1 | Definition

#definition Axler3.2 Linear Map #aka linear transformation A *linear map* from V to W is a function $T:V\to W$ with the following properties:

1.1 | Additivity

$$T(u+v) = Tu + Tv \forall u, v \in V$$

1.2 | Homogenity

$$T(\lambda v) = \lambda(Tv) \forall \lambda \in \mathbb{F}, v \in V$$

2 | Other Notation

2.1 | Set of Maps

#definition Axler3.3 $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

3 | Examples

3.1 | zero (0)

Zero is a function $0:V\to W$ s.t. $0v=0 \forall v\in V$. (It takes all vectors in V and maps them to the additive identity of W)

3.2 | identity (*I*)

The identity maps each from one vector space to itself (in the same vector space):

$$I \in \mathcal{L}(V, V), v \in V : Iv = v$$

$3.3 \mid$ differentiation (D)

$$D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})) : Dp = p'$$

Basically stating that for two polynomials $a,b \in \mathcal{P}(\mathbb{R})$, a'+b'=(a+b)' and with a constant $\lambda \in \mathcal{R}$ $(\lambda a)'=\lambda a'$.

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3.4 | integration

3.5 | multiplication by x^2

$$T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})) : (Tp)(x) = x^2 p(x)$$

is a linear map

3.6 | backward shift

 F^{∞} is the vector space of all sequences of elements in \mathbb{F} .

$$T \in \mathcal{L}\left(\mathbb{F}^{\infty}, \mathbb{F}^{\infty}\right) : T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$$

3.7 |
$$\mathbb{F}^n o \mathbb{F}^m$$

Given a "coefficent matrix" $A:A_{j,k}\in\mathbb{F}\forall j=1,\ldots,m; \forall k=1,\ldots,n,$ define $T\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$:

$$T(x_1,\ldots,x_n)=(A_{1,1}x_1+A_{1,2}x_2+\cdots+A_{1,n}x_n,\ A_{2,1}x_1+\cdots+A_{2,n}x_n,\ \ldots,\ A_{m,1}x_1+\cdots+A_{m,n}x_n)$$

Notice that this is equivalent to taking A as a $m \times n$ matrix and dot producting it with the $n \times 1$ matrix $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

4 | Results

4.1 | Axler3.5 Linear maps and basis of domain

If v_1, \ldots, v_n is a basis of V and $w_1, \ldots, w_n \in W$, then there exists a unique linear map $T: V \to W$ s.t.

$$Tv_j = w_j \forall j \in 1, \dots, n$$

#aka given a basis v of V, there is a unique linear map that maps v to each $w \in W$.

4.1.1 |#careful

- 1. same dimension V and W are both of dimension n.
- 2. same field We defined V and W to both be vector spaces over the same field \mathbb{F} which is either \mathbb{R} or \mathbb{C} .
- 3. v is a basis v_1, \ldots, v_n must be a basis of V (because that fact is used in the proof)

4.1.2 | Questions

1. **DONE** #question what does it mean that "T is uniquely determined on $\mathrm{span}(v_1,\ldots,v_n)$? question There's no ambiguity and so we know exactly which map it's referring to, and thus it is uniquely determined.

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