1 | Thing

First, we will define equations for the distance as a function of t. (Note that both h_0 and θ are parameters but aren't shown as arguments to the function.)

$$\begin{cases} x(t) &= v_0 \cos(\theta) t \\ y(t) &= -\frac{1}{2} g t^2 + v_0 \sin(\theta) t + h_0 \end{cases}$$

Velocity is a function of h_0:

$$v_0 = \sqrt{2g(H - h_0)} \tag{1}$$

We can rewrite $x(t)$ and $y(t)$:

$$\begin{cases} x(t) &= \sqrt{2g(H - h_0)} \cos(\theta) t \\ y(t) &= -\frac{1}{2} g t^2 + \sqrt{2g(H - h_0)} \sin\theta t + h_0 \end{cases}$$

We can get t_f in terms of x:

$$\begin{aligned} t_f &= \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta)} \\ &= \frac{x_f}{v_0 \cos(\theta)} \end{aligned}$$

We can now get y_f in terms of x_f:

$$\begin{aligned} y_f &= -\frac{1}{2} g \left(\frac{x_f}{v_0 \cos(\theta)}\right)^2 + \sqrt{2g(H - h_0)} v_0 \sin(\theta) \frac{x_f}{v_0 \cos(\theta)} + h_0 \\ &= -\frac{g x_f^2}{2 v_0^2 \cos^2(\theta)} + x_f \tan(\theta) + h_0 \end{aligned}$$

We set y_f to equal 0 and differentiate both sides:

$$\begin{aligned} \frac{d}{d\theta}[0] &= \frac{d}{d\theta}\left[-\frac{g x_f^2}{2 v_0^2 \cos^2(\theta)}\right] + \frac{d}{d\theta}[x_f \tan(\theta)] + \frac{d}{d\theta}[h_0] \\ 0 &= -\frac{g}{2 v_0^2}\left(\frac{2\cos^2(\theta) x_f' x_f + 2 x_f^2 \cos(\theta)\sin(\theta)}{\cos^4(\theta)}\right) + x_f' \tan(\theta) + x_f \sec^2(\theta) \end{aligned}$$

We can simplify this into…

$$\begin{aligned} 0 = &-\frac{g}{v_0^2} \cdot x_f' x_f \cos^{-2}(\theta) - \frac{g}{v_0^2} \cdot x_f^2 \sin(\theta) \cos^{-3}(\theta) \\ &+ x_f' \tan(\theta) + x_f \sec^2(\theta) \end{aligned}$$

$$\begin{aligned} x_f' \cdot g v_0^{-2}(x_f \cos^{-2}(\theta)) - x_f' \cdot \tan(\theta) &= x_f \sec^2(\theta) - g v_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta)) \\ x_f'(g v_0^{-2}(x_f \cos^{-2}(\theta)) - \tan(\theta)) &= x_f \sec^2(\theta) - g v_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta)) \end{aligned}$$

We can finally solve for x_f':

$$x_f' = \frac{x_f \sec^2(\theta) - g v_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta))}{g v_0^{-2}(x_f \cos^{-2}(\theta)) - \tan(\theta)}$$

We now set x_f' to zero and solve for theta to get the inflection point, or where the maximum value of theta will be for our original x function. This portion is heavily borrowed from Jack (which doesn't mean much because most of my work is basically Jack's work that I've done without substituting v_0. I feel like Jack deserves most of the credit for my work to be honest.) We will also start using more noughts so this will get messy.

$$
\begin{aligned}
0 &= \frac{x_f \sec^2(\theta_0) - g{v_0}^{-2}(x_f^2 \sin(\theta_0)\cos^{-3}(\theta_0))}{g{v_0}^{-2}(x_f \cos^{-2}(\theta_0)) - \tan(\theta_0)} \\
0 &= x_f \sec^2(\theta_0) - g{v_0}^{-2}(x_f^2 \sin(\theta_0)\cos^{-3}(\theta_0) \\
x_f \sec^2(\theta_0) &= g{v_0}^{-2}(x_f^2 \sin(\theta_0)\cos^{-3}(\theta_0)) \\
\frac{\sec^2(\theta_0)\cos^3(\theta_0)}{\sin(\theta_0)} &= g{v_0}^{-2}x_f \\
\frac{\sin(\theta_0)}{\sec^2(\theta_0)\cos^3(\theta_0)} &= \frac{v_0^2}{gx_f} \\
\tan(\theta_0) &= \frac{v_0^2}{gx_f} \\
\theta_0 &= \arctan(\frac{v_0^2}{gx_f}) \\
&= \arctan(\frac{\sqrt{2g(H-h_0)}^2}{gx_f}) \\
&= \arctan(\frac{2g(H-h_0)}{gx_f}) \\
&= \arctan(\frac{2H-2h_0}{x_f})
\end{aligned}
$$

We can plug this back into the equation for y_f: