1 | Calculus

We know that the gradient of $h(x, y)$ is

$$\nabla h(x, y) = \begin{bmatrix} 2x + 2 \\ 2y - 6 \end{bmatrix}$$

Given this, we know that

$$\nabla h(-1, 3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We know that $(-1, 3)$ is a root of the function. We can also take the second partial derivative:

$$h_{xx}(x, y) = 2$$
$$h_{xy}(x, y) = 0$$
$$h_{yy}(x, y) = 2$$

Because we know that there is no acceleration in the x or y component with respect to their counterparts, and because we know that h_{xx} and h_{yy} are positive, we know that the function is a minimum at $(-1, 3)$.

2 | Non-calculus

We know that $\nabla h(-1, 3) = 0$. Given this, we can factor $h(x, y)$ and prove that the function's absolute minimum is at $(-1, 3)$:

$$\begin{aligned} h(x, y) &= x^2 + y^2 + 2x - 6y + 5 \\ &= (x)(x + 2) + (y)(y - 6) + 5 \end{aligned}$$

After we factor $h(x, y)$, we learn that the function is in reality a sum of two "parabolic" functions with different variables. We can separate $h(x, y)$ into an x component, $(x)(x + 2)$, a y component, $(y)(y - 6)$, and a constant, 5. We know that the minimum of the x component is at the vertex of the parabola given by the expression, or at $x = -1$. We also know that the minimum of the y component is at the vertex of the parabola given by the expression, or at $y = 3$. Now that we know the x value that minimizes the x component and the y value that minimizes the y component, we know that $(-1, 3)$ must be the minimum of $h(x, y)$.

3 | Fur Trading

We will model the price of fox pelts as p_f and the price of seal pelts as p_s. We will also model the quantity of fox pelts sold as q_f and the quantity of seal pelts as q_s. The quantity of fox pelts sold and seal pelts sold can be modeled by the equations

$$q_f = 200 - p_s - 3p_f$$
$$q_s = 150 - 2p_s - p_f$$

respectively. The total revenue can be modeled by the equation

$$\begin{aligned} r(p_f, p_s) &= p_f \cdot q_f + p_s \cdot q_s \\ &= p_f(200 - p_s - 3p_f) + p_s(150 - 2p_s - p_f) \\ &= 200p_f + 150p_s - 2p_f p_s - 3p_f^2 - 2p_s^2 \end{aligned}$$

We can take the gradient of r to find extremas:

$$\nabla r(p_f, p_s) = \begin{bmatrix} 200 - 6p_f - 2p_s \\ 150 - 4p_s - 2p_f \end{bmatrix}$$

We can set both the p_f and p_s component of the vector to zero to get a system of equations and find the extrema:

$$\begin{aligned} 200 - 6p_f - 2p_s &= 0 \\ 150 - 4p_s - 2p_f &= 0 \end{aligned}$$

$$\begin{aligned} 2p_s &= 200 - 6p_f \\ 150 - 2(200 - 6p_f) - 2p_f &= 0 \\ 150 - 400 + 10p_f &= 0 \\ 10p_f &= 250 \\ p_f &= 25 \\ p_s &= 150 - 3p_f \\ &= 100 - 3(25) \\ &= 25 \end{aligned}$$

We have a value for (p_f, p_s) at which the gradient is zero: $(25, 25)$. We only have one extrema, so given the context of the problem, we can assume that it is a maxima. But, because I think Andrew will get mad at me if I don't, I will give a brief explanation as to why this is the minima: If we take the second order partial derviative of r, we get the following:

$$\begin{aligned} r_{p_f p_f}(p_f, p_s) &= -6 \\ r_{p_f p_s}(p_f, p_s) &= -2 \\ r_{p_s p_s}(p_f, p_s) &= -4 \end{aligned}$$

Even though when taking the xy second degree partial we get a value, this doesn't really matter because the product of the xx and yy partials are greater than the square of the xy partial, so it is a maximum or minimum. We know that it is a maximum because both xx and yy partials are negative. Therefore, we know that the optimal price for fox and seal pelts are 25 and 25, respectively.