

1 | 1)

To finish the proof... Given two objects, A and B , with a force F between them, the torque on A and B is given by

$$\tau_A = \vec{r}_A \times \vec{F}_A$$

$$\tau_B = \vec{r}_B \times \vec{F}_B$$

where \vec{F}_A is the force applied by B on A , and vice versa. We know that because of N-3 $\vec{F}_A = -\vec{F}_B$. (We

$$\tau_{AB} = \tau_A + \tau_B$$

also know that the forces point towards each object.) Therefore,

$$\begin{aligned} &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times \vec{F}_B \\ &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A \end{aligned}$$

We know that the direction of the two cross products are orthogonal to the plane that the two objects' position vectors and the origin of the system form.

$$\begin{aligned} \tau_{AB} &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A \\ &= |\vec{r}_A||\vec{F}_A| \sin \theta_A - |\vec{r}_B||\vec{F}_A| \sin \theta_B \\ &= |\vec{r}_A| \sin \theta_A - |\vec{r}_B| \sin \theta_B \end{aligned}$$

The law of sines states that for a triangle $\triangle ABC$, $\frac{\overline{BC}}{\sin \theta_A} = \frac{\overline{AC}}{\sin \theta_B}$. We know that this applies in our particular proof because the objects A , B , and the origin form a triangle. As such,

$$\begin{aligned} |\vec{r}_A| \sin \theta_A &= |\vec{r}_B| \sin \theta_B \\ \tau_{AB} &= 0 \end{aligned}$$

The internal torque of any two objects of a system is zero, so the total internal torque must also be zero.

2 | 2)

$$\vec{r} = R\hat{i} + h\hat{k}$$

$$\vec{L}_1 = \vec{r} \times m\vec{v}$$

We know that for one of the \vec{L} s: $\vec{v} = R\omega\hat{j}$

$$\begin{aligned} \vec{L}_1 &= (R\hat{i} + h\hat{k}) \times mR\omega\hat{j} \\ &= -hmR\omega\hat{i} + mR^2\omega\hat{k} \end{aligned}$$

We know that there are two masses, symmetric about the z-axis, so we know that the angular momentum of the other object can be derived just by multiplying the \hat{i} and \hat{j} terms by -1.

$$\begin{aligned} \vec{L}_2 &= \vec{L}_1 \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= hmR\omega\hat{i} + mR^2\omega\hat{k} \end{aligned}$$

We can add the two to get the aggregate angular momentum of the system:

$$\begin{aligned}
 \vec{L} &= \vec{L}_1 + \vec{L}_2 \\
 &= (-hmR\omega\hat{i} + mR^2\omega\hat{k}) + (hmR\omega\hat{i} + mR^2\omega\hat{k}) \\
 &= 2mR^2\omega\hat{k}
 \end{aligned}$$

3 | 3)

3.1 | a)

We can think of the total angular momentum of an N -mass system as the sum of the z-components of their angular momentum. We only have to worry about their z components because it is given that the masses are symmetric about the center, and as such, the x and y components should cancel out. $\vec{L}_N = \sum_{i=1}^N m_i l_i^2 \cdot \omega \hat{k}$

3.2 | b)

We can think of the total angular momentum as the above sum as N approaches infinity. $\vec{L} = \hat{k}\omega \sum_{i=1}^N m_i l_i^2$

We can think of l as a function of m . We can turn our sum into an integral over the volume:

$$\begin{aligned}
 \vec{L} &= \hat{k}\omega \int_V l^2 dm \\
 dm &= \frac{M}{V_0} dV \\
 \vec{L} &= \hat{k}\omega \int_V l^2 \frac{M}{V_0} dV
 \end{aligned}$$

4 | 4)

We can represent the angular momentum of each point on the rod and integrate over them to find the total

$$L(r) = r\hat{i} \times m_r \vec{v}_r$$

$$\vec{v}_r = r\omega\hat{j}$$

angular momentum. $L(r) = r\hat{i} \times rm_r\omega\hat{j}$

$$= |rm_r\omega||r|\hat{k}$$

$$= r^2 m_r \omega \hat{k}$$

We can now integrate this with respect to r , from $-\frac{L}{2}$ to $\frac{L}{2}$. As we are integrating, we will remove m_r from our function and replace it with λ , as the point mass will be infinitesimally small but will sum to M , and the integral without the mass component will sum to L .

$$\begin{aligned}
\vec{L} &= \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \omega \lambda \hat{k} dr \\
&= \omega \lambda \hat{k} \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 dr \\
&= \omega \lambda \hat{k} \cdot \left[\frac{r^3}{3} \right]_{-\frac{L}{2}}^{\frac{L}{2}} \\
&= \omega \lambda \hat{k} \cdot 2 \frac{L^3}{24} \\
&= \omega \lambda \hat{k} \cdot \frac{L^3}{12} \\
&= \omega \hat{k} \cdot \frac{1}{12} L^2 M \\
&= \frac{1}{12} M L^2 \omega \hat{k}
\end{aligned}$$

5 | 5)

We can first take an equation for the angular momentum of a point, and integrate it polar-ly and then about R .

$$\begin{aligned}
\vec{L} &= \int \int_A \vec{L}_{\theta,r} d\theta dr \\
&= \int_0^R \int_0^{2\pi} r^2 \omega \sigma \hat{k} d\theta dr \\
&= \omega \sigma \hat{k} \int_0^R \int_0^{2\pi} r^2 d\theta dr \\
&= \omega \sigma \hat{k} \int_0^R 2\pi r^2 dr
\end{aligned}$$

We take some inspiration from Problem 4: $\vec{L}_{\theta,r} = r^2 \omega \sigma \hat{k} d\theta dr$ Now we integrate:

$$\begin{aligned}
&= \omega \sigma \hat{k} \cdot 2\pi \frac{R^3}{3} \\
&= \omega \hat{k} \cdot 2\pi \frac{M}{\pi R^2} \cdot \frac{R^3}{3} \\
&= \omega \hat{k} \cdot \frac{2MR}{3}
\end{aligned}$$

Wait, What? I need help on this