1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f.

In this problem, we will define the axis such that the ”lower-left” corner of the wood block (corner sharing x value of the starting position of the marble, but on the ”ground”) as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x-value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive-x, so as the marble rolls, its position’s x value increases. We will define the location of the marble before starting as positive y, and as the marble decreases in height, its position’s y value decreases.

We define the start of the experiment as time t_0, the moment the marble leaves the track and travels as a projectile as t_1, and the end — in the moment when the marble hits the ground — as t_f. We will call the marble m_0.

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0 g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0 g(H - h_0) = m_0 g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0 g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named $\vec{v_0}$.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 \tag{1}$$

$$g(H - h_0) = \frac{1}{2} \vec{v_0}^2 \tag{2}$$

$$2g(H - h_0) = \vec{v_0}^2 \tag{3}$$

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{4}$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} \vec{v_{0x}} = \sqrt{2g(H - h_0)} cos(\theta_0) \\ \vec{v_{0y}} = \sqrt{2g(H - h_0)} sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. $\vec{v_{0y}}$, $\vec{v_{0x}}$, h_0, m_0.

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f.

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as…

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Setup for Solution

We first create a set of equations modeling the location of the marble at t_f.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{-1}{2}g{t_f}^2 + t_f\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f, and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)} \tag{5}$$

Finally, we substitute the definition of t_f into $y(t_f)$.

$$y(t_f) = 0 = \frac{-1}{2}g\frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}^2 + \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \tag{6}$$

We will now proceed to simplify the expression further

$$\begin{aligned} 0 =& \frac{-1}{4}\frac{{x_f}^2}{(H-h_0)cos^2(\theta_0)} + x_f tan(\theta_0) + h_0 && (7) \\ =& \frac{-1}{4}\frac{{x_f}^2}{(H-h_0)}cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 && (8) \\ =& \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 && (9) \end{aligned}$$

3.3 | Solution at $\theta_0 = 0$

We begin by first solving for the expression for ${x_f}^2$ at $\theta_0 = 0$.

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{10}$$

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 + h_0 \tag{11}$$

$$-h_0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 \tag{12}$$

$$4h_0 = \frac{1}{(H-h_0)}{x_f}^2 \tag{13}$$

$${x_f}^2 = 4h_0(H-H_0) \tag{14}$$

3.3.1 | **Finding** $\frac{d{x_f}^2}{dh_0}$

Remember, given the setup of the problem, x_f is optimized when ${x_f}^2$ is optimized. This is due to the fact that x_f could not be negative, and it is representing a maximum travel distance. Hence, we will optimized ${x_f}^2$ for ease of calculation.

$${x_f}^2 = 4h_0(H-H_0) \tag{15}$$

$$\frac{d{x_f}^2}{dh_0} = \frac{d}{dh_0}4h_0(H-h_0) \tag{16}$$

$$= 4\frac{d}{dh_0}h_0(H-h_0) \tag{17}$$

$$= 4((H-h_0)-h_0) \tag{18}$$

$$= 4H - 4h_0 - 4h_0 \tag{19}$$

$$= 4H - 8h_0 \tag{20}$$

3.3.2 | **Optimizing** $\frac{d{x_f}^2}{dh_0}$

The optimization of this statement is fairly simple. We set $\frac{d{x_f}^2}{dh_0} = 0$, and solve for h_0.

$${x_f}^2 = 4H - 8h_0 \tag{21}$$

$$\Rightarrow 0 = 4H - 8h_0 \tag{22}$$

$$\Rightarrow 8h_0 = 4H \tag{23}$$

$$\Rightarrow h_0 = \frac{1}{2}H \tag{24}$$

Hence, the most optimal height at which to launch the marble launcher, given a horizontal launch, is half of the initial height.

3.4 | **Solution at arbitrary** θ_0

3.4.1 | **Solving and optimizing for** $\frac{dx_f}{d\theta_0}$

We need to maximize $\frac{dx_f}{d\theta_0}$ as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for $\frac{h_0}{d\theta_0}$. The position equations above could be leveraged to figure a value for x_f.

1. Finding $\frac{dx_f}{d\theta_0}$ We leverage implicit differentiation to figure a value for $\frac{dx_f}{d\theta_0}$. We set x_f as a differentiable function, and h_0 and H as both constants.

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{25}$$

$$\Rightarrow \frac{d}{d\theta_0}0 = \frac{d}{d\theta_0}(\frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0) \tag{26}$$

$$\Rightarrow 0 = \frac{-1}{4}\frac{1}{(H-h_0)}\frac{d}{d\theta_0}{x_f}^2 cos^{-2}(\theta_0) + \frac{d}{d\theta_0}x_f tan(\theta_0) + \frac{d}{d\theta_0}h_0 \tag{27}$$

$$\Rightarrow 0 = \frac{-1}{4}\frac{1}{(H-h_0)}((\frac{d}{d\theta_0}{x_f}^2)cos^{-2}(\theta_0) + {x_f}^2(\frac{d}{d\theta_0}cos^{-2}(\theta_0))) + \tag{28}$$

$$((\frac{d}{d\theta_0}x_f)tan(\theta_0) + (\frac{d}{d\theta_0}tan(\theta_0))x_f) + 0 \tag{29}$$

$$\Rightarrow 0 = \frac{-1}{4(H-h_0)}((2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + {x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0))) + \tag{30}$$

$$(\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f) \tag{31}$$

$$\Rightarrow 0 = \frac{-1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{-1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) + \tag{32}$$

$$\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f \tag{33}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{34}$$

$$= \frac{-1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{35}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{36}$$

$$= \frac{dx_f}{d\theta_0}\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{37}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{38}$$

$$= \frac{dx_f}{d\theta_0}(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0)) \tag{39}$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{40}$$

2. Optimizing for x_f for θ_0 via $\frac{dx_f}{d\theta_0}$ We now set $\frac{dx_f}{d\theta_0} = 0$ in order to figure critical points for the value of x_f.

$$f\frac{dx_f}{d\theta_0} = \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{41}$$

$$\Rightarrow 0 = \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{42}$$

$$\Rightarrow 0 = \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f \tag{43}$$

$$\Rightarrow sec^2(\theta_0)x_f = \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 \tag{44}$$

$$\Rightarrow sec^2(\theta_0) = \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}x_f \tag{45}$$

$$\Rightarrow 2sec^2(\theta_0)(H-h_0) = (cos^{-3}(\theta_0)sin(\theta_0))x_f \tag{46}$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{sec^2(\theta_0)} \tag{47}$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{sin(\theta_0)}{cos^3(\theta_0)sec^2(\theta_0)} \tag{48}$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{sin(\theta_0)}{cos(\theta_0)} \tag{49}$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = tan(\theta_0) \tag{50}$$

$$\Rightarrow \theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{51}$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize x_f for a given solved x_f. To figure the actual statement that would optimize for both,

3.4.2 | Solving and optimizing for x_f

We will now return to our original expression for the final y-position ($= 0$) to create an expression for x_f.

1. Solving for x_f We first take the previous expression for x_f and supply the expression for θ_0.

$$
\begin{aligned}
0 =& \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 && (52)\\
0 =& \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 sec^2(arctan(\frac{2(H-h_0)}{x_f})) + x_f tan(arctan(\frac{2(H-h_0)}{x_f})) + h_0 && (53)\\
0 =& \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2((\frac{2(H-h_0)}{x_f})^2 + 1) + x_f(\frac{2(H-h_0)}{x_f}) + h_0 && (54)\\
0 =& \frac{-1}{4}\frac{1}{(H-h_0)}((2(H-h_0))^2 + {x_f}^2) + 2(H-h_0) + h_0 && (55)\\
0 =& \frac{-1}{4}((4(H-h_0)) + \frac{{x_f}^2}{(H-h_0)}) + 2(H-h_0) + h_0 && (56)\\
0 =& (-(H-h_0) - \frac{{x_f}^2}{4(H-h_0)}) + 2(H-h_0) + h_0 && (57)\\
\frac{{x_f}^2}{4(H-h_0)} =& -(H-h_0) + 2(H-h_0) + h_0 && (58)\\
{x_f}^2 =& -4(H-h_0)(H-h_0) + 4(H-h_0)2(H-h_0) + 4(H-h_0)h_0 && (59)\\
{x_f}^2 =& -4(H-h_0)^2 + 8(H-h_0)^2 + 4h_0(H-h_0) && (60)\\
{x_f}^2 =& 4(H-h_0)^2 + 4h_0(H-h_0) && (61)\\
& && (62)
\end{aligned}
$$

2. Finding $\frac{d{x_f}^2}{dh_0}$ We know that, by optimizing for ${x_f}^2$, x_f is optimized due to the setup of the problem of the behavior of the length of line.

   Hence, we take the *first* derivative, though of ${x_f}^2$ w.r.t. h_0 and with H held constant.

$$
\begin{aligned}
{x_f}^2 =& 4(H-h_0)^2 + 4h_0(H-h_0) && (63)\\
\Rightarrow \frac{d{x_f}^2}{dh_0} =& \frac{d}{dh_0}(4(H-h_0)^2 + 4h_0(H-h_0)) && (64)\\
=& 4\frac{d}{dh_0}(H-h_0)^2 + 4\frac{d}{dh_0}h_0(H-h_0) && (65)\\
=& 4\frac{d}{dh_0}(H-h_0)^2 + 4((H-h_0)\frac{d}{dh_0}h_0 + h_0\frac{d}{dh_0}(H-h_0)) && (66)\\
=& 4\frac{d}{dh_0}(H-h_0)^2 + 4((H-h_0) - h_0) && (67)\\
=& -8(H-h_0) + 4(H-h_0) - 4h_0 && (68)\\
=& -4(H-h_0) - 4h_0 && (69)\\
=& -4H + 4h_0 - 4h_0 && (70)\\
=& -4H && (71)
\end{aligned}
$$

3. Optimizing for x_f The optimization of x_f requires a little bit more thinking of the scenario of the problem. H must be a positive number, and the expression for ${x_f}^2$ appears to be a straight line with slope $-4H$. Given $H > 0$, $-4H < 0$, and the slope of ${x_f}^2$ is negative — as h_0 increases, ${x_f}^2$ decreases. Given h_0 must be positive, then, $h_{0optim} = 0$.

3.4.3 | Solving for the Actual Optimum of θ_0

We return to following statement:

$$\theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{72}$$

The expression we derived above for x_f under optimal conditions is, per the previous statement:

$$x_f{}^2 = 4(H-h_0)^2 + 4h_0(H-h_0) \tag{73}$$
$$\Rightarrow x_f = \sqrt{4(H-h_0)^2 + 4h_0(H-h_0)} \tag{74}$$

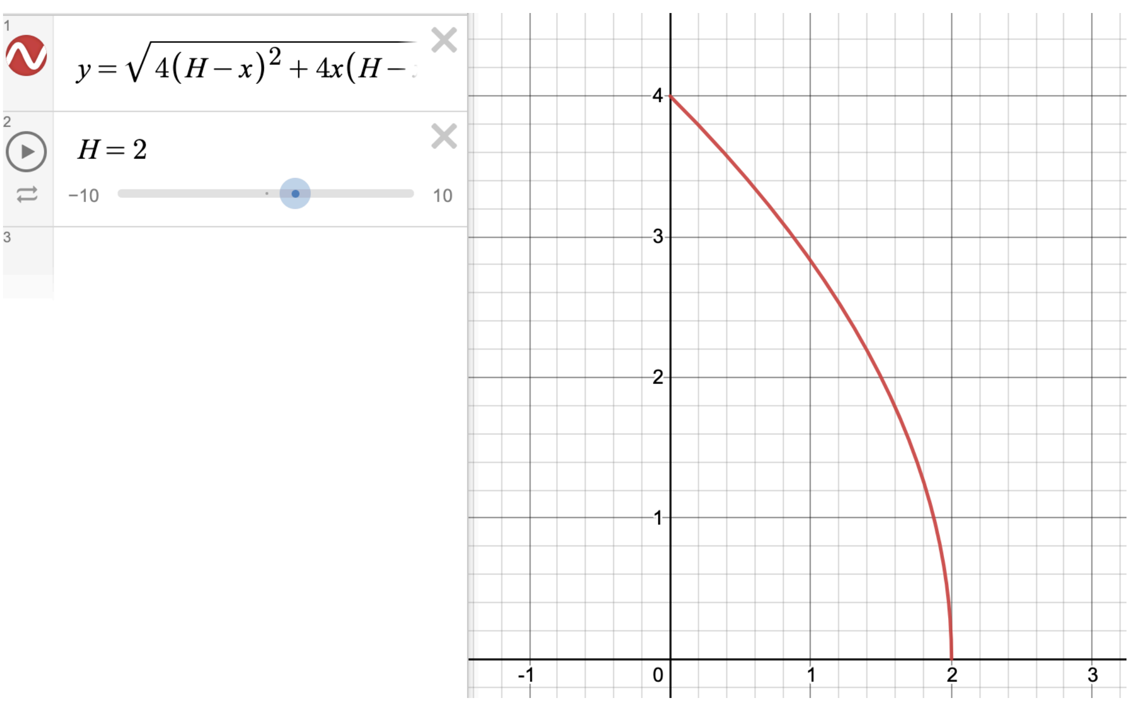
And substituting the statement for x_f back into the expression above, in addition to the optimal value $h_0 = 0$ derived earlier, we derive that

$$\theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{75}$$
$$\Rightarrow \theta_0 = arctan(\frac{2(H-h_0)}{\sqrt{4(H-h_0)^2 + 4h_0(H-h_0)}}) \tag{76}$$
$$= arctan(\frac{2(H)}{\sqrt{4(H)^2}}) \tag{77}$$
$$= arctan(1) \tag{78}$$
$$= 45^\circ \tag{79}$$

And hence, given an arbitrary angle and an arbitrary height to launch, the most optimal scenario is to make the launch point fully on the floor at an 45° angle from the plane of the floor.

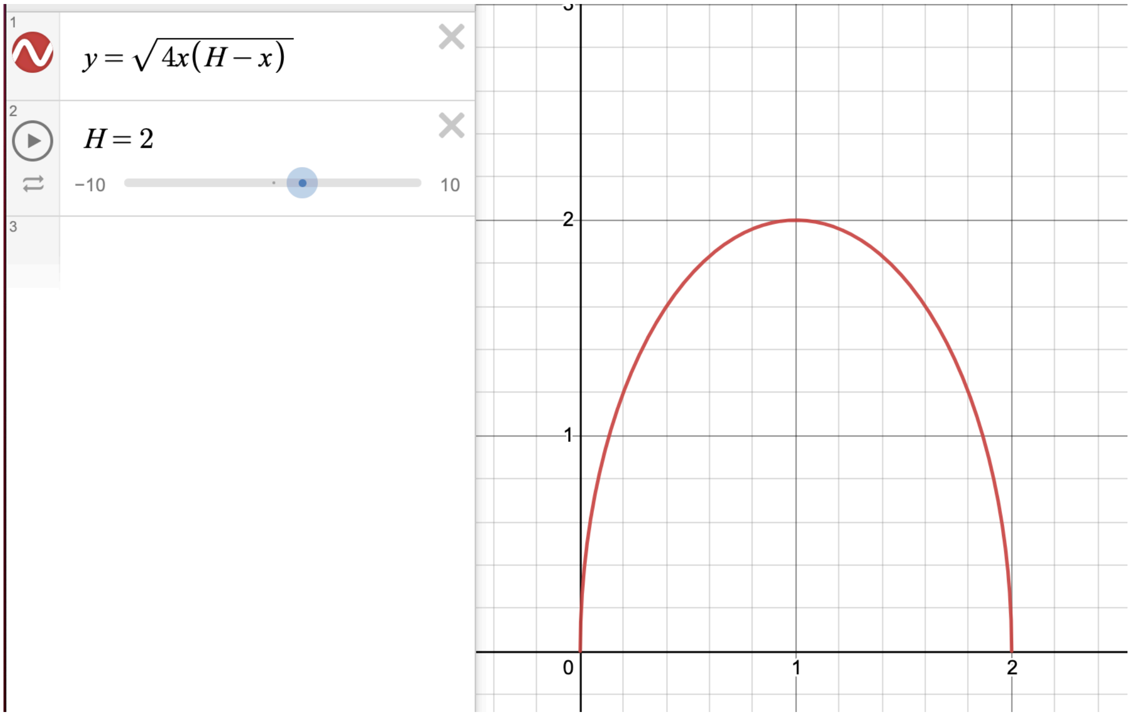
4 | Additional Figures

4.1 | Projected Range w.r.t. varying h_0 at optimal θ_0



This figure is rendered in an Interactive Desmos Plot, the initial launch height, H, is fixed at an arbitrary non-zero value of 2.

4.2 | Projected Range w.r.t. varying h_0 at $\theta_0 = 0$



This figure is rendered in another Interactive Desmos Plot, the initial launch height, H, is again fixed at an arbitrary non-zero value of 2.

As one could see, a parabolic relationship could be seen such that the most optimal value is at exactly $\frac{1}{2}H$.