1 | **Proving the Parallel Axis Theorem**

We will begin with the definition of rotational inertia about an origin:

$$I = \sum_i m_i {l_i}^2 \tag{1}$$

As per defined by the problem $l'_i = x'_i \hat{\imath} + y'_i \hat{\jmath}$, the displacement vector from l_i to the CM.

We also understand that $\vec{R}_{CM} = X_{CM}\hat{\imath} + Y_{CM}\hat{\jmath}$, the components to the location of the center of mass.

Therefore, the actual position $\vec{l_i}$ of the axis of rotation can be expressed as:

$$l_i = \vec{R}_{CM} + \vec{l'_i} \tag{2}$$

Substituting this expression into that for I:

$$\begin{aligned} I &= \sum_i m_i {l_i}^2 &(3)\\ &= \sum_i m_i (\vec{R}_{CM} + \vec{l'_i})^2 &(4)\\ &= \sum_i m_i ((\vec{R}_{CM})^2 + 2(\vec{R}_{CM})(\vec{l'_i}) + (\vec{l'_i})^2) &(5) \end{aligned}$$

We can see that the first and last terms will result in the expression we need, and therefore we are bound to figure why the following expression would result in 0:

$$\begin{aligned} &\sum_i 2(\vec{R}_{CM}) m_i \vec{l'}_i &(6)\\ \Rightarrow\ &2(\vec{R}_{CM}) \sum_i m_i \vec{l'}_i &(7) \end{aligned}$$

Recall that $\vec{l'}_i$ is the location of the distance from a given point mass in the body to the center of mass. Of course, this makes $\sum_i m_i \vec{l'}_i$ the expression for the center of mass *in the center of mass reference frame.*

We know that the coordinate of the center of mass in the center of mass reference frame is 0, making this whole expression 0.

Therefore:

$$\begin{aligned} I &= \sum_i m_i {l_i}^2 &(8)\\ &= \sum_i m_i ((\vec{R}_{CM})^2 + 2(\vec{R}_{CM})(\vec{l'_i}) + (\vec{l'_i})^2) &(9)\\ &= (\vec{R}_{CM})^2 \sum_i m_i + \sum_i m_i (\vec{l'_i})^2 &(10)\\ &= (\vec{R}_{CM})^2 M + \sum_i m_i (\vec{l'_i})^2 &(11)\\ &= D^2 M + I_{CM}\ \blacksquare &(12) \end{aligned}$$

2 | **5-Ring Object**

We will leverage the parallel axis theorem to figure the inertia at the center point.

We can see the distance from the center of mass of each side-sphere to the axis of rotation is $2R$. Furthermore, we can see the mass of the ring is $\frac{1}{5}M$.

As been demonstrated before, the rotational inertia of a ring is:

$$I = MR^2 \tag{13}$$

For the ring with $\frac{1}{5}$ mass then:

$$I_{CM} = \frac{1}{5}MR^2 \tag{14}$$

Applying the parallel axis theorem, then, to each of the four side-objects:

$$\begin{aligned} I &= I_{CM} + \frac{1}{5}MD^2 && (15)\\ &= \frac{1}{5}MR^2 + \frac{4}{5}MR^2 && (16)\\ &= MR^2 && (17) \end{aligned}$$

We repeat this procedure four times, to result in the outer rings' rotational inertia of:

$$4MR^2 \tag{18}$$

The last (”middle”) ring has simply the rotational inertia about its center of origin:

$$\frac{1}{5}MR^2 \tag{19}$$

And therefore, the total rotational inertia is:

$$I = \frac{21}{5}MR^2 \tag{20}$$

3 | **Spinning Top**

3.1 | **Torque**

We are asked to find the magnitude and direction of torque $\vec{\tau}_0$ applied by the string. This is easily achieved with the expression for torque $\vec{r} \times \vec{F_0}$.

We will do this in components. Observing the point at which the string is attached, we noticed that it has two components: one towards the positive y direction, and one towards the positive z direction. That:

$$\vec{r} = R\hat{j} + H\hat{k} \tag{21}$$

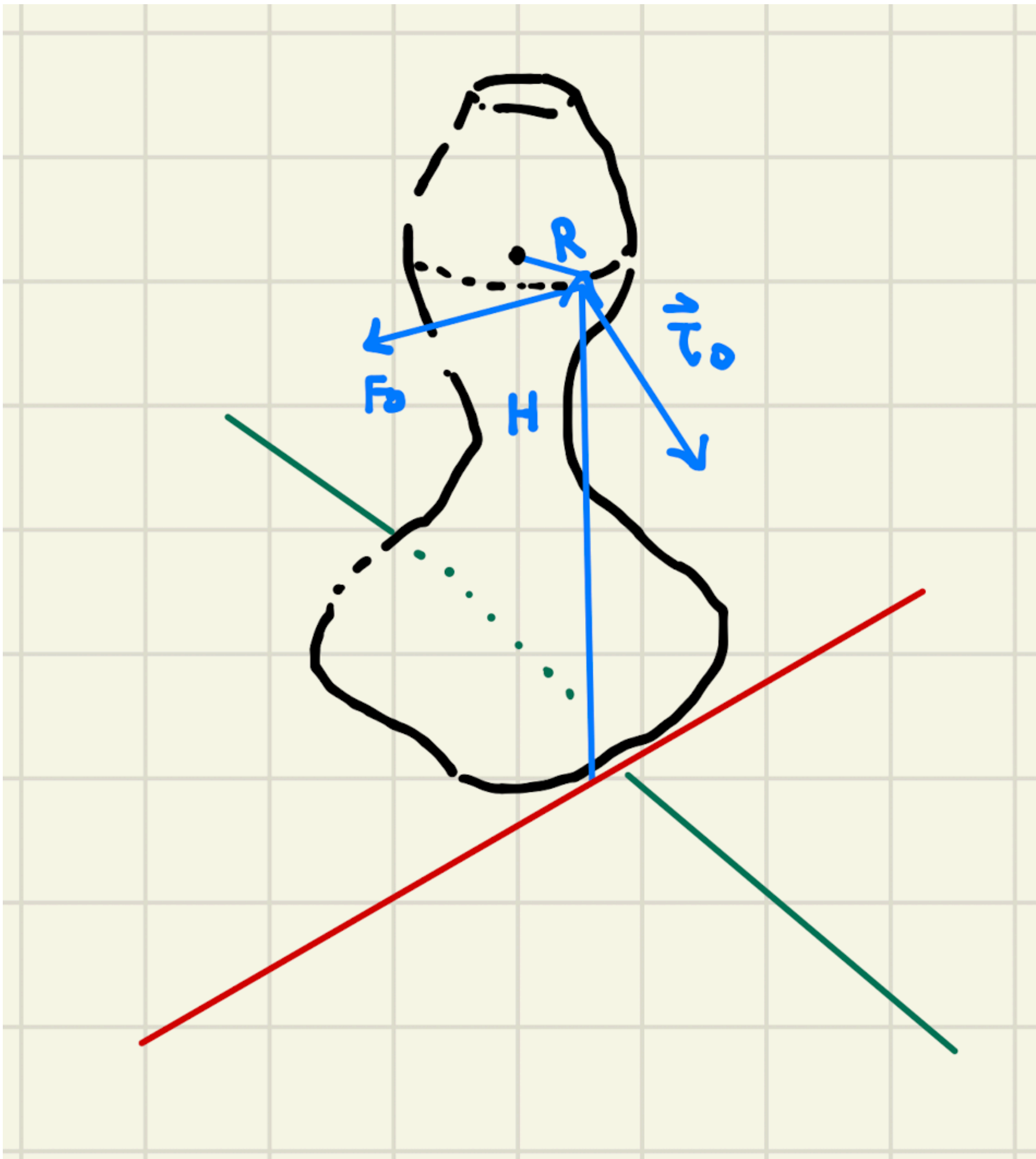
Furthermore, we can see from the graph that $\vec{F_0} = F_0\hat{i}$. Their dot products, therefore, are:

$$\begin{aligned}
\vec{\tau}_0 &= (R\hat{j} + H\hat{k}) \times (F_0\hat{i}) && (22)\\
&= RF_0(\hat{j} \times \hat{i}) + HF_0(\hat{k} \times \hat{i}) && (23)\\
&= -RF_0\hat{k} + HF_0\hat{j} && (24)\\
&= HF_0\hat{j} - RF_0\hat{k} && (25)
\end{aligned}$$

And therefore:

$$\begin{cases}
\tau_{0x} = 0 \\
\tau_{0y} = HF_0 \\
\tau_{0z} = -RF_0
\end{cases} \tag{26}$$

3.2 | **Diagram of $\vec{\tau}_0$**



3.3 | **Balancing Forces**

To figure the net torque on the system, we will need to add the torques contributing to the net torque in the system.

We have already deducted above $\vec{\tau}_0 = HF_0\hat{j} - RF_0\hat{k}$. Given $\vec{F}_1$ is attached to the origin of the system, it contributes no torque. Therefore, to figure out the net torque we only need to deduct that for $\vec{F}_2$.

$$\vec{\tau}_2 = \vec{r_2} \times \vec{F_2} \tag{27}$$
$$= -\frac{H}{2}\hat{k} \times F_2\hat{i} \tag{28}$$
$$= -\frac{HF_2}{2}\hat{j} \tag{29}$$

As we know, the object simply rotates about the $\hat{k}$ axis. Therefore, the net torque along $\hat{j}$ would have to be zero.

Therefore:

$$-\frac{HF_2}{2} + HF_0 = 0 \tag{30}$$

We further understand that the object does not move. This means that it has a net force of 0 as well. That is:

$$F_0 + F_2 - F_1 = 0 \tag{31}$$

We are given F_0. We therefore have two equations for two variables, rendering it suitable for solving.

$$-\frac{HF_2}{2} + HF_0 = 0 \tag{32}$$
$$\Rightarrow 2HF_0 - HF_2 = 0 \tag{33}$$
$$\Rightarrow 2F_0 - F_2 = 0 \tag{34}$$
$$\Rightarrow F_2 = 2F_0 \tag{35}$$
$$F_0 + F_2 - F_1 = 0 \tag{36}$$
$$\Rightarrow F_0 + 2F_0 - F_1 = 0 \tag{37}$$
$$\Rightarrow F_1 = 3F_0 \tag{38}$$

Per the setup of the problem, $\vec{F_1}$ is in the $-\hat{i}$ direction, and $\vec{F_2}$ in the $\hat{i}$ direction. Hence:

$$\begin{cases} \vec{F_1} = -3F_0\hat{i} \\ \vec{F_2} = 2F_0\hat{i} \end{cases} \tag{39}$$

3.4 | **Rotational Inertia**

We have already determined the net torque of the system.

$$\vec{\tau}_2 = -\frac{HF_2}{2}\hat{j} \tag{40}$$
$$\vec{\tau}_0 = HF_0\hat{j} - RF_0\hat{k} \tag{41}$$
$$\vec{\tau}_{net} = (HF_0 - \frac{HF_2}{2})\hat{j} - RF_0\hat{k} \tag{42}$$

We also know from the setup of the problem that the left term works out such that, given the values of F_0 and F_2, it is zero in this scenario. Hence:

$$\vec{\tau}_{net} = -RF_0\hat{k} \tag{43}$$

Based on Newton's Second Law, we understand that:

$$\vec{\tau}_{net} = I\vec{\alpha} \tag{44}$$

If the top has rotational inertia I_0, therefore:

$$I_0\vec{\alpha_0} = -RF_0\vec{k} \tag{45}$$

$$\Rightarrow \vec{\alpha_0} = \frac{-RF_0}{I_0}\vec{k} \tag{46}$$

3.5 | **Kinematics Equations**

Our system is angularity accelerating at an constant angular acceleration of $\vec{\alpha_0}$. As such, we integrate twice to figure the kinematics equations.

First, as derived above:

$$\vec{\alpha}(t) = \frac{-RF_0}{I_0}\vec{k} \tag{47}$$

Taking the first integral of this expression, we get that:

$$\vec{\omega}(t) = \int \vec{\alpha}(t)dt = \frac{-RF_0}{I_0}t\vec{k} + C \tag{48}$$

where, as $\omega = 0$ at $t = 0$:

$$\vec{\omega}(t) = \frac{-RF_0}{I_0}t\vec{k} \tag{49}$$

Performing the integral yet again, we have that:

$$\vec{\theta}(t) = \int \vec{\omega}(t)dt = \frac{-RF_0}{2I_0}t^2\vec{k} + C \tag{50}$$

where again, as $\theta = 0$ at $t = 0$:

$$\vec{\theta}(t) = \frac{-RF_0}{2I_0}t^2\vec{k} \tag{51}$$

4 | **Rectangular Rod**

To find the rotational inertia of a rectangular rod, we need to perform three integrations: building up strips, slices, and finally the inertia of the actual rod.

4.1 | Strip

We begin by recalling that the expression for rotational inertia is:

$$I = \sum_i m_i {l_i}^2 \tag{52}$$

Furthermore, we understand the mass of our entire volume is M. Therefore, the mass density along the object would be $\frac{M}{HWL}$.

To figure the inertia of an infinitesimal strip of point masses, we perform a simple integration along the w axis:

$$\begin{aligned}
I_{strip} &= \int_{-W/2}^{W/2} l^2 dm & (53)\\
&= \int_{-W/2}^{W/2} l^2 \frac{dm}{dl} dl & (54)\\
&= \int_{-W/2}^{W/2} l^2 \frac{M}{HWL} dl & (55)\\
&= \frac{M}{HWL}\left(\frac{(W/2)^3}{3} - \frac{(-W/2)^3}{3}\right) & (56)\\
&= \frac{M}{HWL}\left(\frac{W^3}{24} - \frac{-W^3}{24}\right) & (57)\\
&= \frac{M}{HWL}\left(\frac{W^3}{12}\right) & (58)\\
&= \frac{M}{HL}\left(\frac{W^2}{12}\right) & (59)
\end{aligned}$$

4.2 | Slice

We will now find the inertia of a slice. The procedure is essentially the same, but that there is no longer any non-constant components. Therefore, the inertias, as we are rotating about the same axis of a rigid body together, simply stack ("add"). That is: the rotational inertia of a slice is simply H times that of a strip.

$$\begin{aligned}
I_{slice} &= H\ I_{strip} & (60)\\
&= H\ \frac{M}{HL}\left(\frac{W^2}{12}\right) & (61)\\
&= \frac{M}{L}\left(\frac{W^2}{12}\right) & (62)
\end{aligned}$$

4.3 | Final Rotational Inertia

Finally, we will leverage the parallel axis theorem to deduct rotational inertia of the entire rod.

At every plate i, we note that it will be l_i away from the axis of rotation $\hat{k}$. By the parallel axis theorem:

$$I = I_{cm} + mD^2 \tag{63}$$

where, $l_i = D$ and \$m=M$_{L\$---themassofeachslice.}$

That is, then:

$$I_{slice\ about\ \hat{k}} = \frac{M}{L}\left(\frac{W^2}{12}\right) + m_i(l_i)^2 \tag{64}$$

We aim to find the sum of all such rotational inertia slices about $\vec{k}$ along L, meaning we will figure:

$$I = \sum_L \left(\frac{M}{L}\left(\frac{W^2}{12}\right) + m_i(l_i)^2\right) \tag{65}$$

Splitting this summation into two parts:

$$I = \sum_L \frac{M}{L}\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 \tag{66}$$

We can see that, because the lack of differentials on the left side, the left expression can simply be simplified to route multiplication:

$$I = \sum_L \frac{M}{L}\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 \tag{67}$$

$$= M\left(\frac{W^2}{12}\right) + \sum_L m_i(l_i)^2 \tag{68}$$

The right side, however, requires integration. The actual integral is, fortunately, almost the same procedure as before—summing up differential l_i along L via differential masses m_i. We will leverage the mass density of a slice again: $\frac{M}{L}$ ("total mass divided by all slices").

$$\sum_L m_i(l_i)^2 = \int_{-L/2}^{L/2} l^2 dm \tag{69}$$

$$= \int_{-L/2}^{L/2} l^2 \frac{dm}{dl} dl \tag{70}$$

$$= \int_{-L/2}^{L/2} l^2 \frac{M}{L} dl \tag{71}$$

$$= \frac{M}{L}\int_{-L/2}^{L/2} l^2 dl \tag{72}$$

$$= \frac{M}{L}\int_{-L/2}^{L/2} l^2 dl \tag{73}$$

$$= \frac{M}{L}\left(\frac{l^3}{3}\bigg|_{-L/2}^{L/2}\right) \tag{74}$$

$$= \frac{M}{L}\left(\frac{L^3}{12}\right) \tag{75}$$

$$= M\left(\frac{L^2}{12}\right) \tag{76}$$

Substituting this back into our above expression for I again:

$$
\begin{aligned}
I &= M\left(\frac{W^2}{12}\right) + \sum_{L} m_i(l_i)^2 && (77)\\
&= M\left(\frac{W^2}{12}\right) + M\left(\frac{L^2}{12}\right) && (78)\\
&= \frac{1}{12}M\left(W^2 + L^2\right)\ \blacksquare && (79)
\end{aligned}
$$