1 | Implicit Differentiation unit1::derivatives

1.1 | David's Summary

This is how I understand implicit differentiation.
Say you want to take a derivative of an implicit function like $x^2 + y^2 = 3$.

1. Take the derivative of everything with respect to x: $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}3$ With a little simplification this is just $2x + \frac{d}{dx}y^2 = 0$.
2. Cleverly apply the chain rule to get $\frac{d}{dx}y^2$. Chain rule states that $\frac{d}{du}\frac{du}{dx} = \frac{d}{dx}$. Define $u = y^2$. By chain rule $\frac{du}{dy}\frac{dy}{dx}$ = $\frac{du}{dx}$.
3. Our formula is now $2x + (2y)\frac{dy}{dx} = 0$. Time for some algebra! $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

1.2 | Initial Example

Technique based on the Chain Rule that allows diffrentiation of more functions.
EXAMPLE $\frac{d}{dx}x^a = ax^{a-1}$ This holds true for $a = 0, \pm 1, \pm 2\ldots$ What about fractional powers? Take $a = \frac{m}{n}$
$y = x^{\frac{m}{n}}$ or $y^n = x^m$ We can apply derivative to equation 2 because the methods of diffrentiating the fractional exponent is unknown to us.
$\frac{d}{dx}y^n = \frac{d}{dx}x^m$
$\left(\frac{d}{dy}y^n\right)\frac{dy}{dx} = \frac{d}{dx}x^m$ or... $ny^{n-1}\frac{dy}{dx} = mx^{m-1}$
\$dy$_{\overline{dx=}}$

1.3 | Another Example

$x^2 + y^2 = 1$ is an implicit function, explicitly it is $y = \pm\sqrt{1-x}$ (for convienience limit to positives for now).
Solving it explicitly: $y' = \frac{1}{2}()^{-1/2}(-2x)$ NOTE: $\frac{1}{2}()^{-1/2}$ = $\frac{d}{d()}()^{-1/2}$
Or implicitly:

- Diffrentiate function in the form $x^2 + y^2 = 1$.
- $\frac{d}{dx}(x^2 + y^2 = 1)$ or $2x + 2yy' = 0$
- Solve for y' which is $\frac{-2x}{2y} = \frac{-x}{y}$ (solve algebraically).

Compare $\frac{-x}{y}$ to explicit solution $\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$ and find they are the same as $y = \sqrt{1-x^2}$.

1.4 | A Trickier Example

$y^4 + xy^2 - 2 = 0$

- One can solve it explicitly by using the quadratic equation.
- Implicitly one can apply the product rule and the previous examples to diffrentiate this function.
- Writeup is left as an exercise for the reader.

1.5 | Derivatives of Inverse Functions

EXAMPLE $y = \sqrt{x}, x > 0, y^2 = x$ $f(x) = \sqrt{x}, g(y) = x, g(y) = y^2$ NOTE: If $y = f(x)$ and $g(y) = x$, $g(f(x)) = x$

STATEMENT Implicit differentiation allows computing derivatives of any inverse function provided we know the derivative of the function.

EXAMPLE $y = \tan^{-1} x$ and we'll use the equation $\tan y = x$

- Note that inverse functions are the function reflected over the line $x = y$.
- Recall that derivative of tangent is $\frac{d}{dy}\frac{\sin y}{\cos y} = \frac{1}{\cos^2 y} = \sec^2 y$.
- $\frac{d}{dy} \tan y = 1$ or $\sec^2 y * y' = 1$.
- $y' = \cos^2 y$ which leads to $\frac{d}{dx} \tan^{-1} x = \cos^2(\tan^{-1} x)$.
  - Too complicated!
- Modelling as a right triangle and simplifying more yields $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.

2 | Links

Other techniques for differentiation as well as the topic of logarithms are covered in Exponentials and Logarithms. Further review can be found in MIT SVC Exam Review (Unit 1).