# 1 | Deriving Velocity in Polar Coordinates

In this problem, we aim to determine functions $f_1$, $f_2$, that represents the velocity and acceleration vectors respectively of a circularly movable reference frame.

## 1.1 | Defining First Principles

We begin by defining $\vec{r}$, the vector representing the radius of the rotating frame, as follows:

$$\vec{r} = r\cos(\theta)\hat{i} + r\sin\theta\hat{j} \tag{1}$$

We define $\theta$ as the angle up from the horizontal at which $\vec{r}$ is located, and therefore the vector $\vec{r}$ is simply the magnitude thereof $r$ projected upon that angle $(\cos\theta, \sin\theta)$ into a vector.

Furthermore, we define a unit vector in the direction of $\vec{r}$ as $\hat{r}$. That is:

$$\hat{r} = \cos(\theta)\hat{i} + \sin\theta\hat{j} \tag{2}$$

Lastly, we define a vector $\hat{\theta}$, a unit vector orthogonal to $\vec{r}$. It is defined as such as the direction of $\vec{\theta}$ would, at any given instance, be perpendicular to the direction of $\vec{r}$ and parallel to the direction to its movement.

$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{3}$$

## 1.2 | Determining changes in direction

We now create definitions for changes in ”direction” — the changes present in $\hat{r}$ and $\hat{\theta}$ — which we will leverage later.

$$\begin{align}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{4}\\
&= -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} \tag{5}\\
&= \dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j}) \tag{6}\\
&= \dot{\theta}\hat{\theta} \tag{7}
\end{align}$$

Hence, the change in the direction of $\hat{r}$, aptly and intuitively, could be modeled by the change in the angle $\theta$ times the direction of $\theta$.

$$\begin{align}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j}) \tag{8}\\
&= -\dot{\theta}\cos\theta\hat{i} - \dot{\theta}\sin\theta\hat{j} \tag{9}\\
&= -\dot{\theta}(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{10}\\
&= -\dot{\theta}\hat{r} \tag{11}
\end{align}$$

We now note that, indeed, the change in the direction of $\theta$ is modeled by the direction at which $\vec{r}$ exists, and the angle of $\theta$ as $\theta$ must be orthogonal to $\vec{r}$.

### 1.3 | Solving for $f_1 = \vec{v}$

We now begin to solve for a function $f_1(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, r) = \vec{v}$. We know that the velocity of the frame as a whole could be modeled by the following expression:

$$\vec{v} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt} \tag{12}$$
$$= r\dot{\theta}\hat{\theta} + \hat{r}\dot{r} \tag{13}$$

The derivation of this expression simply follows variable substitution to result in the expression modeling velocity: that the velocity of the point is determined by the radius-scaled velocity in the angle of the object (the velocity that's tangent to the radius), plus the velocity of the radius itself (how fast it grows, the radial velocity.)

### 1.4 | Solving for $f_2 = \vec{a}$.

We here wish to determine a function $f_2(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}, r) = \vec{a} = \frac{d}{dt}\vec{v}$

Deriving the value of this expression, therefore, simply acts as a matter of taking the derivative of the top-derived expression and performing variable substitution for $\dot{\hat{r}}$ and $\dot{\hat{\theta}}$ as determined above as needed.

$$\vec{a} = \frac{d}{dt}(r\dot{\theta}\hat{\theta} + \hat{r}\dot{r}) \tag{14}$$
$$= ((\frac{d}{dt}r)\dot{\theta}\hat{\theta} + ((\frac{d}{dt}\dot{\theta})\hat{\theta} + (\frac{d}{dt}\hat{\theta})\dot{\theta})r) + ((\frac{d}{dt}\hat{r})\dot{r} + (\frac{d}{dt}\dot{r})\hat{r}) \tag{15}$$
$$=((\dot{r})\dot{\theta}\hat{\theta} + ((\ddot{\theta})\hat{\theta} + (-\dot{\theta}\hat{i})\dot{\theta})r) + ((\dot{\theta}\hat{\theta})\dot{r} + (\ddot{r})\hat{r}) \tag{16}$$
$$=(\dot{r}\dot{\theta}\hat{\theta} + \ddot{\theta}\hat{\theta}r - \dot{\theta}^2\hat{r}r) + (\dot{\theta}\hat{\theta}\dot{r} + \ddot{r}\hat{r}) \tag{17}$$
$$=\hat{\theta}(2\dot{r}\dot{\theta} + \ddot{\theta}r) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) \tag{18}$$

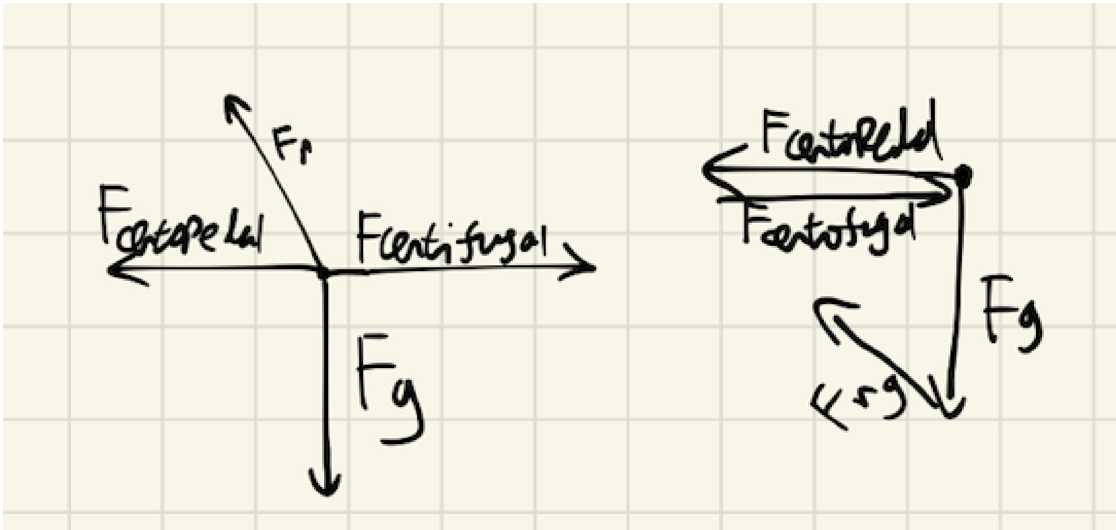
We could now figure that acceleration is actually four terms, scaled by their respective directions ($\hat{\theta}$ and $\hat{r}$). That is:

- $2\dot{r}\dot{\theta}$ is a term representing the coriolis force: a fictitious force pushing objects as either of radius change or spin speed changes.
- $\ddot{\theta}r$ is a term that models the tangential acceleration: how quickly does the object accelerates at a direction tangent to the spin. The "tangential acceleration." We could see this if $\ddot{r} = \dot{\theta} = 0$.
- $\ddot{r}$ is a term that models the acceleration of the radius, as it evidently shows. The "radial acceleration". We could see this if $\dot{\theta} = \ddot{\theta} = 0$.
- $\dot{\theta}^2 r$ is a term that models the inward acceleration that maintains the shape of the circle. It is $\dot{\theta}^2 r = \omega^2 r$, the "centripetal acceleration." We could see this if $\dot{r} = \ddot{r} = \ddot{\theta} = 0$

For a point to stay on the circle, you have to have components of acceleration INTO the circle as well to prevent the side spin from making it fly off. Hence, "tangential acceleration" + "centripetal acceleration."

## 2 | Mass in a Cone

This is probably wrong. TBD: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