Hello, fellow person that comes across this. I have had one brief exposure with Linear Algebra following MATH 21-1 at UCSC. However, Axler is just so cool, so I am trying to learn a bit of linalg on the side to supplement my much more traditional linalg experience at the UC.

A few things of note. This whole thing is very "partial": in the sense that its contents contain many a parts of things omitted which I feel like I have a good grasp on from 21-1 such that I don't need to be reminded again; I only include things that maybe useful to me later either b/c I don't know it or I want to be reminded of it. As such, I don't think this will be helpful for most people.

1 | **1.A**

1.1 | **Things of Note**

- $\lambda \in \mathbb{F}$ is called a "scalar". I mean duh but still.

1.1.1 | **Defining a list**

A list of length n is a collection of n elements (any mathematical object?) separated by commas.

"Identical" lists are established when lists have:

- the same length
- same elements
- in the same order.

Its also called a n-tuple.

n must be a finite non-negative value. Therefore, an "infinitely long list" is not a list.

1.1.2 | **Sets vs Lists**

Lists have order and repetition. In sets, order and repetitions don't matter.

1.1.3 | $\mathbb{F}$

- A set
- Containing 2 elements $0, 1$
- Operators of "addition" and "multiplication" that satisfy the following properties

1. Properties of $\mathbb{F}$ That, with $\alpha, \beta, \lambda \in \mathbb{F}$:
    - **Commutativity** $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
    - **Associativity** $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
    - **Existence of Identities** $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$
    - **Additive Inverse** for every α, $\exists\beta$ s.t. $\alpha + \beta = 0$
    - **Multiplicative Inverse** for every $\alpha \neq 0$, $\exists\beta$ s.t. $\alpha\beta = 1$
    - **Distribution** $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.1.4 | $\mathbb{F}^n$

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F}\ for\ j = 1, \ldots, n\} \tag{1}$$

We say x_j is the j^{th} coordinate of $(x_1, \ldots, x_n)$.
In $\mathbb{F}^n$...

1. Addition
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \tag{2}$$
2. Scalar Multiplication
$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) \tag{3}$$
3. Zero
$$0 = (0, \ldots, 0) \tag{4}$$
4. Additive Inverse ...of $x \in \mathbb{F}^n$:
$$x + (-x) = 0 \tag{5}$$
That:
$$x = (x_1, \ldots, x_n), -x = (-x_1, \ldots, -x_n) \tag{6}$$

1.2 | **In-Text Exercises**

1.2.1 | **Verify that** $i^2 = -1$

$(0 + 1i)(0 + 1i) = (0 + 0 + 0 + ii) = -1$

1.2.2 | **Defining subtraction and division**

$\alpha, \beta \in \mathbb{C}$
Subtraction could be defined in that:

- Let $-\alpha$ be defined as the additive inverse of α
- Subtraction, therefore, is defined $\beta - \alpha = \beta + (-\alpha)$

Division could be defined in that:

- Let $1/\alpha$ be defined as the multiplicative inverse of α
- Subtraction, therefore, is defined $\beta/\alpha = \beta(1/\alpha)$

1.3 | **Actual Exercises**

1: Suppose $a, b \in \mathbb{R}$, $a, b \neq 0$, find $c, d \in \mathbb{R}$ s.t. $\frac{1}{(a+bi)} = c + di$

$$\frac{1}{(a+bi)} = \frac{(a-bi)}{(a+bi)(a-bi)} = \tag{7}$$
$$\Rightarrow \frac{a-bi}{a^2-(bi)^2} = c + di \tag{8}$$
$$\Rightarrow \frac{a-bi}{a^2+b^2} = c + di \tag{9}$$
$$\Rightarrow \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2} = c + di \tag{10}$$

Therefore:

$$c = \frac{a}{a^2+b^2} \tag{11}$$

$$d = \frac{-b}{a^2+b^2} \tag{12}$$

2: Show that $\frac{-1+\sqrt{3}i}{2}$ is the cube root of 1.

$$(\frac{-1+\sqrt{3}i}{2})^3 \tag{13}$$
$$\Rightarrow (\frac{-1+\sqrt{3}i}{2})(\frac{-1+\sqrt{3}i}{2})(\frac{-1+\sqrt{3}i}{2}) \tag{14}$$
$$\Rightarrow \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i)}{8} \tag{15}$$
$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \tag{16}$$
$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \tag{17}$$
$$\Rightarrow \frac{8}{8} = 1 \tag{18}$$

3: Find two distinct square roots of i

?

4: Show that $\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in C$

Let:

$\forall a, b, c, d \in \mathbb{R}$

- $\alpha = (a + bi)$
- $\beta = (c + di)$

$$\alpha + \beta = (a + bi) + (c + di) \tag{19}$$
$$= (a + c) + (b + d)i \tag{20}$$
$$= (c + a) + (d + b)i \tag{21}$$
$$= (c + di) + (a + bi) \tag{22}$$
$$= \beta + \alpha \blacksquare \tag{23}$$

5: Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda), \forall \alpha, \beta, \lambda \in \mathbb{C}$

Let:

$\forall a, b, c, d, e, f \in \mathbb{R}$

- $\alpha = (a + bi)$
- $\beta = (c + di)$
- $\lambda = (e + fi)$

$$(\alpha + \beta) + \lambda = ((a + bi) + (c + di)) + (e + fi) \tag{24}$$
$$= ((a + c) + (b + d)i) + (e + fi) \tag{25}$$
$$= (a + c + e) + (b + d + f)i \tag{26}$$
$$= (a + (c + e)) + (b + (d + f))i \tag{27}$$
$$= (a + bi) + (c + e) + (d + f)i \tag{28}$$
$$= (a + bi) + ((c + di) + (e + fi)) \tag{29}$$
$$= \alpha + (\beta + \lambda) \blacksquare \tag{30}$$

2 | **1.B**

2.1 | **Things of Note**

2.1.1 | **Vector Spaces** V

A "vector"/"point" is a member of a vector space. The exact nature of scalar multiplication depends on which $\mathbb{F}$ we are working it; hence, when being precise, we say that V is a vector space "over $\mathbb{F}$".

1. Motivation
   - Addition is commutative, associative, and has identity
   - Every element has additive inverse
   - Scalar multiplication is associative
   - Addition and scalar multiplication is connected by distribution
2. Basic Operators
   - **Addition** on set V is a function that assigns $u + v \in V$ to each pair $u, v \in V$

- **Scalar Multiplication** on set V is a function that assigns $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and $v \in V$. Note that this is different than a field, because if you can't multiply two different things in and out of the field and expect it to remain. But you could multiply an element in field $\mathbb{F}$ and a vector in vector space V and expect it to stay in V.

Note also "Multiplication" is not defined as 1) there are two and 2) they behave very differently.

3. Properties For $u, v, w \in V$ and $a, b \in \mathbb{F}$.
   - **Commutativity** $u + v = v + u$
   - **Associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$.
   - **Additive Identity** $\exists 0 \in V\ s.t.\ v + 0 = v, \forall v \in V$
   - **Additive Inverse** $\forall v \in V, \exists w \in V\ s.t.\ v + w = 0$
   - **Multiplicative Identity** $1v = v$
   - **Distribution** $a(u + v) = au + av$ and $(a + b)v = av + bv$
4. Unique Additive Identity The additive identity ("zero") in a vector space must be unique. (i.e. there cannot be two distinct zeros 0 and $0'$ which both are $\in V$). This is because:

$$0 = 0 + 0' = 0' + 0 = 0' \tag{31}$$

   That — if both 0 and $0'$ are additive identities, $0 = 0'$.
5. Unique Additive Inverse Every element in a vector space has an unique additive inverse (i.e. there cannot be two distinct additive inverses of $v \in V$ w and w' which both are $\in V$).

   Suppose w and w' are both additive inverses of v, then it holds that:

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w' \tag{32}$$

   That — if both w and w' exists in V, $w = w'$.
6. Zero and Vectors $0v = 0$ for $v \in V$. $a\vec{0} = \vec{0}$ for $a \in \mathbb{F}$.

2.1.2 | $\mathbb{F}^\infty$

Wait but aren't $\mathbb{F}^n$ supposed to be made of lists, which has finite length?

I guess its just sequences of all of everything in F.

$$\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_j \in \mathbb{F}\ for\ j = 1, 2, \ldots\} \tag{33}$$

2.1.3 | $\mathbb{F}^S$

$\mathbb{F}^S$ is defined as the set of functions that maps elements in set S to $\mathbb{F}$. It is a vector space.

1. Addition Addition between $f, g \in \mathbb{F}^S$ is defined by:

$$(f + g)(x) = f(x) + g(x),\ \forall x \in S \tag{34}$$

2. Scalar Multiplication Multiplication between $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^s$, $\lambda f \in \mathbb{F}^s$ is defined as:

$$(\lambda f)(x) = \lambda f(x),\ \forall x \in S \tag{35}$$

3. $\mathbb{F}^n$ and $\mathbb{F}^\infty$ are special cases of $\mathbb{F}^s$ ...this is because a list $\{x_1, x_2, x_3, \ldots, x_n\}$ is actually a bijective mapping between $\{1, 2, 3, \ldots, n\}$ (the indexes) and the values of the list, which are all $\in \mathbb{F}$. so :tada:!

2.2 | In-Text Exercises

2.2.1 | Verify that $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

Not going to write this one out, but:

- Commutativity: via rules addition, commutation (in $\mathbb{F}$), then undoing addition
- Associativity: addition, communication, then undoing addition
- Additive Identity: addition + definition of "zero" in $\mathbb{F}^n$
- Additive Inverse: addition + additive inverse (in $\mathbb{F}$)
- Multiplicative Identity: scalar multiplication (by 1) and then identity (in $\mathbb{F}$)
- Distribution: definition of addition in $\mathbb{F}^n$, scalar multiplication, undoing definition of addition again

2.3 | Actual Exercises

1: Proof that $-(-v) = v,\ \forall v \in V$

Step	Explanation
$v = v + 0$	Additive identity
$v = v + (-v + -(-v))$	Additive inverse
$v = (v + -v) + -(-v)$	Associative property
$v = 0 + -(-v)$	Additive inverse
$v = -(-v)$ ■	Additive Identity

2: Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Proof $a = 0$ or $v = 0$.

Let $a \neq 0$. We define the multiplicative inverse of a as a^{-1}.

Step	Explanation
$v = 1v$	Multiplicative identity
$v = aa^{-1}v$	Multiplicative inverse
$v = ava^{-1}$	Commutativity
$v = 0a^{-1}$	Given
$v = 0$ ■	Number times 0

If $a = 0$, ■.

3: Suppose $v, w \in V$, explain why $\exists$ unique $x \in V$ s.t. $v + 3x = w$

Let $x = \frac{1}{3}(w - v)$; by addition and scalar multiplication, $\exists x \in V$.

Step	Explanation
$v + 3x = w$	Given
$v + 3(\frac{1}{3}(w - v)) = w$	Defined
$v + (w - v) = w$	Multiplication in $\mathbb{F}$
$v - v + w = w$	Commutativity
$w = w$ ■	Additive Inverse

Therefore, $\exists x \in V$ that satisfies the needed property.

Suppose there exists more than 1 x which satisfies this property. We call them x and x'. This would tell us the following equalities:

$v + 3x = w$, $v + 3x' = w$.

It follows from the equalities that:

$3x = w - v$, $3x' = w - v$

Then, it follows that

$3x = 3x'$

Therefore:

$$x = x' \tag{36}$$

Hence, if given that there exists $v + 3x = w$, $v + 3x' = w$, $x = x'$. Hence, there is only one unique x such that $v + 3x = w$.

3 | 1.C

Axler, in his infinite wisdom, has crammed everything that's interesting to note in Chapter 1.c.

3.1 | Things of Note

3.1.1 | Subspaces

(Woo hoo!)

A subset $U \subset V$ is called a "subspace of V" if U is also a vector space using the same addition and scalar multiplication operators.

1. Checking for Subspaces Check for three conditions:

   For $U \subset V$

   - **Additive Identity** $0 \in U$. (also could be defined as "set is nonempty", b/c if nonempty, and its closed under scalar multiplication, multiplying any element by 0 will do the trick. But often showing 0 is in it is actually simpler.)
   - **Closed Under Addition** $u, w \in U$ implies $u + w \in U$
   - **Closed Under Scalar Multiplication** $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

3.1.2 | Summing Subsets

Suppose $U_1, \dots, U_m$ are subsets of V. The "sum" of the subsets ($U_1 + \dots + U_m$) is the set of all possible sums of elements in $U_1, , U_m$. That is:

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\} \tag{37}$$

1. Properties of the Sums of Subspaces Suppose $U_1, \dots, U_m$ are subspaces of V. $U_1 + \dots + U_m$ is the smallest subspace of V containing all of $U_1, \dots, U_m$.

3.1.3 | Direct Sum

Suppose $U_1, \ldots, U_m$ are subspaces of V

- Sum $U_1 + \cdots + U_m$ is a direct sum if every element $u \in U_1 + \cdots + U_m$ can be only written only one way as a sum $u_1 + \cdots + u_m$
- Direct sum is noted as $U_1 \oplus \cdots \oplus U_m$

"Sums is the union, direct sums is the disjoint union".

1. Checking for Direct Sums Suppose $U_1, \ldots, U_m$ are subspaces of V. $U_1 + \cdots + U_m$ is a direct sum iff the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking each u_j equaling to 0. This could be implied from the definition of a direct sum: that there is only one way to write a 0 as $u_1 + \cdots + u_m$, and being closed scalar multiplication means that you could multiply 0 to each subspace individually and they still have to add up to 0.
2. Direct Sum of Two Subspaces Suppose U, W are subspaces of V.

   $U + W$ is a direct sum iff $U \cap W = \{0\}$.

3.2 | In-Text Excercises

3.2.1 | Summing Subspaces, an Example

Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$ and $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Then:

$$U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\} \tag{38}$$

We verify this by writing out the sum.

$$\begin{aligned} U + W &= \{(x_u, x_u, y_u, y_u) + (x_w, x_w, x_w, y_w) : x_u, y_u \in U, x_w, y_w \in W\} && (39)\\ &= \{(x_u + x_w, x_u + x_w, y_u + x_w, y_u + y_w) : x_u, y_u \in U, x_w, y_w \in W\} && (40)\\ &= \{\left(\underbrace{x}_{x_u+x_w}, x, \underbrace{y}_{y_u+x_w}, \underbrace{z}_{y_u+y_w}\right) : x, y, z \in \mathbb{F}\} && (41) \end{aligned}$$

3.2.2 | Verify sums equal to spaces

Suppose U, W are subspaces of $\mathbb{F}^3$.

$$U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \tag{42}$$
$$W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \tag{43}$$

Verify $\mathbb{F}^3 = U \oplus W$

$$U + W = \{(x, y, z) \in F^3 : x, y, z \in F\} = \mathbb{F}^3 \tag{44}$$

:tada:?

3.3 | **Actual Excercises**

1: For each of the following subsets of $\mathbb{F}^3$, determine if it is a subspace of $\mathbb{F}^3$.

$$U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} \tag{45}$$

We could see that $(0, 0, 0) \in U$.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3}), u_2 = (x_3, x_4, \frac{-x_3-2x_4}{3}).u_1, u_2 \in U$.

$u_1 + u_2 = (x_1 + x_3, x_2 + x_4, \frac{-(x_1+x_3)-2(x_2+x_4)}{3})$. Define $x_1 + x_3 = x$, $x_2 + x_4 = y$. Therefore: $u_1 + u_2 = (x, y, \frac{-x-2y}{3})$.

$x + 2y - x - 2y = 0$.

Therefore, U is closed under addition.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3})$. We scale each element by scalar factor λ.

$\lambda u_1 = (\lambda x_1, \lambda x_2, \lambda\frac{-x_1-2x_2}{3})$.

$\lambda x_1 + 2\lambda x_2 - \lambda x_1 - 2\lambda x_2 = 0$. Therefore, U is closed under scalar multiplication.

Therefore, it is a subspace of $\mathbb{F}^3$.

10: Suppose that U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Given U_1 and U_2 are both subspaces, 0 must be in both subsets hence it must be in their intersection.

Let $u_1 = (x_1, y_1, z_1)$. Let $u_2 = (x_2, y_2, z_2)$. Finally, let $u_1, u_2 \in U_1 \cap U_2$. By this last fact, we could derive the fact that $u1, u2 \in U_1, U_2$.

As U_1 is a subspace, U_1 closed under addition. Since $u_1, u_2 \in U_1$, $u_1 + u_2 \in U_1$. As U_2 is a subspace, U_2 closed under addition. Since $u_1, u_2 \in U_2$, $u_1 + u_2 \in U_2$.

As $u_1 + u_2 \in U_1, U_2$, it is additionally $u_1 + u_2 \in U_1 \cap U_2$. $U_1 \cap U_2$ is therefore closed under addition.

By the same token….

As U_1 is a subspace, U_1 is closed under scalar multiplication. Since $u_1 \in U_1$, $\lambda u_1 \in U_1$. As U_2 is a subspace, U_2 is closed under scalar multiplication. Since $u_1 \in U_2$, $\lambda u_1 \in U_2$.

Therefore, as $\lambda u_1 \in U_1, U_2$, it is additionally true that $u_1 \in U_1 \cap U_2$, it is therefore closed under multiplication.

4 | **2.A**

What's with the balancing of these chapters? Like Span and Linear Independence is squished in one, but then Bases gets a whole chapter and so does dimension.

4.1 | **Things of Note**

- Lists of vectors are usually denoted as a list without parentheses

4.1.1 | **Linear Combination**

A linear combination of a list is a sum of vectors in the form:

$$a_1v_1 + \cdots + a_mv_m \tag{46}$$

where, $a_1, \ldots, a_m \in \mathbb{F}$.

1. Verifying Linear Combinations You could note that a linear combination is actually just a linear system of equations. That:

   To check (x, y, z) is a linear combination of (v_1, v_2, v_3), (w_1, w_2, w_3), figure if there exists a pair (a_1, a_2) such that…

   $$\begin{cases} x = a_1v_1 + a_2w_1 \\ y = a_1v_2 + a_2w_2 \\ z = a_1v_3 + a_2w_3 \end{cases} \tag{47}$$

   If so, there exists the requisite scalars such that the linear combination is possible.

4.1.2 | **Linear Span**

The span is the set of all linear combinations of a list of vector. That:

$$span(v_1, \ldots, v_m) = \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbb{F}\} \tag{48}$$

The span of an empty list of vectors $()$ is defined to be $\{0\}$. Due to the fact that any subspace must be closed under scalar multiplication + addition, the span of a list of vectors in V is the smallest subspace of V containing all the vectors in that list.

1. Spanning List If $span(v_1, \ldots, v_m) = V$, we say that $v_1, \ldots, v_m$ spans V.
2. Finite-Dimensional Vector Space A vector space is ”finite-dimensional” if some list of vectors in it could span the space. i.e. there exists a list of vectors that span the space. Otherwise, it is called ”infinite dimensional”.

   Every subspace of a finite dimentional vector space is finite dimentional.

4.1.3 | **Polynomials**

We quickly recap the definition of a polynomial. A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial if $\exists a_0, \ldots, a_m \in \mathbb{F}$ s.t.

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m, \forall z \in \mathbb{F} \tag{49}$$

By the same token, $\mathcal{P}($

1. Degree of a Polynomial A polynomial $p \in \mathcal{P}(\mathbb{F})$ has degree m if $\exists a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that…

   $$p(z) = a_0 + a_1z + \cdots + a_mz^m, \forall z \in \mathbb{F} \tag{50}$$

   If p has degree m, we write $deg\ p = m$. A zero-polynomial is said to have degree $-\infty$.

2. $\mathcal{P}_m(\mathbb{F})$ For a non-negative integer m, $\mathcal{P}_m(\mathbb{F})$ is defined as the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

   You will therefore notice, then, that:

$$\mathcal{P}_m(\mathbb{F}) = span(1, z, \ldots, z^m) \tag{51}$$

4.1.4 | **Linear Independence**

Linear independence exists by a only *unique* choice of scalars $a_1, \ldots, a_m$ to form any given $v \in span(v_1, \ldots, v_m)$. This could also be stated as that:

that the only choice $a_1, \ldots, a_m \in \mathbb{F}$ that makes $a_1v_1 + \cdots + a_mv_m = 0$ is the "trivial" case whereby $a_1 = \cdots = a_m = 0$.

The empty set is also defined as linearly independent.

4.1.5 | **Linear Dependence**

A list of vectors in V is linearly dependent if its not linearly independent.

A list $v_1, \ldots, v_m$ of vectors is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbb{F}$ that's not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.

1. Linear Dependence Lemma Suppose $v_1, \ldots, v_m$ is an linearly dependent list in V. Then, exists a $j \in \{1, 2, \ldots, m\}$ such that:

   (a) $v_j \in span(v_1, \ldots, v_{j-1})$

   (b) If the j^{th} term is removed, the span remains the same

2. Lengths of Lin. Indp List In a finite-dimentional vector space, the length of every linearly independent list is less than or equal to the length of every spanning list.

4.2 | **In-Text Excercises**

4.2.1 | $\mathcal{P}(\mathbb{F})$ **is a subspace of** $\mathbb{F}^{\mathbb{F}}$

Zero exists in the set as, for a polynomial, $(a_0, \ldots, a_m) = (0, \ldots, 0)$ would create a function $f : \mathbb{F} \to 0$.

Due to commutativity, we could group and factor-out input-variable z such that the sum of two polynomials become $(a_{0a} + a_{0b}) + (a_{1a} + a_{1b})z + \cdots + (a_{ma} + a_{mb})z^m$, which would be another polynomial. This would be closed under addition.

Due to distribution, a scalar λ multiplied to a polynomial would just scale every value by λ resulting in $\lambda a_0 + \lambda a_1 z + \cdots + \lambda a_m z^m$, which would be another polynomial. This would be closed under scalar multiplication.

Therefore, the set of polynomials in $\mathbb{F}$ is as subspace.

4.2.2 | **Show that** $\mathcal{P}(\mathbb{F})$ **in infinite-dimensional**

Any list $U \subset \mathcal{P}(\mathbb{F})$ would contain a highest-degree polynomial with degree m. Therefore, the element $z^{m+1} \in \mathcal{P}(\mathbb{F})$ would not be in the span of the list: making the list not span the entire space. Therefore, there could not be a list that spans $\mathcal{P}(\mathbb{F})$, making it infinite-dimentional.

5 | **2.B**

5.1 | **Things of Note**

(wow that was short.)

5.1.1 | **Bases**

A basis of V is a list of vectors in V that's linearly independent **and** spans V. A list $v_1, \ldots, v_n$ could only be a basis if and only if every vector $v \in V$ could be written as a linear combination of $v_1, \ldots, v_n$ uniquely. The "uniquely" part makes linear independence, the "could be written" part makes span.

1. Reducing a Basis Every spanning list in a vector space could be reduced to a basis of the vector space. Basically by removing everything that makes the span linearly dependent.

   And therefore, every finite-dimentional vector space has a basis. Because you could just get a span of the space and keep reducing.

2. Building a Basis The dual of the previous condition is also true. Take a linearly independent list, keep adding vectors that still maintains the linearly independent-ness of the list, and at some point you will get a basis.

   As an correlary to this, we could claim that any subspace U of V is part of the a direct sum that adds up to V. This is because we could expand ("pad") $u_1, \ldots, u_m \in U$ with vectors $w_1, \ldots, w_n$ until we form a basis of V. We could see that $V = U + W$ because the collection of the vectors that make them up is the basis of V. And also because of that they are linearly independent.

   because they are linearly independent, there is only one way to write $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ as a linear combination, making it a direct sum.

5.2 | **In-Text Exercises**

5.2.1 | **The list $1, z, \ldots, z^m$ is a basis for $\mathcal{P}_m(\mathbb{F})$**

We proof first that the sequence spans $\mathcal{P}_m(\mathbb{F})$. This is easy to see, because, definitionally, a polynomial in $\mathcal{P}_m(\mathbb{F})$ a made of a linear combination of $1, z, \ldots, z^m$.

This list is furthermore linearly independent as, the only set of scalars $a_1, \ldots, a_{m+1}$ by which $1, z, \ldots, z^m$ could be scaled to be 0 is in the case by which all scalars are 0, making this list linearly independent.

6 | **2.C**

6.1 | **Things of Note**

6.1.1 | **Dimension!**

$dim\ V$ of a finite-dimentional vector space is the length of any basis of the space. Because the length of linear independent list must be $\leq$ spanning list, there is only one length that's possible in a space such that a list is a basis (linear independent AND spanning.)

- $dim\ \mathbb{F}^n = n$ because the standard basis $\mathbb{F}^n$ as length n

- $dim\ \mathcal{P}_m(\mathbb{F}) = m + 1$ because the basis of the space $1, z, \ldots, z^m$ has $m + 1$ elements

1. Subspace Dimension If V has a finite dimension + if U is a subspace of V, $dim\ U \leq dim\ V$.
2. A John McHugh Special: ”Half is good enough” Linearly independent and/or spanning lists of vectors in V with length $dim\ V$ is a basis of V. So proving two (length, basis, linearly indep., spanning) proves all four.
3. Dimension of a Sum If U_1 and U_2 are finite-dimentional, then:

$$dim(U_1 + U_2) = dim\ U_1 + dim\ U_2 - dim(U_1 \cap U_2) \tag{52}$$

6.2 | In-Text Excercises

6.2.1 | Show that 1, $(x - 5)^2$, $(x - 5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$

Define U, as given by the question:

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\} \tag{53}$$

By thinking a little hard, we could see that $1, (x - 5)^2, (x - 5)^3 \in U$. A linear combination of these elements are shown:

$$a + b(x - 5)^2 + c(x - 5)^3 = 0 \tag{54}$$

We could see that the degree of 0 is 0. Therefore the coefficient for the x^3 term is 0. As the 3rd-degree term in the left expression is cx^3, we therefore deduct that $c = 0$. We repeat this to find $b = 0, a = 0$. Therefore, the list as prescribed is linearly independent. Furthermore, we could see that the dimension of U would have to be ≥ 3 as the length of this linearly independent list is 3. As U is a subspace of $\mathcal{P}_3$, U has a dimension ≤ 4 as the dimension of $\mathcal{P}_3$ is 4 (bases $1, z, \ldots, z^3$).

We further show that, due to the fact that $U \neq \mathcal{P}_3(\mathbb{R})$ (i.e. for instance, $x^2 \in \mathcal{P}_3(\mathbb{R}), x^2 \notin U$), and if $dim\ U = 4$, extending the bases of U towards that for $\mathcal{P}_3(\mathbb{R})$ would exceed $dim\ \mathcal{P}_3(\mathbb{R})$. Therefore, $dim\ U = 3$.

Given the list as given is an linearly independent list with a length of 3 in a subspace of dimension 3, the list is a basis of $\mathcal{P}_3(\mathbb{R})$.

That is, why not? Let’s find out. IDK.