1 | Fifteen Square Puzzle

From the definition of the problem, we have the state (A, B):

- A is a list $(a_1 \ldots a_{15})$ with all the values skipping the empty square
- B is a tuple (X, Y) containing the coordinates of the empty square

We also define ”out of order” as pairs of not-necessarily-continuous values that are not strictly increasing, and ”parity” as $mod\ 2$ of the number of out-of-order pairs plus the row number of the empty square.

1.1 | Defining Transitions

For every single case, there is four possible transitions to make

1. Move empty square up
2. Move empty square down
3. Move empty square left
4. Move empty square right

1.2 | Proving Invariant

We will show that the base state has a specific parity. At $((1 \ldots 15), (4, 4))$, the starting base state, it has parity $0 + 4 = 0(mod\ 2)$.

Let’s declare parity $= 0\ (mod\ 2)$ as the invariant.

1.3 | Proving Invariant through Transitions

Let’s prove that invariant is invariant through all transitions. We will do so in pairs, as the ”moving square” operation is isomorphic by up-down and left-right pairs.

1.3.1 | Moving Up-Down

Moving the empty square up-down constitutes adding/removing three pairs of out-of-order items—shifting a empty square up would result in the item three-items-back to be moved ahead by three items.

Adding three out-of-order pairs, plus subtracting one row from the empty square position, would result in a change in parity of $3-1 = 0\ (mod\ 2)$. It follows that reversing the operation would result in $-3+1 = 0\ (mod\ 2)$.

Shifting up/down does not change the invariant.

1.3.2 | Moving Left-Right

Moving an empty square left-right neither changes the row number for the empty row nor the order of the items. Hence, it does not change the items that constitute the parity—making the parity the same and invariant.

1.4 | Proving Invariant to End

At the final end state, there is

1.5 | Final Proof

By