Angular Momentum & Torque Theorems

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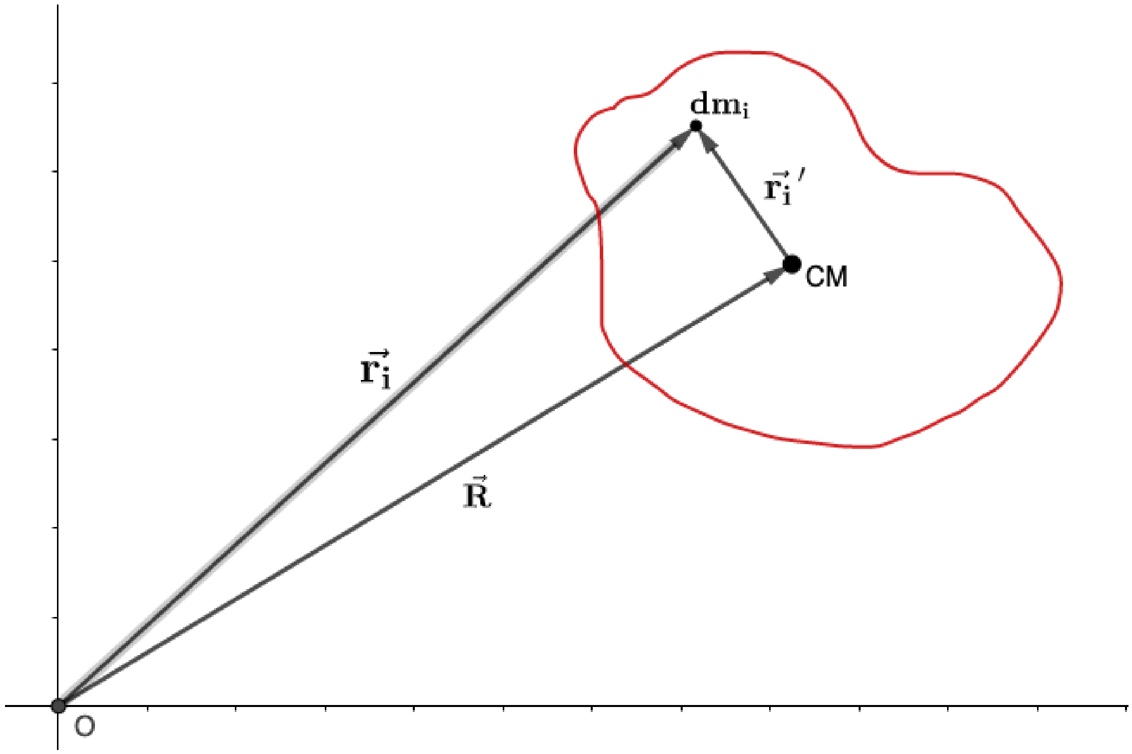
Comparison Chart of Angular & Linear Quantities

Type	Rotational	Linear
Velocity	Angular Velocity $\vec{\omega}$ (rad/s)	Velocity $\vec{v}$ (m/s))
Momentum	Angular Momentum $\vec{L} = \vec{r} \times m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})$	Momentum $\vec{p} = m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1})$
"Force"	Torque $\vec{\tau} = \vec{r} \times \vec{F}$ $(\mathrm{N} \cdot \mathrm{m})$	Force $\vec{F}$ (N)
"Particle 2nd Law"	$\vec{\tau}_{net} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$	$\vec{F}_{net} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$
System 2nd Law	$\vec{\tau}_{net\,ext} = \frac{\mathrm{d}\vec{L}_{sys}}{\mathrm{d}t}$	$\vec{F}_{net\,ext} = \frac{\mathrm{d}\vec{p}_{sys}}{\mathrm{d}t}$
Rigid Body Momentum	$\vec{L}_{sys} = I\vec{\omega}$	$\vec{p}_{sys} = M\vec{v}_{CM}$

where if the system is a *rigid body* rotating with $\vec{\omega} = \omega\hat{z}$, then $I = I_z = \sum_i m_i {\ell_i}^2$, and ℓ_i is the distance from the z-axis. Here I is called the *"Rota-*

tional Inertia" or *"Moment of Inertia"* of the rigid body, and M is called just the *"Inertia"* or *"Mass"* of the system.

Notation



Let O be the origin of an inertial reference frame. Let $\vec{R}$ be the position vector of the CM of a system of masses. The coordinate of a mass m_i is denoted by $\vec{r_i}$ relative to O, and by $\vec{r_i}'$ relative to the CM of the system. Coordinates in the CM-frame are denoted by primes, whereas inertial-frame coordinates are without primes, but coordinate unit vectors remain the same, so that $\hat{x} = \hat{x}\,', \hat{y} = \hat{y}\,', \hat{z} = \hat{z}\,'$. Note that all of the vectors $\vec{R}, \vec{r_i}, \vec{r_i}'$ change over time, and their first and second derivatives are denoted by $\vec{v}$ and $\vec{a}$, respectively. For example, $\vec{V} = \frac{\mathrm{d}\vec{R}}{\mathrm{d}t}, \vec{v_i} = \frac{\mathrm{d}\vec{r_i}}{\mathrm{d}t}, \vec{v_i}' = \frac{\mathrm{d}(\vec{r_i}')}{\mathrm{d}t}$.
The position vector of a point mass m_i is expressed by:

$$\vec{r_i} = \vec{R} + \vec{r_i}'$$

Prove the following theorems:

Lemma 1: Properties of CM reference frame

Prove the following properties of the CM coordinates, for a system consisting of n masses, $\{m_i\}$:

$$\sum_{i=1}^{n} m_i \vec{r}_i{}' = 0$$
$$\sum_{i=1}^{n} m_i \vec{v}_i{}' = 0$$
$$\sum_{i=1}^{n} m_i \vec{a}_i{}' = 0$$

Theorem 1: System Angular Momentum

The System Angular Momentum can be expressed as the sum of two terms: 1) the angular momentum calculated as though all of the mass is placed at the CM, plus 2) the angular momentum of the system measured about the CM as the origin:

$$\boxed{\vec{L}_{sys} = \vec{R} \times M\vec{v}_{cm} + \sum_{i=1}^{n} \vec{r}_i{}' \times m_i \vec{v}_i{}'} \tag{1}$$

(Note: This theorem is true even when the CM is a non-inertial reference frame.)

Theorem 2: Change in System Angular Momentum

The rate of change of the System Angular Momentum can be calculated by the sum of two terms: 1) a term that almost looks like the torque on a point particle located at the CM (i.e., $\vec{R} \times M\vec{a}_{cm}$), plus 2) a term that looks like the sum of torques measured relative to the CM (i.e., $\sum(\vec{r}_i{}' \times m_i\vec{a}_i{}')$).

$$\boxed{\frac{d\vec{L}_{sys}}{dt} = \vec{R} \times M\vec{a}_{cm} + \sum_{i=1}^{n} \vec{r}_i{}' \times m_i\vec{a}_i{}'} \tag{2}$$

Theorem 3: Total Torque on a System

The total torque of a system can be calculated by a combination of torque on the CM plus the torque around the CM:

$$\boxed{\vec{\tau}_{net} = \vec{R} \times \vec{F}_{net} + \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_{i,net\,ext}} \tag{3}$$

where $\vec{F}_{net}$ is the sum of all external forces on the system
and $\vec{F}_{i,net\,ext}$ is the sum of the next external forces on mass m_i

Theorem 4: Net Torque Equals Change of Angular Momentum in the CM-Frame

Combining Theorems (2) and (3) and performing a small amount of algebra, we can obtain an important result that relates torque to angular momentum when measured in the CM reference frame (even though the CM-frame is non-inertial). The relationship is what you would hope, that is:

Net Torque equals Change of Angular Momentum in the CM-Frame

$$\vec{\tau}\,'_{net} = \frac{d}{dt}\vec{L}'_{sys} \tag{4}$$

where $\vec{\tau}\,'_{net} = \sum_{i=1}^{n} \vec{r}_i\,' \times \vec{F}_{i,net\,ext}$ and $\vec{L}'_{sys} = \sum_{i=1}^{n} \vec{r}_i\,' \times m_i \vec{v}_i\,'$

Theorem 5: Angular Momentum of a Spinning Rigid Body in CM-frame

The angular momentum of a rigid body as measured in the (non-inertial) CM-frame is equal to the rotational inertia (I_{CM}) times the angular velocity vector ($\vec{\omega}\,'$). (For more details see note below the equation.)

Angular Momentum in CM-Frame (Rigid Body)

$$\vec{L}\,' = \sum_i \vec{r}_i\,' \times m_i \vec{v}_i\,' = I_{CM}\vec{\omega}\,' \tag{5}$$

(For an axially symmetric rigid body)

(Note: This theorem generally applies only when the rigid body is axially symmetric - i.e. when the CM of each perpendicular slice of the rigid body lies on the axis of rotation. In addition, the angular velocity vector must be aligned with the axis for which the rotational inertia is calculated. If the rigid body is not axially symmetric about the axis of rotation, then this formula yields only the component of the angular momentum that is aligned with the axis of rotation, and there will be other time-varying components of $\vec{L}$ perpendicular to that axis. A more general form of the equation requires use of the rotational inertia tensor).

Theorem 6: Net Torque on a Spinning Rigid Body in CM-frame

In the CM-reference frame, the net torque on a rigid body is equal to the rotational inertia about the CM ($I_{_{CM}}$) times the angular acceleration vector ($\vec{\alpha}\,'$) (Note: $\vec{\tau}\,'$ and $\vec{\alpha}\,'$ must be in the same direction as $\vec{\omega}\,'$, and other requirements apply - see note below).

Net Torque in CM-Frame (Rigid Body)

$$\vec{\tau}\,'_{net} = \sum_i \vec{r}_i\,' \times m_i \vec{a}_i\,' = I_{_{CM}} \vec{\alpha}\,' \tag{6}$$

(For an axially symmetric rigid body)

(Note: This theorem only applies under the same conditions as for Theorem (5). The directions of $\vec{\tau}\,'$, $\vec{\alpha}\,'$ and $\vec{\omega}\,'$ must be parallel (or anti-parallel) and directed along an axis of symmetry as defined earlier for Theorem (5).)