

## 1 | the parabola

A parabola (defined by a focus point and a directrix line) is the set of points

$$\{P : |FP| = \text{distance from } P \text{ to } L\}$$

### 1.1 | polar equation that describes the parabola

$$\begin{aligned} r &= r(\theta) \\ &= d - r \cos \theta \\ r + r \cos \theta &= d \\ r(1 + \cos \theta) &= d \\ r &= \frac{d}{1 + \cos \theta} \end{aligned}$$

### 1.2 | show that cartesian version matches what we expect

Checked using desmos.

$$\begin{aligned} r &= \frac{d}{1 + \cos \theta} \\ x &= \left( \frac{d}{1 + \cos \theta} \right) \cos \theta \\ &= \frac{d \cos \theta}{1 + \cos \theta} \\ y &= \left( \frac{d}{1 + \cos \theta} \right) \sin \theta \\ &= \frac{d \sin \theta}{1 + \cos \theta} \\ x^2 + y^2 &= \frac{d^2}{(1 + \cos \theta)^2} \\ x^2 + y^2 &= \frac{d^2}{\left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right)^2} \\ (x^2 + y^2) \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right)^2 &= d^2 \\ (x^2 + y^2) \left( \frac{x^2}{x^2 + y^2} + \frac{2x}{\sqrt{x^2 + y^2}} + 1 \right) &= d^2 \\ (x^2 + y^2) \left( \frac{x^2}{x^2 + y^2} + \frac{2x\sqrt{x^2 + y^2}}{x^2 + y^2} + 1 \right) &= d^2 \\ x^2 + 2x\sqrt{x^2 + y^2} + \frac{1}{x^2 + y^2} &= d^2 \end{aligned}$$

ugh. too hard.

$$\begin{aligned}
 r &= d - r \cos \theta \\
 &= d - x \\
 \sqrt{x^2 + y^2} &= d - x \\
 x^2 + y^2 &= d^2 + x^2 - 2dx \\
 y^2 &= d^2 - 2dx \\
 y^2 - d^2 &= -2dx \\
 \frac{d^2 - y^2}{2d} &= x \\
 x &= -\frac{y^2}{2d} + \frac{d}{2}
 \end{aligned}$$

more scratch work below:

$$\begin{aligned}
 \frac{d \sin \theta}{1 + \cos \theta} &= \frac{d \cos \theta}{1 + \cos \theta} \\
 (d \sin \theta)(1 + \cos \theta) &= (d \cos \theta)(1 + \cos \theta) \\
 (d \sin \theta) &= (d \cos \theta) \\
 \frac{d}{1 + \cos \theta} &= \sqrt{\frac{d^2 \cos^2 \theta + d^2 \sin^2 \theta}{(1 + \cos \theta)^2}} \\
 d &= \sqrt{d^2 \cos^2 \theta + d^2 \sin^2 \theta}
 \end{aligned}$$

### 1.3 | vertex

$$\left(\frac{d}{2}, 0\right)$$

## 2 | general form

### 2.1 | re-derivation

$$\begin{aligned}
 e &= \frac{r}{d - r \cos \theta} \\
 r &= r(\theta) \\
 &= e(d - r \cos \theta) \\
 &= de - er \cos \theta \\
 r + er \cos \theta &= de \\
 r(1 + e \cos \theta) &= de \\
 r &= \frac{de}{1 + e \cos \theta}
 \end{aligned}$$

Link to the desmos.

2.2 | ellipse when  $0 < e < 1$ 

$$\begin{aligned}
r &= e(d - r \cos \theta) \\
&= e(d - x) \\
\sqrt{x^2 + y^2} &= e(d - x) \\
x^2 + y^2 &= (e(d - x))^2 \\
x^2 + y^2 &= e^2 (d^2 + x^2 - 2dx) \\
x^2 + y^2 &= d^2 e^2 + e^2 x^2 - 2de^2 x \\
-e^2 x^2 + 2de^2 x + x^2 + y^2 &= d^2 e^2 \\
(1 - e^2) x^2 + 2de^2 x + y^2 &= d^2 e^2 \\
(1 - e^2) \left( x^2 + \frac{2de^2}{1 - e^2} x \right) + y^2 &= d^2 e^2 \\
\left( x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} \\
\left( x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} \\
\left( \left( \frac{de^2}{1 - e^2} \right)^2 + x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} + \left( \frac{de^2}{1 - e^2} \right)^2 \\
\left( x + \left( \frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2 (1 - e^2) + d^2 e^4}{(1 - e^2)^2} \\
\left( x + \left( \frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2 - d^2 e^4 + d^2 e^4}{(1 - e^2)^2} \\
\left( x + \left( \frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)^2} \\
\frac{(1 - e^2)^2 \left( x + \left( \frac{de^2}{1 - e^2} \right) \right)^2}{d^2 e^2} + \frac{(1 - e^2) y^2}{d^2 e^2} &= 1 \\
\frac{((1 - e^2)x + de^2)^2}{d^2 e^2} + \frac{(1 - e^2) y^2}{d^2 e^2} &= 1 \\
x_0 &= -\frac{de^2}{1 - e^2} \\
y_0 &= 0 \\
a &= \frac{de}{1 - e^2} \\
b &= \frac{de}{\sqrt{1 - e^2}}
\end{aligned}$$

for  $e < 1$ .

The desmos.

### 2.2.1 | Asymptotes of the hyperbola:

$$\begin{aligned}
 y &= \pm \frac{b}{a} (x - x_0) + y_0 \\
 &= \pm \frac{\frac{de}{\sqrt{1-e^2}}}{\frac{de}{1-e^2}} (x - x_0) + y_0 \\
 &= \pm \frac{de}{\sqrt{1-e^2}} \frac{1-e^2}{de} (x - x_0) + y_0 \\
 &= \pm \frac{1-e^2}{\sqrt{1-e^2}} (x - x_0) + y_0 \\
 &= \pm \sqrt{1-e^2} (x - x_0) + y_0
 \end{aligned}$$

Desmos for the asymptotes.

### 2.3 | hyperbola

Going back to one of the previous equations

$$\begin{aligned}
 \frac{(1-e^2)^2 \left( x + \left( \frac{de^2}{1-e^2} \right) \right)^2}{d^2 e^2} + \frac{(1-e^2)y^2}{d^2 e^2} &= 1 \\
 \frac{(1-e^2)^2 \left( x + \left( \frac{de^2}{1-e^2} \right) \right)^2}{d^2 e^2} - \frac{(e^2-1)y^2}{d^2 e^2} &= 1
 \end{aligned}$$

That makes

$$\begin{aligned}
 x_0 &= -\frac{de^2}{1-e^2} \\
 y_0 &= 0 \\
 a &= \frac{de}{1-e^2} \\
 b &= \frac{de}{\sqrt{e^2-1}}
 \end{aligned}$$

$b$  will be complex unless  $e > 1$ .

The asymptotes:

$$\begin{aligned}
 y = \frac{b}{a} x &= -\frac{e^2-1}{\sqrt{e^2-1}} x = -\sqrt{e^2-1} x \\
 y = -\frac{b}{a} x &= \frac{\sqrt{e^2-1}}{e^2-1} x = \sqrt{e^2-1} x
 \end{aligned}$$

### 2.4 | degenerate ellipse

It is a circle when  $a^2 = b^2$ , aka

$$\begin{aligned}\frac{d^2 e^2}{(1 - e^2)^2} &= \frac{d^2 e^2}{1 - e^2} \\ (1 - e^2)^2 &= 1 - e^2 \\ 1 - e^2 &= 1 \\ e^2 &= 0 \\ e &= 0\end{aligned}$$

You have to take the limit. You get

$$x^2 + y^2 = d^2 e^2 = r^2$$

What could  $d$  possibly be for  $\lim_{e \rightarrow 0} de \neq 0$ ?

$$\begin{aligned}d &= \frac{n}{e} \\ r &= \lim_{e \rightarrow 0} de = \lim_{e \rightarrow 0} \frac{ne}{e}\end{aligned}$$

Plugging back in:

$$\begin{aligned}r &= \lim_{e \rightarrow 0} \frac{de}{1 + e \cos \theta} \\ &= \lim_{e \rightarrow 0} \frac{\frac{n}{e}e}{1 + e \cos \theta} \\ &= \frac{n}{1} \\ &= n\end{aligned}$$

Desmos here. (Had to use  $/j$  because desmos has no limits).