1 | **Problem 1**

The question asks us to figure scalars α and β making:

$$\alpha[\beta[A,B],C] \tag{1}$$

hermitian if A, B, and C is hermitian.

We first expand the expressions in the combinator:

$$\begin{aligned}
&\alpha[\beta[A,B],C] && (2)\\
\Rightarrow\ &\alpha[\beta(AB-BA),C] && (3)\\
\Rightarrow\ &\alpha[\beta(AB-BA),C] && (4)\\
\Rightarrow\ &\alpha\beta[(AB-BA),C] && (5)\\
\Rightarrow\ &\alpha\beta((AB-BA)C-C(AB-BA)) && (6)\\
\Rightarrow\ &\alpha\beta((ABC-BAC)-(CAB-CBA)) && (7)\\
\Rightarrow\ &\alpha\beta(ABC-BAC-CAB+CBA) && (8)
\end{aligned}$$

We wish to figure out values α and β such that the conjugate of the above output equals to itself; that is, so that the following hold:

$$\alpha^*\beta^*(ABC-BAC-CAB+CBA) = \alpha\beta(ABC-BAC-CAB+CBA) \tag{9}$$

We did not need to conjugate the right side as all matricies A, B, C are Hermitian. Let's set:

$$X = ABC-BAC-CAB+CBA \tag{10}$$

We will get, then:

$$\alpha^*\beta^*X = \alpha\beta X \tag{11}$$

Hence, we essentially need to find scalar values α, β such that:

$$\alpha^*\beta^* = \alpha\beta \tag{12}$$

We now expand the above expression:

$$\begin{aligned}
&(a-bi)(c-di) = (a+bi)(c+di) && (13)\\
\Rightarrow\ &ac-(ad+bc)i-bd = ac+(ad+bc)i+bd && (14)\\
\Rightarrow\ &-(ad+bc)i = (ad+bc)i && (15)\\
\Rightarrow\ &-(ad+bc) = (ad+bc) && (16)
\end{aligned}$$

The only value of $ad+bc$ for which this would hold is 0. Hence:

$$\begin{aligned}
&ad+bc = 0 && (17)\\
\Rightarrow\ &ad = -bc && (18)
\end{aligned}$$

If we provide three degrees of freedom, we get that:

$$\begin{cases} \alpha = a + bi \\ \beta = \frac{-ad}{b} + di \end{cases} \tag{19}$$

2 | Problem 2

Considering the Hamiltonian operator:

$$\vec{H} = \frac{\hbar\omega}{2}\sigma\vec{n} \tag{20}$$

Solve:

$$\vec{H}|E_j\rangle = E_j|E_j\rangle \tag{21}$$

where E_j is the \$j\$-th eigenvalue of the hermitian.
Let's first recall the generic expression for σ_n for any normal vector $\vec{n}$:

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix} \tag{22}$$

We will multiply the scalar $\frac{\hbar\omega}{2}$ upon the above expression, resulting in:

$$H = \frac{\hbar\omega}{2}\sigma_n = \begin{pmatrix} \frac{\hbar\omega n_z}{2} & \frac{\hbar\omega(n_x - in_y)}{2} \\ \frac{\hbar\omega(n_x + in_y)}{2} & \frac{-\hbar\omega n_z}{2} \end{pmatrix} \tag{23}$$

Now, we attempt to solve for the eigenvectors of the expression via Sage:

```
# define the normal and pauli_n matrix
nx,ny,nz = var("nx ny nz")
paulin = Matrix([[nz, (nx-i*ny)], [(nx+i*ny), -nz]])

# define the hermitian
hbar, omega = var("hbar omega")
H = ((hbar*omega)/2)*paulin

# we now get the eigenvalues
H.eigenvalues()
H.eigenvectors_right()
```

$$\left(\frac{1}{2}\sqrt{nx^2 + ny^2 + nz^2}hbar\omega, \left[\left(1, -\frac{(nx + i\,ny)nz - \sqrt{nx^2 + ny^2 + nz^2}(nx + i\,ny)}{nx^2 + ny^2}\right)\right], 1\right) \tag{24}$$

The eigenvalues, as per given by Sage, of the Hermitian is:

$$\left[\frac{1}{2}||n||\hbar\omega, \frac{-1}{2}||n||\hbar\omega\right] \tag{25}$$

We will also take one produced eigenvalue:

$$\begin{pmatrix} 1 \\ -\frac{(n_x+n_y\ i)(n_z-||n||)}{{n_x}^2+{n_y}^2} \end{pmatrix} \tag{26}$$

This solution is not extremely useful. However, we can supply a more definite $\vec{n}$ to make this solution more interesting.

Take, for instance, where $\vec{n}$ lies on some θ on the x, y plane. We can take the following parameterization:

$$\begin{cases} n_x = cos\theta \\ n_y = sin\theta \\ n_z = 0 \end{cases} \tag{27}$$

Applying this parameterization to the above eigenvector:

$$\begin{pmatrix} 1 \\ cos\theta + i\ sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \tag{28}$$

The Hamiltonian operator given by $\frac{\hbar\omega}{2}\sigma \cdot \vec{n}$ for a normal vector $\vec{n}$ on the x,y plane is concentrated on a cyclic manner on the $\hat{j}$ direction by:

$$\begin{pmatrix} 1 \\ e^{i} \end{pmatrix} \tag{29}$$