1 | Axler6.45 orthogonal complement, U^{\perp}

def

if U is a subset of V, then the orthogonal complement of U, denoted U^{\perp} , is the set of all vectors in V that are orthogonal to every vector in U:

$$U^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \forall u \in U \}$$

1.1 | results

1.1.1 | Axler6.46 basic properties

- 1. complement is a subspace: if U is a subset of V, then U^{\perp} is a subspace of V
 - (a) zero is orthogonal to each vector, any vector that is the sum of two fully orthogonal vectors or the scalar multiple of an orthogonal vector will still be fully orthogonal.
- 2. $\{0\}^{\perp} = V$
 - (a) zero orthogonal to every vector
- 3. $V^{\perp} = \{0\}$
 - (a) only zero orthogonal to every vector
- 4. If *U* is a subset of *V*, then $U \cap U^{\perp} \subseteq \{0\}$
 - (a) only zero is orthogonal to itself
- 5. If U and W are subsets of V and $U \subseteq W$ then $W^{\perp} \subseteq U^{\perp}$
 - (a) Everything in W^{\perp} is in U^{\perp} , and more.

1.1.2 | Axler6.47 direct sum of a subspace and its orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then,

$$V=U\oplus U^\perp$$

This can be shown by seeing that splitting any vector in V into a U part and a non-U part leads to the non-U being in U^{\perp}

1.1.3 | Axler6.50 dimension of orthogonal complement

Suppose V is finite-dimensional and U is a subpsace of V. Then,

$$\mathbf{U}^\perp = V - U$$

By the dimension of a subspace addition (Axler3.78)

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1.1.4 | Axler6.51 orthogonal complement of orthogonal complement is itself

Suppose ${\it U}$ is a finite-dimensional subspace of ${\it V}$. Then

$$U = (U^{\perp})^{\perp}$$

Because $U \oplus U^{\perp} = V$ is a direct sum and equals V.

The actual proof is by double-inclusion.

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