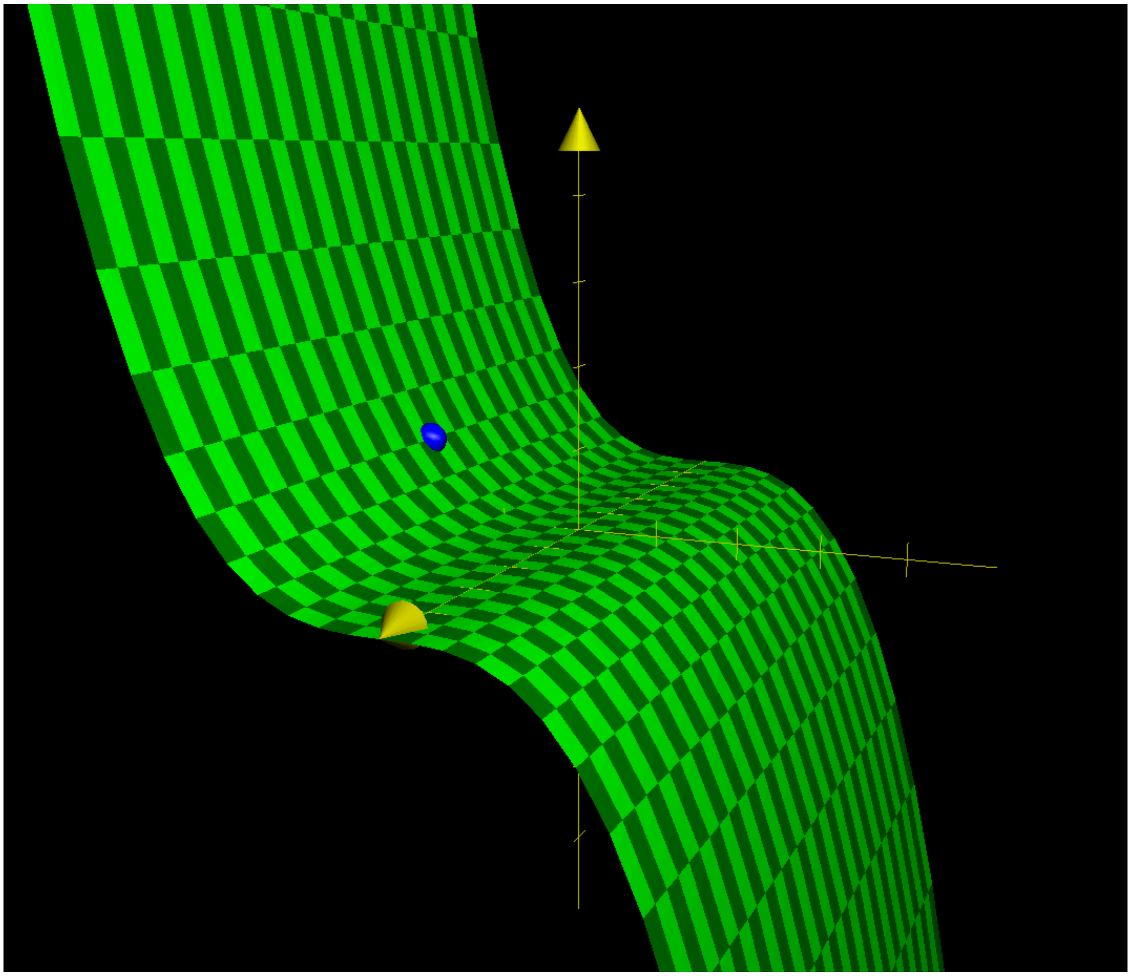
1 | Roofs Number 3

One possible design for one side of the roof would be a simple cubic function; that:

$$f(x, y) = \frac{1}{8}x^3\{0 \leq x \leq 4, -5 \leq y \leq 5\} \tag{1}$$



1.1 | Slope in Middle

The ”middle” of the roof, therefore, is the location $(2, 0)$, as indicated by the blue dot above.
Standing in the middle, and facing the ”ridge” ($+x$) direction, we could calculate the slope of the roof.
The vector facing the ridge of the roof, to the positive x direction, is represented by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2}$$

the gradient of this function is represented by:

$$\begin{bmatrix} \frac{3}{8}x^2 \\ 0 \end{bmatrix} \tag{3}$$

Therefore, at the center point as indicated, the gradient is:

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} \tag{4}$$

Computing the dot product of the direction and the gradient as found, we arrive that — at the center — the slope is:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix} = 6 \tag{5}$$

Therefore, the slope as indicated is $6 \approx 80.5°$.

1.2 | Facing the Peak

We first determine a vector that originates from the center of the roof, and facing towards one of the ridges; that:

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \tag{6}$$

Normalizing this vector, we arrive at:

$$\begin{bmatrix} \frac{2}{\sqrt{29}} \\ \frac{5}{\sqrt{29}} \end{bmatrix} \tag{7}$$

We will then project the gradient at the center point atop this vector:

$$\begin{bmatrix} \frac{2}{\sqrt{29}} \\ \frac{5}{\sqrt{29}} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{12}{\sqrt{29}} \tag{8}$$

Therefore, the slope as indicated is $\frac{12}{\sqrt{29}} \approx 65.83°$.

1.3 | Maximizing the Angle

To face in the steepest direction, we will need to face the direction of the gradient. As the gradient is

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} \tag{9}$$

as derived above, the direction of the gradient is therefore:

$$\begin{bmatrix} \frac{6}{6} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

2 | Spinning Coffee Cups

2.1 | **Deriving Velocity**

The process of acceleration is modeled by the following expression:

$$a(t) = -3cos(t)\hat{i} - 2sin(t)\hat{j} + 0\hat{k} \tag{11}$$

We begin by taking an indefinite integral in each component to result in the expression for velocity.

$$\int a(t)dt = \int -3cos(t)\hat{i} - 2sin(t)\hat{j} + 0\hat{k}\ dt \tag{12}$$

$$= (-3sin(t) + C_1)\hat{i} + (2cos(t) + C_2)\hat{j} + C_3\hat{k}\ dt \tag{13}$$

Though this expression, we supply the base state $(0, 2.1, 1)$ at $t = 0$ to solve for the constants.

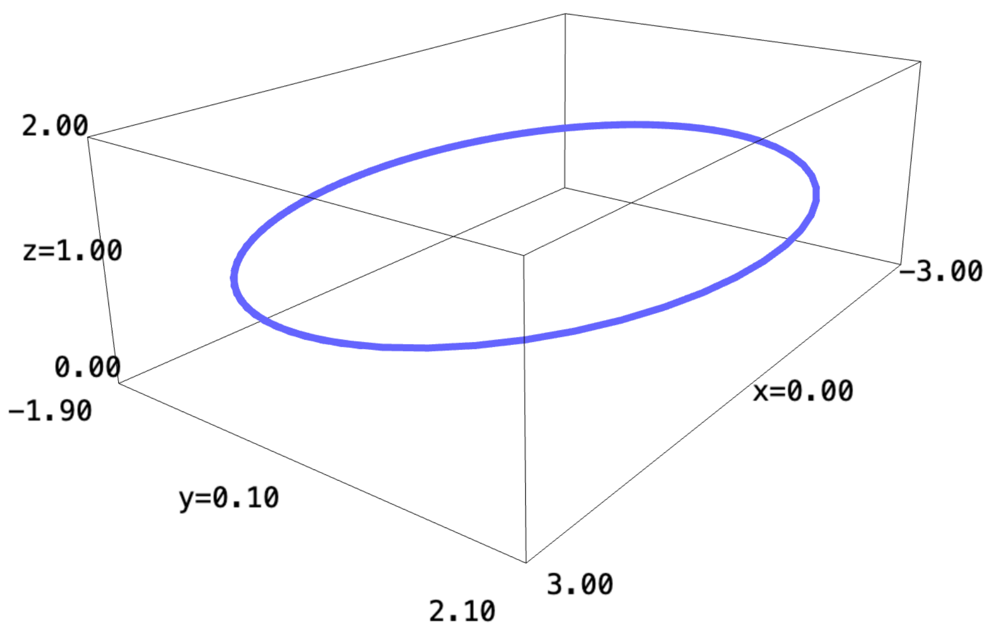
$$(0, 2.1, 1) = C_1\hat{i} + (2 + C_2)\hat{j} + C_3\hat{k} \tag{14}$$

Therefore, we could deduct that $C_1 = 0$, $C_2 = 2.1$, $C_3 = 1$. Hence:

$$v(t) = (-3sin(t))\hat{i} + (2cos(t) + 0.1)\hat{j} + 1\hat{k} \tag{15}$$

We could proceed to graph this expression:

```
f(t) = (-3*sin(t), 2*cos(t)+0.1, 1)
parametric_plot(f, (t, -5, 5), thickness=5)
```



2.2 | **Deriving Position**

The process of position is the indefinite integral of the velocity expression:

$$\int v(t)dt = -3sin(t)\hat{i} + (2cos(t) + 0.1)\hat{j} + \hat{k}\ dt \tag{16}$$
$$= (3cos(t) + C_1)\hat{i} + (2sin(t) + 0.1t + C_2)\hat{j} + (t + C_3)\hat{k} \tag{17}$$

Again, we supply the base state $(3, 0, 12)$ at $t = 0$ to solve for the constants.

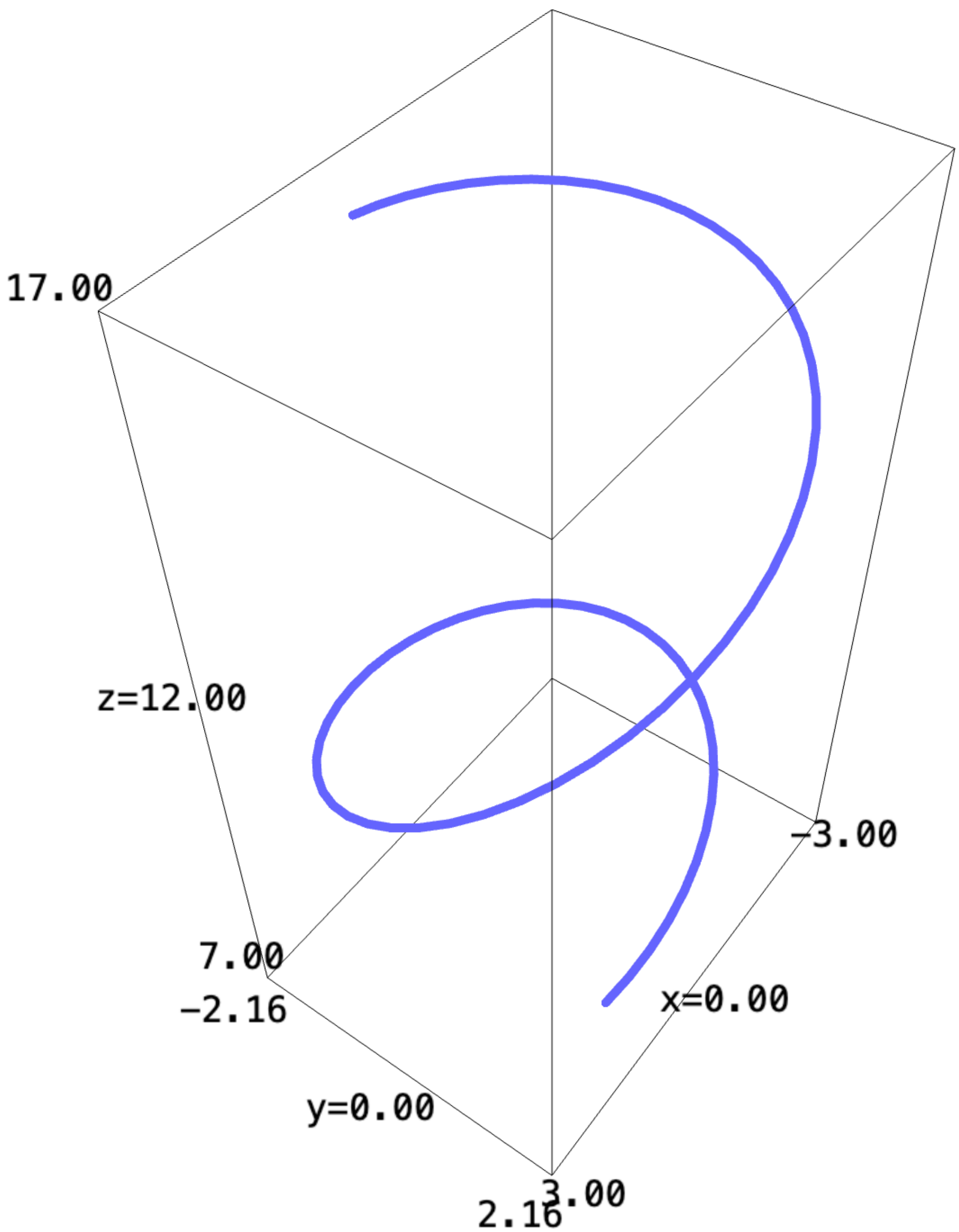
$$(3, 0, 12) = (3 + C_1)\hat{i} + C_2\hat{j} + C_3\hat{k} \tag{18}$$

Therefore, we deduct that $C_1 = 0$, $C_2 = 0$, $C_3 = 12$. Hence:

$$x(t) = 3cos(t)\hat{i} + (2sin(t) + 0.1t)\hat{j} + (t + 12)\hat{k} \tag{19}$$

We could proceed to graph this expression:

```
f(t) = (3*cos(t), (2*sin(t)+0.1*t), t+12)
parametric_plot(f, (t, -5, 5), thickness=5)
```



3 | **Optimizing a Function**

$$h(x, y) = (x^2 - 2y + 7)\hat{i} + (x^2 + y^2)\hat{j} \tag{20}$$

Optimizing a function $h : \mathbb{R}^2 \to \mathbb{R}^2$ requires maximizing-minimizing various dimensions of output. It is possible to optimize along a direction — thereby turning the problem onto one which involves the optimization of two functions $\mathbb{R}^2 \to \mathbb{R}^1$. Or, it is possible to optimize the magnitude.

3.1 | **Optimizing Along a Dimention**

For optimizing the output along $\hat{i}$, we create an expression $g : \mathbb{R}^2 \to \mathbb{R}^1$ such that:

$$g(x, y) = (x^2 - 2y + 7) \tag{21}$$

Taking all the first and second derivatives of this expression, therefore, would be:

$$g_x = 2x \tag{22}$$
$$g_y = -2 \tag{23}$$
$$g_x x = 2 \tag{24}$$
$$g_y y = 0 \tag{25}$$
$$g_x y = 0 \tag{26}$$

Evidently, there is no zeros in (g_x, g_y), meaning there are no minima or maxima in this dimension. Optimizing in the other dimention would follow the same procedure.

For optimizing along $\hat{j}$, we repeat the same procedure. We create an $f : \mathbb{R}^2 \to \mathbb{R}^1$ such that:

$$f(x, y) = x^2 + y^2 \tag{27}$$

Taking, again, all derivatives of this expression:

$$f_x = 2x \tag{28}$$
$$f_y = 2x \tag{29}$$
$$f_{xx} = 2 \tag{30}$$
$$f_{yy} = 2 \tag{31}$$
$$f_{xy} = 0 \tag{32}$$

We could see that, at $(0, 0)$, $\nabla g = 0$. Applying the second partial derivative test at this point, we deduct that:

$$h = 4 - 0 = 4 \tag{33}$$

As h is positive, and both f_{xx} and f_{yy} is positive, we deduct that the point $(0, 0)$ is a local minimum in the $\hat{j}$ dimension.

3.2 | **Optimizing for Magnitude**

The magnitude of the output space of this function, therefore, is:

$$|h(x, y)| = \sqrt{(x^2 - 2y + 7)^2 + (x^2 + y^2)^2} \tag{34}$$

Of course, we could see that, as the output of the square root function is always positive, optimizing this function would be functionally the same as optimizing for $(x^2 - 2y + 7)^2 + (x^2 + y^2)^2$.

We first expand this expression by expanding all squares of expressions:

$$2x^4 + 2x^2y^2 + y^4 - 4x^2y + 14x^2 + 4y^2 - 28y + 49 \tag{35}$$

We then proceed to optimize a function $g : \mathbb{R}^2 \to \mathbb{R}^1$ defined by this expression using the traditional second partial derivative test:

$$g(x, y) = 2x^4 + 2x^2y^2 + y^4 - 4x^2y + 14x^2 + 4y^2 - 28y + 49 \tag{36}$$

To begin with this task, we first take the first and second partial derivatives of this expression:

$$g_x = 8x^3 + 4xy^2 - 8xy + 28x \tag{37}$$
$$g_y = 4x^2y + 4y^3 - 4x^2 + 8y - 28 \tag{38}$$
$$g_{xx} = 24x^2 + 4y^2 - 8y + 28 \tag{39}$$
$$g_{yy} = 4x^2 + 12y^2 + 8 \tag{40}$$
$$g_{xy} = 8xy - 8x \tag{41}$$

To figure the critical points of this expression, we must solve for all values such that:

$$8x^3 + 4xy^2 - 8xy + 28x = 0 \tag{42}$$
$$4x^2y + 4y^3 - 4x^2 + 8y - 28 = 0 \tag{43}$$

Leveraging the first expression, we hold x constant and solve a quadratic expression in y:

$$y = \pm\sqrt{-2x^2 - 6} + 1 \tag{44}$$

Substituting this expression for y onto that below for x, we derive an expression in x for the zero.

$$\pm 4x^2(\sqrt{-2x^2 - 6} + 1) \pm 4(\sqrt{-2x^2 - 6} + 1)^3 - 4x^2 \pm 8(\sqrt{-2x^2 - 6} + 1) - 28 = 0 \tag{45}$$

At this point, we could solve for expressions for x for each combinations of $\pm$ to figure critical points, then apply the second derivative test to figure the optimized points of magnitude. However, this expression, per discussion, is computationally much complex to optimize.