1 | Problem 1

The plane that passes through the vector $\vec{P}_o$ and is perpendicular to $\vec{n}$ is defined by:

$$\{\vec{r} : (\vec{r} - \vec{P}_o) \cdot \vec{n} = 0, \vec{P}_o \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

This works because:

- The $(\vec{r} - \vec{P}_o)$ term is similar to when you subtract some value from x (in a cartisian plane) to shift it over to the right. In a senes, here you are subtracting the postition vector from every single vector on the plane defined by $\vec{r}$, thus shifting the plane to $\vec{P}_o$.
- Setting the dot product between the term above and $\vec{n}$ to 0, ensures that the plane is perpendicular to the normal vector $\vec{n}$

2 | Problem 2

First we can defined:

- $\vec{n} = (n_x, n_y, n_z)$
- $\vec{P}_o = (P_{ox}, P_{oy}, P_{oz})$
- $\vec{r} = (x, y, z)$

Then we can evaluate: $(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$:

$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$
so we see that the cartisian definition of a plane is: $xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$
From this we see:

- $A = n_x$
- $B = n_y$
- $C = n_z$
- $D = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$

Therefore, the normal vector is (A, B, C)

3 | Problem 3

First we can define:

- $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle$
- $\vec{r} = \langle x, y, z \rangle$

Then we can evaluate the equation:

$\hat{n} \cdot \vec{r} = D$
$\Rightarrow \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle \cdot \langle x, y, z \rangle = D$
$\Rightarrow \frac{n_x}{|\vec{n}|} x + \frac{n_y}{|\vec{n}|} y + \frac{n_z}{|\vec{n}|} z = D$

This is the equation for a cartisian definition of a plane, with $\hat{n}$ representing the normal vector and D representing $\frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$ which is the distance from the origin to the plane.

The value of D is found by starting with the orginal vector definition of a plane:

$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow \frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow \vec{r} \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow D = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

Looking at problem 4 we see that this is the distance from the plane to the origin.

4 | **Problem 4**

We can start with this drawing:

IMAGE

In the image we see that there are multple "point $\vec{P}_o$'s" that go from point P_o to the plane. We are trying to solve for d which is the shortest distance from the plane to the point, it can also be defined as the length of the $\vec{P}_o$ that is perpendiclar to the plane. Because all of the $\vec{P}_o$'s come from the same point and $\vec{n}$ is perpendicular to the plane we can find d by finding $comp_{\hat{n}} \vec{P}_o$:

$d = comp_{\hat{n}} \vec{P}_o = 1 \cdot |\vec{P}_o| \cos(\theta)$
$= |\hat{n}||\vec{P}_o| \cos(\theta)$
$= \vec{P}_o \cdot \hat{n}$
$= \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

From problem 2 we know that $\vec{P}_o \cdot \vec{n} = 4$ and that $\vec{n} = \langle 1, 2, 3 \rangle$ and thus:

$$d = \frac{4}{\sqrt{14}}$$

5 | **Problem 5**

We can do something similar to what was done in the problem above in which the $\hat{n}$ component of $\vec{P}_o$ can be used as the distance d, but because $\vec{P}_o$ is center at the origin and the plane may not pass through the origin, the postion vector of the plane, we'll call it $\vec{P}_1$, has to be subtracted from $\vec{P}_o$. To find the length of $\vec{P}_1$ we can do what we did in problem 4.

We'll break this problem into three parts: 1) finding the length of $\vec{P}_o$ parallel to $\hat{n}$ 2) findng the distance of $\vec{P}_1$ parallel to $\hat{n}$ 3) subtracting the two:

5.1 | Finding length of $\vec{P}_o$

From problem 2 we know that $\vec{n} = \langle A, B, C \rangle$. From problem 4 we know that we can find the length of $\vec{P}_o$ parallel to $\hat{n}$ by:

$\vec{P}_o \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$= \frac{\langle x_o, y_o, z_o \rangle \cdot \langle A, B, C \rangle}{|\langle A, B, C \rangle|}$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2+B^2+C^2}}$

5.2 | Finding the length of $\vec{P}_1$

From problem 2 we know that $\vec{n} = \langle A, B, C \rangle$. From problem 4 we know that we can find the length of $\vec{P}_1$ parallel to $\hat{n}$ by:

$$\vec{P}_1 \cdot \hat{n} = \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$$

From problem 2 we know that the dot product between the position vector and the noraml vector is D. Thus:

length of$\vec{P}_1 = \vec{P}_1 \cdot \hat{n}$
$= \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$
$= \frac{D}{|\vec{n}|}$
$= \frac{D}{|\langle A, B, C \rangle|}$
$= \frac{D}{\sqrt{A^2+B^2+C^2}}$

5.3 | Subtratcting the two:

$d = \vec{P}_o - \vec{P}_1$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2+B^2+C^2}} - \frac{D}{\sqrt{A^2+B^2+C^2}}$
$= \frac{Ax_o + By_o + Cz_o - D}{\sqrt{A^2+B^2+C^2}}$

Lastly, you would take the absolute value of the numorator because distances are positive (and the denominator is already positive due to the squaring of A, B, and C). Thus:

$$d = \frac{|Ax_o + By_o + Cz_o - D|}{\sqrt{A^2+B^2+C^2}}$$

6 | Problem 6

First we can define:

- $\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$
- $\vec{B}(t) = B_x(t)\hat{i} + B_y(t)\hat{j} + B_z(t)\hat{k}$

Thus:

$\frac{d}{dt}(\vec{A}(t) \cdot \vec{B}(t)) = \frac{d}{dt}(A_x(t)B_x(t) + A_y(t)B_y(t) + A_z(t)B_z(t))$
$= \frac{d}{dt}(A_x(t)B_x(t)) + \frac{d}{dt}(A_y(t)B_y(t)) + \frac{d}{dt}(A_z(t)B_z(t))$

$= A'_x(t)B_x(t) + A_x(t)B'_x(t) + A'_y(t)B_y(t) + A_y(t)B'_y(t) + A'_z(t)B_z(t) + A_z(t)B'_z(t)$
$= (A'_x(t)B_x(t) + A'_y(t)B_y(t) + A'_z(t)B_z(t)) + (A_x(t)B'_y(t) + A_y(t)B'_y(t) + A_z(t) + B'_z(t))$
$= \vec{A}'(t) \cdot \vec{B}(t) + \vec{A}(t) \cdot \vec{B}'(t)$
$= \frac{d\vec{A}(t)}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}(t)}{dt}$

7 | **Problem 7**

First we can define:

- $\vec{r} = |\vec{r}|\hat{r}$
  - this works because the unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, and so you can multiply both sides by $|\vec{r}|$

We can start by finding the derivative of $\vec{r}$ with the definition above:

$\frac{d}{dt}\vec{r}(t) = \frac{d}{dt}(|\vec{r}(t)|\hat{r}(t))$
$= (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$

We can take this equation and solve for $\frac{d}{dt}|\vec{r}(t)|$:

$\frac{d}{dt}\vec{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \hat{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) \cdot \hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t)) \cdot \hat{r}(t)$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \frac{\vec{r}(t)}{|\vec{r}(t)|} = (\frac{d}{dt}|\vec{r}(t)|) \cdot 1 + |\vec{r}(t)| \cdot 0$
$\Rightarrow \frac{1}{|\vec{r}(t)|}\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}|\vec{r}(t)|$

This proof relied on the fact that the dot product between a vector and it's derivative is zero. This is true because:

$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = C$, where C is some constant.

If we differentiate this then we get:

$\frac{d}{dt}\vec{r}(t)^2 = \frac{d}{dt}C$
$\Rightarrow 2\vec{r}(t) \cdot \vec{r}'(t) = 0$
$\Rightarrow \vec{r}(t) \cdot \vec{r}'(t) = 0$

The chain rule appiles to vectors (show in a different assignment.

Note: for this problem, I got Albert's help with the initial proof, and then I used this source to help with proving that the dot product between a vector and it's derivative is zero: `https://www.reddit.com/r/askmath/comments/aticz8/why_is_a_vector_of_constant_magnitude_always/?scrlybrkr=5f034675`.

8 | **Problem 8**

If we start with the vector equation for a 3D line we get:

$$(x, y, z) = (x_o, y_o, z_o) + t(a, b, c) = (x_o + ta, y_o + tb, z_o + tc)$$

Where (x_o, y_o, z_o) is the position vector of the line (the line passes through this point), and (a, b, c) is the direction in which the line is traveling.

We can take the x, y and z components of the vector form and convert them into the parametric form of the equation of a 3D line:

$(x, y, z) = (x_o + ta, y_o + tb, z_o + tc)$
$\Rightarrow x = x_o + ta$

$\Rightarrow y = y_o + tb$
$\Rightarrow z = z_o + tc$

Lastly, we can solve each of the parameterized components for t and set them equal to each other to get the symmetric form:

$\Rightarrow t = \frac{x-x_o}{a}$
$\Rightarrow t = \frac{y-y_o}{b}$
$\Rightarrow t = \frac{z-z_o}{c}$
$\Rightarrow \frac{x-x_o}{a} = \frac{y-y_o}{b} = \frac{z-z_o}{c}$

With this in mind we can look at the given symmetric equation:

$\frac{x-2}{2} = \frac{y-1}{3} = 2 - z$
$\Rightarrow \frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{-1}$

From this we see that the position vector is $(2, 1, 2)$ and the direction of the line is the same direction as $(2, 3, -1)$

Similar to finding the distance between the plane and the origin we will need the normal vector. The dot product between the normal vector and the vector that describes the directing of the line must equal zero thus:

$\vec{n} \cdot \vec{d_o} = 0$
$(n_x, n_y, n_z) \cdot (2, 3, -1) = 0$
$2n_x + 3n_y - n_z = 0$
$2n_x + 3n_y = n_z$

There are many was that two vectors can be perpendicular, but in this case the normal vector also has to pass through a poin on the line described above. Thus:

$2(2 + 2t) + 3(1 + 3t) = 2 - t$
$\Rightarrow 4 + 4t + 3 + 9t = 2 - t$
$\Rightarrow 14t = -5$
$\Rightarrow t = -\frac{5}{14}$
$\Rightarrow (n_x, n_y, n_z) = (2 - \frac{10}{14}, 1 - \frac{15}{14}, 2 + \frac{5}{14}) = (\frac{18}{14}, \frac{-1}{14}, \frac{33}{14})$

Because we only need the unit vector of the normal vector (see problem 4), we can redefine it with nicer numbers by multiplying it by a constant (nicer numbers but direction is maintained):

$$\vec{n} = (18, -1, 33)$$

Therefore:

$d = \frac{\vec{P_o} \cdot \vec{n}}{|\vec{n}|}$
$= \frac{36-1+66}{\sqrt{18^2+(-1)^2+33^2}}$
$= \frac{101}{\sqrt{1414}}$

Note: for this problem I used this source to see what the different forms of the equation for a 3D line: but I proved that the vector form equaled the symmetric form myself: `https://math.stackexchange.com/questions/404440/what-is-the-equation-for-a-3d-line`.

9 | Problem 9