11 | cubic and a line

11.1 | show tangency

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 - x^3) = 8x - 3x^2\Big|_3$$
$$= 24 - 27 = -3$$
$$\frac{dy}{dx} = \frac{d}{dx}(18 - 3x) = -3$$

11.2 | area between curves

$$\begin{aligned}\int_3^6 18 - 3x - 4x^2 + x^3 dx \to &\frac{1}{4}x^4 - \frac{1}{3}4x^3 - \frac{1}{2}3x^2 + 18x + C\\ &= \frac{1}{4}(6)^4 - \frac{1}{3}4(6)^3 - \frac{1}{2}3(6)^2 + 18(6)\\ &- \frac{1}{4}(3)^4 + \frac{1}{3}4(3)^3 + \frac{1}{2}3(3)^2 - 18(3)\\ &= \boxed{\frac{261}{4}}\end{aligned}$$

12 | estimate area

Right handed Riemann Sum:

$$0.5 + 4 + 10 + 13 + 10 + 0 = 37.5$$

13 | estimate area again

$$4(200 + 2700 + 1100 + 4000 + 200) = 32800$$

14 | area between curves

$$\begin{aligned}\int_0^{10} 2200e^{0.024t}dx - \int_0^{10} 1360e^{0.018t}dx &= \frac{1}{0.024}2200e^{0.024t} - \frac{1}{0.018}1360e^{0.018t}\\ \implies \quad &\frac{1}{0.024}2200e^{0.24} - \frac{1}{0.018}1360e^{0.18} - \frac{1}{0.024}2200 + \frac{1}{0.018}1360 \quad \approx 9964\end{aligned}$$

The area represents the population over those ten years.

15 | **meaning of area**

The shaded region represents the profit made between producing 50 units and 100 units.

16 | **slicing pizza into three using parallel cuts**

The problem of placing slices is the same if we only worry about the top half of the pizza. Thus, we can choose some x for the first slice s.t.

$$
\begin{aligned}
2\int_{-7}^{x}\sqrt{7^2-t^2}dt &= \int_{x}^{7}\sqrt{7^2-t^2}dt \\
2\int_{-7}^{x}\sqrt{7^2-t^2}dt - \int_{x}^{7}\sqrt{7^2-t^2}dt &= 0 \\
2\int_{-7}^{x}\sqrt{7^2-t^2}dt + \int_{7}^{x}\sqrt{7^2-t^2}dt &= 0 \\
2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{-7}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) &= 0 \\
2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) &= 0 \\
2\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt + \int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt &= 0 \\
3\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt &= 0 \\
3\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt &= 0 \\
3\int_{0}^{x}\sqrt{7^2-t^2}dt + \frac{49\pi}{4} &= 0
\end{aligned}
$$

Now, we need to use trigonometric substitution, apparently.

$$x = a\sin\theta, dx = a\cos\theta d\theta$$

Or, you could use David's method, which is just better (cut horizontally instead of vertically)

$$
\begin{aligned}
\int_{-7}^{7}\sqrt{49-x^2} - a dx &= \frac{49\pi}{3} \\
\int_{-7}^{7}\sqrt{49-x^2}dx - \int_{-7}^{7} a dx &= \frac{49\pi}{3} \\
\frac{49\pi}{2} - \int_{-7}^{7} a dx &= \frac{49\pi}{3} \\
\frac{49\pi}{2} - (7a - -7a) &= \frac{49\pi}{3} \\
\frac{49\pi}{6} &= 14a \\
a = \frac{49\pi}{84} &= \frac{7\pi}{12}
\end{aligned}
$$

Since a is the upper half of the center portion, the width of each slice is $2a = \frac{7\pi}{6}$

17 | tractrix

17.1 | derivative

At any moment, if the boat is at (x, y) and the puller is at $(0, h)$, then the velocity of the boat is in the direction

$$\frac{\Delta y}{\Delta x}$$

Where $\Delta x = -x$ and Δy can be found using the Pythagorean Theorem

$$\begin{aligned} & L^2 = \Delta y^2 + x^2 \\ \implies \quad & \Delta y = \sqrt{L^2 - x^2} \end{aligned}$$

Thus, the boat is moving in the direction

$$\frac{\sqrt{L^2 - x^2}}{-x}$$

17.2 | integral

$$\begin{aligned} \int \frac{\sqrt{L^2 - x^2}}{-x} dx &= -\int \frac{1}{x}\sqrt{L^2 - x^2} dx \\ \text{Let } x = L\sin\theta, dx = L\cos\theta d\theta & \\ &= -\int \frac{1}{L\sin\theta}\sqrt{L^2 - L^2\sin^2\theta} dx \\ &= -\int \frac{1}{L\sin\theta} L\sqrt{1 - \sin^2\theta} dx \\ &= -\int \frac{1}{L\sin\theta} L\sqrt{\cos^2\theta} dx \\ &= -\int \frac{L\cos\theta}{L\sin\theta} L\cos\theta d\theta \\ &= -\int L\frac{\cos^2\theta}{\sin\theta} d\theta \\ &= -L\int \frac{1}{\sin\theta} d\theta + L\int \sin\theta d\theta \\ &= L\ln|\csc\theta + \cot\theta| - L\cos\theta + C \\ &= L\ln\left|\frac{L}{x} + \frac{\sqrt{L^2 - x^2}}{x}\right| - \sqrt{L^2 - x^2} + C \end{aligned}$$

18 | TODO water displacement

Plan: find a function $f(r)$ which represents the amount of water displaced for any radius, then take the derivative and find roots.