

# 1 | upper triangular matrix

def

A matrix in which all entries below the diagonal are zero

$$\backslash \left[ \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right] \backslash$$

## 1.1 | results

### 1.1.1 | Axler 5.26 Conditions for upper-triangular matrix

Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_n$  is a basis of  $V$ . The following are equivalent:

- the matrix of  $T$  with respect to  $v_1, \dots, v_n$  is upper triangular
- $Tv_j \in \text{span}\{v_1, \dots, v_j\}$  for each  $j = 1, \dots, n$
- The span of each prefix of the basis is invariant under  $T$ .

### 1.1.2 | Axler 5.27 Over $\mathbb{C}$ , every operator has an upper-triangular matrix

Suppose  $V$  is a finite-dimensional complex vector space and  $T \in \mathcal{L}(V)$ . Then  $T$  has an upper-triangular matrix wrt some basis of  $V$ .

1. intuition There are  $n$  eigenvalues (fundamental theorem of linear algebra) and each one should have a corresponding eigenvector that can sweep out a column? What happens when an eigenvalue has higher multiplicity?
2. proof
  - (a) induction on the dimension of  $V$ . use the fact that the first column can be found, then use the remaining basis vectors as a smaller subspace and do the same thing?

### 1.1.3 | Axler 5.30 Determination of invertibility from upper-triangular matrix

Suppose  $T \in \mathcal{L}(V)$  has an upper-triangular matrix wrt some basis of  $V$ . Then,  $T$  is invertible iff all the entries on the diagonal of the upper-triangular matrix are nonzero.

1. intuition
  - (a) if one of the diagonal vectors is zero, then there is an injectivity/surjectivity problem and the operator is singular
  - (b) proof is by assuming all are nonzero and showing surjective, then by contradiction.

### 1.1.4 | Axler 5.32 Determination of eigenvalues from upper-triangular matrix

Suppose  $T \in \mathcal{L}(V)$  has an upper-triangular matrix wrt some basis of  $V$ . Then the eigenvalues of  $T$  are precisely the entries on the diagonal of that upper-triangular matrix.

1. proof

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix} \quad \mathcal{M}(T - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & * \\ & \lambda_2 - \lambda & \\ & & \ddots \\ 0 & & & \lambda_n - \lambda \end{pmatrix}$$

And that second matrix is only singular when  $\lambda \in \lambda_1, \dots, \lambda_n$