1 | Problem 1

1.1 | Center of Mass, Definition

Per our discussion last semester, and derived via taking a double-integral over the expression for net force of a group of point-masses then dividing by mass, the center of mass' position can be figured via the following expression:

$$r_{\vec{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r_i} \tag{1}$$

where, M is the total mass in the system, m_i the mass of each sub-component, and $\vec{r_i}$ the position of the component.

Therefore, for two objects A, B, with masses M_A, M_B, and center of mass located at $A_{CM} = (A_x, A_y, A_z)$, and $B_{CM} = (B_x, B_y, B_z)$, we can compute the location of the center of mass via the above-given expression; that:

$$r_{\vec{CM}} = \frac{M_A(A_x, A_y, A_z) + M_B(B_x, B_y, B_z)}{M_A + M_B} \tag{2}$$

1.2 | N-Object System

We have shown that the above expression satisfies for two systems A and B. Furthermore, we have the base cases of zero objects, with the center of mass at the origin.

By induction, therefore, the following follows:

$$r_{\vec{CM}} = \frac{M_1(P_{(1,x)}, P_{(1,y)}, P_{(1,z)}) + \cdots + M_N(P_{(1,x)}, P_{(1,y)}, P_{(1,z)})}{M_1 + \cdots + M_N} \tag{3}$$

That: the position of the center of mass is simply the weighted average of all positions.

2 | Big Mountain

Given that the material that makes up Mount Fuji has uniform density, a unit volume will result in a unit mass in any given point. We will define a system where the origin is at the apex of the mountain, and where "downwards" is a positive direction.

Therefore, we can write the center of mass as follows:

$$r_{\vec{CM}} = \frac{\int_0^{h_f} (\pi \tan(65°)h)^2 h \; dh}{\int_0^{h_f} (\pi \tan(65°)h)^2 dh} \tag{4}$$

Where, $(\pi \tan(65°)h)^2$ is the "area" (i.e. mass) at any given slice, and h (outside the expression) the z position of the CoM (as we know the x and y coordinates are $(0,0)$ respectively), $h_f = 3600$ the apex height.

We will note that the constant components immediately cancel:

$$r_{\vec{CM}} = \frac{\int_0^{h_f} (h^2 h = h^3) \; dh}{\int_0^{h_f} h^2 dh} \tag{5}$$

And, therefore, taking the integrals:

$$\vec{r_{CM}} = \frac{\int_0^{h_f} (h^2 h = h^3)\ dh}{\int_0^{h_f} h^2 dh} \tag{6}$$
$$= \frac{\int_0^{h_f} h^3\ dh}{\int_0^{h_f} h^2 dh} \tag{7}$$
$$= \frac{\frac{h^4}{4}\Big|_0^{h_f}}{\frac{h^3}{3}\Big|_0^{h_f}} \tag{8}$$
$$= \frac{\frac{{h_f}^4}{4}}{\frac{{h_f}^3}{3}} \tag{9}$$
$$= \frac{3h_f}{4} \tag{10}$$

We can see that, from the apex, the center of mass will be $\frac{3}{4}$ of the apex height. Therefore, it will be located at 2700 m from the apex, or 900 m from the origin.

3 | **Right Triangular Plate**

We again begin with the expression from first-principles of the center of mass of the object.

$$\vec{r_{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r_i} \tag{11}$$

As per instructed by the problem, for each small strip Δx in the system, the center of mass is exactly halfway up from the middle.

At any given strip, therefore, $(x, \frac{xh}{2b})$ would be the coordinate of its "middle"—its CoM: where x is the x-position at the edge of any given strip, h the given height of the triangle, and b the given base. The right expression is given by the properties of similar triangles to figure the height at the strip, then dividing by 2.

Furthermore, by the same token as leveraging Δx strip above and similar triangles, each strip has a mass of $\Delta x \frac{xh}{b}$ scaled—as the mass is uniform—by some uniform mass scalar M.

We also know that, given the shape of the triangle, the total area (i.e. mass) would be $\frac{1}{2}bh$.

The final center of mass, therefore, can be expressed as two integrals each representing the x and y components of the CoM:

$$\begin{cases} \vec{r_{cmx}} = \frac{2}{bh}\int_0^b \frac{xh}{b}(x)\Delta x \\ \vec{r_{cmy}} = \frac{2}{bh}\int_0^b \frac{xh}{b}\frac{xh}{2b}\Delta x \end{cases} \tag{12}$$

Let us solve each of these integrals!

For the x-component:

$$\frac{2}{bh}\int_0^b \frac{xh}{b} x dx = \frac{2}{bh}\int_0^b \frac{x^2 h}{b} dx \tag{13}$$
$$= \frac{2}{bh}\frac{h}{b}\int_0^b x^2 dx \tag{14}$$
$$= \frac{2}{b^2}\int_0^b x^2 dx \tag{15}$$
$$= \frac{2}{b^2}\left(\frac{x^3}{3}|_0^b\right) \tag{16}$$
$$= \frac{2}{b^2}\left(\frac{b^3}{3}\right) \tag{17}$$
$$= \frac{2}{3}b \tag{18}$$

We conclude, therefore, that the x component of the center of mass of the triangle is located at $\frac{2}{3}$ of its base. For the y-component:

$$\frac{2}{bh}\int_0^b \frac{xh}{b}\frac{xh}{2b} dx = \frac{2}{bh}\int_0^b \frac{(xh)^2}{2b^2} dx \tag{19}$$
$$= \frac{2}{bh}\frac{h^2}{2b^2}\int_0^b x^2 dx \tag{20}$$
$$= \frac{h}{b^3}\int_0^b x^2 dx \tag{21}$$
$$= \frac{h}{b^3}\left(\frac{x^3}{3}|_0^b\right) \tag{22}$$
$$= \frac{h}{b^3}\frac{b^3}{3} \tag{23}$$
$$= \frac{1}{3}h \tag{24}$$

We can further conclude that the y component of the center of mass is located at $\frac{1}{3}$ of its height. Therefore, the position of the center of mass in a triangle like such highlighted is:

$$\left(\frac{2}{3}b, \frac{1}{3}h\right) \tag{25}$$