1 | Slicing into Rectangles

The general idea of Riemann sums is to slice a curve into vertical non-overlapping rectangles to approximate the area between the curve and the x-axis. This can be expressed mathematically as a summation given the function $f(x)$, the range $[a, b]$, and the number of rectangles n:

$$\sum_{k=1}^{n} \frac{b-a}{n} f(a + k\frac{b-a}{n})$$

This can be written more concisely by defining $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$:

$$\sum_{k=1}^{n} \Delta x f(x_i)$$

These estimates all have the right endpoint of the rectangle touching the curve. You could also use the left endpoint, or use the minimum value one and add a triangle to form a trapezoid.

2 | Area Interpretation

Areas under curves can be estimated if you recognize the function. For example:

$$\int_0^1 \sqrt{1-x^2} dx$$

Traces out a quarter of a semicircle, so the area under this curve is $\frac{\pi}{4}$

3 | Upper and Lower Bound

To get an upper and lower bound approximation using a Riemann sum, you cannot always take the left or right edge. Instead, you have to take the minimum or maximum in an interval, usually denoted $f(x_i^*)$.

4 | the Definite Integral

Finally, we can define the definite integral as a limit of Riemann sums.

$$\int_a^b f(x) dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_i)\Delta x$$

Where once again, $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$

These integrals can be evaluated directly with a lot of algebra and some triangular number tricks.