1 | Taylor Series in e^x

Calculate, from the big scary formula, the Taylor series for e^x, centered around $x = 2$.

$$f(x) = e^x = e^2 + e^2(x-2) + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!} \cdots + \frac{e^2(x-2)^n}{n!} \tag{1}$$

2 | Diff. in Higher Dimensions

2.1 | Derivative Matrix 14

Find the derivative matrix of

$$f : \mathbb{R}^4 \to \mathbb{R}^5; f(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1x_3 \\ \tan(x_4) \\ -\ln(x_2) \\ (3x_1 - 2)^4 \\ 1729 \end{bmatrix} \tag{2}$$

$$f'(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_3 & 0 & x_1 & 0 \\ 0 & 0 & 0 & \sec^2(x_4) \\ 0 & \frac{-1}{x_2} & 0 & 0 \\ 12(3x_1 - 2)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

2.2 | Facing an Arbitrary Direction

Suppose you have a function $f(x, y); f : \mathbb{R}^2 \to \mathbb{R}^1$. Imagine you are standing at this function, at the point (x, y), facing θ. What is the slope? For what value is the slope greatest? Upwards? Downwards? Flat?

2.2.1 | Slope at point θ

The slope at angle θ is as follows:

$$f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta) \tag{4}$$

This simply acts to project the components of the gradient of f upon its axis x and y – $\cos(\theta)$ and $\sin(\theta)$ respectively – and sum them into one scalar value.

2.2.2 | Greatest slope Upwards

The process to find the angle θ at which to maximize the slope requires optimizing the above-derived expression. As we know that there exists a θ such that the slope would be maximized, we could perform this by solving for θ on the following expression:

$$\frac{d}{d\theta}(f_x(x,y)\cos(\theta)+f_y(x,y)\sin(\theta))=0 \tag{5}$$

Solution of this expression would therefore be the angle at which the slope is maximized. Hence, we solve for θ in the above expression.

$$\begin{align}
&\frac{d}{d\theta}(f_x(x,y)\cos(\theta)+f_y(x,y)\sin(\theta))=0 \tag{6}\\
\Rightarrow& -f_x(x,y)\sin(\theta)+f_y(x,y)\cos(\theta)=0 \tag{7}\\
\Rightarrow& f_y(x,y)\cos(\theta)=f_x(x,y)\sin(\theta) \tag{8}\\
\Rightarrow& \frac{f_y(x,y)}{f_x(x,y)}=\frac{\sin(\theta)}{\tan(\theta)} \tag{9}\\
\Rightarrow& \frac{f_y(x,y)}{f_x(x,y)}=\tan(\theta) \tag{10}\\
\Rightarrow& \theta=\arctan(\frac{f_y(x,y)}{f_x(x,y)}) \tag{11}
\end{align}$$

The unit direction vector representing this slope, therefore, would be:

$$\begin{bmatrix}\frac{f_x(x,y)}{\sqrt{{f_x}^2(x,y)+{f_y}^2(x,y)}}\\ \frac{f_y(x,y)}{\sqrt{{f_x}^2(x,y)+{f_y}^2(x,y)}}\end{bmatrix} \tag{12}$$

2.2.3 | **Greatest slope Downwards**

Given the max θ as derived above:

$$\pi-\theta \tag{13}$$

The vector orthogonal to the vector direction representing the maximum slope will represent the smallest slope.

The vector representing this expression, therefore, would be

$$\begin{bmatrix}\cos(\pi-\theta)\\ \sin(\pi-\theta)\end{bmatrix} \tag{14}$$

2.2.4 | **Angle of Flat Slope**

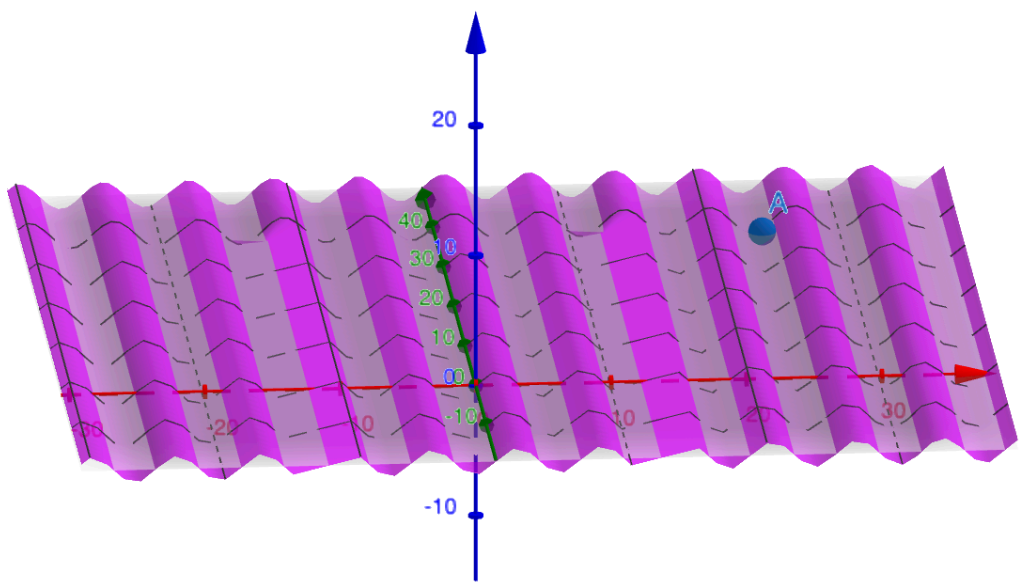
To figure the angle at which a flat slope exists, we simply solve for an expression for θ while setting the above-derived expression for slope-at-angle at 0 as that would represent a flat slope.

$$f_x(x, y)\,\text{cos}(\theta) + f_y(x, y)\,\text{sin}(\theta) = 0 \tag{15}$$
$$\Rightarrow f_y(x, y)\,\text{sin}(\theta) = -f_x(x, y)\,\text{cos}(\theta) \tag{16}$$
$$\Rightarrow \frac{-f_x(x, y)}{f_y(x, y)} = \frac{\text{sin}(\theta)}{\text{cos}(\theta)} \tag{17}$$
$$\Rightarrow \frac{-f_x(x, y)}{f_y(x, y)} = \text{tan}(\theta) \tag{18}$$
$$\Rightarrow \theta = \text{arctan}\,\frac{-f_x(x, y)}{f_y(x, y)} \tag{19}$$

3 | **Sand Dunes**

You are trudging across a field of sand dunes, which the prevailing winds have blown into perfect, parallel, straight lines (or straight ridges, rather). You know that if you walk directly north-northeast, you'll make it to the oasis city of Iskendrebad. The landscape follows the function $f(x, y) = \text{sin}(x)$; you're at the point with x coordinate $23\pi/3$ and y coordinate 37.

3.1 | **Make a Picture of the Situation**



3.2 | **What is your elevation**

At that point, you are at an elevation of $\text{sin}(\frac{23\pi}{3}) = \frac{-\sqrt{3}}{2}$

3.3 | **What does your hike look like?**

"North-northeast" could translate an angle of roughly $68° \approx 0.0174533\ rad$. Slicing though the manifold with a line $y = 2.475x$, which represents the same angle…

We first parameterize the slice equation as follows:

$$y = t$$
$$x = \frac{1}{2.475}t$$

The function at $f(\frac{t}{2.475}, t)$, therefore, is:

$$f(\frac{t}{2.475}, t) = \sin(\frac{t}{2.475}) \tag{20}$$

Hence, the hike will also behave as $f(t) = \sin(\frac{t}{2.475})$.

This could also be understood as the line at which $y = 2.475x$ intersects with $z = \sin(x)$. We could solve for the expression w.r.t. setting x values equal.

$$y = 2.475x \tag{21}$$
$$\Rightarrow x = \frac{y}{2.475} \tag{22}$$
$$\Rightarrow z = \sin(\frac{y}{2.475}) \tag{23}$$

3.4 | **What's the function for the slope along your hike?**

The function for the slope along the hike is the single-variable derivative of the parametrized function above; that is:

$$f'(t) = \frac{d}{dt}\sin(\frac{t}{2.475}) = \frac{1}{2.475}\cos(\frac{t}{2.475}) \tag{24}$$

3.5 | How steep is the sand dune at the point you're standing (in the direction you're hiking)?

As per above, the direction in which we are standing is at $68°$. This would represent a direction vector of:

$$\begin{bmatrix} 0.374606 \\ 0.927183 \end{bmatrix} \tag{25}$$

The gradient of the function at point at $(\frac{23\pi}{3}, 37)$:

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \tag{26}$$

Therefore, the slope at that point is:

$$\begin{bmatrix} 0.374606 \\ 0.927183 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = -0.1873032967 \tag{27}$$

This would amount to a slope of arctan$(-0.1873032967) \approx -10.609°$