1 | Problem

Prove or give a counterexample:

$$\begin{aligned}&\text{dim}(U_1+U_2+U_3)\\ &=\text{dim } U_1+\text{dim } U_2+\text{dim } U_3\\ &\quad-\text{dim}(U_1\cap U_2)-\text{dim}(U_1\cap U_3)-\text{dim}(U_2\cap U_3)\\ &\quad+\text{dim}(U_1\cap U_2\cap U_3)\end{aligned}$$

2 | Reasoning

By Axler2.41 we know that

$$\text{dim}(U_1+U_2)=\text{dim } U_1+\text{dim } U_2-\text{dim}(U_1\cap U_2)$$

By applying this formula to itself, we find that

$$\begin{aligned}&\text{dim}(U_1+U_2+U_3)\\ &=\text{dim}((U_1+U_2)+U_3)\\ &=\text{dim}(U_1+U_2)+\text{dim } U_3-\text{dim}((U_1+U_2)\cap U_3)\\ &=\text{dim } U_1+\text{dim } U_2-\text{dim}(U_1\cap U_2)+\text{dim } U_3-\text{dim}((U_1+U_2)\cap U_3)\end{aligned}$$

To show that the lemma is true, we would have to show that

$$\begin{aligned}&\text{dim } U_1+\text{dim } U_2+\text{dim } U_3-\text{dim}(U_1\cap U_2)-\text{dim}(U_1\cap U_3)-\text{dim}(U_2\cap U_3)+\text{dim}(U_1\cap U_2\cap U_3)\\ &=\text{dim } U_1+\text{dim } U_2-\text{dim}(U_1\cap U_2)+\text{dim } U_3-\text{dim}((U_1+U_2)\cap U_3)\end{aligned}$$

and to provide a counterexample, we just need to find some U_1, U_2, U_3 such that

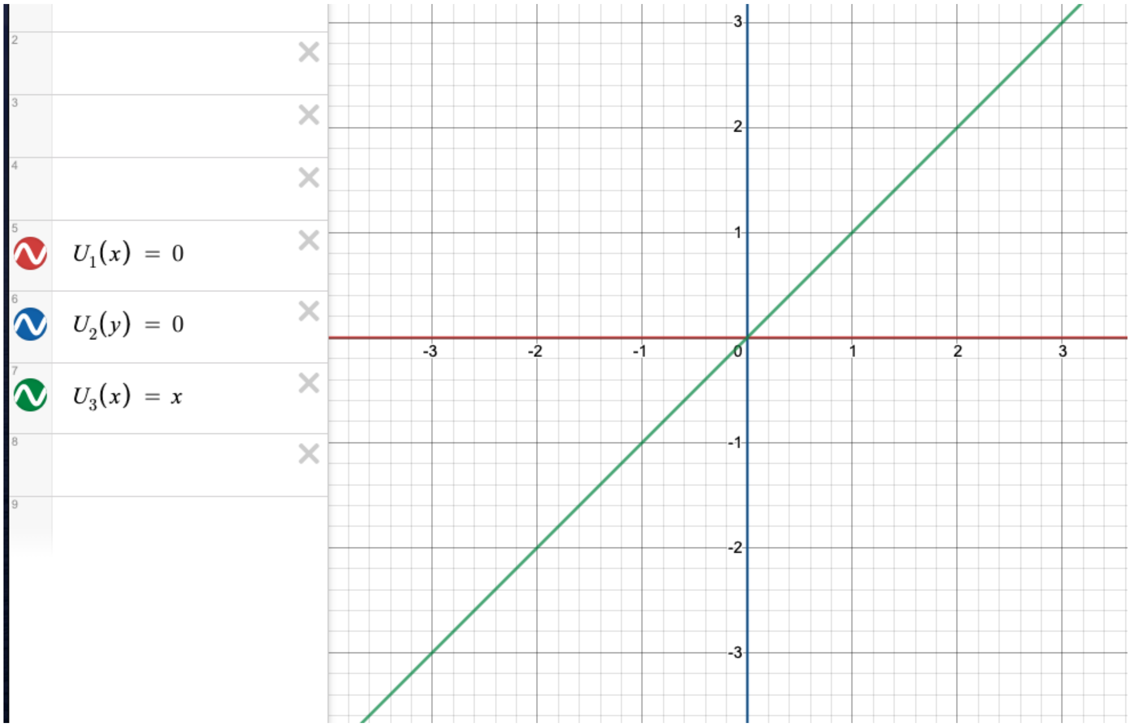
$$\text{dim}(U_1\cap U_3)+\text{dim}(U_2\cap U_3)-\text{dim}(U_1\cap U_2\cap U_3)\neq\text{dim}((U_1+U_2)\cap U_3)$$

3 | Counterexample

If we choose

$$\begin{aligned}U_1&=\left\{\begin{pmatrix}x\\0\end{pmatrix}:x\in\mathbb{R}\right\}\\ U_2&=\left\{\begin{pmatrix}0\\x\end{pmatrix}:x\in\mathbb{R}\right\}\\ U_3&=\left\{\begin{pmatrix}x\\x\end{pmatrix}:x\in\mathbb{R}\right\}\end{aligned}$$

then the graph of the subspaces looks like this:



and the dimesion of each intersection is 0 while the dimension of $(U_1 + U_2) \cap U_3 = 2$. Thus, we have

$$0\text{dim}(U_1 \cap U_3) + 0\text{dim}(U_2 \cap U_3) - 0\text{dim}(U_1 \cap U_2 \cap U_3) \neq \text{dim}((U_1 + U_2) \cap U_3) \\ \implies 0 \neq 2$$

In summary, the sum of these subpsaces is $\mathbb{R}^2$ and the dimension of the sum is 2, but $$\text{dim}(U_1 + U_2 + U_3) = 2 \neq 3 = 1 +$$