1 | **1)**

To finish the proof… Given two objects, A and B, with a force F between them, the torque on A and B is given by

$$\tau_A = \vec{r}_A \times \vec{F}_A$$
$$\tau_B = \vec{r}_B \times \vec{F}_B$$

where $\vec{F}_A$ is the foce applied by B on A, and vice versa. We know that because of N-3 $\vec{F}_A = -\vec{F}_B$. (We also know that the forces point towards each object.) Therefore,

$$\begin{aligned}\tau_{AB} &= \tau_A + \tau_B \\ &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times \vec{F}_B \\ &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A\end{aligned}$$

We know that the direction of the two cross products are orthogonal to the plane that the two objects' position vectors and the origin of the system form.

$$\begin{aligned}\tau_{AB} &= \vec{r}_A \times \vec{F}_A + \vec{r}_B \times -\vec{F}_A \\ &= |\vec{r}_A||\vec{F}_A| \sin\theta_A - |\vec{r}_B||\vec{F}_A| \sin\theta_B \\ &= |\vec{r}_A| \sin\theta_A - |\vec{r}_B| \sin\theta_B\end{aligned}$$

The law of sines states that for a triangle $\triangle ABC$, $\frac{\overline{BC}}{\sin\theta_A} = \frac{\overline{AC}}{\sin\theta_B}$. We know that this applies in our particular proof because the objects A, B, and the origin form a triangle. As such,

$$\begin{aligned}|\vec{r}_A| \sin\theta_A &= |\vec{r}_B| \sin\theta B \\ \tau_{AB} &= 0\end{aligned}$$

The internal torque of any two objects of a system is zero, so the total internal torque must also be zero.

2 | **2)**

We know that for one of the \$\$s:

$$\begin{aligned}\vec{r} &= R\hat{i} + h\hat{k} \\ \vec{L}_1 &= \vec{r} \times m\vec{v} \\ \vec{v} &= R\omega\hat{j} \\ \vec{L}_1 &= (R\hat{i} + h\hat{k}) \times mR\omega\hat{j} \\ &= -hmR\omega\hat{i} + mR^2\omega\hat{k}\end{aligned}$$

We know that there are two masses, symmetric about the z-axis, so we know that the angular momentum of the other object can be derived just by multiplying the $\hat{i}$ and $\hat{j}$ terms by -1.

$$\begin{aligned}\vec{L}_2 &= \vec{L}_1 \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= hmR\omega\hat{i} + mR^2\omega\hat{k}\end{aligned}$$

We can add the two to get the aggregate angular momentum of the system:

$$
\begin{aligned}
\vec{L} &= \vec{L}_1 + \vec{L}_2 \\
&= (-hmR\omega\hat{i} + mR^2\omega\hat{k}) + (hmR\omega\hat{i} + mR^2\omega\hat{k}) \\
&= 2mR^2\omega\hat{k}
\end{aligned}
$$

3 | **3)**

3.1 | **a)**

$\vec{L}_N = \sum_{i=1}^{N} m_i l_i^2 \cdot \omega\hat{k}$

3.2 | **b)**

$\vec{L} = \int_V l^2 (M/V_0)\, dV \cdot \omega\hat{k}$ First, because we know that the object is axially symmetric, the $\hat{i}$ and $\hat{j}$ components of the torque addition will be eliminated by the negative values for those components of the symmetric counterpart, so we will only be left with a $\hat{k}$ component. Given this, we can express the angular momentum as an integral over the volume. WIP

4 | **4)**

We can represent the angular momentum of the two points distance l away from the origin as a function of

$$
\begin{aligned}
\vec{v}_1 &= -\vec{v}_2 \\
\vec{l}_1 &= -\vec{l}_2 \\
\vec{L}(r) &= \vec{r}_1 \times m\vec{v}_1 + vecr_2 \times m\vec{v}_1 \\
&= \vec{r} \times m\vec{v} - \vec{r} \times -m\vec{v}
\end{aligned}
$$

$$
l: \quad
\begin{aligned}
\vec{r} &= r\hat{i} \\
\vec{v} &= L\omega\hat{j} \\
m &= \frac{M}{L}
\end{aligned}
$$

$$
\begin{aligned}
\vec{L}(r) &= r\hat{i} \times M\omega\hat{j} + (-r\hat{i} \times -M\omega\hat{j}) \\
&= 2rM\omega\hat{k}
\end{aligned}
$$

Now that we have a function of the angular velocity in terms of the distance from the center, we can integrate. Keep in mind that we will integrate from 0 to $\frac{L}{2}$, because our angular velocity function is the sum of the two point masses that are r away from the center.

$$\begin{aligned}\vec{L} &= \int_0^{\frac{L}{2}} \vec{L}(r)dr \\ &= \int_0^{\frac{L}{2}} 2rM\omega\hat{k}dr \\ &= [r^2M\omega\hat{k}]_0^{\frac{L}{2}} \\ &= \frac{L^2}{4}M\omega\hat{k}\end{aligned}$$

5 | **5)**

Similarly to problem 4, we will create a function $\vec{L}(\theta)$ that represents the angular momentum of a diameter slice. This will rely on a $\vec{L}(r)$ function defined similarly to problem 4:

$$\begin{aligned}\vec{L}(r) &= 2r\frac{M}{\pi R^2}\omega\hat{k} \\ \vec{L}(\theta) &= \int_0^R \vec{L}(r)\,dr \\ &= \int_0^R 2r\frac{M}{\pi R^2}\omega\hat{k}\,dr \\ &= [r^2\frac{M}{\pi R^2}\omega\hat{k}]_0^R \\ &= \frac{M}{\pi}\omega\hat{k}\end{aligned}$$

Now we can integrate the polar function. When integrating a polar function, we must integrate the function $\frac{f(\theta)^2}{2}$ instead. (Note that we integrate up to π instead of 2π because we include the angular momentum of the symmetric counterpart in the calculation for $\vec{L}(\theta)$)

$$\begin{aligned}\vec{L} &= \int_0^{\pi} \frac{\vec{L}(\theta)^2}{2}\,d\theta \\ &= \int_0^{\pi} \frac{M^2\omega^2}{2\pi^2}\hat{k}\,d\theta \\ &= \frac{M^2\omega^2}{2\pi}\end{aligned}$$