

1 | shoestring loop

$$\begin{aligned}x &= t^2 \\y &= t^3 - 3t \\ \frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 3t^2 - 3 \\ \frac{dy}{dx} &= \frac{3t^2 - 3}{2t}\end{aligned}$$

1.1 | tangents are horizontal or vertical

1.1.1 | horizontal

$$\begin{aligned}3t^2 - 3 &= 0 \\ 3t^2 &= 3 \\ t^2 &= 1 \\ t &= \pm 1\end{aligned}$$

Now, let's find the actual coordinates

$$\begin{aligned}x = t^2 &= 1 \\ y = t^3 - 3t &= -2 \text{ or } 2\end{aligned}$$

and it checks out graphically.

1.1.2 | vertical

$$\begin{aligned}2t &= 0 \\ t &= 0\end{aligned}$$

The actual coordinates

$$(t^2, t^3 - 3t) = (0, 0)$$

and it checks out graphically.

1.2 | concave up

$$\begin{aligned}\frac{d}{dx} \frac{dy}{dx} &= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{2t(6t) - (3t^2 - 3)(2)}{8t^3} \\ &= \frac{6t^2 - 3t^2 + 3}{4t^3} = \frac{3t^2 + 3}{4t^3} > 0 \\ &\therefore \text{concave up for } t > 0\end{aligned}$$

Using similar logic, the curve is concave down for $t \leq 0$.

MVC
21 math 520
assign 2

$r = \sin \theta$

$r = 2 \cos \theta$

$xy = u$

$(r \cos \theta)(r \sin \theta) = u$

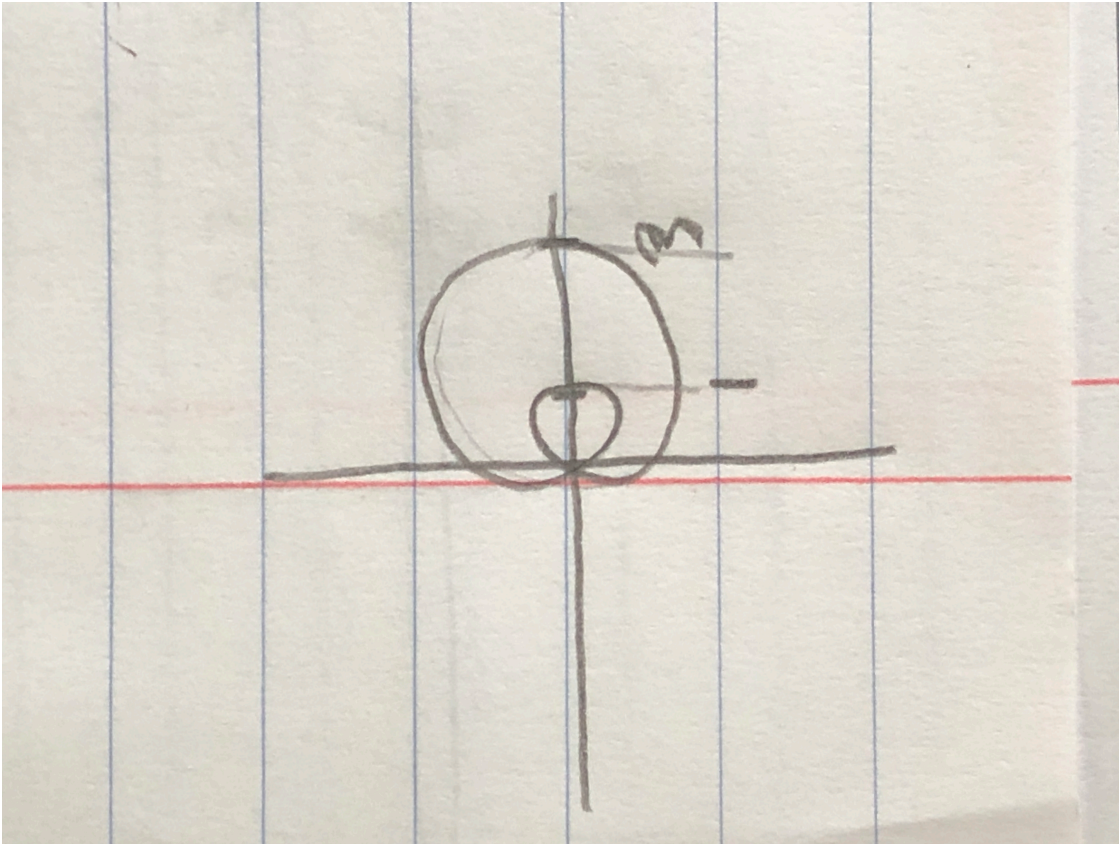
$r^2 \sin \theta \cos \theta = u$

$r = \sqrt{\frac{u}{\sin \theta \cos \theta}}$

! itself once every 2π
because unit circle.

4 | cardioid

4.1 | sketch



4.2 | crosses the origin

Only happens when $\theta = 0$.

$$\begin{aligned}
 r &= 1 + 2 \cos \theta = 0 \\
 2 \cos \theta &= -1 \\
 \cos \theta &= -\frac{1}{2} \\
 \theta &= \cos^{-1}\left(-\frac{1}{2}\right) \\
 &= \frac{2\pi}{3}, -\frac{2\pi}{3}
 \end{aligned}$$

4.3 | derivatives to verify crossing

Let's choose the crossing $\theta = \frac{2\pi}{3}$

$$y = r \sin \theta = (1 + 2 \cos \theta) \sin \theta = \sin \theta + 2 \cos \theta \sin \theta = \sin \theta + \sin 2\theta$$

$$x = r \cos \theta = (1 + 2 \cos \theta) \cos \theta = \cos \theta + 2 \cos^2 \theta$$

$$\frac{dy}{d\theta} = \cos \theta + 2 \cos 2\theta \Rightarrow -\frac{1}{2} + 2 \left(-\frac{1}{2} \right) = -\frac{3}{2}$$

$$\frac{dx}{d\theta} = -\sin \theta - 2(2 \cos \theta) \sin \theta = -\sin \theta - 2 \sin 2\theta = -\frac{\sqrt{3}}{2} - 2 \left(-\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2}$$

$$\frac{d^2y}{d\theta^2} = -\sin \theta - 4 \sin 2\theta = -\frac{\sqrt{3}}{2} - 4 \left(-\frac{\sqrt{3}}{2} \right) = 3 \frac{\sqrt{3}}{2}$$

$$\frac{d^2x}{d\theta^2} = -\cos \theta - 4 \cos 2\theta = \frac{5}{2}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\cos \theta + 2 \cos 2\theta}{\sin \theta + 2 \sin 2\theta} = -\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = -\sqrt{3}$$

$$\frac{dy^2}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} = \frac{\frac{\sqrt{3}}{2} 3 \frac{\sqrt{3}}{2} + \frac{5}{2} \frac{3}{2}}{\left(\frac{\sqrt{3}}{2} \right)^3}$$

$$= \frac{\frac{9}{4} + \frac{15}{4}}{\frac{3\sqrt{3}}{8}}$$

$$= \frac{24}{4} \frac{8}{3\sqrt{3}} = -\frac{16}{\sqrt{3}}$$

4.4 | points where tangent is horizontal or vertical

4.4.1 | horizontal

$$\frac{dy}{dx} = \frac{\cos \theta + 2 \cos 2\theta}{\sin \theta + 2 \sin 2\theta} = 0$$

$$\Rightarrow \cos \theta + 2 \cos 2\theta = 0$$

$$\cos \theta + 2 \cos^2 \theta - 2 \sin^2 \theta = 0$$

$$0 = \cos \theta + 2 \cos^2 \theta - 2 \sin^2 \theta$$

$$0 = \cos \theta + 2 \cos^2 \theta - 2(1 - \cos^2 \theta)$$

$$0 = \cos \theta + 2 \cos^2 \theta - 2 + 2 \cos^2 \theta$$

$$0 = 4 \cos^2 \theta + \cos \theta - 2$$

$$\cos \theta = \frac{-1 \pm \sqrt{1 + 4(4)2}}{8} = \frac{-1 \pm \sqrt{33}}{8}$$

4.4.2 | vertical

Make sure the top isn't also zero when the bottom is zero. Otherwise, by L'hospital rule that $\frac{0}{0}$ could be anything in the world.

$$\begin{aligned} 0 &= \sin \theta + 2 \sin 2\theta \\ &= \sin \theta + 4 \sin \theta \cos \theta \\ &= \sin \theta (1 + 4 \cos \theta) \end{aligned}$$

Either $\theta = 0, \pi$, or

$$0 = 1 + 4 \cos \theta$$

$$-\frac{1}{4} = \cos \theta$$

4.5 | tangent line

$$y = -\sqrt{3}x$$

5 | arclength

$$S = \int_C ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

5.1 | if $y = f(x)$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{??}$$