Angular Momentum & Torque - Part 3

February 18, 2021

Comparison Chart of Angular & Linear Quantities

Type	Rotational	Linear
Velocity	Angular Velocity $\vec{\omega}$ (rad/s)	Velocity $\vec{v}$ (m/s))
Momentum	Angular Momentum $\vec{L} = \vec{r} \times m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1})$	Momentum $\vec{p} = m\vec{v}$ $(\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1})$
"Force"	Torque $\vec{\tau} = \vec{r} \times \vec{F}$ (N m)	Force $\vec{F}$ (N)
"2nd Law"	$\vec{\tau}_{net} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$	$\vec{F}_{net} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$

In part 2 of Angular Momentum + Torque, we added two more theorems: 1) That the "2nd Law" relationship for torque works for a system of masses:

$$\text{System 2nd Law} \quad \vec{\boldsymbol{\tau}}_{net\,ext} = \frac{\mathrm{d}\vec{L}_{sys}}{\mathrm{d}t} \quad \vec{F}_{net\,ext} = \frac{\mathrm{d}\vec{p}_{sys}}{\mathrm{d}t}$$

2) That for a rotating axially symmetric system of masses (i.e. axially symmetric rigid body), rotating about its axis of symmetry ($\vec{z}$), we can express the angular momentum simply as:

$$\text{System Momentum} \quad \vec{L}_{sys} = I\vec{\omega} \quad \vec{p}_{sys} = M\vec{v}_{CM}$$

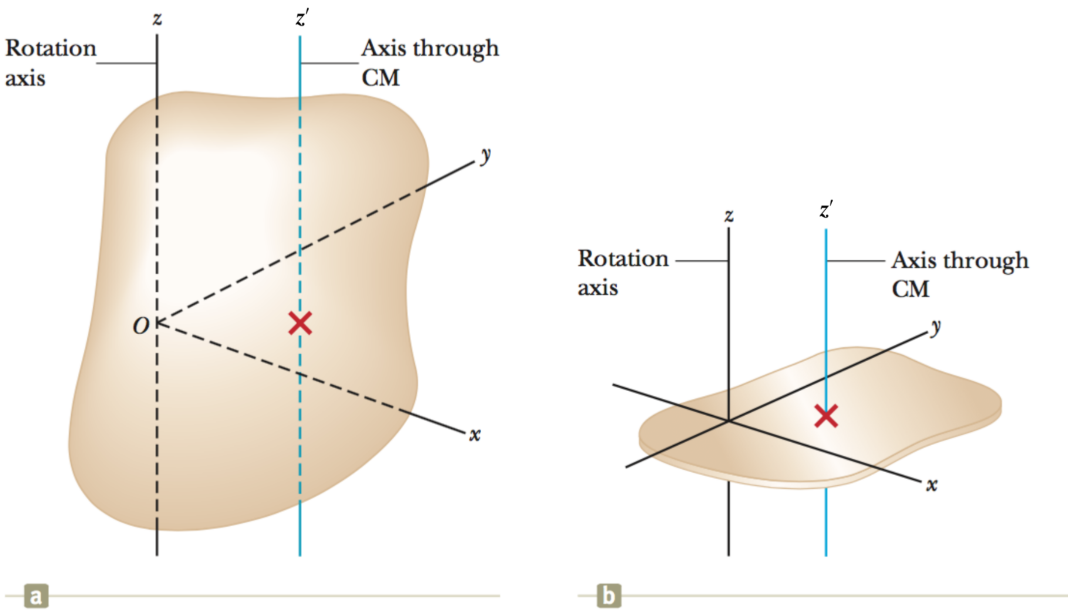
$$\text{where } \vec{\omega} = \omega\hat{z},\ I = \sum_i m_i {\ell_i}^2,\ \text{and } l_i \text{ is the distance from the z-axis}$$

Here I is called the *"Rotational Inertia"* or *"Moment of Inertia"*, and M is called just the *"Inertia"* or *"Mass"* of the system.

Problem 1:

1 Parallel Axis Theorem

In this problem we derive a theorem which will allow us to calculate the rotational inertia of more complex objects by translating and combining simpler smaller objects together. The idea is that if we know the rotational inertia of an object rotating around its CM, then we can calculate the object of the same object rotating around a different axis that is parallel to the first axis.



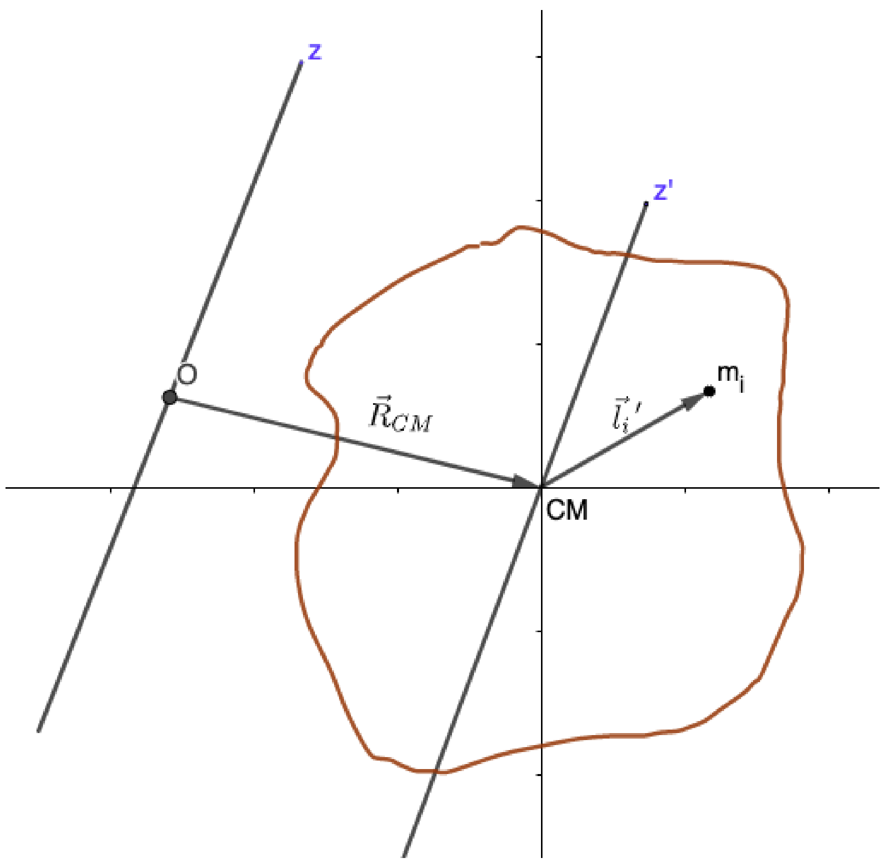
The Parallel Axis Theorem states that the rotational inertia of a rigid body about a given axis z can be computed from the rotational inertia of the object around a different axis, z', where z' is the axis that is parallel to z and passes through the object's center-of-mass.

Let I_{CM} be the rotational inertia of the rigid body rotating about the z'-axis, and let I be the rotational inertia of the object around the z-axis. If D is the distance between the axes z and z', and M is the mass of the rigid body, then the Parallel Axis Theorem states that:

$$I = I_{CM} + MD^2$$

Your assignment is to prove the Parallel Axis Theorem.

[**Hint 1**: As shown in the diagram, select a single flat slice perpendicular to the z- and z'-axes. Let the distance vector $\vec{l_i}$ be expressed as the sum of two vectors $(\vec{R}_{CM} + \vec{l_i}\,')$, where $\vec{R}_{CM} =< X_{CM}, Y_{CM} >$ is the vector from the z-axis origin to the z'-axis origin, and $\vec{l_i}\,' =< x_i', y_i' >$ is the vector from the CM (the origin of the z'-axis) to the mass m_i. Show that when you sum up all of the values $I = \sum_i m_i {l_i}^2$ for the slice, you eventually obtain $I = \sum_i m_i ({x_i'}^2 + {y_i'}^2) + \sum_i m_i D^2$. Show that this reduces to the Parallel Axis Theorem for the slice, and extend it to the Parallel Axis Theorem for the entire rigid body.



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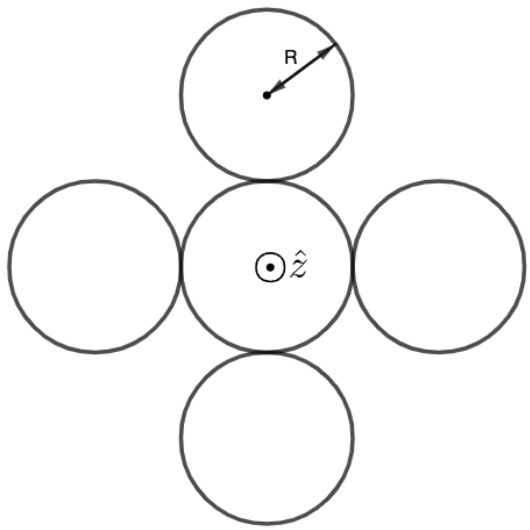
[**Hint 2**: You will obtain intermediate terms like: $X_{CM} \sum_i m_i x_i'$. Consider how the summation relates to the definition of the CM in the prime-coordinate system, and explain why the summation is zero.]

Problem 2:

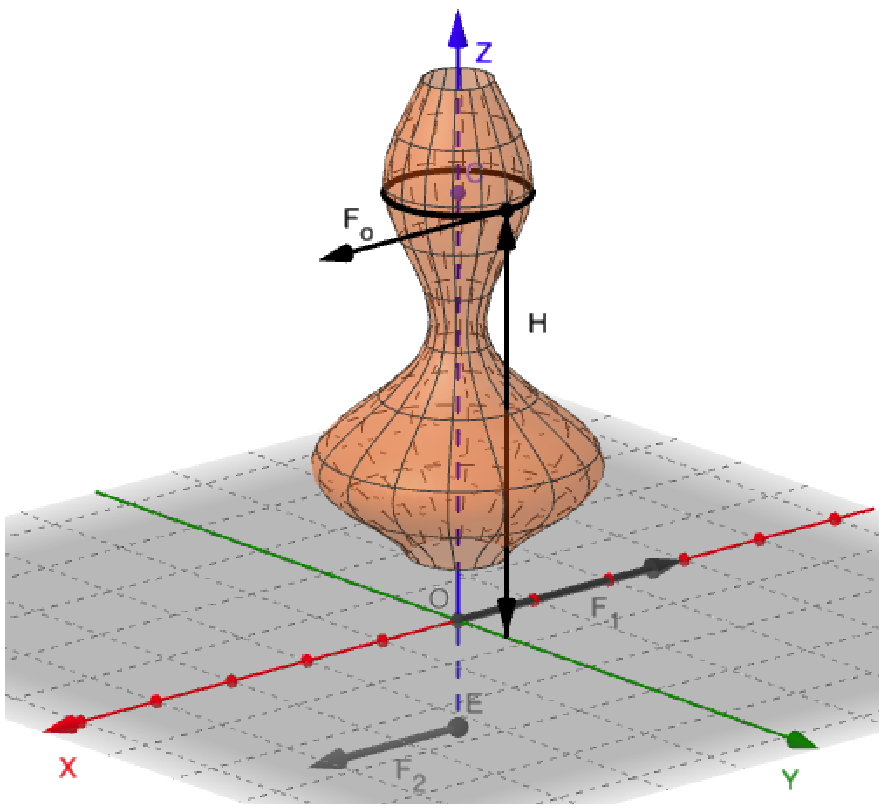
Find the moment of inertia for the 5-ring rigid object shown (diagram below), rotating about the z-axis. Each ring has radius R with equal mass. The **total** mass of the object is M.



[Ans: $\frac{21}{5}MR^2$]

Problem 3:

Consider a rigid body that can rotate freely around its axis of symmetry. For simplicity, let's assume that the object is round like a top so that we can wrap a string around it, which we'll pull to get the top rotating. The axle along the z-axis is made of a massless, frictionless, infinitesimally thin rod which aligns with the object's axis of symmetry.



The origin of the axle is fixed at the xy-plane, but the axle is free to rotate, so imagine that a thin plate (the xy-plane) maintains the position of the origin by providing force $\vec{F_1}$ in the xy-plane against the origin of the axle. We first wrap a string around the top at height $z = H$. When we pull tangentially on the string with a constant force $\vec{F_o} = F_o\hat{x}$, parallel to the xy-plane, at a height H above the origin, a counter-force $\vec{F_2}$ is also applied at $z = (-H/2)$ against the axle to maintain stability of the z-axis by providing a counter-torque on the axle. Let "R" be the radius of the top at $z = H$ (the perpendicular distance from the axle). Find the following:

a) The magnitude and direction of the torque $\vec{\tau}_o$ applied by the string tension $\vec{F}_o$, expressed in terms of $H, R, F_o, \hat{x}, \hat{y}, \hat{z}$. Find all three component vectors of the torque: $\vec{\boldsymbol{\tau}}_{o_x}, \vec{\boldsymbol{\tau}}_{o_y}, \vec{\boldsymbol{\tau}}_{o_z}$. (Hint: this can be done easily using $\vec{\tau} = \vec{r} \times \vec{F}_o$ by breaking $\vec{r}$ into its components.)

b) Sketch the torque vector $\vec{\tau}_o$ on the diagram.

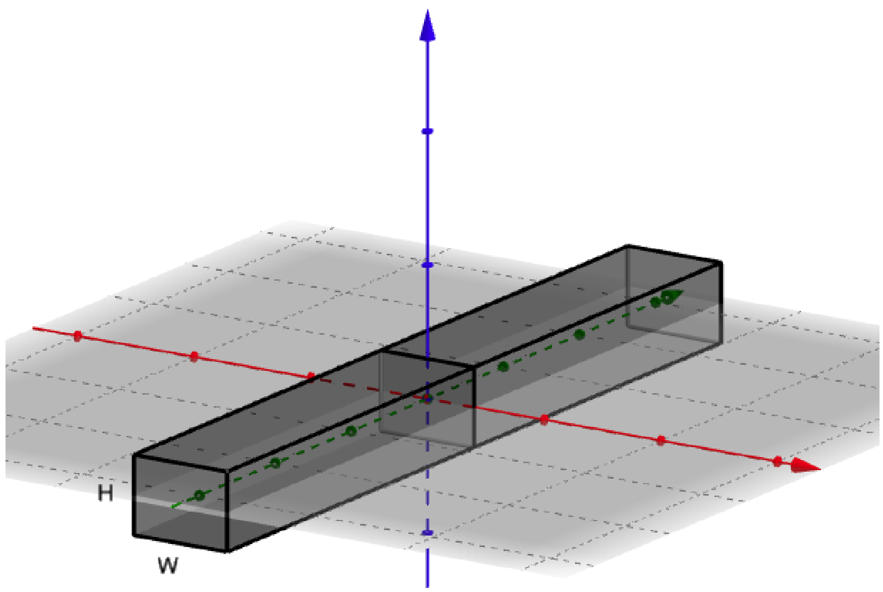
c) The magnitudes and directions of the $\vec{F}_2$ and $\vec{F}_1$ vectors. Explain how you deduced these forces. [Hint: you should be able to deduce what the net torque and net force must be on the system, and from those determine the necessary forces.]

d) If the top has rotational inertia I_o, find the angular acceleration (α_o) of the top in terms of F_o, H, R, I_o.

e) Write the kinematics equations for the angular velocity ($\omega(t)$) and the angular position ($\theta(t)$) of the top as a function of time. Assume that at $t = 0, \theta = 0, \omega = 0$.

Problem 4: Rotational Inertia of a Rectangular Rod

Determine the rotational inertia for a rectangular rod of length L, with cross-section that is W x H, where the axis of rotation passes through the center of mass perpendicular to the L and W axes.



[Hint: Find the rotational inertia of a flat plate of dimension W x H, where the CM axis is perpendicular to the W axis. Then apply the Parallel Axis Theorem to sum up (integrate) cross-sectional slices along the length L.]

[Ans: $\frac{1}{12}M(W^2 + L^2)$]