1 | **Derived Equations of Kinematics**

Just for review:

$$\begin{cases} x(t) = x_0 + v_0 t + \frac{1}{2}a_0 t^2 \\ v(t) = v_0 + a_0 t \\ v^2(x) = v_0{}^2 + 2a(x - x_0) \\ v_{av} = v_{av}\Delta t \end{cases} \tag{1}$$
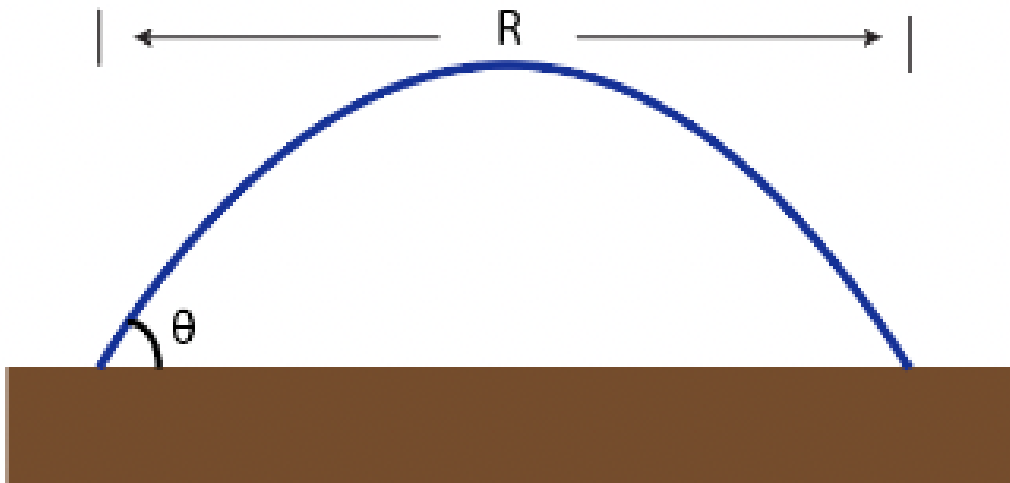
Where,

$$\begin{cases} v_{av} = \frac{1}{2}(v_1 + v_2) \\ \Delta x = x_2 - x_1 \\ \Delta t = t_2 - t_1 \end{cases} \tag{2}$$

2 | **Projectile Problem**

2.1 | **Proving that** $R = \frac{2sin(2\theta)v_0{}^2}{g}$

We begin by establishing the figure by which this model is based on:

We set the velocity at which the launcher launches object is launched from as v_0. Furthermore, we set the initial time at which the object is launched as t_0, the initial position of the trajectory as (x_0, y_0), the final position of the trajectory as (x_f, y_f). We will, as with tradition, set the acceleration due to gravity as g.

To begin, we will create some basic assumptions and statements that we could make given the setup of the problem:

$$\begin{cases} x_0 = 0 \\ y_0 = 0 \\ x_f = R \\ y_f = 0 \end{cases} \tag{3}$$

These assumptions indicate that the object starts by being launched at a position we denote as $(0,0)$ in the system, and travels right (to the positive direction, under current system) such that — after a presumably parabolic trajectory — it reaches position $(R, 0)$: R units out and on the ground $y = 0$ again.

We begin approaching this problem by separating out the third equation of kinematics (relating to position) into a set of parametric equations involving the x and y positions as separate parameters. That is,

$$\begin{cases} x(t) = \frac{-1}{2}g_x t^2 + v_0 \cos\theta t + x_0 \\ y(t) = \frac{-1}{2}g_y t^2 + v_0 \sin\theta t + y_0 \end{cases} \tag{4}$$

Therefore, replacing the assumptions regarding the start positions, we could therefore make the following statements regarding the end position of the projectile:

$$\begin{cases} x_f = R = \frac{-1}{2}g_x {t_f}^2 + v_0 \cos\theta t_f \\ y_f = 0 = \frac{-1}{2}g_y {t_f}^2 + v_0 \sin\theta t_f \end{cases} \tag{5}$$

However, we could simplify this statement even further by the realization that, under the current axis system, there is *no* x-component acceleration due to gravity (that is, $g_x = 0$.) With this claim, we conclude that:

$$\begin{cases} x_f = R = v_0 \cos\theta t_f \\ y_f = 0 = \frac{-1}{2}g {t_f}^2 + v_0 \sin\theta t_f \end{cases} \tag{6}$$

The goal for this deduction is to figure a function R — distance traveled of projectile — given angle θ. Hence, a good method by which this system could be combined into one statement is by performing variable substitution upon t_f, hence eliminating time as a variable in the system and combining the two statements.

We first transpose the first statement such that it is w.r.t. t_f:

$$R = v_0 \cos\theta t_f \tag{7}$$

$$\Rightarrow \frac{R}{v_0 \cos\theta} = t_f \tag{8}$$

Substituting this resulting expression into the statement for y_f, and simplifying, we deduct that:

$$0 = \frac{-1}{2}g\left(\frac{R}{v_0 \cos\theta}\right)^2 + v_0 \sin\theta\left(\frac{R}{v_0 \cos\theta}\right) \tag{9}$$

$$0 = \frac{-1}{2}g\frac{R^2}{{v_0}^2 \cos^2\theta} + \tan\theta R \tag{10}$$

$$0 = R\left(\frac{-1}{2}g\frac{R}{{v_0}^2 \cos^2\theta} + \tan\theta\right) \tag{11}$$

$$\tag{12}$$

Given this expression, we could deduct two statements of the roots of this function w.r.t. R:

$$\begin{cases} 0 = R \\ 0 = \frac{-1}{2}g\frac{R}{v_0{}^2\cos^2\theta} + \tan\theta \end{cases} \tag{13}$$

A R represents the total distance traveled *after* the projectile motion, we disregard the first case and proceed to isolate R in the latter case.

$$0 = \frac{-1}{2}g\frac{R}{v_0{}^2\cos^2\theta} + \tan\theta \tag{14}$$

$$\Rightarrow 2\tan\theta = g\frac{R}{v_0{}^2\cos^2\theta} \tag{15}$$

$$\Rightarrow \frac{2\tan\theta}{g} = \frac{R}{v_0{}^2\cos^2\theta} \tag{16}$$

$$\Rightarrow R = \frac{2\tan\theta v_0{}^2\cos^2\theta}{g} \tag{17}$$

$$\Rightarrow R = \frac{2\frac{\sin\theta}{\cos\theta}\cos^2\theta v_0{}^2}{g} \tag{18}$$

$$\Rightarrow R = \frac{2\sin\theta\cos\theta v_0{}^2}{g} \tag{19}$$

$$\tag{20}$$

At this point, we apply the following Double Angle Formulae:

$$sin(2\theta) = 2\sin\theta\cos\theta \tag{21}$$

Which would simplify the resulting expression to:

$$R = \frac{2sin(2\theta)v_0{}^2}{g} \tag{22}$$

2.2 | **Maximizing** $R(\theta)$

Given…

$$R(\theta) = \frac{2sin(2\theta)v_0{}^2}{g} \tag{23}$$

there are two ways by which $\theta = 45° = \frac{\pi}{4}rad$ could be determined to be a local maximum of $R(\theta)$.

2.2.1 | **…via sinusoidal periods**

The function $R(\theta)$ is predicated upon one independent variable: that of θ. It appears once in the expression for $R(\theta)$, in the term $\sin(2\theta)$.

To maximize $R(\theta)$, therefore, $\sin(2\theta)$ must be maximized. The sine function is sinusoidal (definitely so), and hence its range is $[-1, 1]$. Its maximum value is therefore $\sin(2\theta) = 1$.

Hence, we could figure the value of 2θ provided that we are limited to one period $0 \le 2\theta \le 2\pi$.

$$1 = \sin(2\theta) \tag{24}$$

$$\Rightarrow 2\theta = \arcsin(1) \tag{25}$$

$$\Rightarrow 2\theta = \frac{\pi}{2} \tag{26}$$

$$\Rightarrow \theta = \frac{\pi}{4} \tag{27}$$

Hence, $\theta = \frac{\pi}{4} rad = 45°$ for $R(\theta)$ to be maximized.

2.2.2 | ...via critical points

Local maximums of $R(\theta)$ appears at critical points of $R(\theta)$ where $R'(\theta)$ changes signs from positive to negative. To figure this, we first find the critical points of $R(\theta)$ via the first derivative test.

First, we figure the first derivative of this function:

$$R(\theta) = \frac{2\sin(2\theta)v_0{}^2}{g} \tag{28}$$

$$\Rightarrow \frac{d}{d\theta}R(\theta) = \frac{d}{d\theta}\frac{2sin(2\theta)v_0{}^2}{g} \tag{29}$$

$$\Rightarrow \frac{d}{d\theta}R(\theta) = \frac{2v_0{}^2}{g}\frac{d}{d\theta}\sin(2\theta) \tag{30}$$

$$\Rightarrow \frac{d}{d\theta}R(\theta) = \frac{2v_0{}^2}{g}2\cos(2\theta) \tag{31}$$

$$\tag{32}$$

To apply the first derivative test, we set this expression equal to 0 and solve for θ.

$$R(\theta) = \frac{2v_0{}^2}{g}2\cos(2\theta) \tag{33}$$

$$\Rightarrow 0 = \frac{2v_0{}^2}{g}2\cos(2\theta) \tag{34}$$

$$\Rightarrow 0 = \underbrace{\frac{2v_0{}^2}{g}2}_{not\ involving\ \theta}\cos(2\theta) \tag{35}$$

$$\Rightarrow 0 = \cos(2\theta) \tag{36}$$

$$\tag{37}$$

If we again limit ourselves to one period $0 \leq 2\theta \leq 2\pi$, $cos(2\theta)$ takes on the value of 0 at $\theta = \frac{\pi}{4}$ and $\theta = \frac{-\pi}{4}$.

In the latter case, the second derivative function travels from negative to positive — making it a local *minimum*, which is not the point we are seeking. The former case does satisfy the transition between positive to negative: making it, per the first derivative test, a local maximum.

Hence, $\theta = \frac{\pi}{4} rad = 45°$ for $R(\theta)$ to be maximized.

3 | **Collision Problem**

Prove that, when an object is dropped and another fired, they will collide as long as the projectile is aimed at the target and reaches the vertical path of the target regardless of the launch velocity.

3.1 | **Modeling Collision**

We set the projectile to be m_1, and the vertically dropping object to be m_2. We also define the horizontal distance of the x-components between m_1's start's position and m_2 as x_0. Furthermore, we set the initial height of m_2 as h_0. We set height of the collision location as h.

For the two objects to collide at a time t_1, they have to share x and y coordinate values. Meaning, both components of their equation of kinematics regarding position have to match. To figure such a t_1, we first set up two sets of equations for kinematics that would model the situation.

3.1.1 | **To model** m_1**....**

We begin with the equations of kinematics revised for this situation:

$$\begin{cases} x(t) = v_0 \cos \theta t \\ y(t) = \frac{-1}{2} g t^2 + v_0 \sin \theta t \end{cases} \tag{38}$$

As with the last problem, we notice that $g_x = 0$ because there is no x-component acceleration due to gravity under our system. Furthermore, there are no constants in the initial state of this model due to our definition of the projectile being at $(0,0)$ at the start of the experiment.

Due to the fact that we know m_1 is aimed towards the object, we know that the projectile is aimed such that $\tan(\theta) = \frac{h_0}{x_0}$ ("up towards object, that's right facing x_0 away"). Hence, $\theta = \tan^{-1}(\frac{h_0}{x_0})$.

$$\begin{cases} x(t_1) = v_0 \cos(\tan^{-1}(\frac{h_0}{x_0})) t_1 \\ y(t_1) = \frac{-1}{2} g {t_1}^2 + v_0 \sin(\tan^{-1}(\frac{h_0}{x_0})) t_1 \end{cases} \tag{39}$$

Given we are leveraging an compound trignometric expression in this statement, we leverage the following identity:
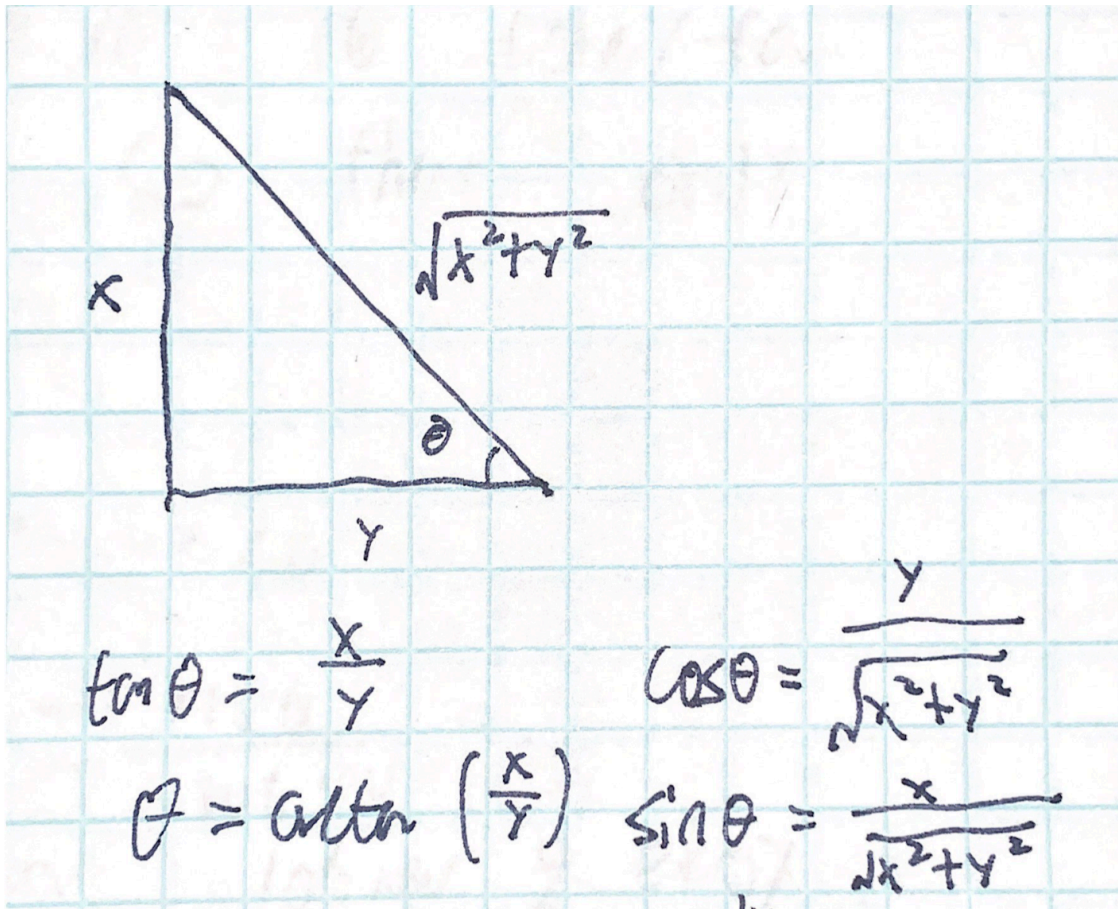
As the arbitrary right triangle models the sides from which the $arctan$ expression could be constructed, which we then leverage to figure its values for sin and cos.

In this case, $x = h_0$, $y = x_0$. Therefore, the expression could be simplified to:

$$\begin{cases} x(t_1) = \frac{v_0 t_1 x_0}{\sqrt{h_0{}^2 + x_0{}^2}} \\ y(t_1) = \frac{-1}{2} g t_1{}^2 + \frac{v_0 t_1 h_0}{\sqrt{h_0{}^2 + x_0{}^2}} \end{cases} \tag{40}$$

3.1.2 | **To model** m_2 ....

As similar to before, we begin with the equations of kinematics for m_2's situation:

$$\begin{cases} x(t) = x_0 \\ y(t) = \frac{-1}{2} g t^2 + y_0 \end{cases} \tag{41}$$

We again notice that $g_x = 0$ due to the same line of reasoning. In fact, as the x-component of position of m_2 never changes, it stays constant at x_0.

We also notice that, as the object is simply in free fall, there is no v_0 to be had initially. Hence, that term is set to 0 as well. Furthermore, as part of the setup of the problem, the falling object starts its fall at a height h.

$$\begin{cases} x(t_1) = x_0 \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + h_0 \end{cases} \tag{42}$$

3.1.3 | **Modeling the process of fall**

Given the two systems determined for m_1 and m_2 respectively:

$$\begin{cases} x(t_1) = \frac{v_0 t_1 x_0}{\sqrt{h_0{}^2 + x_0{}^2}} \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + \frac{v_0 t_1 h_0}{\sqrt{h_0{}^2 + x_0{}^2}} \end{cases} \tag{43}$$

$$\begin{cases} x(t_1) = x_0 \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + h_0 \end{cases} \tag{44}$$

We first focus on the expression for m_1. Provided the setup of the problem, we understand that the launcher is being launched in a constant velocity.

Therefore, the expression $v_0 t_1$ models the expression for the direct distance of travel of the projectile over the time span. This could also be understood as the hypotenuse of the triangle formed by x_0 — the horizontal component of the travel of the projectile — and h_0, the vertical component of the travel.

Hence, $v_0 t_1 = \sqrt{h_0{}^2 + x_0{}^2}$, leveraging pythagoras. Therefore, $\frac{v_0 t_1}{\sqrt{h_0{}^2 + x_0{}^2}} = 1$ at the point of collision.

Substituting this expression onto that of the m_1, we derive that

$$\begin{cases} x(t_1) = 1 x_0 \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + 1 h_0 \end{cases} \tag{45}$$

Which is equivalent to the expression for both x and y components of m_2. Hence the proof of collision is complete.

3.2 | **Height of Collision**

To figure the height of collision h, we need to leverage the following above-deducted expressions of the positions of m_1 and m_2 respectively at the time of collision.

$$\begin{cases} x(t_1) = \frac{v_0 t_1 x_0}{\sqrt{h_0{}^2 + x_0{}^2}} \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + \frac{v_0 t_1 h_0}{\sqrt{h_0{}^2 + x_0{}^2}} \end{cases} \tag{46}$$

$$\begin{cases} x(t_1) = x_0 \\ y(t_1) = \frac{-1}{2}gt_1{}^2 + h_0 \end{cases} \tag{47}$$

We begin by leveraging the former equation of the former system to isolate a value for t_1. As the position of collision happens at x_0 when at t_1, we could set $x(t_1) = x_1$ when at t_1, we could set $x(t_1) = x_0$

$$x_0 = \frac{v_0 t_1 x_0}{\sqrt{h_0{}^2 + x_0{}^2}} \tag{48}$$

$$\Rightarrow t_1 = \frac{\sqrt{h_0{}^2 + x_0{}^2}}{v_0} \tag{49}$$

The height of collision h, therefore, is the y-component of the location of m_2 at t_1, that is, $y(t_1)$ per the second system of position equations.

Performing variable substitution of t_1 on the latter equation of the aforementioned system, we could deduct that:

$$h(v_0, x_0, h_0) = \frac{-1}{2} g \left(\frac{\sqrt{h_0{}^2 + x_0{}^2}}{v_0} \right)^2 + h_0 \tag{50}$$

$$= \frac{-1}{2} g \frac{(h_0{}^2 + x_0{}^2)}{v_0{}^2} + h_0 \tag{51}$$