

## 1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters  $h_0$  and  $\theta_0$  that would best create a maximum launch distance  $x_f$ .

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing  $x$  value of the starting position of the marble, but on the "ground") as  $(-w, 0)$ , where  $w$  is the width of the wooden block. Therefore, we derive the  $x$ -value of the location of the launch of the projectile as  $x = 0$ . We define the direction towards with the marble is launching as positive- $x$ , so as the marble rolls, its position's  $x$  value increases. We will define the location of the marble before starting as positive  $y$ , and as the marble decreases in height, its position's  $y$  value decreases.

We define the start of the experiment as time  $t_0$ , the moment the marble leaves the track and travels as a projectile as  $t_1$ , and the end — in the moment when the marble hits the ground — as  $t_f$ . We will call the marble  $m_0$ .

## 2 | Figuring the Velocity at $t_1$

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that  $m_0$  experiences before  $t_1$  is  $\Delta h = H - h_0$ . Therefore, the potential energy expenditure is  $\Delta PE_{grav} = mg\Delta h = m_0g(H - h_0)$ . Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy  $KE = 0 + m_0g(H - h_0) = m_0g(H - h_0)$ .

For this derivation, for now, we ignore  $KE_{rotational}$ , hence, we could roughly deduct the statement that  $KE_{translational} \approx m_0g(H - h_0)$ .

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at  $t_1$  named  $\vec{v}_0$ .

$$m_0g(H - h_0) = \frac{1}{2}m_0\vec{v}_0^2 \quad (1)$$

$$g(H - h_0) = \frac{1}{2}\vec{v}_0^2 \quad (2)$$

$$2g(H - h_0) = \vec{v}_0^2 \quad (3)$$

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (4)$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} v_{0x} = \sqrt{2g(H - h_0)}\cos(\theta_0) \\ v_{0y} = \sqrt{2g(H - h_0)}\sin(\theta_0) \end{cases}$$

## 3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for  $x_f$  w.r.t.  $v_{0y}$ ,  $v_{0x}$ ,  $h_0$ ,  $m_0$ .

### 3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for  $x_f$ .

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as...

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above,  $x = 0$
- the start y-position of the marble at launch is, as defined above,  $y = h_0$

### 3.2 | Setup for Solution

We first create a set of equations modeling the location of the marble at  $t_f$ .

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f \sqrt{2g(H - h_0)} \cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{-1}{2}gt_f^2 + t_f \sqrt{2g(H - h_0)} \sin(\theta_0) + h_0 \end{cases}$$

We first solve for  $t_f$ , and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \quad (5)$$

Finally, we substitute the definition of  $t_f$  into  $y(t_f)$ .

$$y(t_f) = 0 = \frac{-1}{2}g \left( \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \right)^2 + \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \sqrt{2g(H - h_0)} \sin(\theta_0) + h_0 \quad (6)$$

We will now proceed to simplify the expression further

$$0 = \frac{-1}{4} \frac{x_f^2}{(H - h_0) \cos^2(\theta_0)} + x_f \tan(\theta_0) + h_0 \quad (7)$$

$$= \frac{-1}{4} \frac{x_f^2}{(H - h_0)} \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (8)$$

$$= \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (9)$$

### 3.3 | Solution at $\theta_0 = 0$

We begin by first solving for the expression for  $x_f^2$  at  $\theta_0 = 0$ .

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (10)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 + h_0 \quad (11)$$

$$-h_0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \quad (12)$$

$$4h_0 = \frac{1}{(H - h_0)} x_f^2 \quad (13)$$

$$x_f^2 = 4h_0(H - h_0) \quad (14)$$

### 3.3.1 | Finding $\frac{dx_f^2}{dh_0}$

Remember, given the setup of the problem,  $x_f$  is optimized when  $x_f^2$  is optimized. This is due to the fact that  $x_f$  could not be negative, and it is representing a maximum travel distance. Hence, we will optimize  $x_f^2$  for ease of calculation.

$$x_f^2 = 4h_0(H - h_0) \quad (15)$$

$$\frac{dx_f^2}{dh_0} = \frac{d}{dh_0} 4h_0(H - h_0) \quad (16)$$

$$= 4 \frac{d}{dh_0} h_0(H - h_0) \quad (17)$$

$$= 4((H - h_0) - h_0) \quad (18)$$

$$= 4H - 4h_0 - 4h_0 \quad (19)$$

$$= 4H - 8h_0 \quad (20)$$

### 3.3.2 | Optimizing $\frac{dx_f^2}{dh_0}$

The optimization of this statement is fairly simple. We set  $\frac{dx_f^2}{dh_0} = 0$ , and solve for  $h_0$ .

$$x_f^2 = 4H - 8h_0 \quad (21)$$

$$\Rightarrow 0 = 4H - 8h_0 \quad (22)$$

$$\Rightarrow 8h_0 = 4H \quad (23)$$

$$\Rightarrow h_0 = \frac{1}{2}H \quad (24)$$

Hence, the most optimal height at which to launch the marble launcher, given a horizontal launch, is half of the initial height.

## 3.4 | Solution at arbitrary $\theta_0$

### 3.4.1 | Solving and optimizing for $\frac{dx_f}{d\theta_0}$

We need to maximize  $\frac{dx_f}{d\theta_0}$  as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for  $\frac{h_0}{d\theta_0}$ . The position equations above could

be leveraged to figure a value for  $x_f$ .

1. Finding  $\frac{dx_f}{d\theta_0}$  We leverage implicit differentiation to figure a value for  $\frac{dx_f}{d\theta_0}$ . We set  $x_f$  as a differentiable function, and  $h_0$  and  $H$  as both constants.

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (25)$$

$$\Rightarrow \frac{d}{d\theta_0} 0 = \frac{d}{d\theta_0} \left( \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \right) \quad (26)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{1}{(H - h_0)} \frac{d}{d\theta_0} x_f^2 \cos^{-2}(\theta_0) + \frac{d}{d\theta_0} x_f \tan(\theta_0) + \frac{d}{d\theta_0} h_0 \quad (27)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{1}{(H - h_0)} \left( \left( \frac{d}{d\theta_0} x_f^2 \right) \cos^{-2}(\theta_0) + x_f^2 \left( \frac{d}{d\theta_0} \cos^{-2}(\theta_0) \right) \right) + \quad (28)$$

$$\left( \left( \frac{d}{d\theta_0} x_f \right) \tan(\theta_0) + \left( \frac{d}{d\theta_0} \tan(\theta_0) \right) x_f \right) + 0 \quad (29)$$

$$\Rightarrow 0 = \frac{-1}{4(H - h_0)} \left( (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + x_f^2 (2 \cos^{-3}(\theta_0) \sin(\theta_0)) \right) + \quad (30)$$

$$\left( \frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \right) \quad (31)$$

$$\Rightarrow 0 = \frac{-1}{4(H - h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{-1}{4(H - h_0)} x_f^2 (2 \cos^{-3}(\theta_0) \sin(\theta_0)) + \quad (32)$$

$$\frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \quad (33)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2 \cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (34)$$

$$= \frac{-1}{4(H - h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (35)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2 \cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (36)$$

$$= \frac{dx_f}{d\theta_0} \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (37)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2 \cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (38)$$

$$= \frac{dx_f}{d\theta_0} \left( \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right) \quad (39)$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{\frac{(\cos^{-3}(\theta_0) \sin(\theta_0))}{2(H - h_0)} x_f^2 - \sec^2(\theta_0) x_f}{\left( \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right)} \quad (40)$$

2. Optimizing for  $x_f$  for  $\theta_0$  via  $\frac{dx_f}{d\theta_0}$  We now set  $\frac{dx_f}{d\theta_0} = 0$  in order to figure critical points for the value of  $x_f$ .

$$f \frac{dx_f}{d\theta_0} = \frac{\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{-1}{2(H-h_0)}\right)x_f\cos^{-2}(\theta_0) + \tan(\theta_0)} \quad (41)$$

$$\Rightarrow 0 = \frac{\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{-1}{2(H-h_0)}\right)x_f\cos^{-2}(\theta_0) + \tan(\theta_0)} \quad (42)$$

$$\Rightarrow 0 = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f \quad (43)$$

$$\Rightarrow \sec^2(\theta_0)x_f = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 \quad (44)$$

$$\Rightarrow \sec^2(\theta_0) = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f \quad (45)$$

$$\Rightarrow 2\sec^2(\theta_0)(H-h_0) = (\cos^{-3}(\theta_0)\sin(\theta_0))x_f \quad (46)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{\sec^2(\theta_0)} \quad (47)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos^3(\theta_0)\sec^2(\theta_0)} \quad (48)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos(\theta_0)} \quad (49)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \tan(\theta_0) \quad (50)$$

$$\Rightarrow \theta_0 = \arctan\left(\frac{2(H-h_0)}{x_f}\right) \quad (51)$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize  $x_f$  for a given solved  $x_f$ . To figure the actual statement that would optimize for both,

### 3.4.2 | Solving and optimizing for $x_f$

We will now return to our original expression for the final y-position ( $= 0$ ) to create an expression for  $x_f$ .

1. Solving for  $x_f$  We first take the previous expression for  $x_f$  and supply the expression for  $\theta_0$ .

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (52)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \sec^2(\arctan(\frac{2(H - h_0)}{x_f})) + x_f \tan(\arctan(\frac{2(H - h_0)}{x_f})) + h_0 \quad (53)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 ((\frac{2(H - h_0)}{x_f})^2 + 1) + x_f (\frac{2(H - h_0)}{x_f}) + h_0 \quad (54)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} ((2(H - h_0))^2 + x_f^2) + 2(H - h_0) + h_0 \quad (55)$$

$$0 = \frac{-1}{4} ((4(H - h_0)) + \frac{x_f^2}{(H - h_0)}) + 2(H - h_0) + h_0 \quad (56)$$

$$0 = -(H - h_0) - \frac{x_f^2}{4(H - h_0)} + 2(H - h_0) + h_0 \quad (57)$$

$$\frac{x_f^2}{4(H - h_0)} = -(H - h_0) + 2(H - h_0) + h_0 \quad (58)$$

$$x_f^2 = -4(H - h_0)(H - h_0) + 4(H - h_0)2(H - h_0) + 4(H - h_0)h_0 \quad (59)$$

$$x_f^2 = -4(H - h_0)^2 + 8(H - h_0)^2 + 4h_0(H - h_0) \quad (60)$$

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (61)$$

$$(62)$$

2. Finding  $\frac{dx_f^2}{dh_0}$  We know that, by optimizing for  $x_f^2$ ,  $x_f$  is optimized due to the setup of the problem of the behavior of the length of line.

Hence, we take the *first* derivative, though of  $x_f^2$  w.r.t.  $h_0$  and with  $H$  held constant.

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (63)$$

$$\Rightarrow \frac{dx_f^2}{dh_0} = \frac{d}{dh_0} (4(H - h_0)^2 + 4h_0(H - h_0)) \quad (64)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4 \frac{d}{dh_0} h_0(H - h_0) \quad (65)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4((H - h_0) \frac{d}{dh_0} h_0 + h_0 \frac{d}{dh_0} (H - h_0)) \quad (66)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4((H - h_0) - h_0) \quad (67)$$

$$= -8(H - h_0) + 4(H - h_0) - 4h_0 \quad (68)$$

$$= -4(H - h_0) - 4h_0 \quad (69)$$

$$= -4H + 4h_0 - 4h_0 \quad (70)$$

$$= -4H \quad (71)$$

3. Optimizing for  $x_f$  The optimization of  $x_f$  requires a little bit more thinking of the scenario of the problem.  $H$  must be a positive number, and the expression for  $x_f^2$  appears to be a straight line with slope  $-4H$ . Given  $H > 0$ ,  $-4H < 0$ , and the slope of  $x_f^2$  is negative — as  $h_0$  increases,  $x_f^2$  decreases. Given  $h_0$  must be positive, then,  $h_{0\text{optim}} = 0$ .

### 3.4.3 | Solving for the Actual Optimum of $\theta_0$

We return to following statement:

$$\theta_0 = \arctan\left(\frac{2(H - h_0)}{x_f}\right) \quad (72)$$

The expression we derived above for  $x_f$  under optimal conditions is, per the previous statement:

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (73)$$

$$\Rightarrow x_f = \sqrt{4(H - h_0)^2 + 4h_0(H - h_0)} \quad (74)$$

And substituting the statement for  $x_f$  back into the expression above, in addition to the optimal value  $h_0 = 0$  derived earlier, we derive that

$$\theta_0 = \arctan\left(\frac{2(H - h_0)}{x_f}\right) \quad (75)$$

$$\Rightarrow \theta_0 = \arctan\left(\frac{2(H - h_0)}{\sqrt{4(H - h_0)^2 + 4h_0(H - h_0)}}\right) \quad (76)$$

$$= \arctan\left(\frac{2(H)}{\sqrt{4(H)^2}}\right) \quad (77)$$

$$= \arctan(1) \quad (78)$$

$$= 45^\circ \quad (79)$$

And hence, given an arbitrary angle and an arbitrary height to launch, the most optimal scenario is to make the launch point fully on the floor at an  $45^\circ$  angle from the plane of the floor.