



## 1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters  $h_o$  and  $\theta_0$  that would best create a maximum launch distance  $x_f$ .

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing  $x$  value of the starting position of the marble, but on the "ground") as  $(-w, 0)$ , where  $w$  is the width of the wooden block. Therefore, we derive the  $x$ -value of the location of the launch of the projectile as  $x = 0$ . We define the direction towards which the marble is launching as positive- $x$ , so as the marble rolls, its position's  $x$  value increases. We will define the location of the marble before starting as positive  $y$ , and as the marble decreases in height, its position's  $y$  value decreases.

We define the start of the experiment as time  $t_0$ , the moment the marble leaves the track and travels as a projectile as  $t_1$ , and the end — in the moment when the marble hits the ground — as  $t_f$ . We will call the marble  $m_0$ .

## 2 | Figuring the Velocity at $t_1$

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that  $m_0$  experiences before  $t_1$  is  $\Delta h = H - h_0$ . Therefore, the potential energy expenditure is  $\Delta PE_{grav} = mg\Delta h = m_0g(H - h_0)$ . Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy  $KE = 0 + m_0g(H - h_0) = m_0g(H - h_0)$ .

For this derivation, for now, we ignore  $KE_{rotational}$ , hence, we could roughly deduct the statement that  $KE_{translational} \approx m_0g(H - h_0)$ .

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at  $t_1$  named  $\vec{v}_0$ .

$$m_0g(H - h_0) = \frac{1}{2}m_0\vec{v}_0^2 \quad (1)$$

$$g(H - h_0) = \frac{1}{2}\vec{v}_0^2 \quad (2)$$

$$2g(H - h_0) = \vec{v}_0^2 \quad (3)$$

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (4)$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} v_{0x} = \sqrt{2g(H - h_0)}\cos(\theta_0) \\ v_{0y} = \sqrt{2g(H - h_0)}\sin(\theta_0) \end{cases}$$

## 3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for  $x_f$  w.r.t.  $v_{0y}$ ,  $v_{0x}$ ,  $h_0$ ,  $m_0$ .

### 3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for  $x_f$ .

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = -\frac{1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as...

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above,  $x = 0$
- the start y-position of the marble at launch is, as defined above,  $y = h_0$

### 3.2 | Solving for $x_f$

The position equations above could be leveraged to figure a value for  $x_f$ . We first create a set of equations modeling the location of the marble at  $t_f$ .

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}\cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{-1}{2}gt_f^2 + t_f\sqrt{2g(H-h_0)}\sin(\theta_0) + h_0 \end{cases}$$

To simplify calculations initially, we set  $\sqrt{2g(H-h_0)}$  back as  $\vec{v}_0$  for the ease of initial simplification.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\vec{v}_0\cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{-1}{2}gt_f^2 + t_f\vec{v}_0\sin(\theta_0) + h_0 \end{cases}$$

We first solve for  $t_f$ , and supply it to the second equation.

$$t_f = \frac{-\vec{v}_0\sin(\theta_0) \pm \sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{-g} \quad (5)$$

Given that we know that time is positive in this setup, and subtracting a term will make it even more negative, we could safely ignore the  $-$  term in the  $\pm$  operator.

And, performing variable substitution upon the first equation...

$$x_f = \frac{-\vec{v}_0\sin(\theta_0)\vec{v}_0\cos(\theta_0) + \vec{v}_0\cos(\theta_0)\sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{g} \quad (6)$$

$$= \frac{\frac{-1}{2}\vec{v}_0^2\sin(2\theta_0) + \vec{v}_0\cos(\theta_0)\sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{g} \quad (7)$$

$$= \frac{-\vec{v}_0^2\sin(2\theta_0)}{2g} + \frac{\vec{v}_0\cos(\theta_0)\sqrt{\vec{v}_0^2\sin^2(\theta_0) + 2gh_0}}{g} \quad (8)$$

$$= \frac{\vec{v}_0\cos(\theta_0)\sqrt{\vec{v}_0^2\sin^2(\theta_0) + 2gh_0}}{g} - \frac{\vec{v}_0^2\sin(2\theta_0)}{2g} \quad (9)$$

$$(10)$$

And finally, substituting back the  $\vec{v}_0$  terms...

$$x_f = \frac{\sqrt{2g(H-h_0)}\cos(\theta_0)\sqrt{2g(H-h_0)\sin^2(\theta_0) + 2gh_0}}{g} - \frac{2g(H-h_0)\sin(2\theta_0)}{2g} \quad (11)$$

$$= 2(\sqrt{H-h_0}\cos(\theta_0)\sqrt{(H-h_0)\sin^2(\theta_0) + h_0} - (H-h_0)\sin(2\theta_0)) \quad (12)$$

$$= 2(\cos(\theta_0)\sqrt{(H-h_0)^2\sin^2(\theta_0) + (H-h_0)h_0} - (H-h_0)\sin(2\theta_0)) \quad (13)$$

$$= 2(\cos(\theta_0)\sqrt{H^2\sin^2(\theta_0) - 2Hh_0\sin^2(\theta_0) + h_0^2\sin^2(\theta_0) + Hh_0 - h_0^2} - (H\sin(2\theta_0) - h_0\sin(2\theta_0))) \quad (14)$$

### 3.3 | Optimizing for $x_f$

This would *technically* be a multivariable calculus question. However, we elect to do the following: holding  $h_0$  as constant, and optimizing for  $\theta_0$ , then vs