1 | an example: semicircle revolved around the x-axis to create a sphere

We can make cuts perpendicular to the axis of rotation. In this case, you end up with a bunch of circular disks, where the height of each slice is your semicircle function.

Thus, the volume of the disk is

$$\pi f^2(x_i)\Delta x = (a^2 - x^2)\pi\Delta x$$

This is kinda like a Riemann Sum, but with more stuff added on. We can take the limit of the sum

$$\lim_{n\to\infty} \sum_{k=1}^{n} \pi(a^2 - x_i^2)\Delta x$$

Where $\Delta x = \frac{1}{n}$ and $x_i = -a + \frac{2ak}{n}$

Expressed as an integral:

$$\begin{aligned}
\int_{-a}^{a} \pi(a^2 - x^2)dx &\to \int \pi a^2 dx - \int \pi x^2 dx \\
&= \pi a^2 x - \pi \frac{1}{3} x^3 \\
&\to \pi a^3 - \pi \frac{1}{3} a^3 + \pi a^3 + \pi \frac{1}{3}(-a)^3 \\
&= 2\pi a^3 - \pi \frac{2}{3} a^3 \\
&= \frac{4}{3}\pi a^3
\end{aligned}$$

2 | now lets try a cone

Rotate

$$y = -ax + b$$

Around the y-axis. Then, each circle (which is layed out flat) has thickness dy and radius x or $\frac{y-b}{-a}$

The volume of the disk is then

$$\pi \left(\frac{y-b}{-a}\right)^2 dy$$

Or using r, h as the radius and height of the cone,

$$\pi \left(r - \frac{r}{h}y\right)^2 dy$$

And we can take the integral of that from 0 to h

$$\int_0^h \pi \left(r - \frac{r}{h}y\right)^2 dy \to \pi \int \left(r - \frac{r}{h}y\right)^2 dx$$

Let $u = r - \frac{r}{h}y, du = -\frac{r}{h}dx$

$$= \pi - \frac{h}{r} \int u^2 du$$

$$= -\pi \frac{h}{r} \frac{1}{3} u^3 + C$$

$$= -\pi \frac{h}{r} \frac{1}{3} \left(r - \frac{r}{h}y\right)^3$$

$$\int_0^h \pi \left(r - \frac{r}{h}y\right)^2 dy \to \pi \int r^2 + \left(\frac{r}{h}y\right)^2 - 2r\left(\frac{r}{h}y\right) dx$$

$$= \frac{1}{3}\pi r^3 + \frac{1}{3}\frac{r^2}{h^2}y^3$$