PROBLEM SET #2:
HOW DOES A QUBIT CHANGE?

1) Consider three Hermitian operators, A, B and C. For what values of the complex scalars α and β is the 'double-bracket' $\alpha\left[\beta\left[A,B\right],C\right]$ also Hermitian?

2) Consider a Hamiltonian operator $\mathbf{H} = \frac{\hbar\omega}{2}\sigma\cdot\mathbf{n}$. Solve the time independent Schrödinger equation $\mathbf{H}\left|E_j\right\rangle = E_j\left|Ej\right\rangle$ and interpret the j^{th} solution.

3) Consider a single spin in the state $|u\rangle$ acted on by the Hamiltonian operator $\mathbf{H} = \frac{\hbar\omega}{2}\sigma_z$ and subsequently measured along the x-direction. We let t time units and measure the spin in the y-direction. What are the possible outcomes and what are the probabilities for those outcomes?

4) Show that the Pauli group G_1, consisting of the three Pauli matrices X, Y and Z, and the identity I, generates any 2×2 Hermitian matrix.

5) Consider the set of matrices $G_2 = G_1 \otimes G_1 = \{ A \otimes B |\, A, B \in G_1 \}$, where G_1 is the Pauli group. Can every 4×4 Hermitian matrix be written as a linear combination of matrices in G_2?

6) For any Hermitian operator M, let $\langle M \rangle$ denote its expected value and $\Delta M = \sqrt{\left\langle (M - \langle M \rangle)^2 \right\rangle}$ is its uncertainty (measured as the standard deviation). In the case of a single qubit, use the standard Pauli group basis to represent this expected value and resulting standard deviation in terms of the scalar coefficients of M in that basis.

7) Show that dyadic unitarity isn't closed under composition.

8) Let K_n denote the operator that acts on n qubits by swapping the first qubit with the last one. Use it to show that the Fourier matrix F_N satisfies the following recursion:

$$F_N = \frac{1}{\sqrt{2}} \begin{bmatrix} I^{\otimes(n-1)} & D_{N/2} \\ I^{\otimes(n-1)} & -DN/2 \end{bmatrix} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix} K_n$$

where $N = 2^n$.

9) A *Fredkin gate* swaps 101 and 110 while leaving the other six arguments unchanged. Show that this gate is universal for invertible computation.

10) Consider a universe consisting of only the four points, $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Moreover, endow this universe with periodic boundary conditions so that it wraps around itself. So, moving one step to the right from $(1,0)$ you find yourself back in $(0,0)$. Similarly, moving down from $(0,0)$ you arrive at $(0,1)$ etc. Finally, measure time in increments of $\frac{1}{N}$. Let $\Psi(x,y,t)$ denote the wave function at each point in this toy spacetime. Place a particle at $(0,0)$ at time $t = .0$ and let it propagate freely.

(10.1) Find the probability distribution of the particle's location after two steps.

(10.2) Find the probability distribution of the particle's location after three and four steps.

(10.3) What are the correlations between the particle locations after two, three and four time steps?

(10.4) Find the probability distribution of the particle's location after $2k$ and $2\ell + 1$ steps.

(10.5) Does the correlation between the particle's location depend only on the elapsed time interval?