

1 | 1)

To finish the proof... Given two objects, A and B , with a force F between them, the torque on A and B is given by

$$\tau_A = \vec{r}_A \times \vec{F}_A$$

$$\tau_B = \vec{r}_B \times \vec{F}_B$$

where \vec{F}_A is the force applied by B on A , and vice versa. We know that because of N-3 $\vec{F}_A = -\vec{F}_B$. (We

$$\tau_{AB} = \tau_A + \tau_B$$

also know that the forces point towards each object.) Therefore,

$$= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times \vec{F}_B$$

$$= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A$$

We know that the direction of the two cross products are orthogonal to the plane that the two objects' position vectors and the origin of the system form.

$$\begin{aligned} \tau_{AB} &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A \\ &= |\vec{r}_A||\vec{F}_A| \sin \theta_A - |\vec{r}_B||\vec{F}_A| \sin \theta_B \\ &= |\vec{r}_A| \sin \theta_A - |\vec{r}_B| \sin \theta_B \end{aligned}$$

The law of sines states that for a triangle $\triangle ABC$, $\frac{\overline{BC}}{\sin \theta_A} = \frac{\overline{AC}}{\sin \theta_B}$. We know that this applies in our particular proof because the objects A , B , and the origin form a triangle. As such,

$$|\vec{r}_A| \sin \theta_A = |\vec{r}_B| \sin \theta_B$$

$$\tau_{AB} = 0$$

The internal torque of any two objects of a system is zero, so the total internal torque must also be zero.

2 | 2)

$$\vec{r} = R\hat{i} + h\hat{k}$$

$$\vec{L}_1 = \vec{r} \times m\vec{v}$$

We know that for one of the \$\$\$s: $\vec{v} = R\omega\hat{j}$

$$\begin{aligned} \vec{L}_1 &= (R\hat{i} + h\hat{k}) \times mR\omega\hat{j} \\ &= -h m R \omega \hat{i} + m R^2 \omega \hat{k} \end{aligned}$$

We know that there are two masses, symmetric about the z-axis, so we know that the angular momentum of the other object can be derived just by multiplying the \hat{i} and \hat{j} terms by -1.

$$\begin{aligned} \vec{L}_2 &= \vec{L}_1 \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= h m R \omega \hat{i} + m R^2 \omega \hat{k} \end{aligned}$$

We can add the two to get the aggregate angular momentum of the system:

$$\begin{aligned}
\vec{L} &= \vec{L}_1 + \vec{L}_2 \\
&= (-hmR\omega\hat{i} + mR^2\omega\hat{k}) + (hmR\omega\hat{i} + mR^2\omega\hat{k}) \\
&= 2mR^2\omega\hat{k}
\end{aligned}$$

3 | 3)

3.1 | a)

$$\vec{L}_N = \sum_{i=1}^N m_i l_i^2 \cdot \omega \hat{k}$$

3.2 | b)

$\vec{L} = \int_V l^2 (M/V_0) dV \cdot \omega \hat{k}$ First, because we know that the object is axially symmetric, the \hat{i} and \hat{j} components of the torque addition will be eliminated by the negative values for those components of the symmetric counterpart, so we will only be left with a \hat{k} component. Given this, we can express the angular momentum as an integral over the volume. WIP

4 | 4)

We can represent the angular momentum of the two points distance l away from the origin as a function of

$$\vec{v}_1 = -\vec{v}_2$$

$$\vec{l}_1 = -\vec{l}_2$$

$$\begin{aligned}
\vec{L}(r) &= \vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2 \\
&= \vec{r} \times m\vec{v} - \vec{r} \times -m\vec{v}
\end{aligned}$$

$$\begin{aligned}
l: \quad \vec{r} &= r\hat{i} \\
\vec{v} &= L\omega\hat{j} \\
m &= \frac{M}{L}
\end{aligned}$$

$$\begin{aligned}
\vec{L}(r) &= r\hat{i} \times ML\omega\hat{j} + (-r\hat{i} \times -ML\omega\hat{j}) \\
&= 2rLM\omega\hat{k}
\end{aligned}$$

Now that we have a function of the angular velocity in terms of the distance from the center, we can integrate. Keep in mind that we will integrate from 0 to $\frac{L}{2}$, because our angular velocity function is the sum of the two point masses that are r away from the center.

$$\begin{aligned}\vec{L} &= \int_0^{\frac{L}{2}} \vec{L}(r) dr \\ &= \int_0^{\frac{L}{2}} 2rLM\omega\hat{k} dr \\ &= [r^2LM\omega\hat{k}]_0^{\frac{L}{2}} \\ &= \frac{L^3}{4}M\omega\hat{k}\end{aligned}$$