

1 | inner product

def

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbb{R}$ and has the following properties

- **positivity** $\langle v, v \rangle \geq 0 \forall v \in V$
- **definiteness** $\langle v, v \rangle = 0 \iff v = 0$
- **additivity in first slot** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \forall u, v, w \in V$
- **homogeneity in first slot** $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \forall \lambda \in \mathbb{R}, u, v \in V$
- **conjugate symmetry** $\langle u, v \rangle = \overline{\langle v, u \rangle} \forall u, v \in V$
 - Over the reals, this is equal to $\langle u, v \rangle = \langle v, u \rangle$

2 | motivation

2.1 | **The norm of a complex number $\|z\|$ should be non-negative, so we can define it as**

$$\|z\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$$

Since the square of the absolute value is just a complex number times a conjugate, and because the norm squared should be the inner product of z with itself, maybe the inner product of $w, z \in \mathbb{C}^n$ should equal

$$w_1 \bar{z}_1 + \dots + w_n \bar{z}_n$$

2.2 | **positivity: we want inner product to be the size of the vector, so it should be a positive and real number**

2.3 | **notation**

For a complex scalar $\lambda \in \mathbb{C}$, $\lambda \geq 0$ means λ is real and non-negative

$\langle u, v \rangle$ denotes an inner product.

3 | intuition

3.1 | **additivity/homogeneity in the first slot also implies additivity in the second slot, and 'conjugate homogeneity in the second slot'**

3.2 | **we want the norm to be a real scalar, so we need to take the complex conjugate of the second one**

3.2.1 | **so, the Euclidean inner product is conjugate the second, then dot product**

$$\langle u, v \rangle = u \bar{v}$$