:ROAM$_{\text{REFS}}$: `https://docs.google.com/presentation/d/1rpqDXysh8GJJodXCTbWCj4MmIx87mT_ksyMqHoq-hkk/edit` `https://docs.google.com/presentation/d/1rpqDXysh8GJJodXCTbWCj4MmIx87mT_ksyMqHoq-hkk/edit`

1 | Lecture 1 flow

- A vector space is an algebraic structure like a *group* or *field*.
- $\mathbb{R}$ is the field of real numbers.
- A group is a set and an operation (a two-to-one mapping) where the inputs and outputs are within the set

You can represent a function with inputs/outputs by using the rows as the number of outputs and the columns as the number of inputs. Here, that would mean a 2x1 matrix.

- Groups do not need to be commutative.
- With a second operation, the notion of being *distributive* is introduced.
- Rings are groups with a second operation, but the second operation doesn't have all that the first one does.
  - Integers are an example: group under addition, and then they have multiplication which has identities but not inverses
- Fields are sets that are groups with respect to both operations (i.e. $\mathbb{R}$)
- Field extensions: i.e. $\mathbb{R}[\sqrt{-1}]$, which is real numbers and real numbers as coefficients of i (complex numbers).
  - Vectors can technically be thought of as field extension - since complex numbers are it means you can represent numbers in $\mathbb{R}^2$. This is one possible interpretation of vector multiplication: it's just like multiplying complex numbers.

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad - bc \end{bmatrix}$$
$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad - bc)\sqrt{1}$$

- All of this niceness is only really true for 2 dimensions, although it'll work for 4, 8, and 16 (although without zero divisors) as well.
- Even though this is great, it isn't generalizable, and this was the motivation for vector spaces.
- w:

2 | Lecture 2 flow

2.1 | Linear Combinations

Linear combinations are part of why we call it *linear* algebra.

A linear combination of a list $v_1, ..., v_n$ of vectors in a vector space V is a vector of the form:

$$a_1 v_1 + ... + a_n v_n \tag{1}$$

where $a_1, ..., a_n \in$ **F**.

2.2 | **Spans**

The set of all linear combinations of a list of vectors $v_1, ..., v_n$ in V is called the *span* of $v_1, ..., v_n$. Usually denoted is span$(v_1, ..., v_n)$.

$$\text{span}(v_1, ..., v_n) = \{a_1v_1 + ... + a_nv_n : a_1...a_n \in \mathbf{F}\} \tag{2}$$

Altenratively, the span is the smallest subspace that contains the vectors $v_1, ..., v_n$.

One common question associated with spans is what is *in* a span. Solving equations like following can be challenging:

$$(13, -1, 6) = a_1(2, 1, -1) + a_2(1, -2, 4) \tag{3}$$

What would happen if we added a third vector? It'd be like having three equations and three unknowns, and while intuition makes it seem like its always solvable there are special cases where it isn't.

This can be solved in a easier fashion than guessing and checking through computational methods, but we're focusing on abstract understanding for now.

2.3 | **Linear Independence**

A list $v_1, ..., v_n$ of vectors in V is *linearly independent* if the only choice of coefficients to make $a_1v_1 + ... + a_nv_n$ equal 0 is $a_1 = ... = a_n = 0$. Alternatively. this means that there is no way for these vectors to cancel each other out. n A list $v_1, ..., v_n$ of vectors in V is *linearly dependent* if they are not linearly independent: a.k.a. there is a unique (i.e not everything is 0) choice of $a_1, .., .a_n$ such that $a_1v_1 + ... + a_nv_n = 0$. Alternatively, this means there is a way for these vectors to cancel each other out.

Suppose $v_1, ..., v_n$ is a linearly dependent list in V. There exists a list $j \in \{1, 2, ..., m\}$ such that the following hold:

- $v_j \in$ span$(v_1, ..., v_{j-1}$
- if the jth term is removed from the list , the span of the remaining list equals the span of $v_1, ..., v_n$

Essentially, if an element of the list is a linear combination of other elements in a list you can remove it (since it doesn't contribute to the span).

Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

$$\begin{aligned}
a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 \text{ only when } a_1 = a_2 = a_3 = a_4 = 0 \\
\text{Suppose } b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4(v_4) = 0 \\
b_1v_1 - b_1v_2 + b_2v_2 - b_2v_3 + b_3v_3 - b_3v_4 + b_4v_4) = 0 \\
b_1v_1 - (b_1 - b_2)v_2 - (b_2 - b_3)v_3 - (b_3 - b_4)v_4 = 0 \\
\text{One of } (b_1 - b_2), (b_2 - b_3), (b_3 - b_4) \neq 0 \\
\text{so } v_1, v_2, v_3, v_4 \text{ is linearly dependent.}
\end{aligned}$$

2.4 | **Bases**

Set of polynomials with coefficients in **F** and degree of most n.

A *basis* of V is a set of vectors in V that are linearly independent and spans V. Bases are useful because they give a unique representation of a vector as a linear combination of bases.

Example: $(0,0,1),(0,1,0),(1,0,0)$ is a basis of $\mathbf{F}^3$. Example: The list $(1-1,0),(1,0,-1)$ is a basis of $\{(x,y,z) \in \mathbf{F}^3 : x+y+z=0\}$ (sidenote: this is actually a valid subspace because it forms a plane $z=-x-y$).