1 | Definitions

1.1 | Algebraic Structures

1.1.1 | Group

A set of items and an operation that satisfy closure, identity, inverse, assocativity

1.1.2 | Field

A group and another ”secondary” operation that the set is almost a group under (except the additive identity will have no multiplicative inverse).

1.1.3 | Vector Space

A field and a set of vectors that can be added together or multiplied by scalars from the field, **with the following five properties:**

- commutativity
- assocativity
- additive identity
- additive inverse
- distributive property

1.1.4 | Subspace

A subset of a vector space that is itself a vector space. Only need to show that it:

1. Includes the additive identity (0)
2. Is closed under addition
3. Is closed under scalar multiplication

The subspace must use the same addition and scalar multiplication of its ”superspace”

1.1.5 | Sum

A sum of (**multiple**) **subsets** is all vectors that can be written as the sum of one vector from each sub **set** (or zero).

1.1.6 | Direct Sum

If each element in a sum of (**multiple**) subspaces can be written in only one way (with one summand from each subspace).

1. Results
   (a) Condition for a direct sum **The only way to write zero as sum of one element from each summand space is all zeros iff the sum is a direct sum.**
   (b) Condition for a direct sum of two subspaces The intersection of the two subspaces is zero iff the sum is a direct sum.

1.1.7 | Linear Combination

A linear combination is the sum of some list of vectors with each one multiplied by a coefficient from $\mathbb{F}$

1.1.8 | Linear (In)Dependence

A list of vectors is linearly independent if the only coefficients in a linear combination equal to zero are all zeros. (The only $a_1, \ldots, a_n$ s.t. $a_1v_1 + \cdots + a_nv_n = 0$ is $0, \ldots, 0$) Equivalent: A vector is linearly dependent in a list (and that list is linearly dependent) if it can be written as a linear combination of other vectors in the list. Any list that is not linearly dependent is linearly independent.

1.1.9 | Span

The span of a list is all linear combinations of that list

1.1.10 | Basis

The basis of a vector space is a linearly indepnedent list of the elements in that vector space that spans the vector space (whose span is the vector space). A list of vetors is a basis if there is exactly one way to write every vector as a linear combination of the basis.

1. Results
   (a) All bases of a vector space are the same length
   (b) A linearly indpendent or spanning list of the right length is a basis (buy one get one free)

1.1.11 | Dimension

The dimension of a subspace is the length of it's basis. If the basis does not exist (infinitely long), then the space is infinite dimensional.

1.1.12 | Elementry Matrix

A matrix that applies exactly one valid "row operation": multiply a row, add one row to another, swap row orders.

1.1.13 | Nonsingular / invertible matrix

A non-singular matrix is a matrix that has an inverse, and whose determinant is not zero.

1.2 | Linear Transformations

1.2.1 | Linearity

A transformation is linear if it satisfies additivity (adding inside/outside same) and homogeneity (scalar multiplying inside/outside same).

1.2.2 | Injective

When the outputs being the same implies the inputs were the same. (Mapping is one to one; each element is mapped to atmost once).

1.2.3 | Surjective

When every element in the codomain is in the range (Mapping is onto the codomain; each element mapped to atleast once).

1.2.4 | Linear Map

A map from one vector space to another that is linear (satisfies additivity and homogeniety)

1. Properties
   (a) Linear maps from one space to another is a subspace
   (b) Algebraic Properties
      i. Associative: $T_1\left(T_2T_3\right) = \left(T_1T_2\right)T_3$
      ii. Identity: $IT = TI = T$
      iii. Distributive: $\left(S_1 + S_2\right)T = TS_1 + TS_2$ And the same for the other side, but you have to be careful about whether maps can be multiplied (composed).
2. Product of Linear Map The product ST of two linear maps $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ is the linear map $S(T(u))$ for $u \in U$.

1.2.5 | Image (range, column spac)

Every vector that can be a result of a linear map.

1. Properties
   (a) CHANGES AFTER RREF!
   (b) Surjectivity is the same as the column space being the domain (input space?)

1.2.6 | Kernel (null space)

Every vector that the linear map sends to zero.

1. Properties
   (a) Always includes zero
   (b) Doesn't change after RREF
   (c) Injectivity is the same as the null space being zero