1 | **Asymptotically Analyze**

1.1 | **1.a**

$$f(n) = n^2, g(n) = 3n + 1 \tag{1}$$

$$\lim_{n\to\infty} \frac{n^2}{3n+1} = \infty \tag{2}$$

$$\lim_{n\to\infty} \frac{3n+1}{n^2} = 0 \tag{3}$$

Therefore:

- $g = o(f)$
- $g = O(f)$

1.2 | **1.b**

$$f(n) = \frac{3n-7}{n+4}, g(n) = 4 \tag{4}$$

$$\lim_{n\to\infty} \frac{\frac{3n-7}{n+1}}{4} = \lim_{n\to\infty} \frac{3n-7}{(n+1)4} = \frac{3}{4} \tag{5}$$

$$\lim_{n\to\infty} \frac{4}{\frac{3n-7}{n+1}} = \lim_{n\to\infty} \frac{4(n+1)}{(3n-7)} = \frac{4}{3} \tag{6}$$

Therefore:

- $f = O(g)$
- $g = O(f)$
- $f = \theta(g)$

1.3 | **1.c**

$$f(n) = 4^n, g(n) = 2^n \tag{7}$$

$$\lim_{n\to\infty} \frac{4^n}{2^n} = \lim_{n\to\infty} 2^n = \infty \tag{8}$$

$$\lim_{n\to\infty} \frac{2^n}{4^n} = \lim_{n\to\infty} 2^{-n} = 0 \tag{9}$$

Therefore:

- $g = o(f)$
- $g = O(f)$

1.4 | **1.d**

$$f(n) = n!, g(n) = n^n \tag{10}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \tag{11}$$

$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty \tag{12}$$

Therefore:

- $f = o(g)$
- $f = O(g)$

1.5 | **1.e**

$$f(n) = 2^n, g(n) = 2^{\frac{n}{2}} \tag{13}$$

$$\lim_{n\to\infty} \frac{2^n}{2^{\frac{n}{2}}} = \lim_{n\to\infty} \sqrt{2^n} = \infty \tag{14}$$

$$\lim_{n\to\infty} \frac{2^{\frac{n}{2}}}{2^n} = \lim_{n\to\infty} \frac{1}{\sqrt{2^n}} = 0 \tag{15}$$

Therefore:

- $g = o(f)$
- $g = O(f)$

1.6 | **1.f**

$$f(n) = n^8, g(n) = 1.1^n \tag{16}$$

For this expression, we need to apply l'hospital's rule repeatedly as both the top and bottom evaluates to ∞ until the final n:

$$\lim_{n\to\infty} \frac{n^8}{1.1^n} = \cdots = C \lim_{n\to\infty} \frac{1}{1.1^n} = 0 \tag{17}$$

where C is some constant generated by the power rule.

$$\lim_{n\to\infty} \frac{1.1^n}{n^8} = \cdots = C \lim_{n\to\infty} 1.1^n = \infty \tag{18}$$

Therefore:

- $f = o(g)$
- $f = O(g)$

2 | **Vampires**

The open-form recurrence relationship for vampires is as follows:

- $T(1) = 2$
- $T(n) = 2T(n-1) + 1$

We will first generate a table relating n, T:

n	T(n)	Guess
1	2	2
2	5	5
3	11	11
4	23	23
5	47	47
6	95	95
7	191	191
8	383	383
9	767	767
10	1535	1535

The guessed formula is as follows:

$$f(n) = 2^n + 2^{n-1} - 1 \tag{19}$$

We will now proof this via induction. We will set our inductive hypothesis as, at some $P(n)$ for some n, $P(n) = 2^n + 2^{n-1} - 1 = T(n)$. At our base case $P(1) = 2^1 + 2^0 - 1 = 2 == 2$.

Induction:

- $T(n+1) = 2T(n) + 1$, given
- $T(n+1) = 2(2^n + 2^{n-1} - 1) + 1$, replacing the inductive hypothesis
- $T(n+1) = 2^{n+1} + 2^n - 2 + 1$, simplify
- $T(n+1) = 2^{n+1} + 2^n - 1$, simplify
- $T(n+1) = 2^{(n+1)} + 2^{(n+1)-1} - 1$, simplify

Since $P(n)$ implies $P(n+1)$, and we have proven the base case, by induction this statement is true.

3 | TriSort

3.1 | Three Sorted Lists

To sort two lists, the fastest method is to sort the first two list, then sort the third.

- $(\frac{2}{3}n - 1) + (n - 1)$
- $\frac{5}{3}n - 2$

Therefore, for a three sorted list, each with $\frac{n}{3}$ elements, the worst case runtime would be $\frac{5}{3}n - 2$ to merge the lists together—the worst-case case is to place a pointer on each element to do incremental element comparison.

3.2 | Recurrence Running Time

- $T(1) = 1$
- $T(n) = 3T(\frac{n}{3}) + (\frac{5}{3}n - 2)$

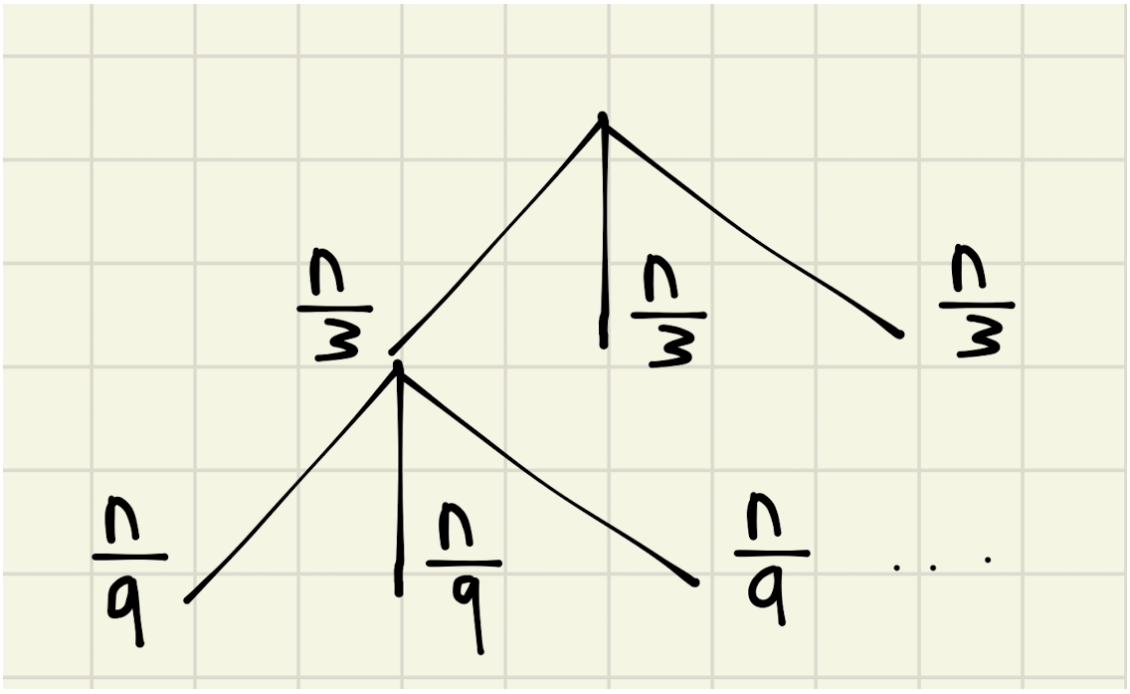
Furthermore:

$$\lim_{n\to\infty} \frac{n}{\frac{5}{3}n - 2} = \frac{1}{2} < \infty \tag{20}$$

Therefore: $\frac{5}{3}n - 2 = \theta(n)$. The resulting expression, therefore is that:

$$T(n) = 3T\left(\frac{n}{3}\right) + \theta(n) \tag{21}$$

3.3 | **Drawing a Recurrence Tree**



3.4 | **Number of Comparisons at Each Layer**

At each layer, there is $a\frac{n}{a}$ comparisons (3 nodes of $\frac{n}{3}$ on the first, 9 nodes of $\frac{n}{9}$, etc.). Therefore, each layer of the tree has $\theta(n)$ comparisons.

3.5 | **Height of the Tree**

WLOG at large values of n, we will assume that the number of elements in the list are a power of 3 to make a balanced tree. To divide n by 3 recursively until we result at 1 (the actual base cases), there will need to be $\log_3(n)$ layers in the tree.

3.6 | **Master Method**

The general recurrence relation for this process is as follows:

$$T(n) = 3T\left(\frac{n}{3}\right) + \theta(n) \tag{22}$$

We can see that $a = 3, b = 3, d = 1$, and the base case is $c = 1$. $b^d = 3$, which is equal to a.

Therefore, by the Master Method, we can say that the runtime for the algorithm is $\theta(nlog(n))$.

3.7 | **Did the student win?**

No. The θ time complexity between Merge and TriMerge is exactly the same, and the constant factor of TriMerge $\frac{5}{3}n - 2$ is roughly $\frac{2}{3}n$ larger than the $n - 1$