1 | Elastic collision

We are given that the object m_1 collides with the rod with velocity v_0, and the rod is floating in free space. Given m_1, v_0, m_2, I_0, and r, we are to figure to the final velocity of m_1 after collision v_f, the velocity of m_2 after collision v_{CM}, and of course the rotation of the rod after collision ω.

We are assuming that this collision elastic.

We have, then, for conservation of linear momentum:

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \tag{1}$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2} m_1 {v_0}^2 + \frac{1}{2} m_1 {v_0}^2 = \left(\frac{1}{2} m_1 {v_f}^2\right) + \left(\frac{1}{2} m_1 {v_f}^2\right) + \left(\frac{1}{2} m_2 {v_{CM}}^2\right) + \left(\frac{1}{2} I_0 \omega^2\right) \tag{2}$$

$$\Rightarrow 2 m_1 {v_0}^2 = \left(2 m_1 {v_f}^2\right) + \left(m_2 {v_{CM}}^2\right) + \left(I_0 \omega^2\right) \tag{3}$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1 r^2 \left(\frac{v_0}{r}\right) = m_1 r^2 \left(\frac{v_f}{r}\right) + I_0 \omega \tag{4}$$

$$\Rightarrow m_1 r v_0 = m_1 r v_f + I_0 \omega \tag{5}$$

We now have a system of three equations that can be combined to solve for three unknowns v_f, v_{CM}, and ω.

Performing the actual solution digitally:

$$v_{cm} = \frac{4 I_0 m_1 v_0}{m_1 m_2 r^2 + I_0 m_1 + 2 I_0 m_2} \tag{6}$$

$$v_f = \frac{(m_1 m_2 r^2 + I_0 m_1 - 2 I_0 m_2) v_0}{m_1 m_2 r^2 + I_0 m_1 + 2 I_0 m_2} \tag{7}$$

and finally, we have

$$\omega = \frac{4 m_1 m_2 r v_0}{m_1 m_2 r^2 + I_0 m_1 + 2 I m_2} \tag{8}$$

2 | Rigid Body Kinetic Energy

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2} m_i {v_i}^2 \tag{9}$$

Because of the fact a point v_i can be defined as a sum of the velocity from the origin plus the displace from from origin ($v_i = v_{CM} + v'_i$), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (v_{CM} + v_i')(v_{CM} + v_i') \tag{10}$$

Now, we shall foil the above expression:

$$\begin{align}
KE &= \sum_i \frac{1}{2} m_i ({v_{CM}}^2 + 2v_{CM}v_i' + {v_i'}^2) \tag{11}\\
&= \sum_i \frac{1}{2} m_i {v_{CM}}^2 + \sum_i m_i v_{CM} v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 \tag{12}\\
&= \frac{1}{2} M_i {v_{CM}}^2 + \sum_i m_i v_{CM} v_i' + \sum_i \frac{1}{2} m_i {v_i'}^2 \tag{13}\\
& \tag{14}
\end{align}$$