1 | Parametric Equations

Consider the curve described by the following parametric equations:

$$x(t) = t^2 \tag{1}$$
$$y(t) = t^3 - ct, c \in \mathbb{R} \tag{2}$$

1.1 | Rectangular Equations

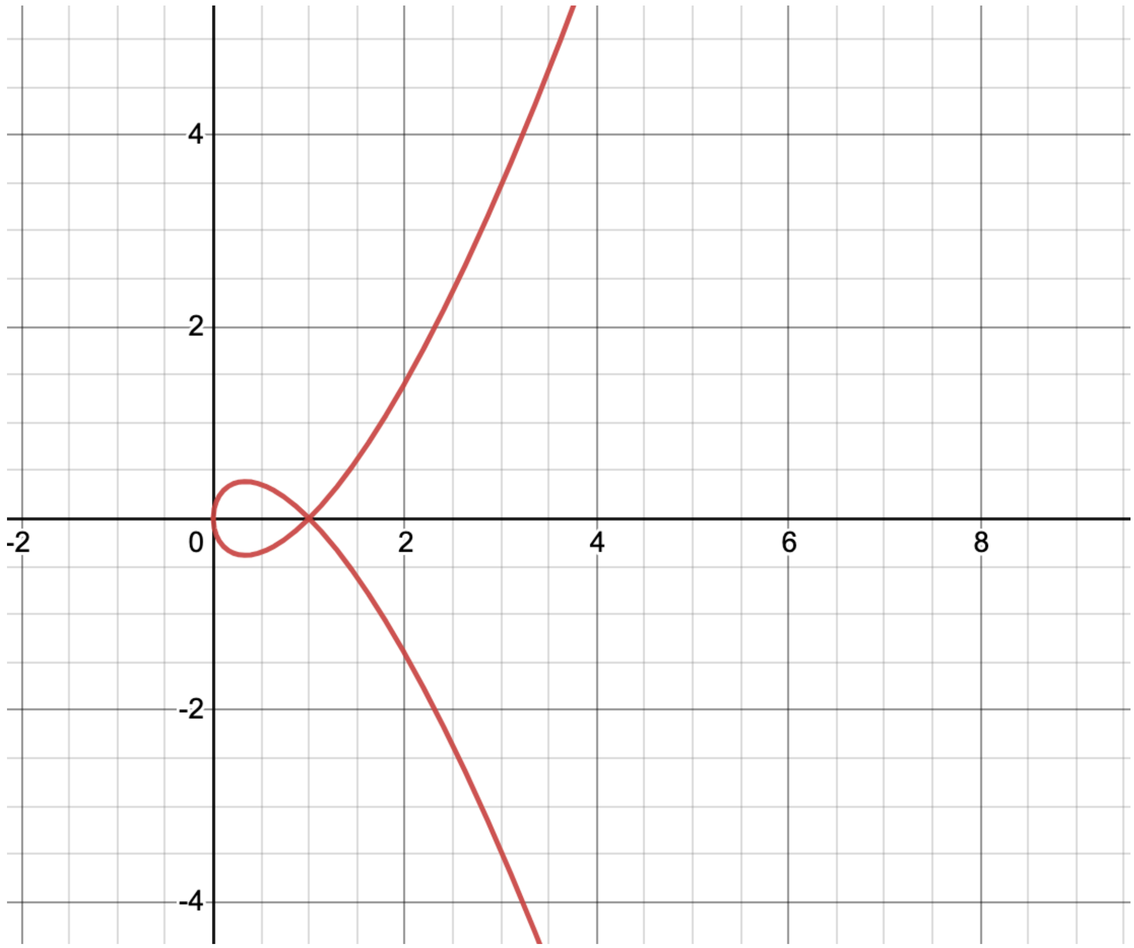
Come up with function — functions, rather — for this curve. In other words, convert it to rectangular form.

Given $x(t) = t^2$, we could figure that $t = \sqrt{x}$. As such, replacing for the definition of t in the second statement, we could derive that:

$$\begin{aligned} y(t) =& t^3 - ct & (3) \\ \Rightarrow y(t) =& (\pm\sqrt{x})^3 - c(\pm\sqrt{x}) & (4) \\ =& \pm x\sqrt{x} \pm c\sqrt{x} & (5) \\ =& \pm \sqrt{x}(x-c) & (6) \end{aligned}$$

1.2 | Sketching the curve

Try sketching it!



1.3 | Particle path and length

Imagine if you are a little particle on this curve, traveling from $t = 5$ to $t = 7$. What is your path, and what's the total distance you travel?

1.3.1 | Start position

$$x(5) = 5^2 = 25 \tag{7}$$
$$y(5) = 5^3 - 5c = 125 - 5c \tag{8}$$

Setting $c = 1$, we derive that...

$$x(5) = 25 \tag{9}$$
$$y(5) = 120 \tag{10}$$

Hence, the start position of the particle is $(25, 120)$.

1.3.2 | **End position**

$$x(7) = 7^2 = 49 \tag{11}$$
$$y(7) = 7^3 - 7 = 336 \tag{12}$$

Hence, the end position of the particle is $(49, 336)$

1.3.3 | **Direction of Travel**

The middle point of the travel is at $t = \frac{12}{2} = 6$.
The derivatives of the parameter equations are as follows:

$$x'(t) = 2t \tag{13}$$
$$y'(t) = 3t^2 - c \tag{14}$$

Therefore, the derivative in the direction of the particle travel is:

$$\frac{dy}{dx} = \frac{3t^2 - c}{2t} \tag{15}$$

At $c = 1$ and $t = 6$, the value is therefore:

$$\frac{108 - 1}{12} \approx 8.9 \tag{16}$$

As the value of the derivative is positive — that as x increases, y increases, we know that at $t = 8$ the particle is traveling in a positive direction as x increases.

1.3.4 | **Total Distance of Travel**

To figure the distance of travel, we need to apply the following expression for arc length:

$$\int_{t=5}^{t=7} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \tag{17}$$
$$\int_{t=5}^{t=7} \sqrt{(2t)^2 + (3t^2 - c)^2} dt \tag{18}$$

At this point, we set $c = 0$ as an example value.

$$\int_{t=5}^{t=7} \sqrt{(2t)^2 + (3t^2)^2} dt \tag{19}$$
$$\int_{t=5}^{t=7} \sqrt{4t^2 + 9t^4} dt \tag{20}$$
$$\int_{t=5}^{t=7} \sqrt{t^2(4 + 9t^2)} dt \tag{21}$$
$$\int_{t=5}^{t=7} t\sqrt{(4 + 9t^2)} dt \tag{22}$$

We now perform u-sub upon this problem to figure the final solution.

$$Let\ u = (4 + 9t^2) \quad (23)$$
$$\frac{du}{dt} = 18t \quad (24)$$
$$dt = \frac{1}{18t}du \quad (25)$$

$$\frac{1}{18}\int_{t=5}^{t=7} u^{\frac{1}{2}}du \quad (26)$$
$$\frac{1}{18}(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \mid_2^3) \quad (27)$$
$$\frac{1}{18}(\frac{2\sqrt{u^3}}{3} \mid_2^3) \quad (28)$$
$$\frac{1}{18}(\frac{2\sqrt{(4+9t^2)^3}}{3} \mid_2^3) \quad (29)$$
$$\frac{1}{18}((\frac{2\sqrt{(4+9(3^2))^3}}{3}) - (\frac{2\sqrt{(4+9(2^2))^3}}{3})) \approx 19.65 \quad (30)$$
$$(31)$$

The particle travels about 19.65 units.

2 | Solids of Revolution

Consider the shape made by taking the function $f(x) = \frac{1}{x}$ from $x = 1$ out to ∞ by spinning it around the x-axis.

2.1 | Surface Area

The surface area of the resulting shape could be deducted by applying the circumference formula for circles produced by rotation. That is —

$$2\pi\int_1^\infty \frac{1}{x}dx \quad (32)$$
$$\Rightarrow 2\pi \lim_{b\to\infty}(ln(x) \mid_1^b)dx \quad (33)$$
$$\Rightarrow \infty \quad (34)$$

2.2 | Volume

The volume calculation is much the same (though, with dramatically different results at infinity), except that the "radius" of the circles are squared and multiplied to π to produce the area of each circular slice.

$$
\begin{aligned}
2\pi \int_1^\infty x^{-2} dx && (35)\\
\Rightarrow \pi \lim_{b\to\infty} \frac{-1}{x} |_1^b \; dx && (36)\\
\Rightarrow -\pi && (37)
\end{aligned}
$$

3 | **Partial Derivatives**

3.1 | $f(x, y) = 7x + 2x^2y^3 + 10y^2$

$$
\begin{aligned}
f_x =& 7 + 4xy^3 && (38)\\
f_y =& 6x^2y^2 + 20y && (39)\\
f_{xx} =& 4y^3 && (40)\\
f_{yy} =& 12x^2y + 20 && (41)\\
f_{xy} =& 12xy^2 && (42)\\
f_{xxx} =& 0 && (43)\\
f_{yyy} =& 12x^2 && (44)\\
f_{xxy} =& 12y^2 && (45)\\
f_{yyx} =& 24yx && (46)
\end{aligned}
$$

3.2 | $f(x_y) = 3xy^3 + 8x^2y^4$

$$
\begin{aligned}
f_x =& 3y^3 + 16xy^4 && (47)\\
f_y =& 9xy^2 + 32x^2y^3 && (48)\\
f_{xx} =& 16y^4 && (49)\\
f_{yy} =& 18xy + 96x^2y^2 && (50)\\
f_{xy} =& 9y^2 + 64xy^3 && (51)\\
f_{xxx} =& 0 && (52)\\
f_{yyy} =& 18x + 192x^2y && (53)\\
f_{xxy} =& 64y^3 && (54)\\
f_{yyx} =& 18y + 192xy^2 && (55)\\
& && (56)
\end{aligned}
$$