PS#22:

Nueva Multivariable Calculus

(Read the solution set to the last pset, obviously! Also read my lengthy conceptual notes on multiple integration.)

1. You are God.

   You are creating a new universe, which, like all your previous universes, will extend infinitely far in all three spatial dimensions. You are designing it such that the energy in the universe is initially densest at the center (or rather, at a particular point, since perhaps it doesn't make sense to talk about the "center" of an infinite region), decreasing the further one gets away from that point. In particular, the energy density of your new universe, at any point (x, y, z), is:

   $$e(x,y,z) = \frac{1}{7^{(x^2+y^2+z^2)^{1.5}}} \quad \substack{\text{yottajoules} \\ \text{per} \\ \text{cubic light-year}}$$

   How much total energy is there in this universe? How does it compare to the total amount of energy in the universe you made for the humans?

   (Neither the physics nor the theology in this problem should be taken seriously.)

   (*You can, and should, work this all out by hand, because you are God, and therefore don't need a computer.*)

2. Find the area of a circle with radius r... USING A DOUBLE INTEGRAL!!!

3. Consider a filled-in unit sphere (a **ball**, in mathematical parlance) of radius 1, centered at the origin:

   $$x^2 + y^2 + z^2 \leq 1$$

   And a cylinder of radius 1/2, parallel to the z-axis, and centered at the origin:

   $$x^2 + y^2 = \left(\frac{1}{2}\right)^2$$

   How much of the ball lies *outside* the cylinder?

   (Physically: imagine you have a two-inch-diameter sphere, and you use the drill press in the i-Lab to drill a one-inch-diameter hole through the center. How much is left?)

4. Last weekend, we found the volume (above a disc/circle of radius 1 centered at the origin) beneath an *infinitely-tall volcano*:

   $$f : \mathbb{R}^2 \to \mathbb{R}^1$$

   $$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

   $$f(r,\theta) = \frac{1}{r}$$

   We found that its volume is finite; in particular, it's 2π:

   $$\underbrace{\int_{y=-1}^{y=+1} \int_{x=-\sqrt{1-y^2}}^{x=+\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy}_{\text{eww rectangular}} = \underbrace{\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \frac{1}{r}\, r dr d\theta}_{\text{mmm polar much better}} = 2\pi$$

(a) Consider a slightly more general version of this volcano:

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

$$f(x,y) = \frac{1}{(x^2+y^2)^k}$$

For what values of k is this volume of this shape infinite? For what values is it finite? (Same base—a circle with radius 1.)

(b) And compare it to the $1D$ case. Suppose we have the function:

$$g : \mathbb{R}^1 \to \mathbb{R}^1$$

$$g(x) = \frac{1}{x^k}$$

For what values of k is $\displaystyle\int_{-1}^{+1} \frac{1}{x^k}\, dx$ finite? For what values is it infinite?

(Perhaps for consistency with the previous and next problem, I should write this as $1/\left(x^2\right)^k$.)