1 | boatman problem

Target displacement: $\langle 3\text{km}, 2\text{km}\rangle$

We are working with the velocities of the boat and the river. The velocity of the river is defined as $r = \langle 0, -3.5\rangle$. We want to find vector $v = \langle v_x, v_y\rangle$ s.t.

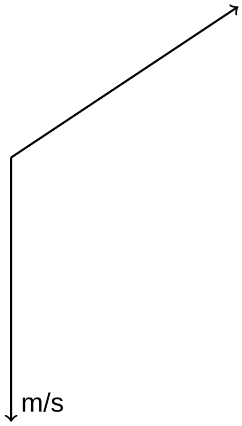
$$|v| = 13 \text{ km/h}$$
$$\lambda(v + r) = \langle 3, 2\rangle$$

Where the trip will take λ hours

$$v_x^2 + v_y^2 = 13^2$$
$$\lambda(v_x + 0) = 3$$
$$\lambda(v_y + -3.5) = 2$$

$$v_x = \frac{3}{\lambda}$$
$$v_y = \frac{2}{\lambda} + 3.5$$
$$\frac{3^2}{\lambda^2} + \left(\frac{2}{\lambda} + 3.5\right)^2 = 13^2$$
$$\frac{3^2}{\lambda^2} + \frac{2^2}{\lambda^2} + 3.5^2 + \frac{4(3.5)}{\lambda} = 13^2$$
$$\frac{3^2 + 2^2}{\lambda^2} + \frac{4(3.5)}{\lambda} = 13^2 - 3.5^2$$
$$3^2 + 2^2 + 4(3.5)\lambda = \lambda^2\left(13^2 - 3.5^2\right)$$
$$13 + 4(3.5)\lambda = \lambda^2\,(156.75)$$
$$-156.75\lambda^2 + 14^2 + 13 = 0$$
$$\frac{-14 \pm \sqrt{14^2 + 4(13)156.75}}{-2(156.75)}$$
$$\frac{-14 + \sqrt{14^2 + 4(13)156.75}}{-2(156.75)} = -0.24676847741$$
$$\frac{-14 - \sqrt{14^2 + 4(13)156.75}}{-2(156.75)} = 0.336082671987$$

Maybe it’s time to do it geometrically



Let θ be the angle difference that you paddle at, and ϕ be the angle that you are aiming for.

$$
\begin{aligned}
3.5^2\lambda^2 &= 13 + 13\lambda - 2(13)13\lambda\cos\theta \\
\tan\phi &= \frac{3}{2} \\
\sin(\theta+\phi) &= \frac{3.5\lambda+2}{13\lambda}
\end{aligned}
$$

attempt 3: after getting help from leonard

$$
\begin{aligned}
\beta = \alpha + \frac{\pi}{2} &= \tan^{-}\frac{2}{3} = 2.158 \\
\frac{\sin\beta}{|v|} &= \frac{\sin\gamma}{3.5} \\
\frac{\sin(2.158)}{13} &= \frac{\sin\gamma}{3.5} \\
3.5\frac{\sin(2.158)}{13} &= \sin\gamma \\
\gamma &= 3.5\frac{\sin(2.158)}{13} = 0.2241 \\
\alpha + \gamma &= 0.588 + 0.2241 = 0.8121 \text{ radians}
\end{aligned}
$$

The speed

$$
\frac{3}{13\cos 0.812} = 0.3353 \text{ hours}
$$

dang it i was actually right the first time. apparently math isn't a democracy.

2 | circular motion

$$
\begin{aligned}
\theta'(t) &= \omega_0 \\
\theta(t) &= \int \omega_0 dt \quad = \omega_0 t + \theta_0
\end{aligned}
$$

2.1 | position

$$\begin{aligned}\vec{r}(t) &= (R\cos\theta(t), R\sin\theta(t)) \\ &= (R\cos(\omega_0 t + \theta_0), R\sin(\omega_0 t + \theta_0))\end{aligned}$$

2.2 | velocity

$$\vec{v}'(t) = (-\omega_0 R\sin(\omega_0 t + \theta_0), \omega_0 R\cos(\omega_0 t + \theta_0))$$

2.3 | acceleration

$$\begin{aligned}\vec{a}(t) &= \vec{v}'(t) \\ &= \frac{d}{dt}(-\omega_0 R\sin(\omega_0 t + \theta_0), \omega_0 R\cos(\omega_0 t + \theta_0)) \\ &= \left(-\omega_0^2 R\cos(\omega_0 t + \theta_0), -\omega_0^2 R\sin(\omega_0 t + \theta_0)\right)\end{aligned}$$

2.4 | perpendicular

$$\begin{aligned}\text{pos:}\ & \frac{\cancel{R}\sin(\omega_0 t + \theta_0)}{\cancel{R}\cos(\omega_0 t + \theta_0)} \\ \text{vel:}\ & \frac{\cancel{\omega_0 R}\cos(\omega_0 t + \theta_0)}{-\cancel{\omega_0 R}\sin(\omega_0 t + \theta_0)}\end{aligned}$$

See? They are negative reciprocals of eachother.

$$\begin{aligned}|\vec{v}| &= \sqrt{\omega_0^2 R^2\sin^2(\omega_0 t + \theta_0) + \omega_0^2 R^2\cos^2(\omega_0 t + \theta 0)} \\ &= \omega_0 R\sqrt{\sin^2(\omega_0 t + \theta_0) + \cos^2(\omega_0 t + \theta 0)} \\ &= \omega_0 R\sqrt{1} \\ &= \omega_0 R\end{aligned}$$

2.5 | acceleration

$$\begin{aligned}\text{accl} &= -\omega_0^2 R(\cos(\omega_0 t + \theta_0), \sin(\omega_0 t + \theta_0)) \\ \text{pos} &= R(\cos(\omega_0 t + \theta_0) + \sin(\omega_0 t + \theta_0))\end{aligned}$$

The acceleration is a negative scalar multiple of the position, and thus points towards the center.

$$|\vec{a}| = -w_0^2 R\sqrt{1} = \frac{w_0^2 R^2}{R} = \frac{|\vec{v}|^2}{R}$$

2.6 | non uniform motion

$$\theta''(t) = a_0$$
$$\theta'(0) = 0$$
$$\theta(0) = 0$$

$$\theta'(t) = a_0 t + C_0 = a_0 t$$

2.7 | angular position

$$\theta(t) = \frac{1}{2}a_0 t^2 + C_1 = \frac{1}{2}a_0 t^2$$

2.8 | position

$$\begin{aligned}\vec{r}(t) &= (R\cos\theta(t), R\sin\theta(t)) \\ &= \left(R\cos\left(\frac{1}{2}a_0t^2\right), R\sin\left(\frac{1}{2}a_0t^2\right)\right)\end{aligned}$$

2.9 | velocity

$$\begin{aligned}\vec{v}(t) &= R\left(-\sin\left(\frac{1}{2}a_0t^2\right)a_0t, \cos\left(\frac{1}{2}a_0t^2\right)a_0t\right) \\ &= Ra_0t\left(-\sin\left(\frac{1}{2}a_0t^2\right), \cos\left(\frac{1}{2}a_0t^2\right)\right)\end{aligned}$$

2.10 | acceleration

$$\begin{aligned}\vec{a}(t) = a_0R\Bigg(&-\cos\left(\frac{1}{2}a_0t^2\right)a_0t^2 - \sin\left(\frac{1}{2}a_0t^2\right), \\ &-\sin\left(\frac{1}{2}a_0t^2\right)a_0t^2 + \cos\left(\frac{1}{2}a_0t^2\right)\Bigg)\end{aligned}$$

2.11 | velocity vector

Both magnitudes are non-zero, so we can use the dot product to show that the position and velocity are perpendicular.

$$\begin{aligned}\vec{r}(t)\cdot\vec{v}(t) &= R\left(-\cos\left(\frac{1}{2}a_0t^2\right)\sin\left(\frac{1}{2}a_0t^2\right)a_0t + \sin\left(\frac{1}{2}a_0t^2\right)\cos\left(\frac{1}{2}a_0t^2\right)a_0t\right) \\ &= R0 = 0\end{aligned}$$

Thus, the velocity is tangent to the circle.

Magnitude:

$$\begin{aligned}\vec{v}(t) &= Ra_0t\sqrt{\sin^2\left(\frac{1}{2}a_0t^2\right)+\cos^2\left(\frac{1}{2}a_0t^2\right)}\\ &= Ra_0t\sqrt{1} = Ra_0t = R\theta'(t)\end{aligned}$$

2.12 | acceleration

We can split it up because we know the $|\vec{v}|^2$ term is the one with $a_0^2t^2$

$$\begin{aligned}\vec{a}(t) &= R\left(-a_0^2t^2\left(\cos\left(\frac{1}{2}a_0t^2\right),\sin\left(\frac{1}{2}a_0t^2\right)\right)+a_0\left(-\sin\left(\frac{1}{2}a_0t^2\right),\cos\left(\frac{1}{2}a_0t^2\right)\right)\right)\\ &= R\left(-\frac{|\vec{v}|^2}{R^2}\frac{\vec{r}}{R}\right)+Ra_0\frac{\vec{v}}{|\vec{v}|}\\ &= \left(-\frac{|\vec{v}|}{R^2}\frac{\vec{r}}{R}\right)+Ra_0\frac{\vec{v}}{|\vec{v}|}\end{aligned}$$

3 | derivative distribution

$$\begin{aligned}\frac{d\vec{A}}{dt} &= \lim_{h\to 0}\frac{\vec{A}(t+h)-\vec{A}(t)}{h}\\ &= \lim_{h\to 0}\frac{(A_x(t+h),A_y(t+h),A_z(t+h))-(A_x(t),A_y(t),A_z(t))}{h}\\ &= \lim_{h\to 0}\frac{(A_x(t+h)-A_x(t),A_y(t+h)-A_y(t),A_z(t+h)-A_z(t))}{h}\\ &= \lim_{h\to 0}\left(\frac{A_x(t+h)-A_x(t)}{h},\frac{A_y(t+h)-A_y(t)}{h},\frac{A_z(t+h)-A_z(t)}{h}\right)\\ &= \left(\lim_{h\to 0}\frac{A_x(t+h)-A_x(t)}{h},\lim_{h\to 0}\frac{A_y(t+h)-A_y(t)}{h},\lim_{h\to 0}\frac{A_z(t+h)-A_z(t)}{h}\right)\\ &= \left(\frac{dA_x(t)}{dt},\frac{dA_y(t)}{dt},\frac{dA_z(t)}{dt}\right)\end{aligned}$$

4 | derivative linearity

$$\begin{aligned}
\frac{d}{dt}\left(\alpha\vec{A}(t)+\beta\vec{B}(t)\right) &= \frac{d}{dt}\left(\alpha A_x(t),\alpha A_y(t)\right)+\left(\beta B_x(t),\beta B_y(t)\right)\\
&= \frac{d}{dt}\left(\alpha A_x(t)+\beta B_x(t),\alpha A_y(t)+\beta B_y(t)\right)\\
&= \left(\frac{d}{dt}\left(\alpha A_x(t)+\beta B_x(t)\right),\frac{d}{dt}\left(\alpha A_y(t)+\beta B_y(t)\right)\right)\\
&= \left(\left(\alpha\frac{d}{dt}A_x(t)+\beta\frac{d}{dt}B_x(t)\right),\left(\alpha\frac{d}{dt}A_y(t)+\beta\frac{d}{dt}B_y(t)\right)\right)\\
&= \left(\left(\alpha\frac{d}{dt}A_x(t),\alpha\frac{d}{dt}A_y(t)\right)+\left(\beta\frac{d}{dt}B_x(t),\beta\frac{d}{dt}B_y(t)\right)\right)\\
&= \alpha\frac{d}{dt}\left(A_x(t),A_y(t)\right)+\beta\frac{d}{dt}\left(B_x(t),B_y(t)\right)\\
&= \alpha\frac{d\vec{A}(t)}{dt}+\beta\frac{d\vec{B}(t)}{dt}
\end{aligned}$$

5 | chain rule for derivatives

$$\begin{aligned}
\frac{d}{dt}\vec{A}(u(t)) &= \left(\frac{d}{dt}A_x(u(t)),\frac{d}{dt}A_y(u(t))\right)\\
&= \left(\frac{dA_x(u)}{du}\frac{du}{dt},\frac{dA_y(u)}{du}\frac{du}{dt}\right)\\
&= \left(\frac{dA_x(u)}{du},\frac{dA_y(u)}{du}\right)\frac{du}{dt}\\
&= \frac{d\vec{A}(u)}{du}\frac{du}{dt}
\end{aligned}$$