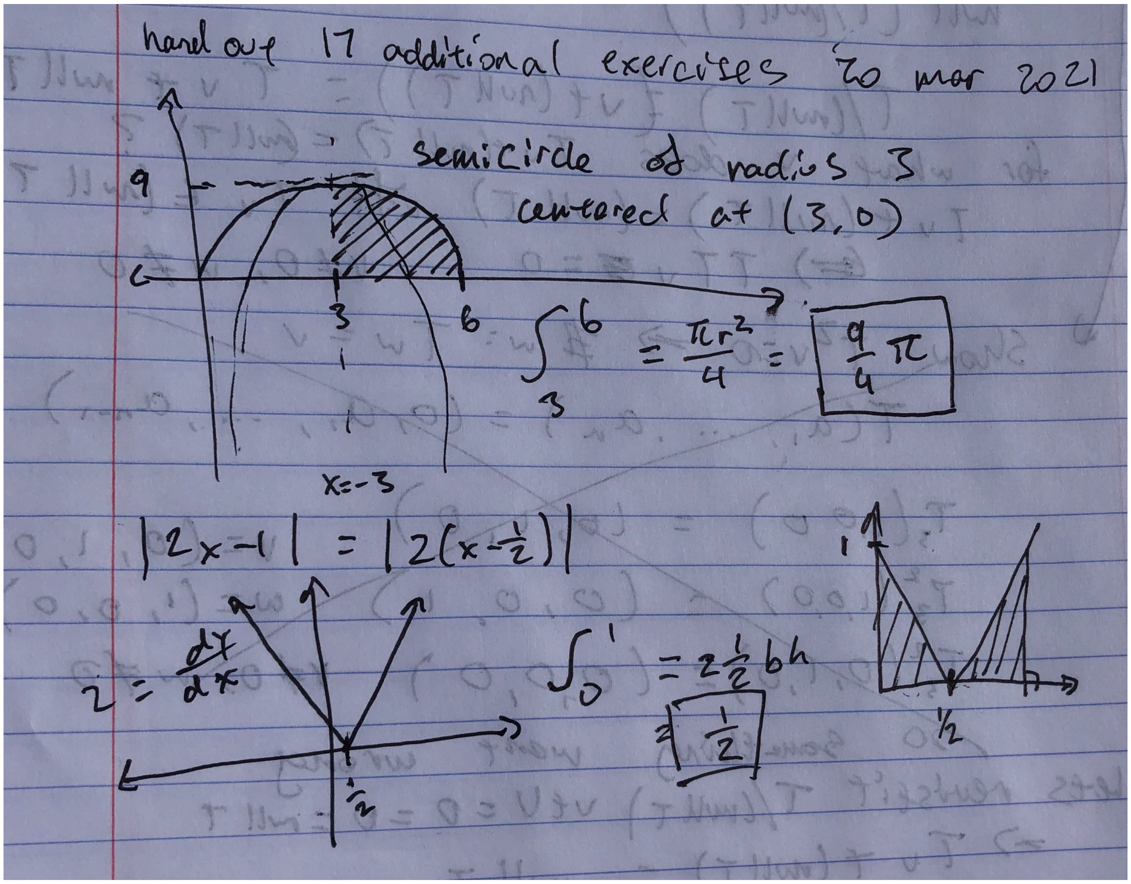
1 | Exercises

1.1 | interpreting in terms of area



1.3 | subtracting integrals

I expect

$$\int_1^2 f(x)dx = \int_1^5 f(x)dx - \int_2^5 f(x)dx = -3 - 4 = -7$$

In fact, I expect

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

1.4 | show $\int_a^b x^2 dx = \frac{b^3-a^3}{3}$

(see attached pages)

Keep in mind

$$\sum_{k=1}^{n} af(x) = a\sum_{k=1}^{n} f(x)$$

$$\sum_{k=1}^{n} (a + f(x)) = an + \sum_{k=1}^{n} f(x)$$

$$
\begin{aligned}
\int_a^b x^2 dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{b-a}{n} \left(a + k\frac{b-a}{n} \right)^2 \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} \left(a + k\frac{b-a}{n} \right)^2 \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} \left(a^2 + \left(k\frac{b-a}{n} \right)^2 + 2ak\frac{b-a}{n} \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} a^2 + \sum_{k=1}^{n} \left(k\frac{b-a}{n} \right)^2 + \sum_{k=1}^{n} 2ak\frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a^2 n + \sum_{k=1}^{n} k^2 \left(\frac{b-a}{n} \right)^2 + 2a\frac{b-a}{n} \sum_{k=1}^{n} k \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a^2 n + \left(\frac{b-a}{n} \right)^2 \sum_{k=1}^{n} k^2 + 2a\frac{b-a}{n} \sum_{k=1}^{n} k \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a^2 n + \left(\frac{b-a}{n} \right)^2 \frac{n(n+1)(2n+1)}{6} + 2a\frac{b-a}{n} \frac{n(n+1)}{2} \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \left(\frac{b-a}{n} \right)^2 \frac{(n+1)(2n+1)}{6} + \cancel{2}a\frac{b-a}{n} \frac{(n+1)}{\cancel{2}} \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n\frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a\frac{b-a}{n}(n+1) \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n\frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a\cancel{n}\frac{b-a}{\cancel{n}} + a\frac{b-a}{n} \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n\frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a(b-a) + a\cancelto{0}{\frac{b-a}{n}} \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^{\cancel{2}}}\cancel{n}\frac{(2n+1)}{6} + \cancelto{0}{\frac{(b-a)^2}{n^2}\frac{(2n+1)}{6}} + a(b-a) \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n}\frac{(2n+1)}{6} + a(b-a) \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{\cancel{n}}{3}\frac{(b-a)^2}{\cancel{n}} + \cancelto{0}{\frac{1}{6}\frac{(b-a)^2}{n}} + a(b-a) \right) \\
&= \lim_{n\to\infty} (b-a) \left(a^2 + \frac{(b-a)^2}{3} + a(b-a) \right) \\
&= a^2(b-a) + \frac{b^2+a^2-2ab}{3}(b-a) + a(b-a)^2 \\
&= \frac{1}{3}\left(3a^2(b-a) + b^2(b-a) + a^2(b-a) - 2ab(b-a) + 3a(b-a)^2 \right) \\
&= \frac{1}{3}\left(3a^2b - 3a^3 + b^3 - ab^2 + a^2b - a^3 - 2ab^2 + 2a^2b + 3a(b^2+a^2-2ab) \right) \\
&= \frac{1}{3}\left(3a^2b - 3a^3 + b^3 - ab^2 + a^2b - a^3 - 2ab^2 + 2a^2b + 3ab^2 + 3a^3 - 6a^2b \right) \\
&= \frac{1}{3}\left(3a^2b - 6a^2b - 3a^3 - a^3 + 3a^3 + b^3 + a^2b + 2a^2b - ab^2 - 2ab^2 + 3ab^2 \right) \\
&= \frac{1}{3}\left(-3a^2b + 3a^2b - a^3 + b^3 \right) \\
&= \frac{1}{3}\left(b^3 - a^3 \right) \\
&= \frac{b^3-a^3}{3}
\end{aligned}
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