1 | **vector equation for a point and normal**

$$\vec{r} : (\vec{r} - \vec{P_0}) \cdot \vec{n} = 0$$

2 | **normal vectors**

In cartesian form:

$$xn_x - p_x n_x + yn_y - p_y n_y + zn_z - p_z n_z = 0$$
$$xn_x + yn_y + zn_z = p_x n_x + p_y n_y + p_z n_z$$
$$xn_x + yn_y + zn_z = \vec{p} \cdot \vec{n}$$

Thus, the normal vector to the equation

$$Ax + By + Cz = D$$

is

$$\vec{n} = \langle A, B, C \rangle$$

3 | **verify that $\vec{n} \cdot \vec{r} = D$ is a plane**

$$\vec{n} \cdot \vec{r} = D$$
$$n_x x + n_y y + n_z z = D$$

Thus, $\vec{n}$ is a vector normal to the plane, and $D = \vec{P_0} \cdot \vec{n}$ where $\vec{P_0}$ is the offset from the origin.

4 | **find the distance from the origin**

The distance vector is some multiple of the normal (because the distance is perpendicular to the plane)
Thus, we just need to find the magnitude of some $\lambda \vec{n}$ such that $\lambda \vec{n}$ lies in the plane. In other words, find

$$\lambda \vec{n} : (\lambda \vec{n} - \vec{p_0}) \cdot \vec{n}$$

We can solve for λ like so:

$$\lambda \vec{n} \cdot \vec{n} - \vec{p_0} \cdot \vec{n} = 0$$
$$\lambda \vec{n} \cdot \vec{n} = \vec{p_0} \cdot \vec{n}$$
$$\lambda = \frac{\vec{p_0} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}$$
$$\lambda = \frac{\vec{p_0} \cdot \vec{n}}{|\vec{n}|^2}$$

Finally, we need to multiply by the magnitude of $\vec{n}$:

$$d = \lambda|\vec{n}|$$
$$= \frac{\vec{p_0} \cdot \vec{n}}{|\vec{n}|}$$

In the original equation, $4 = \vec{n} \cdot \vec{P_0}$ so we can just divide it by the magnitude of the normal

$$d = \frac{4}{|\vec{n}|} = \frac{4}{\sqrt{1+4+9}} = \frac{4}{\sqrt{14}}$$

5 | the distance

Let $\vec{P}$ be a vector that represents the displacement of the plane from the origin. We are looking for the distance from the plane to

$$\vec{P_0} = \langle x_0, y_0, z_0 \rangle$$

which is some other, arbitrary point.

We know the equation for the distance from the origin to the plane is the component of the displacement of the plane $\vec{P}$ on the normal vector $\vec{n}$

$$d_{\text{origin}} = \text{comp}_{\vec{n}}\vec{P} = \frac{\vec{P} \cdot \vec{n}}{|\vec{n}|}$$

We can take the same approach by looking for the projection of the displacement of the point to the plane onto the normal. We know $\vec{P_0} - \vec{P}$ is the displacement from the point to the plane because $\vec{P} + (\vec{P_0} - \vec{P}) = \vec{P_0}$

$$d = \text{comp}_{\vec{n}}\vec{P_0} - \vec{P}$$
$$= \frac{\vec{n} \cdot (\vec{P_0} - \vec{P})}{|\vec{n}|}$$
$$= \frac{\vec{n} \cdot \vec{P_0} - \vec{n} \cdot \vec{P}}{|\vec{n}|}$$
$$= \frac{(n_x P_{0x} + n_y P_{0y} + n_z P_{0z}) - D}{|\vec{n}|}$$
$$= \frac{Ax_0 + By_0 + Cz_0 - D}{|\vec{n}|}$$

6 | show the product rule for vector functions

$$\frac{d}{dt}\left(\vec{A}(t) \cdot \vec{B}(t)\right) = \frac{d}{dt}\left(\vec{A}(t)_x \vec{B}(t)_x + \vec{A}(t)_y \vec{B}(t)_y + \vec{A}(t)_z \vec{B}(t)_z\right)$$
$$= \frac{d}{dt}\left(\vec{A}(t)_x \vec{B}(t)_x\right) + \frac{d}{dt}\left(\vec{A}(t)_y \vec{B}(t)_y\right) + \frac{d}{dt}\left(\vec{A}(t)_z \vec{B}(t)_z\right)$$
$$= \left(\vec{A}'(t)_x \vec{B}(t)_x + \vec{A}(t)_x \vec{B}'(t)_x\right) + \left(\vec{A}'(t)_y \vec{B}(t)_y + \vec{A}(t)_y \vec{B}'(t)_y\right) + \left(\vec{A}'(t)_z \vec{B}(t)_z \vec{A}(t)_z \vec{B}'(t)_z\right)$$
$$= \left(\vec{A}'(t)_x \vec{B}(t)_x + \vec{A}'(t)_y \vec{B}(t)_y + \vec{A}'(t)_z \vec{B}(t)_z\right) + \left(\vec{A}(t)_x \vec{B}'(t)_x + \vec{A}(t)_y \vec{B}'(t)_y + \vec{A}(t)_z \vec{B}'(t)_z\right)$$
$$= \frac{d}{dt}\vec{A}(t) \cdot \vec{B}(t) + \vec{A} \cdot \frac{d}{dt}\vec{B}(t)$$

7 | a space curve with a time parameter

Let $\vec{r} = \vec{r}(t)$

$$
\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{d}{dt}(|\vec{r}|\hat{r}) \\
&= \left(\frac{d}{dt}|\vec{r}|\right)\hat{r} + |\vec{r}|\left(\frac{d}{dt}\hat{r}\right) \qquad \text{quotient rule}
\end{aligned}
$$

Now, we have all the parts that we need

$$
\frac{d\vec{r}}{dt} = \vec{r}\,'(t)
$$
$$
\frac{1}{|\vec{r}(t)|}\vec{r}(t) = \hat{r}
$$
$$
\frac{d}{dt}|\vec{r}(t)| = \frac{d}{dt}|\vec{r}|
$$

So we have an expression for $\vec{r}\,'(t)$, and we want to know what it's equal to when we dot it with $\hat{r}$. Hopefully, we get $\frac{d}{dt}|\vec{r}(t)|$

$$
\begin{aligned}
\hat{r} \cdot \frac{d}{dt}\vec{r}(t) &= \left(\frac{d}{dt}|\vec{r}|\right)\hat{r}\cdot\hat{r} + |\vec{r}|\left(\frac{d}{dt}\hat{r}\right)\hat{r} \\
&= \frac{d}{dt}|\vec{r}|(1) + |\vec{r}|0
\end{aligned}
$$

We know that $\hat{r} \cdot \frac{d\hat{r}}{dt} = 0$ because any change in $\hat{r}$ must maintain $|\hat{r}| = 1$. Thus, there must be no component in $\frac{d\hat{r}}{dt}$ in the same direction of $\hat{r}$.

Thus,

$$
\hat{r} \cdot \frac{d}{dt}\vec{r} = \frac{d}{dt}|\vec{r}|
$$