

## 1 | Problem: Axler 3.E exercise 18

Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subspace of  $V$ . Let  $\pi$  denote the quotient map from  $V$  onto  $V/U$ . Prove that there exists  $S \in \mathcal{L}(V/U, W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

Intuitively, if we mod out part of the null  $T$ , then we should still be able to have a map that does what  $T$  would do. If we are able to do what  $T$  would do, then when modding out  $U$  we only removed part of null  $T$  and lost no information.

## 2 | Forward Direction

Intuitively, we can treat  $S \circ \pi$  as a single map and take a basis of  $V$  to the same place that  $T$  would, and the maps would be equal.

Let  $S$  be a relation between  $V/U$  and  $W$  defined by

$$S(U + v) = Tv$$

If  $S$  is well defined (every element in  $V/U$  is mapped to exactly one place), then  $S$  will inherit additivity and homogeneity from  $T$  and  $S \circ \pi$  will equal  $T$ .

Let  $v \in V$  and  $v' \in V/U$  s.t.  $v' = \pi v$  ( $v'$  is where  $\pi$  takes  $v$ ). Then, to show that  $S$  is well defined, we must show that  $v$  has atleast one and at most one image through  $S \circ \pi$ .

Because  $\pi v$  is well defined, and  $U + v$  was arbitrary in the definition of  $S$ , each  $v$  must have atleast one image in  $W$ .

Take  $S$  to be an arbitrary linear map. The only restriction on  $S$  that could cause  $S(U + v) \neq Tv$  is  $S(0) = 0$  (this statement is not watertight). Thus,  $S$  is defined if  $\forall U + v = U = 0, Tv = 0$ . Equivalently,  $S$  is defined if  $U \subseteq \text{null } T$ , which is given in the problem.

Thus,  $S$  is well defined. To show that it inherits additivity and homogeneity:

$$S(U + u) + S(U + v) = Tu + Tv = T(u + v) = S(U + u + U + v) = S(U + (u + v))$$

$$\lambda(S(U + v)) = \lambda Tv = T(\lambda v) = S(U + (\lambda v))$$

Thus,  $S$  is linear, and  $S \circ \pi = T$  if  $U \subseteq \text{null } T$ .

## 3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what  $T$  would do.

Assume  $U \not\subseteq \text{null } T$ . There exists  $v \in U$  s.t.  $Tv \neq 0$ . This is some of the "information" that was "lost". Because  $v \in U$ ,

$$\pi v = U + v = U$$

Because  $U$  is the additive identity (0) in  $V/U$ , and because linear maps take zero to zero,  $S \in \mathcal{L}(V/U, W)$  must take  $\pi v = 0$  to zero. Thus, either  $S(\pi v) \neq Tv$  or  $S$  is not a linear map, both of which are contradictions.

This shows that if  $U \not\subseteq \text{null } T$ , then  $S \notin \mathcal{L}(V/U, W)$  or  $T \neq S \circ \pi$ . Thus, if  $S \in \mathcal{L}(V/U, W)$  and  $T = S \circ \pi$ , then  $U \subseteq \text{null } T$ .