1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(T)$ is injective if and only if $(T) \cap (T) = \{0\}$

2 | Proof

2.1 | Condition Manipulation

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | Left Condition

The left-hand side "$T/(T)$ is injective" is equivalent to:

$$\begin{aligned} (T/U\,(v+U)=0) &\implies (v+U=0) && \text{(alternate definition of injective)} \\ Tv+U=T &\implies v+U=T && (T/U(v+U) \text{ is defined as } Tv+U) \\ Tv+(T)=T &\implies v+(T)=T && (U=T) \\ Tv\in T &\implies v\in T \\ T^2v=0 &\implies v\in T \end{aligned}$$

2.1.2 | Right Condition

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in T$ then $w \notin T$" and "if $w \in T$ then $w \notin T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned} (\exists v : Tv = w) &\implies (Tw \neq 0) && \text{(definitions of null space and range)} \\ v \notin T &\implies T^2v \neq 0 && (w \neq 0) \\ T^2v = 0 &\implies v \in T && \text{(contrapositive)} \end{aligned}$$

2.2 | Proof

The statements are equivalent. ■