1 | **Probelem 1**

1.1 | **1-1**

Because all of the terms $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$ are all crossed with $\vec{A}$, all of the vectors are in the same plane which is perpendicular to $\vec{A}$. If we set $\vec{A} = (0, 0, A_z)$ to be on the z axis, then $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$ would all be in the xy plane.

1.2 | **1-2**

Because $\vec{B}_{\perp \vec{A}}$, $\vec{C}_{\perp \vec{A}}$, $(\vec{B} + \vec{C})_{\perp \vec{A}}$ all are perpendicular to $\vec{A}$, they all must be coplaner, in the plane that is perpendicular to $\vec{A}$, which happens to be in the same plane as $\vec{A} \times \vec{B}$, $\vec{A} \times \vec{C}$, and $\vec{A} \times (\vec{B} + \vec{C})$.

1.3 | **1-3**

Because of the definition of the cross product, we know that:

$$(\vec{A} \times \vec{B}) \perp \vec{B}$$

$$(\vec{A} \times \vec{C}) \perp \vec{C}$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp (\vec{B} + \vec{C})$$

In tandem with the information form part 1-1 (all of the terms are perpendicular to $\vec{A}$), we know that:
$(\vec{A} \times \vec{B}) \perp \vec{A}\vec{B}$ plane
$(\vec{A} \times \vec{C}) \perp \vec{A}\vec{C}$ plane
$(\vec{A} \times (\vec{B} + \vec{C})) \perp \vec{A}(\vec{B} + \vec{C})$ plane
Thus, if we want to show that:

$$(\vec{A} \times \vec{B}) \perp (\vec{B}_{\perp \vec{A}})$$

$$(\vec{A} \times \vec{C}) \perp (\vec{C}_{\perp \vec{A}})$$

$$(\vec{A} \times (\vec{B} + \vec{C})) \perp (\vec{B} + \vec{C})_{\perp \vec{A}}$$

Then you need to show that:
$(\vec{B}_{\perp \vec{A}}) \in \vec{A}\vec{B}$ plane
$(\vec{C}_{\perp \vec{A}}) \in \vec{A}\vec{C}$ plane
$(\vec{B} + \vec{C})_{\perp \vec{A}} \in \vec{A}(\vec{B} + \vec{C})$ plane
To do this we can use the linear algebra definition of a plane. This definition states that a plane is defined as the locus of points that can be described by the linear combinatition of two vectors. In this case the two vectors are:

$\vec{A}$ and $\vec{B}$
$\vec{A}$ and $\vec{C}$
$\vec{A}$ and $(\vec{B}+\vec{C})$
We have defined:

$$\vec{A}=(0,0,A_z)$$

And can define:

$$\vec{B}=(B_x,B_y,B_z)$$

$$\vec{C}=(C_x,C_y,C_z)$$

Thus:

$$(\vec{B}+\vec{C})=(B_x+C_x,B_y+C_y,B_z+C_z)$$

Because $\vec{B}_{\perp\vec{A}}$, $\vec{C}_{\perp\vec{A}}$, and $(\vec{B}+\vec{C})_{\perp\vec{A}}$ are the projections of $\vec{B}$, $\vec{C}$, and $(\vec{B}+\vec{C})$ onto xy plane, respectively, they are defined as:

$$\vec{B}_{\perp\vec{A}}=(B_x,B_y,0)=(B_x,B_y,B_z)+n(0,0,A_z)=\vec{B}+n\vec{A}$$

$$\vec{C}_{\perp\vec{A}}=(C_x,C_y,0)=(C_x,C_y,C_z)+m(0,0,A_z)=\vec{C}+m\vec{A}$$

$(\vec{B}+\vec{C})_{\perp\vec{A}}=(B_x+C_x,B_y+C_y,0)$
$=(B_x+C_x,B_y+C_y,B_z+C_z)+p(0,0,A_z)=(\vec{B}+\vec{C})+p\vec{A}$
Thus, by the linear algebra definition of a plane:
$(\vec{B}_{\perp\vec{A}})\in\vec{A}\vec{B}$ plane
$(\vec{C}_{\perp\vec{A}})\in\vec{A}\vec{C}$ plane
$(\vec{B}+\vec{C})_{\perp\vec{A}}\in\vec{A}(\vec{B}+\vec{C})$ plane
Therefore:

$$(\vec{A}\times\vec{B})\perp(\vec{B}_{\perp\vec{A}})$$

$$(\vec{A}\times\vec{C})\perp(\vec{C}_{\perp\vec{A}})$$

$$(\vec{A}\times(\vec{B}+\vec{C}))\perp(\vec{B}+\vec{C})_{\perp\vec{A}}$$

1.4 | 1-4

$(\vec{A}\times\vec{B})$ points in the direction of $\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})$
$(\vec{A}\times\vec{C})$ points in the direction of $\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}})$
$\vec{A}\times(\vec{B}+\vec{C})$ points in the direction of $\vec{R}_{90^\circ}(\vec{B}+\vec{C})_{\perp\vec{A}}$

1.5 | **1-5**

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin(\theta) \cdot \frac{\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})}{|\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})|}$$

The last "term" is there just to set the direction (from part 1-4).

The middle part, $|\vec{B}| \sin(\theta)$ is magnitude of component of of $\vec{B}$ that is perpendicular to $\vec{A}$.

We know that $\vec{B}_{\perp \vec{A}}$ is component of $\vec{B}$ that is perpendicular to $\vec{A}$ from the definition of projection and the facts that:

- $\vec{B}_{\perp \vec{A}}$ is in the xy plane
- $\vec{A}$ is perpendicular to the xy plane

Thus we know that:

$$|\vec{B}| \sin(\theta) = |\vec{B}_{\perp \vec{A}}|$$

Because $R_{90^\circ}()$ rotates the vector by 90 degrees:

$|\vec{B}_{\perp \vec{A}}| = |\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})|$

Therefore:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin(\theta) \cdot \frac{\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})}{|\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})|} = |\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp \vec{A}})$$

1.6 | **1-6**

As stated in problem 1-5:

$$\vec{B}_{\perp \vec{A}} = (B_x, B_y, 0)$$

$$\vec{C}_{\perp \vec{A}} = (C_x, C_y, 0)$$

$$(\vec{B} + \vec{C})_{\perp \vec{A}} = (B_x + C_x, B_y + C_y, 0)$$

This works by definition of projection, and the fact that the x values cannot influence the y or z values of a vector:

Therefore:

$$\vec{B}_{\perp \vec{A}} + \vec{C}_{\perp \vec{A}} = (\vec{B} + \vec{C})_{\perp \vec{A}}$$

1.7 | **1-7**

IMAGE

The rotation opperation is defined as:

$\vec{t} = (x, y, 0)$

$$\vec{R}_{90^\circ}(\vec{t}) = (-y, x, 0)$$

Where $\vec{t}$ is rotated 90 degrees counter clockwise in the xy plane.
Thus:

$$\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) = (-B_y, B_x, 0)$$

$$\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = (-C_y, C_x, 0)$$

$$\vec{R}_{90^\circ}((\vec{B} + C)_{\perp\vec{A}}) = (-(B_y + C_y), B_x + C_x, 0) = (-B_y - C_y, B_x + C_x, 0)$$

From this we see that:

$$\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + \vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = \vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

1.8 | **1-8**

If we multiply both sides of the equation with a scalar, $|\vec{A}|$, then we get:

$$|\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

From step 1-5 we know that:

$$\vec{A} \times \vec{B} = |\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}})$$

and by similar means we can show that:

$$\vec{A} \times \vec{C} = |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}})$$

$$\vec{A} \times (\vec{B} + \vec{C}) = |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$$

Thus we can substitue:
$|\vec{A}|\vec{R}_{90^\circ}(\vec{B}_{\perp\vec{A}}) + |\vec{A}|\vec{R}_{90^\circ}(\vec{C}_{\perp\vec{A}}) = |\vec{A}|\vec{R}_{90^\circ}((\vec{B} + \vec{C})_{\perp\vec{A}})$
$\Rightarrow \vec{A} \times \vec{B} + \vec{A} \times \vec{B} = \vec{A} \times (\vec{B} + \vec{C})$
Therefore the cross product is distributive over addition.

2 | **Problem 2**

To start, we can define:

$$\vec{A} = (A_x, A_y, A_z) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = (B_x, B_y, B_z) = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Next, using the fact that the cross product is distributive across addition (shown in problem 1):

$\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_Z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$
$= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k}$
$+ A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} + A_yB_z\hat{\jmath} \times \hat{k}$
$+ A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k}$

From the definition of a cross product, we know that the cross product between any two vectors that are parallel is zero. This is because the sin$(0) = 0$. Thus all of the terms that take the cross product between two of the same unit vectors can be replace with 0:

$= A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k}$
$+ A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_z\hat{\jmath} \times \hat{k}$
$+ A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath}$

Next, $\hat{\imath} \times \hat{\jmath}$ would yield a vector length one in the direction of a vector that is perpendicular to both $\hat{\imath}$ and $\hat{\jmath}$, which would be $\hat{k}$ (using the right hand rule to get the correct direction). Conversly, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$. We can use the right hand rule for all possible combinations of unit vector cross products (excluding of the same kind):

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

$$\hat{\imath} \times \hat{k} = -\hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}$$

$$\hat{\jmath} \times \hat{k} = \hat{\imath}$$

$$\hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{k} \times \hat{\jmath} = -\hat{\imath}$$

Therefore:

$= A_xB_y\hat{k} - A_xB_z\hat{\jmath}$
$- A_yB_x\hat{k} + A_yB_z\hat{\imath}$
$+ A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}$
$= A_xB_y\hat{k} - A_yB_x\hat{k}$
$+ A_yB_z\hat{\imath} - A_zB_y\hat{\imath}$
$- A_xB_z\hat{\jmath} + A_zB_x\hat{\jmath}$
$= (A_xB_y - A_yB_x)\hat{k} + (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath}$
$= (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$

3 | **Problem 3**

3.1 | **3-1**

First let's define the following:

$$\vec{P} = (P_x, P_y, P_z)$$

$$\vec{Q} = (Q_x, Q_y, Q_z)$$

$$\vec{R} = (R_x, R_y, R_z)$$

Using the three position vectors we can get two vectors that are in the plane:

$$\vec{V_1} = \vec{P} - \vec{Q}$$

$$\vec{V_2} = \vec{R} - \vec{Q}$$

$\vec{V_1}$ and $\vec{V_2}$ are both in the the same plane as $\vec{P}$, $\vec{Q}$, and $\vec{R}$. This is because, their tails meet at $\vec{Q}$, because $\vec{Q}$ is negative and the tips are at $\vec{P}$ and $\vec{R}$ respectively. To find a vector normal to this plane we can use the cross product:

$$\vec{n} = \vec{V_1} \times \vec{V_2} = (\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q})$$

Now we can use the vector definition of a plane:

$$\{\vec{r} : (\vec{r} - \vec{P_o}) \cdot \vec{n} = 0, \vec{P_o} \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

We can define:

$\vec{P_o} = \vec{P}$

though it could be any of the given vectors.

Thus, the vetor equation of the plane containing the three points is:

$$\{\vec{r} : (\vec{r} - \vec{P}) \cdot ((\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q})) = 0, \vec{P} \in \mathbb{R}^3, \vec{Q} \in \mathbb{R}^3, \vec{R} \in \mathbb{R}^3\}$$

3.2 | 3-2

First we have to find the cartisian equation of a plane given three points. We can use what we found in part 3-1 to know that:

$\vec{n} = \vec{V_1} \times \vec{V_2}$
$= (\vec{P} - \vec{Q}) \times (\vec{R} - \vec{Q})$
$= (P_x - Q_x, P_y - Q_y, P_z - Q_z) \times (R_x - Q_x, R_y - Q_y, R_z - Q_z)$
$= ((P_y - Q_y)(R_z - Q_z) - (P_z - Q_z)(R_y - Q_y),$
$(P_z - Q_z)(R_x - Q_x) - (P_x - Q_x)(R_z - Q_z),$
$(P_x - Q_x)(R_y - Q_y) - (P_y - Q_y)(R_x - Q_x))$
$= ((P_yR_z - Q_yR_z - Q_zP_y + Q_zQ_y) - (P_zR_y - Q_zR_y - Q_yP_z + Q_yQ_z),$
$(P_zR_x - Q_zR_x - Q_xP_z + Q_xQ_z) - (P_xR_z - Q_xR_z - Q_zP_x + Q_zQ_x),$
$(P_xR_y - Q_xR_y - Q_yP_x + Q_xQ_y) - (P_yR_x - Q_yR_x - Q_xP_y + Q_xQ_y))$
$= (P_yR_x - Q_yR_z - Q_zP_y - P_zR_y + Q_zR_y + Q_yP_z,$
$P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x,$
$P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$

Once we have the normal vetor we can use the vector definition of a plane to get the cartisian equation of the plane:

$$\{\vec{r} : (\vec{r} - \vec{P_o}) \cdot \vec{n} = 0, \vec{P_o} \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P_o} \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P_o} \cdot \vec{n}$

We can define:

$$\vec{r} = (x, y, z)$$

$$\vec{P_o} = \vec{P}$$

though it could be any of the given vectors.

Evalutating the left side of the equation:

$\vec{r} \cdot \vec{n} = (x, y, z) \cdot (P_yR_x - Q_zR_x - Q_xP_y - P_zR_y + Q_zR_y + Q_yP_z,$
$P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x, P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$
$= x(P_yR_x - Q_yR_z - Q_zP_y - P_zR_y + Q_zR_y + Q_yP_z)$
$+ y(P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x)$
$+ z(P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$

Evalutating the right side of the equation:

$\vec{P_o} \cdot \vec{n} = (P_x, P_y, P_z) \cdot (P_yR_x - Q_zR_x - Q_xP_y - P_zR_y + Q_zR_y + Q_yP_z,$
$P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x, P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$
$= P_x(P_yR_x - Q_yR_z - Q_zP_y - P_zR_y + Q_zR_y + Q_yP_z)$
$+ P_y(P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x)$
$+ P_z(P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$

Thus the cartisian equation for the plane is the following:

$x(P_yR_x - Q_yR_z - Q_zP_y - P_zR_y + Q_zR_y + Q_yP_z)$
$+ y(P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x)$
$+ z(P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$
$=$
$P_x(P_yR_x - Q_yR_z - Q_zP_y - P_zR_y + Q_zR_y + Q_yP_z)$
$+ P_y(P_zR_x - Q_zR_x - Q_xP_z - P_xR_z + Q_xR_z + Q_zP_x)$
$+ P_z(P_xR_y - Q_xR_y - Q_yP_x - P_yR_x + Q_yR_x + Q_xP_y)$

Now that we have the equation we just need to plug the numbers in:

$x(0 - 4 - 0 - 2 - 6 - 1)$
$+ y(0 - 0 - 0 - 8 + 0 + 6)$
$+ z(-4 - 0 - 2 - 0 + 0 + 0)$
$=$
$2(0 - 4 - 0 - 2 - 6 - 1)$
$+ 0(0 - 0 - 0 - 8 + 0 + 6)$
$- 1(-4 - 0 - 2 - 0 + 0 + 0)$
$\Rightarrow -13x - 2y - 6z = -26 + 6$
$\Rightarrow -13x - 2y - 6z = -20$

We can check this by substituting all 3 points into the equation:

The first point is $(2, 0, -1)$:

$-13(2) - 2(0) - 6(-1) \stackrel{?}{=} -20$
$- 26 - 0 + 6 \stackrel{?}{=} -20$
$- 20 = -20$

The second point is $(0, 1, 3)$:

$-13(0) - 2(1) - 6(3) \stackrel{?}{=} -20$
$0 - 2 - 18 \stackrel{?}{=} -20$
$-20 = -20$

The third point is $(0, -2, 4)$:

$-13(0) - 2(-2) - 6(4) \stackrel{?}{=} -20$
$0 + 4 + 24 \stackrel{?}{=} -20$
$-20 = -20$

Thus $-13x - 2y - 6z = -20$ is the cartisian equation of the plane that contains the three points $(2, 0, -1)$, $(0, 1, 3)$, and $(0, -2, 4)$