1 | **0**

- Done

2 | **1**

First partial derivatives: $<\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}>=<8xy^5+9x^2y^2, 20x^2y^4+6x^3y>$ Second partial derivatives: $<\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}>=<8y^5+18xy^2, 40xy^4+18x^2y, 80x^2y^3+6x^3>$ Third partial derivatives: $<\frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial x^2 y}, \frac{\partial^3 f}{\partial xy^2}, \frac{\partial^3 f}{\partial y^3}>=<18y^2, 40y^4+36xy, 160xy^3+18x^2, 240x^2y^2>$

3 | **2**

3.1 | **(a)**

$$\begin{bmatrix} \frac{-x^2y+y^3}{x^4+2x^2y^2+y^4} & \frac{-xy^2+x^3}{x^4+2x^2y^2+y^4} \end{bmatrix}$$

3.2 | **(b)**

$$\begin{bmatrix} y & x+2z & 2y \\ 4xy^2 & 4x^2y & 0 \end{bmatrix}$$

4 | **3**

There are 2 1st partial derivatives. There are 3 2nd partial derivatives. There are 4 3rd partial derivatives. There are k+1 k-degree partial derivatives.

5 | **4**

To get the slope of the function at point $(1, 2)$, we must first evaluate the partial derivative of the function with respect to both x and y:

$$\frac{\partial f}{\partial x} = 4xy^3 + 7$$
$$\frac{\partial f}{\partial y} = 6x^2y^2 + 20y$$

From this, we can evaluate the derivative of the function at point $(1, 2)$:

$$f'(1,2) = \begin{bmatrix} 4 \cdot (2)^3 + 7 \\ 6 \cdot (2)^2 + 20 \cdot (2) \end{bmatrix} = \begin{bmatrix} 39 \\ 64 \end{bmatrix}$$

Now that we have the derivative of f at point $(1, 2)$, we need to find the slope of f at $(1, 2)$ towards the direction represented by the vector $<-\frac{\sqrt{3}}{2}, \frac{1}{2}>$. To find the slope of a function at a point given a direction vector, we need to first extract the angle between this vector and the x axis—actually, a more rigorous definition would be the amount of radians we need to rotate $\hat{i}$ counterclockwise to make it point in hte same direction of that vector. We can do this by taking the dot product of the vector (let's represent it with $\vec{v}$) and the x-axis basis vector, $\hat{i}$, and then dividing by the products of their magnitude to extract θ:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\hat{i}}{||\vec{v}||\cdot||\hat{i}||}\right)\\ &= \cos^{-1}\left(\frac{-\sqrt{3}}{2\cdot(1)\cdot(1)}\right)\\ &= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\\ &= \frac{5\pi}{6}\end{aligned}$$

With theta, we can now get the slope in the direction of $\vec{v}$ with the equation

$$s(\theta) = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta) \tag{1}$$

When we plug in the values we got earlier, we get

$$\begin{aligned}s(\frac{5\pi}{6}) &= 39\cos(\frac{5\pi}{6}) + 64\sin(\frac{5\pi}{6})\\ &= -1.77\end{aligned}$$

This is the answer to the slope of f at point $(1,2)$ towards the direction $< -\frac{\sqrt{3}}{2}, \frac{1}{2} >$. But this is just one slope towards one direction. What if we wanted the direction that maximizes the slope at $(1,2)$? (Note: The direction that maximizes the slope is π radians away from (i.e. opposite direction of) the direction that minimizes the slope. The proof of this is trivial (seriously) and is left as an exercise to the reader.) We will find the value for θ that maximizes the slope using a little bit of calculus. We already have the equation for the slope as a function of θ:

$$f(\theta) = 39\cos(\theta) + 64\sin(\theta) \tag{2}$$

We can take the derivative with respect of θ and find the values of theta at which the derivative is 0 to find the local maxima:

$$\begin{aligned}\frac{df}{d\theta} &= -39\sin(\theta) + 64\cos(\theta) = 0\\ \frac{64}{39} &= \frac{\sin(\theta)}{\cos(\theta)}\\ &= \tan(\theta)\\ \theta &= \tan^{-1}(\frac{64}{39})\end{aligned}$$

We found the value of θ that maximizes the slope, but what if we wanted to find the value of theta where the slope is zero? We can just set $f(\theta)$ to 0 and solve for theta:

$$
\begin{aligned}
f(\theta) &= 0 \\
39\cos(\theta) + 64\sin(\theta) &= 0 \\
64\sin(\theta) &= -39\cos(\theta) \\
\frac{\sin(\theta)}{\cos(\theta)} &= -\frac{39}{64} \\
\tan(\theta) &= -\frac{39}{64} \\
\theta &= \tan^{-1}(-\frac{39}{64})
\end{aligned}
$$