1 | Notes from Within the Lecture

$$CM = \frac{\sum m_i \vec{r_i}}{\sum m_i} = \frac{1}{M} \sum m_i \vec{r_i} \tag{1}$$

i.e.: the centre of mass is the weighted average of the centers. "First moment."

$$I = M \frac{\sum m_i {r_i}^2}{M} = \sum m_i {r_i}^2 \tag{2}$$

i.e.: the rotational inertia is M times the weighted average, squared. "Second moment."

- 1st moment: Mean
- 2nd moment: Standard Deviation
- 3rd moment: Skew
- 4th moment: Kurtosis

If I take an axis, and shift the direction of the axis, we can compute the rotational inertia about the center of mass and then move it

- Disk: $I = \frac{1}{2}MR$.
- Ring: $I = MR^2$
- Strip: $I = \frac{1}{12}Mx^2$
- Rectangle: $I = \frac{1}{12}Mx^2$, yes, x along the axis of rotation. And yes, y does not matter.

2 | Perpendicular Axis Theorem

$$I_z = I_x + I_y \tag{3}$$

for a laminar object—infinitesimally thin, in the X-Y plane (perpendicular to the z axis).

Why?

$$\begin{align}
I_z &= \sum_i m_i {l_i}^2 \tag{4}\\
&= \sum_i m_i {x_i}^2 + \sum_i m_i {y_i}^2 \tag{5}\\
&= {l_i}^2 = {x_i}^2 {y_i}^2 \tag{6}
\end{align}$$

3 | **Proving the Parallel Axis Theorem**

We will begin with the definition of rotational inertia about an origin:

$$I = \sum_i m_i {l_i}^2 \tag{7}$$

As per defined by the problem $l'_i = x'_i\hat{\imath} + y'_i\hat{\jmath}$, the displacement vector from l_i to the CM.

We also understand that $\vec{R}_{CM} = X_{CM}\hat{\imath} + Y_{CM}\hat{\jmath}$, the components to the location of the center of mass.

Therefore, the actual position $\vec{l_i}$ of the axis of rotation can be expressed as:

$$l_i = \vec{R}_{CM} + \vec{l'_i} \tag{8}$$

Substituting this expression into that for I:

$$\begin{aligned} I &= \sum_i m_i {l_i}^2 && (9) \\ &= \sum_i m_i(\vec{R}_{CM} + \vec{l'_i})^2 && (10) \\ &= \sum_i m_i((\vec{R}_{CM})^2 + 2(\vec{R}_{CM})(\vec{l'_i}) + (\vec{l'_i})^2) && (11) \end{aligned}$$

We can see that the first and last terms will result in the expression we need, and therefore we are bound to figure why the following expression would result in 0:

$$\begin{aligned} &\sum_i 2(\vec{R}_{CM}) m_i \vec{l'}_i && (12) \\ &\Rightarrow 2(\vec{R}_{CM}) \sum_i m_i \vec{l'}_i && (13) \end{aligned}$$

Recall that $\vec{l'}_i$ is the location of the distance from a given point mass in the body to the center of mass. Of course, this makes $\sum_i m_i \vec{l'}_i$ the expression for the center of mass *in the center of mass reference frame.*

We know that the coordinate of the center of mass in the center of mass reference frame is 0, making this whole expression 0.

Therefore:

$$\begin{aligned} I &= \sum_i m_i {l_i}^2 && (14) \\ &= \sum_i m_i((\vec{R}_{CM})^2 + 2(\vec{R}_{CM})(\vec{l'_i}) + (\vec{l'_i})^2) && (15) \\ &= (\vec{R}_{CM})^2 \sum_i m_i + \sum_i m_i(\vec{l'_i})^2 && (16) \\ &= (\vec{R}_{CM})^2 M + \sum_i m_i(\vec{l'_i})^2 && (17) \\ &= D^2 M + I_{CM} \blacksquare && (18) \end{aligned}$$

4 | 5-Ring Object

We will leverage the parallel axis theorem to figure the inertia at the center point.

We can see the distance from the center of mass of each side-sphere to the axis of rotation is $2R$. Furthermore, we can see the mass of the ring is $\frac{1}{5}M$.

As been demonstrated before, the rotational inertia of a ring is:

$$I = MR^2 \tag{19}$$

For the ring with $\frac{1}{5}$ mass then:

$$I_{CM} = \frac{1}{5}MR^2 \tag{20}$$

Applying the parallel axis theorem, then, to each of the four side-objects:

$$\begin{aligned} I &= I_{CM} + \frac{1}{5}MD^2 && (21) \\ &= \frac{1}{5}MR^2 + \frac{4}{5}MR^2 && (22) \\ &= MR^2 && (23) \end{aligned}$$

We repeat this procedure four times, to result in the outer rings' rotational inertia of:

$$4MR^2 \tag{24}$$

The last (”middle”) ring has simply the rotational inertia about its center of origin:

$$\frac{1}{5}MR^2 \tag{25}$$

And therefore, the total rotational inertia is:

$$I = \frac{21}{5}MR^2 \tag{26}$$

5 | Spinning Top

5.1 | Torque

We are asked to find the magnitude and direction of torque $\vec{\tau}_0$ applied by the string. This is easily achieved with the expression for torque $\vec{r} \times \vec{F_0}$.

We will do this in components. Observing the point at which the string is attached, we noticed that it has two components: one towards the positive y direction, and one towards the positive z direction. That:

$$\vec{r} = R\hat{j} + H\hat{k} \tag{27}$$

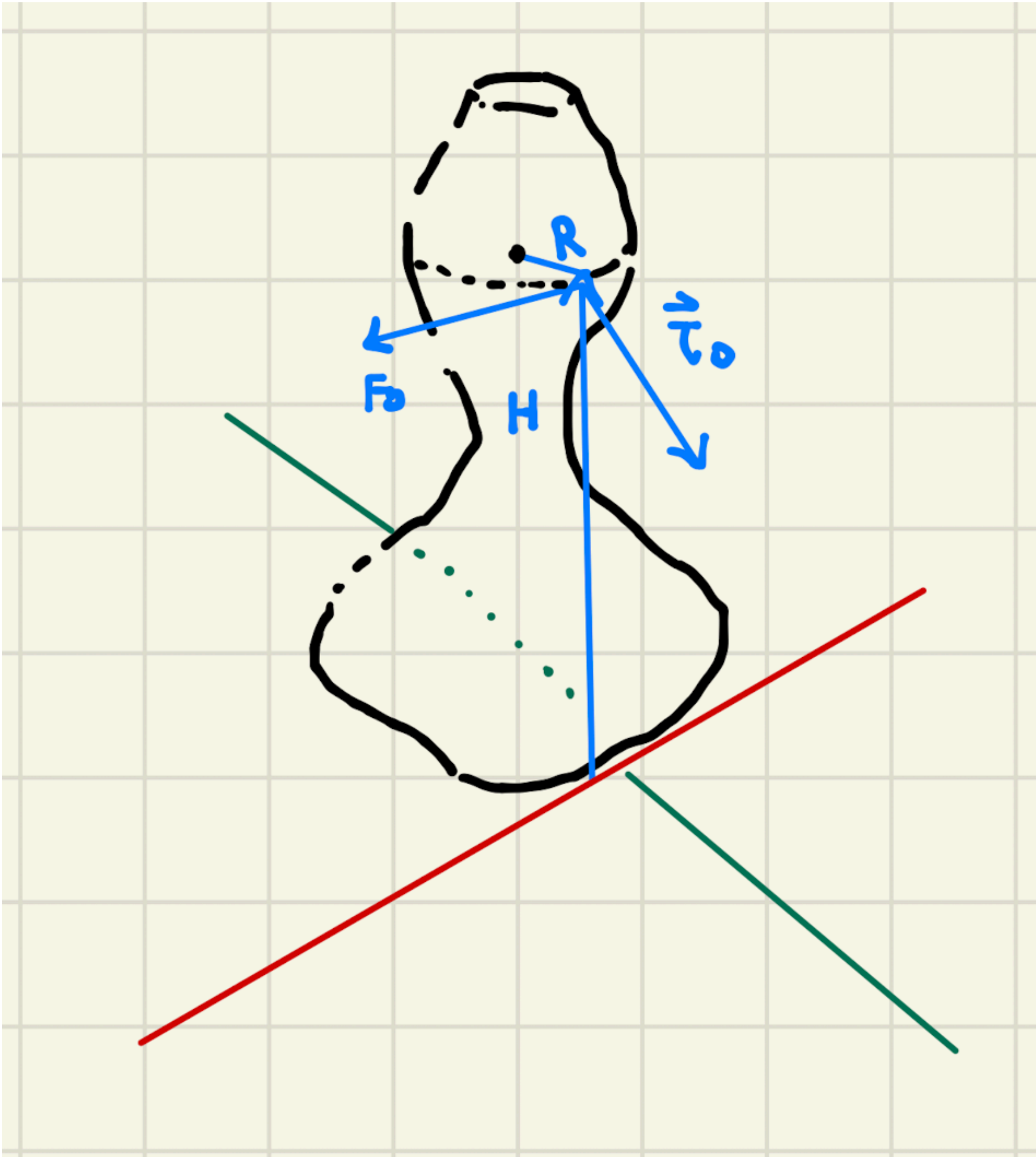
Furthermore, we can see from the graph that $\vec{F_0} = F_0\hat{i}$. Their dot products, therefore, are:

$$
\begin{aligned}
\vec{\tau}_0 &= (R\hat{j} + H\hat{k}) \times (F_0\hat{i}) && (28)\\
&= RF_0(\hat{j} \times \hat{i}) + HF_0(\hat{k} \times \hat{i}) && (29)\\
&= -RF_0\hat{k} + HF_0\hat{j} && (30)\\
&= HF_0\hat{j} - RF_0\hat{k} && (31)
\end{aligned}
$$

And therefore:

$$
\begin{cases}
\tau_{0x} = 0\\
\tau_{0y} = HF_0\\
\tau_{0z} = -RF_0
\end{cases}
\tag{32}
$$

5.2 | Diagram of $\vec{\tau}_0$



5.3 | Balancing Forces

To figure the net torque on the system, we will need to add the torques contributing to the net torque in the system.

We have already deducted above $\vec{\tau}_0 = HF_0\hat{j} - RF_0\hat{k}$. Given $\vec{F_1}$ is attached to the origin of the system, it contributes no torque. Therefore, to figure out the net torque we only need to deduct that for $\vec{F_2}$.

$$\vec{\tau}_2 = \vec{r_2} \times \vec{F_2} \tag{33}$$
$$= -\frac{H}{2}\hat{k} \times F_2\hat{i} \tag{34}$$
$$= -\frac{HF_2}{2}\hat{j} \tag{35}$$

As we know, the object simply rotates about the $\hat{k}$ axis. Therefore, the net torque along $\hat{j}$ would have to be zero.

Therefore:

$$-\frac{HF_2}{2} + HF_0 = 0 \tag{36}$$

We further understand that the object does not move. This means that it has a net force of 0 as well. That is:

$$F_0 + F_2 - F_1 = 0 \tag{37}$$

We are given F_0. We therefore have two equations for two variables, rendering it suitable for solving.

$$-\frac{HF_2}{2} + HF_0 = 0 \tag{38}$$
$$\Rightarrow 2HF_0 - HF_2 = 0 \tag{39}$$
$$\Rightarrow 2F_0 - F_2 = 0 \tag{40}$$
$$\Rightarrow F_2 = 2F_0 \tag{41}$$
$$F_0 + F_2 - F_1 = 0 \tag{42}$$
$$\Rightarrow F_0 + 2F_0 - F_1 = 0 \tag{43}$$
$$\Rightarrow F_1 = 3F_0 \tag{44}$$

Per the setup of the problem, $\vec{F_1}$ is in the $-\hat{i}$ direction, and $\vec{F_2}$ in the $\hat{i}$ direction. Hence:

$$\begin{cases} \vec{F_1} = -3F_0\hat{i} \\ \vec{F_2} = 2F_0\hat{i} \end{cases} \tag{45}$$

5.4 | Rotational Inertia

We have already determined the net torque of the system.

$$\vec{\tau}_2 = -\frac{HF_2}{2}\hat{j} \tag{46}$$
$$\vec{\tau}_0 = HF_0\hat{j} - RF_0\hat{k} \tag{47}$$
$$\vec{\tau}_{net} = (HF_0 - \frac{HF_2}{2})\hat{j} - RF_0\hat{k} \tag{48}$$

We also know from the setup of the problem that the left term works out such that, given the values of F_0 and F_2, it is zero in this scenario. Hence:

$$\vec{\tau}_{net} = -RF_0\hat{k} \tag{49}$$

Based on Newton's Second Law, we understand that:

$$\vec{\tau}_{net} = I\vec{\alpha} \tag{50}$$

If the top has rotational inertia I_0, therefore:

$$I_0\vec{\alpha_0} = -RF_0\vec{k} \tag{51}$$

$$\Rightarrow \vec{\alpha_0} = \frac{-RF_0}{I_0}\vec{k} \tag{52}$$

5.5 | Kinematics Equations

Our system is angularity accelerating at an constant angular acceleration of $\vec{\alpha_0}$. As such, we integrate twice to figure the kinematics equations.

First, as derived above:

$$\vec{\alpha}(t) = \frac{-RF_0}{I_0}\vec{k} \tag{53}$$

Taking the first integral of this expression, we get that:

$$\vec{\omega}(t) = \int \vec{\alpha}(t)dt = \frac{-RF_0}{I_0}t\vec{k} + C \tag{54}$$

where, as $\omega = 0$ at $t = 0$:

$$\vec{\omega}(t) = \frac{-RF_0}{I_0}t\vec{k} \tag{55}$$

Performing the integral yet again, we have that:

$$\vec{\theta}(t) = \int \vec{\omega}(t)dt = \frac{-RF_0}{2I_0}t^2\vec{k} + C \tag{56}$$

where again, as $\theta = 0$ at $t = 0$:

$$\vec{\theta}(t) = \frac{-RF_0}{2I_0}t^2\vec{k} \tag{57}$$

6 | Rectangular Rod

To find the rotational inertia of a rectangular rod, we need to perform three integrations: building up strips, slices, and finally the inertia of the actual rod.

6.1 | Strip

We begin by recalling that the expression for rotational inertia is:

$$I = \sum_i m_i {l_i}^2 \tag{58}$$

Furthermore, we understand the mass of our entire volume is M. Therefore, the mass density along the object would be $\frac{M}{HWL}$.

To figure the inertia of an infinitesimal strip of point masses, we perform a simple integration along the w axis:

$$\begin{aligned}
I_{strip} &= \int_{-W/2}^{W/2} l^2 dm &(59)\\
&= \int_{-W/2}^{W/2} l^2 \frac{dm}{dl} dl &(60)\\
&= \int_{-W/2}^{W/2} l^2 \frac{M}{HWL} dl &(61)\\
&= \frac{M}{HWL}\left(\frac{(W/2)^3}{3} - \frac{(-W/2)^3}{3}\right) &(62)\\
&= \frac{M}{HWL}\left(\frac{W^3}{24} - \frac{-W^3}{24}\right) &(63)\\
&= \frac{M}{HWL}\left(\frac{W^3}{12}\right) &(64)\\
&= \frac{M}{HL}\left(\frac{W^2}{12}\right) &(65)
\end{aligned}$$

6.2 | Slice

We will now find the inertia of a slice. The procedure is essentially the same, but that there is no longer any non-constant components. Therefore, the inertias, as we are rotating about the same axis of a rigid body together, simply stack ("add"). That is: the rotational inertia of a slice is simply H times that of a strip.

$$\begin{aligned}
I_{slice} &= H\ I_{strip} &(66)\\
&= H\ \frac{M}{HL}\left(\frac{W^2}{12}\right) &(67)\\
&= \frac{M}{L}\left(\frac{W^2}{12}\right) &(68)
\end{aligned}$$

6.3 | Final Rotational Inertia

Finally, we will leverage the parallel axis theorem to deduct rotational inertia of the entire rod.

At every plate i, we note that it will be l_i away from the axis of rotation $\hat{k}$. By the parallel axis theorem:

$$I = I_{cm} + mD^2 \tag{69}$$

where, $l_i = D$ and \$m=M$_{\overline{L\$---themassofeachslice.}}$

That is, then:

$$I_{slice\ about\ \hat{k}} = \frac{M}{L}\left(\frac{W^2}{12}\right) + m_i(l_i)^2 \tag{70}$$

We aim to find the sum of all such rotational inertia slices about $\vec{k}$ along L, meaning we will figure:

$$I = \sum_L \left(\frac{M}{L}\left(\frac{W^2}{12}\right) + m_i(l_i)^2\right) \tag{71}$$

Splitting this summation into two parts:

$$I = \sum_L \frac{M}{L}\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 \tag{72}$$

We can see that, because the lack of differentials on the left side, the left expression can simply be simplified to route multiplication:

$$\begin{aligned} I &= \sum_L \frac{M}{L}\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 && (73)\\ &= M\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 && (74) \end{aligned}$$

The right side, however, requires integration. The actual integral is, fortunately, almost the same procedure as before—summing up differential l_i along L via differential masses m_i. We will leverage the mass density of a slice again: $\frac{M}{L}$ ("total mass divided by all slices").

$$\begin{aligned} \sum_L m_i(l_i)^2 &= \int_{-L/2}^{L/2} l^2 dm && (75)\\ &= \int_{-L/2}^{L/2} l^2 \frac{dm}{dl} dl && (76)\\ &= \int_{-L/2}^{L/2} l^2 \frac{M}{L} dl && (77)\\ &= \frac{M}{L}\int_{-L/2}^{L/2} l^2 dl && (78)\\ &= \frac{M}{L}\int_{-L/2}^{L/2} l^2 dl && (79)\\ &= \frac{M}{L}\left(\left.\frac{l^3}{3}\right|_{-L/2}^{L/2}\right) && (80)\\ &= \frac{M}{L}\left(\frac{L^3}{12}\right) && (81)\\ &= M\left(\frac{L^2}{12}\right) && (82) \end{aligned}$$

Substituting this back into our above expression for I again:

$$I = M\left(\frac{W^2}{12}\right) + \sum_{L} m_i(l_i)^2 \tag{83}$$

$$= M\left(\frac{W^2}{12}\right) + M\left(\frac{L^2}{12}\right) \tag{84}$$

$$= \frac{1}{12}M\left(W^2 + L^2\right) \blacksquare \tag{85}$$