PS#30: THE MASTER FORMULA!!!

Nueva Multivariable Calculus

1. Using the method of taking the determinant of the derivative matrix (the "Jacobian determinant"), verify the differential correction factor for a polar double integral, $dA = r\,dr\,d\theta$. You can re-watch the video we watched in class, or Google other resources for "change of variables jacobian determinant" for some guidance. We derived this correction factor using some *ad hoc* geometry; now we get to see how this more general and more powerful method of using a Jacobian determinant gets us the same answer, mechanically, algebraically, automatically!

2. Likewise, verify the differential correction formula for a spherical triple integral, $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$!

3. We've found two separate correction formulas for calculating the surface area of a curvy surface. However, neither of our two formulations for the surface area differential element involve polar/cylindrical coordinates. One involves rectangular coordinates:

   given a curvy surface $z = f(x, y)$:

$$dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

   and the other involves a parameterization:

   given a curvy surface parameterized by $\vec{v}(t, u)$:

$$dA = \left|\frac{\partial \vec{v}}{\partial t} \times \frac{\partial \vec{v}}{\partial s}\right| dt\,ds$$

   What if we want to find the surface area of a curvy surface that's described in polar coordinates? Can we find a formula for the surface area correction factor *in polar coordinates*??? In other words, what if we have $z = f(r, \theta)$ (okay, I guess this is really cylindrical coordinates)—what's the curvy-surface correction factor for that??

   (Note that it's not $dA = r\,dr\,d\theta$. That's the correction factor for a *flat* surface described in polar coordinates. We need the correction factor for a potentially-curvy surface! Similarly, $dA = dx\,dy$ is the "correction factor" for a flat surface in rectangular coordinates (though we wouldn't even call it that). For a curvy surface described in rectangular, we need the big fancy formula above. It reduces to just $dx\,dy$ when the surface is flat, because then $\partial z/\partial x$ is zero and $\partial z/\partial y$ is zero, so this becomes:

$$dA = \sqrt{1 + \underbrace{\left(\frac{\partial z}{\partial x}\right)^2}_{=0} + \underbrace{\left(\frac{\partial z}{\partial y}\right)^2}_{=0}}\,dx\,dy$$

$$= \sqrt{1}dx\,dy$$

$$= dx\,dy$$

   I.e., just a normal rectangular double integral, over a flat, non-curvy region.)