1 | Deriving Rotational KE and Inertia

Given m_i, mass, $\vec{r_i}'$, location of the center of mass, l_i, ω, the angular velocity, figure a $KE_{tot,rot}$.

Because of the fact that the value ω is in units $\frac{d\theta}{dt}$, the rate of radians change, and we know of a radius of the spin l_i, we could figure the velocity at which it is moving by simply scaling the change in radians up to a circle of radius l_i, that is:

$$V_i' = l_i\omega \tag{1}$$

(note that, to understand this, radians $\frac{arclength}{radius}$)

And so, substituting into the statement of $\sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2$

$$KE_{rot} = \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 \tag{2}$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i(l_i\omega)^2 \tag{3}$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i l_i^2\omega^2 \tag{4}$$

$$= \frac{1}{2}\omega^2\sum_{i=1}^{N}(m_i l_i^2) \tag{5}$$

1.1 | Rotational Inertia

The right sum — the mass times the distance away from maxis of rotation ($\sum_{i=1}^{N}(m_i l_i^2)$) — is defined as the rotational (moment) of inertia (spinny mass). That is,

$$I = \sum_{i=1}^{N}(m_i l_i^2) \tag{6}$$

Replacing that value in the prior statement, the statement of KE_{rot} is defined as:

$$KE_{rot} = \frac{1}{2}\omega^2 I \tag{7}$$

1.2 | Rotational Inertia for a Ring

For a ring (that's perfectly circular) rotating on an axis perpendicular to the plane of the ring, the l_i — distance from axis of rotation — is the same value: namely, the radius R as the radius of a circle is the same for all positions. Meaning,

$$l_i = R \tag{8}$$

regardless of which value i.

Hence, the value of KE_{rot} would be evaluated as…

$$
\begin{aligned}
KE_{rot} =& \sum_{i=1}^{N}(m_i l_i^2) && (9)\\
=& \sum_{i=1}^{N}(m_i R^2) && (10)\\
=& R^2 \sum_{i=1}^{N} m_i && (11)\\
& && (12)
\end{aligned}
$$

Substituting M as the sum of all masses in the ring ($M = \sum_{i=1}^{N} m_i$), the statement is therefore:

$$KE_{rot} = MR^2 \tag{13}$$

1.3 | Rotational Inertia of a Solid Sphere

I believe that the rotational inertia of I_{sphere} to be less than I_{disk}. This is because, as the dimension of the object increases, it would be easier to change its velocity (a disk is easier to spin than a ring, etc.). Hence, my intuition states that I_{sphere} would be lower than I_{disk}.

Mathematically, as M is staying at the same value, in the disk case has more mass closer to the axis of rotation — meaning that the $m_i R^2$ term would be smaller in more of the point masses than that of an object at a lower dimension. Hence, the sphere would have more points with lower $m_i R^2$ terms than that of disk; hence, I_{sphere} would be less than I_{disk}.

2 | Kinematics Equations

Given $a = a_0$, initial velocity v_0, and position y_0, we derive the kinematics equations.

$$
\begin{aligned}
a(t) =& a_0 && (14)\\
\int a(t)dt =& \int a_0 dt && (15)\\
v(t) =& a_0 t + C && (16)
\end{aligned}
$$

We are given that $v(0) = v_0$. $v(0) = C = v_0$, hence, $C = v_0$. The velocity statement is therefore,

$$v(t) = a_0 t + v_0 \tag{17}$$

Continuing with integration:

$$
\begin{aligned}
v(t) =& a_0 t + v_0 && (18)\\
\int v(t) =& \int a_0 t + v_0 dt && (19)\\
y(t) =& \frac{1}{2} a_0 t^2 + v_0 t + C && (20)\\
& && (21)
\end{aligned}
$$

Again, substituting $C = y_0$ by the same logic above — $y(0) = C = y_0$, we derive the statement for the position equation.

$$y(t) = \frac{1}{2}a_0t^2 + v_0t + y_0 \tag{22}$$

2.1 | Proving $v^2(t) = v_0^2 + 2a_0(y(t) - y_0)$

We start at the statement for $v(t)$, squaring it, and substituting the necessary statements.

$$v(t) = a_0t + v_0 \tag{23}$$
$$\Rightarrow v^2(t) = {a_0}^2t^2 + 2a_0v_0t + v_0^2 \tag{24}$$
$$v^2(t) = {v_0}^2 + 2a_0(\frac{1}{2}a_0t^2 + v_0t) \tag{25}$$
$$v^2(t) = {v_0}^2 + 2a_0(\frac{1}{2}a_0t^2 + v_0t + y_0 - y_0) \tag{26}$$
$$v^2(t) = {v_0}^2 + 2a_0(y(t) - y_0) \tag{27}$$

It is therefore shown that:

$$v^2(t) = {v_0}^2 + 2a_0(y(t) - y_0) \tag{28}$$

2.2 | Proving $\Delta y = \frac{v(t_1)+v(t_2)}{2}\Delta t$

Showing $\Delta y = \frac{v(t_1)+v(t_2)}{2}\Delta t$, defining $\Delta y = y(t_2) - y(t_1)$ and $\Delta t = t_2 - t_1$. Substituting the appropriate values for $v(t)$, Δy, Δt and solving…

$$\Delta y = \frac{v(t_1) + v(t_2)}{2}\Delta t \tag{29}$$
$$y(t_2) - y(t_1) = \frac{v(t_1) + v(t_2)}{2}t_2 - t_1 \tag{30}$$
$$y(t_2) - y(t_1) = \frac{((a_0t_1 + v_0) + (a_0t_2 + v_0))}{2}t_2 - t_1 \tag{31}$$
$$y(t_2) - y(t_1) = \frac{((a_0t_1t_2 + v_0t_2) - (a_0{t_1}^2 + v_0t_1) + (a_0{t_2}^2 + v_0t_2) - (a_0t_1t_2 + v_0t_1))}{2} \tag{32}$$
$$y(t_2) - y(t_1) = \frac{(a_0{t_2}^2 + 2v_0t_2) - (a_0{t_1}^2 + 2v_0t_1)}{2} \tag{33}$$
$$y(t_2) - y(t_1) = \frac{(a_0{t_2}^2 + 2v_0t_2 + 2y_0) - (a_0{t_1}^2 + 2v_0t_1 + 2y_0)}{2} \tag{34}$$
$$y(t_2) - y(t_1) = \frac{1}{2}a_0{t_2}^2 + v_0t_2 + y_0 - \frac{1}{2}a_0{t_1}^2 + v_0t_1 + y_0 \tag{35}$$
$$y(t_2) - y(t_1) = y(t_2) - y(t_1) \tag{36}$$
$$\tag{37}$$

Hence, it is demonstrated that:

$$\Delta y = \frac{v(t_1) + v(t_2)}{2}\Delta t \tag{38}$$

3 | Question regarding signage of the equations demonstrated above.