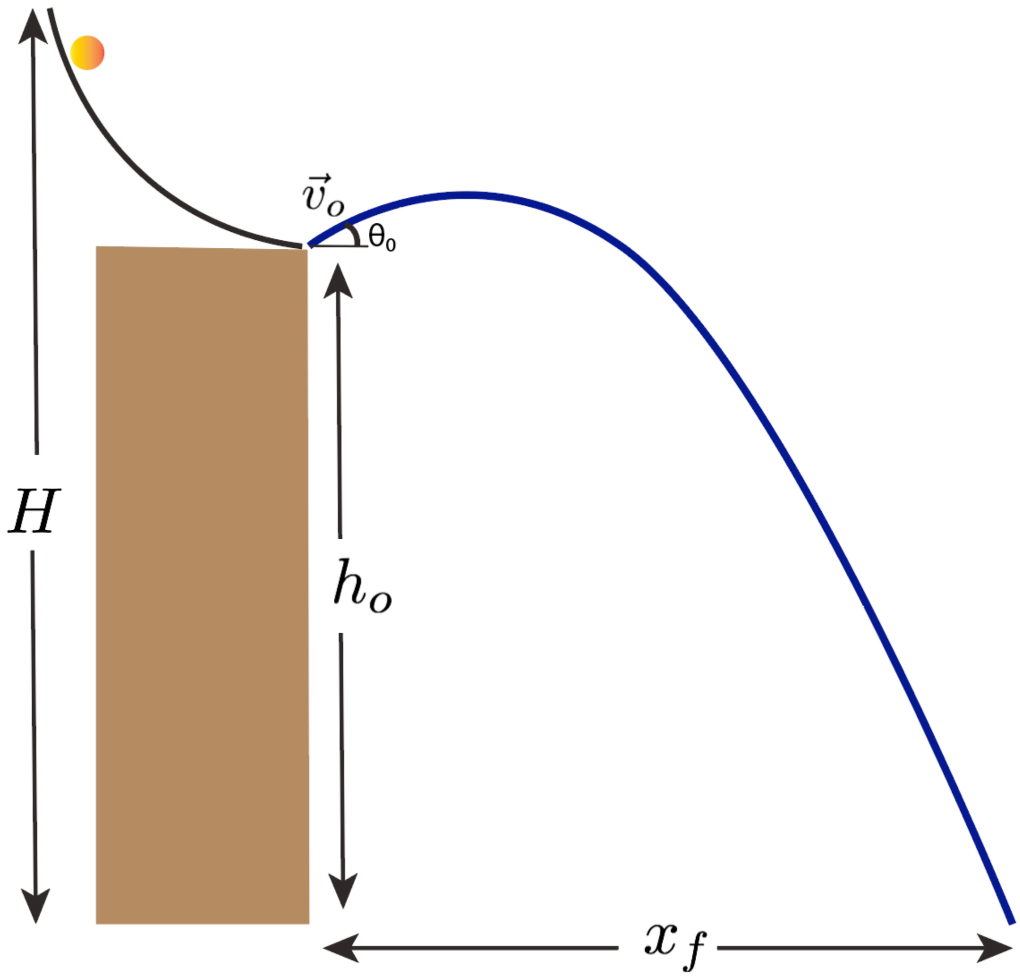

1 | **Setup**

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f.

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x-value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive-x, so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y, and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0, the moment the marble leaves the track and travels as a projectile as t_1, and the end — in the moment when the marble hits the ground — as t_f. We will call the marble m_0.

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0 g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0 g(H - h_0) = m_0 g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0 g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named $\vec{v_0}$.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 \tag{1}$$

$$g(H - h_0) = \frac{1}{2} \vec{v_0}^2 \tag{2}$$

$$2g(H - h_0) = \vec{v_0}^2 \tag{3}$$

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{4}$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} \vec{v_{0x}} = \sqrt{2g(H - h_0)} cos(\theta_0) \\ \vec{v_{0y}} = \sqrt{2g(H - h_0)} sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. $\vec{v_{0y}}$, $\vec{v_{0x}}$, h_0, m_0.

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f.

$$\begin{cases} x(t) = \frac{1}{2} a_{0x} t^2 + v_{0x} t + x_0 \\ y(t) = \frac{1}{2} a_{0y} t^2 + v_{0y} t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x} t \\ y(t) = \frac{-1}{2} g t^2 + v_{0y} t + h_0 \end{cases}$$

as…

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | **Solving for x_f**

The position equations above could be leveraged to figure a value for x_f. We first create a set of equations modeling the location of the marble at t_f.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{1}{2}g{t_f}^2 + t_f\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \end{cases}$$

To simplify calculations initially, we set $\sqrt{2g(H-h_0)}$ back as $\vec{v_0}$ for the ease of initial simplification.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\vec{v_0}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{-1}{2}g{t_f}^2 + t_f\vec{v_0}sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f, and supply it to the second equation.

$$t_f = \frac{-\vec{v_0}sin(\theta_0) \pm \sqrt{(\vec{v_0}sin(\theta_0))^2 + 2gh_0}}{-g} \tag{5}$$

Given that we know that time is positive in this setup, and subtracting a term will make it even more negative, we could safely ignore the $+$ term in the $\pm$ operator.

And, performing variable substitution upon the first equation…

$$\begin{aligned} x_f =& \frac{-\vec{v_0}sin(\theta_0)\vec{v_0}cos(\theta_0) - \vec{v_0}cos(\theta_0)\sqrt{(\vec{v_0}sin(\theta_0))^2 + 2gh_0}}{-g} && (6) \\ =& \frac{\frac{-1}{2}{\vec{v_0}}^2 sin(2\theta_0) - \vec{v_0}cos(\theta_0)\sqrt{(\vec{v_0}sin(\theta_0))^2 + 2gh_0}}{-g} && (7) \\ =& \frac{-{\vec{v_0}}^2 sin(2\theta_0)}{-2g} - \frac{\vec{v_0}cos(\theta_0)\sqrt{{\vec{v_0}}^2 sin^2(\theta_0) + 2gh_0}}{-g} && (8) \\ =& \frac{-\vec{v_0}cos(\theta_0)\sqrt{{\vec{v_0}}^2 sin^2(\theta_0) + 2gh_0}}{-g} - \frac{{\vec{v_0}}^2 sin(2\theta_0)}{-2g} && (9) \\ =& \frac{\vec{v_0}cos(\theta_0)\sqrt{{\vec{v_0}}^2 sin^2(\theta_0) + 2gh_0}}{g} + \frac{{\vec{v_0}}^2 sin(2\theta_0)}{2g} && (10) \\ & && (11) \end{aligned}$$

And finally, substituting back the $\vec{v_0}$ terms…

$$\begin{aligned} x_f =& \frac{\sqrt{2g(H-h_0)}cos(\theta_0)\sqrt{2g(H-h_0)sin^2(\theta_0) + 2gh_0}}{g} + \frac{2g(H-h_0)sin(2\theta_0)}{2g} && (12) \\ =& 2(\sqrt{H-h_0}cos(\theta_0)\sqrt{(H-h_0)sin^2(\theta_0) + h_0}) + (H-h_0)sin(2\theta_0) && (13) \\ =& 2(cos(\theta_0)\sqrt{(H-h_0)^2 sin^2(\theta_0) + (H-h_0)h_0}) + (H-h_0)sin(2\theta_0) && (14) \\ =& 2(cos(\theta_0)\sqrt{H^2 sin^2(\theta_0) - 2Hh_0 sin^2(\theta_0) + {h_0}^2 sin^2(\theta_0) + Hh_0 - {h_0}^2}) + (Hsin(2\theta_0) - h_0 sin(2\theta_0)) && (15) \end{aligned}$$

3.3 | **Optimizing for x_f**

This would *technically* be a multivariable calculus question. However, we elect to do the following: holding h_0 as constant, and optimizing for θ_0, and finally substituting the optimized result and derivation again.