# 1 | **Problem 1**

## 1.1 | **a)**

Inorder for the boat to be going in the right direction we know that $\vec{C} + \vec{S} = \alpha \vec{D}$, where $\vec{C}$ is the current of the river, $\vec{S}$ is the speed of the boat, $\alpha$ is some scalar and $\vec{D}$ is the vector that goes from the boatman's starting point to their desired endpoint.

We can set the boatman's start point as $(0, 0)$, and thus $\vec{D} = \langle 3, 2 \rangle$. We also know that $\vec{C} = \langle 0, -3.5 \rangle$. Lastly, $\vec{S} = \langle 13 \sin(\theta), 13 \cos(\theta) \rangle$, where $\theta$ is the angle between the side of the river and $\vec{S}$.

We can then plug in these values into the equation written above:

$\vec{C} + \vec{S} = \alpha \vec{D}$
$\Rightarrow \langle 0, -3.5 \rangle + \langle 13 \sin(\theta), 13 \cos(\theta) \rangle = \alpha \langle 3, 2 \rangle$
$\Rightarrow \langle 13 \sin(\theta), -3.5 + 13 \cos(\theta) \rangle = \langle \alpha 3, \alpha 2 \rangle$
$\Rightarrow 13 \sin(\theta) = \alpha 3, -3.5 + 13 \cos(\theta) = \alpha 2$
$\Rightarrow 6\alpha = 26 \sin(\theta), 6\alpha = -10.5 + 39 \cos(\theta)$
$\Rightarrow 26 \sin(\theta) = -10.5 + 39 \cos(\theta)$

plug it into wolfram alpha:

$\theta \approx 0.75686$ radians or $\approx 43.36°$

## 1.2 | **b)**

The net velocity of the boat is $\vec{S} + \vec{C} = \langle 13 \sin(\theta), 13 \cos(\theta) - 3.5 \rangle$, where $\theta$ is the answer to part a. To get the speed of the boat we find the magnitude of this vector:

$|\vec{S} + \vec{C}| = \sqrt{(13 \sin(\theta))^2 + (13 \cos(\theta) - 3.5)} \approx 10.7282$ km/h

Now we need to find the distance traveled by the boat, which should be the magnitude of $\vec{D}$:

$|\vec{D}| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.60555$ km

To get the time it took to take the trip we divide the distace by the speed:

$\frac{3.60555}{10.7282} = 0.336$ hours, which is $20.2$ minutes

# 2 | **Problem 2**

## 2.1 | **a)**

$\vec{r}(t) = (R \cos(\omega_o t), R \sin(\omega_o t))$

This is because the x coordinate is defined as $r \cos(\theta)$ and the y coordinate is defined as $r \sin(\theta)$. In this case $r$ or the radius is $R$ and $\theta$ is $\omega_o t$, because $\omega_o = \frac{\theta}{t}$ (definition of angular velocity.

## 2.2 | **b)**

$\vec{v}(t) = \vec{r}'(t) = (\frac{d}{dt} R \cos(\omega_o t), \frac{d}{dt} R \sin(\omega_o t)) = (-R\omega_o \sin(\omega_o t), R\omega_o \cos(\omega_o t))$
Answer: $\vec{v}(t) = (-R\omega_o \sin(\omega_o t), R\omega_o \cos(\omega_o t))$

## 2.3 | **c)**

$\vec{a}(t) = \vec{r}''(t) = (\frac{d}{dt}(-R\omega_o \sin(\omega_o t)), \frac{d}{dt} R\omega_o \cos(\omega_o t)) = (-R\omega_o^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t))$

Answer: $\vec{a}(t) = (-R\omega_o^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t))$

## 2.4 | **d)**

The tangent:

Because $\vec{v}(t)$ is a vector it can be placed anywhere on the plane, so the only requierment for $\vec{v}(t)$ is that it has to be perpendicular to $\vec{r}(t)$, which is the radius of the circle. This means that the slope of $\vec{v}(t)$ has to be the opposite reciprocol of $\vec{r}(t)$:

Slope of $\vec{r}(t) = \frac{\Delta y}{\Delta x} = \frac{R\sin(\omega_o t) - 0}{R\cos(\omega_o t) - 0} = \frac{\sin(\omega_o t)}{\cos(\omega_o t)}$

Slope of $\vec{v}(t) = \frac{\Delta y}{\Delta x} = \frac{R\omega_o \cos(\omega_o t) - 0}{-R\omega_o \sin(\omega_o t) - 0} = -\frac{\cos(\omega_o t)}{\sin(\omega_o t)}$

The slopes are opposite reciprocols, thus $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$, thus $\vec{v}(t)$ is tangent to the circle.

The magnitude:

$|\vec{v}(t)| = \sqrt{(-R\omega_o \cos(\omega_o t))^2 + (R\omega_o \sin(\omega_o t))^2} = \sqrt{R^2\omega_o^2 \cos^2(\omega_o t + R^2\omega_o^2 \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^2(\cos^2(\omega_o t) + \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^2(1)} = \sqrt{R^2\omega_o^2} = R\omega_o$

## 2.5 | **e)**

Point towards the center of the circle:

$\vec{a}(t)$ is a scalar multiple of $\vec{r}(t)$: $\vec{a}(t) = -\omega_o^2 \cdot \vec{r}(t) = -\omega_o^2(R\cos(\omega_o t), R\sin(\omega_o t)) = (-R\omega^2 \cos(\omega_o t), -R\omega_o^2 \sin(\omega_o t)) = \vec{a}(t)$

because the scalar multiple is negative $\vec{v}(t)$ points in the opposite direction of $\vec{r}(t)$, which is towards the center of the circle because $\vec{r}(t)$ points from the center of the circle outwards.

The magnitude:

$|\vec{a}(t)| = \sqrt{(-R\omega^2 \cos(\omega_o t))^2 + (-R\omega_o^2 \sin(\omega_o t))^2} = \sqrt{R^2\omega_o^4 \cos^2(\omega_o t) + R^2\omega_o^4 \sin^2(\omega_o t)} = \sqrt{R^2\omega_o^4(\cos^2(\omega_o t) + \sin^2(\omega_o t))} = \sqrt{R^2\omega_o^4(1)} = \sqrt{R^2\omega_o^4} = R\omega_o^2$

$\frac{|\vec{v}(t)|^2}{R} = \frac{(R\omega_o)^2}{R} = \frac{R^2\omega_o^2}{R} = R\omega_o^2$

## 2.6 | **f)**

$\theta'(t) = \int \theta''(t)dt = \int \alpha_o dt = \alpha_o t + c$, where $c$ is the constant of integration

Answer: $\alpha_o t + c$

## 2.7 | **g)**

$\theta(t) = \int \theta'(t)dt = \int(\alpha_o t + c)dt = \int \alpha_o t dt + \int c dt = \frac{\alpha_o t^2}{2} + ct + c'$ where c' is another constant of integration.

Answer: $\frac{\alpha_o t^2}{2} + ct + c'$

### 2.8 | **h)**

For simplicity's sake, I am going to assume that $c = 0$ and $c' = 0$:

$\vec{r}(t) = (R\cos(\frac{\alpha_o t^2}{2}), R\sin(\frac{\alpha_o t^2}{2}))$

### 2.9 | **i)**

$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = (\frac{d}{dt}R\cos(\frac{\alpha_o t^2}{2}), \frac{d}{dt}R\sin(\frac{\alpha_o t^2}{2})) = (-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$

Answer: $\vec{v}(t) = (-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$

### 2.10 | **j)**

$\vec{a}(t) = \vec{r}''(t) = (\frac{d}{dt} - R\alpha_o t\sin(\frac{\alpha_o t^2}{2}), \frac{d}{dt}R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))$

It will be easier to visualize by doing the math in it's x and y component:

x component:

$\frac{d}{dt} - R\alpha_o t\sin(\frac{\alpha_o t^2}{2}) = -R\alpha_o(\sin(\frac{\alpha_o t^2}{2}) + \alpha_o t^2\cos(\frac{\alpha_o t^2}{2})) = -R\alpha_o\sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2})$

y component:

$\frac{d}{dt}R\alpha_o t\cos(\frac{\alpha_o t^2}{2}) = R\alpha_o(\cos(\frac{\alpha_o t^2}{2}) - \alpha_o t^2\sin(\frac{\alpha_o t^2}{2})) = R\alpha_o\cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2})$

Putting these two together we get:

$\vec{a}(t) = (-R\alpha_o\sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2}), R\alpha_o\cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2}))$

### 2.11 | **k)**

The tangent:

As stated above, because $\vec{v}(t)$ is a vector, it can be placed anywhere (for example, the point where $\vec{r}(t)$ intersects with the circle). Thus we only need to show that $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$ in order to show that $\vec{v}(t)$ is tangent to the circle:

Slope of $\vec{r}(t) = \frac{\Delta y}{\Delta x} = \frac{R\sin(\frac{\alpha_o t^2}{2})}{R\cos(\frac{\alpha_o t^2}{2})} = \frac{\sin(\frac{\alpha_o t^2}{2})}{\cos(\frac{\alpha_o t^2}{2})}$

Slope of $\vec{v}(t) = \frac{\Delta y}{\Delta x} = \frac{R\alpha_o t\cos(\frac{\alpha_o t^2}{2})}{-R\alpha_o t\sin(\frac{\alpha_o t^2}{2})} = -\frac{\cos(\frac{\alpha_o t^2}{2})}{\sin(\frac{\alpha_o t^2}{2})}$

The slopse are opposite reciprocals, thus $\vec{v}(t)$ is perpendicular to $\vec{r}(t)$, thus $\vec{v}(t)$ is tangent to the circle

The magnitude:

$|\vec{v}(t)| = \sqrt{(-R\alpha_o t\sin(\frac{\alpha_o t^2}{2}))^2 + (R\alpha_o t\cos(\frac{\alpha_o t^2}{2}))^2} = \sqrt{R^2\alpha_o^2 t^2(\cos^2(\frac{\alpha_o t^2}{2}) + \sin^2(\frac{\alpha_o t^2}{2}))} = \sqrt{R^2\alpha_o^2 t^2(1)} = R\alpha_o t = R \cdot \theta'(t)$

### 2.12 | **l)**

This problem would be better visualized as broken down into terms:

The first term:

$-(\frac{|\vec{v}|^2}{R} \cdot \frac{\vec{r}}{R}) = -(\frac{(R\alpha_o t)^2}{R} \cdot \frac{(R\cos(\frac{\alpha_o t^2}{2}), R\sin(\frac{\alpha_o t^2}{2}))}{R}) = -(R\alpha_o^2 t^2 \cdot (\cos(\frac{\alpha_o t^2}{2}), \sin(\frac{\alpha_o t^2}{2}))) = (-R\alpha_o^2 t^2\cos(\frac{\alpha_o t^2}{2}), -R\alpha_o^2 t^2\sin(\frac{\alpha_o t^2}{2}))$

The second term:

$(R\alpha_o \cdot \frac{\vec{v}}{|\vec{v}|}) = R\alpha_o \cdot \frac{(-R\alpha_o t \sin(\frac{\alpha_o t^2}{2}), R\alpha_o t \cos(\frac{\alpha_o t^2}{2}))}{R\alpha_o t} = R\alpha_o \cdot (-\sin(\frac{\alpha_o t^2}{2}), \cos(\frac{\alpha_o t^2}{2})) = (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2}))$

Adding the two terms together:

$-(\frac{|\vec{v}|^2}{R} \cdot \frac{\vec{r}}{R}) + (R\alpha_o \cdot \frac{\vec{v}}{|\vec{v}|}) = (-R\alpha_o^2 t^2 \cos(\frac{\alpha_o t^2}{2}), -R\alpha_o^2 t^2 \sin(\frac{\alpha_o t^2}{2})) + (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2})) = (-R\alpha_o \sin(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2 \cos(\frac{\alpha_o t^2}{2}), R\alpha_o \cos(\frac{\alpha_o t^2}{2}) - R\alpha_o^2 t^2 \sin(\frac{\alpha_o t^2}{2})) = \vec{a}(t)$

# 3 | **Problem 3**

First let's start with the definition of a derivative:

assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can rewrite $\vec{A}(t)$ as:

$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$

We can plug this definition of $\vec{A}(t)$ into the deriviative equation:

$\vec{A}'(t) = \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t)\hat{i} + A_y(t+\Delta t)\hat{j} + A_z(t+\Delta t)\hat{k} - A_x(t)\hat{i} - A_y(t)\hat{j} - A_z(t)\hat{k}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t)\hat{i} - A_x(t)\hat{i} + A_y(t+\Delta t)\hat{j} - A_y(t)\hat{j} + A_z(t+\Delta t)\hat{k} - A_z(t)\hat{k}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{\hat{i}(A_x(t+\Delta t) - A_x(t)) + \hat{j}(A_y(t+\Delta t) - A_y(t)) + \hat{k}(A_z(t+\Delta t) - A_z(t))}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{A_x(t+\Delta t) - A_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j} + \lim_{\Delta t \to 0} \frac{A_z(t+\Delta t) - A_z(t)}{\Delta t}\hat{k}$
$= (\lim_{\Delta t \to 0} \frac{A_x(t+\Delta t) - A_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_z(t+\Delta t) - A_z(t)}{\Delta t})$
$= (\frac{dA_x(t)}{dt}, \frac{dA_y(t)}{dt}, \frac{dA_z(t)}{dt})$

# 4 | **Problem 4**

Again, starting with the definition of a derivative:

assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can define the $\vec{A}(t)$ and $\vec{B}(t)$ as:

$\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j}$

$\vec{B}(t) = B_x(t)\hat{i} + B_y(t)\hat{j}$

And thus:

$\frac{d}{dt}(\alpha\vec{A}(t) + \beta\vec{B}(t)) = \frac{d}{dt}(\alpha(A_x(t)\hat{i} + A_y(t)\hat{j}) + \beta(B_x(t)\hat{i} + B_y(t)\hat{j}))$
$= \frac{d}{dt}(\alpha A_x(t)\hat{i} + \alpha A_y(t)\hat{j} + \beta B_x(t)\hat{i} + \beta B_y(t)\hat{j})$
$= \lim_{\Delta t \to 0} \frac{\alpha A_x(t+\Delta t)\hat{i} + \alpha A_y(t+\Delta t)\hat{j} + \beta B_x(t+\Delta t)\hat{i} + \beta B_y(t+\Delta t)\hat{j} - \alpha A_x(t)\hat{i} - \alpha A_y(t)\hat{j} - \beta B_x(t)\hat{i} - \beta B_y(t)\hat{j}}{\Delta t}$
$= \lim_{\Delta t \to 0} \frac{\alpha A_x(t+\Delta t)\hat{i} - \alpha A_x(t)\hat{i} + \alpha A_y(t+\Delta)\hat{j} - \alpha A_y(t)\hat{j} + \beta B_x(t+\Delta t)\hat{i} - \beta B_x(t)\hat{i} + \beta B_y(t+\Delta t)\hat{j} - \beta B_y(t)\hat{j}}{\Delta t}$
$= \alpha \lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}\hat{i} + \alpha \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j} + \beta \lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}\hat{i} + \beta \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t}\hat{j}$
$= \alpha(\lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}\hat{j}) + \beta(\lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t}\hat{j})$
$= \alpha(\lim_{\Delta t \to 0} \frac{A_x(t+\Delta_t) - A_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{A_y(t+\Delta t) - A_y(t)}{\Delta t}) + \beta(\lim_{\Delta t \to 0} \frac{B_x(t+\Delta_t) - B_x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{B_y(t+\Delta t) - B_y(t)}{\Delta t})$
$= \alpha(\frac{dA_x(t)}{dt}, \frac{dA_y(t)}{dt}) + \beta(\frac{dB_x(t)}{dt}, \frac{dB_y(t)}{dt})$

$= \alpha \frac{d\vec{A}(t)}{dt} + \beta \frac{d\vec{B}(t)}{dt}$

# 5 | **Problem 5**

We can start with the definition of a derivative:

Assuming that $\vec{r}(t)$ is a generic vector function:

$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Next we can define $\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j}$

Therefore:

$\frac{d}{dt}(\vec{A}(u(t))) = \lim_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t)\hat{i} + A_y(u(t)+\Delta t)\hat{j} - A_x(u(t))\hat{i} - A_y(u(t))\hat{j}}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t)\hat{i} - A_x(u(t))\hat{i} + A_y(u(t)+\Delta t)\hat{j} - A_y(u(t))\hat{j}}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{A_x(u(t)+\Delta t) - A_x(u(t))}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{A_y(u(t)+\Delta t) - A_y(u(t))}{\Delta t}\hat{j}$

$= \left(\frac{dA_x(u(t))}{dt}, \frac{dA_y(u(t))}{dt}\right)$

Here we can apply the chain rule for normal functions (which we proved in single variable calculus)

$= (A_x'(u(t)) \cdot u'(t), A_y'(u(t)) \cdot u'(t))$

$= u'(t) \cdot (A_x'(u(t)), A_y'(u(t)))$

$= \frac{du(t)}{dt} \cdot (A_x'(u(t)), A_y'(u(t)))$

$= \frac{du(t)}{dt} \cdot \vec{A}'(u(t))$

$= \frac{du(t)}{dt} \frac{d\vec{A}(u)}{du}$

$= \frac{d\vec{A}(u)}{du} \frac{du(t)}{dt}$