

## 1 | Slicing into Rectangles

The general idea of Riemann sums is to slice a curve into vertical non-overlapping rectangles to approximate the area between the curve and the x-axis. This can be expressed mathematically as a summation given the function  $f(x)$ , the range  $[a, b]$ , and the number of rectangles  $n$ :

$$\sum_{k=1}^n \frac{b-a}{n} f\left(a + k \frac{b-a}{n}\right)$$

This can be written more concisely by defining  $\Delta x = \frac{b-a}{n}$  and  $x_k = a + k\Delta x$ :

$$\sum_{k=1}^n \Delta x f(x_k)$$

These estimates all have the right endpoint of the rectangle touching the curve. You could also use the left endpoint, or use the minimum value one and add a triangle to form a trapezoid.

## 2 | Area Interpretation

Areas under curves can be estimated if you recognize the function. For example:

$$\int_0^1 \sqrt{1-x^2} dx$$

Traces out a quarter of a semicircle, so the area under this curve is  $\frac{\pi}{4}$

## 3 | Upper and Lower Bound

To get an upper and lower bound approximation using a Riemann sum, you cannot always take the left or right edge. Instead, you have to take the minimum or maximum in an interval, usually denoted  $f(x_i^*)$ .

## 4 | the Definite Integral

Finally, we can define the definite integral as a limit of Riemann sums.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x$$

Where once again,  $\Delta x = \frac{b-a}{n}$  and  $x_k = a + k\Delta x$

These integrals can be evaluated directly with a lot of algebra and some triangular number tricks.