

Hello, fellow person that comes across this. I have had one brief exposure with Linear Algebra following MATH 21-1 at UCSC. However, Axler is just so cool, so I am trying to learn a bit of linalg on the side to supplement my much more traditional linalg experience at the UC.

A few things of note. This whole thing is very "partial": in the sense that its contents contain many a parts of things omitted which I feel like I have a good grasp on from 21-1 such that I don't need to be reminded again; I only include things that maybe useful to me later either b/c I don't know it or I want to be reminded of it. As such, I don't think this will be helpful for most people.

1 | 1.A

1.1 | Things of Note

- $\lambda \in \mathbb{F}$ is called a "scalar". I mean duh but still.

1.1.1 | Defining a list

A list of length n is a collection of n elements (any mathematical object?) separated by commas.

"Identical" lists are established when lists have:

- the same length
- same elements
- in the same order.

Its also called a n -tuple.

n must be a finite non-negative value. Therefore, an "infinitely long list" is not a list.

1.1.2 | Sets vs Lists

Lists have order and repetition. In sets, order and repetitions don't matter.

1.1.3 | \mathbb{F}

- A set
- Containing 2 elements 0, 1
- Operators of "addition" and "multiplication" that satisfy the following properties

1. Properties of \mathbb{F} That, with $\alpha, \beta, \lambda \in \mathbb{F}$:

- **Commutativity** $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
- **Associativity** $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
- **Existence of Identities** $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$
- **Additive Inverse** for every α , $\exists \beta$ s.t. $\alpha + \beta = 0$
- **Multiplicative Inverse** for every $\alpha \neq 0$, $\exists \beta$ s.t. $\alpha\beta = 1$
- **Distribution** $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.1.4 | \mathbb{F}^n

$$\mathbb{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n\} \quad (1)$$

We say x_j is the j^{th} coordinate of (x_1, \dots, x_n) .

In \mathbb{F}^n ...

1. Addition

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) \quad (2)$$

2. Scalar Multiplication

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n) \quad (3)$$

3. Zero

$$0 = (0, \dots, 0) \quad (4)$$

4. Additive Inverse ...of $x \in \mathbb{F}^n$:

$$x + (-x) = 0 \quad (5)$$

That:

$$x = (x_1, \dots, x_n), -x = (-x_1, \dots, -x_n) \quad (6)$$

1.2 | In-Text Exercises

1.2.1 | Verify that $i^2 = -1$

$$(0 + 1i)(0 + 1i) = (0 + 0 + 0 + ii) = -1$$

1.2.2 | Defining subtraction and division

$$\alpha, \beta \in \mathbb{C}$$

Subtraction could be defined in that:

- Let $-\alpha$ be defined as the additive inverse of α
- Subtraction, therefore, is defined $\beta - \alpha = \beta + (-\alpha)$

Division could be defined in that:

- Let $1/\alpha$ be defined as the multiplicative inverse of α
- Subtraction, therefore, is defined $\beta/\alpha = \beta(1/\alpha)$

1.3 | Actual Exercises

1: Suppose $a, b \in \mathbb{R}$, $a, b \neq 0$, find $c, d \in \mathbb{R}$ s.t. $\frac{1}{a+bi} = c + di$

$$\frac{1}{a+bi} = \frac{(a-bi)}{(a+bi)(a-bi)} = \quad (7)$$

$$\Rightarrow \frac{a-bi}{a^2 - (bi)^2} = c + di \quad (8)$$

$$\Rightarrow \frac{a-bi}{a^2 + b^2} = c + di \quad (9)$$

$$\Rightarrow \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2} = c + di \quad (10)$$

Therefore:

$$c = \frac{a}{a^2 + b^2} \quad (11)$$

$$d = \frac{-b}{a^2 + b^2} \quad (12)$$

2: Show that $\frac{-1+\sqrt{3}i}{2}$ is the cube root of 1.

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 \quad (13)$$

$$\Rightarrow \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) \quad (14)$$

$$\Rightarrow \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i)}{8} \quad (15)$$

$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \quad (16)$$

$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \quad (17)$$

$$\Rightarrow \frac{8}{8} = 1 \quad (18)$$

3: Find two distinct square roots of i

?

4: Show that $\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in \mathbb{C}$

Let:

$$\forall a, b, c, d \in \mathbb{R}$$

- $\alpha = (a + bi)$
- $\beta = (c + di)$

$$\alpha + \beta = (a + bi) + (c + di) \quad (19)$$

$$= (a + c) + (b + d)i \quad (20)$$

$$= (c + a) + (d + b)i \quad (21)$$

$$= (c + di) + (a + bi) \quad (22)$$

$$= \beta + \alpha \blacksquare \quad (23)$$

5: Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda), \forall \alpha, \beta, \lambda \in \mathbb{C}$

Let:

$$\forall a, b, c, d, e, f \in \mathbb{R}$$

- $\alpha = (a + bi)$
- $\beta = (c + di)$
- $\lambda = (e + fi)$

$$(\alpha + \beta) + \lambda = ((a + bi) + (c + di)) + (e + fi) \quad (24)$$

$$= ((a + c) + (b + d)i) + (e + fi) \quad (25)$$

$$= (a + c + e) + (b + d + f)i \quad (26)$$

$$= (a + (c + e)) + (b + (d + f))i \quad (27)$$

$$= (a + bi) + (c + e) + (d + f)i \quad (28)$$

$$= (a + bi) + ((c + di) + (e + fi)) \quad (29)$$

$$= \alpha + (\beta + \lambda) \blacksquare \quad (30)$$

2 | 1.B

2.1 | Things of Note

2.1.1 | Vector Spaces V

A "vector"/"point" is a member of a vector space. The exact nature of scalar multiplication depends on which \mathbb{F} we are working in; hence, when being precise, we say that V is a vector space "over \mathbb{F} ".

1. Motivation

- Addition is commutative, associative, and has identity
- Every element has additive inverse
- Scalar multiplication is associative
- Addition and scalar multiplication is connected by distribution

2. Basic Operators

- **Addition** on set V is a function that assigns $u + v \in V$ to each pair $u, v \in V$

- **Scalar Multiplication** on set V is a function that assigns $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and $v \in V$. Note that this is different than a field, because if you can't multiply two different things in and out of the field and expect it to remain. But you could multiply an element in field \mathbb{F} and a vector in vector space V and expect it to stay in V .

Note also "Multiplication" is not defined as 1) there are two and 2) they behave very differently.

3. Properties For $u, v, w \in V$ and $a, b \in \mathbb{F}$.

- **Commutativity** $u + v = v + u$
- **Associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$.
- **Additive Identity** $\exists 0 \in V$ s.t. $v + 0 = v, \forall v \in V$
- **Additive Inverse** $\forall v \in V, \exists w \in V$ s.t. $v + w = 0$
- **Multiplicative Identity** $1v = v$
- **Distribution** $a(u + v) = au + av$ and $(a + b)v = av + bv$

4. Unique Additive Identity The additive identity ("zero") in a vector space must be unique. (i.e. there cannot be two distinct zeros 0 and $0'$ which both are $\in V$). This is because:

$$0 = 0 + 0' = 0' + 0 = 0' \quad (31)$$

That — if both 0 and $0'$ are additive identities, $0 = 0'$.

5. Unique Additive Inverse Every element in a vector space has an unique additive inverse (i.e. there cannot be two distinct additive inverses of $v \in V$ w and w' which both are $\in V$).

Suppose w and w' are both additive inverses of v , then it holds that:

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w' \quad (32)$$

That — if both w and w' exists in V , $w = w'$.

6. Zero and Vectors $0v = 0$ for $v \in V$. $a\vec{0} = \vec{0}$ for $a \in \mathbb{F}$.

2.1.2 $|\mathbb{F}^\infty$

Wait but aren't \mathbb{F}^n supposed to be made of lists, which has finite length?

I guess its just sequences of all of everything in F .

$$\mathbb{F}^\infty = \{(x_1, x_2, \dots) : x_j \in \mathbb{F} \text{ for } j = 1, 2, \dots\} \quad (33)$$

2.1.3 $|\mathbb{F}^S$

\mathbb{F}^S is defined as the set of functions that maps elements in set S to \mathbb{F} . It is a vector space.

1. Addition Addition between $f, g \in \mathbb{F}^S$ is defined by:

$$(f + g)(x) = f(x) + g(x), \forall x \in S \quad (34)$$

2. Scalar Multiplication Multiplication between $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, $\lambda f \in \mathbb{F}^S$ is defined as:

$$(\lambda f)(x) = \lambda f(x), \forall x \in S \quad (35)$$

3. \mathbb{F}^n and \mathbb{F}^∞ are special cases of \mathbb{F}^S ...this is because a list $\{x_1, x_2, x_3, \dots, x_n\}$ is actually a bijective mapping between $\{1, 2, 3, \dots, n\}$ (the indexes) and the values of the list, which are all $\in \mathbb{F}$. so :tada:!

2.2 | In-Text Exercises

2.2.1 | Verify that \mathbb{F}^n is a vector space over \mathbb{F}

Not going to write this one out, but:

- Commutativity: via rules addition, commutation (in \mathbb{F}), then undoing addition
- Associativity: addition, communication, then undoing addition
- Additive Identity: addition + definition of "zero" in \mathbb{F}^n
- Additive Inverse: addition + additive inverse (in \mathbb{F})
- Multiplicative Identity: scalar multiplication (by 1) and then identity (in \mathbb{F})
- Distribution: definition of addition in \mathbb{F}^n , scalar multiplication, undoing definition of addition again

2.3 | Actual Exercises

1: Proof that $-(-v) = v$, $\forall v \in V$

Step	Explanation
$v = v + 0$	Additive identity
$v = v + (-v + -(-v))$	Additive inverse
$v = (v + -v) + -(-v)$	Associative property
$v = 0 + -(-v)$	Additive inverse
$v = -(-v)$ ■	Additive Identity

2: Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Proof $a = 0$ or $v = 0$.

Let $a \neq 0$. We define the multiplicative inverse of a as a^{-1} .

Step	Explanation
$v = 1v$	Multiplicative identity
$v = aa^{-1}v$	Multiplicative inverse
$v = av a^{-1}$	Commutativity
$v = 0a^{-1}$	Given
$v = 0$ ■	Number times 0

If $a = 0$, ■.

3: Suppose $v, w \in V$, explain why \exists unique $x \in V$ s.t. $v + 3x = w$

Let $x = \frac{1}{3}(w - v)$; by addition and scalar multiplication, $\exists x \in V$.

Step	Explanation
$v + 3x = w$	Given
$v + 3(\frac{1}{3}(w - v)) = w$	Defined
$v + (w - v) = w$	Multiplication in \mathbb{F}
$v - v + w = w$	Commutativity
$w = w$ ■	Additive Inverse

Therefore, $\exists x \in V$ that satisfies the needed property.

Suppose there exists more than 1 x which satisfies this property. We call them x and x' . This would tell us the following equalities:

$$v + 3x = w, v + 3x' = w.$$

It follows from the equalities that:

$$3x = w - v, 3x' = w - v$$

Then, it follows that

$$3x = 3x'$$

Therefore:

$$x = x' \quad (36)$$

Hence, if given that there exists $v + 3x = w, v + 3x' = w, x = x'$. Hence, there is only one unique x such that $v + 3x = w$.

3 | 1.C

Axler, in his infinite wisdom, has crammed everything that's interesting to note in Chapter 1.c.

3.1 | Things of Note

3.1.1 | Subspaces

(Woo hoo!)

A subset $U \subset V$ is called a "subspace of V " if U is also a vector space using the same addition and scalar multiplication operators.

1. Checking for Subspaces Check for three conditions:

For $U \subset V$

- **Additive Identity** $0 \in U$. (also could be defined as "set is nonempty", b/c if nonempty, and its closed under scalar multiplication, multiplying any element by 0 will do the trick. But often showing 0 is in it is actually simpler.)
- **Closed Under Addition** $u, w \in U$ implies $u + w \in U$
- **Closed Under Scalar Multiplication** $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

3.1.2 | Summing Subsets

Suppose U_1, \dots, U_m are subsets of V . The "sum" of the subsets $(U_1 + \dots + U_m)$ is the set of all possible sums of elements in U_1, \dots, U_m . That is:

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\} \quad (37)$$

1. Properties of the Sums of Subspaces Suppose U_1, \dots, U_m are subspaces of V . $U_1 + \dots + U_m$ is the smallest subspace of V containing all of U_1, \dots, U_m .

3.1.3 | Direct Sum

Suppose U_1, \dots, U_m are subspaces of V

- Sum $U_1 + \dots + U_m$ is a direct sum if $U_1 + \dots + U_m$ can be only written only one way as a sum $u_1 + \dots + u_m$
- Direct sum is noted as $U_1 \oplus \dots \oplus U_m$

"Sums is the union, direct sums is the disjoint union".

1. Checking for Direct Sums Suppose U_1, \dots, U_m are subspaces of V . $U_1 + \dots + U_m$ is a direct sum iff the only way to write 0 as a sum $u_1 + \dots + u_m$ is by taking each u_j equaling to 0. This could be implied from the definition of a direct sum: that there is only one way to write a 0 as $u_1 + \dots + u_m$, and being closed scalar multiplication means that you could multiply 0 to each subspace individually and they still have to add up to 0.
2. Direct Sum of Two Subspaces Suppose U, W are subspaces of V .
 $U + W$ is a direct sum iff $U \cap W = \{0\}$.

3.2 | In-Text Exercises

3.2.1 | Summing Subspaces, an Example

Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$ and $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Then:

$$U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\} \quad (38)$$

We verify this by writing out the sum.

$$U + W = \{(x_u, x_u, y_u, y_u) + (x_w, x_w, x_w, y_w) : x_u, y_u \in U, x_w, y_w \in W\} \quad (39)$$

$$= \{(x_u + x_w, x_u + x_w, y_u + x_w, y_u + y_w) : x_u, y_u \in U, x_w, y_w \in W\} \quad (40)$$

$$= \left\{ \left(\underbrace{x}_{x_u + x_w}, x, \underbrace{y}_{y_u + x_w}, \underbrace{z}_{y_u + y_w} \right) : x, y, z \in \mathbb{F} \right\} \quad (41)$$

3.2.2 | Verify sums equal to spaces

Suppose U, W are subspaces of \mathbb{F}^3 .

$$U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \quad (42)$$

$$W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \quad (43)$$

Verify $\mathbb{F}^3 = U \oplus W$

$$U + W = \{(x, y, z) \in \mathbb{F}^3 : x, y, z \in \mathbb{F}\} = \mathbb{F}^3 \quad (44)$$

:tada:?

3.3 | Actual Exercises

1: For each of the following subsets of \mathbb{F}^3 , determine if it is a subspace of \mathbb{F}^3 .

$$U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} \quad (45)$$

We could see that $(0, 0, 0) \in U$.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3})$, $u_2 = (x_3, x_4, \frac{-x_3-2x_4}{3})$. $u_1, u_2 \in U$.

$u_1 + u_2 = (x_1 + x_3, x_2 + x_4, \frac{-(x_1+x_3)-2(x_2+x_4)}{3})$. Define $x_1 + x_3 = x$, $x_2 + x_4 = y$. Therefore: $u_1 + u_2 = (x, y, \frac{-x-2y}{3})$.

$$x + 2y - x - 2y = 0.$$

Therefore, U is closed under addition.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3})$. We scale each element by scalar factor λ .

$$\lambda u_1 = (\lambda x_1, \lambda x_2, \lambda \frac{-x_1-2x_2}{3}).$$

$\lambda x_1 + 2\lambda x_2 - \lambda x_1 - 2\lambda x_2 = 0$. Therefore, U is closed under scalar multiplication.

Therefore, it is a subspace of \mathbb{F}^3 .

10: Suppose that U_1 and U_2 are subspaces of V . Prove that the intersection $U_1 \cap U_2$ is a subspace of V .

Given U_1 and U_2 are both subspaces, 0 must be in both subsets hence it must be in their intersection.

Let $u_1 = (x_1, y_1, z_1)$. Let $u_2 = (x_2, y_2, z_2)$. Finally, let $u_1, u_2 \in U_1 \cap U_2$. By this last fact, we could derive the fact that $u_1, u_2 \in U_1, U_2$.

As U_1 is a subspace, U_1 closed under addition. Since $u_1, u_2 \in U_1$, $u_1 + u_2 \in U_1$. As U_2 is a subspace, U_2 closed under addition. Since $u_1, u_2 \in U_2$, $u_1 + u_2 \in U_2$.

As $u_1 + u_2 \in U_1, U_2$, it is additionally $u_1 + u_2 \in U_1 \cap U_2$. $U_1 \cap U_2$ is therefore closed under addition.

By the same token....

As U_1 is a subspace, U_1 is closed under scalar multiplication. Since $u_1 \in U_1$, $\lambda u_1 \in U_1$. As U_2 is a subspace, U_2 is closed under scalar multiplication. Since $u_1 \in U_2$, $\lambda u_1 \in U_2$.

Therefore, as $\lambda u_1 \in U_1, U_2$, it is additionally true that $u_1 \in U_1 \cap U_2$, it is therefore closed under multiplication.