## 1 | Problem: Axler 3.E exercise 18

Suppose  $T \in \mathcal{L}(V,W)$  and U is a subspace of V. Let  $\pi$  denote the quotient map from V onto V/U. Prove that there exists  $S \in \mathcal{L}(V/U,W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

## 2 | Forward Direction

Intuitively, we can treat  $S \circ \pi$  as a single map and take a basis of V to the same place that T would, and the maps would be equal.

Let S be a relation between V/U and W defined by

$$S(U+v) = Tv$$

If S is well defined (every element in V/U is mapped to exactly one place), then S will inherit additivity and homogeneity from T and  $S \circ \pi$  will equal T.

Let  $v \in V$  and  $v' \in V/U$  s.t.  $v' = \pi v$  (v' is where  $\pi$  takes v). Then, to show that S is well defined, we must show that v has atleast one and at most one image through  $S \circ \pi$ .

Because  $\pi v$  is well defined, and U+v was arbitrary in the definition of S, each v must have atleast one image in W.

Take S to be an arbitrary linear map. The only restriction on S that could cause  $S(U+v) \neq Tv$  is S(0) = 0 (this statement is not watertight). Thus, S is defined if  $\forall U+v=U=0$ , Tv=0. Equivalently, S is defined if  $U\subseteq \mathsf{null}\ T$ , which is given in the problem.

Thus, S is well defined. To show that it inherits additivity and homogeneity:

$$S(U+u) + S(U+v) = Tu + Tv = T(u+v) = S(U+u+U+v) = S(U+(u+v))$$

$$\lambda \left( S(U+v) \right) = \lambda Tv = T(\lambda v) = S(U+(\lambda v))$$

Thus, S is linear, and  $S \circ \pi = T$  if  $U \subseteq \text{null } T$ .

## 3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume  $U \nsubseteq \text{null } T$ . There exists  $v \in U$  s.t.  $Tv \ne 0$ . This is some of the "information" that was "lost". Because  $v \in U$ ,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero,  $S \in \mathcal{L}(V/U,W)$  must take  $\pi v = 0$  to zero. Thus, either  $S(\pi v) \neq Tv$  or S is not a linear map, both of which are contradictions.

This shows that if  $U \nsubseteq \text{null } T$ , then  $S \notin \mathcal{L}(V/U,W)$  or  $T \neq S \circ \pi$ . Thus, if  $S \in \mathcal{L}(V/U,W)$  and  $T = S \circ \pi$ , then  $U \subseteq \text{null } T$ .

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