

## 1 | Position of $m_i$

In a rigid body consisting of  $N$  point masses, the vector to the position of  $m_i$  is defined as  $\vec{r}_i(t)$ , which is defined as follows:

$$\vec{r}_i(t) = \vec{R}_{CM}(t) + \vec{r}_i'(t) \quad (1)$$

whereas,  $\vec{R}_{CM}(t)$  is the position vector of the center of mass of the rigid body as a whole, and  $\vec{r}_i'(t)$  the vector from the center of mass to  $m_i$ .

## 2 | Velocity of $m_i$

The velocity of  $m_i$  is simply determined by the first derivative of the position equation as per above. Namely, that:

$$\vec{v}_i(t) = \vec{V}_{CM}(t) + \vec{v}_i'(t) \quad (2)$$

where,  $\vec{v}_i(t)$  is the velocity vector of  $m_i$ , and  $\vec{V}_{CM}(t)$  is the velocity vector of the center of mass of the rigid body, and  $\vec{v}_i'(t)$  is the velocity vector from center of mass to  $m_i$ .

## 3 | Deriving $KE_{total}$

### 3.1 | Setting up

From definition of  $KE_{total}$  itself,  $KE_{total}$  is the sum of all energies of each point mass in the rigid body.

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 \quad (3)$$

### 3.2 | Derivation, part 1

Expanding this equation and substituting the value of  $v_i$ , and additionally setting  $M = \sum m_i$  (namely, that  $M$  represents the total mass of the rigid body) we could derive:

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \sum_{i=1}^N \frac{1}{2} m_i (v_i \cdot v_i) \quad (4)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i ((\vec{V}_{CM} + \vec{v}_i') \cdot (\vec{V}_{CM} + \vec{v}_i')) \quad (5)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i (\vec{V}_{CM}^2 + 2 \times (\vec{v}_i' \cdot \vec{V}_{CM}) + \vec{v}_i'^2) \quad (6)$$

$$= \sum_{i=1}^N \frac{1}{2} m_i \vec{V}_{CM}^2 + \sum_{i=1}^N m_i \times (\vec{v}_i' \cdot \vec{V}_{CM}) + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (7)$$

$$= \frac{1}{2} \vec{V}_{CM}^2 \sum_{i=1}^N m_i + \vec{V}_{CM} \sum_{i=1}^N m_i \vec{v}_i' + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (8)$$

### 3.3 | Dealing with the Middle Term

At this point, we must note that  $\sum_{i=1}^N m_i \vec{v}_i' = 0$ . Per the definition of the center of mass, the following holds:

$$r_{CM} = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i \quad (9)$$

Changing reference frame to that of the center of mass itself, this equation therefore becomes:

$$r_{CM}' = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (10)$$

It is important to realize here that  $r_{CM}' = 0$  because of the fact that — at the reference point of the center of mass, the center of mass is at a zero-vector distance away from itself.

In order to figure a statement with respect to the *velocity* of  $r_i'$ , we take the derivative of the previous equation with respect to time.

$$0 = \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (11)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{M}\right) \sum_i m_i \vec{r}_i' \quad (12)$$

$$= \left(\frac{1}{M}\right) \sum_i m_i \vec{v}_i' \quad (13)$$

Given that  $\frac{1}{M}$  could not be zero for an object with non-zero mass, it is concluded therefore that  $\sum_i m_i \vec{v}_i' = 0$ .

### 3.4 | Derivation, part 2

As  $\sum_i m_i \vec{v}_i' = 0$ , the  $KE_{total}$  work-in-progress equation's middle term (which contains the statement  $\sum_i m_i \vec{v}_i'$ ) is therefore zero. Substituting that in and removing the term, we therefore result in:

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \frac{1}{2} V_{CM}^2 \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (14)$$

Replacing the definition of  $M = \sum m_i$ , we result in

$$\sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \frac{1}{2} M V_{CM}^2 + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (15)$$

$$KE_{total} = \frac{1}{2} M V_{CM}^2 + \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i'^2 \quad (16)$$

The left term of this equation ( $\frac{1}{2} M V_{CM}^2$ ) is the clear original statement for  $KE_{translational}$ . As component masses of a rigid body cannot experience translational motion about its center of origin, the second term is therefore rotational only and so  $KE_{rotational}$ .

Therefore:

$$KE_{total} = KE_{translational} + KE_{rotational} \quad (17)$$