1 | Jacobian Determinant for Polar

We are to determine (pun not intended) the polar correction factor for a double integral, $dA = r\ dr\ d\theta$. To do this, we will have to first figure the change of bases expressions such that we can take:

$$f(x,y) = g(r,\theta) \tag{1}$$

Fortunately, this is already derived to use from before.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

Therefore, we have that:

$$f(x,y) = f(r\cos\theta, r\sin\theta) \tag{3}$$

And therefore, we can figure $J_{r,\theta}$:

$$J = \begin{bmatrix} cos\theta & -r\sin\theta \\ sin\theta & r\cos\theta \end{bmatrix} \tag{4}$$

Taking its determinant, then:

$$det(J) = r\cos^2\theta + r\sin^2\theta = r \tag{5}$$

And therefore, the change-of-basis result would be:

$$dx\ dy = r\ dr\ d\theta \tag{6}$$

2 | Jacobian Determinant for Spherical

We again need to figure a correction factor for $dx\ dy\ dz = \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi$. We therefore have to figure a change of bases for the expression:

$$f(x,y,z) = g(\rho,\theta,\phi) \tag{7}$$

We can leverage the shape of the object to determine the parameterization:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \tag{8}$$

We will now figure the matrix for $J_{\rho,\theta,\phi}$:

$$J = \begin{bmatrix} sin\phi\cos\theta & -\rho\ sin\phi\sin\theta & \rho\ cos\phi\cos\theta \\ sin\phi\sin\theta & \rho\ sin\phi\cos\theta & \rho\ cos\phi\sin\theta \\ cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \tag{9}$$

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var("rho phi theta")
M = matrix([[sin(phi)*cos(theta), -rho*sin(phi)*sin(theta), rho*cos(phi)*cos(theta)], [sin(phi)*sin(the
M
M.det().full_simplify()
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Not quite sure why Sage didn't simply $(-\rho)^2$ into ρ^2, but, we can see that:

$$dx\ dy\ dz = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi \tag{10}$$

3 | Surface area in polar

Given the fact that:

$$dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dx\ dy \tag{11}$$

We are to figure the corresponding for a function in polar.

I suppose we can work this out as if we are doing traditional u-substitution: that is, we are to find a function that corrects for the correction factor as well as the dx and dy components.

Recall again, that:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{12}$$

Furthermore: per the chain and total derivative rule, we have that:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} \tag{13}$$

and,

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} \tag{14}$$

To actually figure this, then, we have to find expressions for each of the rightward partials.

Take, first, $\frac{\partial \theta}{\partial x}$; we have:

$$x = r\ cos\theta \tag{15}$$

$$\Rightarrow \frac{\partial}{\partial x} x = \frac{\partial}{\partial x} r\ cos\theta \tag{16}$$

$$\Rightarrow 1 = -r\ sin\theta \frac{\partial \theta}{\partial x} \tag{17}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-1}{r\ sin\theta} \tag{18}$$

Furthermore, for $\frac{\partial r}{\partial x}$

We have, trivially:

$$\frac{\partial r}{\partial x} = \frac{1}{cos\theta} \tag{19}$$

Therefore:

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial \theta} \cdot \frac{1}{r \sin\theta} + \frac{\partial f}{\partial r} \cdot \frac{1}{\cos\theta} \tag{20}$$

We can repeat this for y:

For $\frac{\partial \theta}{\partial y}$; we have:

$$y = r\ sin\theta \tag{21}$$

$$\Rightarrow \frac{\partial}{\partial x} x = \frac{\partial}{\partial x} r\ cos\theta \tag{22}$$

$$\Rightarrow 1 = r\ cos\theta \frac{\partial \theta}{\partial x} \tag{23}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{1}{r\ cos\theta} \tag{24}$$

Furthermore, for $\frac{\partial r}{\partial y}$

We have, trivially:

$$\frac{\partial r}{\partial x} = \frac{1}{sin\theta} \tag{25}$$

Therefore:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \theta} \cdot \frac{1}{r \cos\theta} + \frac{\partial f}{\partial r} \cdot \frac{1}{\sin\theta} \tag{26}$$

Lastly, we recall that $dx\ dy = r\ dr\ d\theta$

Finally, putting it all together:

$$dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dx\ dy \tag{27}$$

$$= \sqrt{1 + \left(\frac{\partial f}{\partial r} \cdot \frac{1}{\cos\theta} - \frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial f}{\partial \theta} \cdot \frac{1}{r \cos\theta} + \frac{\partial f}{\partial r} \cdot \frac{1}{\sin\theta}\right)^2}\ dx\ dy \tag{28}$$

(29)