1 | **Del Operator**

We know from before that the divergence of $\vec{F}$ in normal Cartesian coordinates is equal to:

$$div\ \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z} \tag{1}$$

We will define an operator ∇ with the following property:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{2}$$

Which, given its shape, we can actually express $div\ \vec{F}$ in a much simpler way:

$$\nabla \cdot \vec{F} \tag{3}$$

which would result in the same expression for divergence as defined above.

2 | **Divergence Theorem**

We will now begin exploring a proof for the Divergence Theorem: that the sum of divergence over a volume is equal to the surface integral of that surface.

We first clam that, per intuition, that the surface integral of an object is the sum of the surface integrals of all of its surfaces:

$$\iint_S \vec{F} \cdot \hat{n}\ dS = \sum_{l=1}^{N} \iint_{S_i} \vec{F} \cdot \hat{n}\ dS \tag{4}$$

Any adjacent subfaces that are not part of the outer face S_1, S_2 will have $\hat{n_1} = -\hat{n_2}$, whose sums of surface will cancel each out. Therefore, the above expression holds.

We will now expand the above expression by multiplying and diving by the differential volume at each step:

$$\sum_{l=1}^{N} \iint_{S_i} \vec{F} \cdot \hat{n}\ dS = \sum_{l=1}^{N} \left[\frac{1}{\Delta V_l} \iint_{S_i} \vec{F} \cdot \hat{n}\ dS\right] \Delta V_l \tag{5}$$

We furthermore show that, for the expression in the bracket, we literally have the formal definition of divergence at some surface i. Therefore:

$$\sum_{l=1}^{N} \left[\frac{1}{\Delta V_l} \iint_{S_i} \vec{F} \cdot \hat{n}\ dS\right] \Delta V_l = \sum_{l=1}^{N} \left[\nabla \cdot \vec{F}\right]_i \Delta V_l \tag{6}$$

And lastly, as $N \to \infty$ and $\Delta V_l \to 0$ (as we are working at a point)—per the definition of a surface integral at a point:

$$\lim_{N \to \infty, \Delta V_i \to 0} \sum_{l=1}^{N} \left[\nabla \cdot \vec{F}\right]_i \Delta V_l = \iiint_V \nabla \cdot \vec{F}\ dV \tag{7}$$

We can see that, essentially, the sum becomes a normal volume integral! Hence, it turns out that its much easier to calculate the surface integral by calculating its divergence, then summing; that:

$$\iint_S \vec{F} \cdot \hat{n} \; dS = \iiint_V \nabla \cdot \vec{F} \; dV \tag{8}$$