PS#21: Triple integrals, but in more coordinate systems!!!

Nueva Multivariable Calculus

(Read the solution set to the last pset, obviously, and also maybe read my very bad/very sketchy notes on triple integrals in different coordinate systems I wrote last year, linked on Canvas?)

1. Now that you've finished painting the walls of your Pringle-shaped house, you're inside it, at a makeshift desk, trying to figure out how much paint you need to paint the roof, as well as how much tile you need to cover the floor. Phew! These calculations are so exhausting that you keep having to wipe the sweat from your brow... or, actually, maybe that's just because it's intolerably hot inside. I guess you shouldn't have painted the walls matte black! And maybe you should have included some windows! Oops.

   It's clearly way too difficult to work out these hard calculations until you get the temperature down to a reasonable double-digit number. So you turn to another house-building task: the HVAC system. You want to buy an air-conditioning unit, but to do that, you need to know how much air to condition—A/C systems are sold based on the volume of space they can effectively cool.

2. Consider the function:
$$f(x, y) = \sqrt{x^2 + y^2}$$
   What's its average value over the region bounded by the polar curve $r = 3 \sin 2\theta$, where θ runs from 0 to $\pi/2$? (Give a picture(s) of the shape, too, of course!)

3. Suppose you have an upside-down ice cream cone, described in **cylindrical coordinates** by the equation:
$$r = 1 - z$$
   (There's no θ, so it's radially symmetric, like a good ice cream cone.) How tall is it? How wide is it at its mouth? Draw a picture. Suppose its density is given by:
$$d(r, \theta, z) = z$$
   What's the total mass of this cone?

4. **Sam Timinsky's Special Sphere**. Sam Timinsky comes to you with an artifact he's showing his historiography-of-the-atomic-bomb class: it's a sphere of plutonium, intended to be the core of an atomic bomb!!! Unfortunately, due to a production error, the mass is *just* subcritical, which is why the core still exists, and why you and Sam aren't both dead.

   Sam wants to know how heavy it is. He doesn't know how to use basic measuring tools like a scale or a triple beam balance, but he *does* know that the density of the sphere, which (due to the alloy and whatnot) drops off somewhat with distance from the center. The sphere is four inches in diameter, and its density, as a function of the distance ρ from its center, is:
$$d(\rho, \theta, \phi) = 100 - \rho^2 \quad \text{kilograms/cubic inch}$$
   How heavy is Sam's scary sphere?