

# 1 | Cauchy-Schwarz Inequality

**important**

'One of the most important inequalities in mathematics'

Suppose  $u, v \in V$  (where  $V$  is an inner product space). Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

The inequality is an equality iff one of  $u, v$  is a scalar multiple of the other.

## 1.1 | intuition

For the Euclidean inner product, this is true because  $\langle u, v \rangle = \|u\| \|v\| \cos \theta$ . However, the Cauchy-Schwarz inequality works for all inner product spaces, using the generalized Pythagorean theorem (instead of the law of cosines).

## 1.2 | proof is by the orthogonal decomposition

By homogeneity of norms,

$$\left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 = \left| \frac{\langle u, v \rangle}{\|v\|^2} \right|^2 \|v\|^2$$

## 1.3 | results

### 1.3.1 | triangle inequality

Suppose  $u, v \in V$ . Then

$$\|u + v\| \leq \|u\| + \|v\|$$

The inequality is an equality if and only if one of  $u, v$  is a non-negative multiple of the other (degenerate triangle)

This is proven by noticing that  $\langle u, v \rangle + \langle v, u \rangle = \langle u, v \rangle \overline{\langle v, u \rangle} = 2\operatorname{Re}\langle u, v \rangle \leq 2|\langle u, v \rangle| \leq 2\|u\| \|v\|$  by conjugate symmetry and Cauchy-Schwarz.

### 1.3.2 | Parallelogram Equality

Suppose  $u, v \in V$ . Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$