1 | External Torque

Proof:

$$\vec{\tau}_{total\ ext} = \frac{d\vec{L}_{system}}{dt} \tag{1}$$

According to the work that is already done, we have that:

$$\frac{d\vec{L}_{system}}{dt} = \vec{\tau}_{tot\ ext} + \sum_{1 \leq i < j \leq N} \left[(\vec{r}_i \times \vec{F}_{j \to i}) + (\vec{r}_j \times \vec{F}_{i \to j}) \right] \tag{2}$$

The left expression $\vec{\tau}_{tot\ ext}$ already achieves what we want in terms of the final expression, so the right expression must cancel to 0.

We first begin by recognizing, by Newton's Third Law, $\vec{F}_{i \to j} = -\vec{F}_{j \to i}$ as long as both objects are within the system. Therefore, we will first begin by making this substitution below:

$$(\vec{r}_i \times \vec{F}_{j \to i}) + (\vec{r}_j \times \vec{F}_{i \to j}) = (\vec{r}_j \times \vec{F}_{i \to j}) - (\vec{r}_i \times \vec{F}_{i \to j}) \tag{3}$$

We further understand that the cross-product is distributive across addition; therefore:

$$(\vec{r}_i \times \vec{F}_{j \to i}) + (\vec{r}_j \times \vec{F}_{i \to j}) = (\vec{r}_j \times \vec{F}_{i \to j}) - (\vec{r}_i \times \vec{F}_{i \to j}) \tag{4}$$

$$= (\vec{r}_j - \vec{r}_i) \times \vec{F}_{i \to j} \tag{5}$$

At this point, we realise that a vector $\vec{r_i}$ subtracted from $\vec{r_j}$ is a vector $i \to j$, which would be parallel to $\vec{F}_{i \to j}$. As such, their cross products would be zero.

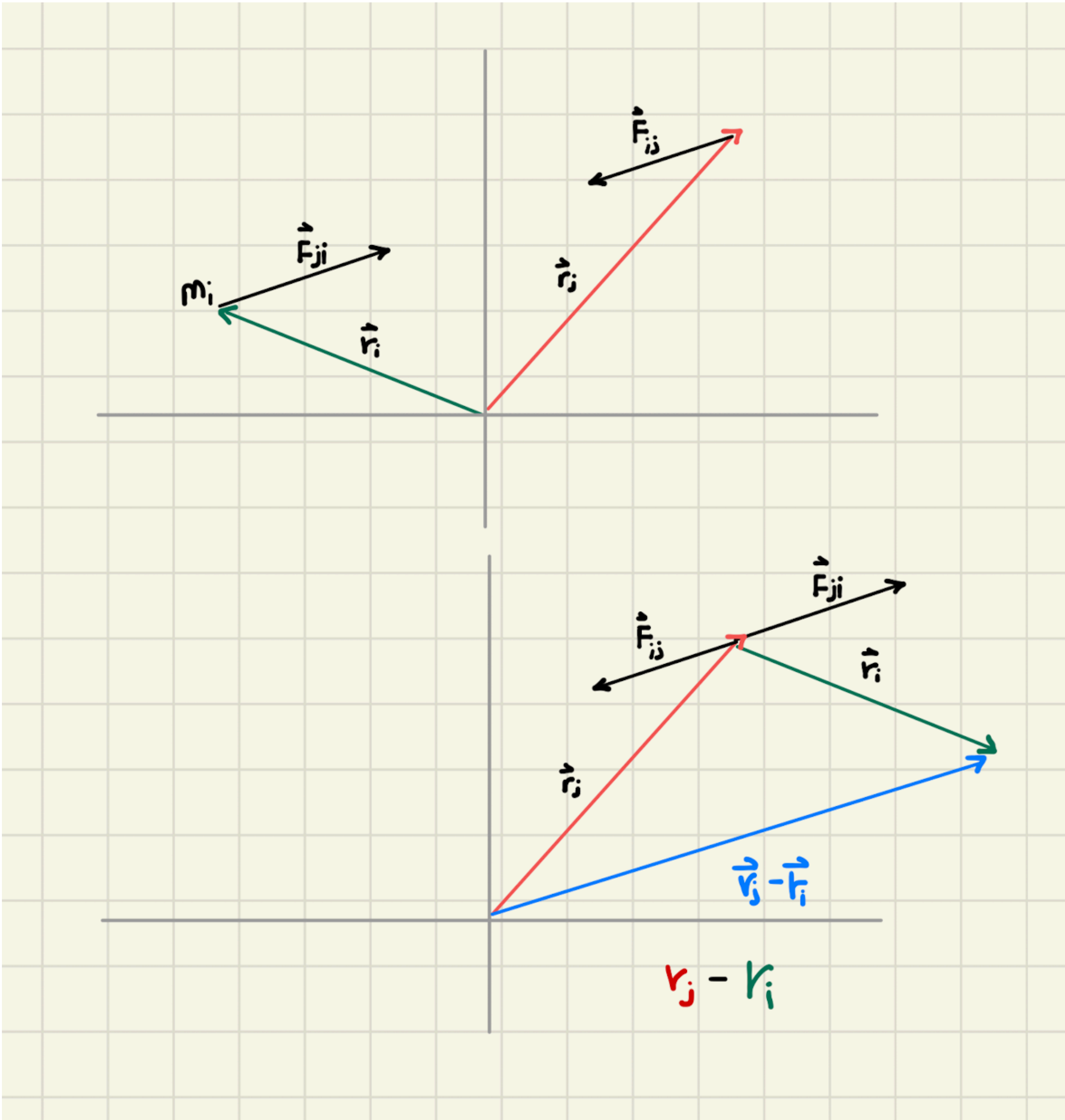
Completing the proof, then:

$$\frac{d\vec{L}_{system}}{dt} = \vec{\tau}_{tot\ ext} + \sum_{1 \leq i < j \leq N} \left[(\vec{r}_i \times \vec{F}_{j \to i}) + (\vec{r}_j \times \vec{F}_{i \to j}) \right] \tag{6}$$

$$= \vec{\tau}_{tot\ ext} + \sum_{1 \leq i < j \leq N} 0 \tag{7}$$

$$= \vec{\tau}_{tot\ ext} + 0 \tag{8}$$

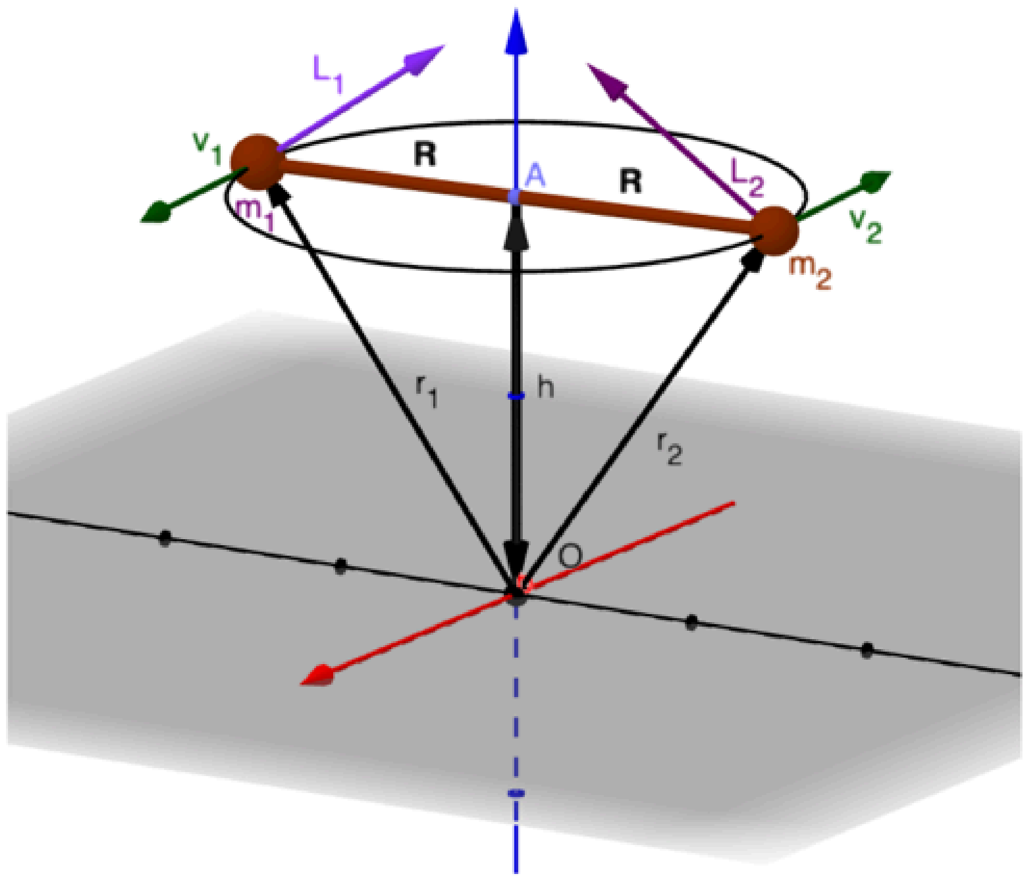
$$= \vec{\tau}_{tot\ ext} \blacksquare \tag{9}$$



2 | Two Rotating Point Masses

We do this problem by calculating the angular momentum of each of the $m_{\{1,2\}}$, and adding the angular accelerations together.

The system is given by this figure:



We define a coordinate such that the ”figure”'s left is the negative y direction, the ”figure”'s right is the positive y direction. The side ”outside” the page is the positive x direction, and the side ”inside” the page is the negative x direction.

2.1 | Angular Momentum from m_1

We begin with the expression for tangential velocity, which is simply $R\omega$.

Therefore:

$$\begin{aligned}\vec{V} &= (R\hat{j})(\omega\hat{k}) && (10)\\ &= (R\omega)(\hat{j}\hat{k}) && (11)\\ &= R\omega\hat{i} && (12)\end{aligned}$$

We will supply this expression for that of the angular momentum: $\vec{R} \times m\vec{V}$:

$$\vec{L} = \vec{r} \times m\vec{V} \tag{13}$$
$$= (-R\hat{j} + h\hat{k}) \times m(R\omega\hat{i}) \tag{14}$$
$$= -R\hat{j} \times mR\omega\vec{i} + h\hat{k} \times mR\omega\vec{i} \tag{15}$$
$$= -mR^2\omega(\hat{j} \times \hat{i}) + hmR\omega(\hat{k} \times \hat{i}) \tag{16}$$
$$= mR^2\omega\hat{k} + hmR\omega\hat{j} \tag{17}$$

2.2 | Angular Momentum from m_2

The calculations here is essentially almost the same. We just need to take a few sign changes as R in m_2 is in the opposite direction as R in m_1.

Therefore:

$$\vec{V} = (-R\hat{j})(\omega\hat{k}) \tag{18}$$
$$= (-R\omega)(\hat{j}\hat{k}) \tag{19}$$
$$= -R\omega\hat{i} \tag{20}$$

And, as with before, we supply it to the previous expression.

$$\vec{L} = \vec{r} \times m\vec{V} \tag{21}$$
$$= (R\hat{j} + h\hat{k}) \times m(-R\omega\hat{i}) \tag{22}$$
$$= R\hat{j} \times -mR\omega\vec{i} + h\hat{k} \times -mR\omega\vec{i} \tag{23}$$
$$= -mR^2\omega(\hat{j} \times \vec{i}) - hmR\omega(\hat{k} \times \vec{i}) \tag{24}$$
$$= mR^2\omega\hat{k} - hmR\omega\hat{j} \tag{25}$$

2.3 | Summing the $\vec{L_i}$ s

We will now sub the two $\vec{L_i}$ s together:

$$\vec{L}_1 + \vec{L}_2 = mR^2\omega\hat{k} + hmR\omega\hat{j} + mR^2\omega\hat{k} - hmR\omega\hat{j} \tag{26}$$
$$= mR^2\omega\hat{k} + mR^2\omega\hat{k} + hmR\omega\hat{j} - hmR\omega\hat{j} \tag{27}$$
$$= 2mR^2\omega\hat{k} + 0 \tag{28}$$
$$= 2mR^2\omega\hat{k} \blacksquare \tag{29}$$

3 | Rigid Body Generalization

3.1 | N-Mass Formula

We will think about a single mass m_i in isolation. We will use ℓ_i to represent the perpendicular distance from m_i to $\hat{k}$.

We begin by analyzing some m_i, and add all m_i together.

3.2 | The angular momentum of m_i

For every object m_i, we take it being arranged at some θ between $\hat{i}$ and $\hat{j}$. We will perform the same procedures as with above to calculate angular momentum.

ℓ_i is angled between $\hat{i}$ and $\hat{j}$. Specifically, it is located at:

$$\ell_i = cos(\theta)\ell_i\hat{i} + sin(\theta)\ell_i\hat{j} \tag{30}$$

We will calculate the tangential velocity, as usual:

$$\begin{aligned}
\vec{V} &= (cos(\theta)\ell_i\hat{i} + sin(\theta)\ell_i\hat{j})(\omega\hat{k}) && (31)\\
&= cos(\theta)\ell_i(\omega\hat{k})\hat{i} + sin(\theta)\ell_i(\omega\hat{k})\hat{j} && (32)\\
&= cos(\theta)\ell_i\omega(\hat{k}\hat{i}) + sin(\theta)\ell_i\omega(\hat{k}\hat{j}) && (33)\\
&= cos(\theta)\ell_i\omega\hat{j} - sin(\theta)\ell_i\omega\hat{i} && (34)
\end{aligned}$$

And, as with before, we will supply it to the expression for angular momentum.

$$\begin{aligned}
\vec{L} &= \vec{r} \times m\vec{V} && (35)\\
&= (cos(\theta)\ell_i\hat{i} + sin(\theta)\ell_i\hat{j} + h\hat{k}) \times m(cos(\theta)\ell_i\omega\hat{j} - sin(\theta)\ell_i\omega\hat{i}) && (36)\\
&= (cos(\theta)\ell_i\hat{i} + sin(\theta)\ell_i\hat{j} + h\hat{k}) \times (mcos(\theta)\ell_i\omega\hat{j} - msin(\theta)\ell_i\omega\hat{i}) && (37)
\end{aligned}$$

We will now compute the actual cross product:

$$\begin{aligned}
\vec{L} =&(hmcos\theta\ell_i\omega)\hat{i}+ && (38)\\
&(hmsin\theta\ell_i\omega)\hat{j}+ && (39)\\
&(m\omega(cos\theta\ell_i)^2 + m\omega(sin\theta\ell_i)^2)\hat{k} && (40)
\end{aligned}$$

Let's expand the last term slightly:

$$\begin{aligned}
&m\omega(cos\theta\ell_i)^2 + m\omega(sin\theta\ell_i)^2 && (41)\\
=&m\omega(cos\theta\ell_i)^2 + m\omega(sin\theta\ell_i)^2 && (42)\\
=&m\omega\ cos^2\theta{\ell_i}^2 + m\omega\ sin^2\theta{\ell_i}^2 && (43)\\
=&m\omega(1 - sin^2\theta){\ell_i}^2 + m\omega\ sin^2\theta{\ell_i}^2 && (44)\\
=&m\omega{\ell_i}^2 - m\omega{\ell_i}^2 sin^2\theta + m\omega\ sin^2\theta{\ell_i}^2 && (45)\\
=&m\omega{\ell_i}^2 && (46)
\end{aligned}$$

And therefore, we can re-write the following expression for $\vec{L}$:

$$\begin{aligned}
\vec{L} =&(hmcos\theta\ell_i\omega)\hat{i}+ && (47)\\
&(hmsin\theta\ell_i\omega)\hat{j}+ && (48)\\
&m\omega{\ell_i}^2\hat{k} && (49)
\end{aligned}$$

3.3 | **Angular momentum of the system**

Therefore, the total momentum of the system would be:

$$\vec{L} = \sum_{i=1}^{N} \left(hm\ cos \left(\frac{2\pi i}{N} \right) \ell_i \omega \right) \hat{i} + \left(hm\ sin \left(\frac{2\pi i}{N} \right) \ell_i \omega \right) \hat{j} + m\omega {\ell_i}^2 \hat{k} \tag{50}$$

At this point, we need to remember the fact that the points in the generalized formula we are to derive are axially symmetric. This means that the components $cos\theta\hat{i}$ and $sin(\theta)\hat{j}$ will, pairwise, cancel each other out.

$$\begin{aligned}
\vec{L} &= \sum_{i=1}^{N} \left(hm\ cos \left(\frac{2\pi i}{N} \right) \ell_i \omega \right) \hat{i} + \left(hm\ sin \left(\frac{2\pi i}{N} \right) \ell_i \omega \right) \hat{j} + m\omega {\ell_i}^2 \hat{k} &(51)\\
&= \sum_{i=1}^{N} 0 + m\omega {\ell_i}^2 \hat{k} &(52)\\
&= \sum_{i=1}^{N} m\omega {\ell_i}^2 \hat{k} &(53)\\
&= \hat{k}\omega \sum_{i=1}^{N} m {\ell_i}^2 &(54)
\end{aligned}$$

3.4 | **Actual Integral Expression**

Our object has total mass M and volume V_0. We will now determine an integral expression for its angular momentum.

So far, we have that:

$$\vec{L} = \hat{k}\omega \sum_{i=1}^{N} m {\ell_i}^2 \tag{55}$$

We will convert $\ell_i = \ell(m)$, a function in terms of differential mass m that maps mass to the distance from z axis ℓ.

Hence:

$$\begin{aligned}
\vec{L} &= \hat{k}\omega \sum_{i=1}^{N} m {\ell_i}^2 &(56)\\
&= \hat{k}\omega \int_V \ell^2 dm &(57)\\
&= \hat{k}\omega \int_V \ell^2 \frac{dm}{dv} dV &(58)\\
&= \hat{k}\omega \int_V \ell^2 \frac{M}{V_0} dV &(59)
\end{aligned}$$

In our last step, we can see that the ratio between differential $\frac{dm}{dv}$ can be equivalent—as all components of this system are point masses—to the overall $\frac{M}{V_0}$.

4 | **The Rod**

In this problem, we essentially have a line of aligned infinitesimal point masses around the -center. For every point i, we understand that its tangential velocity can be modeled by:

$$\begin{aligned}\vec{V} &= (\ell_i\hat{j})(\omega\hat{k}) && (60)\\ &= (\ell_i\omega)(\hat{j}\hat{k}) && (61)\\ &= \ell_i\omega\hat{i} && (62)\end{aligned}$$

We again supply this expression into that for $\vec{L}$:

$$\begin{aligned}\vec{L_i} &= \vec{r}\times m_i\vec{V} && (63)\\ &= (\ell_i\hat{j}+h\hat{k})\times m_i\ell_i\omega\hat{i} && (64)\\ &= \ell_i m_i\ell_i\omega(\hat{j}\times\hat{i}) + hm_i\ell_i\omega(\hat{k}\times\hat{i}) && (65)\\ &= -hm_i\ell_i\omega\hat{j} + {\ell_i}^2 m_i\omega\hat{k} && (66)\end{aligned}$$

To figure the actual momentum, we will have to sum the momentums for all i:

$$\vec{L} = \sum_i \vec{L_i} = \sum_i \left(-hm_i\ell_i\omega\hat{j} + {\ell_i}^2 m_i\omega\hat{k}\right) \tag{67}$$

Of course, to actually perform the summation, we integrate over each differential mass:

$$\begin{aligned}&\int_L \left(-h\ell\omega\hat{j} + \ell^2\omega\hat{k}\right) dm && (68)\\ =&\omega\int_L \left(-h\ell\hat{j} + \ell^2\hat{k}\right) dm && (69)\\ =&\omega\int_L \left(-h\ell\hat{j} + \ell^2\hat{k}\right) \frac{dm}{dl}dl && (70)\\ =&\omega\int_L \left(-h\ell\hat{j} + \ell^2\hat{k}\right) \lambda dl && (71)\\ =&\omega\lambda\int_L \left(-h\ell\hat{j} + \ell^2\hat{k}\right) dl && (72)\end{aligned}$$

We now supply the integral with actual bounds: $[-\frac{L}{2}, \frac{L}{2}]$.

$$
\begin{aligned}
&\omega\lambda \int_L \left(-h\ell\hat{j} + \ell^2\hat{k}\right) dl && (73)\\
=&\omega\lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(-h\ell\hat{j} + \ell^2\hat{k}\right) dl && (74)\\
=&\omega\lambda \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} -h\ell\hat{j}dl + \int_{-\frac{L}{2}}^{\frac{L}{2}} \ell^2\hat{k}dl\right) && (75)\\
=&\omega\lambda \left(h\hat{j}\int_{-\frac{L}{2}}^{\frac{L}{2}} \ell dl - \hat{k}\int_{-\frac{L}{2}}^{\frac{L}{2}} \ell^2 dl\right) && (76)\\
=&\omega\lambda \left(-h\hat{j}\frac{\ell^2}{2} + \hat{k}\frac{\ell^3}{3}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} && (77)\\
=&\omega\lambda \left(-h\hat{j}\frac{\frac{L}{2}^2}{2} + \hat{k}\frac{\frac{L}{2}^3}{3}\right) - \left(-h\hat{j}\frac{\frac{-L}{2}^2}{2} + \hat{k}\frac{\frac{-L}{2}^3}{3}\right) && (78)\\
=&2\frac{L^3}{24}\omega\lambda\hat{k} && (79)\\
=&\frac{L^3}{12}\omega\lambda\hat{k} && (80)
\end{aligned}
$$

We continue be realizing that $\lambda = \frac{M}{L}$. Supplying that to the equation:

$$
\begin{aligned}
&\frac{L^3}{12}\omega\lambda\hat{k} && (81)\\
=&\frac{L^3}{12}\omega\frac{M}{L}\hat{k} && (82)\\
=&\frac{L^2}{12}M\omega\hat{k} && (83)\\
=&\frac{1}{12}L^2M\omega\hat{k} && (84)
\end{aligned}
$$

5 | **The Disk**

Our original expression for the momentum in the system is the same.

$$
\begin{aligned}
\vec{L_i} &= \vec{r} \times m_i\vec{V} && (85)\\
&= m\omega{\ell_i}^2\hat{k} && (86)
\end{aligned}
$$

We will have to add the angular momentum in two directions to get that of a disk:

$$\vec{L} = \sum_i \vec{L_i} = \sum_A \vec{L_i} \tag{87}$$

where, A is the area of the surface.

To begin doing this, we will construct the differential operator for a disk. We understand the circumference of the circle is $2\pi r$ for some radius r. Furthermore, for a small ring with infinitesimal thickness dr, it has an area of $2\pi rdr$.

Therefore:

$$da = 2\pi r dr \tag{88}$$

We finally want to find the expression for differential mass. Of course, it is simply the mass density $\frac{dm}{da}$ multiplied by the value of da.

$$\begin{aligned} dm &= \frac{dm}{da}\ da && (89)\\ &= \sigma\ da && (90)\\ &= \frac{M}{\pi R^2} da && (91)\\ &= \frac{M}{\pi R^2} 2\pi r dr && (92)\\ &= \frac{2rM}{R^2} dr && (93) \end{aligned}$$

We can now finally take the integral:

$$\begin{aligned} &\int_0^R \vec{L}_i && (94)\\ \Rightarrow &\int_0^R \omega r^2 \hat{k}\ dm && (95)\\ \Rightarrow &\int_0^R \omega r^2 \hat{k} \frac{2rM}{R^2} dr && (96)\\ \Rightarrow &\int_0^R \omega \hat{k} \frac{2r^3 M}{R^2} dr && (97)\\ \Rightarrow &\frac{2M\omega}{R^2} \hat{k} \int_0^R r^3 dr && (98)\\ \Rightarrow &\frac{2M\omega}{R^2} \hat{k} \left(\left. \frac{r^4}{4} \right|_0^R \right) && (99)\\ \Rightarrow &\frac{2M\omega}{R^2} \hat{k} \frac{R^4}{4} && (100)\\ \Rightarrow &\frac{1}{2} M R^2 \omega \hat{k}\ \blacksquare && (101) \end{aligned}$$