2 + \sum_i m_i V_{CM} v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 && (52)\\
&= \frac{1}{2} M {V_{CM}}^2 + V_{CM} \sum_i m_i v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 && (53)
\end{aligned}$$

At which point, we realize that we have in the middle arrived at the definition of the center of mass in the reference frame of the center of mass—meaning that it is indeed 0 because the center of mass is at the origin of the center of mass. Moving on, then:

$$\begin{aligned}
KE &= \frac{1}{2} M {V_{CM}}^2 + V_{CM} \sum_i m_i v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 && (54)\\
&= \frac{1}{2} M {V_{CM}}^2 + \frac{1}{2} \sum_i m_i {v_i'}^2 && (55)
\end{aligned}$$

We now note that this is a rigid body. Therefore, each point mass will not "slip" from the center of mass—they remain the same distance relative each other. Therefore, we can leverage the no slipping assumption to claim that $v_i' = r_i' \omega$.

$$KE = \frac{1}{2}MV_{CM}{}^2 + \frac{1}{2}\sum_i m_i(r_i'\omega)^2 \tag{56}$$

$$= \frac{1}{2}MV_{CM}{}^2 + \frac{1}{2}\sum_i m_i r_i'^2\omega^2 \tag{57}$$

$$= \frac{1}{2}MV_{CM}{}^2 + \frac{1}{2}\omega^2\sum_i m_i r_i'^2 \tag{58}$$

$$= \frac{1}{2}MV_{CM}{}^2 + \frac{1}{2}I\omega^2 \ \blacksquare \tag{59}$$