1 | Evaluating a Cylindrical Integral

Considering the function:

$$f(x, y, z) = \sqrt{x^2 + y^2} \tag{1}$$

To evaluate the integral, we will convert it to cylindrical coordinates. We note first that the integral is to be evaluated inside the cylinder of $x^2 + y^2 = 16$, which means that we wish to evaluate it in a circle with center at the origin with radius 4.

Furthermore, we understand that the bounds of the function are to be evaluated between $[-5, -4]$.

If we set up the integral, we will get:

$$\int_{-5}^{-4} \int_C \sqrt{x^2 + y^2}\ dx\ dy\ dz \tag{2}$$

This is convenient. We can evaluate the inner integral first like in $\mathbb{R}^2 \to \mathbb{R}^1$, then simply evaluate the other integral after.

Let's do so.

Note that the inner integral is a normal cylindrical coordinate setup. Therefore, we can take the following substitution:

$$\sqrt{x^2 + y^2} = r \tag{3}$$

Furthermore, that:

$$dx\ dy = dr\ d\theta \tag{4}$$

With the appropriate bounds, then:

$$\int_0^{2\pi} \int_0^4 r\ dr\ d\theta \tag{5}$$

$$\Rightarrow \int_0^{2\pi} \frac{r^2}{2}\bigg|_0^4 d\theta \tag{6}$$

$$\Rightarrow \int_0^{2\pi} 8\ d\theta \tag{7}$$

$$\Rightarrow 16\pi \tag{8}$$

Finally, we will take the integral of this value dz:

$$\int_{-5}^{-4} 16\pi\ dz = 16\pi \tag{9}$$

Therefore, the value of the integral is 16π.

2 | Uselessly Spherical Integral

We first recall that the differential volume can be written as:

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \tag{10}$$

To take this integral, then, we have to figure the distance ρ to a rectangle for every point (ϕ, θ).

We will take this integral w.r.t. one quadrant of the four parts of the rectangle, then we will scale it up by 4. Each of the quadrants are divided into again two equally-sized pieces: one sweeping from the baseline until halfway up, the other sweeping halfway up until the center line. These are the areas where either the adjacent side is consistently $\frac{b}{2}$ or the opposite side is consistently $\frac{a}{2}$.

The switch happens exactly at the triangle where the bottom side has length $\frac{b}{2}$ and the right side has length $\frac{a}{2}$. This triangle, per the definition of arctan, has an angle of:

$$\theta = \tan^{-1}\left(\frac{a}{b}\right) \tag{11}$$

Therefore, from $\theta = [0, \tan^{-1}\left(\frac{a}{b}\right)]$, we have sidelength $\frac{b}{2\cos\theta}$. From $\theta = [\tan^{-1}\left(\frac{a}{b}\right), \frac{\pi}{2}]$, we have sidelength $\frac{a}{2\sin\theta}$.

By the same token, for the vertical declination, from $\theta = [0, \tan^{-1}\left(\frac{a}{b}\right)]$, $\phi = [0, \tan^{-1}\left(\frac{c}{b}\cos\theta\right)]$, we have $\rho = \frac{b}{2\cos\theta\cos\phi}$. From $\theta = [\tan^{-1}\left(\frac{a}{b}\right), \frac{\pi}{2}]$, $\phi = [0, \tan^{-1}\left(\frac{c}{a}\sin\theta\right)]$, we have $\rho = \frac{a}{2\sin\theta\cos\phi}$. For all other ϕ, $\phi = [\tan^{-1}\left(\frac{c}{b}\cos\theta\right), \frac{\pi}{2}]$, we have the $\frac{c}{2}$ term taking precedence and hence $\rho = \frac{c}{2\sin\phi}$.

Therefore, we will have to take the quarter-integral in three parts:

$$\int_0^{\tan^{-1}\left(\frac{a}{b}\right)} \int_0^{\tan^{-1}\left(\frac{c}{b}\cos\theta\right)} \frac{b}{2\cos\theta\cos\phi} \; d\phi \; d\theta \tag{12}$$

$$\int_0^{\tan^{-1}\left(\frac{a}{b}\right)} \frac{b}{2\cos\theta} \int_0^{\tan^{-1}\left(\frac{c}{b}\cos\theta\right)} \frac{1}{\cos\phi} \; d\phi \; d\theta \tag{13}$$

$$\int_0^{\tan^{-1}\left(\frac{a}{b}\right)} \frac{b}{2\cos\theta} \left.\frac{\sin\phi}{\cos^2\phi}\right|_0^{\tan^{-1}\left(\frac{c}{b}\cos\theta\right)} d\theta \tag{14}$$

$$\int_0^{\tan^{-1}\left(\frac{a}{b}\right)} \frac{1}{2} c \sqrt{\frac{c^2\cos(\theta)^2 + b^2}{b^2}} \cos(\theta)^2 \; d\theta \tag{15}$$

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var("a b c")
f(theta) = 1/2*c*sqrt((c^2*cos(theta)^2 + b^2)/b^2)*cos(theta)^2
# assume((a*b)>0)
f.integrate(theta, 0, arctan(a/b))
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$$\frac{1}{2} c \sqrt{\frac{c^2\cos(\theta)^2 + b^2}{b^2}} \cos(\theta)^2 \tag{16}$$