

1 | Problem 1

1.1 | 1-1

Because all of the terms $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$ are all crossed with \vec{A} , all of the vectors are in the same plane which is perpendicular to \vec{A} . If we set $\vec{A} = (0, 0, A_z)$ to be on the z axis, then $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$ would all be in the xy plane.

1.2 | 1-2

Because $\vec{B}_{\perp \vec{A}}$, $\vec{C}_{\perp \vec{A}}$, $(\vec{B} + \vec{C})_{\perp \vec{A}}$ all are perpendicular to \vec{A} , they all must be coplaner, in the plane that is perpendicular to \vec{A} , which happens to be in the same plane as $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$.

1.3 | 1-3

Because of the definition of the cross product, we know that:

$$(\vec{A} \times \vec{B}) \perp \vec{B}$$

$$(\vec{A} \times \vec{C}) \perp \vec{C}$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp (\vec{B} + \vec{C})$$

In tandem with the information from part 1-1 (all of the terms are perpendicular to \vec{A}), we know that:

$$(\vec{A} \times \vec{B}) \perp \vec{A}\vec{B} \text{ plane}$$

$$(\vec{A} \times \vec{C}) \perp \vec{A}\vec{C} \text{ plane}$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp \vec{A}(\vec{B} + \vec{C}) \text{ plane}$$

Thus, if we want to show that:

$$(\vec{A} \times \vec{B}) \perp (\vec{B}_{\perp \vec{A}})$$

$$(\vec{A} \times \vec{C}) \perp (\vec{C}_{\perp \vec{A}})$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp (\vec{B} + \vec{C})_{\perp \vec{A}}$$

Then you need to show that:

$$(\vec{B}_{\perp \vec{A}}) \in \vec{A}\vec{B} \text{ plane}$$

$$(\vec{C}_{\perp \vec{A}}) \in \vec{A}\vec{C} \text{ plane}$$

$$(\vec{B} + \vec{C})_{\perp \vec{A}} \in \vec{A}(\vec{B} + \vec{C}) \text{ plane}$$

To do this we can use the linear algebra definition of a plane. This definition states that a plane is defined as the locus of points that can be described by the linear combination of two vectors. In this case the two vectors are:

\vec{A} and \vec{B}

\vec{A} and \vec{C}

\vec{A} and $(\vec{B} + \vec{C})$

We have defined:

$$\vec{A} = (0, 0, A_z)$$

And can define:

$$\vec{B} = (B_x, B_y, B_z)$$

$$\vec{C} = (C_x, C_y, C_z)$$

Thus:

$$(\vec{B} + \vec{C}) = (B_x + C_x, B_y + C_y, B_z + C_z)$$

Because $\vec{B}_{\perp\vec{A}}$, $\vec{C}_{\perp\vec{A}}$, and $(\vec{B} + \vec{C})_{\perp\vec{A}}$ are the projections of \vec{B} , \vec{C} , and $(\vec{B} + \vec{C})$ onto xy plane, respectively, they are defined as:

$$\vec{B}_{\perp\vec{A}} = (B_x, B_y, 0) = (B_x, B_y, B_z) + n(0, 0, A_z) = \vec{B} + n\vec{A}$$

$$\vec{C}_{\perp\vec{A}} = (C_x, C_y, 0) = (C_x, C_y, C_z) + m(0, 0, A_z) = \vec{C} + m\vec{A}$$

$$\begin{aligned} (\vec{B} + \vec{C})_{\perp\vec{A}} &= (B_x + C_x, B_y + C_y, 0) \\ &= (B_x + C_x, B_y + C_y, B_z + C_z) + p(0, 0, A_z) = (\vec{B} + \vec{C}) + p\vec{A} \end{aligned}$$

Thus, by the linear algebra definition of a plane:

$$(\vec{B}_{\perp\vec{A}}) \in \vec{A}\vec{B} \text{ plane}$$

$$(\vec{C}_{\perp\vec{A}}) \in \vec{A}\vec{C} \text{ plane}$$

$$(\vec{B} + \vec{C})_{\perp\vec{A}} \in \vec{A}(\vec{B} + \vec{C}) \text{ plane}$$

Therefore:

$$(\vec{A} \times \vec{B}) \perp (\vec{B}_{\perp\vec{A}})$$

$$(\vec{A} \times \vec{C}) \perp (\vec{C}_{\perp\vec{A}})$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp (\vec{B} + \vec{C})_{\perp\vec{A}}$$

1.4 | 1-4

$(\vec{A} \times \vec{B})$ points in the direction of $\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})$

$(\vec{A} \times \vec{C})$ points in the direction of $\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}})$

$\vec{A} \times (\vec{B} + \vec{C})$ points in the direction of $\vec{R}_{90^\circ}(\vec{B} + \vec{C})_{\perp\vec{A}}$

1.5 | 1-5

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin(\theta) \cdot \frac{\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})}{|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})|}$$

The last "term" is there just to set the direction (from part 1-4).

The middle part, $|\vec{B}| \sin(\theta)$ is magnitude of component of \vec{B} that is perpendicular to \vec{A} .

We know that $\vec{B}_{\perp\vec{A}}$ is component of \vec{B} that is perpendicular to \vec{A} from the definition of projection and the facts that:

- $\vec{B}_{\perp\vec{A}}$ is in the xy plane
- \vec{A} is perpendicular to the xy plane

Thus we know that:

$$|\vec{B}| \sin(\theta) = |\vec{B}_{\perp\vec{A}}|$$

Because $R_{90^\circ}()$ rotates the vector by 90 degrees:

$$|\vec{B}_{\perp\vec{A}}| = |\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})|$$

Therefore:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin(\theta) \cdot \frac{\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})}{|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})|} = |\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})$$

1.6 | 1-6

As stated in problem 1-5:

$$\vec{B}_{\perp\vec{A}} = (B_x, B_y, 0)$$

$$\vec{C}_{\perp\vec{A}} = (C_x, C_y, 0)$$

$$(\vec{B} + \vec{C})_{\perp\vec{A}} = (B_x + C_x, B_y + C_y, 0)$$

This works by definition of projection, and the fact that the x values cannot influence the y or z values of a vector:

Therefore:

$$\vec{B}_{\perp\vec{A}} + \vec{C}_{\perp\vec{A}} = (\vec{B} + \vec{C})_{\perp\vec{A}}$$

1.7 | 1-7

<https://www.desmos.com/calculator/zppshmkmmni>

- the red line is $\vec{B}_{\perp\vec{A}}$

- the blue line is $\vec{C}_{\perp\vec{A}}$
- the green line is $(\vec{B} + \vec{C})_{\perp\vec{A}}$
- the purple line is
- the black line is
- the orange line is

The rotation operation is defined as:

$$\vec{t} = (x, y, 0)$$

$$\vec{R}_{90^\circ}(\vec{t}) = (-y, x, 0)$$

Where \vec{t} is rotated 90 degrees counter clockwise in the xy plane.

Thus:

$$\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) = (-B_y, B_x, 0)$$

$$\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = (-C_y, C_x, 0)$$

$$\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}}) = (-(B_y + C_y), B_x + C_x, 0) = (-B_y - C_y, B_x + C_x, 0)$$

From this we see that:

$$\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + \vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = \vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

1.8 | 1-8

If we multiply both sides of the equation with a scalar, $|\vec{A}|$, then we get:

$$|\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

From step 1-5 we know that:

$$\vec{A} \times \vec{B} = |\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})$$

and by similar means we can show that:

$$\vec{A} \times \vec{C} = |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}})$$

$$\vec{A} \times (\vec{B} + \vec{C}) = |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

Thus we can substitute:

$$\begin{aligned} |\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) &= |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}}) \\ \Rightarrow \vec{A} \times \vec{B} + \vec{A} \times \vec{C} &= \vec{A} \times (\vec{B} + \vec{C}) \end{aligned}$$

Therefore the cross product is distributive over addition.

2 | Problem 2

To start, we can define:

$$\vec{A} = (A_x, A_y, A_z) = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = (B_x, B_y, B_z) = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Next, using the fact that the cross product is distributive across addition (shown in problem 1):

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ &= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k} \\ &\quad + A_y B_x \hat{j} \times \hat{i} + A_y B_y \hat{j} \times \hat{j} + A_y B_z \hat{j} \times \hat{k} \\ &\quad + A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j} + A_z B_z \hat{k} \times \hat{k} \end{aligned}$$

From the definition of a cross product, we know that the cross product between any two vectors that are parallel is zero. This is because the $\sin(0) = 0$. Thus all of the terms that take the cross product between two of the same unit vectors can be replaced with 0:

$$\begin{aligned} &= A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k} \\ &\quad + A_y B_x \hat{j} \times \hat{i} + A_y B_z \hat{j} \times \hat{k} \\ &\quad + A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j} \end{aligned}$$

Next, $\hat{i} \times \hat{j}$ would yield a vector length one in the direction of a vector that is perpendicular to both \hat{i} and \hat{j} , which would be \hat{k} (using the right hand rule to get the correct direction). Conversely, $\hat{j} \times \hat{i} = -\hat{k}$. We can use the right hand rule for all possible combinations of unit vector cross products (excluding of the same kind):

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$

Therefore:

$$\begin{aligned} &= A_x B_y \hat{k} - A_x B_z \hat{j} \\ &\quad - A_y B_x \hat{k} + A_y B_z \hat{i} \\ &\quad + A_z B_x \hat{j} - A_z B_y \hat{i} \\ &= A_x B_y \hat{k} - A_y B_x \hat{k} \\ &\quad + A_y B_z \hat{i} - A_z B_y \hat{i} \\ &\quad - A_x B_z \hat{j} + A_z B_x \hat{j} \\ &= (A_x B_y - A_y B_x) \hat{k} + (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} \\ &= (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \end{aligned}$$

3 | Problem 3

3.1 | 3-1

First let's define the following:

$$\vec{P} = (P_x, P_y, P_z)$$

$$\vec{Q} = (Q_x, Q_y, Q_z)$$

$$\vec{R} = (R_x, R_y, R_z)$$

Using the three position vectors we can get two vectors that are in the plane:

$$\vec{V}_1 = \vec{P} - \vec{Q}$$

$$\vec{V}_2 = \vec{R} - \vec{Q}$$

\vec{V}_1 and \vec{V}_2 are both in the the same plane as \vec{P} , \vec{Q} , and \vec{R} . This is because, their tails meet at \vec{Q} , because \vec{Q} is negative and the tips are at \vec{P} and \vec{R} respectively. To find a vector normal to this plane we can use the cross product:

$$\vec{n} = \vec{V}_1 \times \vec{V}_2 = (\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q})$$

Now we can use the vector definition of a plane:

$$\{\vec{r} : (\vec{r} - \vec{P}_o) \cdot \vec{n} = 0, \vec{P}_o \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

We can define:

$$\vec{P}_o = \vec{P}$$

though it could be any of the given vectors.

Thus, the vector equation of the plane containing the three points is:

$$\{\vec{r} : (\vec{r} - \vec{P}) \cdot ((\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q})) = 0, \vec{P} \in \mathbb{R}^3, \vec{Q} \in \mathbb{R}^3, \vec{R} \in \mathbb{R}^3\}$$

3.2 | 3-2

First we have to find the cartesian equation of a plane given three points. We can use what we found in part 3-1 to know that:

$$\begin{aligned} \vec{n} &= \vec{V}_1 \times \vec{V}_2 \\ &= (\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q}) \\ &= (P_x - Q_x, P_y - Q_y, P_z - Q_z) \times (R_x - Q_x, R_y - Q_y, R_z - Q_z) \\ &= ((P_y - Q_y)(R_z - Q_z) - (P_z - Q_z)(R_y - Q_y), \\ &\quad (P_z - Q_z)(R_x - Q_x) - (P_x - Q_x)(R_z - Q_z), \\ &\quad (P_x - Q_x)(R_y - Q_y) - (P_y - Q_y)(R_x - Q_x)) \\ &= ((P_y R_z - Q_y R_z - Q_z P_y + Q_z Q_y) - (P_z R_y - Q_z R_y - Q_y P_z + Q_y Q_z), \\ &\quad (P_z R_x - Q_z R_x - Q_x P_z + Q_x Q_z) - (P_x R_z - Q_x R_z - Q_z P_x + Q_z Q_x), \end{aligned}$$

$$\begin{aligned}
& (P_x R_y - Q_x R_y - Q_y P_x + Q_x Q_y) - (P_y R_x - Q_y R_x - Q_x P_y + Q_x Q_y) \\
&= (P_y R_x - Q_y R_z - Q_z P_y - P_z R_y + Q_z R_y + Q_y P_z, \\
& P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x, \\
& P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y)
\end{aligned}$$

Once we have the normal vector we can use the vector definition of a plane to get the cartesian equation of the plane:

$$\{\vec{r} : (\vec{r} - \vec{P}_o) \cdot \vec{n} = 0, \vec{P}_o \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

$$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$$

$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$$

We can define:

$$\vec{r} = (x, y, z)$$

$$\vec{P}_o = \vec{P}$$

though it could be any of the given vectors.

Evaluating the left side of the equation:

$$\begin{aligned}
\vec{r} \cdot \vec{n} &= (x, y, z) \cdot (P_y R_x - Q_z R_x - Q_x P_y - P_z R_y + Q_z R_y + Q_y P_z, \\
& P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x, P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y) \\
&= x(P_y R_x - Q_y R_z - Q_z P_y - P_z R_y + Q_z R_y + Q_y P_z) \\
&+ y(P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x) \\
&+ z(P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y)
\end{aligned}$$

Evaluating the right side of the equation:

$$\begin{aligned}
\vec{P}_o \cdot \vec{n} &= (P_x, P_y, P_z) \cdot (P_y R_x - Q_z R_x - Q_x P_y - P_z R_y + Q_z R_y + Q_y P_z, \\
& P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x, P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y) \\
&= P_x(P_y R_x - Q_y R_z - Q_z P_y - P_z R_y + Q_z R_y + Q_y P_z) \\
&+ P_y(P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x) \\
&+ P_z(P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y)
\end{aligned}$$

Thus the cartesian equation for the plane is the following:

$$\begin{aligned}
& x(P_y R_x - Q_y R_z - Q_z P_y - P_z R_y + Q_z R_y + Q_y P_z) \\
&+ y(P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x) \\
&+ z(P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y) \\
&= \\
& P_x(P_y R_x - Q_y R_z - Q_z P_y - P_z R_y + Q_z R_y + Q_y P_z) \\
&+ P_y(P_z R_x - Q_z R_x - Q_x P_z - P_x R_z + Q_x R_z + Q_z P_x) \\
&+ P_z(P_x R_y - Q_x R_y - Q_y P_x - P_y R_x + Q_y R_x + Q_x P_y)
\end{aligned}$$

Now that we have the equation we just need to plug the numbers in:

$$\begin{aligned}
& x(0 - 4 - 0 - 2 - 6 - 1) \\
&+ y(0 - 0 - 0 - 8 + 0 + 6) \\
&+ z(-4 - 0 - 2 - 0 + 0 + 0) \\
&= \\
& 2(0 - 4 - 0 - 2 - 6 - 1) \\
&+ 0(0 - 0 - 0 - 8 + 0 + 6) \\
&- 1(-4 - 0 - 2 - 0 + 0 + 0) \\
&\Rightarrow -13x - 2y - 6z = -26 + 6 \\
&\Rightarrow -13x - 2y - 6z = -20
\end{aligned}$$

We can check this by substituting all 3 points into the equation:

The first point is $(2, 0, -1)$:

$$-13(2) - 2(0) - 6(-1) \stackrel{?}{=} -20$$

$$-26 - 0 + 6 \stackrel{?}{=} -20$$

$$-20 = -20$$

The second point is $(0, 1, 3)$:

$$-13(0) - 2(1) - 6(3) \stackrel{?}{=} -20$$

$$0 - 2 - 18 \stackrel{?}{=} -20$$

$$-20 = -20$$

The third point is $(0, -2, 4)$:

$$-13(0) - 2(-2) - 6(4) \stackrel{?}{=} -20$$

$$0 + 4 + 24 \stackrel{?}{=} -20$$

$$-20 = -20$$

Thus $-13x - 2y - 6z = -20$ is the cartesian equation of the plane that contains the three points $(2, 0, -1)$, $(0, 1, 3)$, and $(0, -2, 4)$