1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f.

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x-value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive-x, so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y, and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0, the moment the marble leaves the track and travels as a projectile as t_1, and the end — in the moment when the marble hits the ground — as t_f. We will call the marble m_0.

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0 g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0 g(H - h_0) = m_0 g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0 g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named $\vec{v_0}$.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 \tag{1}$$

$$g(H - h_0) = \frac{1}{2} \vec{v_0}^2 \tag{2}$$

$$2g(H - h_0) = \vec{v_0}^2 \tag{3}$$

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{4}$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} \vec{v_{0x}} = \sqrt{2g(H - h_0)} cos(\theta_0) \\ \vec{v_{0y}} = \sqrt{2g(H - h_0)} sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. $\vec{v_{0y}}$, $\vec{v_{0x}}$, h_0, m_0.

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f.

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as,

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Setup for Solution

We first create a set of equations modeling the location of the marble at t_f.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{-1}{2}g{t_f}^2 + t_f\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f, and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)} \tag{5}$$

Finally, we substitute the definition of t_f into $y(t_f)$.

$$y(t_f) = 0 = \frac{-1}{2}g\frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}^2 + \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \tag{6}$$

We will now proceed to simplify the expression further

$$\begin{aligned} 0 =& \frac{-1}{4}\frac{{x_f}^2}{(H-h_0)cos^2(\theta_0)} + x_f tan(\theta_0) + h_0 \quad (7) \\ =& \frac{-1}{4}\frac{{x_f}^2}{(H-h_0)}cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \quad (8) \\ =& \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \quad (9) \end{aligned}$$

3.3 | Solution at $\theta_0 = 0$

We begin by first solving for the expression for ${x_f}^2$ at $\theta_0 = 0$.

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{10}$$

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 + h_0 \tag{11}$$

$$-h_0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 \tag{12}$$

$$4h_0 = \frac{1}{(H-h_0)}{x_f}^2 \tag{13}$$

$${x_f}^2 = 4h_0(H-H_0) \tag{14}$$

3.3.1 | Finding $\frac{d{x_f}^2}{dh_0}$

Remember, given the setup of the problem, x_f is optimized when ${x_f}^2$ is optimized. This is due to the fact that x_f could not be negative, and it is representing a maximum travel distance. Hence, we will optimized ${x_f}^2$ for ease of calculation.

$${x_f}^2 = 4h_0(H-H_0) \tag{15}$$

$$\frac{d{x_f}^2}{dh_0} = \frac{d}{dh_0}4h_0(H-h_0) \tag{16}$$

$$= 4\frac{d}{dh_0}h_0(H-h_0) \tag{17}$$

$$= 4((H-h_0)-h_0) \tag{18}$$

$$= 4H - 4h_0 - 4h_0 \tag{19}$$

$$= 4H - 8h_0 \tag{20}$$

3.3.2 | Optimizing $\frac{d{x_f}^2}{dh_0}$

The optimization of this statement is fairly simple. We set $\frac{d{x_f}^2}{dh_0} = 0$, and solve for h_0.

$${x_f}^2 = 4H - 8h_0 \tag{21}$$

$$\Rightarrow 0 = 4H - 8h_0 \tag{22}$$

$$\Rightarrow 8h_0 = 4H \tag{23}$$

$$\Rightarrow h_0 = \frac{1}{2}H \tag{24}$$

Hence, the most optimal height at which to launch the marble launcher, given a horizontal launch, is half of the initial height.

3.4 | Solution at arbitrary θ_0

3.4.1 | Solving and optimizing for $\frac{dx_f}{d\theta_0}$

We need to maximize $\frac{dx_f}{d\theta_0}$ as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for $\frac{h_0}{d\theta_0}$. The position equations above could be leveraged to figure a value for x_f.

1. Finding $\frac{dx_f}{d\theta_0}$ We leverage implicit differentiation to figure a value for $\frac{dx_f}{d\theta_0}$. We set x_f as a differentiable function, and h_0 and H as both constants.

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{25}$$

$$\Rightarrow \frac{d}{d\theta_0}0 = \frac{d}{d\theta_0}(\frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0) \tag{26}$$

$$\Rightarrow 0 = \frac{-1}{4}\frac{1}{(H-h_0)}\frac{d}{d\theta_0}{x_f}^2 cos^{-2}(\theta_0) + \frac{d}{d\theta_0}x_f tan(\theta_0) + \frac{d}{d\theta_0}h_0 \tag{27}$$

$$\Rightarrow 0 = \frac{-1}{4}\frac{1}{(H-h_0)}((\frac{d}{d\theta_0}{x_f}^2)cos^{-2}(\theta_0) + {x_f}^2(\frac{d}{d\theta_0}cos^{-2}(\theta_0)))+ \tag{28}$$

$$((\frac{d}{d\theta_0}x_f)tan(\theta_0) + (\frac{d}{d\theta_0}tan(\theta_0))x_f) + 0 \tag{29}$$

$$\Rightarrow 0 = \frac{-1}{4(H-h_0)}((2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + {x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)))+ \tag{30}$$

$$(\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f) \tag{31}$$

$$\Rightarrow 0 = \frac{-1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{-1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0))+ \tag{32}$$

$$\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f \tag{33}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{34}$$

$$= \frac{-1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{35}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{36}$$

$$= \frac{dx_f}{d\theta_0}\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{37}$$

$$\Rightarrow \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{38}$$

$$= \frac{dx_f}{d\theta_0}(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0)) \tag{39}$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{40}$$

2. Optimizing for x_f for θ_0 via $\frac{dx_f}{d\theta_0}$ We now set $\frac{dx_f}{d\theta_0} = 0$ in order to figure critical points for the value of x_f.

$$
\begin{aligned}
f\frac{dx_f}{d\theta_0} &= \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} && (41)\\
\Rightarrow 0 &= \frac{\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{-1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} && (42)\\
\Rightarrow 0 &= \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f && (43)\\
\Rightarrow sec^2(\theta_0)x_f &= \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 && (44)\\
\Rightarrow sec^2(\theta_0) &= \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}x_f && (45)\\
\Rightarrow 2sec^2(\theta_0)(H-h_0) &= (cos^{-3}(\theta_0)sin(\theta_0))x_f && (46)\\
\Rightarrow \frac{2(H-h_0)}{x_f} &= \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{sec^2(\theta_0)} && (47)\\
\Rightarrow \frac{2(H-h_0)}{x_f} &= \frac{sin(\theta_0)}{cos^3(\theta_0)sec^2(\theta_0)} && (48)\\
\Rightarrow \frac{2(H-h_0)}{x_f} &= \frac{sin(\theta_0)}{cos(\theta_0)} && (49)\\
\Rightarrow \frac{2(H-h_0)}{x_f} &= tan(\theta_0) && (50)\\
\Rightarrow \theta_0 &= arctan(\frac{2(H-h_0)}{x_f}) && (51)
\end{aligned}
$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize x_f for a given solved x_f. To figure the actual statement that would optimize for both,

3.4.2 | Solving and optimizing for x_f

We will now return to our original expression for the final y-position ($= 0$) to create an expression for x_f.

1. Solving for x_f We first take the previous expression for x_f and supply the expression for θ_0.

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \quad (52)$$

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2 sec^2(arctan(\frac{2(H-h_0)}{x_f})) + x_f tan(arctan(\frac{2(H-h_0)}{x_f})) + h_0 \quad (53)$$

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}{x_f}^2((\frac{2(H-h_0)}{x_f})^2 + 1) + x_f(\frac{2(H-h_0)}{x_f}) + h_0 \quad (54)$$

$$0 = \frac{-1}{4}\frac{1}{(H-h_0)}((2(H-h_0))^2 + {x_f}^2) + 2(H-h_0) + h_0 \quad (55)$$

$$0 = \frac{-1}{4}((4(H-h_0)) + \frac{{x_f}^2}{(H-h_0)}) + 2(H-h_0) + h_0 \quad (56)$$

$$0 = (-(H-h_0) - \frac{{x_f}^2}{4(H-h_0)}) + 2(H-h_0) + h_0 \quad (57)$$

$$\frac{{x_f}^2}{4(H-h_0)} = -(H-h_0) + 2(H-h_0) + h_0 \quad (58)$$

$${x_f}^2 = -4(H-h_0)(H-h_0) + 4(H-h_0)2(H-h_0) + 4(H-h_0)h_0 \quad (59)$$

$${x_f}^2 = -4(H-h_0)^2 + 8(H-h_0)^2 + 4h_0(H-h_0) \quad (60)$$

$${x_f}^2 = 4(H-h_0)^2 + 4h_0(H-h_0) \quad (61)$$

$$\quad (62)$$

2. Finding $\frac{d{x_f}^2}{dh_0}$ We know that, by optimizing for ${x_f}^2$, x_f is optimized due to the setup of the problem of the behavior of the length of line.

   Hence, we take the *first* derivative, though of ${x_f}^2$ w.r.t. h_0 and with H held constant.

$${x_f}^2 = 4(H-h_0)^2 + 4h_0(H-h_0) \quad (63)$$

$$\Rightarrow \frac{d{x_f}^2}{dh_0} = \frac{d}{dh_0}(4(H-h_0)^2 + 4h_0(H-h_0)) \quad (64)$$

$$= 4\frac{d}{dh_0}(H-h_0)^2 + 4\frac{d}{dh_0}h_0(H-h_0) \quad (65)$$

$$= 4\frac{d}{dh_0}(H-h_0)^2 + 4((H-h_0)\frac{d}{dh_0}h_0 + h_0\frac{d}{dh_0}(H-h_0)) \quad (66)$$

$$= 4\frac{d}{dh_0}(H-h_0)^2 + 4((H-h_0) - h_0) \quad (67)$$

$$= -8(H-h_0) + 4(H-h_0) - 4h_0 \quad (68)$$

$$= -4(H-h_0) - 4h_0 \quad (69)$$

$$= -4H + 4h_0 - 4h_0 \quad (70)$$

$$= -4H \quad (71)$$

3. Optimizing for x_f The optimization of x_f requires a little bit more thinking of the scenario of the problem. H must be a positive number, and the expression for ${x_f}^2$ appears to be a straight line with slope $-4H$. Given $H > 0$, $-4H < 0$, and the slope of ${x_f}^2$ is negative — as h_0 increases, ${x_f}^2$ decreases. Given h_0 must be positive, then, $h_{0optim} = 0$.

3.4.3 | Solving for the Actual Optimum of θ_0

We return to following statement:

$$\theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{72}$$

The expression we derived above for x_f under optimal conditions is, per the previous statement:

$$x_f{}^2 = 4(H-h_0)^2 + 4h_0(H-h_0) \tag{73}$$

$$\Rightarrow x_f = \sqrt{4(H-h_0)^2 + 4h_0(H-h_0)} \tag{74}$$

And substituting the statement for x_f back into the expression above, in addition to the optimal value $h_0 = 0$ derived earlier, we derive that

$$\theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{75}$$

$$\Rightarrow \theta_0 = arctan(\frac{2(H-h_0)}{\sqrt{4(H-h_0)^2 + 4h_0(H-h_0)}}) \tag{76}$$

$$= arctan(\frac{2(H)}{\sqrt{4(H)^2}}) \tag{77}$$

$$= arctan(1) \tag{78}$$

$$= 45^\circ \tag{79}$$

And hence, given an arbitrary angle and an arbitrary height to launch, the most optimal scenario is to make the launch point fully on the floor at an 45° angle from the plane of the floor.

4 | **Pure Roll**

Distance by which the center of mass moves is the same distance that is traversed in the circumference. The conventional model of friction does not work to a rolling object.

5 | **Kinematics Considering Rotation**

We know, for a sphere, the rotational inertia is $I = \frac{2}{5}MR^2$. And hence, actually considering the fact that the marble rolls down the initial ramp, and some energy from the height change is actually contributed to the process of roll.

From before, we know that the process of the initial fall creates kinetic energy $m_0 g(H-h_0)$.

Instead of setting that equivalent to simply kinetic energy, we now revise it with the definition of rotational inertia included.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 + \frac{1}{2}(\frac{2}{5} m_0 R^2)\omega^2 \tag{80}$$
$$\Rightarrow 2g(H - h_0) = \vec{v_0}^2 + (\frac{2}{5} R^2)\omega^2 \tag{81}$$
$$\Rightarrow 2g(H - h_0) = \vec{v_0}^2 + (\frac{2}{5} R^2)(\frac{\vec{v_0}}{R})^2 \tag{82}$$
$$\Rightarrow 2g(H - h_0) = \vec{v_0}^2 + (\frac{2}{5} \vec{v_0}^2) \tag{83}$$
$$\Rightarrow 2g(H - h_0) = \frac{7}{5} \vec{v_0}^2 \tag{84}$$
$$\Rightarrow \vec{v_0}^2 = \frac{10}{7} g(H - h_0) \tag{85}$$
$$\Rightarrow \vec{v_0} = \sqrt{\frac{10}{7} g(H - h_0)} \tag{86}$$

The translational velocity at the point of launch, with consideration to rotational kinetic energy, is shown to be slower than that without consideration to rotation by ($(\sqrt{2} - \sqrt{\frac{10}{7}})(g(H - h_0))$).

This decrease would, also, therefore result a slightly shorter total launch distance — though the effect would be reasonably minimal (by a value of $(\sqrt{2} - \sqrt{\frac{10}{7}}) \approx 0.21898$.

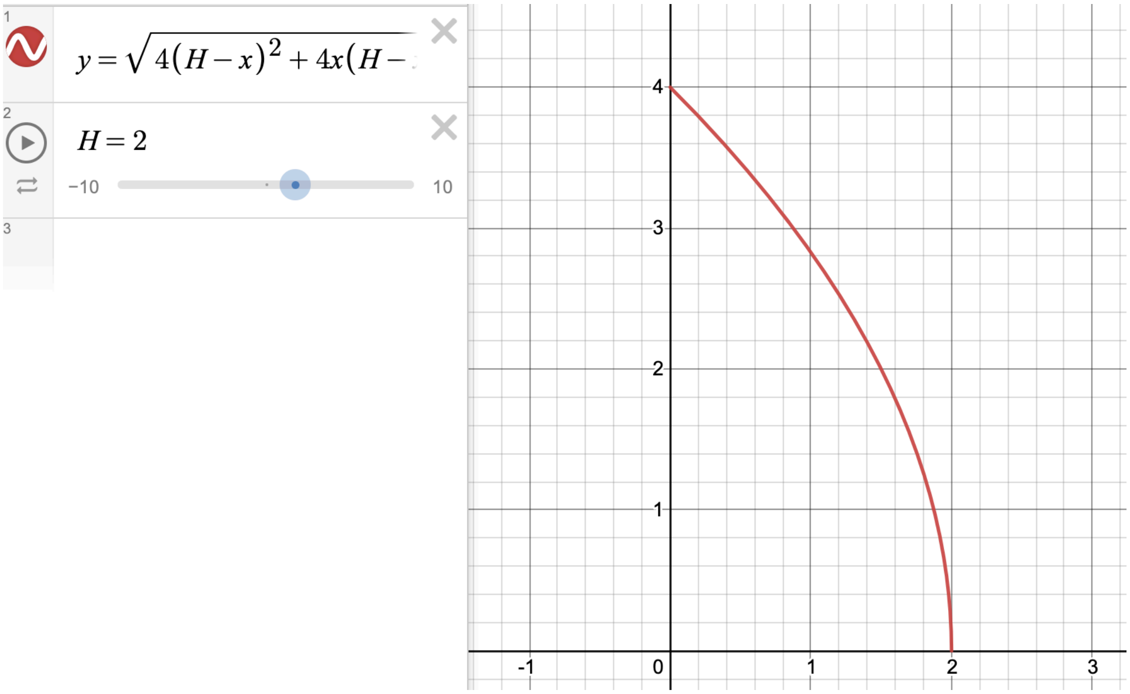
Given the value of error in $\vec{v_0}$, we could then isolate impact of the error in h_0 by simply solving for it in the expression for $\vec{v_0}$ as all other variables in the system is treated constantly. We use the ”original” expression for $\vec{v_0}$ here as the optimal h_0 is deducted using that value.

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{87}$$
$$\Rightarrow 0.219 = \sqrt{2g(H - h_0)} \tag{88}$$
$$\Rightarrow 0.048 = 2g(H - h_0) \tag{89}$$
$$\Rightarrow 2.446 \times 10^{-3} = (H - h_0) \tag{90}$$
$$\Rightarrow -2.446 \times 10^{-3} = h_0 \tag{91}$$

Hence, we note that — as we consider the rotational inertia — the value that results for h_0 would require a correction to be -2.446 centimeters shorter.

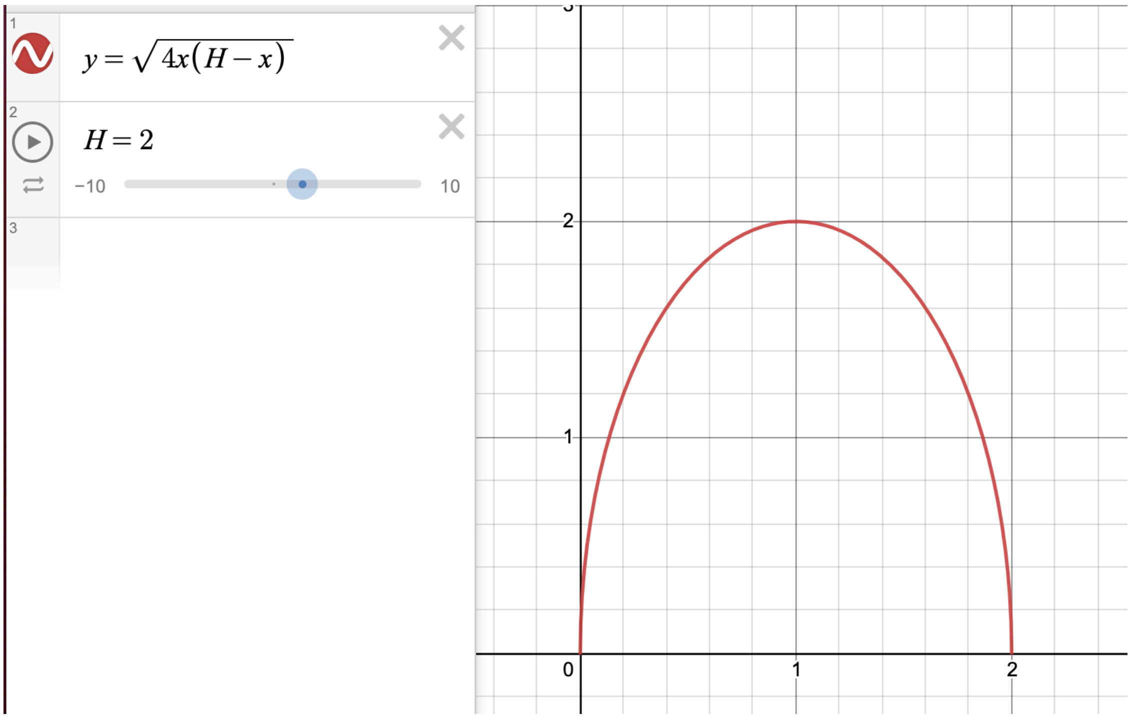
6 | Additional Figures

6.1 | **Projected Range w.r.t. varying h_0 at optimal θ_0**



This figure is rendered in an Interactive Desmos Plot, the initial launch height, H, is fixed at an arbitrary non-zero value of 2.

6.2 | **Projected Range w.r.t. varying h_0 at $\theta_0 = 0$**



This figure is rendered in another Interactive Desmos Plot, the initial launch height, H, is again fixed at an arbitrary non-zero value of 2.

As one could see, a parabolic relationship could be seen such that the most optimal value is at exactly $\frac{1}{2}H$.

6.3 | Optimal θ changes w.r.t h

We quickly return to the following expression:

$$\theta_0 = arctan(\frac{2(H-h_0)}{x_f}) \tag{92}$$

We write an expression of x_f, that:

$$tan(\theta_0) = \frac{2(H-h_0)}{x_f} \tag{93}$$

$$\Rightarrow x_f = \frac{2(H-h_0)}{tan(\theta_0)} \tag{94}$$

We then optimize x_f w.r.t. θ_0.

7 | Error Analysis

We have derived the fact that:

$$x_f = \sqrt{\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}h^2} \tag{95}$$

To deduct an expression for the potential sources of error, we take the partial derivatives of x_f w.r.t. H and h_0.

$$\frac{\partial x_f}{\partial H} = \frac{-(\frac{200}{49}H - \frac{60}{49}h_0)}{2\sqrt{\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2}} \tag{96}$$

$$\frac{\partial x_f}{\partial h_0} = \frac{-(-\frac{60}{49}H - \frac{80}{49}h_0)}{2\sqrt{\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2}} \tag{97}$$

Therefore, we could deduct that…

$$\begin{aligned}
\Delta x_f =& \sqrt{(\frac{\partial x_f}{\partial H}\Delta H)^2 + (\frac{\partial x_f}{\partial h_0}\Delta h_0)^2} && (98)\\
=& \sqrt{(\frac{-(\frac{200}{49}H - \frac{60}{49}h_0)}{2\sqrt{\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2}}\Delta H)^2 + (\frac{-(-\frac{60}{49}H - \frac{80}{49}h_0)}{2\sqrt{\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2}}\Delta h_0)^2} && (99)\\
=& \sqrt{\frac{(\frac{200}{49}H - \frac{60}{49}h_0)^2}{4(\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2)}(\Delta H)^2 + \frac{(\frac{60}{49}H + \frac{80}{49}h_0)^2}{4(\frac{100}{49}H^2 - \frac{60}{49}Hh_0 - \frac{40}{49}{h_0}^2)}(\Delta h_0)^2} && (100)
\end{aligned}$$