1 | shoestring loop

$$
\begin{aligned}
x &= t^2 \\
y &= t^3 - 3t \\
\frac{dx}{dt} &= 2t \\
\frac{dy}{dt} &= 3t^2 - 3 \\
\frac{dy}{dx} &= \frac{3t^2 - 3}{2t}
\end{aligned}
$$

1.1 | tangents are horizontal or vertical

1.1.1 | horizontal

$$
\begin{aligned}
3t^2 - 3 &= 0 \\
3t^2 &= 3 \\
t^2 &= 1 \\
t &= \pm 1
\end{aligned}
$$

Now, let's find the actual coordinates

$$
\begin{aligned}
x &= t^2 && = 1 \\
y &= t^3 - 3t && = -2 \text{ or } 2
\end{aligned}
$$

and it checks out graphically.

1.1.2 | vertical

$$
\begin{aligned}
2t &= 0 \\
t &= 0
\end{aligned}
$$

The actual coordinates

$$
(t^2, t^3 - 3t) = (0, 0)
$$

and it checks out graphically.

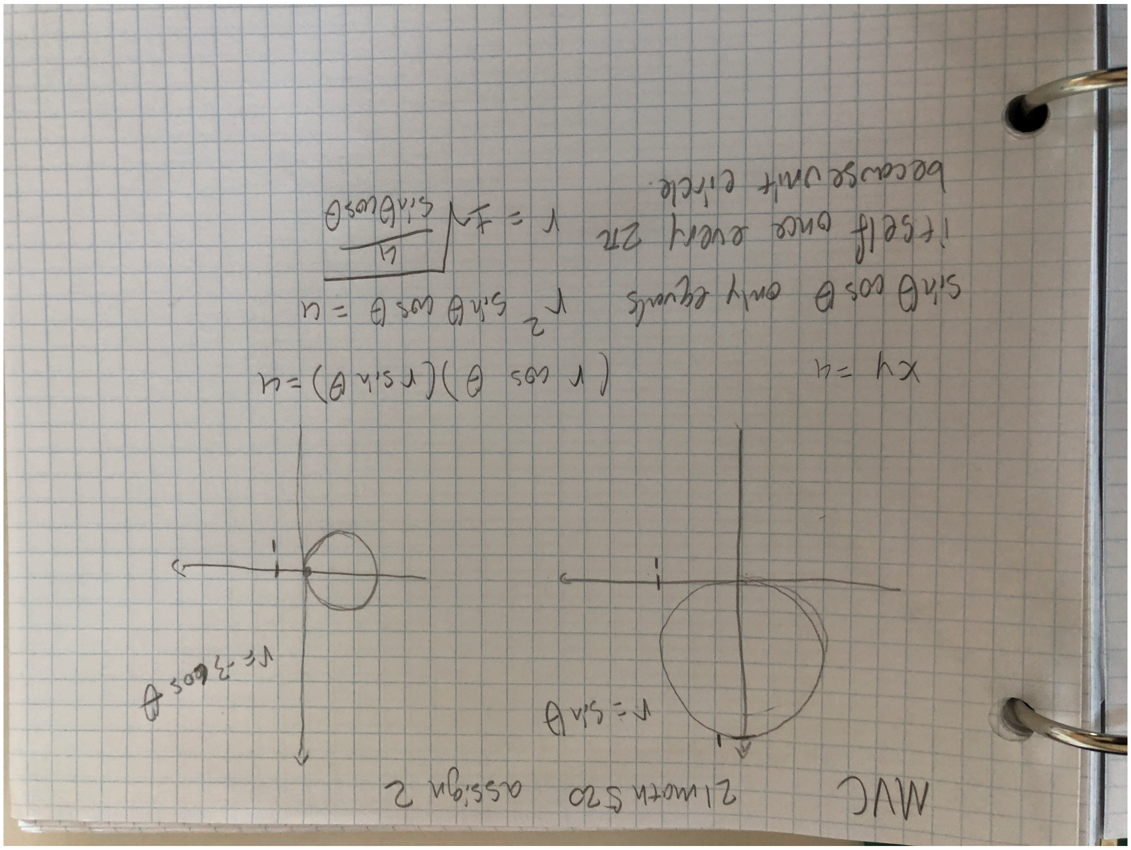
1.2 | concave up

$$
\begin{aligned}
\frac{d}{dx}\frac{dy}{dx} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} &= \frac{2t(6t) - (3t^2 - 3)(2)}{8t^3} \\
&= \frac{6t^2 - 3t^2 + 3}{4t^3} = \frac{3t^2 + 3}{4t^3} > 0 \\
&\therefore \text{ concave up for } t > 0
\end{aligned}
$$

1.3 | concave down

Using similar logic, the curve is concave down for $t \leq 0$.

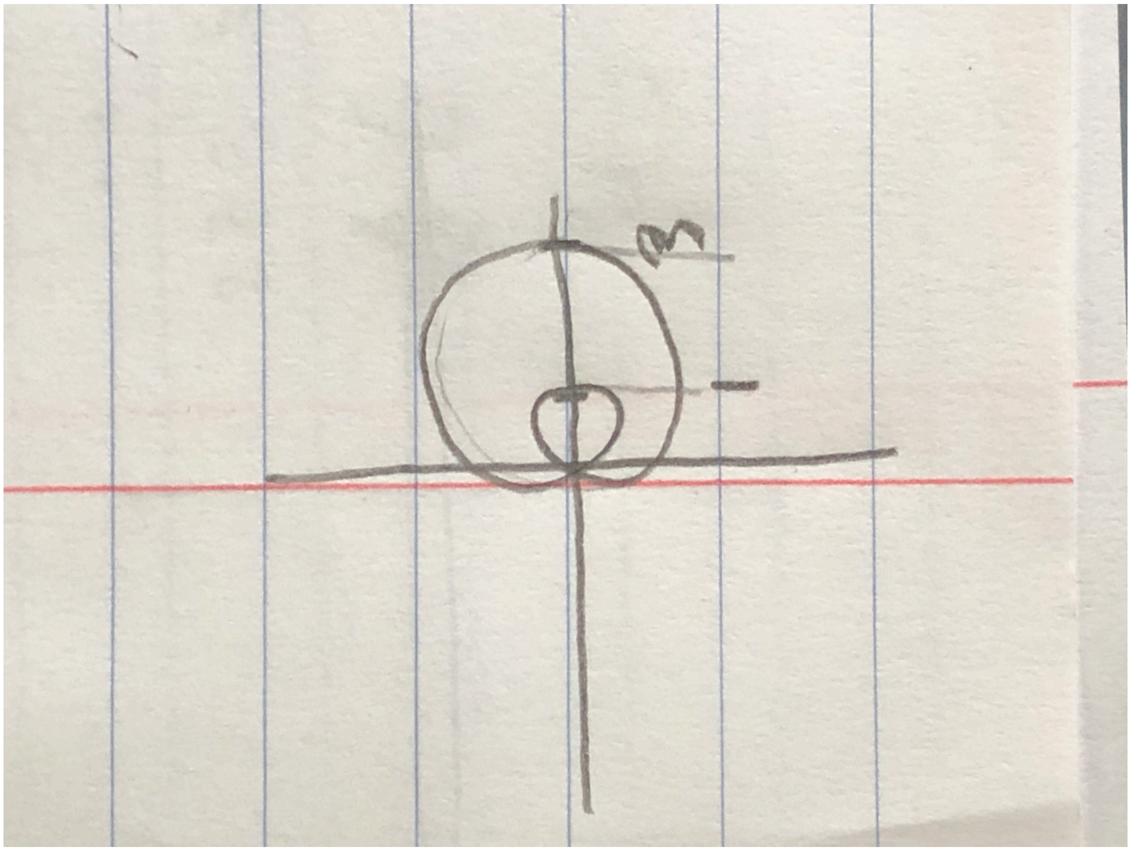
2 | polar curves + converting to cartesian



Also see the (updated) desmos.

4 | cardiod

4.1 | sketch



4.2 | crosses the origin

Only hapens when $\theta = 0$.

$$\begin{aligned} r = 1 + 2\cos\theta &= 0 \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \cos^{-}\left(-\frac{1}{2}\right) \\ &= \frac{2\pi}{3}, -\frac{2\pi}{3} \end{aligned}$$

4.3 | derivatives to verify crossing

Let's choose the crossing $\theta = \frac{2\pi}{3}$

$$
\begin{aligned}
y &= r\sin\theta = (1+2\cos\theta)\sin\theta = \sin\theta + 2\cos\theta\sin\theta = \sin\theta + \sin 2\theta \\
x &= r\cos\theta = (1+2\cos\theta)\cos\theta = \cos\theta + 2\cos^2\theta \\
\frac{dy}{d\theta} &= \cos\theta + 2\cos 2\theta \implies -\frac{1}{2} + 2\left(-\frac{1}{2}\right) = -\frac{3}{2} \\
\frac{dx}{d\theta} &= -\sin\theta - 2(2\cos\theta)\sin\theta = -\sin\theta - 2\sin 2\theta = -\frac{\sqrt{3}}{2} - 2(-\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2} \\
\frac{d^2y}{d\theta^2} &= -\sin\theta - 4\sin 2\theta = -\frac{\sqrt{3}}{2} - 4\left(-\frac{\sqrt{3}}{2}\right) = 3\frac{\sqrt{3}}{2} \\
\frac{d^2x}{d\theta^2} &= -\cos\theta - 4\cos 2\theta = \frac{5}{2} \\
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = -\frac{\cos\theta + 2\cos 2\theta}{\sin\theta + 2\sin 2\theta} = -\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = -\sqrt{3} \\
\frac{dy^2}{dx^2} &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} = \frac{\frac{\sqrt{3}}{2}3\frac{\sqrt{3}}{2} + \frac{5}{2}\frac{3}{2}}{\left(\frac{\sqrt{3}}{2}\right)^3} \\
&= \frac{\frac{9}{4} + \frac{15}{4}}{\frac{3\sqrt{3}}{8}} && = \frac{24}{4}\frac{8}{3\sqrt{3}} = -\frac{16}{\sqrt{3}}
\end{aligned}
$$

4.4 | points where tangent is horizontal or vertical

4.4.1 | horizontal

$$
\begin{aligned}
\frac{dy}{dx} = \frac{\cos\theta + 2\cos 2\theta}{\sin\theta + 2\sin 2\theta} &= 0 \\
\implies \cos\theta + 2\cos 2\theta &= 0 \\
\cos\theta + 2\cos^2\theta - 2\sin^2\theta &= 0
\end{aligned}
$$

$$
\begin{aligned}
0 &= \cos\theta + 2\cos^2\theta - 2\sin^2\theta \\
0 &= \cos\theta + 2\cos^2\theta - 2\left(1 - \cos^2\theta\right) \\
0 &= \cos\theta + 2\cos^2\theta - 2 + 2\cos^2\theta \\
0 &= 4\cos^2\theta + \cos\theta - 2 \\
\cos\theta &= \frac{-1 \pm \sqrt{1+4(4)2}}{8} = \frac{-1 \pm \sqrt{33}}{8}
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{-1+\sqrt{33}}{8}\right), \cos^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) \\
&= 0.936, 3.709, 2.574, 5.347 \\
&= (1.296, 1.76), (0.579, 0.369), (0.579, 0.369), (1.296, -1.76)
\end{aligned}
$$

4.4.2 | vertical

$$\begin{aligned} \frac{dx}{dy} &= \frac{\sin\theta + 2\sin 2\theta}{\cos\theta + 2\cos 2\theta} \\ \implies &\quad \sin\theta + 2\sin 2\theta = 0 \end{aligned}$$

Make sure the top isn't also zero when the bottom is zero. Otherwise, by L'hospital rule that $\frac{0}{0}$ could be anything in the world.

$$\begin{aligned} 0 &= \sin\theta + 2\sin 2\theta \\ &= \sin\theta + 4\sin\theta\cos\theta \\ &= \sin\theta(1 + 4\cos\theta) \end{aligned}$$

Either $\theta = 0, \pi$, or

$$\begin{aligned} 0 &= 1 + 4\cos\theta \\ -\frac{1}{4} &= \cos\theta \\ \theta &= 1.823, 4.46 \end{aligned}$$

The locations are at $(1, 0), (3, 0), (-0.125, -0.4847), (-0.125, 0.4847)$.

4.5 | tangent line

$$y = -\sqrt{3}x$$

5 | arclength

$$S = \int_C ds = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

Also note that

$$\boxed{\frac{dy}{dx} \triangleq \lim_{\Delta x\to 0} \frac{\delta y}{\delta x}}$$

5.1 | if $y = f(x)$

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}\right) dx^2} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} dx \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

5.2 | **if** $y = y(t), x = x(t)$

$$
\begin{aligned}
\Delta S &= \sqrt{\Delta x^2 + \Delta y^2}\frac{\Delta t}{\Delta t} \\
\implies & \quad \sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2}}\Delta t \\
\implies & \quad \int \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}}dt
\end{aligned}
$$