# Angular Momentum & Torque - Part 1

February 15, 2022

## Definitions of Angular Momentum & Torque

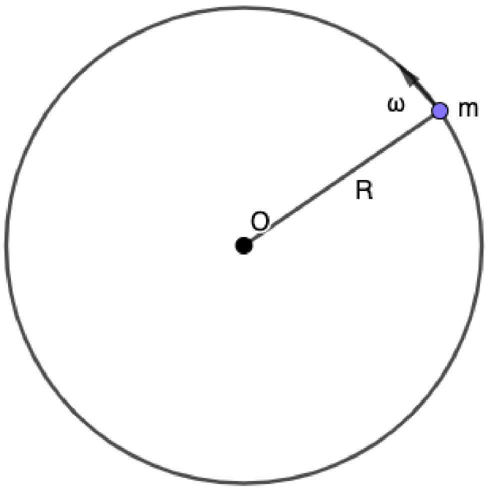
In class, we defined torque and angular momentum as follows:

$$\text{Torque: } \vec{\tau} = \vec{r} \times \vec{F}$$
$$\text{Angular Momentum: } \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

where $\vec{r}$ is the position vector of the point mass where the force $\vec{F}$ is applied, and $\vec{p}$ is the momentum of the point mass. Note that the torque and angular momentum are both coordinate system dependent, as their values change depending on the origin (because the position vector $\vec{r}$ depends on the location of the origin).

The following problems are designed to help you develop some intuition about the meaning of the angular momentum:

**Problem 1:**




Find the angular momentum of a ball of mass $m$ moving on a circle of radius $R$ about the center point $O$ at a constant rotational velocity of $\omega$ radians per second.

[Hint: to find the velocity of the mass in terms of omega, calculate the length of the circumference, and divide by the amount of time it takes to travel the circumference in terms of $\omega$.]

**Problem 2:**

A ball of mass $m$ travels with constant velocity $\vec{v}_o$ through space. At a point in time, its position is given by the position vector $\vec{R}_o$ relative to the origin of the coordinate system. Assuming there is no net force on the ball, does the angular momentum of the object change over time? Explain.

## Problem 3: Derive the relationship between torque and angular momentum

We want to show that torque and angular momentum are related by the following formula:

$$\vec{\tau}_{net} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{1}$$

Prove the formula for a single point mass m by one of the following methods:

1) Calculus method: Take the derivative of $\vec{L}$ and show that it results in $\vec{r} \times \vec{F}$, where $\vec{F}$ is the net force acting on the point mass and $\vec{r}$ is the position vector of the mass from the origin. You may assume that the derivative product rule applies for the cross product, as long as you maintain the order of the vectors. Be sure to explain why one of the terms resulting from the

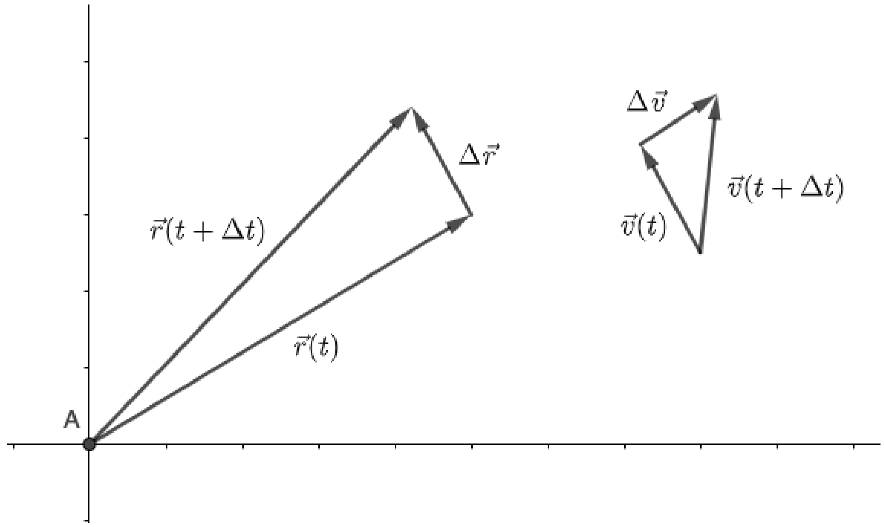
2) You may avoid using the product rule by deriving the derivative geometrically. Use the limit definition of the derivative, and use geometry to expand the angular momentum vector into displacement, velocity and accelerations. You may follow the hints below:

[Hint 1: write the definition of the time derivative of $\vec{L}$:

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\vec{L}(t + \Delta t) - \vec{L}(t)}{\Delta t}$$

]

[Hint 2: expand the derivative with $\vec{L} = \vec{r} \times m\vec{v}$, using $\vec{r}(t + \Delta t) = \vec{r} + \Delta\vec{r}$ and $\vec{v}(t + \Delta t) = \vec{v} + \Delta\vec{v}$. For a visual description, see image below]




[Hint 3: take the limit as $\Delta t \to 0, \Delta r \to 0, \Delta v \to 0$]

[Hint 4: remember that $\vec{v} \times \vec{v} = 0$]