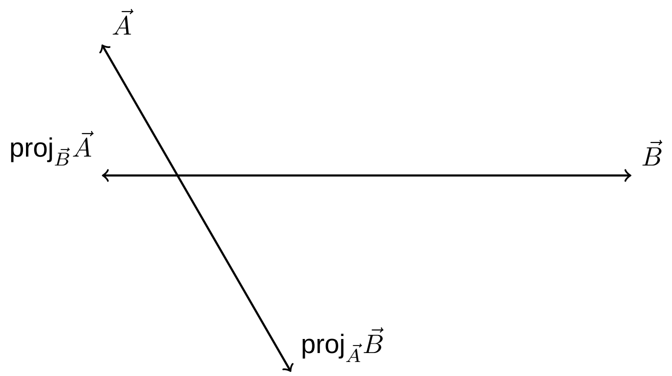
1 | vectors at an angle

1.1 | a sketch



1.2 | components

$$\begin{aligned}\text{comp}_{\vec{A}}\vec{B} &= 6\cos 120^\circ \\ \text{comp}_{\vec{B}}\vec{A} &= 2\cos 120^\circ\end{aligned}$$

1.3 | dot product

$$\begin{aligned}\vec{A}\cdot\vec{B} &= |A||B|\cos\theta \\ &= 2\cdot 6\cdot\cos 120 = -6\end{aligned}$$

2 | proving expression for component

Lets redefine the coordinate axis so that $\vec{A}$ lies along the x-axis. Then,

$$\begin{aligned}\text{comp}_{\vec{A}}\vec{B} &= |\vec{B}|\cos\theta \\ &= \frac{|\vec{A}||\vec{B}|\cos\theta}{|\vec{A}|} \\ &= \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}\end{aligned}$$

3 | expression for projection

The projection is just a vector with length comp$_{\vec{A}}\vec{B}$ in the direction of $\vec{A}$.

$$\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}\right)\frac{\vec{A}}{|\vec{A}|}$$

4 | expression for perpendicular

The part of $\vec{A}$ that is perpendicular to $\vec{B}$ is just the whole vector minus the part that is parallel:

$$\begin{aligned}\vec{A}_{\perp\vec{B}} &= \vec{A} - \text{proj}_{\vec{B}}\vec{A} \\ &= \vec{A} - \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)\vec{B}\end{aligned}$$

Checking using the dot product:

$$\begin{aligned}\left(\vec{A} - \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)\vec{B}\right)\cdot\vec{B} &= \vec{A}\cdot\vec{B} - \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)\vec{B}\cdot\vec{B} \\ &= \vec{A}\cdot\vec{B} - \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)|\vec{B}|^2 \\ &= \vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{B} \\ &= 0\end{aligned}$$

5 | find angle using dot product

Well, the dot product already includes the angle, so let's just solve for that

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

The angle between:

$$\begin{aligned}\theta &= \cos^{-}\left(\frac{3+2-4}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+1^2+2^2}}\right) \\ &= \cos^{-}\left((3+2-4)/(3*\sqrt{14})\right) = \cos^{-}(0.08908708) \quad \approx 84.8^\circ\end{aligned}$$

6 | distributivity of dot product across vector addition

The dot product is defined as

$$\begin{aligned}\vec{A}\cdot\vec{B} &= |\vec{A}||\vec{B}|\cos\theta \\ &= |\vec{A}|\text{comp}_{\vec{A}}\vec{B}\end{aligned}$$

We want to show that

$$\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

$$\begin{aligned}\vec{A}\cdot\left(\vec{B}+\vec{C}\right) &= |\vec{A}|\text{comp}_{\vec{A}}\left(\vec{B}+\vec{C}\right) \\ &= |\vec{A}|\left(\text{comp}_{\vec{A}}\vec{B}+\text{comp}_{\vec{A}}\vec{C}\right) && \text{distributivity of components over addition} \\ &= |\vec{A}|\text{comp}_{\vec{A}}\vec{B}+|\vec{A}|\text{comp}_{\vec{A}}\vec{C} \\ &= \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}\end{aligned}$$

Distributivity of component addition can be seen by imagining a new coordinate system where $\vec{A}$ is one of the axes. The amount that each vector "moves a spot" along a direction combined is the same as if you did the moves separately.

7 | line through a point

$$\vec{r}(t)=\vec{P_0}+\vec{v_0}t$$

This way, $\vec{r}(t)=\vec{P_0}$ at $t=0$, and the direction of the line is the velocity vector ($\frac{d}{dt}\vec{r}=\vec{v_0}$).

8 | vector equation that passes through the points

The vectors are

$$\begin{aligned}\vec{a}&=\langle -1,4,1\rangle \\ \vec{b}&=\langle 2,-5,-3\rangle\end{aligned}$$

Let's choose

$$\vec{r}(t)=\vec{p}+\vec{v}t$$

and make sure that $\vec{r}(0)=\vec{a}$, and $\vec{r}(1)=\vec{b}$. We can do this by setting

$$\begin{aligned}\vec{p}&=\vec{a}=\langle -1,4,1\rangle \\ \vec{v}&=\vec{b}-\vec{a}=\langle 3,-9,-4\rangle\end{aligned}$$

Thus,

$$\boxed{\vec{r}(t)=\langle -1,4,1\rangle+\langle 3,-9,-4\rangle t}$$

This way,

$$\begin{aligned}\vec{r}(0)&=\vec{p}=\vec{a} \\ \vec{r}(1)&=\vec{p}+\vec{v}=\vec{a}+(\vec{b}-\vec{a})=\vec{b}\end{aligned}$$