1 | Axler 6.A exercise 9

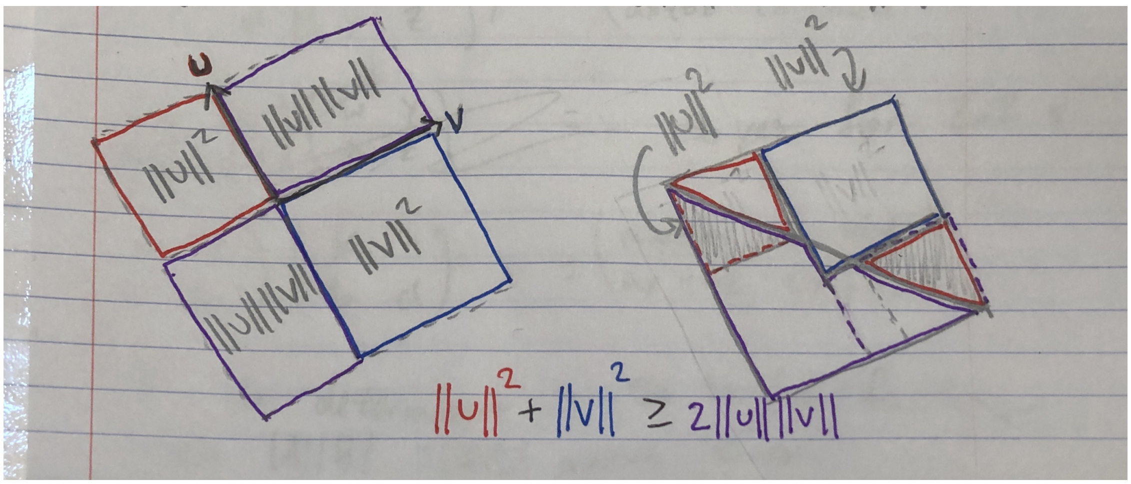
Suppose $u, v \in V$ and $\|u\| \le 1$ and $\|v\| \le 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|u\|^2} \le 1 - |\langle u, v\rangle|$$

2 | Proof

2.1 | Useful Lemma

$$\|u\|^2 + \|v\|^2 \ge 2\|u\|\|v\|$$



This proof is only valid for inner product spaces over $\mathbb{F}^n$ and the Euclidean norm. An algebraic proof would be better.

2.2 | Cauchy-Schwarz Corollary

$$|\langle u, v\rangle| \le \|u\|\|v\| \implies 1 - \|u\|\|v\| \quad \le 1 - |\langle u, v\rangle|$$

2.3 | Main Proof

Now, to show that the square of the left hand side is less than or equal to the square of the right hand side,

$$
\begin{aligned}
(1 - \|u\|^2)(1 - \|v\|^2) &= 1 - \|u\|^2 - \|v\|^2 + \|u\|^2\|v\|^2 \\
&= 1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2 \\
\text{\textbackslash[} \qquad &\le 1 - 2\|u\|\|v\| + \|u\|^2\|v\|^2 && \text{by the earlier lemma} \qquad \text{\textbackslash]} \\
&= (1 - \|u\|\|v\|)^2 \\
&\le (1 - |\langle u, v\rangle|)^2 && \text{by the Cauchy-Schwarz corollary}
\end{aligned}
$$

Taking square roots of both sides proves the desired result. ■