1 | Problem

Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, w)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\text{dim null } ST \leq \text{dim null } S + \text{dim null } T.$$

2 | Proof

All vectors $v \in$ null ST must have been nulled by T or S, and therefore either it must be in null T or Tv in range $T \cap$ null S. Notationally,

$$\text{null } ST = \text{null } T \cup \{v : Tv \in (\text{range } T \cap \text{null } S)\}$$

Note that because this union is equal to null ST, it is a vector space. Because no vector can be in both null T and $\{v : Tv \in (\text{range } T \cap \text{null } S)\}$, the dimension of the union is

$$\text{dim null } ST = \text{dim null } T + \text{dim } (\{v : Tv \in (\text{range } T \cap \text{null } S)\})$$

Every value of w that satisfies $w \in (\text{range } T \cap \text{null } S)$ will the output of Tv for some v, because the range is defined as all the outputs of Tv.

$$\text{dim null } ST = \text{dim null } T + \text{dim } (\text{range } T \cap \text{null } S)$$

An intersection can only make the dimension of a set smaller, so dim $(\text{range } T \cap \text{null } S) \leq$ dim null S and

$$\text{dim range } ST \leq \text{dim null } S, \text{dim null } T$$