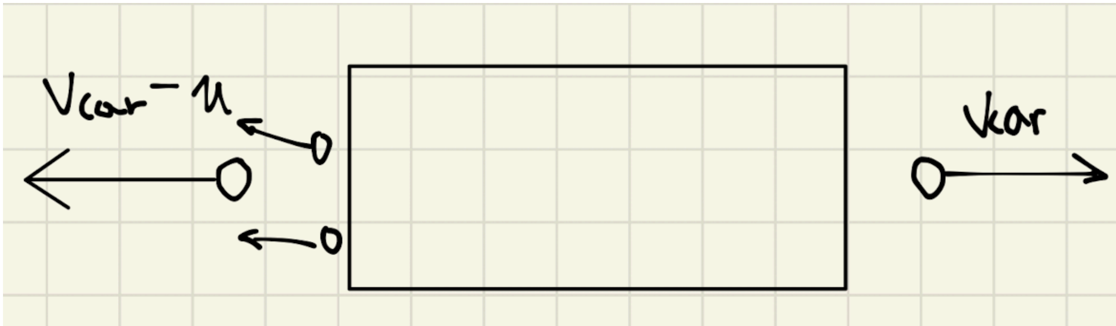
1 | Draw a picture to describe this



2 | All Armadillos at Once

- The mass of all armadillos is: Nm
- The mass of the flatcar: M

2.1 | Pre jump

Therefore, the mass ”on car” pre-jump: $Nm + M$, and the mass ”on car” post-jump: Nm.

The velocity of the car at the moment that the armadillo jumps is $\vec{v}_{car}$, and therefore, the net momentum pre-jump is simply the total mass times the velocity of the car-armadillo combination: $(Nm + M)\vec{v}_{car}$.

2.2 | Post jump

We will use $\vec{v_f}_{car}$ to represent the velocity of the flatcar after all armadillo jumps. We further have a given value $(\vec{v}_{car} - \vec{u})$ for the velocity of the armadillo block chunk. The masses of all sub-components do not change.

Therefore, the net momentum post-jump is: $(\vec{v}_{car} - \vec{u})Nm + \vec{v_f}_{car}M$.

2.3 | Conserving Momentum

Based on the law of the conservation of momentum, these two values shall be equal. For the car originally at rest (or, as we are thinking in the reference frame of the original car) that:

$$(Nm + M)0 = (\vec{v}_{car} - \vec{u})Nm + \vec{v_f}_{car}M \tag{1}$$

We will now leverage this expression to solve for the value of $\vec{v_f}_{car}$.

$$
\begin{aligned}
0 &= (\vec{v}_{car} - \vec{u})Nm + \vec{v_f}_{car}M && (2)\\
\Rightarrow 0 &= \vec{v}_{car}Nm - \vec{u}Nm + \vec{v_f}_{car}M && (3)\\
\Rightarrow 0 &= \vec{v}_{car}(Nm + M) - \vec{u}Nm && (4)\\
\Rightarrow \vec{v}_{car}(Nm + M) &= \vec{u}Nm && (5)\\
\Rightarrow \vec{v}_{car} &= \frac{N\vec{u}m}{Nm + M} && (6)
\end{aligned}
$$

3 | Groups of Armadillos

In this problem, the mass of the flatcar remains the same. We will perform calculation for a few singular armadillo and then generalise to a larger expression.

3.1 | Armadillo number one

For each armadillo, their mass is m. The net mass pre-jump, therefore, is all of the armadillos on-car + the mass of the car: that $Nm + M$. Therefore, the left side of the expression does not change: $(Nm + M)0$.

However, the right side of the expression, in this case, is only inclusive of one armadillo m plus one less from the originally; that: $(\vec{v}_{car} - \vec{u})m + \vec{v}_{car}((N-1)m + M)$.

$$0 = (\vec{v}_{car} - \vec{u})m + \vec{v_f}_{car}((N-1)m + M) \tag{7}$$

Expanding this expression and solving for $\vec{v_f}$...

$$
\begin{aligned}
0 &= (\vec{v}_{car} - \vec{u})m + \vec{v}_{car}((N-1)m + M) && (8)\\
&= \vec{v}_{car}m - \vec{u}m + \vec{v}_{car}((N-1)m + M) && (9)\\
&= \vec{v}_{car}(m + ((N-1)m + M)) - \vec{u}m && (10)\\
&= \vec{v}_{car}(Nm + M) - \vec{u}m && (11)\\
\Rightarrow \vec{v}_{car} &= \frac{\vec{u}m}{(Nm + M)} && (12)
\end{aligned}
$$

3.2 | Armadillo number two

For amadillo number 2, we essentially repeat the calculations again. But, we will use $\vec{v_0}_{car}$ to represent the velocity of the car *before* number 2 jumps (i.e. $\vec{v}_{car}$ in the previous step), and $\vec{v}_{car}$ to represent the result of jumping for armadillo number 2.

Therefore:

$$((N-1)m + M)\vec{v_0}_{car} = (\vec{v}_{car} - \vec{u})m + \vec{v}_{car}((N-2)m + M) \tag{13}$$

Performing the same set of algebraic manipulations as above, we arrive at:

$$((N-1)m + M)\vec{v_0}_{car} = \vec{v}_{car}((N-1)m + M) - \vec{u}m \tag{14}$$

Computing further to isolate $\vec{v}_{car}$...

$$((N-1)m+M)\vec{v}_{0car} = \vec{v}_{car}(m+((N-1)m+M)) - \vec{u}m \tag{15}$$
$$\Rightarrow \vec{v}_{car}((N-1)m+M) = ((N-1)m+M)\vec{v}_{0car} + \vec{u}m \tag{16}$$
$$\Rightarrow \vec{v}_{car} = \frac{((N-1)m+M)\vec{v}_{0car} + \vec{u}m}{((N-1)m+M)} \tag{17}$$
$$\Rightarrow \vec{v}_{car} = \vec{v}_{0car} + \frac{\vec{u}m}{((N-1)m+M)} \tag{18}$$

Substituting the original $\vec{v}_{car}$ upon the current $\vec{v}_{0car}$, we get that:

$$\vec{v}_{car} = \frac{\vec{u}m}{(Nm+M)} + \frac{\vec{u}m}{((N-1)m+M)} \tag{19}$$

3.3 | Armadillo number N

It is clear that the numerator is the mass thrown out ($\vec{u}m$), and the denominator is the mass remaining at each point. Expanding the previous series towards N items, we arrive the following finite series:

$$\vec{v}_{car} = \frac{\vec{u}m}{(Nm+M)} + \frac{\vec{u}m}{((N-1)m+M)} + \dots + \frac{\vec{u}m}{((N-N)m+M)} \tag{20}$$

Simplifying this expression slightly, we deduce that:

$$\vec{v}_{car} = m\vec{u}\left(\frac{1}{(Nm+M)} + \frac{1}{((N-1)m+M)} + \dots + \frac{1}{((N-N)m+M)}\right) \tag{21}$$

And finally, writing the expression into sum notation, we have:

$$\vec{v}_{car} = m\vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)m+M} \tag{22}$$

4 | Zero-Mass Car

Let's see $M \to 0$.

$$\lim_{M\to 0}\vec{v}_{car} = \lim_{M\to 0} m\vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)m+M} \tag{23}$$

We will perform some algebraic manipulations to first simplify this expression:

$$\lim_{M\to 0} m\vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)m+M} \tag{24}$$
$$\Rightarrow m\vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)m} \tag{25}$$
$$\Rightarrow \vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)} \tag{26}$$

At this point, we notice that the right side—due to the commutativity of addition—is equivalent to the following expression:

$$\vec{u}\sum_{i=0}^{N-1}\frac{1}{i}=\vec{u}\sum_{i=1}^{N}\frac{1}{i} \tag{27}$$

Therefore, as the mass of the flatcar approaches 0, the final velocity after throwing off all armadillos will approach the 1st degree harmonic series. As $N \to \infty$, this series will diverge.

5 | Jump as a Block vs Jump by one

The final velocity of jumping as a block is as follows:

$$\vec{v}_{carb}=\frac{Nm\vec{u}}{Nm+M} \tag{28}$$

The final velocity of jumping individual is as follows:

$$\vec{v}_{cars}=m\vec{u}\sum_{i=0}^{N-1}\frac{1}{(N-i)m+M} \tag{29}$$

As opposed to the case with an entire-block jump, the ”jumping individual” case actually resolves to increasing-smaller $(N-i)$ denominators at once — resulting in larger individual terms unlike the constant-sized $\frac{m\vec{u}}{Nm+M}$. In the individual-armadillo case, the velocity of the car builds up slowly as each individual gets thrown off.

6 | Mitosis

Before the mitosis starts, for N:

$$\vec{v}_{car}=m'\vec{u}\sum_{i=0}^{N'-1}\frac{1}{(N'-i)m'+M} \tag{30}$$

When the process of mitosis begins, our $m'=\frac{m}{2}$ and our $N'=2N$. We will perform this substitution to the above expression:

$$\vec{v}_{car}=\frac{m}{2}\vec{u}\sum_{i=0}^{2N-1}\frac{1}{(2N-i)\frac{m}{2}+M} \tag{31}$$

$$=m\vec{u}\sum_{i=0}^{2N-1}\frac{1}{2((2N-i)\frac{m}{2}+M)} \tag{32}$$

$$=m\vec{u}\sum_{i=0}^{2N-1}\frac{1}{(2N-i)m+2M} \tag{33}$$

After mitosis, therefore, the final expression for $\vec{v}_{car}$ is as follows:

$$\vec{v}_{car} = m\vec{u} \sum_{i=0}^{2N-1} \frac{1}{(2N-i)m + 2M} \tag{34}$$

We see that there is two times the amount of components on the post-mitoses expression than there is pre-mitoses. Hence, we can ensure that the prior sum is smaller by ensuring that every latter component is larger than at least *half* of each of the prior ones.

Pairwise subtracting $\frac{1}{2}$ of the left side at every N, then, it is evident that for any (N, i):

$$\frac{1}{(2N-2i)m + 2M} < \frac{1}{(2N-i)m + 2M} \tag{35}$$

We can see that, for every pair, there is i more subtracted from $2N$. This renders the post-mitosis result in a faster train.

7 | **Differential Momentum Flow**

We are given the following expression:

$$M\frac{dv}{dt} = \vec{u}\frac{dm}{dt} \tag{36}$$

The expression tells us that the mass of the "train", multiplied by its instantaneous acceleration, is equal to the differential mass ejected, multiplied by the velocity of ejection.

This expression helps us figure, if the ejected mass is fully smooth (completely mitothesized armadillo mash, a.k.a. rocket fuel), the acceleration which the act of ejection provides to the source.

We will now figure an expression for the velocity of the car given the mass ejected

$$M\frac{dv}{dt} = \vec{u}\frac{dm}{dt} \tag{37}$$

$$\Rightarrow M\frac{dv}{dm}\frac{dm}{dt} = \vec{u}\frac{dm}{dt} \tag{38}$$

$$\Rightarrow M\frac{dv}{dm} = \vec{u}\frac{dm}{dt}\frac{dt}{dm} \tag{39}$$

$$\Rightarrow M\frac{dv}{dm} = \vec{u} \tag{40}$$

$$\Rightarrow M dv = \vec{u} dm \tag{41}$$

$$\Rightarrow dv = \frac{\vec{u}}{M} dm \tag{42}$$

$$\Rightarrow \int_{v_0}^{v} dv = \vec{u} \int_{m_0}^{m} \frac{1}{M} dm \tag{43}$$

At this point, we will utilize the fact that dm is in the opposite direction of dM (mass ejected is the mass that is been ejected, resulting in less mass.) Hence, $dm = -dM$.

$$\int_{v_0}^{v} dv = \vec{u} \int_{m_0}^{m} \frac{1}{M} dm \tag{44}$$

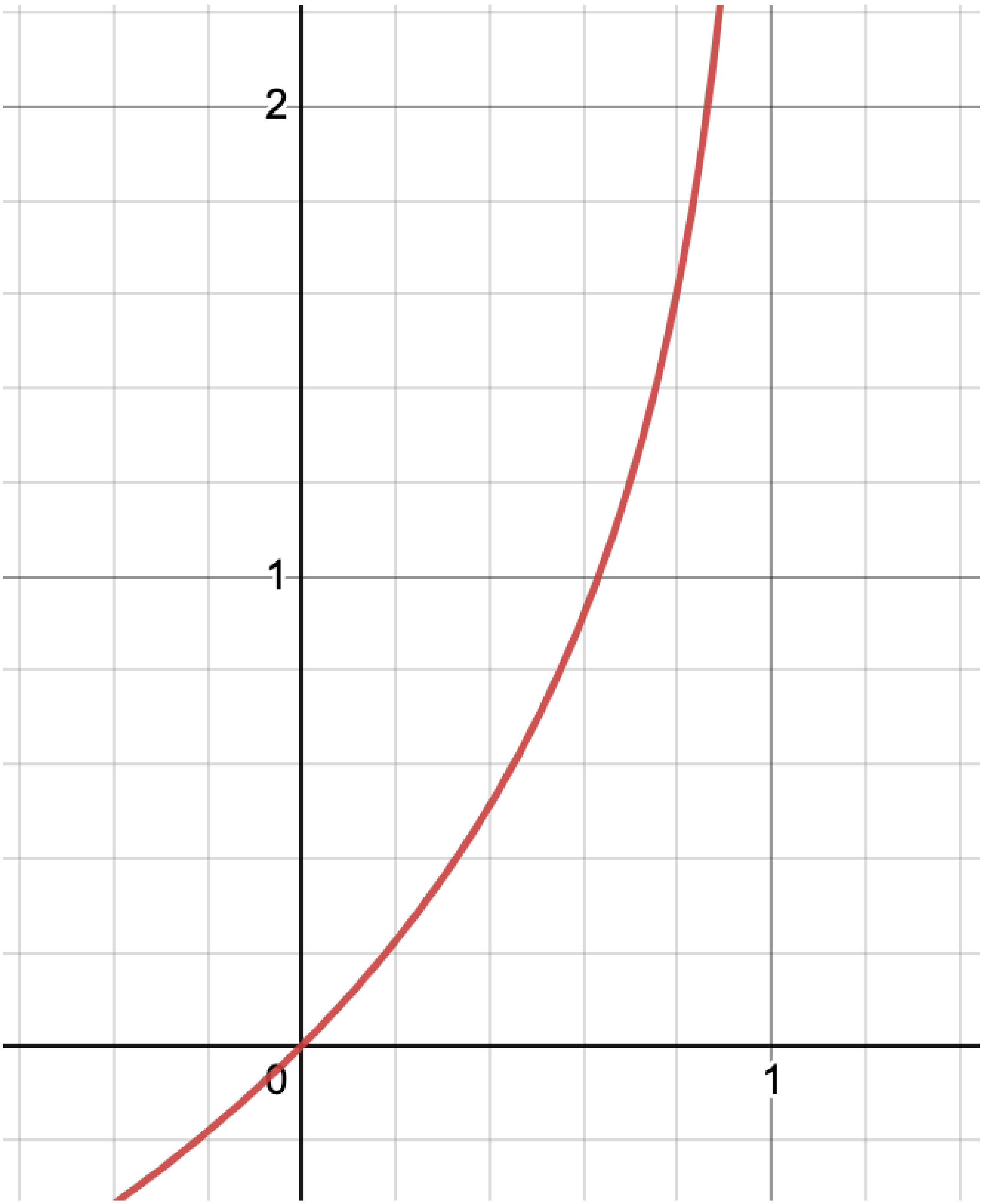
$$\Rightarrow \int_{v_0}^{v} dv = -\vec{u} \int_{m_0}^{m} \frac{1}{M} dM \tag{45}$$

$$\Rightarrow v|_{v_0}^{v} = -\vec{u} \ln(m)|_{m_0}^{m} \tag{46}$$

At this point, we note that $v_0 = 0$ as the initial velocity is 0. Furthermore, we will define the value of m_e as mass ejected. Note that we are integrating over dM, w.r.t. mass *remaining*, so the "initial" state is M, nothing ejected, and the "final" state $M - m_e$, no more in the tank.

$$v = \vec{u}(\ln(M) - \ln(M - m_e)) = \vec{u} \ln(\frac{M}{M - m_e}) \tag{47}$$

We will pick $1 = m$, as well as $1 = \vec{u}$ for the following graph:



Let's take the limit as $M - m_e \to 0$!

$$\lim_{(M-m_e)\to 0} \vec{u}\ln(\frac{M}{M-m_e}) \tag{48}$$

$$\Rightarrow \infty \tag{49}$$

This is the exact same conclusion as the diverging behavior of the infinite $p = 1$ harmonic series as determined in part d — as $\lim_{n\to\infty}$ in that case, which is the same for the differential mass here.