1 | Exponentials and Logarithms

unit1::derivatives

1.1 | Exponential Functions

Goal: Calculate $\frac{d}{dx}a^x$

$$\lim_{\Delta x \to 0} \tfrac{a^{x+\Delta x}-a^x}{\Delta x}$$

As
$$a^{x+\Delta x}=a^xa^{\Delta x}$$
, $\lim_{\Delta x \to 0}a^x \frac{a^{\Delta x}-1}{\Delta x}$

 a^x is a constant so $a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

$$M(a) := \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

With new definition $\frac{d}{dx}a^x=M(a)a^x$. Plug in x=0 to get $\frac{d}{dx}a^0=M(a)$, showing M(a) is the slope at 0.

What is M(a)?

- Define base e as the unique number such that M(e) = 1
- If this is the case $\frac{d}{dx}e^x = e^x$ (as $\frac{d}{dx}a^x = M(a)a^x$).

Why does e exist?

• Take example f(x) = 2, f'(0) = M(2) and stretch by constant k.

$$f(kx) = 2^{kx} = (2^k)^x = b^x$$
, where $b = 2^k$.

• As k is increased the slope of the function gets steeper. $\frac{d}{dx}b^x=kf'(kx)$

– At 0,
$$\frac{d}{dx}b^x=kf'(kx)=kf'(0)=kM(2)$$
 so $b=e$ when $k=\frac{1}{M(2)}$

1.2 | The Natural Log

Recall that $\ln x_1 x_2 = \ln x_1 + \ln x_2$ and $\ln 1 = 0$ and $\ln e = 1$.

Differentiate $w = \ln x$ implicitly in the form $e^w = x$:

- $\frac{d}{dx}e^w = \frac{d}{dx}x = 1$
- $\frac{d}{dw}e^w\frac{dw}{dx}=1$ or $e^w\frac{dw}{dx}=1$
- Algebra yields $\frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$

1.3 | Back to The Exponential

 $\underline{\mathsf{Method}\,\mathbf{1}}\,\mathsf{Use}\;\mathsf{base}\;e=(e^{\ln a})^x=e^{x\ln a}.$

Just as the derivative of e^{3x} is $3e^{3x}$ by chain rule, $\frac{d}{dx}e^{x \ln a} = (\ln a)e^{x \ln a}$. So, $\frac{d}{dx}a^x = (\ln a)a^x$.

NOTE: No matter what our base (2 or 10 or something else) the derivative is still $(\ln a)a^x$ and that's one reason why it's the "natural" log as it comes up naturally.

Method 2

HEY I NEED REVISITING REWATCH THE LAST 15 MIN OF THIS

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Logarithmic differentiation. Chain rule + differentiation of logarithm. $(\ln u)' = \frac{u'}{u}$

EXAMPLE

$$v = x^x \, \ln v = x \ln x \, (\ln v)' = \ln x + x \tfrac{1}{x} \, \tfrac{v'}{v} = 1 + \ln x \, v' = v (1 + \ln x) \, \tfrac{d}{dx} x^x = x^x (1 + \ln x)$$

2 | Links

Further review can be found in MIT SVC Exam Review (Unit 1).

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