1 | orthogonal decomposition

An orthogonal decomposition is a way of writing some vector $v \neq 0 \in V$ as the scaled other vector $u \in V$ plus an orthogonal component

> Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u,v \rangle}{\|v\|^2}$ and $w = u - cv$. Then,
>
> $$\langle w, v \rangle = 0 \text{ and } u = cv + w$$

The important algebra is just setting up a system of equations and noticing that orthogonality implies

$$\begin{aligned}
&0 = \langle u - cv, v \rangle \\
\implies &0 = \langle u - cv, v \rangle = \langle u, v \rangle - \langle cv, v \rangle \\
&= \langle u, v \rangle - c\langle v, v \rangle \\
&= \langle u, v \rangle - c\|v\|^2
\end{aligned}$$

which can then be solved for c

2 | motivation

If we have some vector b which is not in the column space of A (there does not exist $x : Ax = b$) but we still want the best "approximation", then we want to take the "closest" approximation. Suppose $\hat{b}$ is such an approximation, then we want the norm of the difference ($b - \hat{b}$) to be minimal. Thus, we want $b - \hat{b}$ to be orthogonal to the column space of A. This motivates orthogonal decomposition.