1 | Jacobian Determinant for Polar

We are to determine (pun not intended) the polar correction factor for a double integral, $dA = r\ dr\ d\theta$.
To do this, we will have to first figure the change of bases expressions such that we can take:

$$f(x, y) = g(r, \theta) \tag{1}$$

Fortunately, this is already derived to use from before.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

Therefore, we have that:

$$f(x, y) = f(r\cos\theta, r\sin\theta) \tag{3}$$

And therefore, we can figure $J_{r,\theta}$:

$$J = \begin{bmatrix} cos\theta & -r\sin\theta \\ sin\theta & r\cos\theta \end{bmatrix} \tag{4}$$

Taking its determinant, then:

$$det(J) = r\cos^2\theta + r\sin^2\theta = r \tag{5}$$

And therefore, the change-of-basis result would be:

$$dx\ dy = r\ dr\ d\theta \tag{6}$$

2 | Jacobian Determinant for Spherical

We again need to figure a correction factor for $dx\ dy\ dz = \rho^2\ \sin\phi\ d\rho\ d\theta\ d\phi$.
We therefore have to figure a change of bases for the expression:

$$f(x, y, z) = g(\rho, \theta, \phi) \tag{7}$$

We can leverage the shape of the object to determine the parameterization:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \tag{8}$$

We will now figure the matrix for $J_{\rho,\theta,\phi}$:

$$J = \begin{bmatrix} sin\phi\cos\theta & -\rho\ sin\phi\sin\theta \\ sin\phi\sin\theta & \rho\ sin\phi\cos\theta \\ cos\phi & \end{bmatrix} \tag{9}$$