

1 | Classic 1D Optimization

What's the rectangle with largest area that has a perimeter of 100 units? Give the dimensions and the area. You can do this using just the basic tricks of single-variable calculus—but do it using the method of Lagrange multipliers.

1.1 | Without math

Intuitively, the rectangle with the largest area is a square. Therefore, the dimensions of the shape would be $25 \times 25 = 625$ units.

1.2 | Solve using 1D calc

We will set the length on one side as x , and represent all other sides in terms of x . The area of the shape, therefore, would be:

$$A(x) = x \left(\frac{100 - 2x}{2} \right) \quad (1)$$

$$= x (50 - x) \quad (2)$$

$$= 50x - x^2 \quad (3)$$

We will use the first derivative test to figure critical points and optimize.

$$A'(x) = 50 - 2x \quad (4)$$

Solving for $A'(x) = 0$, we get that $x = 25$. We can see that, about this point, as x increases, A' becomes smaller—and vice versa; therefore this is the maximum of $A(x)$.

Given one side of this shape is 25 units long, the other must be as well. Therefore, the square is the largest possible area by which this can be the case.

1.3 | Unnecessarily Solving Using Lagrange Multipliers

We will then proceed to use 2D calculus to solve for this same are. We will now construct area as a $\mathbb{R}^2 \rightarrow \mathbb{R}^1$ function, that:

$$A(x, y) = xy \quad (5)$$

Furthermore, we will construct the perimeter of the square as:

$$g(x, y) = 100 == 2(x + y) \quad (6)$$

as per given in the problem.

To figure the Lagrange multiplier, we will figure some set of points where:

$$\begin{cases} \nabla A = \lambda \nabla g \\ g = 0 \end{cases} \quad (7)$$

In order to do so, we first take ∇A and ∇g :

$$\nabla A = \begin{pmatrix} y \\ x \end{pmatrix} \quad (8)$$

$$\nabla g = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (9)$$

Finally, we understand that:

$$\nabla A = \lambda \nabla g \quad (10)$$

$$\Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (11)$$

From this, we arrive at three expressions:

$$\begin{cases} y = 2\lambda \\ x = 2\lambda \\ 100 = 2x + 2y \end{cases} \quad (12)$$

Solving these expressions pairwise, we arrive at:

$$\lambda = \frac{25}{2}, y = x \quad (13)$$

Substituting this to the final expression, we essentially get the same result:

$$100 = 4\{y, x\} \quad (14)$$

Therefore, $x = y = 25$.

2 | Aquarium Optimization

We will set up an aquarium with three side lengths, (x, y, z) . We will take that one unit area of the bottom is cost of one unit, and therefore the side glasses would have cost of 2 units.

The total cost C , therefore, can be modeled by the following expression:

$$C(x, y, z) = xy + 4xz + 4yz \quad (15)$$

(this can be scaled by some ρ which is the cost density, but that will not be a factor as scaling would be optimized out of the function)

Furthermore, we are given that there is some constant V which is the volume that the glass must satisfy, that:

$$g(x, y, z) = V == xyz \quad (16)$$

Applying the same principle as the Lagrange multiplier, we will first figure ∇C and ∇g .

$$\nabla C = \begin{pmatrix} y + 4z \\ x + 4z \\ 4x + 4y \end{pmatrix} \quad (17)$$

and

$$\nabla g = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \quad (18)$$

Lastly, based on the Lagrange multiplier principle, we want to find (x, y, z) such that:

$$\begin{pmatrix} y + 4z \\ x + 4z \\ 4x + 4y \end{pmatrix} = \lambda \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \quad (19)$$

This will bring us four expressions for four unknowns:

$$\begin{cases} y + 4z = \lambda(yz) \\ x + 4z = \lambda(xz) \\ 4x + 4y = \lambda(xy) \\ 0 = xyz - V \end{cases} \quad (20)$$

We will first take the top three expressions and divide the right side to result in expressions all in terms of λ .

$$\begin{cases} \frac{1}{z} + \frac{4}{y} = \lambda \\ \frac{1}{z} + \frac{4}{x} = \lambda \\ \frac{4}{y} + \frac{4}{x} = \lambda \end{cases} \quad (21)$$

Observing the top two expressions, we see that $y = x$. Therefore, we will replace in these expressions every statement of y with x .

$$\begin{cases} \frac{1}{z} + \frac{4}{x} = \lambda \\ \frac{8}{x} = \lambda \end{cases} \quad (22)$$

Setting these two expressions equal:

$$\frac{1}{z} + \frac{4}{x} = \frac{8}{x} \quad (23)$$

$$\Rightarrow \frac{1}{z} = \frac{4}{x} \quad (24)$$

$$\Rightarrow x = 4z \quad (25)$$

Awesome, now we can solve for x via the last unused expression from our original four equations.

$$0 = xyz - V \quad (26)$$

$$\Rightarrow 0 = x(x) \left(\frac{1}{4}x \right) - V \quad (27)$$

$$\Rightarrow 0 = \frac{1}{4}x^3 - V \quad (28)$$

$$\Rightarrow V = \frac{1}{4}x^3 \quad (29)$$

$$\Rightarrow 4V = x^3 \quad (30)$$

$$\Rightarrow x = \sqrt[3]{4V} \quad (31)$$

And taking our original expressions for y and z , we see that:

$$\begin{cases} x = \sqrt[3]{4V} \\ y = \sqrt[3]{4V} \\ z = \frac{1}{4}\sqrt[3]{4V} \end{cases} \quad (32)$$

Though its not useful here (as the above answer is the optima), we will finally solve for λ :

$$\frac{8}{\sqrt[3]{4V}} = \lambda \quad (33)$$

3 | Temperature in Space

Say the temperature is given by:

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^1 \quad (34)$$

$$f(x, y, z) = x^2 + y^5 - 2z \quad (35)$$

Firstly, at $(-1, -1, -1)$, the temperature would be -2 Kelvin, which is really hot such that the gas particles attract each other and is indeed possible! This fact is courtesy of Albert H. who sent it along.

3.1 | Temperature at that Point

The temperature at that point is given by $f(3, 2, 7)$, which is:

$$f(3, 2, 7) = 27 \quad (36)$$

3.2 | Is it hot?

27 kelvin would be considered really cold!

3.3 | Jet pack Thruster

To figure the direction of optimization, we find ∇f , which is:

$$\nabla f = \begin{pmatrix} 2x \\ 5y^4 \\ -2 \end{pmatrix} \quad (37)$$

At $(3, 2, 7)$, the gradient vector is:

$$\begin{pmatrix} 6 \\ 80 \\ -2 \end{pmatrix} \quad (38)$$

Normalizing this into a direction:

$$\begin{pmatrix} \frac{3}{\sqrt{1610}} \\ \frac{40}{\sqrt{1610}} \\ \frac{-1}{\sqrt{1610}} \end{pmatrix} \quad (39)$$

To maximize the temperature as quickly as possible (which is what we want to do), go in this direction.

3.4 | Broken Thruster

If the thruster is broken, the direction of fastest temperature ascent is in the y direction as we have shown in the above gradient vector. For every unit traveled in the y direction, the temperature increases by 80 kelvin.

3.5 | Fixed Thruster

If the thruster is fixed again, for every unit traveled in that direction, the temperature will change by the magnitude of that vector:

$$\text{norm} \begin{pmatrix} 6 \\ 80 \\ -2 \end{pmatrix} = 2\sqrt{1610} \approx 80.25 \quad (40)$$