1 | Axler5.21 Complex Vector Spaces have atleast one eigenvalue

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

2 | intuition

2.1 | by the fundamental theorem of algebra, the characteristic polynomial will have roots and thus there will be eigenvalues.

3 | proof

3.1 | by factoring, we turn the polynomial of maps into a composition of linear maps of the form $(T - \lambda I)$ and the input vector has to go to all of them. We choose a v s.t. it should be equal to zero, which means that one of the maps needs to send the v to zero (and that map will be injective and that lambda will be an eigenvalue).

3.2 | to formalize the "one of the maps sends the input to zero," you can just use a prev proof "if a chain of maps is not injective, then one of the maps is not injective" or induct because there is a finite number of maps.