Slopes in higher dimensions!

Nueva Multivariable Calculus

1. Consider a two-dimesional **paraboloid**:

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

$$f(x, y) = x^2 + y^2$$

Let's think about its slope!!! (Use a 3D graphing tool to see what it looks like first, if you don't already know.)

(a) Suppose you're standing at the origin, facing in the $+x$-direction. How steep is the function, at that point, in that direction?

(b) Suppose you're standing at $(0, 12)$, facing in the $+y$-direction. How steep is the function, at that point, in that direction?

(c) Suppose you're standing at $(-3, 0)$, facing in the $-x$-direction. How steep is the function, at that point, in that direction?

(d) Suppose you're standing at $(7, 7)$, facing in the direction of a $45°$ angle on the xy-plane, a.k.a. the unit vector $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. How steep is the function, at that point, in that direction?

(e) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$, facing in the $+60°$-direction on the xy-plane. How steep is the function, at that point, in that direction?

(f) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$, facing in the $+45°$-direction on the xy-plane. How steep is the function, at that point, in that direction?

(g) Suppose you're standing at the point (a, b), looking in the direction of the angle θ on the xy-plane. How steep is the function, at that point, in that direction?

(h) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$. When is the slope biggest? Make a guess, knowing the geometry and shape of this function. Then, using your 1D calc optimization skills, calculate it! (In other words, verify your guess.) What direction do you need to face in to make the slope the biggest/the function the steepest? Facing in that direction, how steep is the slope? (Give the steepness both in normal slope units, and as an angle up from the xy-plane.)

Likewise: when is the slope the biggest, but going *downwards*? Make a guess, and then verify it, using the same procedure. What's the direction you need to turn such that the slope is greatest (i.e., the function steepest)? What's the direction you need to turn such that the function is flattest (i.e., such that its slope is zero, or closest to it)? (You can probably make a decent guess at the answer, from knowing the geometry of a paraboloid, but try to make a reasoned argument using calculus.)

(i) Most generally: suppose you're standing on the function at some point, facing in some direction. What's the slope? What direction do you need to turn in for it to be the steepest? What direction do you need to turn in for it to be the flattest (closest to zero)?

Draw pictures of all of these!!!! 3D pictures are hard to draw, sure, but at the very least you could draw a top-down, bird's eye view of the situation, maybe with some **contour lines**, like on a topographical map!

2. Okay, all the same questions, but this time, for a **hyperbolic paraboloid**, which is a much weirder shape:

$$g : \mathbb{R}^2 \to \mathbb{R}^1$$
$$g(x, y) = x^2 - y^2$$

Use a 3D graphing tool or something to make a pretty graph of this, to see what it looks like! And then, same (similar?) questions:

(a) Suppose you're standing at the origin, facing in the $+x$-direction. How steep is the function, at that point, in that direction?

(b) Suppose you're standing at $(0, 12)$, facing in the $+y$-direction. How steep is the function, at that point, in that direction?

(c) Suppose you're standing at $(-3, 0)$, facing in the $-x$-direction. How steep is the function, at that point, in that direction?

(d) Suppose you're standing at $(7, 7)$, facing in the direction of a $45°$ angle on the xy-plane, a.k.a. the unit vector $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. How steep is the function, at that point, in that direction?

(e) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$, facing in the $+60°$-direction on the xy-plane. How steep is the function, at that point, in that direction?

(f) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$, facing in the $+45°$-direction on the xy-plane. How steep is the function, at that point, in that direction?

(g) Suppose you're standing at the point (a, b), looking in the direction of the angle θ on the xy-plane. How steep is the function, at that point, in that direction?

(h) Suppose you're standing at $\left(5, 5\sqrt{3}\right)$. What's the direction you need to turn such that the slope is greatest (i.e., the function steepest)? What's the direction you need to turn such that the function is flattest (i.e., such that its slope is zero, or closest to it)?

1. Consider the parabaloid:

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$
$$f(x, y) = x^2 + y^2$$

(a) Draw this! I mean, you should probably already know that it looks like a normal 1D parabola, but spun around the z-axis, so it looks like some sort of infinite bullet shape or something. Sketch it, or make a computer-generated graph anyway! Calculus is

(b) What are the partials

(c) suppose you're at the point (a, b)

what's the slope

in the direction of a particular unit vector

in the direction of $\langle -1, 0\rangle$

in the direction of a non-unit vector

in the direction of some angle

in the direction of a general angle

(d) When is the slope biggest? Make a guess, knowing the geometry and shape of this function. (If that's too hard to visualize, try replacing a and b with actual numbers, and doing the rest o this problem thusly.)

Then, using your 1D calc optimization skills, calculate it! (In other words, verify your guess.) What direction do you need to face in to make the slope the biggest/the function the steepest? Facing in that direction, how steep is the slope? (Give the steepness both in normal slope units, and as an angle up from the xy-plane.)

Likewise: when is the slope the biggest, but going *downwards*? Make a guess, and then verify it, using the same procedure.

(e) When is the slope 0? In other words, if you're at the point (a, b) and want to gingerly walk along a flat contour, moving along the surface but neither going up nor down, what direction ought you walk in? guess. verify!

(f) On your graph of the parabaloid, make an attempt to label all of these different slopes! (I say "attempt" because... well, it's a lot of them, and I don't have any great ideas about how to visualize these all simultaneously without it being a total mess. But maybe you'll be able to do it!)

2. Consider the hyperbolic parabaloid:

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$
$$f(x, y) = x^2 - y^2$$

(a) draw

(b)