> You are building a swimming pool for outside of your Pringles-shaped house. It's going to be circular, with a 40 meter diameter. The depth will be constant latitudinally (i.e., along east-west lines), and will increase linearly from two meters at the south end to seven meters at the north end. How much water will you need to fill it? Calculate this both using a double integral a) in rectangular and b) in polar. How much will all that water cost? (Seriously! Look up whatever the prevailing water rates are in your municipality, or ask your parents about their water bill, and calculate it!) (Pictures!)

We know that the shape upon which are integrating would be circular, with radius 20. The shape of our pool would be latitudinally inclides along the north-south direction. If we define the latitudinal axis as x, and the longitudinal ones as y, we understand that the shape would be a function in y.

At -20, we will have a height of 2. At 20 (a whole diameter later), we will have a height of 7. Figuring the slope of this, we get that for a change in location by 8 units, we will have a change in height of 1 unit.

Supplying the expression into the slope of $\frac{1}{8}$, to maintain $y = -20$, and $z = 2$, we undrestand that the intercept ("bias") in the function would be 4.5.

Hence, the expression needed would be:

$$f : \mathbb{R}^2 \to \mathbb{R}^1 \tag{1}$$

$$f(x,y) = \frac{1}{8}y + 4.5 \tag{2}$$

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z = var("z")
def bound_fn(x,y,z):
    return x^2 + y^2 < 20^2
f(x,y) = (1/8)*y+4.5

implicit_plot3d(f-z, (x, -20,20), (y, -20, 20), (z, 2,7), region=bound_fn, plot_points=100)
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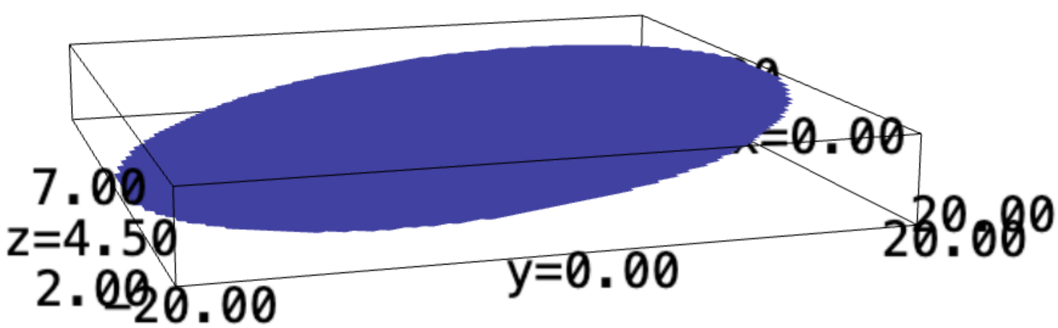

We will now take this expression in two ways: in both Cartesian and Cylindrical coordinates.

1 | Cartesian Coordinates

Our function, expression again, would be:

$$f(x,y) = \frac{1}{8}y + 4.5 \tag{3}$$

Furthermore, our bound function can be expressed as:

$$x^2 + y^2 = 20^2 \tag{4}$$

At every point x, then, we can see that our bound renders it as $y = \pm\sqrt{20^2 - x^2}$ (the top and bottom half of circle). We will take this as our bound on y and integrate the function as such. We will integrate along the y dimension first, then integrate along x.

$$\int_{-20}^{20}\int_{-\sqrt{400-x^2}}^{\sqrt{400-x^2}} \frac{1}{8}y + 4.5\ dy\ dx \tag{5}$$

$$\Rightarrow \int_{-20}^{20} \left(\frac{1}{8}\frac{y^2}{2} + 4.5y\Big|_{-\sqrt{400-x^2}}^{\sqrt{400-x^2}} \right) dx \tag{6}$$

$$\Rightarrow \int_{-20}^{20} 9\sqrt{20^2 - x^2}\ dx \tag{7}$$

At this point, we perform a trig substitution. We will substitute $x = 20\sin\theta$, and therefore $dx = 20\cos\theta d\theta$. Performing the actual substitution, then:

$$9\sqrt{20^2 - 20^2\sin^2\theta}\ 20\cos\theta\ d\theta \tag{8}$$

$$\Rightarrow 9\sqrt{20^2(1 - \sin^2\theta)}\ 20\cos\theta\ d\theta \tag{9}$$

$$\Rightarrow 9\sqrt{20^2\cos^2\theta}\ 20\cos\theta\ d\theta \tag{10}$$

$$\Rightarrow 9(20^2\cos^2\theta)d\theta \tag{11}$$

$$\Rightarrow 3600\cos^2\theta d\theta \tag{12}$$

Let's further transform the bounds of the expression:

$$20 = 20\sin\theta \tag{13}$$

$$\Rightarrow 1 = \sin\theta \tag{14}$$

$$\Rightarrow \theta = \frac{\pi}{2} \tag{15}$$

$$-20 = 20\sin\theta \tag{16}$$

$$\Rightarrow -1 = \sin\theta \tag{17}$$

$$\Rightarrow \theta = \frac{3\pi}{2} \tag{18}$$

Supplying this back to the integral, we get that:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3600\ cos^2\theta d\theta \tag{19}$$

We will transform this using the double-angle formula:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3600\ cos^2\theta d\theta \tag{20}$$
$$\Rightarrow \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3600\ \frac{cos\ 2\theta + 1}{2} d\theta \tag{21}$$
$$\Rightarrow \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1800\ (cos\ 2\theta + 1) d\theta \tag{22}$$
$$\Rightarrow 1800 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (cos\ 2\theta + 1) d\theta \tag{23}$$
$$\Rightarrow 1800 \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} cos\ 2\theta\ d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \right) \tag{24}$$
$$\Rightarrow 1800 \left(\frac{1}{2}\ sin\ 2\theta\ |_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \pi \right) \tag{25}$$
$$\Rightarrow 1800\pi \tag{26}$$

We can see that the area under this shape is 1800π.

2 | Cylindrical Coordinates

Our function, at this point, is:

$$f(x, y) = \frac{1}{8}y + 4.5 \tag{27}$$

We will parameterize the function based on the regular parameterization scheme for polar/Cartesian conversion, based on triangles on the unit circle:

$$\begin{cases} y = r\ \text{sin}(\theta) \\ x = r\ \text{cos}(\theta) \end{cases} \tag{28}$$

Performing the actual parameterization, then:

$$f(r, \theta) = \frac{1}{8} r\ \text{sin}(\theta) + 4.5 \tag{29}$$

We understand that, at area point $d\theta$, the circumference of a ring upon which we are integrating is $rd\theta$. Therefore:

$$dA = r\ d\theta\ dr \tag{30}$$

Taking the integral over a full circle with radius 20, then:

$$
\begin{aligned}
&\int_0^{20} \int_0^{2\pi} r \left(\frac{1}{8} r \sin(\theta) + 4.5\right) d\theta\ dr && (31)\\
\Rightarrow &\int_0^{20} \int_0^{2\pi} \left(\frac{1}{8} r^2 \sin(\theta) + 4.5r\right) d\theta\ dr && (32)\\
\Rightarrow &\int_0^{20} \left(\frac{-1}{8} r^2 \cos(\theta) + 4.5r\theta\right)\Bigg|_0^{2\pi}\ dr && (33)\\
\Rightarrow &\int_0^{20} 9r\pi\ dr && (34)\\
\Rightarrow &\frac{9}{2} r^2 \pi\Bigg|_0^{20} && (35)\\
\Rightarrow &1800\pi && (36)
\end{aligned}
$$

We can see that the value of the function under the circle is 1800π. This agrees with our previous derivation.