

1 | Elastic collision

We are given that the object m_1 collides with the rod with velocity v_0 , and the rod is floating in free space. Given m_1 , v_0 , m_2 , I_0 , and r , we are to figure out the final velocity of m_1 after collision v_f , the velocity of m_2 after collision v_{CM} , and of course the rotation of the rod after collision ω .

We are assuming that this collision is elastic.

We have, then, for conservation of linear momentum:

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \quad (1)$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2} m_1 v_0^2 = \left(\frac{1}{2} m_1 v_f^2 \right) + \left(\frac{1}{2} m_2 v_{CM}^2 \right) + \left(\frac{1}{2} I_0 \omega^2 \right) \quad (2)$$

$$\Rightarrow m_1 v_0^2 = (m_1 v_f^2) + (m_2 v_{CM}^2) + (I_0 \omega^2) \quad (3)$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1 r^2 \left(\frac{v_0}{r} \right) = m_1 r^2 \left(\frac{v_f}{r} \right) + I_0 \omega \quad (4)$$

$$\Rightarrow m_1 r v_0 = m_1 r v_f + I_0 \omega \quad (5)$$

We now have a system of three equations that can be combined to solve for three unknowns v_f , v_{CM} , and ω .

We will begin with the kinetic energy expression:

$$m_1 v_0^2 = (m_1 v_f^2) + (m_2 v_{CM}^2) + (I_0 \omega^2) \quad (6)$$

Now, this last term seems a little concerning to deal with. Therefore, we will use the final expression to simplify it:

$$m_1 r v_0 = m_1 r v_f + I_0 \omega \quad (7)$$

$$\Rightarrow m_1 r v_0 - m_1 r v_f = I_0 \omega \quad (8)$$

and also:

$$\frac{m_1 r v_0 - m_1 r v_f}{I_0} = \omega \quad (9)$$

Therefore, $I_0 \omega^2$ would be:

$$\frac{(m_1 r v_0 - m_1 r v_f)^2}{I_0} \quad (10)$$

Supplying that back:

$$m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + (I_0 \omega^2) \quad (11)$$

$$\Rightarrow m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + \frac{(m_1 r v_0 - m_1 r v_f)^2}{I_0} \quad (12)$$

$$\Rightarrow m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + \frac{(r(m_1 v_0 - m_1 v_f))^2}{I_0} \quad (13)$$

$$\Rightarrow m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + \frac{r^2(m_1 v_0 - m_1 v_f)^2}{I_0} \quad (14)$$

$$\Rightarrow m_1 v_0^2 - m_1 v_f^2 = m_2 v_{CM}^2 + \frac{r^2(m_1 v_0 - m_1 v_f)^2}{I_0} \quad (15)$$

$$\Rightarrow m_1 I_0 v_0^2 - m_1 I_0 v_f^2 = m_2 I_0 v_{CM}^2 + r^2(m_1 v_0 - m_1 v_f)^2 \quad (16)$$

At this point, we realize that we haven't use the first expression yet. Therefore, we will endeavor to use it to rid of the one v_{CM} we have here.

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \quad (17)$$

$$\Rightarrow m_1 v_0 - m_1 v_f = m_2 v_{CM} \quad (18)$$

$$\Rightarrow \frac{m_1 v_0 - m_1 v_f}{m_2} = v_{CM} \quad (19)$$

Ok, so now, supplying all of this back:

$$m_1 I_0 v_0^2 - m_1 I_0 v_f^2 = m_2 I_0 v_{CM}^2 + r^2(m_1 v_0 - m_1 v_f)^2 \quad (20)$$

$$\Rightarrow m_1 I_0 v_0^2 - m_1 I_0 v_f^2 = m_2 I_0 \left(\frac{m_1 v_0 - m_1 v_f}{m_2} \right)^2 + r^2(m_1 v_0 - m_1 v_f)^2 \quad (21)$$

$$\Rightarrow m_1 I_0 v_0^2 - m_1 I_0 v_f^2 = m_2 I_0 \frac{(m_1 v_0 - m_1 v_f)^2}{m_2^2} + r^2(m_1 v_0 - m_1 v_f)^2 \quad (22)$$

$$\Rightarrow m_1 m_2 I_0 v_0^2 - m_1 m_2 I_0 v_f^2 = m_2 I_0 (m_1 v_0 - m_1 v_f)^2 + m_2 r^2 (m_1 v_0 - m_1 v_f)^2 \quad (23)$$

$$\Rightarrow m_1 m_2 I_0 v_0^2 - m_1 m_2 I_0 v_f^2 = m_2 I_0 ((m_1 v_0)^2 - 2((m_1)^2 v_f v_0) + (m_1 v_f)^2) \quad (24)$$

$$+ (m_1 v_f)^2) + m_2 r^2 ((m_1 v_0)^2 - 2((m_1)^2 v_f v_0) + (m_1 v_f)^2) \quad (25)$$

$$\Rightarrow I_0 v_0^2 - I_0 v_f^2 = m_1 I_0 (v_0^2 - 2v_f v_0 + v_f^2) + m_1 r^2 (v_0^2 - 2v_f v_0 + v_f^2) \quad (26)$$

$$\Rightarrow I_0 v_0^2 - I_0 v_f^2 = (m_1 I_0 v_0^2 - 2m_1 I_0 v_f v_0 + m_1 I_0 v_f^2) + (m_1 r^2 v_0^2 - 2m_1 r^2 v_f v_0 + m_1 r^2 v_f^2) \quad (27)$$

$$\Rightarrow -I_0 v_f^2 + 2m_1 I_0 v_f v_0 - m_1 I_0 v_f^2 + 2m_1 r^2 v_f v_0 - m_1 r^2 v_f^2 = (m_1 I_0 v_0^2) + (m_1 r^2 v_0^2) - I_0 v_0^2 \quad (28)$$

$$\Rightarrow -(I_0 + m_1 I_0 + m_1 r^2) v_f^2 + 2m_1 v_0 (I_0 + r^2) v_f = (m_1 I_0 v_0^2) + (m_1 r^2 v_0^2) - I_0 v_0^2 \quad (29)$$

$$\Rightarrow -(I_0 + m_1 I_0 + m_1 r^2) v_f^2 + 2m_1 v_0 (I_0 + r^2) v_f = (m_1 I_0 + m_1 r^2 - I_0) v_0^2 \quad (30)$$

$$\Rightarrow -(I_0 + m_1 I_0 + m_1 r^2) v_f^2 + 2m_1 v_0 (I_0 + r^2) v_f - (m_1 I_0 + m_1 r^2 - I_0) v_0^2 = 0 \quad (31)$$

There are probably more elegant ways to do this, but now, its quadratic equation time!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (32)$$

$$\begin{cases} a = -(I_0 + m_1 I_0 + m_1 r^2) \\ b = 2m_1 v_0 (I_0 + r^2) \\ c = -(m_1 I_0 + m_1 r^2 - I_0) v_0^2 \\ x = v_f \end{cases} \quad (33)$$

Let's tackle that middle term first.

$$b^2 = (2m_1v_0(I_0 + r^2))^2 \quad (34)$$

$$= 4m_1^2v_0^2(I_0 + r^2)^2 \quad (35)$$

$$= 4m_1^2v_0^2(I_0^2 + 2I_0r^2 + r^4) \quad (36)$$

$$= (4m_1^2v_0^2I_0^2 + 8m_1^2v_0^2I_0r^2 + 4m_1^2v_0^2r^4) \quad (37)$$

and,

$$4ac = 4(I_0 + m_1I_0 + m_1r^2)(m_1I_0 + m_1r^2 - I_0)v_0^2 \quad (38)$$

$$= 4v_0^2((m_1I_0^2 + m_1I_0r^2 - I_0^2) + ((m_1I_0)^2 + m_1^2I_0r^2 - m_1I_0^2) + (m_1^2r^2I_0 + (m_1r^2)^2 - m_1I_0r^2)) \quad (39)$$

$$= 4m_1^2v_0^2I_0^2 + 8m_1^2v_0^2I_0r^2 + 4m_1^2v_0^2r^4 - 4I_0^2v_0^2 \quad (40)$$

Therefore, subtracting the two terms, we will note that they are only different by:

$$b^2 - 4ac = 4I_0^2v_0^2 \quad (41)$$

Finally, then; performing the square root:

$$\sqrt{b^2 - 4ac} = 2I_0v_0 \quad (42)$$

Subtracting negative b to this expression:

$$- 2m_1v_0(I_0 + r^2) - 2I_0v_0 \quad (43)$$

$$\Rightarrow - 2m_1v_0I_0 - 2I_0v_0 - 2m_1v_0r^2 \quad (44)$$

And finally, dividing by $2a$:

$$\frac{-2m_1v_0I_0 - 2I_0v_0 - 2m_1v_0r^2}{-2(I_0 + m_1I_0 + m_1r^2)} = \frac{m_1v_0I_0 + I_0v_0 + m_1v_0r^2}{(I_0 + m_1I_0 + m_1r^2)} \quad (45)$$

$$= v_0 \quad (46)$$

Ok, I suppose this is a very valid solution, but not a very exciting one ("nothing happened"). Let's rewind and add negative b instead.

Subtracting negative b to this expression:

$$- 2m_1v_0(I_0 + r^2) + 2I_0v_0 \quad (47)$$

$$\Rightarrow - 2m_1v_0I_0 - 2I_0v_0 + 2m_1v_0r^2 \quad (48)$$

And finally, dividing by $2a$:

$$\frac{-2m_1v_0I_0 - 2I_0v_0 + 2m_1v_0r^2}{-2(I_0 + m_1I_0 + m_1r^2)} = \frac{m_1v_0I_0 + I_0v_0 - m_1v_0r^2}{(I_0 + m_1I_0 + m_1r^2)} \quad (49)$$

$$= \frac{(m_1I_0 + I_0 - m_1r^2)v_0}{(I_0 + m_1I_0 + m_1r^2)} \blacksquare \quad (50)$$

Phew. Now, we simply have to leverage the top expression to figure the rest of the expressions.

We have that:

$$\frac{m_1 v_0 - m_1 v_f}{m_2} = v_{CM} \quad (51)$$

from the previous figuring of v_f , therefore:

$$\frac{m_1 v_0 - m_1 v_f}{m_2} = v_{CM} \quad (52)$$

$$\Rightarrow \frac{m_1 v_0 - m_1 \frac{(m_1 I_0 + I_0 - m_1 r^2) v_0}{(I_0 + m_1 I_0 + m_1 r^2)}}{m_2} = v_{CM} \quad (53)$$

$$\Rightarrow m_1 v_0 - m_1 \frac{(m_1 I_0 + I_0 - m_1 r^2) v_0}{(I_0 + m_1 I_0 + m_1 r^2)} = m_2 v_{CM} \quad (54)$$

$$\Rightarrow \frac{m_1 v_0 (I_0 + m_1 I_0 + m_1 r^2)}{(I_0 + m_1 I_0 + m_1 r^2)} - \frac{m_1 v_0 (m_1 I_0 + I_0 - m_1 r^2)}{(I_0 + m_1 I_0 + m_1 r^2)} = m_2 v_{CM} \quad (55)$$

$$\Rightarrow \frac{2m_1 v_0 m_1 r^2}{(I_0 + m_1 I_0 + m_1 r^2)} = m_2 v_{CM} \quad (56)$$

$$\Rightarrow v_{CM} = \frac{2m_1 v_0 m_1 r^2}{m_2 (I_0 + m_1 I_0 + m_1 r^2)} \blacksquare \quad (57)$$

We finally have, to figure ω :

$$m_1 r v_0 - m_1 r v_f = I_0 \omega \quad (58)$$

Therefore:

$$m_1 r v_0 - m_1 r v_f = I_0 \omega \quad (59)$$

$$\Rightarrow \frac{r(m_1 v_0 - m_1 v_f)}{I_0} = \omega \quad (60)$$

$$\Rightarrow (m_1 v_0 - m_1 v_f) = \frac{I_0}{r} \omega \quad (61)$$

From halfway through the expression block before, we have figured this result, that:

$$(m_1 v_0 - m_1 v_f) = \frac{2m_1 v_0 m_1 r^2}{(I_0 + m_1 I_0 + m_1 r^2)} \quad (62)$$

Therefore, we have finally that:

$$\omega = \frac{2m_1 v_0 m_1 r}{(I_0 + m_1 I_0 + m_1 r^2) I_0} \blacksquare \quad (63)$$

And we are done.

2 | Rigid Body Kinetic Energy

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2} m_i v_i^2 \quad (64)$$

Because of the fact a point v_i can be defined as a sum of the velocity from the origin plus the displace from origin ($v_i = v_{CM} + v'_i$), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM} + v'_i)(V_{CM} + v'_i) \quad (65)$$

Now, we shall foil the above expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM}^2 + 2v_{CM}v'_i + v_i'^2) \quad (66)$$

$$= \sum_i \frac{1}{2} m_i V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (67)$$

$$= \frac{1}{2} M V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (68)$$

$$= \frac{1}{2} M V_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (69)$$

At which point, we realize that we have in the middle arrived at the definition of the center of mass in the reference frame of the center of mass—meaning that it is indeed 0 because the center of mass is at the origin of the center of mass. Moving on, then:

$$KE = \frac{1}{2} M V_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (70)$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i v_i'^2 \quad (71)$$

We now note that this is a rigid body. Therefore, each point mass will not "slip" from the center of mass—they remain the same distance relative each other. Therefore, we can leverage the no slipping assumption to claim that $v'_i = r'_i \omega$.

$$KE = \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i (r'_i \omega)^2 \quad (72)$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i r_i'^2 \omega^2 \quad (73)$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \omega^2 \sum_i m_i r_i'^2 \quad (74)$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} I \omega^2 \blacksquare \quad (75)$$