1 | **Problem 1**

The plane that passes through the vector $\vec{P}_o$ and is perpendicular to $\vec{n}$ is defined by:

$$\{\vec{r} : (\vec{r} - \vec{P}_o) \cdot \vec{n} = 0, \vec{P}_o \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

This works because:

- The $(\vec{r} - \vec{P}_o)$ term is similar to when you subtract some value from x (in a cartisian plane) to shift it over to the right. In a senes, here you are subtracting the postition vector from every single vector on the plane defined by $\vec{r}$, thus shifting the plane to $\vec{P}_o$.
- Setting the dot product between the term above and $\vec{n}$ to 0, ensures that the plane is perpendicular to the normal vector $\vec{n}$

2 | **Problem 2**

First we can defined:

- $\vec{n} = (n_x, n_y, n_z)$
- $\vec{P}_o = (P_{ox}, P_{oy}, P_{oz})$
- $\vec{r} = (x, y, z)$

Then we can evaluate: $(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$:
$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$
so we see that the cartisian definition of a plane is: $xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$
From this we see:

- $A = n_x$
- $B = n_y$
- $C = n_z$
- $D = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$

Therefore, the normal vector is (A, B, C)

3 | **Problem 3**

First we can define:

- $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle$
- $\vec{r} = \langle x, y, z \rangle$

Then we can evaluate the equation:

$\hat{n} \cdot \vec{r} = D$
$\Rightarrow \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle \cdot \langle x, y, z \rangle = D$
$\Rightarrow \frac{n_x}{|\vec{n}|} x + \frac{n_y}{|\vec{n}|} y + \frac{n_z}{|\vec{n}|} z = D$

This is the equation for a cartisian definition of a plane, with $\hat{n}$ representing the normal vector and D representing $\frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$ which is the distance from the origin to the plane.

The value of D is found by starting with the orginal vector definition of a plane:

$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow \frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow \vec{r} \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow D = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

Looking at problem 4 we see that this is the distance from the plane to the origin.

4 | **Problem 4**

We can start with this drawing:

IMAGE

In the image we see that there are multple "point $\vec{P}_o$'s" that go from point P_o to the plane. We are trying to solve for d which is the shortest distance from the plane to the point, it can also be defined as the length of the $\vec{P}_o$ that is perpendiclar to the plane. Because all of the $\vec{P}_o$'s come from the same point and $\vec{n}$ is perpendicular to the plane we can find d by finding $comp_{\hat{n}} \vec{P}_o$:

$d = comp_{\hat{n}} \vec{P}_o = 1 \cdot |\vec{P}_o| \cos(\theta)$
$= |\hat{n}||\vec{P}_o| \cos(\theta)$
$= \vec{P}_o \cdot \hat{n}$
$= \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

From problem 2 we know that $\vec{P}_o \cdot \vec{n} = 4$ and that $\vec{n} = \langle 1, 2, 3 \rangle$ and thus:

$$d = \frac{4}{\sqrt{14}}$$

5 | **Problem 5**

We can do something similar to what was done in the problem above in which the $\hat{n}$ component of $\vec{P}_o$ can be used as the distance d, but because $\vec{P}_o$ is center at the origin and the plane may not pass through the origin, the postion vector of the plane, we'll call it $\vec{P}_1$, has to be subtracted from $\vec{P}_o$. To find the length of $\vec{P}_1$ we can do what we did in problem 4.

We'll break this problem into three parts: 1) finding the length of $\vec{P}_o$ parallel to $\hat{n}$ 2) findng the distance of $\vec{P}_1$ parallel to $\hat{n}$ 3) subtracting the two:

5.1 | Finding length of $\vec{P}_o$

From problem 2 we know that $\vec{n} = \langle A, B, C\rangle$. From problem 4 we know that we can find the length of $\vec{P}_o$ parallel to $\hat{n}$ by:

$\vec{P}_o \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$= \frac{\langle x_o, y_o, z_o\rangle \cdot \langle A,B,C\rangle}{|\langle A,B,C\rangle|}$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2+B^2+C^2}}$

5.2 | Finding the length of $\vec{P}_1$

From problem 2 we know that $\vec{n} = \langle A, B, C\rangle$. From problem 4 we know that we can find the length of $\vec{P}_1$ parallel to $\hat{n}$ by:

$$\vec{P}_1 \cdot \hat{n} = \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$$

From problem 2 we know that the dot product between the position vector and the noraml vector is D. Thus:

length of$\vec{P}_1 = \vec{P}_1 \cdot \hat{n}$
$= \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$
$= \frac{D}{|\vec{n}|}$
$= \frac{D}{|\langle A,B,C\rangle|}$
$= \frac{D}{\sqrt{A^2+B^2+C^2}}$

5.3 | Subtratcting the two:

$d = \vec{P}_o - \vec{P}_1$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2+B^2+C^2}} - \frac{D}{\sqrt{A^2+B^2+C^2}}$
$= \frac{Ax_o + By_o + Cz_o - D}{\sqrt{A^2+B^2+C^2}}$

Lastly, you would take the absolute value of the numorator because distances are positive (and the denominator is already positive due to the squaring of A, B, and C). Thus:

$$d = \frac{|Ax_o + By_o + Cz_o - D|}{\sqrt{A^2+B^2+C^2}}$$

6 | Problem 6

First we can define:

- $\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$
- $\vec{B}(t) = B_x(t)\hat{i} + B_y(t)\hat{j} + B_z(t)\hat{k}$

Thus:

$\frac{d}{dt}(\vec{A}(t) \cdot \vec{B}(t)) = \frac{d}{dt}(A_x(t)B_x(t) + A_y(t)B_y(t) + A_z(t)B_z(t))$
$= \frac{d}{dt}(A_x(t)B_x(t)) + \frac{d}{dt}(A_y(t)B_y(t)) + \frac{d}{dt}(A_z(t)B_z(t))$

$= A'_x(t)B_x(t) + A_x(t)B'_x(t) + A'_y(t)B_y(t) + A_y(t)B'_y(t) + A'_z(t)B_z(t) + A_z(t)B'_z(t)$
$= (A'_x(t)B_x(t) + A'_y(t)B_y(t) + A'_z(t)B_z(t)) + (A_x(t)B'_y(t) + A_y(t)B'_y(t) + A_z(t) + B'_z(t))$
$= \vec{A}'(t) \cdot \vec{B}(t) + \vec{A}(t) \cdot \vec{B}'(t)$
$= \frac{d\vec{A}(t)}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}(t)}{dt}$

7 | Problem 7

First we can define:

- $\vec{r} = |\vec{r}|\hat{r}$
  - this works because the unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, and so you can multiply both sides by $|\vec{r}|$

We can start by finding the derivative of $\vec{r}$ with the definition above:

$\frac{d}{dt}\vec{r}(t) = \frac{d}{dt}(|\vec{r}(t)|\hat{r}(t))$
$= (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$

We can take this equation and solve for $\frac{d}{dt}|\vec{r}(t)|$:

$\frac{d}{dt}\vec{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \hat{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) \cdot \hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t)) \cdot \hat{r}(t)$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \frac{\vec{r}(t)}{|\vec{r}(t)|} = (\frac{d}{dt}|\vec{r}(t)|) \cdot 1 + |\vec{r}(t)| \cdot 0$
$\Rightarrow \frac{1}{|\vec{r}(t)|}\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}|\vec{r}(t)|$

This proof relied on the fact that the dot product between a vector and it's derivative is zero. This is true because:

$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = C$, where C is some constant.

If we differentiate this then we get:

$\frac{d}{dt}\vec{r}(t)^2 = \frac{d}{dt}C$
$\Rightarrow 2\vec{r}(t) \cdot \vec{r}'(t) = 0$
$\Rightarrow \vec{r}(t) \cdot \vec{r}'(t) = 0$

The chain rule appiles to vectors (show in a different assignment.

Note: for this problem, I got Albert's help with the initial proof, and then I used this source to help with proving that the dot product between a vector and it's derivative is zero: `https://www.reddit.com/r/askmath/comments/aticz8/why_is_a_vector_of_constant_magnitude_always/?scrlybrkr=5f034675`.

8 | Problem 8

If we start with the vector equation for a 3D line we get:

$$(x, y, z) = (x_o, y_o, z_o) + t(a, b, c) = (x_o + ta, y_o + tb, z_o + tc)$$

Where (x_o, y_o, z_o) is the position vector of the line (the line passes through this point), and (a, b, c) is the direction in which the line is traveling.

We can take the x, y and z components of the vector form and convert them into the parametric form of the equation of a 3D line:

$(x, y, z) = (x_o + ta, y_o + tb, z_o + tc)$
$\Rightarrow x = x_o + ta$

$\Rightarrow y = y_o + tb$
$\Rightarrow z = z_o + tc$

Lastly, we can solve each of the parameterized components for t and set them equal to each other to get the symmetric form:

$\Rightarrow t = \frac{x-x_o}{a}$
$\Rightarrow t = \frac{y-y_o}{b}$
$\Rightarrow t = \frac{z-z_o}{c}$
$\Rightarrow \frac{x-x_o}{a} = \frac{y-y_o}{b} = \frac{z-z_o}{c}$

With this in mind we can look at the given symmetric equation:

$\frac{x-2}{2} = \frac{y-1}{3} = 2 - z$
$\Rightarrow \frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{-1}$

From this we see that the position vector is $(2, 1, 2)$ and the direction of the line is the same direction as $(2, 3, -1)$

Note: for this problem I used this source to see what the different forms of the equation for a 3D line: but I proved that the vector form equaled the symmetric form myself: `https://math.stackexchange.com/questions/404440/what-is-the-equation-for-a-3d-line`.