1 | Thing

First, we will define equations for the distance as a function of t. (Note that both h_0 and θ are parameters but aren't shown as arguments to the function.)

$$\begin{cases} x(t) &= v_0 \cos(\theta) t \\ y(t) &= -\frac{1}{2} g t^2 + v_0 \sin(\theta) t + h_0 \end{cases}$$

Velocity is a function of h_0:

$$v_0 = \sqrt{2g(H - h_0)} \tag{1}$$

We can rewrite $x(t)$ and $y(t)$:

$$\begin{cases} x(t) &= \sqrt{2g(H - h_0)} \cos(\theta) t \\ y(t) &= -\frac{1}{2} g t^2 + \sqrt{2g(H - h_0)} \sin\theta t + h_0 \end{cases}$$

We can get t_f in terms of x:

$$\begin{aligned} t_f &= \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta)} \\ &= \frac{x_f}{v_0 \cos(\theta)} \end{aligned}$$

We can now get y_f in terms of x_f:

$$\begin{aligned} y_f &= -\frac{1}{2} g \left(\frac{x_f}{v_0 \cos(\theta)}\right)^2 + v_0 \sin(\theta) \frac{x_f}{v_0 \cos(\theta)} + h_0 \\ &= -\frac{g x_f^2}{2 v_0^2 \cos^2(\theta)} + x_f \tan(\theta) + h_0 \end{aligned}$$

We set y_f to equal 0 and differentiate both sides:

$$\begin{aligned} \frac{d}{d\theta}[0] &= \frac{d}{d\theta}\left[-\frac{g x_f^2}{2 v_0^2 \cos^2(\theta)}\right] + \frac{d}{d\theta}[x_f \tan(\theta)] + \frac{d}{d\theta}[h_0] \\ 0 &= -\frac{g}{2 v_0}\left(\frac{2\cos^2(\theta) x_f' x_f + 2 x_f^2 \cos(\theta)\sin(\theta)}{\cos^4(\theta)}\right) + x_f' \tan(\theta) + x_f \sec^2(\theta) \end{aligned}$$

We can simplify this further:

$$0 = -\frac{g x_f' x_f}{v_0 \cos^2(\theta)} - \frac{g x_f^2 \tan(\theta)}{v_0 \cos^2(\theta)} + x_f' \tan(\theta) + \frac{x_f}{\cos^2(\theta)}$$

Solving for $\tan(\theta)$:

$$\begin{aligned} \tan(\theta)\left(x_f' - \frac{g x_f^2}{v_0 \cos^2(\theta)}\right) &= \frac{g x_f' x_f - v_0 x_f}{v_0 \cos^2(\theta)} \\ \tan(\theta)\left(\frac{v_0 x_f' \cos^2(\theta) - g x_f^2}{v_0 \cos^2(\theta)}\right) &= \frac{g x_f' x_f - v_0 x_f}{v_0 \cos^2(\theta)} \end{aligned}$$