In this problem, we aim to determine functions f_1, f_2, that represents the velocity and acceleration vectors respectively of a circularly movable reference frame.

1 | Defining First Principles

We begin by defining $\vec{r}$, the vector representing the radius of the rotating frame, as follows:

$$\vec{r} = r\cos(\theta)\hat{i} + r\sin\theta\hat{j} \tag{1}$$

We define θ as the angle up from the horizontal at which $\vec{r}$ is located, and therefore the vector $\vec{r}$ is simply the magnitude thereof r projected upon that angle $(\cos\theta, \sin\theta)$ into a vector.

Furthermore, we define a unit vector in the direction of $\vec{r}$ as $\hat{r}$. That is:

$$\hat{r} = \cos(\theta)\hat{i} + \sin\theta\hat{j} \tag{2}$$

Lastly, we define a vector $\hat{\theta}$, a unit vector orthogonal to $\vec{r}$. It is defined as such as the direction of $\vec{\theta}$ would, at any given instance, be perpendicular to the direction of $\vec{r}$ and parallel to the direction to its movement.

$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{3}$$

2 | Determining changes in direction

We now create definitions for changes in "direction" — the changes present in $\hat{r}$ and $\hat{\theta}$ — which we will leverage later.

$$\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j}) && (4)\\
&= -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} && (5)\\
&= \dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j}) && (6)\\
&= \dot{\theta}\hat{\theta} && (7)
\end{aligned}$$

Hence, the change in the direction of $\hat{r}$, aptly and intuitively, could be modeled by the change in the angle θ times the direction of θ.

$$\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j}) && (8)\\
&= -\dot{\theta}\cos\theta\hat{i} - \dot{\theta}\sin\theta\hat{j} && (9)\\
&= -\dot{\theta}(\cos\theta\hat{i} + \sin\theta\hat{j}) && (10)\\
&= -\dot{\theta}\hat{r} && (11)
\end{aligned}$$

We now note that, indeed, the change in the direction of θ is modeled by the direction at which $\vec{r}$ exists, and the angle of θ as θ must be orthogonal to $\vec{r}$.

3 | Solving for $f_1 = \vec{v}$

We now begin to solve for a function $f_1(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, r) = \vec{v}$. We know that the velocity of the frame as a whole could be modeled by the following expression:

$$\vec{v} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt} \tag{12}$$

$$= r\dot{\theta}\hat{\theta} + \hat{r}\dot{r} \tag{13}$$

The derivation of this expression simply follows variable substitution to result in the expression modeling velocity: that the velocity of the point is determined by the radius-scaled velocity in the angle of the object (how fast it spins), plus the velocity of the radius itself (how fast it grows.)

4 | Solving for $f_2 = \vec{a}$.

We here wish to determine a function $f_2(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}, r) = \vec{a} = \frac{d}{dt}\vec{v}$

Deriving the value of this expression, therefore, simply acts as a matter of taking the derivative of the top-derived expression and performing variable substitution for $\dot{\hat{r}}$ and $\dot{\hat{\theta}}$ as determined above as needed.

$$\vec{a} = \frac{d}{dt}(r\dot{\theta}\hat{\theta} + \hat{r}\dot{r}) \tag{14}$$

$$= ((\frac{d}{dt}r)\dot{\theta}\hat{\theta} + ((\frac{d}{dt}\dot{\theta})\hat{\theta} + (\frac{d}{dt}\hat{\theta})\dot{\theta})r) + ((\frac{d}{dt}\hat{r})\dot{r} + (\frac{d}{dt}\dot{r})\hat{r}) \tag{15}$$

$$=((\dot{r})\dot{\theta}\hat{\theta} + ((\ddot{\theta})\hat{\theta} + (-\dot{\theta}\hat{i})\dot{\theta})r) + ((\dot{\theta}\hat{\theta})\dot{r} + (\ddot{r})\hat{r}) \tag{16}$$

$$=(\dot{r}\dot{\theta}\hat{\theta} + \ddot{\theta}\hat{\theta}r - \dot{\theta}^2\hat{r}r) + (\dot{\theta}\hat{\theta}\dot{r} + \ddot{r}\hat{r}) \tag{17}$$

$$=\hat{\theta}(2\dot{r}\dot{\theta} + \ddot{\theta}r) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) \tag{18}$$

We could now figure that acceleration is actually four terms, scaled by their respective directions ($\hat{\theta}$ and $\hat{r}$). That is:

- $2\dot{r}\dot{\theta}$ is a term representing the coriolis effect: a fictitious force pushing objects as either of radius change or spin speed changes
- $\ddot{\theta}r$ is a term that models the tangential acceleration: how quickly does the object accelerates at a direction tangent to the spin
- $\ddot{r}$ is a term that models the acceleration of the radius, as it evidently shows
- $\dot{\theta}^2 r$ is a term that models the inward acceleration that maintains the shape of the circle. It is $\dot{\theta}^2 r = \omega^2 r$