$$\begin{split} \Delta \mathsf{PE} &= \mathsf{PE}_H - \mathsf{PE}_{h_0} \\ \mathsf{PE}_H &= mgH \\ \mathsf{PE}_{h_0} &= mgh_0 \\ \Delta \mathsf{PE} &= mg(H-h_0) \end{split}$$

$$\Delta \text{PE} = \text{KE}$$
 
$$mg(H - h_0) = \frac{1}{2}mv^2$$
 
$$g(H - h_0) = \frac{1}{2}v^2$$
 
$$\sqrt{2g(H - h_0)} = v$$

$$\begin{cases} x(t_f) = x_f = v_0 \cos \theta t_f = \sqrt{2g(H-h_0)} \cos \theta t_f \\ y(t_f) = 0 = v_0 \sin \theta t_f - \frac{1}{2}gt_f^2 + h_0 = \sqrt{2g(H-h_0)} \sin \theta t_f - \frac{1}{2}gt_f^2 + h_0 \\ \frac{x_f}{v_0 \cos \theta} = t_f \end{cases}$$

We can then plug this in and apply implicit differentiation to get  $\frac{dx_f}{d\theta}$ :

$$0 = -\frac{1}{2}g\left(\frac{x_f}{v_0\cos\theta}\right)^2 + v_0\sin\theta\frac{x_f}{v_0\cos\theta} + h_0$$
 
$$\frac{d}{d\theta}0 = \frac{d}{d\theta}\left(-\frac{1}{2}g\left(\frac{x_f}{v_0\cos\theta}\right)^2\right) + \frac{d}{d\theta}v_0\sin\theta\frac{x_f}{v_0\cos\theta} + \frac{d}{d\theta}h_0$$
 
$$0 = \frac{d}{d\theta}\left(-\frac{1}{2}g\left(\frac{x_f}{v_0\cos\theta}\right)^2\right) + \frac{d}{d\theta}\tan\theta x_f + 0$$
 
$$0 = \frac{d}{d\theta}\left(-\frac{1}{2}g\frac{x_f^2}{v_0^2\cos^2\theta}\right) + \frac{d}{d\theta}\tan\theta x_f + 0$$
 
$$0 = \frac{d}{d\theta}\left(-\frac{1}{2v_0^2}gx_f^2\frac{1}{\cos^2\theta}\right) + \frac{d}{d\theta}\tan\theta x_f$$
 
$$0 = -\frac{g}{2v_0^2}\frac{d}{d\theta}\left(x_f^2\frac{1}{\cos^2\theta}\right) + \frac{d}{d\theta}\tan\theta x_f$$
 
$$0 = -\frac{g}{2v_0^2}\left(2x_f\frac{dx_f}{d\theta}\frac{1}{\cos^2\theta} + x_f^22\tan\theta\sec^2\theta\right)\frac{1}{\cos^2\theta}\sec^2\theta x_f + \tan\theta\frac{dx_f}{d\theta}$$

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$$\begin{split} 0 &= -\frac{g}{2v_0^2} 2x_f \frac{dx_f}{d\theta} \frac{1}{\cos^2\theta} - \frac{g}{2v_0^2} x_f^2 2\tan\theta \sec^2\theta + \sec^2\theta x_f + \tan\theta \frac{dx_f}{d\theta} \\ &\frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta - \sec^2\theta x_f = -\frac{g}{2v_0^2} 2x_f \frac{dx_f}{d\theta} \frac{1}{\cos^2\theta} + v_0 \tan\theta \frac{dx_f}{d\theta} \\ &\frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta - \sec^2\theta x_f = \frac{dx_f}{d\theta} \left( v_0 \sec^2\theta x_f - \frac{g}{2v_0^2} 2x_f \frac{1}{\cos^2\theta} \right) \\ &\frac{\frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta - \sec^2\theta x_f}{v_0 \sec^2\theta x_f - \frac{g}{2v_0^2} 2x_f \frac{1}{\cos^2\theta}} = \frac{dx_f}{d\theta} \end{split}$$

We can now optimize this monstrosity:

$$\frac{\frac{g}{2v_0^2}2x_f^2\tan\theta\sec^2\theta-\sec^2\theta x_f}{v_0\sec^2\theta x_f-\frac{g}{2v_0^2}2x_f\frac{1}{\cos^2\theta}}=0$$

$$\begin{split} \frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta - \sec^2\theta x_f &= 0 \\ \frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta - \sec^2\theta x_f &= 0 \\ \frac{g}{2v_0^2} 2x_f^2 \tan\theta \sec^2\theta &= \sec^2\theta x_f \\ \frac{g}{v_0^2} x_f \tan\theta &= 1 \\ \tan\theta &= \frac{1}{x_f} \frac{v_0^2}{g} \\ \tan\theta &= \frac{v_0^2}{gx_f} \end{split}$$

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$$\begin{split} 0 &= -\frac{1}{2}g\left(\frac{x_f}{v_0\cos\theta}\right)^2 + v_0\sin\theta\frac{x_f}{v_0\cos\theta} + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2}\frac{1}{\cos^2\theta} + x_f\tan\theta + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2}(1+\tan^2\theta) + x_f\tan\theta + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2}\left(1+\frac{v_0^4}{g^2x_f^2}\right) + x_f\frac{v_0^2}{gx_f} + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2} - \frac{1}{2}g\frac{x_f^2}{v_0^2}\frac{v_0^4}{g^2x_f^2} + \frac{v_0^2}{g} + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2} - \frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2} - \frac{v_0^2}{2g} + \frac{v_0^2}{g} + h_0 \\ 0 &= -\frac{1}{2}g\frac{x_f^2}{v_0^2} - \frac{v_0^2}{2g} + \frac{v_0^2}{g} + h_0 \end{split}$$

$$v_0 = \sqrt{2g(H - h_0)}$$

$$v_0^2 = 2g(H - h_0)$$

$$0 = -\frac{1}{2}g\frac{x_f^2}{v_0^2} - \frac{v_0^2}{2g} + \frac{v_0^2}{g} + h_0$$