1 | Axler6.45 orthogonal complement, $U^{\perp}$ def

if U is a subset of V, then the orthogonal complement of U, denoted $U^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^{\perp} = \{v \in V : \langle v, u \rangle = 0 \forall u \in U\}$$

1.1 | results

1.1.1 | Axler6.46 basic properties

1. complement is a subspace: if U is a subset of V, then $U^{\perp}$ is a subspace of V
   (a) zero is orthogonal to each vector, any vector that is the sum of two fully orthogonal vectors or the scalar multiple of an orthogonal vector will still be fully orthogonal.
2. $\{0\}^{\perp} = V$
   (a) zero orthogonal to every vector
3. $V^{\perp} = \{0\}$
   (a) only zero orthogonal to every vector
4. If U is a subset of V, then $U \cap U^{\perp} \subseteq \{0\}$
   (a) only zero is orthogonal to itself
5. If U and W are subsets of V and $U \subseteq W$ then $W^{\perp} \subseteq U^{\perp}$
   (a) Everything in $W^{\perp}$ is in $U^{\perp}$, and more.

1.1.2 | Axler6.47 direct sum of a subspace and its orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then,

$$V = U \oplus U^{\perp}$$

This can be shown by seeing that splitting any vector in V into a U part and a non-U part leads to the non-U being in $U^{\perp}$

1.1.3 | Axler6.50 dimension of orthogonal complement

Suppose V is finite-dimensional and U is a subpsace of V. Then,

$$U^{\perp} = V - U$$

By the dimension of a subspace addition (Axler3.78)

1.1.4 | Axler6.51 orthogonal complement of orthogonal complement is itself

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}$$

Because $U \oplus U^{\perp} = V$ is a direct sum and equals V.
The actual proof is by double-inclusion.