1 | cross product is distributive across addition

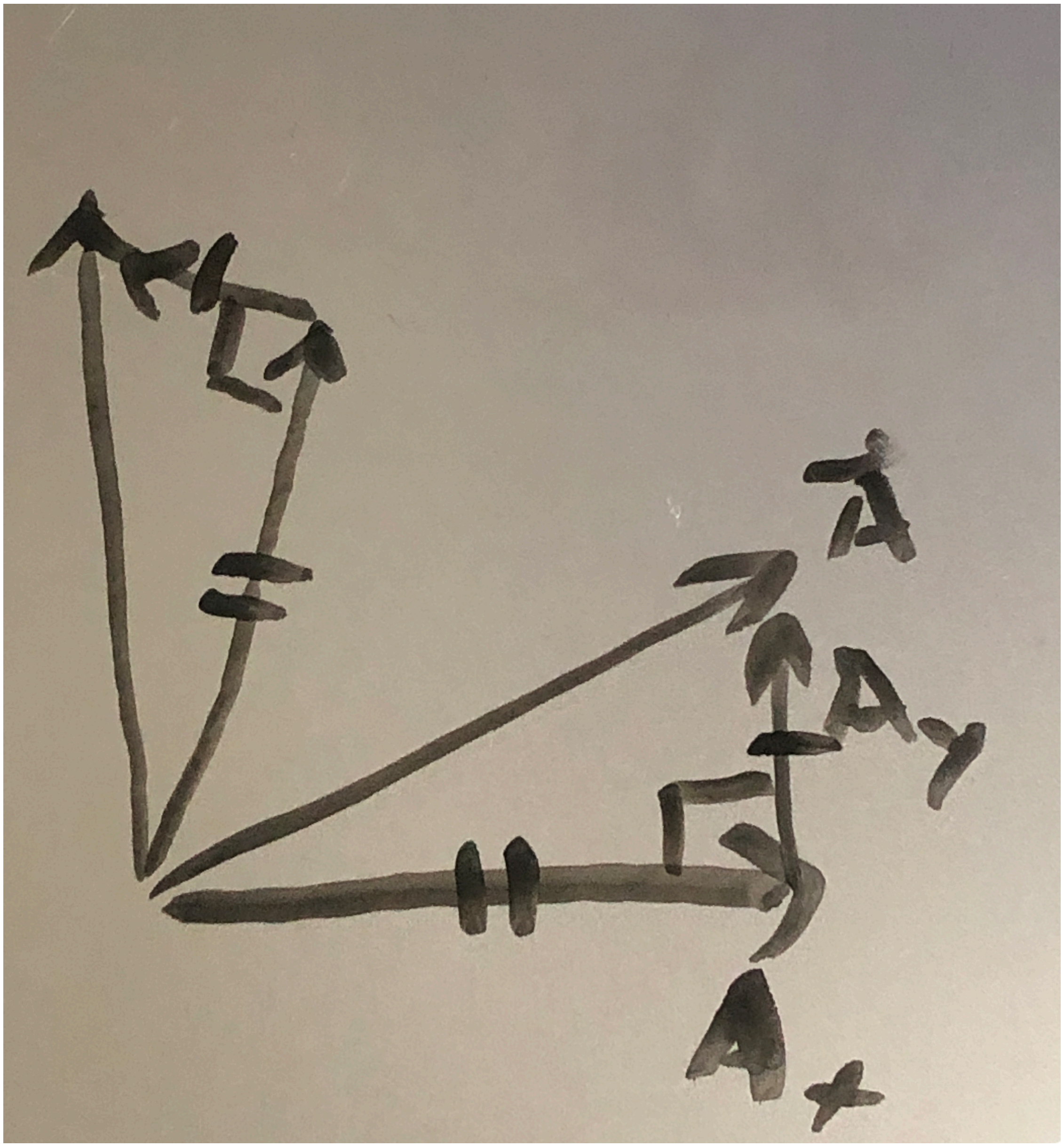
Show that
$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

First, notice that each term in the previous equation is perpendicular to $\vec{A}$. Thus, we can consider compress this 3d problem into two dimensions. Let $\vec{A}$ point out of the page. Then, to show that the direction of proj$_{\vec{A}}\vec{B} + \vec{C}$ is the same as proj$_{\vec{A}}\vec{B} +$ proj$_{\vec{A}}\vec{C}$.

They are the same because rotation is linear, ie. if R_{90} is the rotation matrix (the result of $\vec{A}$ when looking at projections onto the plane), then

$$R_{90}\left(\vec{A} + \vec{B}\right) = R_{90}\left(\vec{A}\right) + R_{90}\left(\vec{B}\right)$$

1.1 | **proof that rotation is additive**



Algebraically, R_{90} can be thought of as multiplying by the corresponding rotation matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and matrix multiplication is linear, and thus additive.

$$
\begin{aligned}
R_{90}\left(\vec{A}+\vec{B}\right) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_x + B_x \\ A_y + B_y \end{pmatrix} \\
&= \begin{pmatrix} A_y + B_y \\ -A_x - B_x \end{pmatrix} \\
&= \begin{pmatrix} A_y \\ -A_x \end{pmatrix} + \begin{pmatrix} B_y \\ -B_x \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} \\
&= R_{90}(\vec{A}) + R_{90}(\vec{B})
\end{aligned}
$$

We use the clockwise rotation because we imagine looking at the plane s.t. $\vec{A}$ points towards us.

1.2 | magnitude of cross product distributivity

For the magnitude

$$
\begin{aligned}
\left|\vec{A}\times\left(\vec{B}+\vec{C}\right)\right| &= |\vec{A}||\vec{B}+\vec{C}|\sin\theta_{\vec{A},\vec{B}+\vec{C}} \\
&= |\vec{A}|\left|\left(\vec{B}+\vec{C}\right)_{\perp A}\right| \\
&= |\vec{A}|\left(\left|\vec{B}_{\perp\vec{A}}\right| + \left|\vec{C}_{\perp\vec{A}}\right|\right) && \text{(additivity of components)} \\
&= |\vec{A}|\left|\vec{B}_{\perp\vec{A}}\right| + |\vec{A}|\left|\vec{C}_{\perp\vec{A}}\right| \\
&= |\vec{A}||\vec{B}|\sin\theta_{\vec{A},\vec{B}} + |\vec{A}||\vec{C}|\sin\theta_{\vec{A},\vec{C}} \\
&= \left|\vec{A}\times\vec{B}\right| + \left|\vec{A}\times\vec{C}\right|
\end{aligned}
$$

Geometrically, taking the cross product constitutes a rotation of the projections of $\vec{B}$, $\vec{C}$, and $\vec{B}+\vec{C}$ on the plane normal to $\vec{A}$, and we established above that rotation is additive.

2 | use distributivity to derive the algebraic form

$$
\vec{A}\times\vec{B} = \left(\vec{A}_x\hat{\imath} + \vec{A}_y\hat{\jmath} + \vec{A}_z\hat{k}\right) \times \left(\vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k}\right)
$$

$$
\begin{aligned}
&= \vec{A}_x\hat{\imath}\times\vec{B}_x\hat{\imath} && + \vec{A}_x\hat{\imath}\times\vec{B}_y\hat{\jmath} && + \vec{A}_x\hat{\imath}\times\vec{B}_z\hat{k} \\
&+ \vec{A}_y\hat{\jmath}\times\vec{B}_x\hat{\imath} && + \vec{A}_y\hat{\jmath}\times\vec{B}_y\hat{\jmath} && + \vec{A}_y\hat{\jmath}\times\vec{B}_z\hat{k} \\
&+ \vec{A}_z\hat{k}\times\vec{B}_x\hat{\imath} && + \vec{A}_z\hat{k}\times\vec{B}_y\hat{\jmath} && + \vec{A}_z\hat{k}\times\vec{B}_z\hat{k} \\
&= 0 && + \vec{A}_x\vec{B}_y\hat{k} && - \vec{A}_x\vec{B}_z\hat{\jmath} \\
&- \vec{A}_y\vec{B}_x\hat{k} && + 0 && + \vec{A}_y\vec{B}_z\hat{\imath} \\
&+ \vec{A}_z\vec{B}_x\hat{\jmath} && - \vec{A}_z\vec{B}_y\hat{\imath} && + 0
\end{aligned}
$$

$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$ because $\sin\theta$ between a vector and itself is zero, and the other vectors are defined using the right hand rule and by $\hat{\imath}\times\hat{\jmath} = \hat{k}$.

$$
= \left(\vec{A}_y\vec{B}_z - \vec{A}_z\vec{B}_y\right)\hat{\imath} \quad + \left(\vec{A}_x\vec{B}_z - \vec{A}_y\vec{B}_x\right)\hat{\jmath} + \left(\vec{A}_x\vec{B}_y - \vec{A}_z\vec{B}_y\right)\hat{k}
$$

Leonard's amazing mnemonic: ijkijkijk

3 | **plane equation**

Plan: Perpendicular to the cross product of the differences.
The points being $\vec{P}_1, \vec{P}_2, \vec{P}_3$
Perpendicular to the perpendicular

$$\vec{n} = \left(\vec{P}_1 - \vec{P}_2\right) \times \left(\vec{P}_1 - \vec{P}_3\right)$$

We know the plane is perpendicular to that normal $\vec{n}$, and offset by one of the P_i

$$\vec{r} = \vec{r} : \left(\vec{r} - \vec{P}_1\right) \cdot \vec{n} = 0$$

Plugging in our definition of $\vec{n}$,

$$\begin{aligned}
\vec{r} = \vec{r} : \left(\vec{r} - \vec{P}_1\right) \cdot \left(\left(\vec{P}_1 - \vec{P}_2\right) \times \left(\vec{P}_1 - \vec{P}_3\right)\right) = 0 \\
= \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_1 \times \vec{P}_1 + \vec{P}_2 \times \vec{P}_3 - \vec{P}_1 \times \vec{P}_2 - \vec{P}_1 \times \vec{P}_3\right) \\
= \left(\vec{r} - \vec{P}_1\right) \cdot \left(0 + \vec{P}_2 \times \vec{P}_3 - \vec{P}_1 \times \vec{P}_2 - \vec{P}_1 \times \vec{P}_3\right) \\
= \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_2 \times \vec{P}_3 - \vec{P}_1 \times \vec{P}_2 - \vec{P}_1 \times \vec{P}_3\right) \\
= \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_2 \times \vec{P}_3\right) + \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_1 \times \vec{P}_2\right) + \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_1 \times \vec{P}_3\right) \\
= \left(\vec{r} - \vec{P}_1\right) \cdot \left(\vec{P}_2 \times \vec{P}_3\right) + \vec{r} \cdot \left(\vec{P}_1 \times \vec{P}_2\right) + \vec{r} \cdot \left(\vec{P}_1 \times \vec{P}_3\right) = 0
\end{aligned}$$

4 | **trying it with a set of numbers**

$$\begin{aligned}
\vec{P} &= (2, 0, -1) \\
\vec{Q} &= (0, 1, 3) \\
\vec{R} &= (0, -2, 4) \\
\vec{Q} - \vec{R} &= (0, 3, -1) \\
\vec{P} - \vec{R} &= (2, 2, -5)
\end{aligned}$$

Cross product time

$$\begin{aligned}
n &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 2 & -5 \end{vmatrix} \\
&= (-15 + 2)\hat{i} + (-2)\hat{j} + (-6)\hat{k} \\
&= (-13, -2, -6)
\end{aligned}$$

$$\begin{aligned}
\vec{r} \cdot \vec{n} &= \vec{P} \cdot \vec{n} \\
-13x - 2y - 6z &= -26 + 6 \quad = -20 \\
13x + 2y + 6z &= 20
\end{aligned}$$