1 | Position of m_i

In a rigid body consisting of N point masses, the vector to the position of m_i is defined as $\vec{r_i(t)}$, which is defined as follows:

$$\vec{r_i(t)} = \vec{R_{CM}}(t) + \vec{r_i}'(t) \tag{1}$$

whereas, $\vec{R_{CM}}(t)$ is the position vector of the center of mass of the rigid body as a whole, and $\vec{r_i}'(t)$ the vector from the center of mass to m_i.

2 | Velocity of m_i

The velocity of m_i is simply determined by the first derivative of the position equation as per above. Namely, that:

$$\vec{v_i(t)} = \vec{V_{CM}}(t) + \vec{v_i}'(t) \tag{2}$$

where, $\vec{v_i(t)}$ is the velocity vector of m_i, and $\vec{V_{CM}}(t)$ is the velocity vector of the center of mass of the rigid body, and $\vec{v_i}'(t)$ is the velocity vector from center of mass to m_i.

3 | Deriving KE_{total}

3.1 | Setting up

From definition of KE_{total} itself, KE_{total} is the sum of all energies of each point mass in the rigid body.

$$\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \tag{3}$$

3.2 | Derivation, part 1

Expanding this equation and substituting the value of v_i, and additionally setting $M = \sum m_i$ (namely, that M represents the total mass of the rigid body) we could derive:

$$\sum_{i=1}^{N}\frac{1}{2}m_iv_i^2 = \sum_{i=1}^{N}\frac{1}{2}m_i(v_i \cdot v_i) \tag{4}$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i((\vec{V_{CM}} + \vec{v_i}') \cdot (\vec{V_{CM}} + \vec{v_i}')) \tag{5}$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i(\vec{V_{CM}}^2 + 2 \times (\vec{v_i}' \cdot \vec{V_{CM}}) + \vec{v_i}'^2)) \tag{6}$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i\vec{V_{CM}}^2 + \sum_{i=1}^{N}m_i \times (\vec{v_i}' \cdot \vec{V_{CM}}) + \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{7}$$

$$= \frac{1}{2}\vec{V_{CM}}^2\sum_{i=1}^{N}m_i + \vec{V_{CM}}\sum_{i=1}^{N}m_i\vec{v_i}' + \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{8}$$

3.3 | Dealing with the Middle Term

At this point, we must note that $\sum_{i=1}^{N} m_i\vec{v_i}' = 0$. Per the definition of the center of mass, the following holds:

$$r\vec{_{CM}} = (\frac{1}{M})\sum_i m_i\vec{r_i} \tag{9}$$

Changing reference frame to that of the center of mass itself, this equation therefore becomes:

$$r\vec{_{CM}}' = (\frac{1}{M})\sum_i m_i\vec{r_i}' \tag{10}$$

It is important to realize here that $r\vec{_{CM}}' = 0$ because of the fact that — at the reference point of the center of mass, the center of mass is at a zero-vector distance away from itself.

In order to figure a statement with respect to the *velocity* of r_i', we take the derivative of the previous equation with respect to time.

$$0 = (\frac{1}{M})\sum_i m_i\vec{r_i}' \tag{11}$$

$$\Rightarrow \frac{d}{dt}(\frac{1}{M})\sum_i m_i\vec{r_i}' \tag{12}$$

$$= (\frac{1}{M})\sum_i m_i\vec{v_i}' \tag{13}$$

Given that $\frac{1}{M}$ could not be zero for an object with non-zero mass, it is concluded therefore that $\sum_i m_i\vec{v_i}' = 0$.

3.4 | Derivation, part 2

As $\sum_i m_i\vec{v_i}' = 0$, the KE_{total} work-in-progress equation's middle term (which contains the statement $\sum_i m_i\vec{v_i}'$) is therefore zero. Substituting that in and removing the term, we therefore result in:

$$\sum_{i=1}^{N}\frac{1}{2}m_iv_i^2 = \frac{1}{2}\vec{V_{CM}}^2\sum_{i=1}^{N}m_i + \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{14}$$

Replacing the definition of $M = \sum m_i$, we result in

$$\sum_{i=1}^{N}\frac{1}{2}m_iv_i^2 = \frac{1}{2}M\vec{V_{CM}}^2 + \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{15}$$

$$KE_{total} = \frac{1}{2}M\vec{V_{CM}}^2 + \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{16}$$

The left term of this equation ($\frac{1}{2}M\vec{V_{CM}}^2$) is the clear original statement for $KE_{translational}$. As component masses of a rigid body cannot experience translational motion about its center of origin, the second term is therefore rotational only and so $KE_{rotational}$.

Therefore:

$$KE_{total} = KE_{translational} + KE_{rotational} \tag{17}$$