1 | vector equation for a point and normal

$$\vec{r} : (\vec{r} - \vec{P_0}) \cdot \vec{n} = 0$$

The plane is all vectors s.t. when offset by $\vec{P_0}$, they are orthogonal to $\vec{n}$. It can be thought of as a transformation of the "parent plane" $\vec{r} \cdot \vec{n} = 0$, just as $y - k = m(x - h)$ is a transformation of the parent line $y = mx$.

2 | normal vectors

In cartesian form:

$$\begin{aligned} xn_x - p_x n_x + yn_y - p_y n_y + zn_z - p_z n_z &= 0 \\ xn_x + yn_y + zn_z &= p_x n_x + p_y n_y + p_z n_z \\ xn_x + yn_y + zn_z &= \vec{p} \cdot \vec{n} \end{aligned}$$

Thus, the normal vector to the equation

$$Ax + By + Cz = D$$

is

$$\vec{n} = \langle A, B, C \rangle$$

3 | verify that $\hat{n} \cdot \vec{r} = D$ is a plane

$$\begin{aligned} \hat{n} \cdot \vec{r} &= D \\ n_x x + n_y y + n_z z &= D \end{aligned}$$

Thus, $\hat{n}$ is a vector normal to the plane, and $D = \vec{P_0} \cdot \hat{n}$ where $\vec{P_0}$ is the offset from the origin.

D is the distance from the plane to the origin, because it is the component of the offset vector onto the normal vector, and we know that the closest offset will be normal to the plane.

4 | find the distance from the origin

The distance vector is some multiple of the normal (because the distance is perpendicular to the plane)

Thus, we just need to find the magnitude of some $\lambda\vec{n}$ such that $\lambda\vec{n}$ lies in the plane. In other words, find

$$\lambda\vec{n} : (\lambda\vec{n} - \vec{p_0}) \cdot \vec{n}$$

We can solve for λ like so:

$$\begin{aligned}
\lambda\vec{n}\cdot\vec{n} - \vec{p_0}\cdot\vec{n} &= 0 \\
\lambda\vec{n}\cdot\vec{n} &= \vec{p_0}\cdot\vec{n} \\
\lambda &= \frac{\vec{p_0}\cdot\vec{n}}{\vec{n}\cdot\vec{n}} \\
\lambda &= \frac{\vec{p_0}\cdot\vec{n}}{|\vec{n}|^2}
\end{aligned}$$

Finally, we need to multiply by the magnitude of $\vec{n}$:

$$\begin{aligned}
d &= \lambda|\vec{n}| \\
&= \frac{\vec{p_0}\cdot\vec{n}}{|\vec{n}|}
\end{aligned}$$

In the original equation, $4 = \vec{n}\cdot\vec{P_0}$ so we can just divide it by the magnitude of the normal

$$d = \frac{4}{|\vec{n}|} = \frac{4}{\sqrt{1+4+9}} = \frac{4}{\sqrt{14}}$$

5 | **the distance**

Let $\vec{P}$ be a vector that represents the displacement of the plane from the origin. This can be any point on the plane, because all displacements that lie on the plane are equivalent. This way, we know that the equation for the plane is $\vec{r} : \vec{r}\cdot\vec{n} = \vec{P}\cdot\vec{n} = D$

We are looking for the distance from the plane to

$$\vec{P_0} = \langle x_0, y_0, z_0\rangle$$

which is some other, arbitrary point.

We know the equation for the distance from the origin to the plane is the component of the displacement of the plane $\vec{P}$ on the normal vector $\vec{n}$

$$d_{\text{origin}} = \text{comp}_{\vec{n}}\vec{P} = \frac{\vec{P}\cdot\vec{n}}{|\vec{n}|}$$

We can take the same approach by looking for the projection of the displacement of the point to the plane onto the normal. We know $\vec{P_0} - \vec{P}$ is the displacement from the point to the plane because $\vec{P} + (\vec{P_0} - \vec{P}) = \vec{P_0}$

$$\begin{aligned}
d &= \left|\text{comp}_{\vec{n}}\left(\vec{P_0} - \vec{P}\right)\right| \\
&= \left|\frac{\vec{n}\cdot(\vec{P_0} - \vec{P})}{|\vec{n}|}\right| \\
&= \left|\frac{\vec{n}\cdot\vec{P_0} - \vec{n}\cdot\vec{P}}{|\vec{n}|}\right| \\
&= \left|\frac{(n_xP_{0x} + n_yP_{0y} + n_zP_{0z}) - D}{|\vec{n}|}\right| \\
&= \frac{|Ax_0 + By_0 + Cz_0 - D|}{|\vec{n}|}
\end{aligned}$$

6 | show the product rule for vector functions

$$\begin{aligned}
\frac{d}{dt}\left(\vec{A}(t)\cdot\vec{B}(t)\right) &= \frac{d}{dt}\left(\vec{A}(t)_x\vec{B}(t)_x + \vec{A}(t)_y\vec{B}(t)_y + \vec{A}(t)_z\vec{B}(t)_z\right) \\
&= \frac{d}{dt}\left(\vec{A}(t)_x\vec{B}(t)_x\right) + \frac{d}{dt}\left(\vec{A}(t)_y\vec{B}(t)_y\right) + \frac{d}{dt}\left(\vec{A}(t)_z\vec{B}(t)_z\right) \\
&= \left(\vec{A}'(t)_x\vec{B}(t)_x + \vec{A}(t)_x\vec{B}'(t)_x\right) + \left(\vec{A}'(t)_y\vec{B}(t)_y + \vec{A}(t)_y\vec{B}'(t)_y\right) + \left(\vec{A}'(t)_z\vec{B}(t)_z\vec{A}(t)_z\vec{B}'(t)_z\right) \\
&= \left(\vec{A}'(t)_x\vec{B}(t)_x + \vec{A}'(t)_y\vec{B}(t)_y + \vec{A}'(t)_z\vec{B}(t)_z\right) + \left(\vec{A}(t)_x\vec{B}'(t)_x + \vec{A}(t)_y\vec{B}'(t)_y + \vec{A}(t)_z\vec{B}'(t)_z\right) \\
&= \frac{d}{dt}\vec{A}(t)\cdot\vec{B}(t) + \vec{A}\cdot\frac{d}{dt}\vec{B}(t)
\end{aligned}$$

7 | a space curve with a time parameter

Let $\vec{r} = \vec{r}(t)$

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{d}{dt}(|\vec{r}|\hat{r}) \\
&= \left(\frac{d}{dt}|\vec{r}|\right)\hat{r} + |\vec{r}|\left(\frac{d}{dt}\hat{r}\right) \qquad \text{product rule}
\end{aligned}$$

Now, we have all the parts that we need

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \vec{r}\,'(t) \\
\frac{1}{|\vec{r}(t)|}\vec{r}(t) &= \hat{r} \\
\frac{d}{dt}\,|\vec{r}(t)| &= \frac{d}{dt}|\vec{r}|
\end{aligned}$$

So we have an expression for $\vec{r}\,'(t)$, and we want to know what it's equal to when we dot it with $\hat{r}$. Hopefully, we get $\frac{d}{dt}|\vec{r}(t)|$

$$\begin{aligned}
\hat{r}\cdot\frac{d}{dt}\vec{r}(t) &= \left(\frac{d}{dt}|\vec{r}|\right)\hat{r}\cdot\hat{r} + |\vec{r}|\left(\frac{d}{dt}\hat{r}\right)\hat{r} \\
&= \frac{d}{dt}|\vec{r}|(1) + |\vec{r}|0
\end{aligned}$$

We know that $\hat{r}\cdot\frac{d\hat{r}}{dt} = 0$ because any change in $\hat{r}$ must maintain $|\hat{r}| = 1$. Thus, there must be no component in $\frac{d\hat{r}}{dt}$ in the same direction of $\hat{r}$.

Thus,

$$\hat{r}\cdot\frac{d}{dt}\vec{r} = \frac{d}{dt}|\vec{r}|$$