1 | Exponentials and Logarithms unit1::derivatives

1.1 | Exponential Functions

Goal: Calculate $\frac{d}{dx}a^x$

- $\lim\limits_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$

As $a^{x+\Delta x} = a^x a^{\Delta x}$, $\lim\limits_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x}$

a^x is a constant so $a^x \lim\limits_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

$M(a) := \lim\limits_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

With new definition $\frac{d}{dx}a^x = M(a)a^x$. Plug in $x = 0$ to get $\frac{d}{dx}a^0 = M(a)$, showing $M(a)$ is the slope at 0.

What is M(a)?

- Define base e as the unique number such that $M(e) = 1$
- If this is the case $\frac{d}{dx}e^x = e^x$ (as $\frac{d}{dx}a^x = M(a)a^x$).

Why does e exist?

- Take example $f(x) = 2$, $f'(0) = M(2)$ and stretch by constant k.

$f(kx) = 2^{kx} = (2^k)^x = b^x$, where $b = 2^k$.

- As k is increased the slope of the function gets steeper. $\frac{d}{dx}b^x = kf'(kx)$
  - At 0, $\frac{d}{dx}b^x = kf'(kx) = kf'(0) = kM(2)$ so $b = e$ when $k = \frac{1}{M(2)}$

1.2 | The Natural Log

Recall that $\ln x_1 x_2 = \ln x_1 + \ln x_2$ and $\ln 1 = 0$ and $\ln e = 1$.

Differentiate $w = \ln x$ implicitly in the form $e^w = x$:

- $\frac{d}{dx}e^w = \frac{d}{dx}x = 1$
- $\frac{d}{dw}e^w \frac{dw}{dx} = 1$ or $e^w \frac{dw}{dx} = 1$
- Algebra yields $\frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$

1.3 | Back to The Exponential

Method 1 Use base e = $(e^{\ln a})^x = e^{x \ln a}$.

Just as the derivative of e^{3x} is $3e^{3x}$ by chain rule, $\frac{d}{dx}e^{x \ln a} = (\ln a)e^{x \ln a}$. So, $\frac{d}{dx}a^x = (\ln a)a^x$.

NOTE: No matter what our base (2 or 10 or something else) the derivative is still $(\ln a)a^x$ and that's one reason why it's the "natural" log as it comes up naturally.

Method 2

HEY I NEED REVISITING REWATCH THE LAST 15 MIN OF THIS

Logarithmic differentiation. Chain rule + differentiation of logarithm. $(\ln u)' = \frac{u'}{u}$

EXAMPLE

$v = x^x$ $\ln v = x \ln x$ $(\ln v)' = \ln x + x\frac{1}{x}$ $\frac{v'}{v} = 1 + \ln x$ $v' = v(1 + \ln x)$ $\frac{d}{dx}x^x = x^x(1 + \ln x)$

2 | Links

Further review can be found in MIT SVC Exam Review (Unit 1).