

1 | Problem

$$\begin{aligned}
 &\dim(U_1 + U_2 + U_3) \\
 &= \dim U_1 + \dim U_2 + \dim U_3 \\
 &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\
 &\quad + \dim(U_1 \cap U_2 \cap U_3)
 \end{aligned}$$

Prove or give a counterexample: \$\$

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2 | Reasoning

By Axler 2.41 we know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

By applying this formula to itself, we find that

$$\begin{aligned}
 &\dim(U_1 + U_2 + U_3) \\
 &= \dim((U_1 + U_2) + U_3) \\
 &= \dim(U_1 + U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3) \\
 &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3)
 \end{aligned}$$

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To show that the lemma is true, we would have to show that

$$\begin{aligned}
 &\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) \\
 &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3)
 \end{aligned}$$

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and to provide a counterexample, we just need to find some U_1, U_2, U_3 such that

$$\dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3) \neq \dim((U_1 + U_2) \cap U_3)$$

3 | Counterexample

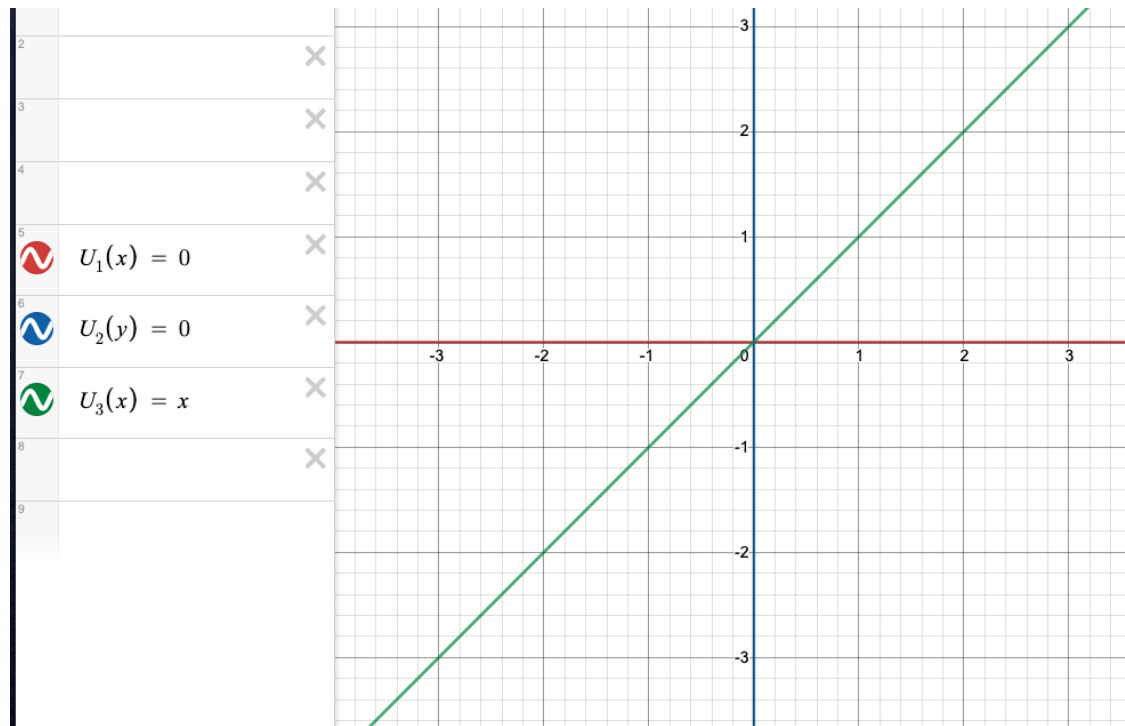
If we choose

$$\begin{aligned}
 U_1 &= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} \\
 U_2 &= \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \\
 U_3 &= \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R} \right\}
 \end{aligned}$$

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then the graph of the subspaces looks like this:



and the dimension of each intersection is 0 while the dimension of $(U_1 + U_2) \cap U_3 = 2$. Thus, we have

$$\begin{aligned} \dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3) &\neq \dim((U_1 + U_2) \cap U_3) \\ \implies 0 &\neq 2 \end{aligned}$$

In summary, the sum of these subspaces is \mathbb{R}^2 and the dimension of the sum is 2, but $\dim(U_1 + U_2 + U_3) = 2 \neq 3 = 1 + 1 + 1$