Angular Momentum & Torque - Part 2

March 12, 2021

Comparison Chart of Angular & Linear Quantities

Type	Rotational	Linear
Velocity	Angular Velocity $\vec{\omega}$ (rad/s)	Velocity $\vec{v}$ (m/s))
Momentum	Angular Momentum $\vec{L} = \vec{r} \times m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})$	Momentum $\vec{p} = m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1})$
"Force"	Torque $\vec{\tau} = \vec{r} \times \vec{F}$ (N m)	Force $\vec{F}$ (N)
"2nd Law"	$\vec{\tau}_{net} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$	$\vec{F}_{net} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$

We already derived the "2nd Law" Torque-Angular Momentum relationship for point masses. What we require are some tools to work with systems of masses and with solid objects.

Problem 1:

Finish the proof of the following theorem:

Theorem: External Torque on a System
For a system of N point mass particles, the total external torque is equal to the time derivative of the total angular momentum of the system

$$\boldsymbol{\vec{\tau}}_{total\,ext} = \frac{\mathrm{d}\vec{L}_{system}}{\mathrm{d}t} \tag{1}$$

Proof:
Starting work from the right side of eqn (1):

$$\frac{\mathrm{d}\vec{L}_{system}}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\mathrm{d}\vec{L}_i}{\mathrm{d}t} = \sum_i \vec{\tau}_{i\,net} \tag{2}$$

where $\vec{\tau}_{i\,net}$ is the net torque on the i^{th} particle

$$= \sum_1^N \vec{r}_i \times \vec{F}_{i\,net} \tag{3}$$

$$= \overbrace{\sum_1^N \vec{r}_i \times \vec{F}_{i\,net\,ext}}^{\text{torque from external forces}} + \overbrace{\sum_1^N \vec{r}_i \times \vec{F}_{i\,net\,int}}^{\text{torque from internal forces}} \tag{4}$$

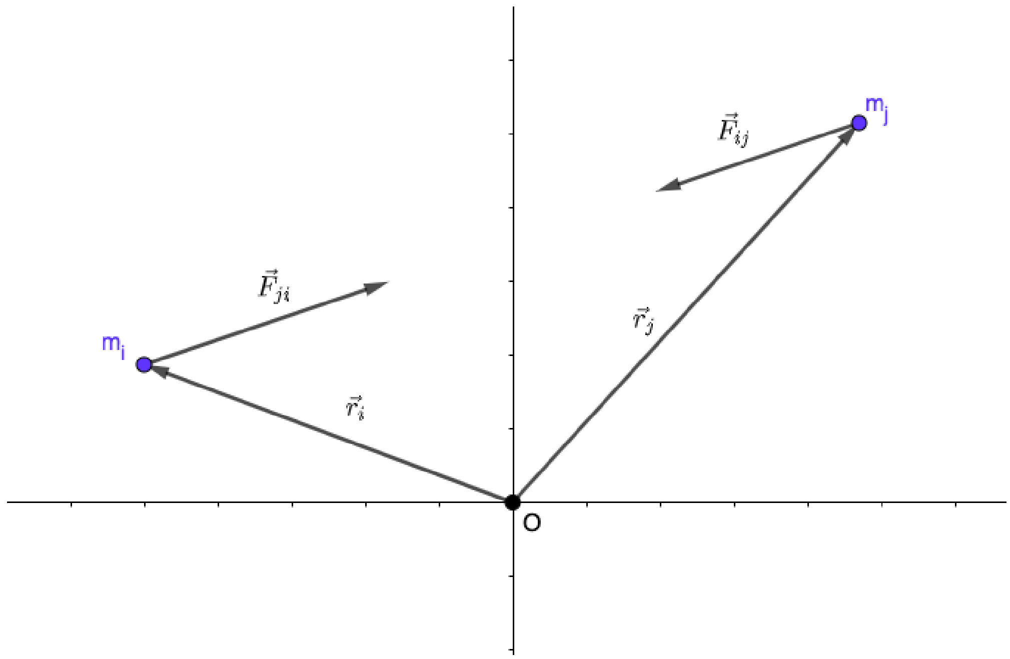
$$= \sum_1^N \vec{\tau}_{i\,net\,ext} + \sum_i \overbrace{\sum_{j\neq i} \vec{r}_i \times \underbrace{\vec{F}_{j\to i}}_{\text{force that j exerts on i}}}^{\text{torque from all particles j on particle i}} \tag{5}$$

Reordering the double summation to group the paired N-3 forces:

$$\frac{\mathrm{d}\vec{L}_{system}}{\mathrm{d}t} = \vec{\tau}_{tot\,ext} + \underbrace{\sum_{1\le i<j\le N}}_{\text{this makes sure that each i,j pair is included exactly once}} [\ \overbrace{(\vec{r}_i \times \vec{F}_{j\to i})}^{\text{torque on i from j}} + \overbrace{(\vec{r}_j \times \vec{F}_{i\to j})}^{\text{torque on j from i}}\ ] \tag{6}$$

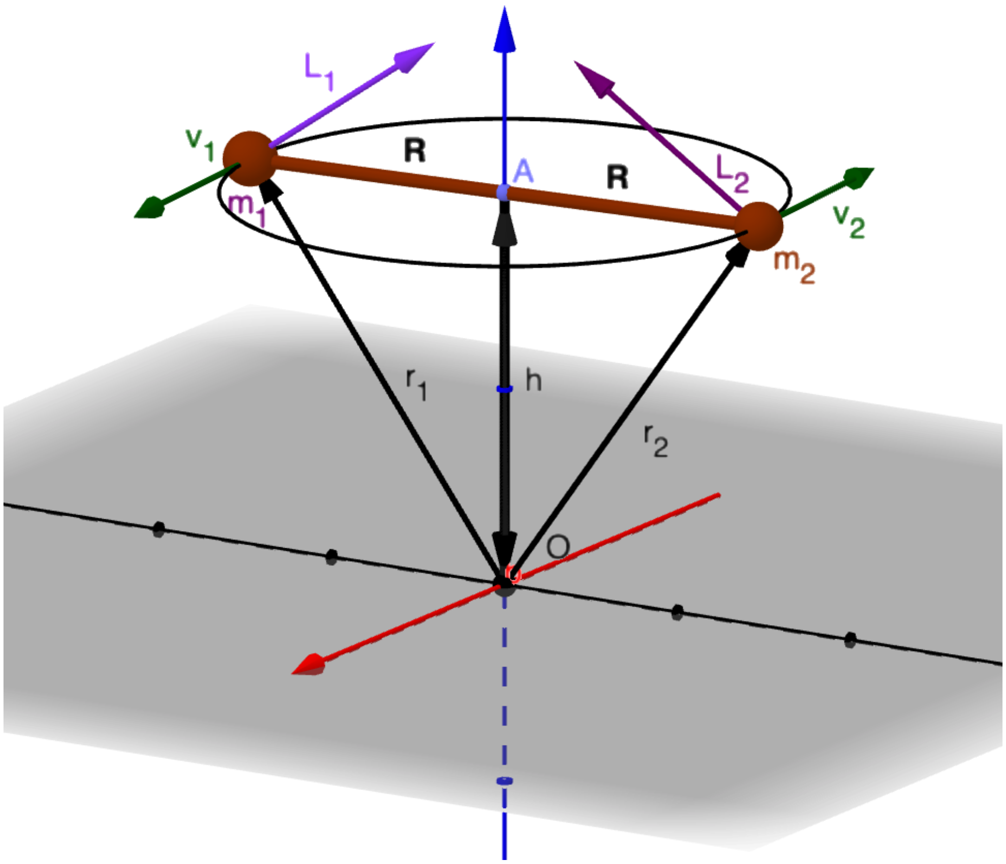
The first term is the desired answer, so we need to show that the summation on the right is zero. Using the fact that according to N-3, $\vec{F}_{j\rightarrow i} = -\vec{F}_{i\rightarrow j}$, and that all internal forces are central forces, we hope to show that for each i,j pair, the contents between the brackets is zero.

Draw your own diagram and explain why the two terms in the brackets in eqn (6) cancel each other. Here is an example drawing to get you started:



Problem 2: Rotating Symmetric Point Masses:

Calculate the angular momentum of the following rotating system, spinning around the z-axis:



A massless stick with point masses at each end spins with constant angular velocity $\vec{\omega}$ pointing in the upward z-axis direction ($\vec{\omega} = \omega\hat{k}$). The stick is spinning around its center at a height h above the xy-plane and the length of the stick is $2R$. The masses on the ends of the stick have equal mass, $m_1 = m_2 = m$, so the system is symmetric about the z-axis.

(Reminder: the unit vectors that point along the x-axis, y-axis, and z-axis are notated as $\hat{i}, \hat{j}, \hat{k}$, or alternatively, $\hat{x}, \hat{y}, \hat{z}$, respectively.)

Be sure to find both the direction and magnitude of $\vec{L}$ for the system.

(Ans: $\vec{L} = 2mR^2\omega\hat{k}$)

Problem 3: Generalization beyond 2 Point Masses

a) From what you learned in problem (2), write a generalized formula for how to calculate the angular momentum of an axially symmetric system of N point masses rotating with angular velocity $\vec{\omega} = \omega\hat{z}$ around the z-axis. In your answer, use the symbol "ℓ_i" to represent the (perpendicular) distance from a point mass m_i to the z-axis.

b) Convert your generalization into an integral for how to calculate the angular momentum of an axially symmetric solid object rotating about the z-axis. Assume the object has mass M and volume V_o. Each mass element will have distance from the z=axis of $\ell(m)$ (i.e., ℓ is a function of each piece of mass dm).

(Ans: a. $\vec{L} = \hat{k}\omega \sum_{i=1}^{N} m_i \ell_i^2$ b. $\vec{L} = \hat{k}\omega \int_V \ell^2 dm = \hat{k}\omega \int_V \ell^2 \left(\frac{M}{V_o}\right) dV$)

Problem 4: Symmetric Rigid Bodies - The Rod

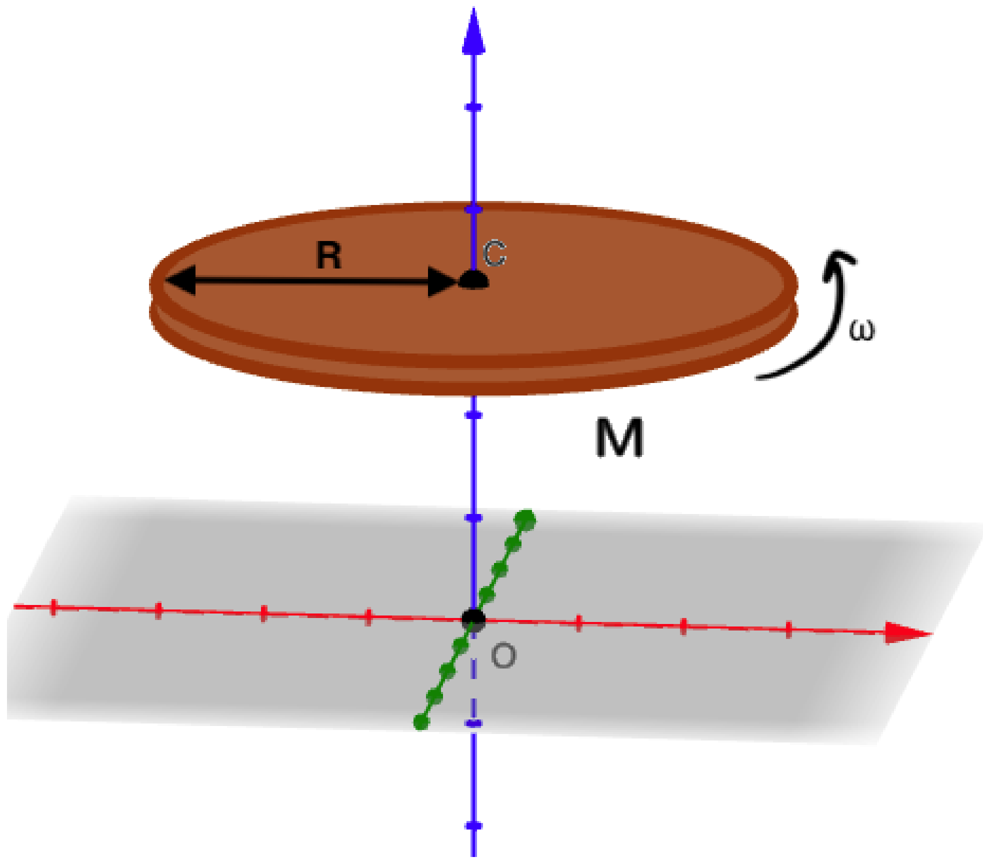
Now let's use what you learned in problem (2) to find the angular momentum of a rigid body with symmetry around the axis of rotation.

Find the angular momentum of a thin rod with uniform density, having mass M, with length L and infinitesimal cross-section. The rod rotates with angular velocity $\vec{\omega} = \omega\hat{z}$. (You can think of the rod as having linear density $\lambda = M/L$.) The rod spins around its center as in the diagram given for problem (2).

(Ans: $\vec{L} = \frac{1}{12} M L^2 \omega \hat{z}$)

Problem 5: Symmetric Rigid Bodies - The Disk

Extending our knowledge to laminar objects: Find the angular momentum of a solid disk of mass M with uniform density, infinitesimal thickness, and radius R, rotating about the z-axis, with the z-axis perpendicular to the disk's surface and passing through the center of the disk. The disk rotates with angular velocity $\vec{\omega} = \omega\hat{z}$. (You can think of the disk as having uniform density of $\sigma = \frac{M}{\pi R^2}$).



(Ans: $\vec{L} = \frac{1}{2}MR^2\omega\hat{z}$)