1 | Problem

Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V,w)$ and $T \in \mathcal{L}(U,V)$. Prove that

 $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$

2 | **Proof**

All vectors $v \in \text{null } ST$ must have been nulled by T or S, and therefore either it must be in null T or Tv in range $T \cap \text{null } S$. Notationally,

$$\mathsf{null}\ ST = \mathsf{null}\ T \cup \{v : Tv \in (\mathsf{range}\ T \cap \mathsf{null}\ S)\}$$

Note that because this union is equal to null ST, it is a vector space. Because no vector can be in both null T and $\{v: Tv \in (\text{range } T \cap \text{null } S)\}$, the dimension of the union is

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\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim (\{v : Tv \in (\operatorname{range} T \cap \operatorname{null} S)\})
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Every value of w that satisfies $w \in (\text{range } T \cap \text{null } S)$ will the output of Tv for some v, because the range is defined as all the outputs of Tv.

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\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim (\operatorname{range} T \cap \operatorname{null} S)
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An intersection can only make the dimension of a set smaller, so dim (range $T \cap \text{null } S$) $\leq \text{dim null } S$ and

 $\dim \operatorname{range} ST \leq \dim \operatorname{null} S, \dim \operatorname{null} T$

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