More applications of Calculating Derivatives.

1 | Linear and Quadratic Approximations

1.1 | Linear Approximations

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

Tangent line is near the function curve at values close to x_0 so it serves as an approximation there. Think back to $\lim_{x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{x-x_0}$ and realize that the approximation $\frac{\Delta f}{\Delta x} \approx f'(x_0)$ arises. This is the same relationship as the first equation!

Why?

- Manipulate second approximation to yield the relation $\Delta f \approx f'(x_0)\Delta x$.
- Substitute to get $f(x) - f(x_0) \approx f'(x_0)(x - x_0)$.
- Put constant on the other side to get the original equation. $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

1.2 | Useful Examples

To simplify, base point (x_0) will be 0. Formula becomes $f(x) \approx f(0) + f'(0)x$ with $x_0 = 0$.

NOTE: This formula and the examples based off of it only approximate for values of x near 0.

For $x \approx 0$:

- $\sin x$: $f(x) \approx f(0) + f'(0)x$ so $\sin x \approx x$
- $\cos x$: $f(x) \approx f(0) + f'(0)x$ so $\cos x \approx 1$
- e^x: $f(x) \approx f(0) + f'(0)x$ so $e^x \approx 1 + x$
- $\ln(1 + x)$: $f(x) \approx f(0) + f'(0)x$ so $\ln(x + 1) \approx x$
- $(1 + x)^r$: $f(x) \approx f(0) + f'(0)x$ so $(1 + x)^r \approx 1 + rx$

Linear approximations greatly simplify what can sometimes be more complicated functions (i.e. logarithms which would require a calculator, while 1+x is much simpler).

Example 3 *For $x \approx 0$, find a linear approximation of $\frac{e^{-3x}}{\sqrt{1+x}}$.*

- Remember the earlier approximations of e^x and $(1 + x)^r$.
- Rewrite as product: $e^{-3x}(1 + x)^{-1/2} \approx (1 - 3x)(1 - \frac{1}{2}x)$.
- Expand: $e^{-3x}(1 + x)^{-1/2} \approx 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2$.
- Goal is a linear approximation so loosely approximate to $e^{-3x}(1+x)^{-1/2} \approx 1-\frac{7}{2}x$ (sum up coefficients).
  - Drop the x^2 and higher terms as they get small as x is near zero.

1.3 | **Quadratic Approximations**

Serves as an *extension* of the linear approximation formula.

$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$

Where does that last term come from? Here's one way to sort of be more comfortable with it: since it's a quadratic approximation it must be able to perfectly recreate a quadratic function. With $f(x) = a + bx + cx^2$, $f'(x) = b + 2cx$ and $f''(x) = 2c$. Plugging that in for a value like 0, we get $f(0) = a$, $f'(0) = b$, and $f''(0) = 2c$. In order to recover all necessary terms, we need to halve the second derivative. There's no other way it could be true.

Example 2 *Compute* ln 1.1 *with a quadratic approximation.*

- Algebra yields ln $1 + x \approx x - \frac{x^2}{2}$.
- Plug in: ln$(1.1) \approx \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2$ or 0.95

Quadratic is not always more helpful, as in approximations like that of sin x, quadratic term vanishes.

Here's an example of where it *is* helpful: cos(x).

```
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
import math
x = np.arange(-math.pi/2, math.pi/2, 0.1);
fig=plt.figure(figsize=(3,2))
plt.plot(x,np.cos(x),label="cos(x)")
plt.plot(x, [1 for i in x],label="1") # lin approx
plt.plot(x, [1 - i**2 / 2 for i in x], label="1-(x^2)/2") # quad approx
fig.tight_layout()
plt.legend()
fname = 'images/myfig.png'
plt.savefig(fname)
fname # return this to org-mode
```