

## 1 | Definition

#definition Axler3.2 Linear Map #aka linear transformation A *linear map* from  $V$  to  $W$  is a function  $T : V \rightarrow W$  with the following properties:

### 1.1 | Additivity

$$T(u + v) = Tu + Tv \forall u, v \in V$$

### 1.2 | Homogeneity

$$T(\lambda v) = \lambda(Tv) \forall \lambda \in \mathbb{F}, v \in V$$

## 2 | Other Notation

### 2.1 | Set of Maps

#definition Axler3.3  $\mathcal{L}(V, W)$

The set of all linear maps from  $V$  to  $W$  is denoted  $\mathcal{L}(V, W)$ .

## 3 | Examples

### 3.1 | zero ( $0$ )

Zero is a function  $0 : V \rightarrow W$  s.t.  $0v = 0 \forall v \in V$ . (It takes all vectors in  $V$  and maps them to the additive identity of  $W$ )

### 3.2 | identity ( $I$ )

The identity maps each from one vector space to itself (in the same vector space):

$$I \in \mathcal{L}(V, V), v \in V : Iv = v$$

### 3.3 | differentiation ( $D$ )

$$D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})) : Dp = p'$$

Basically stating that for two polynomials  $a, b \in \mathcal{P}(\mathbb{R})$ ,  $a' + b' = (a + b)'$  and with a constant  $\lambda \in \mathcal{R}$   $(\lambda a)' = \lambda a'$ .

### 3.4 | integration

### 3.5 | multiplication by $x^2$

$$T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})) : (Tp)(x) = x^2 p(x)$$

is a linear map

### 3.6 | backward shift

$\mathbb{F}^\infty$  is the vector space of all sequences of elements in  $\mathbb{F}$ .

$$T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty) : T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

### 3.7 | $\mathbb{F}^n \rightarrow \mathbb{F}^m$

Given a "coefficient matrix"  $A : A_{j,k} \in \mathbb{F} \forall j = 1, \dots, m; \forall k = 1, \dots, n$ , define  $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ :

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + A_{1,2}x_2 + \dots + A_{1,n}x_n, A_{2,1}x_1 + \dots + A_{2,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

Notice that this is equivalent to taking  $A$  as a  $m \times n$  matrix and dot producting it with the  $n \times 1$  matrix  $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ .

## 4 | Results

### 4.1 | Axler3.5 Linear maps and basis of domain

If  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ , then there exists a unique linear map  $T : V \rightarrow W$  s.t.

$$Tv_j = w_j \forall j \in 1, \dots, n$$

#aka given a basis  $v$  of  $V$ , there is a unique linear map that maps  $v$  to each  $w \in W$ .

#### 4.1.1 | #careful

1. same dimension  $V$  and  $W$  are both of dimension  $n$ .
2. same field We defined  $V$  and  $W$  to both be vector spaces over the same field  $\mathbb{F}$  which is either  $\mathbb{R}$  or  $\mathbb{C}$ .
3.  $v$  is a basis  $v_1, \dots, v_n$  must be a basis of  $V$  (because that fact is used in the proof)

#### 4.1.2 | Questions

1. **DONE** #question what does it mean that " $T$  is uniquely determined on  $\text{span}(v_1, \dots, v_n)$ "? question There's no ambiguity and so we know exactly which map it's referring to, and thus it is uniquely determined.