

1 | Elastic collision

We are given that the object m_1 collides with the rod with velocity v_0 , and the rod is floating in free space. Given m_1 , v_0 , m_2 , I_0 , and r , we are to figure out the final velocity of m_1 after collision v_f , the velocity of m_2 after collision v_{CM} , and of course the rotation of the rod after collision ω .

We are assuming that this collision is elastic.

We have, then, for conservation of linear momentum:

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \quad (1)$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2} m_1 v_0^2 + \frac{1}{2} m_1 v_0^2 = \left(\frac{1}{2} m_1 v_f^2 \right) + \left(\frac{1}{2} m_1 v_f^2 \right) + \left(\frac{1}{2} m_2 v_{CM}^2 \right) + \left(\frac{1}{2} I_0 \omega^2 \right) \quad (2)$$

$$\Rightarrow 2m_1 v_0^2 = (2m_1 v_f^2) + (m_2 v_{CM}^2) + (I_0 \omega^2) \quad (3)$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1 r^2 \left(\frac{v_0}{r} \right) = m_1 r^2 \left(\frac{v_f}{r} \right) + I_0 \omega \quad (4)$$

$$\Rightarrow m_1 r v_0 = m_1 r v_f + I_0 \omega \quad (5)$$

We now have a system of three equations that can be combined to solve for three unknowns v_f , v_{CM} , and ω .

Performing the actual solution digitally:

$$v_{cm} = \frac{4I_0 m_1 v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I_0 m_2} \quad (6)$$

$$v_f = \frac{(m_1 m_2 r^2 + I_0 m_1 - 2I_0 m_2) v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I_0 m_2} \quad (7)$$

and finally, we have

$$\omega = \frac{4m_1 m_2 r v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I_0 m_2} \quad (8)$$

2 | Rigid Body Kinetic Energy

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2} m_i v_i^2 \quad (9)$$

Because of the fact a point v_i can be defined as a sum of the velocity from the origin plus the displacement from origin ($v_i = v_{CM} + v'_i$), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (v_{CM} + v'_i)(v_{CM} + v'_i) \quad (10)$$