

## 1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters  $h_0$  and  $\theta_0$  that would best create a maximum launch distance  $x_f$ .

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing  $x$  value of the starting position of the marble, but on the "ground") as  $(-w, 0)$ , where  $w$  is the width of the wooden block. Therefore, we derive the  $x$ -value of the location of the launch of the projectile as  $x = 0$ . We define the direction towards with the marble is launching as positive- $x$ , so as the marble rolls, its position's  $x$  value increases. We will define the location of the marble before starting as positive  $y$ , and as the marble decreases in height, its position's  $y$  value decreases.

We define the start of the experiment as time  $t_0$ , the moment the marble leaves the track and travels as a projectile as  $t_1$ , and the end — in the moment when the marble hits the ground — as  $t_f$ . We will call the marble  $m_0$ .

## 2 | Figuring the Velocity at $t_1$

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that  $m_0$  experiences before  $t_1$  is  $\Delta h = H - h_0$ . Therefore, the potential energy expenditure is  $\Delta PE_{grav} = mg\Delta h = m_0g(H - h_0)$ . Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy  $KE = 0 + m_0g(H - h_0) = m_0g(H - h_0)$ .

For this derivation, for now, we ignore  $KE_{rotational}$ , hence, we could roughly deduct the statement that  $KE_{translational} \approx m_0g(H - h_0)$ .

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at  $t_1$  named  $\vec{v}_0$ .

$$m_0g(H - h_0) = \frac{1}{2}m_0\vec{v}_0^2 \quad (1)$$

$$g(H - h_0) = \frac{1}{2}\vec{v}_0^2 \quad (2)$$

$$2g(H - h_0) = \vec{v}_0^2 \quad (3)$$

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (4)$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} v_{0x} = \sqrt{2g(H - h_0)}\cos(\theta_0) \\ v_{0y} = \sqrt{2g(H - h_0)}\sin(\theta_0) \end{cases}$$

## 3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for  $x_f$  w.r.t.  $v_{0y}$ ,  $v_{0x}$ ,  $h_0$ ,  $m_0$ .

### 3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for  $x_f$ .

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as,

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above,  $x = 0$
- the start y-position of the marble at launch is, as defined above,  $y = h_0$

### 3.2 | Setup for Solution

We first create a set of equations modeling the location of the marble at  $t_f$ .

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f \sqrt{2g(H - h_0)} \cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{-1}{2}gt_f^2 + t_f \sqrt{2g(H - h_0)} \sin(\theta_0) + h_0 \end{cases}$$

We first solve for  $t_f$ , and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \quad (5)$$

Finally, we substitute the definition of  $t_f$  into  $y(t_f)$ .

$$y(t_f) = 0 = \frac{-1}{2}g \left( \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \right)^2 + \frac{x_f}{\sqrt{2g(H - h_0)} \cos(\theta_0)} \sqrt{2g(H - h_0)} \sin(\theta_0) + h_0 \quad (6)$$

We will now proceed to simplify the expression further

$$0 = \frac{-1}{4} \frac{x_f^2}{(H - h_0) \cos^2(\theta_0)} + x_f \tan(\theta_0) + h_0 \quad (7)$$

$$= \frac{-1}{4} \frac{x_f^2}{(H - h_0)} \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (8)$$

$$= \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (9)$$

### 3.3 | Solution at $\theta_0 = 0$

We begin by first solving for the expression for  $x_f^2$  at  $\theta_0 = 0$ .

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (10)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 + h_0 \quad (11)$$

$$-h_0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \quad (12)$$

$$4h_0 = \frac{1}{(H - h_0)} x_f^2 \quad (13)$$

$$x_f^2 = 4h_0(H - h_0) \quad (14)$$

### 3.3.1 | Finding $\frac{dx_f^2}{dh_0}$

Remember, given the setup of the problem,  $x_f$  is optimized when  $x_f^2$  is optimized. This is due to the fact that  $x_f$  could not be negative, and it is representing a maximum travel distance. Hence, we will optimize  $x_f^2$  for ease of calculation.

$$x_f^2 = 4h_0(H - h_0) \quad (15)$$

$$\frac{dx_f^2}{dh_0} = \frac{d}{dh_0} 4h_0(H - h_0) \quad (16)$$

$$= 4 \frac{d}{dh_0} h_0(H - h_0) \quad (17)$$

$$= 4((H - h_0) - h_0) \quad (18)$$

$$= 4H - 4h_0 - 4h_0 \quad (19)$$

$$= 4H - 8h_0 \quad (20)$$

### 3.3.2 | Optimizing $\frac{dx_f^2}{dh_0}$

The optimization of this statement is fairly simple. We set  $\frac{dx_f^2}{dh_0} = 0$ , and solve for  $h_0$ .

$$x_f^2 = 4H - 8h_0 \quad (21)$$

$$\Rightarrow 0 = 4H - 8h_0 \quad (22)$$

$$\Rightarrow 8h_0 = 4H \quad (23)$$

$$\Rightarrow h_0 = \frac{1}{2}H \quad (24)$$

Hence, the most optimal height at which to launch the marble launcher, given a horizontal launch, is half of the initial height.

## 3.4 | Solution at arbitrary $\theta_0$

### 3.4.1 | Solving and optimizing for $\frac{dx_f}{d\theta_0}$

We need to maximize  $\frac{dx_f}{d\theta_0}$  as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for  $\frac{h_0}{d\theta_0}$ . The position equations above could

be leveraged to figure a value for  $x_f$ .

1. Finding  $\frac{dx_f}{d\theta_0}$  We leverage implicit differentiation to figure a value for  $\frac{dx_f}{d\theta_0}$ . We set  $x_f$  as a differentiable function, and  $h_0$  and  $H$  as both constants.

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (25)$$

$$\Rightarrow \frac{d}{d\theta_0} 0 = \frac{d}{d\theta_0} \left( \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \right) \quad (26)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{1}{(H - h_0)} \frac{d}{d\theta_0} x_f^2 \cos^{-2}(\theta_0) + \frac{d}{d\theta_0} x_f \tan(\theta_0) + \frac{d}{d\theta_0} h_0 \quad (27)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{1}{(H - h_0)} \left( \left( \frac{d}{d\theta_0} x_f^2 \right) \cos^{-2}(\theta_0) + x_f^2 \left( \frac{d}{d\theta_0} \cos^{-2}(\theta_0) \right) \right) + \quad (28)$$

$$\left( \left( \frac{d}{d\theta_0} x_f \right) \tan(\theta_0) + \left( \frac{d}{d\theta_0} \tan(\theta_0) \right) x_f \right) + 0 \quad (29)$$

$$\Rightarrow 0 = \frac{-1}{4(H - h_0)} \left( (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) \right) + \quad (30)$$

$$\left( \frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \right) \quad (31)$$

$$\Rightarrow 0 = \frac{-1}{4(H - h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{-1}{4(H - h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) + \quad (32)$$

$$\frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \quad (33)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (34)$$

$$= \frac{-1}{4(H - h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (35)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (36)$$

$$= \frac{dx_f}{d\theta_0} \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (37)$$

$$\Rightarrow \frac{1}{4(H - h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (38)$$

$$= \frac{dx_f}{d\theta_0} \left( \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right) \quad (39)$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{\frac{(\cos^{-3}(\theta_0) \sin(\theta_0))}{2(H - h_0)} x_f^2 - \sec^2(\theta_0) x_f}{\left( \frac{-1}{2(H - h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right)} \quad (40)$$

2. Optimizing for  $x_f$  for  $\theta_0$  via  $\frac{dx_f}{d\theta_0}$  We now set  $\frac{dx_f}{d\theta_0} = 0$  in order to figure critical points for the value of  $x_f$ .

$$f \frac{dx_f}{d\theta_0} = \frac{\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{-1}{2(H-h_0)}\right)x_f\cos^{-2}(\theta_0) + \tan(\theta_0)} \quad (41)$$

$$\Rightarrow 0 = \frac{\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{-1}{2(H-h_0)}\right)x_f\cos^{-2}(\theta_0) + \tan(\theta_0)} \quad (42)$$

$$\Rightarrow 0 = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f \quad (43)$$

$$\Rightarrow \sec^2(\theta_0)x_f = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 \quad (44)$$

$$\Rightarrow \sec^2(\theta_0) = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f \quad (45)$$

$$\Rightarrow 2\sec^2(\theta_0)(H-h_0) = (\cos^{-3}(\theta_0)\sin(\theta_0))x_f \quad (46)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{\sec^2(\theta_0)} \quad (47)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos^3(\theta_0)\sec^2(\theta_0)} \quad (48)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos(\theta_0)} \quad (49)$$

$$\Rightarrow \frac{2(H-h_0)}{x_f} = \tan(\theta_0) \quad (50)$$

$$\Rightarrow \theta_0 = \arctan\left(\frac{2(H-h_0)}{x_f}\right) \quad (51)$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize  $x_f$  for a given solved  $x_f$ . To figure the actual statement that would optimize for both,

### 3.4.2 | Solving and optimizing for $x_f$

We will now return to our original expression for the final y-position ( $= 0$ ) to create an expression for  $x_f$ .

1. Solving for  $x_f$  We first take the previous expression for  $x_f$  and supply the expression for  $\theta_0$ .

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (52)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 \sec^2(\arctan(\frac{2(H - h_0)}{x_f})) + x_f \tan(\arctan(\frac{2(H - h_0)}{x_f})) + h_0 \quad (53)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} x_f^2 ((\frac{2(H - h_0)}{x_f})^2 + 1) + x_f (\frac{2(H - h_0)}{x_f}) + h_0 \quad (54)$$

$$0 = \frac{-1}{4} \frac{1}{(H - h_0)} ((2(H - h_0))^2 + x_f^2) + 2(H - h_0) + h_0 \quad (55)$$

$$0 = \frac{-1}{4} ((4(H - h_0)) + \frac{x_f^2}{(H - h_0)}) + 2(H - h_0) + h_0 \quad (56)$$

$$0 = -(H - h_0) - \frac{x_f^2}{4(H - h_0)} + 2(H - h_0) + h_0 \quad (57)$$

$$\frac{x_f^2}{4(H - h_0)} = -(H - h_0) + 2(H - h_0) + h_0 \quad (58)$$

$$x_f^2 = -4(H - h_0)(H - h_0) + 4(H - h_0)2(H - h_0) + 4(H - h_0)h_0 \quad (59)$$

$$x_f^2 = -4(H - h_0)^2 + 8(H - h_0)^2 + 4h_0(H - h_0) \quad (60)$$

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (61)$$

$$(62)$$

2. Finding  $\frac{dx_f^2}{dh_0}$  We know that, by optimizing for  $x_f^2$ ,  $x_f$  is optimized due to the setup of the problem of the behavior of the length of line.

Hence, we take the *first* derivative, though of  $x_f^2$  w.r.t.  $h_0$  and with  $H$  held constant.

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (63)$$

$$\Rightarrow \frac{dx_f^2}{dh_0} = \frac{d}{dh_0} (4(H - h_0)^2 + 4h_0(H - h_0)) \quad (64)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4 \frac{d}{dh_0} h_0(H - h_0) \quad (65)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4((H - h_0) \frac{d}{dh_0} h_0 + h_0 \frac{d}{dh_0} (H - h_0)) \quad (66)$$

$$= 4 \frac{d}{dh_0} (H - h_0)^2 + 4((H - h_0) - h_0) \quad (67)$$

$$= -8(H - h_0) + 4(H - h_0) - 4h_0 \quad (68)$$

$$= -4(H - h_0) - 4h_0 \quad (69)$$

$$= -4H + 4h_0 - 4h_0 \quad (70)$$

$$= -4H \quad (71)$$

3. Optimizing for  $x_f$  The optimization of  $x_f$  requires a little bit more thinking of the scenario of the problem.  $H$  must be a positive number, and the expression for  $x_f^2$  appears to be a straight line with slope  $-4H$ . Given  $H > 0$ ,  $-4H < 0$ , and the slope of  $x_f^2$  is negative — as  $h_0$  increases,  $x_f^2$  decreases. Given  $h_0$  must be positive, then,  $h_{0\text{optim}} = 0$ .

### 3.4.3 | Solving for the Actual Optimum of $\theta_0$

We return to following statement:

$$\theta_0 = \arctan\left(\frac{2(H - h_0)}{x_f}\right) \quad (72)$$

The expression we derived above for  $x_f$  under optimal conditions is, per the previous statement:

$$x_f^2 = 4(H - h_0)^2 + 4h_0(H - h_0) \quad (73)$$

$$\Rightarrow x_f = \sqrt{4(H - h_0)^2 + 4h_0(H - h_0)} \quad (74)$$

And substituting the statement for  $x_f$  back into the expression above, in addition to the optimal value  $h_0 = 0$  derived earlier, we derive that

$$\theta_0 = \arctan\left(\frac{2(H - h_0)}{x_f}\right) \quad (75)$$

$$\Rightarrow \theta_0 = \arctan\left(\frac{2(H - h_0)}{\sqrt{4(H - h_0)^2 + 4h_0(H - h_0)}}\right) \quad (76)$$

$$= \arctan\left(\frac{2(H)}{\sqrt{4(H)^2}}\right) \quad (77)$$

$$= \arctan(1) \quad (78)$$

$$= 45^\circ \quad (79)$$

And hence, given an arbitrary angle and an arbitrary height to launch, the most optimal scenario is to make the launch point fully on the floor at an  $45^\circ$  angle from the plane of the floor.

## 4 | Pure Roll

Distance by which the center of mass moves is the same distance that is traversed in the circumference. The conventional model of friction does not work to a rolling object.

## 5 | Kinematics Considering Rotation

We know, for a sphere, the rotational inertia is  $I = \frac{2}{5}MR^2$ . And hence, actually considering the fact that the marble rolls down the initial ramp, and some energy from the height change is actually contributed to the process of roll.

From before, we know that the process of the initial fall creates kinetic energy  $m_0g(H - h_0)$ .

Instead of setting that equivalent to simply kinetic energy, we now revise it with the definition of rotational inertia included.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v}_0^2 + \frac{1}{2} \left( \frac{2}{5} m_0 R^2 \right) \omega^2 \quad (80)$$

$$\Rightarrow 2g(H - h_0) = \vec{v}_0^2 + \left( \frac{2}{5} R^2 \right) \omega^2 \quad (81)$$

$$\Rightarrow 2g(H - h_0) = \vec{v}_0^2 + \left( \frac{2}{5} R^2 \right) \left( \frac{\vec{v}_0}{R} \right)^2 \quad (82)$$

$$\Rightarrow 2g(H - h_0) = \vec{v}_0^2 + \left( \frac{2}{5} \vec{v}_0^2 \right) \quad (83)$$

$$\Rightarrow 2g(H - h_0) = \frac{7}{5} \vec{v}_0^2 \quad (84)$$

$$\Rightarrow \vec{v}_0^2 = \frac{10}{7} g(H - h_0) \quad (85)$$

$$\Rightarrow \vec{v}_0 = \sqrt{\frac{10}{7} g(H - h_0)} \quad (86)$$

The translational velocity at the point of launch, with consideration to rotational kinetic energy, is shown to be slower than that without consideration to rotation by  $((\sqrt{2} - \sqrt{\frac{10}{7}})(g(H - h_0)))$ .

This decrease would, also, therefore result a slightly shorter total launch distance — though the effect would be reasonably minimal (by a value of  $(\sqrt{2} - \sqrt{\frac{10}{7}}) \approx 0.21898$ ).

Given the value of error in  $\vec{v}_0$ , we could then isolate impact of the error in  $h_0$  by simply solving for it in the expression for  $\vec{v}_0$  as all other variables in the system is treated constantly. We use the "original" expression for  $\vec{v}_0$  here as the optimal  $h_0$  is deducted using that value.

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (87)$$

$$\Rightarrow 0.219 = \sqrt{2g(H - h_0)} \quad (88)$$

$$\Rightarrow 0.048 = 2g(H - h_0) \quad (89)$$

$$\Rightarrow 2.446 \times 10^{-3} = (H - h_0) \quad (90)$$

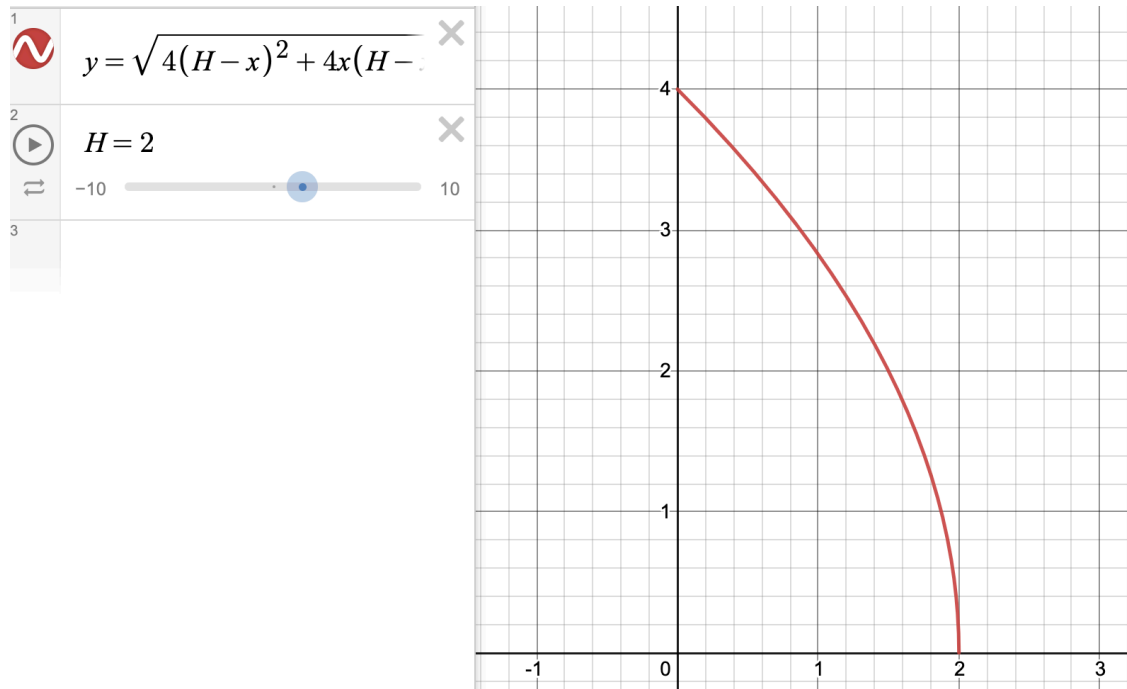
$$\Rightarrow -2.446 \times 10^{-3} = h_0 \quad (91)$$

Hence, we note that — as we consider the rotational inertia — the value that results for  $h_0$  would require a correction to be  $-2.446$  centimeters shorter.

## 6 | Additional Figures

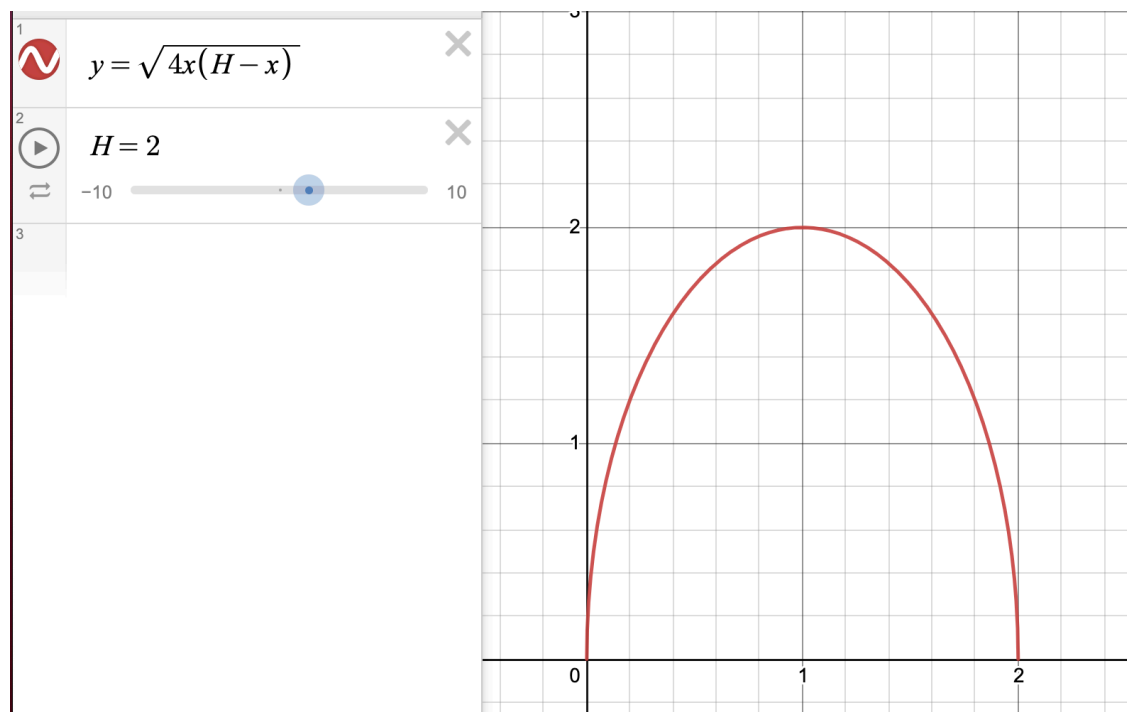


### 6.1 | Projected Range w.r.t. varying $h_0$ at optimal $\theta_0$



This figure is rendered in an Interactive Desmos Plot, the initial launch height,  $H$ , is fixed at an arbitrary non-zero value of 2.

### 6.2 | Projected Range w.r.t. varying $h_0$ at $\theta_0 = 0$



This figure is rendered in another Interactive Desmos Plot, the initial launch height,  $H$ , is again fixed at an arbitrary non-zero value of 2.

As one could see, a parabolic relationship could be seen such that the most optimal value is at exactly  $\frac{1}{2}H$ .

### 6.3 | Optimal $\theta$ changes w.r.t $h$

We quickly return to the following, optimized, expression for  $x_f$ :

We graph the expression for optimal  $\theta = \arctan\left(\frac{\sqrt{2g(H-h_0)}}{\sqrt{2gH}}\right)$

$$\theta = \arctan\left(\frac{\sqrt{2g(H-h_0)}}{\sqrt{2gH}}\right) \quad (92)$$

$$= \arctan\left(\frac{\sqrt{H-h_0}}{\sqrt{H}}\right) \quad (93)$$

