1 | Problem 1

1.1 | a)

Inorder for the boat to be going in the right direction we know that $\vec{C}+\vec{S}=\alpha\vec{D}$, where $\vec{C}$ is the current of the river, $\vec{S}$ is the speed of the boat, α is some scalar and $\vec{D}$ is the vector that goes from the boatman's starting point to their desired endpoint.

We can set the boatman's start point as $(0,0)$, and thus $\vec{D}=\langle 3,2\rangle$. We also know that $\vec{C}=\langle 0,-3.5\rangle$. Lastly, $\vec{S}=\langle 13\sin(\theta),13\cos(\theta)\rangle$, where θ is the angle between the side of the river and $\vec{S}$.

We can then plug in these values into the equation written above:

$\vec{C}+\vec{S}=\alpha\vec{D}$
$\Rightarrow \langle 0,-3.5\rangle+\langle 13\sin(\theta),13\cos(\theta)\rangle=\alpha\langle 3,2\rangle$
$\Rightarrow \langle 13\sin(\theta),-3.5+13\cos(\theta)\rangle=\langle \alpha 3,\alpha 2\rangle$
$\Rightarrow 13\sin(\theta)=\alpha 3,-3.5+13\cos(\theta)=\alpha 2$
$\Rightarrow 6\alpha=26\sin(\theta),6\alpha=-10.5+39\cos(\theta)$
$\Rightarrow 26\sin(\theta)=-10.5+39\cos(\theta)$

plug it into wolfram alpha:

$\theta\approx 0.75686$ radians or $\approx 43.36°$

1.2 | b)

The net velocity of the boat is $\vec{S}+\vec{C}=\langle 13\sin(\theta),13\cos(\theta)-3.5\rangle$, where θ is the answer to part a. To get the speed of the boat we find the magnitude of this vector:

$|\vec{S}+\vec{C}|=\sqrt{(13\sin(\theta))^2+(13\cos(\theta)-3.5)}\approx 10.7282$ km/h

Now we need to find the distance traveled by the boat, which should be the magnitude of $\vec{D}$:

$|\vec{D}|=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}\approx 3.60555$ km

To get the time it took to take the trip we divide the distace by the speed:

$\frac{3.60555}{10.7282}=0.336$ hours, which is 20.2 minutes

2 | Problem 2

2.1 | a)

$\vec{r}(t)=(R\cos(\omega_o t),R\sin(\omega_o t))$

This is because the x coordinate is defined as $r\cos(\theta)$ and the y coordinate is defined as $r\sin(\theta)$. In this case r or the radius is R and θ is $\omega_o t$, because $\omega_o=\frac{\theta}{t}$ (definition of angular velocity.

2.2 | b)

$\vec{v}(t)=\vec{r}'(t)=(\frac{d}{dt}R\cos(\omega_o t),\frac{d}{dt}R\sin(\omega_o t))=(-R\omega_o\sin(\omega_o t),R\omega_o\cos(\omega_o t))$

Answer: $\vec{v}(t)=(-R\omega_o\sin(\omega_o t),R\omega_o\cos(\omega_o t))$

2.3 | c)

$\vec{a}(t) = \vec{r}''(t) = (\frac{d}{dt}(-R\omega_o \sin(\omega_o t)), \frac{d}{dt} R\omega_o \cos(\omega_o t)) = (-R\omega_o^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t))$
Answer: $\vec{a}(t) = (-R\omega_o^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t))$

2.4 | d)

The tangent:

Because $\vec{v}(t)$ is a vector it can be placed anywhere on the plane, so the only requierment for $\vec{v}(t)$ is that it has to be perpendicular to $\vec{r}(t)$, which is the radius of the circle. This means that the slope of $\vec{v}(t)$ has to be the opposite reciprocol of $\vec{r}(t)$:

Slope of $\vec{r}(t) = \frac{\Delta y}{\Delta x} = \frac{R\sin(\omega_o t)-0}{R\cos(\omega_o t)-0} = \frac{\sin(\omega_o t)}{\cos(\omega_o t)}$

Slope of $\vec{v}(t) = \frac{\Delta y}{\Delta x} = \frac{R\omega_o\cos(\omega_o t)-0}{-R\omega_o\sin(\omega_o t)-0} = -\frac{\cos(\omega_o t)}{\sin(\omega_o t)}$

The slopes are opposite reciprocols, thus $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$, thus $\vec{v}(t)$ is tangent to the circle.

The magnitude:

$|\vec{v}(t)| = \sqrt{(-R\omega_o \cos(\omega_o t))^2 + (R\omega_o \sin(\omega_o t))^2} = \sqrt{R^2\omega_o^2 \cos^2(\omega_o t + R^2\omega_o^2 \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^2(\cos^2(\omega_o t) + \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^2(1)} = \sqrt{R^2\omega_o^2} = R\omega_o$

2.5 | e)

Point towards the center of the circle:

$\vec{a}(t)$ is a scalar multiple of $\vec{r}(t)$: $\vec{a}(t) = -\omega_o^2 \cdot \vec{r}(t) = -\omega_o^2(R\cos(\omega_o t), R\sin(\omega_o t)) = (-R\omega^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t)) = \vec{a}(t)$

because the scalar multiple is negative $\vec{r}(t)$ points in the opposite direction of $\vec{r}(t)$, which is towards the center of the circle because $\vec{r}(t)$ points from the center of the circle outwards.

The magnitude:

$|\vec{a}(t)| = \sqrt{(-R\omega^2 \cos(\omega_o t))^2 + (-R\omega_o^2 \sin(\omega_o t))^2} = \sqrt{R^2\omega_o^4 \cos^2(\omega_o t) + R^2\omega_o^4 \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^4(\cos^2(\omega_o t) + \sin^2(\omega_o t))} = \sqrt{R^2\omega_o^4(1)} = \sqrt{R^2\omega_o^4} = R\omega_o^2$

$\frac{|\vec{v}(t)|^2}{R} = \frac{(R\omega_o)^2}{R} = \frac{R^2\omega_o^2}{R} = R\omega_o^2$

2.6 | f)

$\theta'(t) = \int \theta''(t)dt = \int \alpha_o dt = \alpha_o t + c$, where c is the constant of integration

Answer: $\alpha_o t + c$

2.7 | g)

$\theta(t) = \int \theta'(t)dt = \int(\alpha_o t + c)dt = \int \alpha_o t dt + \int c dt = \frac{\alpha_o t^2}{2} + ct + c'$ where c' is another constant of integration.

Answer: $\frac{\alpha_o t^2}{2} + ct + c'$

2.8 | h)

Given that $\vec{r}(0) = (R, 0)$, we know that $c = 0$ and $c' = 0$:
$\vec{r}(t) = (R\cos(\frac{\alpha_o t^2}{2}), R\sin(\frac{\alpha_o t^2}{2}))$

2.9 | i)

$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = (\frac{d}{dt}R\cos(\frac{\alpha_o t^2}{2}, \frac{d}{dt}R\sin(\frac{\alpha_o t^2}{2})) = (-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$
Answer: $\vec{v}(t) = (-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$

2.10 | j)

$\vec{a}(t) = \vec{r}''(t) = (-\frac{d}{dt}R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), \frac{d}{dt}R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$

It will be easier to visualize by doing the math in it's x and y component:

x component:

$\frac{d}{dt} - R\alpha_o t\sin(\frac{\alpha_o t^2}{2}) = -R\alpha_o(\sin(\frac{\alpha_o t^2}{2}) + \alpha_o t^2\cos(\frac{\alpha_o t^2}{2})) = -R\alpha_o\sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2})$

y component:

$\frac{d}{dt}R\alpha_o t\cos(\frac{\alpha_o t^2}{2}) = R\alpha_o(\cos(\frac{\alpha_o t^2}{2}) - \alpha_o t^2\sin(\frac{\alpha_o t^2}{2})) = R\alpha_o\cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2})$

Putting these two together we get:

$\vec{a}(t) = (-R\alpha_o\sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2}), R\alpha_o\cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2}))$

2.11 | k)

The tangent:

As stated above, because $\vec{v}(t)$ is a vector, it can be placed anywhere (for example, the point where $\vec{r}(t)$ intersects with the circle). Thus we only need to show that $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$ in order to show that $\vec{v}(t)$ is tangent to the circle:

Slope of $\vec{r}(t) = \frac{\Delta y}{\Delta x} = \frac{R\sin(\frac{\alpha_o t^2}{2})}{R\cos(\frac{\alpha_o t^2}{2})} = \frac{\sin(\frac{\alpha_o t^2}{2})}{\cos(\frac{\alpha_o t^2}{2})}$

Slope of $\vec{v}(t) = \frac{\Delta y}{\Delta x} = \frac{R\alpha_o t\cos(\frac{\alpha_o t^2}{2})}{-R\alpha_o t\sin(\frac{\alpha_o t^2}{2})} = -\frac{\cos(\frac{\alpha_o t^2}{2})}{\sin(\frac{\alpha_o t^2}{2})}$

The slopse are opposite reciprocals, thus $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$, thus $\vec{v}(t)$ is tangent to the circle

The magnitude:

$|\vec{v}(t)| = \sqrt{(-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}))^2 + (R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))^2} = \sqrt{R^2\alpha_o^2 t^2(\cos^2(\frac{\alpha_o t^2}{2}) + \sin^2(\frac{\alpha_o t^2}{2}))} = \sqrt{R^2\alpha_o^2 t^2(1)} = R\alpha_o t = R\cdot\theta'(t)$

2.12 | l)

This problem would be better visualized as broken down into terms:

The first term:

$-(\frac{|\vec{v}|^2}{R}\cdot\frac{\vec{r}}{R}) = -(\frac{(R\alpha_o t)^2}{R}\cdot\frac{(R\cos(\frac{\alpha_o t^2}{2}), R\sin(\frac{\alpha_o t^2}{2}))}{R}) = -(R\alpha_o^2 t^2\cdot(\cos(\frac{\alpha_o t^2}{2}), \sin(\frac{\alpha_o t^2}{2}))) = (-R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2}), -R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2}))$

The second term:

$(R\alpha_o \cdot \frac{\vec{v}}{|\vec{v}|}) = R\alpha_o \cdot \frac{(-R\alpha_o t \sin(\frac{\alpha_o t^2}{2}), R\alpha_o t \cos(\frac{\alpha_o t^2}{2}))}{R\alpha_o t} = R\alpha_o \cdot (-\sin(\frac{\alpha_o t^2}{2}), \cos(\frac{\alpha_o t^2}{2})) = (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2}))$

Adding the two terms together:

$-(\frac{|\vec{v}|^2}{R} \cdot \frac{\vec{r}}{R}) + (R\alpha_o \cdot \frac{\vec{v}}{|\vec{v}|}) = (-R\alpha_o^2 t^2 \cos(\frac{\alpha_o t^2}{2}), -R\alpha_o^2 t^2 \sin(\frac{\alpha_o t^2}{2})) + (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2})) = (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2 \cos(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2 \sin(\frac{\alpha_o t^2}{2})) = \vec{a}(t)$

3 | Problem 3

First let's start with the definition of a derivative:

assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can rewrite $\vec{A}(t)$ as:

$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$

We can plug this definition of $\vec{A}(t)$ into the deriviative equation:

$\vec{A}'(t) = \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t)\hat{i} + A_y(t+\Delta t)\hat{j} + A_z(t+\Delta t)\hat{k} - A_x(t)\hat{i} - A_y(t)\hat{j} - A_z(t)\hat{k}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t)\hat{i} - A_x(t)\hat{i} + A_y(t+\Delta t)\hat{j} - A_y(t)\hat{j} + A_z(t+\Delta t)\hat{k} - A_z(t)\hat{k}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{\hat{i}(A_x(t+\Delta t) - A_x(t)) + \hat{j}(A_y(t+\Delta t) - A_y(t)) + \hat{k}(A_z(t+\Delta t) - A_z(t))}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t) - A_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j} + \lim_{\Delta t \to 0} \frac{A_z(t+\Delta t) - A_z(t)}{\Delta t}\hat{k}$
$= (\lim_{\Delta t \to 0} \frac{A_x(t+\Delta t) - A_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_z(t+\Delta t) - A_z(t)}{\Delta t})$
$= (\frac{dA_x(t)}{dt}, \frac{dA_y(t)}{dt}, \frac{dA_z(t)}{dt})$

4 | Problem 4

Again, starting with the definition of a derivative:

assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can define the $\vec{A}(t)$ and $\vec{B}(t)$ as:

$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j}$

$\vec{B}(t) = B_x(t)\hat{i} + B_y(t)\hat{j}$

And thus:

$\frac{d}{dt}(\alpha\vec{A}(t) + \beta\vec{B}(t)) = \frac{d}{dt}(\alpha(A_x(t)\hat{i} + A_y(t)\hat{j}) + \beta(B_x(t)\hat{i} + B_y(t)\hat{j}))$
$= \frac{d}{dt}(\alpha A_x(t)\hat{i} + \alpha A_y(t)\hat{j} + \beta B_x(t)\hat{i} + \beta B_y(t)\hat{j})$
$= \lim_{\Delta t \to 0} \frac{\alpha A_x(t+\Delta t)\hat{i} + \alpha A_y(t+\Delta t)\hat{j} + \beta B_x(t+\Delta t)\hat{i} + \beta B_y(t+\Delta t)\hat{j} - \alpha A_x(t)\hat{i} - \alpha A_y(t)\hat{j} - \beta B_x(t)\hat{i} - \beta B_y(t)\hat{j}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{\alpha A_x(t+\Delta t)\hat{i} - \alpha A_x(t)\hat{i} + \alpha A_y(t+\Delta)\hat{j} - \alpha A_y(t)\hat{j} + \beta B_x(t+\Delta t)\hat{i} - \beta B_x(t)\hat{i} + \beta B_y(t+\Delta t)\hat{j} - \beta B_y(t)\hat{j}}{\Delta t}$
$= \alpha \lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}\hat{i} + \alpha \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j} + \beta \lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}\hat{i} + \beta \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t}\hat{j}$
$= \alpha(\lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j}) + \beta(\lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t}\hat{j})$
$= \alpha(\lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}) + \beta(\lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t})$
$= \alpha(\frac{dA_x(t)}{dt}, \frac{dA_y(t)}{dt}) + \beta(\frac{dB_x(t)}{dt}, \frac{dB_y(t)}{dt})$

$= \alpha \frac{d\vec{A}(t)}{dt} + \beta \frac{d\vec{B}(t)}{dt}$

5 | **Problem 5**

We can start with the definition of a derivative:

Assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \text{lim}_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can define $\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j}$

Therefore:

$\frac{d}{dt}(\vec{A}(u(t))) = \text{lim}_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t)\hat{i} + A_y(u(t)+\Delta t)\hat{j} - A_x(u(t))\hat{i} - A_y(u(t))\hat{j}}{\Delta t}$
$= \text{lim}_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t)\hat{i} - A_x(u(t))\hat{i} + A_y(u(t)+\Delta t)\hat{j} - A_y(u(t))\hat{j}}{\Delta t}$
$= \text{lim}_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t) - A_x(u(t))}{\Delta t}\hat{i} + \text{lim}_{\Delta t \to 0} \frac{A_y(u(t)+\Delta t) - A_y(u(t))}{\Delta t}\hat{j}$
$= (\frac{dA_x(u(t))}{dt}, \frac{dA_y(u(t))}{dt})$

Here we can apply the chain rule for normal functions (which we proved in single variable calculus)

$= (A_x'(u(t)) \cdot u'(t), A_y'(u(t)) \cdot u'(t))$
$= u'(t) \cdot (A_x'(u(t)), A_y'(u(t)))$
$= \frac{du(t)}{dt} \cdot (A_x'(u(t)), A_y'(u(t)))$
$= \frac{du(t)}{dt} \cdot \vec{A}'(u(t))$
$= \frac{du(t)}{dt} \frac{d\vec{A}(u)}{du}$
$= \frac{d\vec{A}(u)}{du} \frac{du(t)}{dt}$