1 | We Begin at Polar 2D Motion

Let's think about a particle with some velocity $\vec{v}$, some acceleration $\vec{a}$, and some location $\vec{r}$.
Parameterize $\vec{r}$, we figure that:

$$\vec{r} = (x(t), y(t)) \tag{1}$$

Therefore, the velocity would be:

$$\dot{\vec{r}} = \vec{v} = (\dot{x}, \dot{y}) = \dot{x}\hat{i} + \dot{y}\hat{j} \tag{2}$$

By the same token:

$$\vec{a} = \ddot{\vec{r}} \tag{3}$$

1.1 | A weird coordinate

We create a new coordinate system

$$\hat{r} = \hat{r}(t) \tag{4}$$

and

$$\hat{\theta} = \hat{\theta}(t) \tag{5}$$

and also

$$\hat{r} \perp \hat{\theta} \tag{6}$$

1.2 | Relating the new systems

$$\vec{r} = r\hat{r} \tag{7}$$

"the value of the vector r is the direction of r times the magnitude of r."

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt} \tag{8}$$

Now, we note that because $\hat{r} \perp \hat{\theta}$, $\frac{d\hat{r}}{dt}$, the tiny instant change of the direction must be in the $\hat{\theta}$ direction.
Our job today is to write a function $\vec{v} = f_1(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, r)$, $\vec{v} = f_2(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}, r)$