1 | Motivation: Gauss' Law

The original electric field expression is cool:

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\vec{r})\hat{u}\vec{r'}}{|\vec{r}-\vec{r'}|^2}dV' \tag{1}$$

but it required the use of a triple-integral over a whole volume of space, per charge, which is very inefficient to compute.

Gauss' Law provides a much more practical solution to the same problem, that:

$$\iint_s \vec{E}\cdot\vec{n}dS = \frac{q}{\epsilon_0} \tag{2}$$

This expression involves a "unit normal vector", $\vec{n}$, as well as a "surface integral" $\iint_s$ over S. In this chapter, we will explore both of those things.

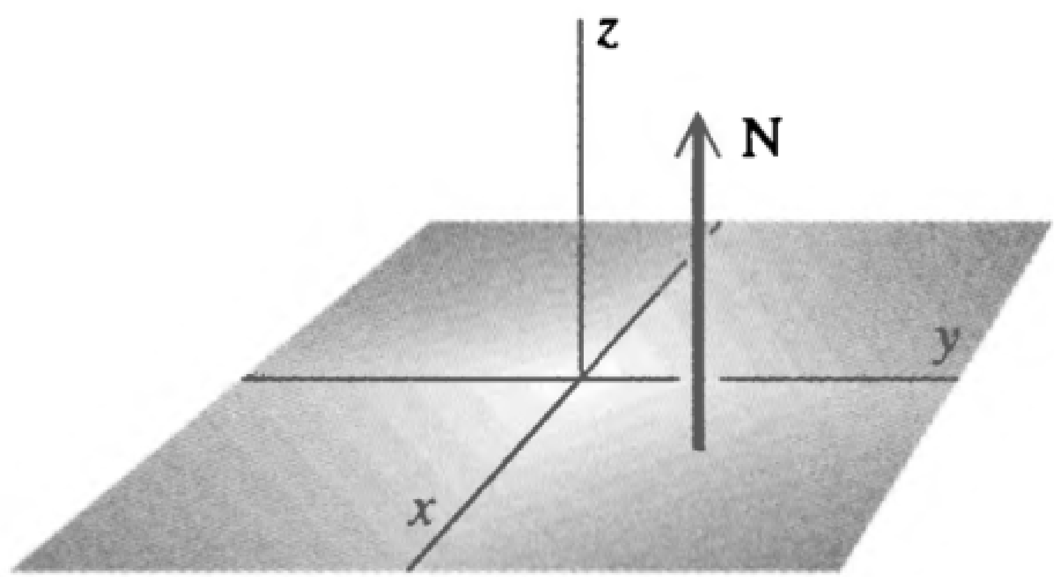
2 | Unit Normal Vector

The normal unit vector is a vector orthogonal to a certain surface. It is likely part of most of the surface integrals we will see.

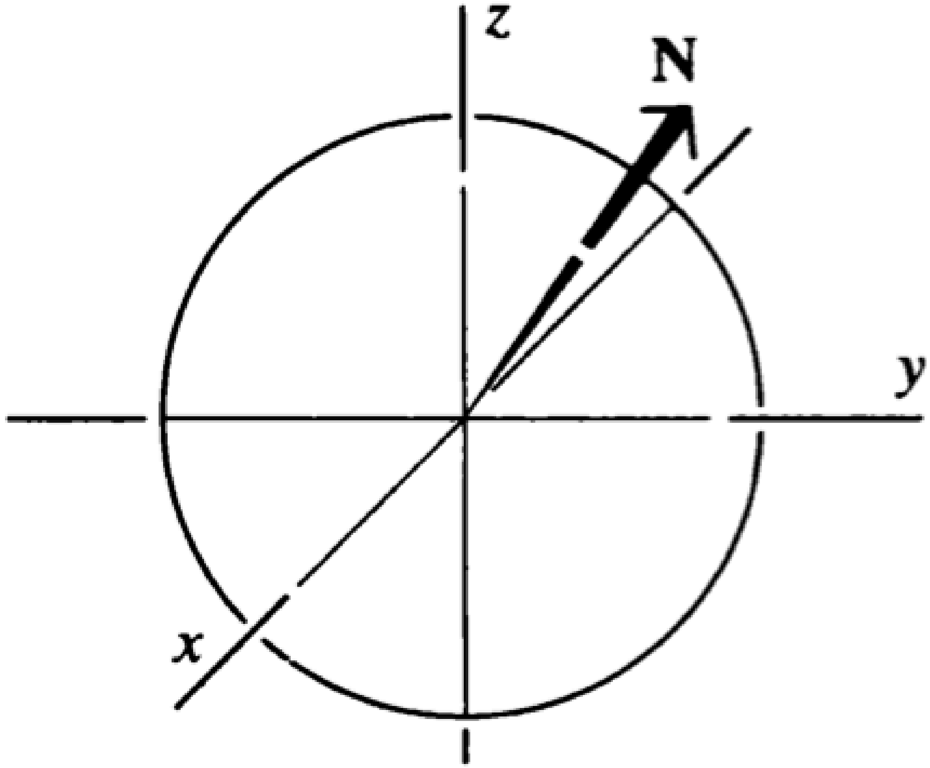
2.1 | Intuition

The intuition of a unit vector orthogonal to a direction is pretty intuitive.

If "to a plane", it is orthogonal to the plane:

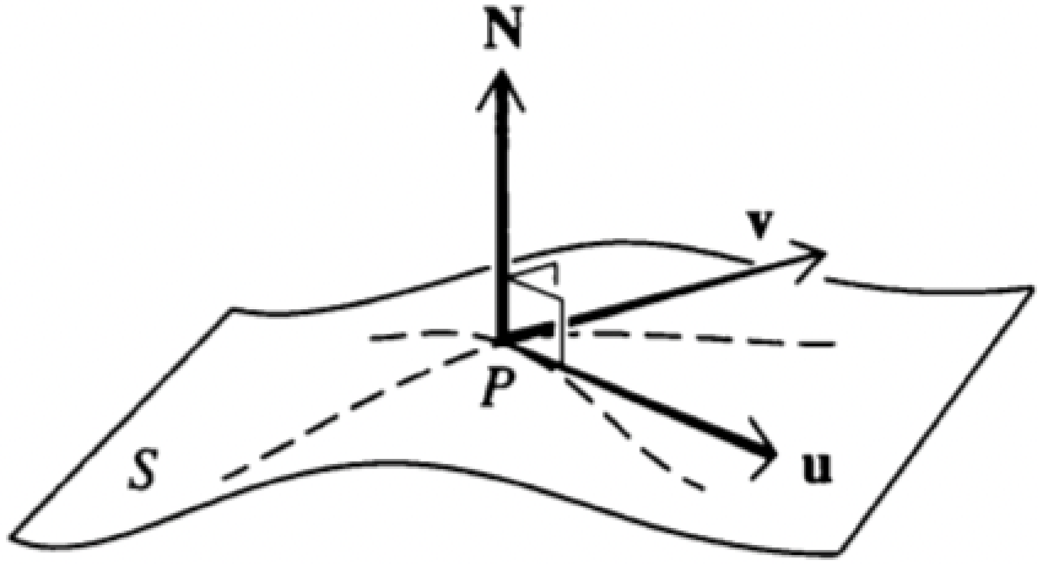


If "to a spherical surface", it is a vector in the radial direction to a slice of the surface:



2.2 | Formal definition of a Normal Unit Vector

Consider an arbitrary surface S. Construct two noncollinear vector $\vec{u}$ and $\vec{u}$ tangent to S at some point P. Some vector $\vec{N}$ which is orthogonal to both $\vec{u}$ and $\vec{u}$ with also an origin at P is, by definition, normal to S at P.



That is:

$$\hat{n} = \frac{\vec{N}}{||N||} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} \tag{3}$$

For some $\vec{u}$, $\vec{v}$ both tangent to point P.

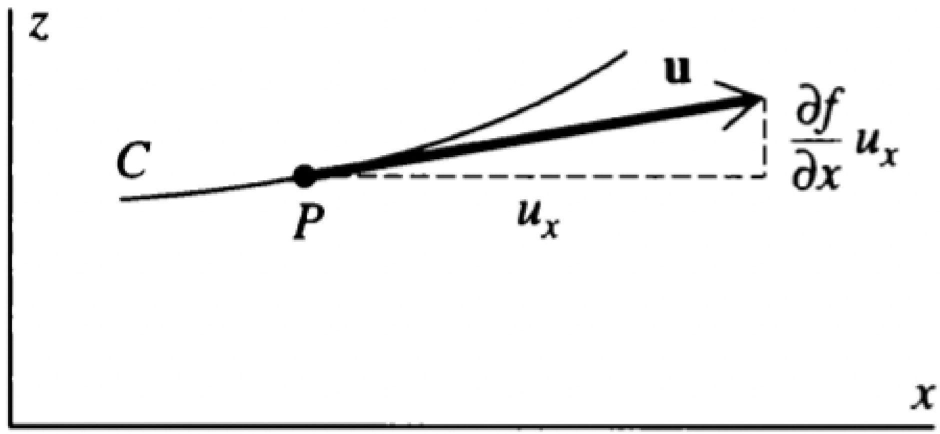
3 | **Finding an Expression for $\hat{n}$**

To find an expression for $\hat{n}$, we will consider some surface S by the eqn. $f(x, y)$. Our goal is to identify some $\vec{u}$ and $\vec{v}$ which will yield the result in the needed vector $\vec{n}$.

We will construct a plane P perpendicular to the floor, parallel to the x axis, on S which is parallel to the xz plane. The plane will intersect with our surface at some curve C. (This is bit like parameterization.)

Let's take some vector $\vec{u}$ tangent to C at P. We will set its projection upon the x component as, per convention, $\vec{u}_x$.

We will now have a visual proof for the z component from the book:



Given some $\vec{u}_x$, and the function f, we know that multiplying $\frac{\partial f}{\partial x}$ by $\vec{u}_x$, will yield ∂f, the change in the z component. Therefore:

$$\vec{u} = \hat{i}\vec{u}_x + \hat{k}\left(\frac{\partial f}{\partial x}\vec{u}_x\right) = \left[\hat{i} + \hat{k}\left(\frac{\partial f}{\partial x}\right)\right]\vec{u}_x \tag{4}$$

Now, we will construct v. The simplest way to do this is to construct another plane P', but it is now parallel to the y axis, on S which is parallel to the yz plane. It should be trivial to see that, if we construct a vector $\vec{v}$ from the trace curve C' of the second plane P' upon S.

Doing the same thing, we will end up with:

$$\vec{v} = \hat{j}\vec{v}_y + \hat{k}\left(\frac{\partial f}{\partial y}\vec{u}_y\right) = \left[\hat{j} + \hat{k}\left(\frac{\partial f}{\partial y}\right)\right]\vec{u}_y \tag{5}$$

And now, we consider their cross product.

$$\begin{array}{cccc} & \hat{i} & \hat{j} & \hat{k} \\ \vec{u} & u_x & 0 & \left(\frac{\partial f}{\partial x}\right)\vec{u}_x \\ \vec{v} & 0 & u_y & \left(\frac{\partial f}{\partial y}\right)\vec{u}_y \end{array}$$

The actual work of computing it is left as an exercise to the reader. However, we will get:

$$\vec{u} \times \vec{v} = \left[-\hat{i}\left(\frac{\partial f}{\partial x}\right) - \hat{j}(\frac{\partial f}{\partial y}\right) + \hat{k}u_x u_y \tag{6}$$