# 1 | Implicit Differentiation

### unit1::derivatives

### 1.1 | David's Summary

This is how I understand implicit differentiation.

Say you want to take a derivative of an implicit function like  $x^2 + y^2 = 3$ .

- 1. Take the derivative of everything with respect to x:  $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}3$  With a little simplification this is just  $2x + \frac{d}{dx}y^2 = 0$ .
- 2. Cleverly apply the chain rule to get  $\frac{d}{dx}y^2$ . Chain rule states that  $\frac{d}{du}\frac{du}{dx}=\frac{d}{dx}$ . Define  $u=y^2$ . By chain rule  $\frac{du}{dy}\frac{dy}{dx}=\frac{du}{dx}$ .
- 3. Our formula is now  $2x+(2y)\frac{dy}{dx}=0$ . Time for some algebra!  $\frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$

### 1.2 | Initial Example

Technique based on the Chain Rule that allows diffrentiation of more functions.

**EXAMPLE**  $\frac{d}{dx}x^a=ax^{a-1}$  This holds true for  $a=0,\pm 1,\pm 2...$  What about fractional powers? Take  $a=\frac{m}{n}$   $y=x^{\frac{m}{n}}$  or  $y^n=x^m$  We can apply derivative to equation 2 because the methods of diffrentiating the fractional exponent is unknown to us.

$$\begin{array}{l} \frac{d}{dx}y^n=\frac{d}{dx}x^m\\ \left(\frac{d}{dy}y^n\right)\frac{dy}{dx}=\frac{d}{dx}x^m \text{ or ... } ny^{n-1}\frac{dy}{dx}=mx^{m-1}\\ \text{$dy$}_{\overline{dx=}} \end{array}$$

## 1.3 | Another Example

 $x^2+y^2=1$  is an implicit function, explicitly it is  $y=\pm\sqrt{1-x}$  (for convienience limit to positives for now). Solving it explicitly:  $y'=\frac{1}{2}()^{-1/2}(-2x)$  NOTE:  $\frac{1}{2}()^{-1/2}=\frac{d}{d()}()^{-1/2}$ 

Or implicitly:

- Diffrentiate function in the form  $x^2 + y^2 = 1$ .
- $\frac{d}{dx}(x^2+y^2=1)$  or 2x+2yy'=0
- Solve for y' which is  $\frac{-2x}{2y} = \frac{-x}{y}$  (solve algebraically).

Compare  $\frac{-x}{y}$  to explicit solution  $\frac{1}{2}(1-x^2)^{-1/2}(-2x)=\frac{-x}{\sqrt{1-x^2}}$  and find they are the same as  $y=\sqrt{1-x^2}$ .

### 1.4 | A Trickier Example

$$y^4 + xy^2 - 2 = 0$$

- One can solve it explicitly by using the quadratic equation.
- Implicitly one can apply the product rule and the previous examples to diffrentiate this function.
- Writeup is left as an exercise for the reader.

Taproot • 2021-2022 Page 1

#### 1.5 | Derivatives of Inverse Functions

**EXAMPLE** 
$$y=\sqrt{x}, x>0, y^2=x$$
  $f(x)=\sqrt{x}, g(y)=x, g(y)=y^2$  NOTE: If  $y=f(x)$  and  $g(y)=x$ ,  $g(f(x))=x$ 

STATEMENT Implicit differentiation allows computing derivatives of any inverse function provided we know the derivative of the function.

**EXAMPLE**  $y = \tan^{-1} x$  and we'll use the equation  $\tan y = x$ 

- Note that inverse functions are the function reflected over the line x=y.
- Recall that derivative of tangent is  $\frac{d}{dy}\frac{\sin y}{\cos y}=\frac{1}{\cos^2 y}=\sec^2 y.$
- $\frac{d}{dy}\tan y = 1$  or  $\sec^2 y * y' = 1$ .
- $y' = \cos^2 y$  which leads to  $\frac{d}{dx} \tan^{-1} x = \cos^2 (\tan^{-1} x)$ .
  - Too complicated!
- Modelling as a right triangle and simplifying more yields  $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ .

### 2 | **Links**

Other techniques for differentiation as well as the topic of logarithms are covered in Exponentials and Logarithms. Further review can be found in MIT SVC Exam Review (Unit 1).

Taproot • 2021-2022 Page 2