1 | the parabola

A parabola (defined by a focus point and a directrix line) is the set of points

$$\{P : |FP| = \text{ distance from } P \text{ to } L\}$$

1.1 | polar equation that describes the parabola

$$\begin{aligned} r &= r(\theta) \\ &= d - r\cos\theta \\ r + r\cos\theta &= d \\ r(1+\cos\theta) &= d \\ r &= \frac{d}{1+\cos\theta} \end{aligned}$$

1.2 | show that cartesian version matches what we expect

Checked using desmos.

$$\begin{aligned} r &= \frac{d}{1+\cos\theta} \\ x &= \left(\frac{d}{1+\cos\theta}\right)\cos\theta \\ &= \frac{d\cos\theta}{1+\cos\theta} \\ y &= \left(\frac{d}{1+\cos\theta}\right)\sin\theta \\ &= \frac{d\sin\theta}{1+\cos\theta} \\ x^2+y^2 &= \frac{d^2}{(1+\cos\theta)^2} \\ x^2+y^2 &= \frac{d^2}{\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)^2} \\ (x^2+y^2)\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)^2 &= d^2 \\ (x^2+y^2)\left(\frac{x^2}{x^2+y^2}+\frac{2x}{\sqrt{x^2+y^2}}+1\right) &= d^2 \\ (x^2+y^2)\left(\frac{x^2}{x^2+y^2}+\frac{2x\sqrt{x^2+y^2}}{x^2+y^2}+1\right) &= d^2 \\ x^2+2x\sqrt{x^2+y^2}+\frac{1}{x^2+y^2} &= d^2 \end{aligned}$$

ugh. too hard.

$$\begin{aligned}
r &= d - r\cos\theta \\
&= d - x \\
\sqrt{x^2+y^2} &= d - x \\
x^2 + y^2 &= d^2 + x^2 - 2dx \\
y^2 &= d^2 - 2dx \\
y^2 - d^2 &= -2dx \\
\frac{d^2 - y^2}{2d} &= x \\
x &= -\frac{y^2}{2d} + \frac{d}{2}
\end{aligned}$$

more scratch work below:

$$\begin{aligned}
\frac{d\sin\theta}{1+\cos\theta} &= \frac{d\cos\theta}{1+\cos\theta} \\
(d\sin\theta)(1+\cos\theta) &= (d\cos\theta)(1+\cos\theta) \\
(d\sin\theta) &= (d\cos\theta) \\
\frac{d}{1+\cos\theta} &= \sqrt{\frac{d^2\cos^2\theta + d^2\sin^2\theta}{(1+\cos\theta)^2}} \\
d &= \sqrt{d^2\cos^2\theta + d^2\sin^2\theta}
\end{aligned}$$

1.3 | vertex

$$\left(\frac{d}{2}, 0\right)$$

2 | general form

2.1 | re-derivation

$$\begin{aligned}
e &= \frac{r}{d - r\cos\theta} \\
r &= r(\theta) \\
&= e(d - r\cos\theta) \\
&= de - er\cos\theta \\
r + er\cos\theta &= de \\
r(1 + e\cos\theta) &= de \\
r &= \frac{de}{1 + e\cos\theta}
\end{aligned}$$

Link to the desmos.

2.2 | elipse when $0 < e < 1$

$$\begin{aligned}
r &= e(d - r\cos\theta) \\
&= e(d - x) \\
\sqrt{x^2 + y^2} &= e(d - x) \\
x^2 + y^2 &= (e(d - x))^2 \\
x^2 + y^2 &= e^2\left(d^2 + x^2 - 2dx\right) \\
x^2 + y^2 &= d^2e^2 + e^2x^2 - 2de^2x \\
-e^2x^2 + 2de^2x + x^2 + y^2 &= d^2e^2 \\
\left(1 - e^2\right)x^2 + 2de^2x + y^2 &= d^2e^2 \\
\left(1 - e^2\right)\left(x^2 + \frac{2de^2}{1 - e^2}x\right) + y^2 &= d^2e^2 \\
\left(x^2 + \frac{2de^2}{1 - e^2}x\right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2}{(1 - e^2)} \\
\left(x^2 + \frac{2de^2}{1 - e^2}x\right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2}{(1 - e^2)} \\
\left(\left(\frac{de^2}{1 - e^2}\right)^2 + x^2 + \frac{2de^2}{1 - e^2}x\right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2}{(1 - e^2)} + \left(\frac{de^2}{1 - e^2}\right)^2 \\
\left(x + \left(\frac{de^2}{1 - e^2}\right)\right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2(1 - e^2) + d^2e^4}{(1 - e^2)^2} \\
\left(x + \left(\frac{de^2}{1 - e^2}\right)\right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2 \cancel{- d^2e^4} \cancel{+ d^2e^4}}{(1 - e^2)^2} \\
\left(x + \left(\frac{de^2}{1 - e^2}\right)\right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2e^2}{(1 - e^2)^2} \\
\frac{(1 - e^2)^2\left(x + \left(\frac{de^2}{1 - e^2}\right)\right)^2}{d^2e^2} + \frac{(1 - e^2)y^2}{d^2e^2} &= 1 \\
\frac{\left((1 - e^2)x + de^2\right)^2}{d^2e^2} + \frac{(1 - e^2)y^2}{d^2e^2} &= 1
\end{aligned}$$

$$\begin{aligned}
x_0 &= -\frac{de^2}{1 - e^2} \\
y_0 &= 0 \\
a &= \frac{de}{1 - e^2} \\
b &= \frac{de}{\sqrt{1 - e^2}}
\end{aligned}$$

for $e < 1$.

The desmos.

2.3 | hyperbola

Going back to one of the preivous equations

$$\frac{(1-e^2)^2\left(x+\left(\frac{de^2}{1-e^2}\right)\right)^2}{d^2e^2}+\frac{(1-e^2)y^2}{d^2e^2}=1$$
$$\frac{(1-e^2)^2\left(x+\left(\frac{de^2}{1-e^2}\right)\right)^2}{d^2e^2}-\frac{(e^2-1)y^2}{d^2e^2}=1$$

That makes

$$\begin{aligned}
x_0 &= -\frac{de^2}{1-e^2}\\
y_0 &= 0\\
a &= \frac{de}{1-e^2}\\
b &= \frac{de}{\sqrt{e^2-1}}
\end{aligned}$$

b will be complex unless $e>1$.

2.4 | degenerate ellipse

It is a circle when $a^2=b^2$, aka

$$\begin{aligned}
\frac{d^2e^2}{\left(1-e^2\right)^2} &= \frac{d^2e^2}{1-e^2}\\
\left(1-e^2\right)^2 &= 1-e^2\\
1-e^2 &= 1\\
e^2 &= 0\\
e &= 0
\end{aligned}$$

You have to take the limit. You get

$$x^2y^2=d^2e^2 \quad =r^2$$

What could d possibly be for $\lim\limits_{e\to 0} de \neq 0$?

$$\begin{aligned}
d &= \frac{n}{e}\\
\lim_{e\to 0} de &= \lim_{e\to 0}\frac{ne}{e}
\end{aligned}$$