

## 1 | orthogonal decomposition

An orthogonal decomposition is a way of writing some vector  $v \neq 0 \in V$  as the scaled other vector  $u \in V$  plus an orthogonal component

Suppose  $u, v \in V$ , with  $v \neq 0$ . Set  $c = \frac{\langle u, v \rangle}{\|v\|^2}$  and  $w = u - cv$ . Then,

$$\langle w, v \rangle = 0 \text{ and } u = cv + w$$

The important algebra is just setting up a system of equations and noticing that orthogonality implies

$$\begin{aligned} 0 &= \langle u - cv, v \rangle \\ \Rightarrow 0 &= \langle u - cv, v \rangle = \langle u, v \rangle - \langle cv, v \rangle \\ &= \langle u, v \rangle - c\langle v, v \rangle \\ &= \langle u, v \rangle - c\|v\|^2 \end{aligned}$$

which can then be solved for  $c$

## 2 | motivation

If we have some vector  $b$  which is not in the column space of  $A$  (there does not exist  $x : Ax = b$ ) but we still want the best "approximation", then we want to take the "closest" approximation. Suppose  $\hat{b}$  is such an approximation, then we want the norm of the difference  $(b - \hat{b})$  to be minimal. Thus, we want  $b - \hat{b}$  to be orthogonal to the column space of  $A$ . This motivates orthogonal decomposition.