

1 | Problem 1

We can start by modeling the mass through the following parameterized equations:

$$\begin{cases} x(t) = x_0 + v_0 \cos \theta t + \frac{1}{2} 0 t^2 \\ y(t) = y_0 + v_0 \sin \theta t - \frac{1}{2} g t^2 \end{cases}$$

We are interested in calculating R which is equal to $x(t_f)$ where t_f is the time when the projectile hits the ground. We can then use our parametric equations to get R alongside the knowledge that $y(t_f) = 0$ (as the projectile has hit the floor). Additionally, we can set the origin in our coordinate system to be (x_0, y_0) and so the x_0 and y_0 are therefore redefined to be 0.

$$\begin{cases} x(t_f) = R = v_0 \cos \theta t_f \\ y(t_f) = 0 = v_0 \sin \theta t_f - \frac{1}{2} g t_f^2 \end{cases}$$

We can then use the equation for $x(t_f)$ to solve for t_f .

$$\frac{R}{v_0 \cos \theta} = t_f$$

This can then be plugged into the equation for $y(t_f)$ and then solved for R .

$$\begin{aligned} 0 &= v_0 \sin \theta \frac{R}{v_0 \cos \theta} - \frac{1}{2} g \left(\frac{R}{v_0 \cos \theta} \right)^2 \\ 0 &= v_0 \sin \theta \frac{R}{v_0 \cos \theta} - \frac{1}{2} g \left(\frac{R}{v_0 \cos \theta} \right)^2 \\ 0 &= \tan \theta R - \frac{1}{2} g \left(\frac{R}{v_0 \cos \theta} \right)^2 \\ 0 &= R \left(\tan \theta - \frac{1}{2} g \frac{R}{v_0^2 \cos^2 \theta} \right) \end{aligned}$$

From there we can split this into another system of equations:

$$\begin{cases} 0 = R \\ 0 = \tan \theta - \frac{1}{2} g \frac{R}{v_0^2 \cos^2 \theta} \end{cases}$$

We can then solve one of the equations in our system to get our answer:

$$0 = \tan \theta - \frac{1}{2}g \frac{R}{v_0^2 \cos^2 \theta}$$

$$\frac{1}{2}g \frac{R}{v_0^2 \cos^2 \theta} = \tan \theta$$

$$\frac{1}{2}gR = \tan \theta v_0^2 \cos^2 \theta$$

$$R = \frac{2 \tan \theta v_0^2 \cos^2 \theta}{g}$$

$$R = \frac{v_0^2 2 \sin \theta \cos \theta}{g}$$

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

1.1 | Subproblem A

The maximum of the sine function is $\sin(90^\circ) = 1$ and since the numerator of the range equation has the term $\sin(2\theta)$ the angle yielding the maximum range is 45° .

1.2 | Subproblem B

We can differentiate the range equation with respect to theta to get the critical points of the function, then use the second derivative test to show that the critical point is a maximum.

$$\frac{d}{d\theta} \frac{v_0^2}{g} \sin(2\theta)$$

$$\frac{v_0^2}{g} \frac{d}{d\theta} \sin(2\theta)$$

$$\frac{v_0^2}{g} (2 \cos(2\theta))$$

$$\frac{2v_0^2}{g} \cos(2\theta)$$

To get the critical points we find the where this expression is equal to 0.

$$0 = \frac{2v_0^2}{g} \cos(2\theta)$$

By limiting the domain to $0 \leq \theta \leq 2\pi$ we can find a critical point at $\theta = \frac{\pi}{4}$.

We can then differentiate once more to determine if the critical point is a maximum.

$$\begin{aligned} \frac{d}{d\theta} \frac{2v_0^2}{g} \cos(2\theta) &= -\frac{4v_0^2}{g} \sin(2\theta) \\ -\frac{4v_0^2}{g} \sin\left(\frac{\pi}{2}\right) &< 0, \text{ therefore the point is a local maximum.} \end{aligned}$$

2 | Problem 2

To begin, we can define the origin to be the location of the cannon (a.k.a. m_1 's initial position). The mass being fired out of the cannon, m_1 , can then be modeled by the following equations:

$$\begin{cases} x(t) = v_0 \cos \theta t \\ y(t) = v_0 \sin \theta t - \frac{1}{2}gt^2 \end{cases}$$

We are interested in these equations at t_f , the time of the collision. Additionally, from the problem we know that $y(t_f) = h$ and $x(t_f) = x_0$.

$$\begin{cases} x(t_f) = x_0 = v_0 \cos \theta t_f \\ y(t_f) = h = v_0 \sin \theta t_f - \frac{1}{2}gt_f^2 \end{cases}$$

We can then model the falling target, the mass m_2 , through the following equations and similarly evaluate them at time of collision t_f :

$$\begin{cases} x(t) = x_0 \\ y(t) = h_0 - \frac{1}{2}gt^2 \end{cases}$$

$$\begin{cases} x(t_f) = x_0 \\ y(t_f) = h = h_0 - \frac{1}{2}gt_f^2 \end{cases}$$

Knowing that these masses collide at the time of collision, we can then set both sets of equations equal to one another.

$$\begin{cases} x_0 = v_0 \cos \theta t_f \\ h_0 - \frac{1}{2} g t_f^2 = v_0 \sin \theta t_f - \frac{1}{2} g t_f^2 \end{cases}$$

We can then replace θ with $\tan^{-1}(\frac{h_0}{x_0})$ since the cannon is aimed directly at the target.

$$\begin{cases} x_0 = v_0 \cos(\tan^{-1}(\frac{h_0}{x_0})) t_f \\ h_0 - \frac{1}{2} g t_f^2 = v_0 \sin(\tan^{-1}(\frac{h_0}{x_0})) t_f - \frac{1}{2} g t_f^2 \end{cases}$$

$$\begin{cases} x_0 = v_0 \cos(\tan^{-1}(\frac{h_0}{x_0})) t_f \\ h_0 = v_0 \sin(\tan^{-1}(\frac{h_0}{x_0})) t_f \end{cases}$$

One can then solve the system of equations:

$$\frac{x_0}{\cos(\tan^{-1}(\frac{h_0}{x_0})) t_f} = v_0$$

$$h_0 = \frac{x_0 \sin(\tan^{-1}(\frac{h_0}{x_0})) t_f}{\cos(\tan^{-1}(\frac{h_0}{x_0})) t_f}$$

$$h_0 = x_0 \tan(\tan^{-1}(\frac{h_0}{x_0}))$$

$$h_0 = x_0 \frac{h_0}{x_0}$$

$$h_0 = h_0$$

Because we get an identity, we know that the two systems of equations for the masses at time t_f are in fact equal and they the masses will always collide.

2.1 | Expression for Height

We can use the already determined systems for m_1 and m_2 as well as the value of θ to determine h .

$$\begin{cases} x(t_f) = x_0 = v_0 \cos \tan^{-1}(\frac{h_0}{x_0}) t_f \\ y(t_f) = h = v_0 \sin \tan^{-1}(\frac{h_0}{x_0}) t_f - \frac{1}{2} g t_f^2 \end{cases}$$

$$\begin{cases} x(t_f) = x_0 \\ y(t_f) = h = h_0 - \frac{1}{2} g t_f^2 \end{cases}$$

We can begin by solving for t_f through the first system:

$$x_0 = v_0 \cos \tan^{-1}\left(\frac{h_0}{x_0}\right) t_f$$

$$\frac{x_0}{v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))} = t_f$$

We can then plug this into the second to get h .

$$h = h_0 - \frac{1}{2} g t_f^2$$

$$h = h_0 - \frac{1}{2} g \left(\frac{x_0}{v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))} \right)^2$$

3 | Problem 3

With an initial velocity, our system of equations modeling the heights of the falling target is now the following:

$$\begin{cases} y_{\text{projectile}}(t_f) = v_0 \sin \theta t_f - \frac{1}{2} g t_f^2 \\ y_{\text{target}}(t_f) = h_0 + v_1 t_f - \frac{1}{2} g t_f^2 \end{cases}$$

$$\begin{cases} x_{\text{projectile}}(t_f) = v_0 \cos \theta t_f \\ x_{\text{target}}(t_f) = v_0 \cos \theta t_f \end{cases}$$

$$\begin{cases} v_0 \sin \theta t_f - \frac{1}{2} g t_f^2 = h_0 + v_1 t_f - \frac{1}{2} g t_f^2 \\ x_0 = v \cos \theta t_f \end{cases}$$

$$\begin{cases} v_0 \sin \theta t_f = h_0 + v_1 t_f \\ x_0 = v \cos \theta t_f \end{cases}$$

$$x_0 = v \cos \theta t_f$$

$$\frac{x_0}{v_0 \cos \theta} = t_f$$

$$v_0 \sin \theta \frac{x_0}{v_0 \cos \theta} = h_0 + v_1 \frac{x_0}{v_0 \cos \theta}$$

$$\sin \theta \frac{x_0}{\cos \theta} = h_0 + v_1 \frac{x_0}{v_0 \cos \theta}$$

$$x_0 \tan \theta = h_0 + v_1 \frac{x_0}{v_0 \cos \theta}$$