



1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_o and θ_0 that would best create a maximum launch distance x_f .

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x -value of the location of the launch of the projectile as $x = 0$. We define the direction towards which the marble is launching as positive- x , so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y , and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0 , the moment the marble leaves the track and travels as a projectile as t_1 , and the end — in the moment when the marble hits the ground — as t_f . We will call the marble m_0 .

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0g(H - h_0) = m_0g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named \vec{v}_0 .

$$m_0g(H - h_0) = \frac{1}{2}m_0\vec{v}_0^2 \quad (1)$$

$$g(H - h_0) = \frac{1}{2}\vec{v}_0^2 \quad (2)$$

$$2g(H - h_0) = \vec{v}_0^2 \quad (3)$$

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (4)$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} v_{0x} = \sqrt{2g(H - h_0)}\cos(\theta_0) \\ v_{0y} = \sqrt{2g(H - h_0)}\sin(\theta_0) \end{cases} \quad (5)$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. v_{0y} , v_{0x} , h_0 , m_0 .

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f .

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases} \quad (6)$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases} \quad (7)$$

as...

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity

- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Solving for x_f

The position equations above could be leveraged to figure a value for x_f . We first create a set of equations modeling the location of the marble at t_f .

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}\cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{1}{2}gt_f^2 + t_f\sqrt{2g(H-h_0)}\sin(\theta_0) + h_0 \end{cases} \quad (8)$$

To simplify calculations initially, we set $\sqrt{2g(H-h_0)}$ back as \vec{v}_0 for the ease of initial simplification.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\vec{v}_0\cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}gt_f^2 + v_{0y}t_f + h_0 = \frac{1}{2}gt_f^2 + t_f\vec{v}_0\sin(\theta_0) + h_0 \end{cases} \quad (9)$$

We first solve for t_f , and supply it to the second equation.

$$t_f = \frac{-\vec{v}_0\sin(\theta_0) \pm \sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{g} \quad (10)$$

Given that we know that time is positive in this setup, and subtracting a term will make it even more negative, we could safely ignore the $-$ term in the \pm operator.

And, performing variable substitution upon the first equation...

$$x_f = \frac{-\vec{v}_0\sin(\theta_0)\vec{v}_0\cos(\theta_0) + \vec{v}_0\cos(\theta_0)\sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{g} \quad (11)$$

$$= \frac{\frac{-1}{2}\vec{v}_0^2\sin(2\theta_0) + \vec{v}_0\cos(\theta_0)\sqrt{(\vec{v}_0\sin(\theta_0))^2 + 2gh_0}}{g} \quad (12)$$

$$= \frac{-\vec{v}_0^2\sin(2\theta_0)}{2g} + \frac{\vec{v}_0\cos(\theta_0)\sqrt{\vec{v}_0^2\sin^2(\theta_0) + 2gh_0}}{g} \quad (13)$$

$$= \frac{\vec{v}_0\cos(\theta_0)\sqrt{\vec{v}_0^2\sin^2(\theta_0) + 2gh_0}}{g} - \frac{\vec{v}_0^2\sin(2\theta_0)}{2g} \quad (14)$$

$$(15)$$

And finally, substituting back the \vec{v}_0 terms...

$$x_f = \frac{\sqrt{2g(H-h_0)}\cos(\theta_0)\sqrt{2g(H-h_0)\sin^2(\theta_0) + 2gh_0}}{g} - \frac{2g(H-h_0)\sin(2\theta_0)}{2g} \quad (16)$$

$$= \frac{\sqrt{2g(H-h_0)}\cos(\theta_0)\sqrt{2g(H-h_0)\sin^2(\theta_0) + 2gh_0}}{g} - (H-h_0)\sin(2\theta_0) \quad (17)$$

$$x_f = 2\sqrt{h_0(H-h_0)}$$