

## 1 | Axler6.56 Minimizing the distance to a subspace

Suppose  $U$  is a finite-dimensional subspace of  $V$ ,  $v \in V$ , and  $u \in U$ . Then,

$$\|v - P_U v\| \leq \|v - u\|$$

Because we often end up having to find the minimal  $v - u$  where  $u \in U$ , this result makes linear algebra applicable to numerous real-world applications.

### 1.1 | Proof

$$\begin{aligned} \|v - P_U v\|^2 &\leq \|v - P_U v\|^2 + \|P_U v - u\|^2 && \text{by } 0 \leq \|P_U v - u\|^2 \\ &= \|(v - P_U v) + (P_U v - u)\|^2 && \text{by the Pythagorean Theorem} \\ &= \|v - u\|^2 \end{aligned}$$

Inequality is an equality only when  $u = P_U v$ .

### 1.2 | An example

First define an inner product that will be our cost function. In this case, they use the integral of  $f(x)g(x)$  on the range  $[-\pi, \pi]$ . Then, orthonormalize a basis of the polynomials up to degree 6 (using the Gram-Schmidt procedure) and take the orthonormal projection using the same inner product. This ends up with roughly

$$u(x) = 0.987862x - 0.155271x^3 + 0.00564312x^5$$

Which ends up being a better approximation for the range than the corresponding 5-th degree Taylor polynomial.