$h : \mathbb{R}^2 \to \mathbb{R}^1$

$h(x, y) = x^2 + y^2 + 2x - 6y + 5$

The point $(-1, +3, h(-1, +3))$ (work out what the z-coordinate at $x = -1, y = +3$ is) is a minimum of this function! I guarantee it! Prove this, in two ways:

- Using the gross ugly don't-memorize-it multivariable second derivative test. (Feel free to look it up; do summarize/explain in your solution how to use it.)
- Without using calculus *at all*! (Let alone multivariable calculus!) (I have a hint for you if you need it.)

1 | Calculus

We know that the gradient of $h(x, y)$ is

$\nabla h(x, y) = \begin{bmatrix} 2x + 2 \\ 2y - 6 \end{bmatrix}$ Given this, we know that $\nabla h(-1, 3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ We know that $(-1, 3)$ is a root of the function. We can also take the second partial derivative: $\begin{aligned} h_{xx}(x, y) &= 2 \\ h_{xy}(x, y) &= 0 \\ h_{yy}(x, y) &= 2 \end{aligned}$ Because we know that there is no acceleration in the x or y component with respect to their counterparts, and because we know that h_{xx} and h_{yy} are positive, we know that the function is a minimum at $(-1, 3)$.

2 | Non-calculus

We know that $\nabla h(-1, 3) = 0$. We also know that $h(-1, 3) = -5$. Given this, we can solve for $h(x, y)$ and prove that the function cannot be less than -5: $\begin{aligned} h(x, y) &= x^2 + y^2 + 2x - 6y + 5 \\ &= (x)(x + 2) + (y)(y - 6) + 5 \end{aligned}$