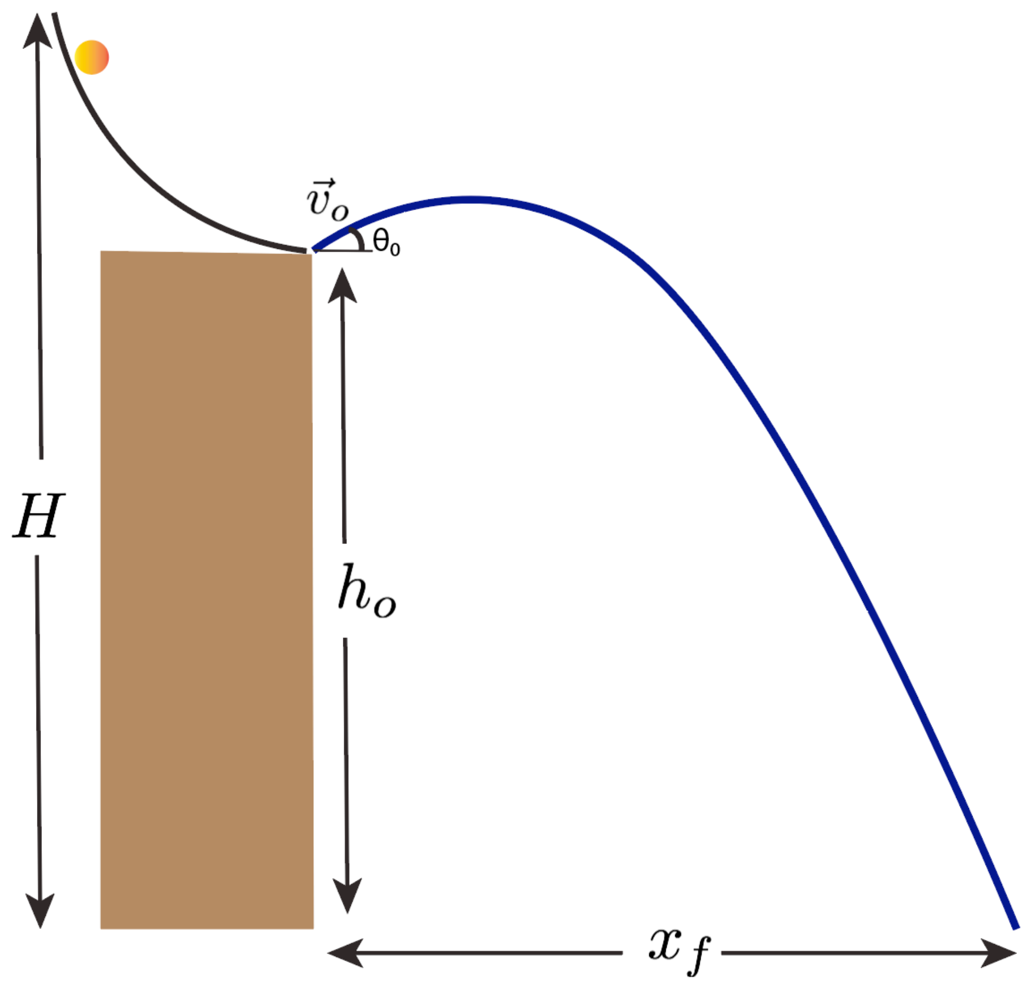

1 | **Setup**

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f.

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x-value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive-x, so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y, and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0, the moment the marble leaves the track and travels as a projectile as t_1, and the end — in the moment when the marble hits the ground — as t_f. We will call the marble m_0.

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0 g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0 g(H - h_0) = m_0 g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0 g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named $\vec{v_0}$.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 \tag{1}$$

$$g(H - h_0) = \frac{1}{2} \vec{v_0}^2 \tag{2}$$

$$2g(H - h_0) = \vec{v_0}^2 \tag{3}$$

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{4}$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} \vec{v_{0x}} = \sqrt{2g(H - h_0)} cos(\theta_0) \\ \vec{v_{0y}} = \sqrt{2g(H - h_0)} sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. $\vec{v_{0y}}$, $\vec{v_{0x}}$, h_0, m_0.

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f.

$$\begin{cases} x(t) = \frac{1}{2} a_{0x} t^2 + v_{0x} t + x_0 \\ y(t) = \frac{1}{2} a_{0y} t^2 + v_{0y} t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x} t \\ y(t) = \frac{-1}{2} g t^2 + v_{0y} t + h_0 \end{cases}$$

as…

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Solving for $\frac{dx_f}{d\theta_0}$

We need to maximize $\frac{dx_f}{d\theta_0}$ as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for $\frac{h_0}{d\theta_0}$. The position equations above could be leveraged to figure a value for x_f.

3.2.1 | Setup for Solution

We first create a set of equations modeling the location of the marble at t_f.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{-1}{2}g{t_f}^2 + t_f\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f, and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)} \tag{5}$$

Finally, we substitute the definition of t_f into $y(t_f)$.

$$y(t_f) = 0 = \frac{-1}{2}g\frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}^2 + \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \tag{6}$$

We will now proceed to simplify the expression further

$$0 = \frac{-1}{4}\frac{-{x_f}^2}{(H-h_0)cos^2(\theta_0)} + x_f tan(\theta_0) + h_0 \tag{7}$$

$$= \frac{-1}{4}\frac{-{x_f}^2}{(H-h_0)}cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{8}$$

$$= \frac{-1}{4}\frac{-1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{9}$$

3.2.2 | Finding $\frac{dx_f}{d\theta_0}$

We leverage implicit differentiation to figure a value for $\frac{dx_f}{d\theta_0}$. We set x_f as a differentiable function, and h_0 and H as both constants.

$$
\begin{align}
0 =& \frac{-1}{4}\frac{-1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{10}\\
\Rightarrow \frac{d}{d\theta_0}0 =& \frac{d}{d\theta_0}(\frac{-1}{4}\frac{-1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0) \tag{11}\\
\Rightarrow 0 =& \frac{-1}{4}\frac{-1}{(H-h_0)}\frac{d}{d\theta_0}{x_f}^2 cos^{-2}(\theta_0) + \frac{d}{d\theta_0}x_f tan(\theta_0) + \frac{d}{d\theta_0}h_0 \tag{12}\\
\Rightarrow 0 =& \frac{-1}{4}\frac{-1}{(H-h_0)}((\frac{d}{d\theta_0}{x_f}^2)cos^{-2}(\theta_0) + {x_f}^2(\frac{d}{d\theta_0}cos^{-2}(\theta_0)))+ \tag{13}\\
& ((\frac{d}{d\theta_0}x_f)tan(\theta_0) + (\frac{d}{d\theta_0}tan(\theta_0))x_f) + 0 \tag{14}\\
\Rightarrow 0 =& \frac{-1}{4}\frac{-1}{(H-h_0)}((2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + {x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)))+ \tag{15}\\
& ((\frac{dtan(\theta_0)}{d\theta_0}x_f) + (\frac{d}{d\theta_0}tan(\theta_0))x_f) \tag{16}\\
& \tag{17}
\end{align}
$$