1 | Definitions

1.1 | Algebraic Structures

1.1.1 | Group

A set of items and an operation that satisfy closure, identity, inverse, assocativity

1.1.2 | Field

A group and another "secondary" operation that the set is almost a group under (except the additive identity will have no multiplicative inverse).

1.1.3 | Vector Space

A field and a set of vectors that can be added together or multiplied by scalars from the field, with the following five properties:

- · commutativity
- assocativity
- · additive identity
- · additive inverse
- · distributive property

1.1.4 | Subspace

A subset of a vector space that is itself a vector space. Only need to show that it:

- 1. Includes the additive identity (0)
- 2. Is closed under addition
- 3. Is closed under scalar multiplication

The subspace must use the same addition and scalar multiplication of its "superspace"

1.1.5 | Sum

A sum of (multiple) subsets is all vectors that can be written as the sum of one vector from each sub set (or zero).

1.1.6 | Direct Sum

If each element in a sum of (**multiple**) subspaces can be written in only one way (with one summand from each subspace).

ExrOn • 2021-2022 Page 1

1. Results

- (a) Condition for a direct sum The only way to write zero as sum of one element from each summand space is all zeros iff the sum is a direct sum.
- (b) Condition for a direct sum of two subspaces The intersection of the two subspaces is zero iff the sum is a direct sum.

1.1.7 | Linear Combination

A linear combination is the sum of some list of vectors with each one multiplied by a coefficient from $\mathbb F$

1.1.8 | Linear (In)Dependence

A list of vectors is linearly independent if the only coefficients in a linear combination equal to zero are all zeros. (The only a_1, \ldots, a_n s.t. $a_1v_1 + \cdots + a_nv_n = 0$ is $0, \ldots, 0$) Equivalent: A vector is linearly dependent in a list (and that list is linearly dependent) if it can be written as a linear combination of other vectors in the list. Any list that is not linearly dependent is linearly independent.

1.1.9 | Span

The span of a list is all linear combinations of that list

1.1.10 | Basis

The basis of a vector space is a linearly independent list of the elements in that vector space that spans the vector space (whose span is the vector space). A list of vetors is a basis if there is exactly one way to write every vector as a linear combination of the basis.

1. Results

- (a) All bases of a vector space are the same length
- (b) A linearly indpendent or spanning list of the right length is a basis (buy one get one free)

1.1.11 | Dimension

The dimension of a subspace is the length of it's basis. If the basis does not exist (infinitely long), then the space is infinite dimensional.

1.1.12 | Elementry Matrix

A matrix that applies exactly one valid "row operation": multiply a row, add one row to another, swap row orders.

1.1.13 | Nonsingular / invertible matrix

A non-singular matrix is a matrix that has an inverse, and whose determinant is not zero.

ExrOn • 2021-2022 Page 2

1.2 | Linear Transformations

1.2.1 | Linearity

A transformation is linear if it satisfies additivity (adding inside/outside same) and homogeneity (scalar multiplying inside/outside same).

1.2.2 | Injective

When the outputs being the same implies the inputs were the same. (Mapping is one to one; each element is mapped to atmost once).

1.2.3 | Surjective

When every element in the codomain is in the range (Mapping is onto the codomain; each element mapped to atleast once).

1.2.4 | Linear Map

A map from one vector space to another that is linear (satisfies additivity and homogeniety)

- 1. Properties
 - (a) Linear maps from one space to another is a subspace
 - (b) Algebraic Properties
 - i. Associative: $T_1(T_2T_3) = (T_1T_2)T_3$
 - ii. Identity: IT = TI = T
 - iii. Distributive: $(S_1 + S_2)T = TS_1 + TS_2$ And the same for the other side, but you have to be careful about whether maps can be multiplied (composed).
- 2. Product of Linear Map The product ST of two linear maps $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ is the linear map S(T(u)) for $u \in U$.

1.2.5 | Image (range, column spac)

Every vector that can be a result of a linear map.

- 1. Properties
 - (a) CHANGES AFTER RREF!
 - (b) Surjectivity is the same as the column space being the domain (input space?)

1.2.6 | Kernel (null space)

Every vector that the linear map sends to zero.

- 1. Properties
 - (a) Always includes zero
 - (b) Doesn't change after RREF
 - (c) Injectivity is the same as the null space being zero

ExrOn • 2021-2022 Page 3