

## 1 | Dot product:

- Name: dot product
- Result: Scalar
- Interpretation (what it measures): parallelity
  - the more parallel the larger the dot product
- Magnitude (with sign):  $|\vec{a}||\vec{b}|\cos(\theta)$
- Geometric magnitude:  $|\vec{a}||\vec{b}_{\parallel\vec{a}}|$
- Direction: no direction
- Algebraic form:  $a_xb_x + a_yb_y + a_zb_z$
- Algebraic properties:
  - commutative
  - associative
  - distributive across addition

## 2 | Cross product:

- Name: Cross product
- Result: Vector
- Interpretation (what it measures): Orthgonality
  - the more orthogonal the longer the cross product
- Magnitude (with sign):  $|\vec{a}||\vec{b}|\sin(\theta)$
- Geometric Magnitude:  $|\vec{a}||\vec{b}_{\perp\vec{a}}|$
- Direction: perpendicular to the two vectors
  - by the right hand rule by rotating the first vector into the second vector
- Albraic form:  $\langle a_yb_z - a_zb_y, a_xb_z - a_zb_x, a_xb_y - a_yb_x \rangle$
- Algebraic properties:
  - Anticommutative
  - $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
  - $(\vec{A} \times \vec{B}) \perp \vec{A}$
  - $(\vec{A} \times \vec{B}) \perp \vec{B}$
  - Antiassociative

### 3 | Application of cross product:

- In physics there is something called torque, notated  $\tau$ 
  - Torque is the net force of things that rotate, so:
    - \*  $F_{net} = ma$
    - \*  $\tau_{net} = I\omega$
- Somethings to note about  $\tau$ :
  - It increases with a longer lever
  - It increases with a greater force
    - \* that is perpendicular to the lever
- Given these requirements we can make a formula:
  - $|\tau| = |\vec{r}||\vec{F}_{\perp\vec{r}}|$ , where  $\vec{F}$  is the force applied to the door, and  $\vec{r}$  is the radius of the lever.
  - this, the right side of the equation, can be described using the dot product:  $|\tau| = \vec{r} \times \vec{F}$

### 4 | Derivation of cross product algebraic form:

To start, we can define:

$$\vec{A} = (A_x, A_y, A_z) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = (B_x, B_y, B_z) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Next we have to assume that the dot product is distributive across addition:

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x\hat{i} \times \hat{i} + A_xB_y\hat{i} \times \hat{j} + A_xB_z\hat{i} \times \hat{k} \\ &\quad + A_yB_x\hat{j} \times \hat{i} + A_yB_y\hat{j} \times \hat{j} + A_yB_z\hat{j} \times \hat{k} \\ &\quad + A_zB_x\hat{k} \times \hat{i} + A_zB_y\hat{k} \times \hat{j} + A_zB_z\hat{k} \times \hat{k}\end{aligned}$$

From the definition of a cross product, we know that the cross product between any two vectors that are parallel is zero, thus:

$$\begin{aligned}&= A_xB_y\hat{i} \times \hat{j} + A_xB_z\hat{i} \times \hat{k} \\ &\quad + A_yB_x\hat{j} \times \hat{i} + A_yB_z\hat{j} \times \hat{k} \\ &\quad + A_zB_x\hat{k} \times \hat{i} + A_zB_y\hat{k} \times \hat{j}\end{aligned}$$

$\hat{i} \times \hat{j}$  would yield a vector length one in the direction of a vector that is perpendicular to both  $\hat{i}$  and  $\hat{j}$ , which would be  $\hat{k}$ . Conversely,  $\hat{i} \times \hat{j} = -\hat{k}$ . Therefore:

$$\begin{aligned}&= A_xB_y\hat{k} + A_xB_z\hat{j} \\ &\quad - A_yB_x\hat{k} + A_yB_z\hat{i} \\ &\quad - A_zB_x\hat{j} - A_zB_y\hat{i} \\ &= A_xB_y\hat{k} - A_yB_x\hat{k} \\ &\quad + A_yB_z\hat{i} - A_zB_y\hat{i} \\ &\quad + A_xB_z\hat{j} - A_zB_x\hat{j} \\ &= (A_xB_y - A_yB_x)\hat{k} + (A_yB_z - A_zB_y)\hat{i} + (A_xB_z - A_zB_x)\hat{j} \\ &= (A_yB_z - A_zB_y, A_xB_z - A_zB_x, A_xB_y - A_yB_x)\end{aligned}$$