1 | **Problem 1**

We can start by modeling the mass through the following parameterized equations:

$$\begin{cases} x(t) = x_0 + v_0 \cos\theta t + \frac{1}{2}0t^2 \\ y(t) = y_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \end{cases}$$

We are interested in calculating R which is equal to $x(t_f)$ where t_f is the time when the projectile hits the ground. We can then use our parametric equations to get R alongside the knowledge that $y(t_f) = 0$ (as the projectile has hit the floor). Additionally, we can set the origin in our coordinate system to be (x_0, y_0) and so the x_0 and y_0 are therefore redefined to be 0.

$$\begin{cases} x(t_f) = R = v_0 \cos\theta t_f \\ y(t_f) = 0 = v_0 \sin\theta t_f - \frac{1}{2}gt_f^2 \end{cases}$$

We can then use the equation for $x(t_f)$ to solve for t_f.

$\frac{R}{v_0 \cos\theta} = t_f$

This can then be plugged into the equation for $y(t_f)$ and then solved for R.

$$\begin{aligned} 0 &= v_0 \sin\theta \frac{R}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{R}{v_0 \cos\theta}\right)^2 \\ 0 &= v_0 \sin\theta \frac{R}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{R}{v_0 \cos\theta}\right)^2 \\ 0 &= \tan\theta R - \frac{1}{2}g\left(\frac{R}{v_0 \cos\theta}\right)^2 \\ 0 &= R\left(\tan\theta - \frac{1}{2}g\frac{R}{v_0^2 \cos^2\theta}\right) \end{aligned}$$

From there we can split this into another system of equations:

$$\begin{cases} 0 = R \\ 0 = \tan\theta - \frac{1}{2}g\frac{R}{v_0^2 \cos^2\theta} \end{cases}$$

We can then solve one of the equations in our system to get our answer:

$$\begin{aligned} 0 &= \tan\theta - \frac{1}{2}g\frac{R}{v_0^2 \cos^2\theta} \\ \frac{1}{2}g\frac{R}{v_0^2 \cos^2\theta} &= \tan\theta \\ \frac{1}{2}gR &= \tan\theta v_0^2 \cos^2\theta \\ R &= \frac{2\tan\theta v_0^2 \cos^2\theta}{g} \\ R &= \frac{v_0^2 2\sin\theta\cos\theta}{g} \\ &\boxed{R = \frac{v_0^2 \sin(2\theta)}{g}} \end{aligned}$$

1.1 | Subproblem A

The maximum of the sine function is $\sin(90°) = 1$ and since the numerator of the range equation has the term $\sin(2\theta)$ the angle yielding the maximum range is $45°$.

1.2 | Subproblem B

We can differentiate the range equation with respect to theta to get the critical points of the function, then use the second derivative test to show that the critical point is a maximum.

$$\frac{d}{d\theta}\frac{v_0^2}{g}\sin(2\theta)$$
$$\frac{v_0^2}{g}\frac{d}{d\theta}\sin(2\theta)$$
$$\frac{v_0^2}{g}(2\cos(2\theta))$$
$$\frac{2v_0^2}{g}\cos(2\theta)$$

To get the critical points we find the where this expression is equal to 0.

$$0 = \frac{2v_0^2}{g}\cos(2\theta)$$

By limiting the domain to $0 \le \theta \le 2\pi$ we can find a critical point at $\theta = \frac{\pi}{4}$.
We can then differentiate once more to determine if the critical point is a maximum.

$$\frac{d}{d\theta}\frac{2v_0^2}{g}\cos(2\theta) = -\frac{4v_0^2}{g}\sin(2\theta)$$
$$-\frac{4v_0^2}{g}\sin(\frac{\pi}{2}) < 0\text{, therefore the point is a local maximum.}$$

2 | Problem 2

To begin, we can define the origin to be the location of the cannon (a.k.a. m_1's initial position). The mass being fired out of the cannon, m_1, can then be modeled by the following equations:

$$\begin{cases} x(t) = v_0\cos\theta t \\ y(t) = v_0\sin\theta t - \frac{1}{2}gt^2 \end{cases}$$

We are interested in these equations at t_f, the time of the collision. Additionally, from the problem we know that $y(t_f) = h$ and $x(t_f) = x_0$.

$$\begin{cases} x(t_f) = x_0 = v_0\cos\theta t_f \\ y(t_f) = h = v_0\sin\theta t_f - \frac{1}{2}gt_f^2 \end{cases}$$

We can then model the falling target, the mass m_2, through the following equations and similarly evaluate them at time of collision t_f:

$$\begin{cases} x(t) = x_0 \\ y(t) = h_0 - \frac{1}{2}gt^2 \end{cases}$$

$$\begin{cases} x(t_f) = x_0 \\ y(t_f) = h = h_0 - \frac{1}{2}gt_f^2 \end{cases}$$

Knowing that these masses collide at the time of collision, we can then set both sets of equations equal to one another.

$$\begin{cases} x_0 = v_0 \cos\theta t_f \\ h_0 - \frac{1}{2}gt_f^2 = v_0 \sin\theta t_f - \frac{1}{2}gt_f^2 \end{cases}$$

We can then replace θ with $\tan^{-1}(\frac{h_0}{x_0})$ since the cannon is aimed directly at the target.

$$\begin{cases} x_0 = v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))t_f \\ h_0 - \frac{1}{2}gt_f^2 = v_0 \sin(\tan^{-1}(\frac{h_0}{x_0}))t_f - \frac{1}{2}gt_f^2 \end{cases}$$

$$\begin{cases} x_0 = v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))t_f \\ h_0 = v_0 \sin(\tan^{-1}(\frac{h_0}{x_0}))t_f \end{cases}$$

One can then solve the system of equations:

$$\begin{aligned} \frac{x_0}{\cos(\tan^{-1}(\frac{h_0}{x_0}))t_f} &= v_0 \\ h_0 &= \frac{x_0 \sin(\tan^{-1}(\frac{h_0}{x_0}))t_f}{\cos(\tan^{-1}(\frac{h_0}{x_0}))t_f} \\ h_0 &= x_0 \tan(\tan^{-1}(\frac{h_0}{x_0})) \\ h_0 &= x_0 \frac{h_0}{x_0} \\ h_0 &= h_0 \end{aligned}$$

Because we get an identity, we know that the two systems of equations for the masses at time t_f are in fact equal and they the masses will always collide.

2.1 | Expression for Height

We can use the already determined systems for m_1 and m_2 as well as the value of θ to determine h.

$$\begin{cases} x(t_f) = x_0 = v_0 \cos\tan^{-1}(\frac{h_0}{x_0})t_f \\ y(t_f) = h = v_0 \sin\tan^{-1}(\frac{h_0}{x_0})t_f - \frac{1}{2}gt_f^2 \end{cases}$$

$$\begin{cases} x(t_f) = x_0 \\ y(t_f) = h = h_0 - \frac{1}{2}gt_f^2 \end{cases}$$

We can begin by solving for t_f through the first system:

$$x_0 = v_0 \cos \tan^{-1}(\frac{h_0}{x_0})t_f$$
$$\frac{x_0}{v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))} = t_f$$

We can then plug this into the second to get h.

$$h = h_0 - \frac{1}{2}gt_f^2$$
$$h = h_0 - \frac{1}{2}g\left(\frac{x_0}{v_0 \cos(\tan^{-1}(\frac{h_0}{x_0}))}\right)^2$$

3 | **Problem 3**

Testing.