

1 | the parabola

A parabola (defined by a focus point and a directrix line) is the set of points

$$\{P : |FP| = \text{distance from } P \text{ to } L\}$$

1.1 | polar equation that describes the parabola

$$\begin{aligned} r &= r(\theta) \\ &= d - r \cos \theta \\ r + r \cos \theta &= d \\ r(1 + \cos \theta) &= d \\ r &= \frac{d}{1 + \cos \theta} \end{aligned}$$

1.2 | show that cartesian version matches what we expect

Checked using desmos.

$$\begin{aligned} r &= \frac{d}{1 + \cos \theta} \\ x &= \left(\frac{d}{1 + \cos \theta} \right) \cos \theta \\ &= \frac{d \cos \theta}{1 + \cos \theta} \\ y &= \left(\frac{d}{1 + \cos \theta} \right) \sin \theta \\ &= \frac{d \sin \theta}{1 + \cos \theta} \\ x^2 + y^2 &= \frac{d^2}{(1 + \cos \theta)^2} \\ x^2 + y^2 &= \frac{d^2}{\left(1 + \frac{x}{\sqrt{x^2 + y^2}} \right)^2} \\ (x^2 + y^2) \left(1 + \frac{x}{\sqrt{x^2 + y^2}} \right)^2 &= d^2 \\ (x^2 + y^2) \left(\frac{x^2}{x^2 + y^2} + \frac{2x}{\sqrt{x^2 + y^2}} + 1 \right) &= d^2 \\ (x^2 + y^2) \left(\frac{x^2}{x^2 + y^2} + \frac{2x\sqrt{x^2 + y^2}}{x^2 + y^2} + 1 \right) &= d^2 \\ x^2 + 2x\sqrt{x^2 + y^2} + \frac{1}{x^2 + y^2} &= d^2 \end{aligned}$$

ugh. too hard.

$$\begin{aligned}
 r &= d - r \cos \theta \\
 &= d - x \\
 \sqrt{x^2 + y^2} &= d - x \\
 x^2 + y^2 &= d^2 + x^2 - 2dx \\
 y^2 &= d^2 - 2dx \\
 y^2 - d^2 &= -2dx \\
 \frac{d^2 - y^2}{2d} &= x \\
 x &= -\frac{y^2}{2d} + \frac{d}{2}
 \end{aligned}$$

more scratch work below:

$$\begin{aligned}
 \frac{d \sin \theta}{1 + \cos \theta} &= \frac{d \cos \theta}{1 + \cos \theta} \\
 (d \sin \theta)(1 + \cos \theta) &= (d \cos \theta)(1 + \cos \theta) \\
 (d \sin \theta) &= (d \cos \theta) \\
 \frac{d}{1 + \cos \theta} &= \sqrt{\frac{d^2 \cos^2 \theta + d^2 \sin^2 \theta}{(1 + \cos \theta)^2}} \\
 d &= \sqrt{d^2 \cos^2 \theta + d^2 \sin^2 \theta}
 \end{aligned}$$

1.3 | vertex

$$\left(\frac{d}{2}, 0\right)$$

2 | general form

2.1 | re-derivation

$$\begin{aligned}
 e &= \frac{r}{d - r \cos \theta} \\
 r &= r(\theta) \\
 &= e(d - r \cos \theta) \\
 &= de - er \cos \theta \\
 r + er \cos \theta &= de \\
 r(1 + e \cos \theta) &= de \\
 r &= \frac{de}{1 + e \cos \theta}
 \end{aligned}$$

Link to the desmos.

2.2 | ellipse when $0 < e < 1$

$$\begin{aligned}
r &= e(d - r \cos \theta) \\
&= e(d - x) \\
\sqrt{x^2 + y^2} &= e(d - x) \\
x^2 + y^2 &= (e(d - x))^2 \\
x^2 + y^2 &= e^2 (d^2 + x^2 - 2dx) \\
x^2 + y^2 &= d^2 e^2 + e^2 x^2 - 2de^2 x \\
-e^2 x^2 + 2de^2 x + x^2 + y^2 &= d^2 e^2 \\
(1 - e^2) x^2 + 2de^2 x + y^2 &= d^2 e^2 \\
(1 - e^2) \left(x^2 + \frac{2de^2}{1 - e^2} x \right) + y^2 &= d^2 e^2 \\
\left(x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} \\
\left(x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} \\
\left(\left(\frac{de^2}{1 - e^2} \right)^2 + x^2 + \frac{2de^2}{1 - e^2} x \right) + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)} + \left(\frac{de^2}{1 - e^2} \right)^2 \\
\left(x + \left(\frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2 (1 - e^2) + d^2 e^4}{(1 - e^2)^2} \\
\left(x + \left(\frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2 - d^2 e^4 + d^2 e^4}{(1 - e^2)^2} \\
\left(x + \left(\frac{de^2}{1 - e^2} \right) \right)^2 + \frac{y^2}{(1 - e^2)} &= \frac{d^2 e^2}{(1 - e^2)^2} \\
\frac{(1 - e^2)^2 \left(x + \left(\frac{de^2}{1 - e^2} \right) \right)^2}{d^2 e^2} + \frac{(1 - e^2) y^2}{d^2 e^2} &= 1 \\
\frac{((1 - e^2)x + de^2)^2}{d^2 e^2} + \frac{(1 - e^2) y^2}{d^2 e^2} &= 1 \\
x_0 &= -\frac{de^2}{1 - e^2} \\
y_0 &= 0 \\
a &= \frac{de}{1 - e^2} \\
b &= \frac{de}{\sqrt{1 - e^2}}
\end{aligned}$$

for $e < 1$.

The desmos.

2.3 | hyperbola

Going back to one of the previous equations

$$\frac{(1-e^2)^2 \left(x + \left(\frac{de^2}{1-e^2} \right) \right)^2}{d^2 e^2} + \frac{(1-e^2)y^2}{d^2 e^2} = 1$$

$$\frac{(1-e^2)^2 \left(x + \left(\frac{de^2}{1-e^2} \right) \right)^2}{d^2 e^2} - \frac{(e^2-1)y^2}{d^2 e^2} = 1$$

That makes

$$x_0 = -\frac{de^2}{1-e^2}$$

$$y_0 = 0$$

$$a = \frac{de}{1-e^2}$$

$$b = \frac{de}{\sqrt{e^2-1}}$$

b will be complex unless $e > 1$.

2.4 | degenerate ellipse

It is a circle when $a^2 = b^2$, aka

$$\frac{d^2 e^2}{(1-e^2)^2} = \frac{d^2 e^2}{1-e^2}$$

$$(1-e^2)^2 = 1-e^2$$

$$1-e^2 = 1$$

$$e^2 = 0$$

$$e = 0$$

You have to take the limit. You get

$$x^2 y^2 = d^2 e^2 = r^2$$

What could d possibly be for $\lim_{e \rightarrow 0} de \neq 0$?