1 | Deriving Rotational KE and Inertia

Given m_i, mass, $\vec{r_i}'$, location of the center of mass, l_i, ω, the angular velocity, figure a $KE_{tot,rot}$.

Because of the fact that the value ω is in units $\frac{d\theta}{dt}$, the rate of radians change, and we know of a radius of the spin l_i, we could figure the velocity at which it is moving by simply scaling the change in radians up to a circle of radius l_i, that is:

$$V_i' = l_i\omega \tag{1}$$

(note that, to understand this, radians $\frac{arclength}{radius}$)

And so, substituting into the statement of $\sum_{i=1}^{N} \frac{1}{2} m_i \vec{v_i}'^2$

$$\begin{aligned}
KE_{rot} &= \sum_{i=1}^{N} \frac{1}{2} m_i \vec{v_i}'^2 && (2)\\
&= \sum_{i=1}^{N} \frac{1}{2} m_i (l_i\omega)^2 && (3)\\
&= \sum_{i=1}^{N} \frac{1}{2} m_i l_i^2 \omega^2 && (4)\\
&= \frac{1}{2}\omega^2 \sum_{i=1}^{N} (m_i l_i^2) && (5)
\end{aligned}$$

1.1 | Rotational Inertia

The right sum — the mass times the distance away from maxis of rotation ($\sum_{i=1}^{N}(m_i l_i^2)$) — is defined as the rotational (moment) of inertia (spinny mass). That is,

$$I = \sum_{i=1}^{N} (m_i l_i^2) \tag{6}$$

Replacing that value in the prior statement, the statement of KE_{rot} is defined as:

$$KE_{rot} = \frac{1}{2}\omega^2 I \tag{7}$$

1.2 | Rotational Inertia for a Ring

For a ring (that's perfectly circular) rotating on an axis perpendicular to the plane of the ring, the l_i — distance from axis of rotation — is the same value: namely, the radius R as the radius of a circle is the same for all positions. Meaning,

$$l_i = R \tag{8}$$

regardless of which value i.

Hence, the value of KE_{rot} would be evaluated as…

$$KE_{rot} = \sum_{i=1}^{N}(m_i l_i^2) \tag{9}$$
$$= \sum_{i=1}^{N}(m_i R^2) \tag{10}$$
$$= R^2 \sum_{i=1}^{N} m_i \tag{11}$$
$$\tag{12}$$

Substituting M as the sum of all masses in the ring ($M = \sum_{i=1}^{N} m_i$), the statement is therefore:

$$KE_{rot} = MR^2 \tag{13}$$

1.3 | Rotational Inertia of a Solid Sphere

I believe that the rotational inertia of I_{sphere} to be less than I_{disk}. This is because, as the dimension of the object increases, it would be easier to change its velocity (a disk is easier to spin than a ring, etc.). Hence, my intuition states that I_{sphere} would be lower than I_{disk}.

Mathematically, as M is staying at the same value, in the disk case has more mass closer to the axis of rotation — meaning that the $m_i R^2$ term would be smaller in more of the point masses than that of an object at a lower dimension. Hence, the sphere would have more points with lower $m_i R^2$ terms than that of disk; hence, I_{sphere} would be less than I_{disk}.

2 | Kinematics Equations

Given $a = a_0$, initial velocity v_0, and position y_0, we derive the kinematics equations.

$$a(t) = a_0 \tag{14}$$
$$\int a(t)dt = \int a_0 dt \tag{15}$$
$$v(t) = a_0 t + C \tag{16}$$

We are given that $v(0) = v_0$. $v(0) = C = v_0$, hence, $C = v_0$. The velocity statement is therefore,

$$v(t) = a_0 x + v_0 \tag{17}$$

Continuing with integration:

$$v(t) = a_0 x + v_0 \tag{18}$$
$$\int v(t) = \int a_0 x + v_0 dt \tag{19}$$
$$y(t) = \frac{1}{2} a_0 x^2 + v_0 x + C \tag{20}$$
$$\tag{21}$$

Again, substituting $C = y_0$ by the same logic above — $y(0) = C = y_0$, we derive the statement for the position equation.

$$y(t) = \frac{1}{2}a_0x^2 + v_0x + y_0 \tag{22}$$

2.1 | Proving $v^2(t) = v_0^2 + 2a_0(y(t) - y_0)$

We start at the statement for $v(t)$, squaring it, and substituting the necessary statements.

$$\begin{aligned}
v(t) =& a_0x + v_0 && (23)\\
\Rightarrow v^2(t) =& a_0^2x^2 + 2a_0v_0x + v_0^2 && (24)\\
v^2(t) =& v_0^2 + 2a_0(\frac{1}{2}a_0x^2 + v_0x) && (25)\\
v^2(t) =& v_0^2 + 2a_0(\frac{1}{2}a_0x^2 + v_0x + y_0 - y_0) && (26)\\
v^2(t) =& v_0^2 + 2a_0(y(t) - y_0) && (27)
\end{aligned}$$

It is therefore shown that:

$$v^2(t) = v_0^2 + 2a_0(y(t) - y_0) \tag{28}$$