

1 | Thing

First, we will define equations for the distance as a function of t . (Note that both h_0 and θ are parameters but aren't shown as arguments to the function.)

$$\begin{cases} x(t) = v_0 \cos(\theta)t \\ y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t + h_0 \end{cases}$$

Velocity is a function of h_0 :

$$v_0 = \sqrt{2g(H - h_0)} \quad (1)$$

We can rewrite $x(t)$ and $y(t)$:

$$\begin{cases} x(t) = \sqrt{2g(H - h_0)} \cos(\theta)t \\ y(t) = -\frac{1}{2}gt^2 + \sqrt{2g(H - h_0)} \sin(\theta)t + h_0 \end{cases}$$

We can get t_f in terms of x :

$$\begin{aligned} t_f &= \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta)} \\ &= \frac{x_f}{v_0 \cos(\theta)} \end{aligned}$$

We can now get y_f in terms of x_f :

$$\begin{aligned} y_f &= -\frac{1}{2}g\left(\frac{x_f}{v_0 \cos(\theta)}\right)^2 + \sqrt{2g(H - h_0)} v_0 \sin(\theta) \frac{x_f}{v_0 \cos(\theta)} + h_0 \\ &= -\frac{gx_f^2}{2v_0^2 \cos^2(\theta)} + x_f \tan(\theta) + h_0 \end{aligned}$$

We set y_f to equal 0 and differentiate both sides:

$$\begin{aligned} \frac{d}{d\theta}[0] &= \frac{d}{d\theta}\left[-\frac{gx_f^2}{2v_0^2 \cos^2(\theta)}\right] + \frac{d}{d\theta}[x_f \tan(\theta)] + \frac{d}{d\theta}[h_0] \\ 0 &= -\frac{g}{2v_0^2} \left(\frac{2 \cos^2(\theta) x'_f x_f + 2x_f^2 \cos(\theta) \sin(\theta)}{\cos^4(\theta)} \right) + x'_f \tan(\theta) + x_f \sec^2(\theta) \end{aligned}$$

We can simplify this into...

$$\begin{aligned} 0 &= -\frac{g}{v_0^2} \cdot x'_f x_f \cos^{-2}(\theta) - \frac{g}{v_0^2} \cdot x_f^2 \sin(\theta) \cos^{-3}(\theta) \\ &\quad + x'_f \tan(\theta) + x_f \sec^2(\theta) \\ x'_f \cdot gv_0^{-2}(x_f \cos^{-2}(\theta)) - x'_f \cdot \tan(\theta) &= x_f \sec^2(\theta) - gv_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta)) \\ x'_f(gv_0^{-2}(x_f \cos^{-2}(\theta)) - \tan(\theta)) &= x_f \sec^2(\theta) - gv_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta)) \end{aligned}$$

We can finally solve for x'_f :

$$x'_f = \frac{x_f \sec^2(\theta) - gv_0^{-2}(x_f^2 \sin(\theta) \cos^{-3}(\theta))}{gv_0^{-2}(x_f \cos^{-2}(\theta)) - \tan(\theta)}$$

We now set x'_f to zero and solve for theta to get the inflection point, or where the maximum value of theta will be for our original x function. This portion is heavily borrowed from Jack (which doesn't mean much because most of my work is basically Jack's work that I've done without substituting v_0 . I feel like Jack deserves most of the credit for my work to be honest.) We will also start using more noughts so this will get messy.

$$0 = \frac{x_f \sec^2(\theta_0) - gv_0^{-2}(x_f^2 \sin(\theta_0) \cos^{-3}(\theta_0))}{gv_0^{-2}(x_f \cos^{-2}(\theta_0)) - \tan(\theta_0)}$$

$$0 = x_f \sec^2(\theta_0) - gv_0^{-2}(x_f^2 \sin(\theta_0) \cos^{-3}(\theta_0))$$

$$x_f \sec^2(\theta_0) = gv_0^{-2}(x_f^2 \sin(\theta_0) \cos^{-3}(\theta_0))$$

$$\frac{\sec^2(\theta_0) \cos^3(\theta_0)}{\sin(\theta_0)} = gv_0^{-2} x_f$$

$$\frac{\sin(\theta_0)}{\sec^2(\theta_0) \cos^3(\theta_0)} = \frac{v_0^2}{gx_f}$$

$$\tan(\theta_0) = \frac{v_0^2}{gx_f}$$

$$\theta_0 = \arctan\left(\frac{v_0^2}{gx_f}\right)$$

$$= \arctan\left(\frac{\sqrt{2g(H - h_0)}^2}{gx_f}\right)$$

$$= \arctan\left(\frac{2g(H - h_0)}{gx_f}\right)$$

$$= \arctan\left(\frac{2H - 2h_0}{x_f}\right)$$

We can plug this back into the equation for y_f :