1 | Triple Osward's Box

To solve this problem, we will need to take a triple integral along each of the dimensions to add up the energy inside the box.

We know that the box is modeled by the function:

$$e(x, y, z) = x^2y + 11z + 13 \tag{1}$$

We further understand that one corner of the box is located at the origin and the other, at $(3, 7, 4)$.

We will now take the triple integral, one along each dimension:

$$\int_0^4 \int_0^7 \int_0^3 x^2y + 11z + 13 \; dx \; dy \; dz \tag{2}$$

We will take parts of this integral at a time.

$$\int_0^3 x^2y + 11z + 13 \; dx \tag{3}$$

$$= \frac{x^3y}{3} + 11zx + 13x \Big|_0^3 \tag{4}$$

$$= 9y + 33z + 39 \tag{5}$$

And now, we do this again for the second integral.

$$\int_0^7 9y + 33z + 39dy \tag{6}$$

$$= \frac{9y^2}{2} + 33zy + 39y \Big|_0^7 \tag{7}$$

$$= \frac{441}{2} + 231z + 273 \tag{8}$$

$$= 493.5 + 231z \tag{9}$$

And finally, we take the third integral:

$$\int_0^4 493.5 + 231z \; dz \tag{10}$$

$$= 493.5z + \frac{231z^2}{2} \Big|_0^4 \tag{11}$$

$$= 1974 + 1848 \tag{12}$$

$$= 3822 \tag{13}$$

We can see there is 3822 total units of energy in the box.

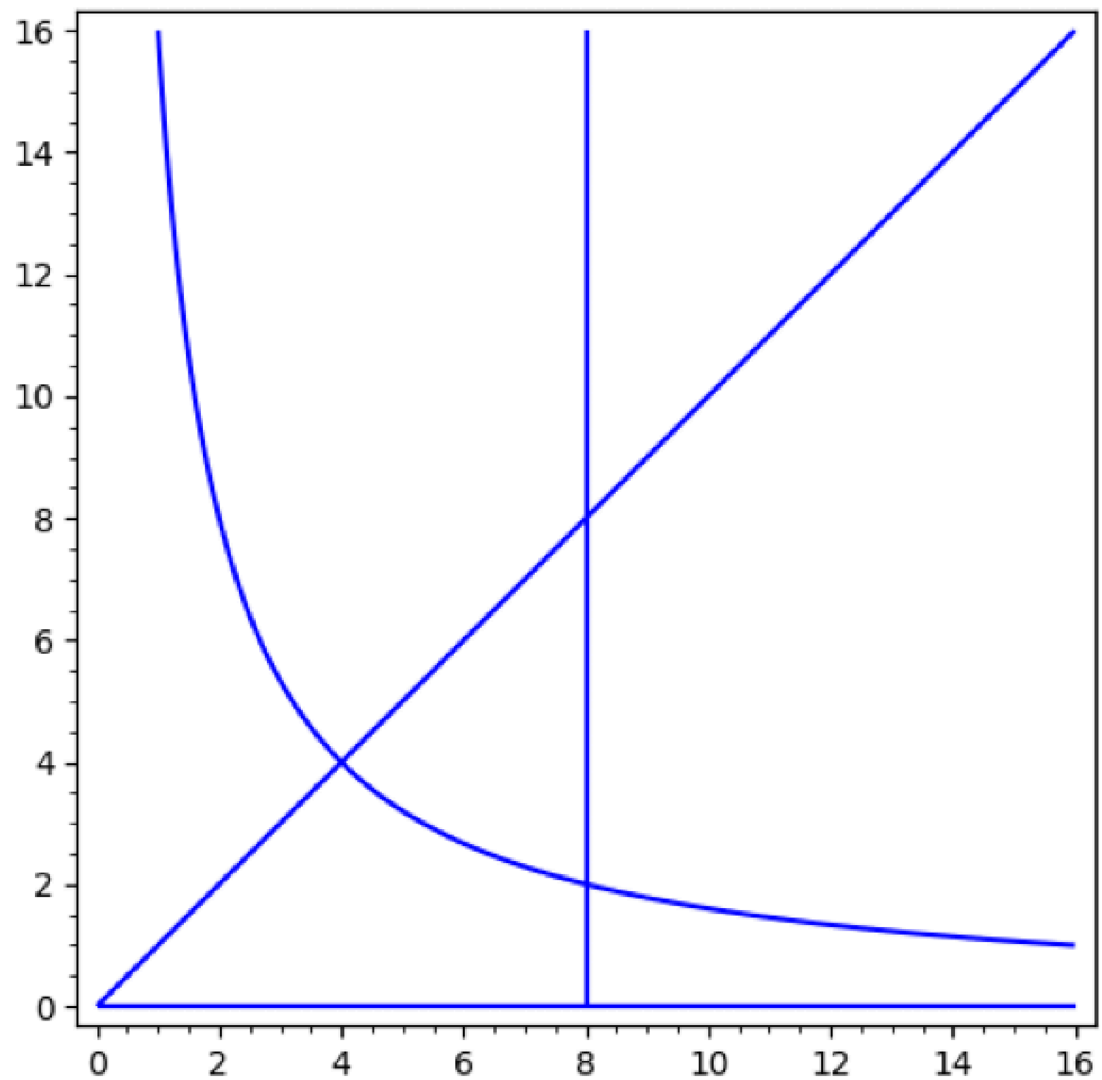
2 | Custom-Bound Integration

We are to find to volume beneath $z = x^2$ via the bounds of $xy = 16$, $y = x$, $y = 0$, and $x = 8$.

From the bound functions of y, it is evident that the parameter y is bound by $[0, x]$. For the bounds of x, we can figure that its bounded in the right by $x = 8$, and the right by $x = \frac{16}{y}$.

Plotting the bounds together:

```
x,y = var("x y")
eqns = [y==0, y==x,x==8, x*y==16]
sum(implicit_plot(i, (x,0,16), (y,0,16)) for i in eqns)
```



As per (not actually given) by the problem, we wish to find the "lower-left" region bound.

2.1 | y-dimension first

Let's perform the integral along the y dimension first, reducing it to a function in x.

We see that the top bound is by two piece-wise functions, each in one dimension of x. We will perform the integral between bounds $[0, 4]$.

$$\int_0^x x^2 dy \tag{14}$$
$$\Rightarrow x^3 \tag{15}$$

And, integrating between $[0,4]$, we have that:

$$\int_0^4 x^3 dx \tag{16}$$
$$\Rightarrow 64 \tag{17}$$

We find the integral now between $[4,8]$, with the bound:

$$\int_0^{16/x} x^2 dy \tag{18}$$
$$\Rightarrow \frac{16x^2}{x} \tag{19}$$
$$\Rightarrow 16x \tag{20}$$

And then, integrating along $[4,8]$:

$$\int_4^8 16x dx \tag{21}$$
$$\Rightarrow \left. \frac{16x^2}{2} \right|_4^8 \tag{22}$$
$$\Rightarrow 384 \tag{23}$$

The total area under the curve, then, is 448.

2.2 | x-dimension first

We can do this again, but rotated. We can perform the integral along the x dimension, reducing it into a function in y.

We see that the right is bound again by two piece-wise functions, each in one dimension of y. We will perform the integral between the bounds $[0,2]$.

$$\int_y^8 x^2 dx \tag{24}$$
$$\Rightarrow \left. \frac{x^3}{3} \right|_y^8 \tag{25}$$
$$\Rightarrow \frac{512}{3} - \frac{y^3}{3} \tag{26}$$

And then, we have to integrate between the bounds $[0,2]$.

$$\int_0^2 \left(\frac{512}{3} - \frac{y^3}{3}\right)\, dy \tag{27}$$
$$\Rightarrow \frac{512y}{3} - \frac{y^4}{12}\bigg|_0^2\, dy \tag{28}$$
$$\Rightarrow \frac{1024}{3} - \frac{16}{12} \tag{29}$$
$$\Rightarrow \frac{4080}{12} = 340 \tag{30}$$

We can perform the integration again $[2,4]$.

$$\int_y^{16/y} x^2 dx \tag{31}$$
$$\Rightarrow \frac{x^3}{3}\bigg|_y^{16/y} \tag{32}$$
$$\Rightarrow \frac{4096}{3y^3} - \frac{y^3}{3} \tag{33}$$
$$\Rightarrow \frac{4096y^{-3}}{3} - \frac{y^3}{3} \tag{34}$$

And then, we will integrate by the bounds $[2,4]$:

$$\int_2^4 \frac{4096y^{-3}}{3} - \frac{y^3}{3}\, dy \tag{35}$$
$$\Rightarrow \frac{4096y^{-2}}{-6} - \frac{y^4}{4}\bigg|_2^4 \tag{36}$$
$$\Rightarrow 108 \tag{37}$$

Hence, the total area under the curve would also be 448 units.

As we can see, the two integrals agree with respect to the area in that corner.

3 | Cylindrical Coordinates

To find the volume upon $f(x,y) = 7+x+y$ above a circle of radius 5, we can leverage cylindrical coordinates.

```
x,y,z = var("x y z")
f(x,y) = 7+x+y
implicit_plot3d(7+x+y-z, (x,-10,10), (y,-10,10), (z,-10,10), region=lambda x,y,z:(x**2+y**2)<5, plot_poi
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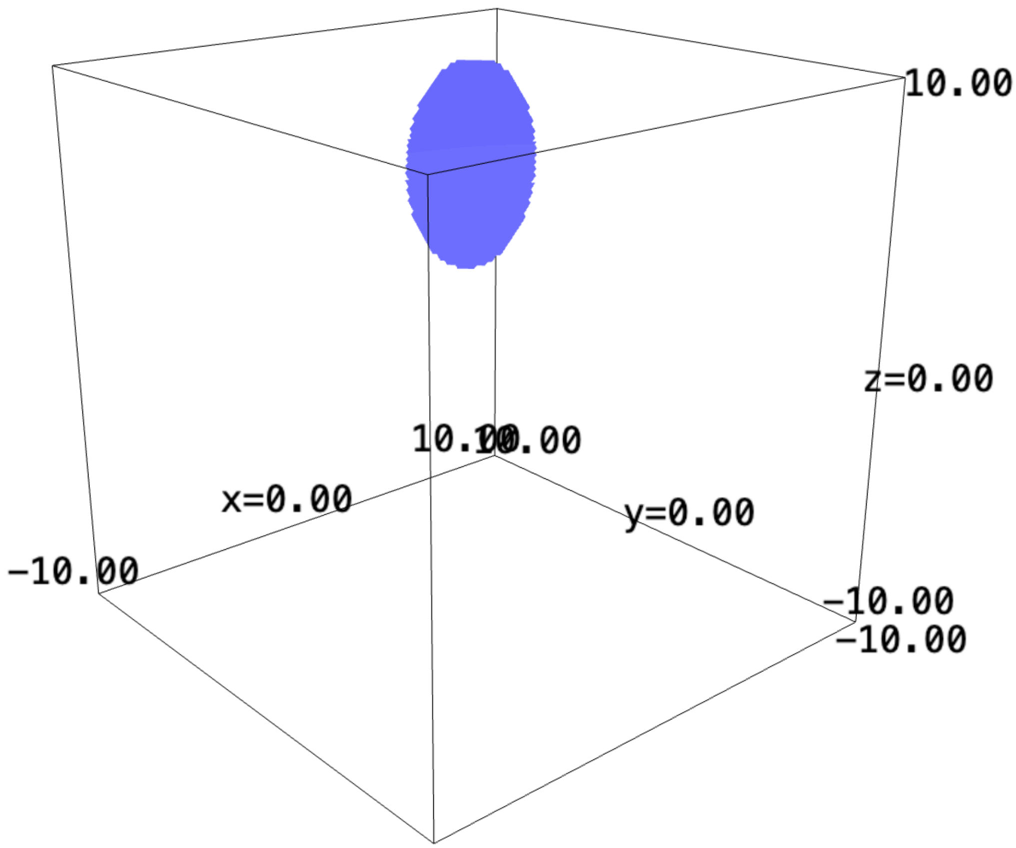

Recall that, given an unit circle, we have:

$$\begin{cases} x = r\ cos(\theta) \\ y = r\ sin(\theta) \end{cases} \tag{38}$$

Supplying the radius of 5 to the expression, we have the following parameterization:

$$\begin{cases} x = 5\ cos(t) \\ y = 5\ sin(t) \end{cases} \tag{39}$$

We can further figure the change of the function along t. That is:

$$\begin{cases} \frac{dx}{dt} = -5\ sin(t) \\ \frac{dy}{dt} = 5\ cos(t) \end{cases} \tag{40}$$

And furthermore, we can see that $\frac{df}{dt}$ is:

$$\frac{df}{dt} = 5 \tag{41}$$

To take the parameterization, we finally get that:

$$f(t) = 7 + 5\ cos(t) + 5\ sin(t) \tag{42}$$

Finally the integral between 0 and 2π:

$$\int_0^{2\pi} 5f(t)\ dt \tag{43}$$

$$\Rightarrow \int_0^{2\pi} 5(7 + 5\ cos(t) + 5\ sin(t))\ dt \tag{44}$$

$$\Rightarrow \int_0^{2\pi} 35 + 25\ cos(t) + 25\ sin(t)\ dt \tag{45}$$

$$\Rightarrow\ 35t + 25\ sin(t) - 25\ cos(t)\ |_0^{2\pi} \tag{46}$$

$$\Rightarrow\ 70\pi - 25 \tag{47}$$

The line integral under the curve is $70\pi - 25$.