1 | Deriving Rotational KE and Inertia

Given m_i, mass, $\vec{r_i}'$, location of the center of mass, l_i, ω, the angular velocity, figure a $KE_{tot,rot}$.

Because of the fact that the value ω is in units $\frac{d\theta}{dt}$, the rate of radians change, and we know of a radius of the spin l_i, we could figure the velocity at which it is moving by simply scaling the change in radians up to a circle of radius l_i, that is:

$$V_i' = l_i\omega \tag{1}$$

(note that, to understand this, radians $\frac{arclength}{radius}$)

And so, substituting into the statement of $\sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2$

$$\begin{aligned} KE_{rot} &= \sum_{i=1}^{N}\frac{1}{2}m_i\vec{v_i}'^2 && (2)\\ &= \sum_{i=1}^{N}\frac{1}{2}m_i(l_i\omega)^2 && (3)\\ &= \sum_{i=1}^{N}\frac{1}{2}m_il_i^2\omega^2 && (4)\\ &= \frac{1}{2}\omega^2\sum_{i=1}^{N}(m_il_i^2) && (5) \end{aligned}$$

1.1 | Rotational Inertia

The right sum — the mass times the distance away from maxis of rotation ($\sum_{i=1}^{N}(m_il_i^2)$) — is defined as the rotational (moment) of inertia (spinny mass). That is,

$$I = \sum_{i=1}^{N}(m_il_i^2) \tag{6}$$

Replacing that value in the prior statement, the statement of KE_{rot} is defined as:

$$KE_{rot} = \frac{1}{2}\omega^2 I \tag{7}$$

1.2 | Rotational Inertia for a Ring

For a ring (that's perfectly circular) rotating on an axis perpendicular to the plane of the ring, the l_i — distance from axis of rotation — is the same value: namely, the radius R as the radius of a circle is the same for all positions. Meaning,

$$l_i = R \tag{8}$$

regardless of which value i.

Hence, the value of KE_{rot} would be evaluated as…

$$KE_{rot} = \sum_{i=1}^{N}(m_i l_i^2) \tag{9}$$
$$= \sum_{i=1}^{N}(m_i R^2) \tag{10}$$
$$= R^2 \sum_{i=1}^{N} m_i \tag{11}$$
$$\tag{12}$$

Substituting M as the sum of all masses in the ring ($M = \sum_{i=1}^{N} m_i$), the statement is therefore:

$$KE_{rot} = MR^2 \tag{13}$$

1.3 | **Rotational Inertia of a Solid Sphere**

I believe that the rotational inertia of I_{sphere} to be less than I_{disk}. This is because, as the dimension of the object increases, it would be easier to change its velocity (a disk is easier to spin than a ring, etc.). Hence, my intuition states that I_{sphere} would be lower than I_{disk}.

Mathematically, as M is staying at the same value, in the disk case has more mass closer to the axis of rotation — meaning that the $m_i R^2$ term would be smaller in more of the point masses than that of an object at a lower dimension. Hence, the sphere would have more points with lower $m_i R^2$ terms than that of disk; hence, I_{sphere} would be less than I_{disk}.

2 | **Kinematics Equations**

We begin by defining a coordinate system such that positive values are pointed ”downwards”. That is, as values increase in the positive direction, their corresponding vectors are pointed further towards the ”down” direction.

Given $a = a_0$, initial velocity v_0, and position y_0, we derive the kinematics equations.

$$a(t) = a_0 \tag{14}$$
$$\int a(t)dt = \int a_0 dt \tag{15}$$
$$v(t) = a_0 t + C \tag{16}$$

We are given that $v(0) = v_0$. $v(0) = C = v_0$, hence, $C = v_0$. The velocity statement is therefore,

$$v(t) = a_0 t + v_0 \tag{17}$$

Continuing with integration:

$$v(t) = a_0 t + v_0 \tag{18}$$
$$\int v(t) = \int a_0 t + v_0 dt \tag{19}$$
$$y(t) = \frac{1}{2} a_0 t^2 + v_0 t + C \tag{20}$$
$$\tag{21}$$

Again, substituting $C = y_0$ by the same logic above — $y(0) = C = y_0$, we derive the statement for the position equation.

$$y(t) = \frac{1}{2} a_0 t^2 + v_0 t + y_0 \tag{22}$$

2.1 | Proving $v^2(t) = v_0^2 + 2a_0(y(t) - y_0)$

We start at the statement for $v(t)$, squaring it, and substituting the necessary statements.

$$v(t) = a_0 t + v_0 \tag{23}$$
$$\Rightarrow v^2(t) = {a_0}^2 t^2 + 2a_0 v_0 t + v_0^2 \tag{24}$$
$$v^2(t) = {v_0}^2 + 2a_0(\frac{1}{2} a_0 t^2 + v_0 t) \tag{25}$$
$$v^2(t) = {v_0}^2 + 2a_0(\frac{1}{2} a_0 t^2 + v_0 t + y_0 - y_0) \tag{26}$$
$$v^2(t) = {v_0}^2 + 2a_0(y(t) - y_0) \tag{27}$$

It is therefore shown that:

$$v^2(t) = {v_0}^2 + 2a_0(y(t) - y_0) \tag{28}$$

2.2 | Proving $\Delta y = \frac{v(t_1)+v(t_2)}{2} \Delta t$

Showing $\Delta y = \frac{v(t_1)+v(t_2)}{2} \Delta t$, defining $\Delta y = y(t_2) - y(t_1)$ and $\Delta t = t_2 - t_1$. Substituting the appropriate values for $v(t)$, Δy, Δt and solving…

$$\Delta y = \frac{v(t_1) + v(t_2)}{2} \Delta t \tag{29}$$

$$y(t_2) - y(t_1) = \frac{v(t_1) + v(t_2)}{2} t_2 - t_1 \tag{30}$$

$$y(t_2) - y(t_1) = \frac{((a_0 t_1 + v_0) + (a_0 t_2 + v_0))}{2} t_2 - t_1 \tag{31}$$

$$y(t_2) - y(t_1) = \frac{((a_0 t_1 t_2 + v_0 t_2) - (a_0 {t_1}^2 + v_0 t_1) + (a_0 {t_2}^2 + v_0 t_2) - (a_0 t_1 t_2 + v_0 t_1))}{2} \tag{32}$$

$$y(t_2) - y(t_1) = \frac{(a_0 {t_2}^2 + 2 v_0 t_2) - (a_0 {t_1}^2 + 2 v_0 t_1)}{2} \tag{33}$$

$$y(t_2) - y(t_1) = \frac{(a_0 {t_2}^2 + 2 v_0 t_2 + 2 y_0) - (a_0 {t_1}^2 + 2 v_0 t_1 + 2 y_0)}{2} \tag{34}$$

$$y(t_2) - y(t_1) = \frac{1}{2} a_0 {t_2}^2 + v_0 t_2 + y_0 - \frac{1}{2} a_0 {t_1}^2 + v_0 t_1 + y_0 \tag{35}$$

$$y(t_2) - y(t_1) = y(t_2) - y(t_1) \tag{36}$$

$$\tag{37}$$

Hence, it is demonstrated that:

$$\Delta y = \frac{v(t_1) + v(t_2)}{2} \Delta t \tag{38}$$

3 | Question regarding signage

The Kinematics Equations derivations above relied on the fact that the coordinate system was defined as ”positive downwards”. Were this not to be the case, constants would have to be redefined and the signs of most terms would be flipped:

Given $a = -a_0$, initial velocity $-v_0$, and position $-y_0$, we (re)derive the kinematics equations.

$$a(t) = -a_0 \tag{39}$$

$$\int a(t) dt = \int -a_0 dt \tag{40}$$

$$v(t) = -a_0 t + C \tag{41}$$

We are given that $v(0) = -v_0$. $v(0) = C = -v_0$, hence, $C = -v_0$. The velocity statement is therefore,

$$v(t) = -a_0 t - v_0 \tag{42}$$

Continuing with integration:

$$v(t) = -a_0 t - v_0 \tag{43}$$

$$\int v(t) = \int -a_0 t - v_0 dt \tag{44}$$

$$y(t) = \frac{-1}{2} a_0 t^2 - v_0 t + C \tag{45}$$

$$\tag{46}$$

Again, substituting $C = -y_0$ by the same logic above — $y(0) = C = -y_0$, we derive the statement for the position equation.

$$y(t) = \frac{-1}{2}a_0t^2 - v_0t - y_0 \tag{47}$$

As such, if the signage were flipped, terms of the kinematics equation would therefore be flipped.

4 | Proving the Third Equation for V^2

$$v^2(t) \stackrel{?}{=} {v_0}^2 + 2a_0(y(t) - r_0) \tag{48}$$

We begin by deriving an equation of $a(y)$ — acceleration as a function of position.

Based on first principles, the following is true:

$$a = \frac{dv}{dt} \tag{49}$$

Apply the chain rule, we derive the following:

$$a = \frac{dv}{dt} \tag{50}$$

$$\Rightarrow a = \frac{dv}{dy}\frac{dy}{dt} \Rightarrow a = \frac{dv}{dy}v \tag{51}$$