

## 1 | Jacobian Determinant for Polar

We are to determine (pun not intended) the polar correction factor for a double integral,  $dA = r dr d\theta$ .

To do this, we will have to first figure the change of bases expressions such that we can take:

$$f(x, y) = g(r, \theta) \quad (1)$$

Fortunately, this is already derived to use from before.

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad (2)$$

Therefore, we have that:

$$f(x, y) = f(r \cos \theta, r \sin \theta) \quad (3)$$

And therefore, we can figure  $J_{r,\theta}$ :

$$J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \quad (4)$$

Taking its determinant, then:

$$\det(J) = r \cos^2 \theta + r \sin^2 \theta = r \quad (5)$$

And therefore, the change-of-basis result would be:

$$dx dy = r dr d\theta \quad (6)$$

## 2 | Jacobian Determinant for Spherical

We again need to figure a correction factor for  $dx dy dz = \rho^2 \sin \phi d\rho d\theta d\phi$ .

We therefore have to figure a change of bases for the expression:

$$f(x, y, z) = g(\rho, \theta, \phi) \quad (7)$$

We can leverage the shape of the object to determine the parameterization:

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad (8)$$

We will now figure the matrix for  $J_{\rho,\theta,\phi}$ :

$$J = \begin{bmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{bmatrix} \quad (9)$$

```

var("rho phi theta")
M = matrix([[sin(phi)*cos(theta), -rho*sin(phi)*sin(theta), rho*cos(phi)*cos(theta)], [sin(phi)*sin(theta), rho*sin(phi)*cos(theta), -rho*cos(phi)*sin(theta)], [rho*cos(phi)*sin(theta), rho*sin(phi)*sin(theta), rho*cos(phi)*cos(theta)]]
M
M.det().full_simplify()

```

Not quite sure why Sage didn't simply  $(-\rho)^2$  into  $\rho^2$ , but, we can see that:

$$dx \, dy \, dz = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \quad (10)$$

### 3 | Surface area in polar

Given the fact that:

$$dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx \, dy \quad (11)$$

We are to figure the corresponding for a function in polar.

I suppose we can work this out as if we are doing traditional u-substitution: that is, we are to find a function that corrects for the correction factor as well as the  $dx$  and  $dy$  components.

Recall again, that:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad (12)$$

Furthermore: per the chain and total derivative rule, we have that:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} \quad (13)$$

and,

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} \quad (14)$$

To actually figure this, then, we have to find expressions for each of the rightward partials.

Take, first,  $\frac{\partial \theta}{\partial x}$ ; we have:

$$x = r \cos \theta \quad (15)$$

$$\Rightarrow \theta = \arccos\left(\frac{x}{r}\right) \quad (16)$$

Taking its partial derivative by  $x$ :

$$f(x, r) = \arccos(x/r)$$