1 | Differentiation in high dimensions

1.1 | 14)

$$\nabla f = \begin{bmatrix} x_3 & 0 & x_1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sec^2(x_2)} \\ 0 & -\frac{1}{x_2} & 0 & 0 \\ 12(x_1-2)^3 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2 | 23)

The slope, given a function f at a point (x, y), in the direction θ, is given by

$$s(\theta) = \frac{\partial}{\partial x} f(x,y) \cdot \cos(\theta) + \frac{\partial}{\partial y} f(x,y) \cdot \sin(\theta)$$

Note that both $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are constants and will be treated as constants, because x and y stay constant.

Given this function, we can find the value of theta that maximizes this function:

$max(s)$ = θ for which $s'(\theta) = 0$ and $s''(\theta) < 0$

We need to know the derivative of $s(\theta)$ of the first and second degree:

$$s'(\theta) = -\frac{\partial}{\partial x} f(x,y) \cdot \sin(\theta) + \frac{\partial}{\partial y} f(x,y) \cdot \cos(\theta)$$
$$s''(\theta) = -\frac{\partial}{\partial x} f(x,y) \cdot \cos(\theta) - \frac{\partial}{\partial y} f(x,y) \cdot \sin(\theta)$$

We can now set $s'(\theta) = 0$ and solve for θ:

$$s'(\theta) = 0 = -\frac{\partial}{\partial x} f(x,y) \cdot \sin(\theta) + \frac{\partial}{\partial y} f(x,y) \cdot \cos(\theta)$$
$$\frac{\partial}{\partial x} f(x,y) \cdot \sin(\theta) = \frac{\partial}{\partial y} f(x,y) \cdot \cos(\theta)$$
$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\partial x}{\partial y}$$
$$\tan(\theta) = \partial x \partial y$$
$$\theta = \tan^{-1}\left(\frac{\partial x}{\partial y}\right)$$

Note that $\frac{\partial x}{\partial y}$ is just $\frac{\frac{\partial}{\partial x} f(x,y)}{\frac{\partial}{\partial y} f(x,y)}$.

2 | Sand Dunes

Our function for the sand dunes is $f(x, y) = \sin(x)$. The oasis city is directly north-northeast, which means that, given that the vector $\hat{i}$ is pointing in the east direction, the angle of north-northeast will be $\theta = \frac{3\pi}{8}$. We also know that we are at the coordinate $(\frac{23\pi}{3}, 32)$. Based on this, the gradient of $f(x, y)$ can be given by:

$$\nabla f(x,y) = \begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} \tag{1}$$

The slope of $f(x, y)$ in the direction θ can be modeled as:

$$s(x, y) = -\sin\left(\frac{3\pi}{8}\right)\cos(x) \tag{2}$$

We essentially have a derivative of the sand dunes in 3D. We want to turn this into a 2D function (as in, R1 -> R1). We can do this by rewriting the equation as a function of x, integrating, and then multiplying by a constant to reflect the additional distance we are covering (because we are moving in a diagonal trajectory).

$$s(x) = -\sin\left(\frac{3\pi}{8}\right)\cos(x)$$

$$S_{proto}(x) = -\sin\left(\frac{3\pi}{8}\right)\sin(x) + C$$

We know that our initial position (on the sand dunes) was at $(\frac{23\pi}{3}, 32)$, and $f(\frac{23\pi}{3}, 32) = -\frac{\sqrt{3}}{2}$. That means that $S_{proto}(\frac{23\pi}{3}) = -\frac{\sqrt{3}}{2}$, or it should, ideally. We can tweak C to be that way:

$$-\frac{\sqrt{3}}{2} = -\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{23\pi}{3}\right) + C$$

$$-\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \cdot -\frac{\sqrt{2+\sqrt{2}}}{2} + C$$

$$C = -\frac{1}{4}\sqrt{3}\left(2 + \sqrt{2+\sqrt{2}}\right)$$

With $S_{proto}(x)$, we are assuming that our initial position in the sand dune is $x = \frac{23\pi}{3}$. Instead, we should model the initial position as being $x = 0$: $S_{proto}(x) = -\sin\left(\frac{3\pi}{8}\right)\sin\left(x - \frac{23\pi}{3}\right) - \frac{1}{4}\sqrt{3}\left(2 + \right.$

We are almost at the end. Currently, x, is in the direction of $\hat{i}$. Instead, it should be in the direction of θ. We can change this! We just need to divide x by $(\cos\theta)$ $(= \cos(\frac{3\pi}{8}))$. Simple as.

$$S(x) = -\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{1}{\cos(\frac{3\pi}{8})}\left(x - \frac{23\pi}{3}\right)\right) - \frac{1}{4}\sqrt{3}\left(2 + \sqrt{2+\sqrt{2}}\right)$$

This is really complicated and is probably wrong, but it is good enough for me.