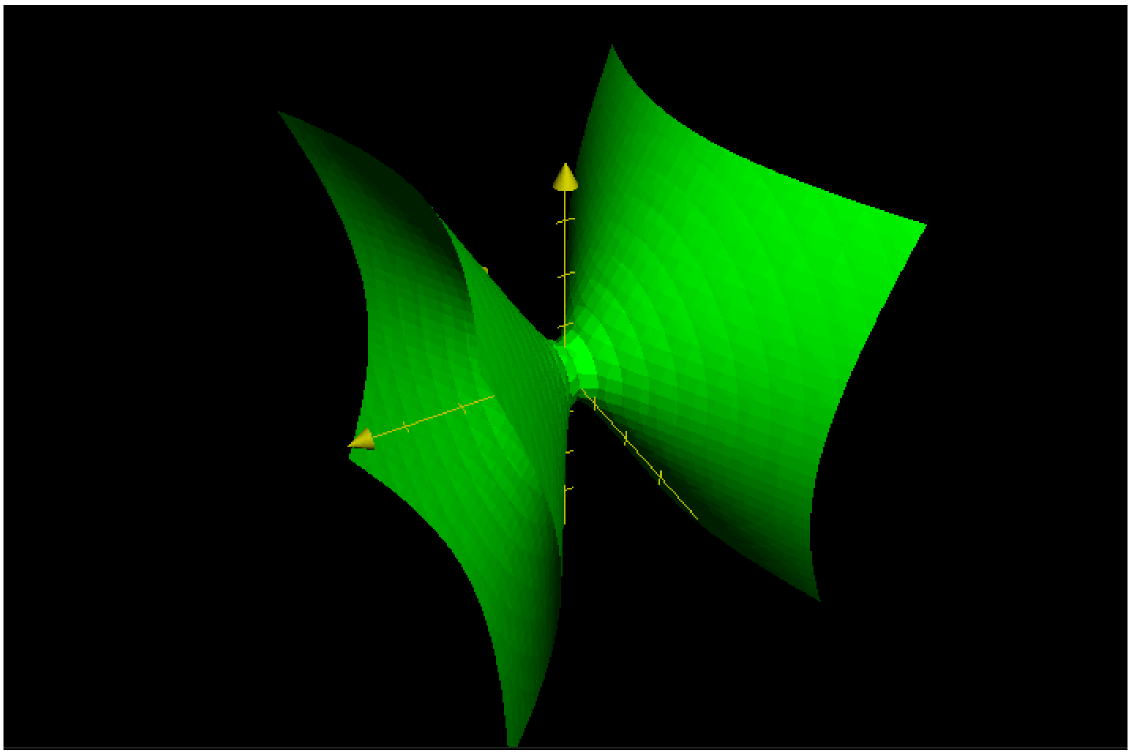
1 | Plane Tangent to a Surface

Here's a surface:

$$x^2 - 3y^2 + z^2 = 7 \tag{1}$$

1.1 | Graph It!



1.2 | Find the Equation of the Plane Tangent to it at $(1, 1, 3)$

We will first establish an equation for the statement w.r.t. z:

$$x^2 - 3y^2 + z^2 = 7 \tag{2}$$
$$\Rightarrow z^2 = 7 - x^2 + 3y^2 \tag{3}$$
$$\Rightarrow z = \sqrt{7 - x^2 + 3y^2} \tag{4}$$

Then, we find the components of the slope in each of the dimensions.

$$\frac{\partial}{\partial x}\sqrt{7 - x^2 + 3y^2} \tag{5}$$
$$\Rightarrow \frac{-2x}{2\sqrt{7 - x^2 + 3y^2}} \tag{6}$$
$$\Rightarrow \frac{-x}{\sqrt{7 - x^2 + 3y^2}} \tag{7}$$

$$\frac{\partial}{\partial y}\sqrt{7-x^2+3y^2} \tag{8}$$
$$\Rightarrow \frac{6y}{2\sqrt{7-x^2+3y^2}} \tag{9}$$
$$\Rightarrow \frac{3y}{\sqrt{7-x^2+3y^2}} \tag{10}$$

Therefore, at point $(1,1,3)$, the gradient is as follows

$$\begin{bmatrix} \frac{-1}{3} \\ 1 \end{bmatrix} \tag{11}$$

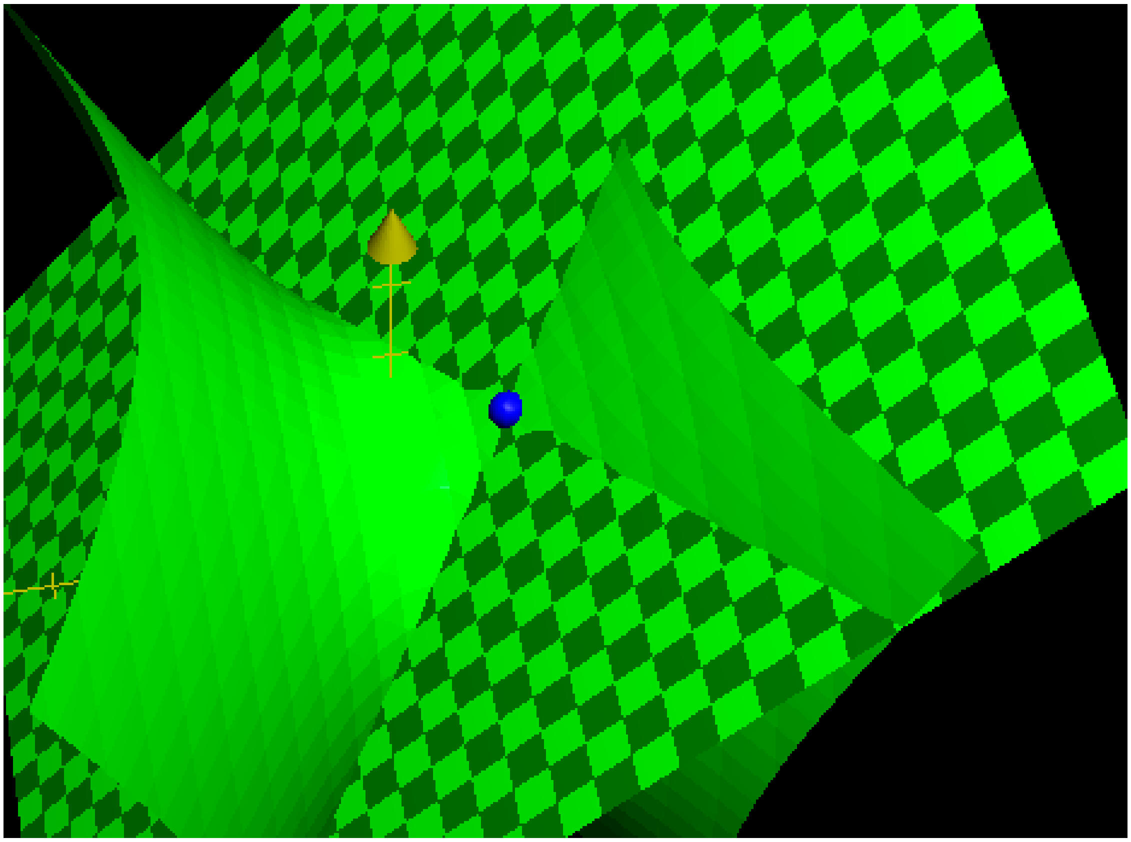
The basic equation for the plane, then, would be:

$$z = \frac{-1}{3}x + 1y + b \tag{12}$$

As the plane is tangent to $(1,1,3)$, we will supply these values and calculate the necessary constant.

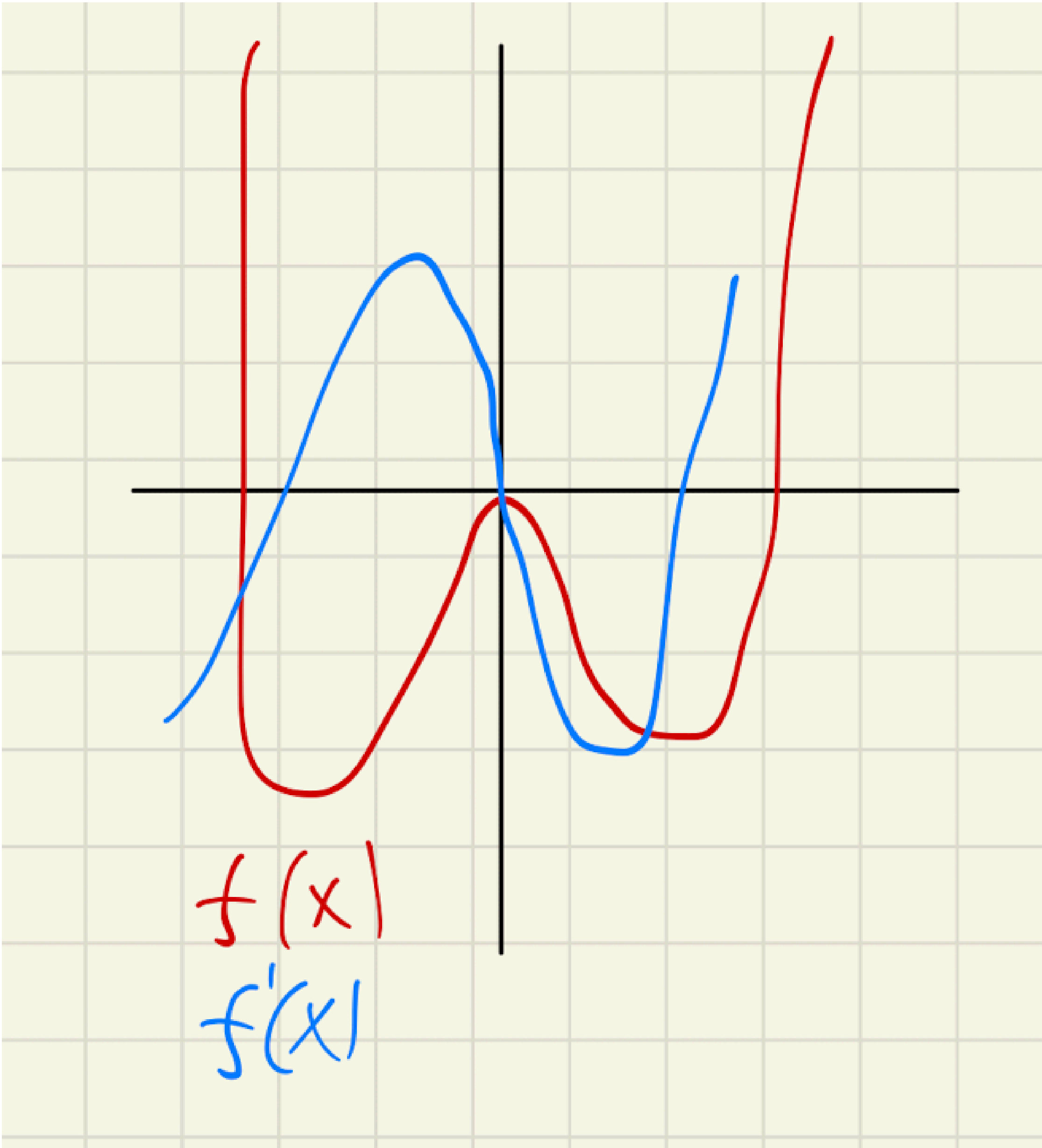
$$3 = -\frac{1}{3} + 1 + b \tag{13}$$
$$\Rightarrow 3 = \frac{2}{3} + b \tag{14}$$
$$\Rightarrow b = \frac{7}{3} \tag{15}$$

1.3 | Graph the surface and its tangent plane!



2 | Optimization Functions

2.1 | $f(x) = x^4 - 9x^2$



We first will figure the first and seconds derivatives of this function to check and verify its critical points.

$$f'(x) = 4x^3 - 18x \tag{16}$$

$$f''(x) = 12x^2 - 18 \tag{17}$$

To figure the critical points (maxima, minima, inflection), we first solve for the points in which $f'(x)$ is 0.

$$\begin{aligned} 0 &= 4x^3 - 18x && (18) \\ \Rightarrow 0 &= x(4x^2 - 18) && (19) \\ \Rightarrow x &= \{0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\} && (20) \end{aligned}$$

We then figure the second derivatives of the function at these two points.

$$f''(\{0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\}) = \{-18, 36, 36\} \quad (21)$$

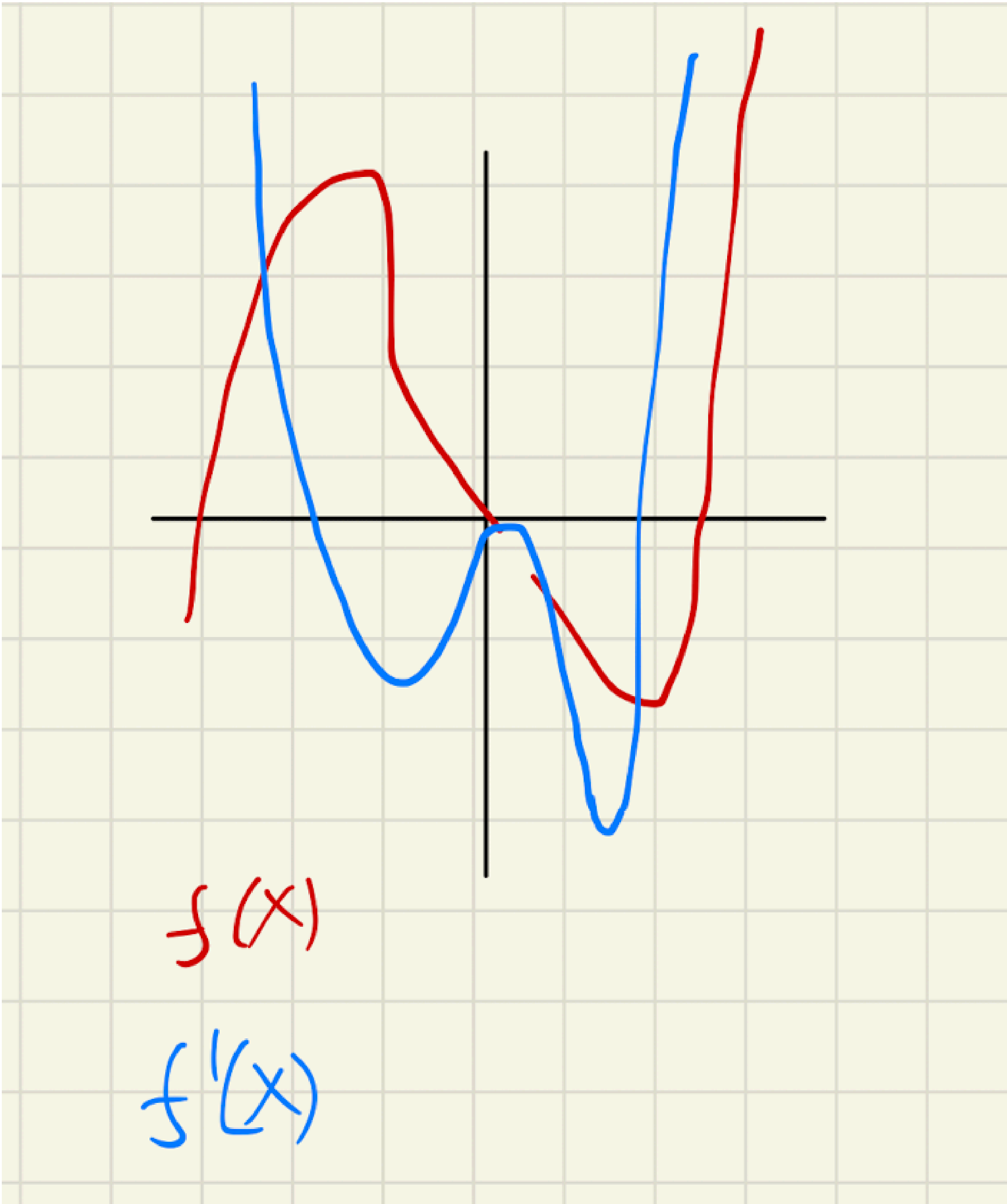
At the first critical point, the second derivative is negative. By the second-derivative test, therefore, this point serves as a local maxima is the function, after that point, would begin to exhibit a negative slope and then proceed to decrease.

At the second and third critical points, the second derivative is positive. By the second-derivative test, therefore, this point serves as local minima as the function, after that point, would begin to exhibit a positive slope and then proceed to increase.

The coordinates of the critical points are, therefore, as follows:

Type	x	y
maxima	0	0
minima	3sqrt(2)/2	-20.25
minima	-3sqrt(2)/2	-20.25

2.2 | $f(x) = x^5 - x^4 - 6x^3$



We first will figure the first and seconds derivatives of this function to check and verify its critical points.

$$f'(x) = 5x^4 - 4x^3 - 18x^2 \tag{22}$$

$$f''(x) = 20x^3 - 12x^2 - 36x \tag{23}$$

To figure the critical points (maxima, minima, inflection), we first solve for the points in which $f'(x)$ is 0.

$$0 = 5x^4 - 4x^3 - 18x^2 \tag{24}$$
$$\Rightarrow 0 = x^2(5x^2 - 4x^1 - 18) \tag{25}$$
$$\Rightarrow x = \{0, \frac{4+\sqrt{376}}{10}, \frac{4-\sqrt{376}}{10}\} \tag{26}$$

We then figure the second derivatives of the function at these two points.

$$f''(\{0, \frac{4+\sqrt{376}}{10}, \frac{4-\sqrt{376}}{10}\}) \approx \{0, 106.09, -45.93\} \tag{27}$$

At the first critical point, the second derivative is zero. By the second-derivative test, this would mean that that point is an inflection point — the function changes increasing/decreasing status at that point..

At the second critical point, the second derivative is positive. By the second-derivative test, therefore, this point serves as local minima as the function, after that point, would begin to exhibit a positive slope and then proceed to increase.

At the third critical point, the second derivative is negative. By the second-derivative test, therefore, this point serves as a local maxima is the function, after that point, would begin to exhibit a negative slope and then proceed to decrease.

The coordinates of the critical points are, therefore, as follows:

Type	x	y
inflection	0	0
minima	(4+sqrt(346))/10	-36.7
maxima	(4-sqrt(346))/10	7.63