

## 1 | Elastic collision

We are given that the object  $m_1$  collides with the rod with velocity  $v_0$ , and the rod is floating in free space. Given  $m_1$ ,  $v_0$ ,  $m_2$ ,  $I_0$ , and  $r$ , we are to figure out the final velocity of  $m_1$  after collision  $v_f$ , the velocity of  $m_2$  after collision  $v_{CM}$ , and of course the rotation of the rod after collision  $\omega$ .

We are assuming that this collision is elastic.

We have, then, for conservation of linear momentum:

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \quad (1)$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2} m_1 v_0^2 = \left( \frac{1}{2} m_1 v_f^2 \right) + \left( \frac{1}{2} m_2 v_{CM}^2 \right) + \left( \frac{1}{2} I_0 \omega^2 \right) \quad (2)$$

$$\Rightarrow m_1 v_0^2 = (m_1 v_f^2) + (m_2 v_{CM}^2) + (I_0 \omega^2) \quad (3)$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1 r^2 \left( \frac{v_0}{r} \right) = m_1 r^2 \left( \frac{v_f}{r} \right) + I_0 \omega \quad (4)$$

$$\Rightarrow m_1 r v_0 = m_1 r v_f + I_0 \omega \quad (5)$$

We now have a system of three equations that can be combined to solve for three unknowns  $v_f$ ,  $v_{CM}$ , and  $\omega$ .

We will begin with the kinetic energy expression:

$$m_1 v_0^2 = (m_1 v_f^2) + (m_2 v_{CM}^2) + (I_0 \omega^2) \quad (6)$$

Now, this last term seems a little concerning to deal with. Therefore, we will use the final expression to simplify it:

$$m_1 r v_0 = m_1 r v_f + I_0 \omega \quad (7)$$

$$\Rightarrow m_1 r v_0 - m_1 r v_f = I_0 \omega \quad (8)$$

and also:

$$\frac{m_1 r v_0 - m_1 r v_f}{I_0} = \omega \quad (9)$$

Therefore,  $I_0 \omega^2$  would be:

$$\frac{(m_1 r v_0 - m_1 r v_f)^2}{I_0} \quad (10)$$

Supplying that back:

$$m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + (I_0 \omega^2) \quad (11)$$

$$\Rightarrow m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + \frac{(m_1 r v_0 - m_1 r v_f)^2}{I_0} \quad (12)$$

$$\Rightarrow m_1 v_0^2 = m_1 v_f^2 + m_2 v_{CM}^2 + \frac{r^2 (m_1 v_0 - m_1 v_f)^2}{I_0} \quad (13)$$

$$\Rightarrow m_1 v_0^2 - m_1 v_f^2 = m_2 v_{CM}^2 + \frac{r^2 (m_1 v_0 - m_1 v_f)^2}{I_0} \quad (14)$$

$$\Rightarrow m_1 I_0 v_0^2 - m_1 I_0 v_f^2 = m_2 I_0 v_{CM}^2 + r^2 (m_1 v_0 - m_1 v_f)^2 \quad (15)$$

At this point, we realize that we haven't use the first expression yet. Therefore, we will endeavour to use it to rid of the one  $v_{CM}$  we have here.

Performing the actual solution digitally:

$$v_{cm} = \frac{2I_0 m_1 v_0}{m_1 m_2 r^2 + I_0 m_1 + I_0 m_2} \quad (16)$$

$$v_f = \frac{(m_1 m_2 r^2 + I_0 m_1 - I_0 m_2) v_0}{m_1 m_2 r^2 + I_0 m_1 + I_0 m_2} \quad (17)$$

and finally, we have

$$\omega = \frac{2m_1 m_2 r v_0}{m_1 m_2 r^2 + I_0 m_1 + I m_2} \quad (18)$$

## 2 | Rigid Body Kinetic Energy

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2} m_i v_i^2 \quad (19)$$

Because of the fact a point  $v_i$  can be defined as a sum of the velocity from the origin plus the displace from origin ( $v_i = v_{CM} + v'_i$ ), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM} + v'_i)(V_{CM} + v'_i) \quad (20)$$

Now, we shall foil the above expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM}^2 + 2v_{CM} v'_i + v_i'^2) \quad (21)$$

$$= \sum_i \frac{1}{2} m_i V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (22)$$

$$= \frac{1}{2} M V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (23)$$

$$= \frac{1}{2} M V_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \quad (24)$$

At which point, we realize that we have in the middle arrived at the definition of the center of mass in the reference frame of the center of mass—meaning that it is indeed 0 because the center of mass is at the origin of the center of mass. Moving on, then:

$$KE = \frac{1}{2}MV_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2}m_i v'^2_i \quad (25)$$

$$= \frac{1}{2}MV_{CM}^2 + \frac{1}{2} \sum_i m_i v'^2_i \quad (26)$$

We now note that this is a rigid body. Therefore, each point mass will not "slip" from the center of mass—they remain the same distance relative each other. Therefore, we can leverage the no slipping assumption to claim that  $v'_i = r'_i \omega$ .

$$KE = \frac{1}{2}MV_{CM}^2 + \frac{1}{2} \sum_i m_i (r'_i \omega)^2 \quad (27)$$

$$= \frac{1}{2}MV_{CM}^2 + \frac{1}{2} \sum_i m_i r'^2_i \omega^2 \quad (28)$$

$$= \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\omega^2 \sum_i m_i r'^2_i \quad (29)$$

$$= \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I\omega^2 \blacksquare \quad (30)$$