1 | **Triple Osward's Box**

To solve this problem, we will need to take a triple integral along each of the dimensions to add up the energy inside the box.

We know that the box is modeled by the function:

$$e(x, y, z) = x^2y + 11z + 13 \tag{1}$$

We further understand that one corner of the box is located at the origin and the other, at $(3, 7, 4)$.

We will now take the triple integral, one along each dimension:

$$\int_0^4 \int_0^7 \int_0^3 x^2y + 11z + 13\ dx\ dy\ dz \tag{2}$$

We will take parts of this integral at a time.

$$\int_0^3 x^2y + 11z + 13\ dx \tag{3}$$

$$= \frac{x^3y}{3} + 11zx + 13x\bigg|_0^3 \tag{4}$$

$$=9y + 33z + 39 \tag{5}$$

And now, we do this again for the second integral.

$$\int_0^7 9y + 33z + 39dy \tag{6}$$

$$= \frac{9y^2}{2} + 33zy + 39y\bigg|_0^7 \tag{7}$$

$$=\frac{441}{2} + 231z + 273 \tag{8}$$

$$=493.5 + 231z \tag{9}$$

And finally, we take the third integral:

$$\int_0^4 493.5 + 231z\ dz \tag{10}$$

$$= 493.5z + \frac{231z^2}{2}\bigg|_0^4 \tag{11}$$

$$=1974 + 1848 \tag{12}$$

$$=3822 \tag{13}$$

We can see there is 3822 total units of energy in the box.

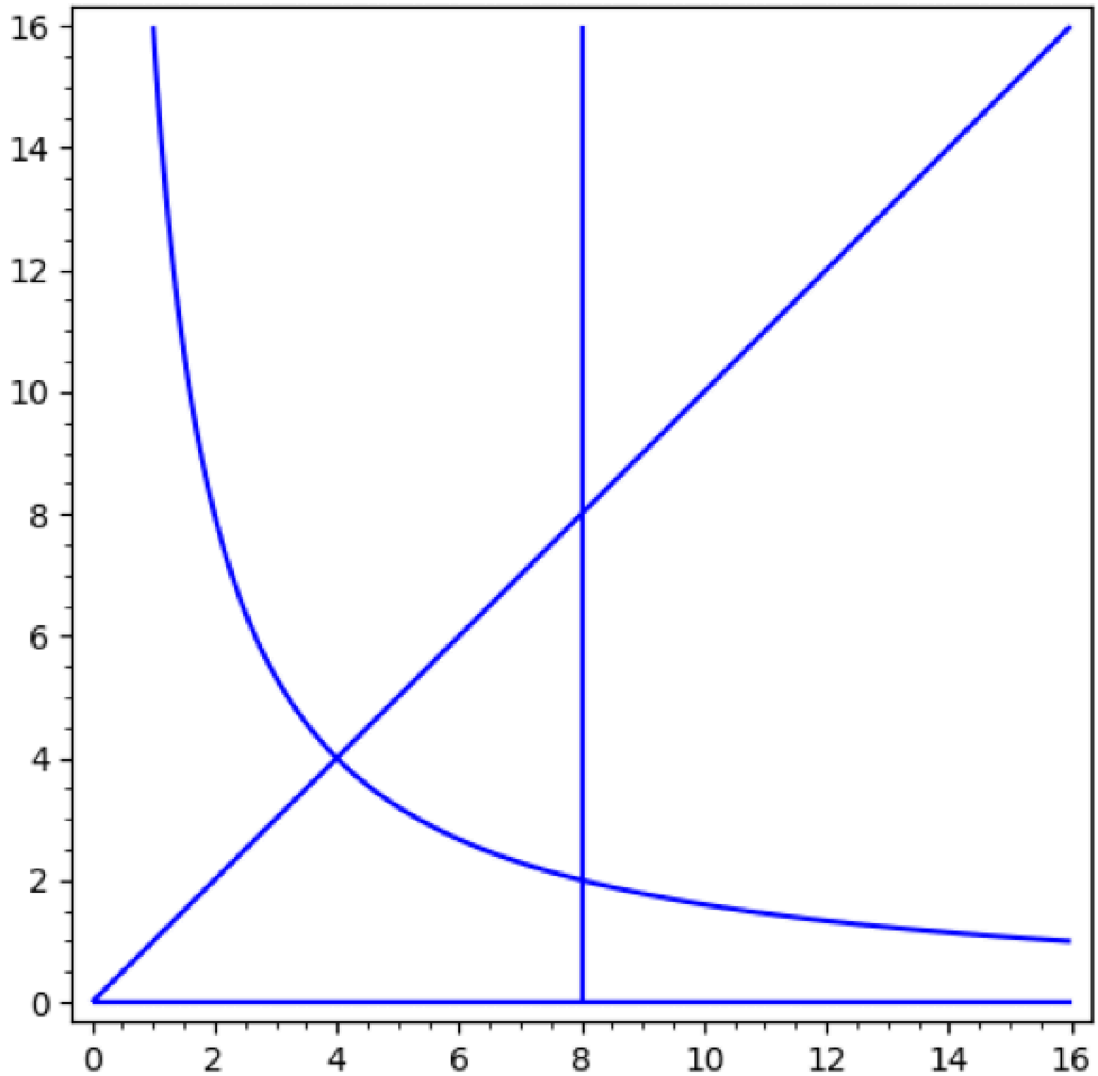
2 | **Custom-Bound Integration**

We are to find to volume beneath $z = x^2$ via the bounds of $xy = 16$, $y = x$, $y = 0$, and $x = 8$.

From the bound functions of y, it is evident that the parameter y is bound by $[0, x]$. For the bounds of x, we can figure that its bounded in the right by $x = 8$, and the right by $x = \frac{16}{y}$.

Plotting the bounds together:

```
x,y = var("x y")
eqns = [y==0, y==x,x==8, x*y==16]
sum(implicit_plot(i, (x,0,16), (y,0,16)) for i in eqns)
```



As per (not actually given) by the problem, we wish to find the "lower-left" region bound.

2.1 | y-dimension first

Let's perform the integral along the y dimension first, reducing it to a function in x.

We see that the top bound is by two piece-wise functions, each in one dimension of x. We will perform the integral between bounds $[0, 4]$.

$$\int_0^x x^2 dy \tag{14}$$
$$\Rightarrow x^3 \tag{15}$$

And, integrating between $[0, 4]$, we have that:

$$\int_0^4 x^3 dx \tag{16}$$
$$\Rightarrow 64 \tag{17}$$

We find the integral now between $[4, 8]$, with the bound:

$$\int_0^{16/x} x^2 dy \tag{18}$$
$$\Rightarrow \frac{16x^2}{x} \tag{19}$$
$$\Rightarrow 16x \tag{20}$$

And then, integrating along $[4, 8]$:

$$\int_4^8 16x dx \tag{21}$$
$$\Rightarrow \left. \frac{16x^2}{2} \right|_4^8 \tag{22}$$
$$\Rightarrow 384 \tag{23}$$

The total area under the curve, then, is 448.

2.2 | x-dimension first

We can do this again, but rotated. We can perform the integral along the x dimension, reducing it into a function in y.

We see that the right is bound again by two piece-wise functions, each in one dimension of y. We will perform the integral between the bounds $[0, 2]$.

$$\int_y^8 x^2 dx \tag{24}$$
$$\Rightarrow \left. \frac{x^3}{3} \right|_y^8 \tag{25}$$
$$\Rightarrow \frac{512}{3} - \frac{y^3}{3} \tag{26}$$

And then, we have to integrate between the bounds $[0, 2]$.

$$
\begin{aligned}
&\int_0^2 \left(\frac{512}{3} - \frac{y^3}{3}\right)\ dy && (27)\\
\Rightarrow\ &\frac{512y}{3} - \frac{y^4}{12}\bigg|_0^2\ dy && (28)\\
\Rightarrow\ &\frac{1024}{3} - \frac{16}{12} && (29)\\
\Rightarrow\ &\frac{4080}{12} = 340 && (30)
\end{aligned}
$$

We can perform the integration again $[2, 4]$.

$$
\begin{aligned}
&\int_y^{16/y} x^2 dx && (31)\\
\Rightarrow\ &\frac{x^3}{3}\bigg|_y^{16/y} && (32)\\
\Rightarrow\ &\frac{4096}{3y^3} - \frac{y^3}{3} && (33)\\
\Rightarrow\ &\frac{4096y^{-3}}{3} - \frac{y^3}{3} && (34)
\end{aligned}
$$

And then, we will integrate by the bounds $[2, 4]$:

$$
\begin{aligned}
&\int_2^4 \frac{4096y^{-3}}{3} - \frac{y^3}{3}\ dy && (35)\\
\Rightarrow\ &\frac{4096y^{-2}}{-6} - \frac{y^4}{4}\bigg|_2^4 && (36)\\
\Rightarrow &108 && (37)
\end{aligned}
$$

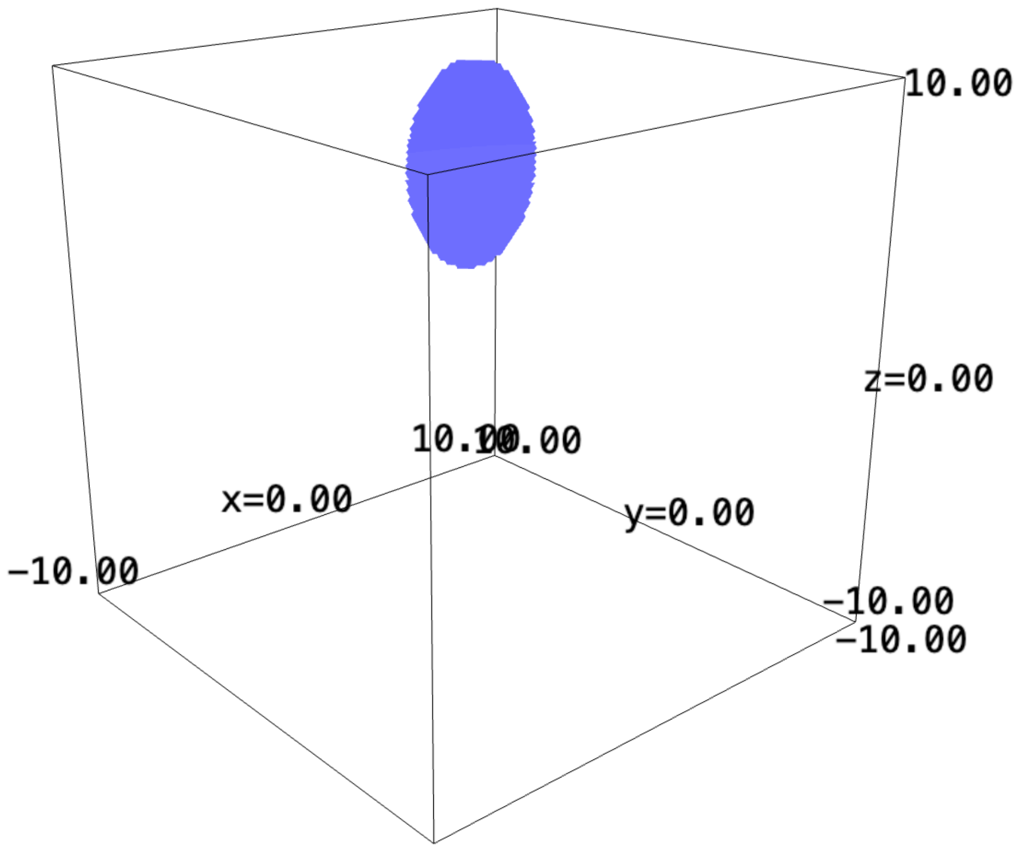
Hence, the total area under the curve would also be 448 units.

As we can see, the two integrals agree with respect to the area in that corner.

3 | Cylindrical Coordinates

To find the volume beneath $f(x, y) = 7 + x + y$ above a circle of radius 5, we can leverage cylindrical coordinates.

```
x,y,z = var("x y z")
f(x,y) = 7+x+y
implicit_plot3d(7+x+y-z, (x,-10,10), (y,-10,10), (z,-10,10), region=lambda x,y,z:(x**2+y**2)<5, plot_po
```



Recall that, given an unit circle, we have:

$$\begin{cases} x = r\ cos(\theta) \\ y = r\ sin(\theta) \end{cases} \tag{38}$$

and the z value stays Cartesian. Therefore, we can change the expression into that in r and θ:

$$f(r, \theta) = 7 + r\ cos(\theta) + r\ sin(\theta) \tag{39}$$

The "shell" of each circle in the expression would have circumference $r\ d\theta$, therefore, the integral to form a circle of radius 5 and $\theta \in [0, 2\pi]$ would be:

$$\int_0^5 \int_0^{2\pi} r(7 + r\ cos(\theta) + r\ sin(\theta))\ d\theta\ dr \tag{40}$$

$$\Rightarrow \int_0^5 r(7\theta + r\ sin(\theta) - r\ cos(\theta)|_0^{2\pi}\ )dr \tag{41}$$

$$\Rightarrow \int_0^5 (14\pi - r) - (-r)\ dr \tag{42}$$

$$\Rightarrow \int_0^5 14\pi\ r\ dr \tag{43}$$

$$\Rightarrow 175\pi \tag{44}$$

We can check this value by taking vertical slices of the function. A circle of radius five centered about the origin can be modeled by the expression:

$$y = \pm\sqrt{5^2 - x^2} \tag{45}$$

We will take the integral along the y dimension first, bounded by this function; then, we will sum the results along $[-5, 5]$ for the x dimension.

Taking the integral in y:

$$\int_0^{\sqrt{25-x^2}} 7 + x + y\ dy + \int_{-\sqrt{25-x^2}}^0 7 + x + y\ dy \tag{46}$$

$$\Rightarrow \left(7y + xy + \frac{y^2}{2}\right)\Big|_0^{\sqrt{25-x^2}} + \left(7y + xy + \frac{y^2}{2}\right)\Big|_{-\sqrt{25-x^2}}^0 \tag{47}$$

$$\Rightarrow \left(7\sqrt{25-x^2} + x\sqrt{25-x^2} + \frac{25-x^2}{2}\right) + \left(7\sqrt{25-x^2} + x\sqrt{25-x^2} - \frac{25-x^2}{2}\right) \tag{48}$$

$$\Rightarrow 14\sqrt{25-x^2} + 2x\sqrt{25-x^2} \tag{49}$$

We will leverage Sage to take the integral in x.

$$\int_{-5}^5 14\sqrt{25-x^2} + 2x\sqrt{25-x^2}\ dx \tag{50}$$

```
f(x,y)=7+x+y
(f.integrate(y, 0, sqrt(25-x^2)) + f.integrate(y, -sqrt(25-x^2), 0)).integrate(x,-5,5)
```

As we can see, the result from both expressions are the same, 175π.