

1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f .

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x -value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive- x , so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y , and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0 , the moment the marble leaves the track and travels as a projectile as t_1 , and the end — in the moment when the marble hits the ground — as t_f . We will call the marble m_0 .

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0g(H - h_0) = m_0g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named \vec{v}_0 .

$$m_0g(H - h_0) = \frac{1}{2}m_0\vec{v}_0^2 \quad (1)$$

$$g(H - h_0) = \frac{1}{2}\vec{v}_0^2 \quad (2)$$

$$2g(H - h_0) = \vec{v}_0^2 \quad (3)$$

$$\vec{v}_0 = \sqrt{2g(H - h_0)} \quad (4)$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} v_{0x} = \sqrt{2g(H - h_0)}\cos(\theta_0) \\ v_{0y} = \sqrt{2g(H - h_0)}\sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. v_{0y} , v_{0x} , h_0 , m_0 .

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f .

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = -\frac{1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as...

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Solving and optimizing for $\frac{dx_f}{d\theta_0}$

We need to maximize $\frac{dx_f}{d\theta_0}$ as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for $\frac{h_0}{d\theta_0}$. The position equations above could be leveraged to figure a value for x_f .

3.2.1 | Setup for Solution

We first create a set of equations modeling the location of the marble at t_f .

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f \sqrt{2g(H - h_0)}\cos(\theta_0) \\ y(t_f) = 0 = -\frac{1}{2}gt_f^2 + v_{0y}t_f + h_0 = -\frac{1}{2}gt_f^2 + t_f \sqrt{2g(H - h_0)}\sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f , and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H - h_0)}\cos(\theta_0)} \quad (5)$$

Finally, we substitute the definition of t_f into $y(t_f)$.

$$y(t_f) = 0 = -\frac{1}{2}g \frac{x_f^2}{2g(H - h_0)\cos^2(\theta_0)} + \frac{x_f}{\sqrt{2g(H - h_0)}\cos(\theta_0)} \sqrt{2g(H - h_0)}\sin(\theta_0) + h_0 \quad (6)$$

We will now proceed to simplify the expression further

$$0 = \frac{-1}{4} \frac{-x_f^2}{(H - h_0)\cos^2(\theta_0)} + x_f \tan(\theta_0) + h_0 \quad (7)$$

$$= \frac{-1}{4} \frac{-x_f^2}{(H - h_0)} \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (8)$$

$$= \frac{-1}{4} \frac{-1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (9)$$

3.2.2 | Finding $\frac{dx_f}{d\theta_0}$

We leverage implicit differentiation to figure a value for $\frac{dx_f}{d\theta_0}$. We set x_f as a differentiable function, and h_0 and H as both constants.

$$0 = \frac{-1}{4} \frac{-1}{(H-h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (10)$$

$$\Rightarrow \frac{d}{d\theta_0} 0 = \frac{d}{d\theta_0} \left(\frac{1}{4} \frac{1}{(H-h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \right) \quad (11)$$

$$\Rightarrow 0 = \frac{1}{4} \frac{1}{(H-h_0)} \frac{d}{d\theta_0} x_f^2 \cos^{-2}(\theta_0) + \frac{d}{d\theta_0} x_f \tan(\theta_0) + \frac{d}{d\theta_0} h_0 \quad (12)$$

$$\Rightarrow 0 = \frac{1}{4} \frac{1}{(H-h_0)} \left(\left(\frac{d}{d\theta_0} x_f^2 \right) \cos^{-2}(\theta_0) + x_f^2 \left(\frac{d}{d\theta_0} \cos^{-2}(\theta_0) \right) \right) + \quad (13)$$

$$\left(\left(\frac{d}{d\theta_0} x_f \right) \tan(\theta_0) + \left(\frac{d}{d\theta_0} \tan(\theta_0) \right) x_f \right) + 0 \quad (14)$$

$$\Rightarrow 0 = \frac{1}{4(H-h_0)} \left((2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) \right) + \quad (15)$$

$$\left(\frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \right) \quad (16)$$

$$\Rightarrow 0 = \frac{1}{4(H-h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{1}{4(H-h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) + \quad (17)$$

$$\frac{dx_f}{d\theta_0} \tan(\theta_0) + \sec^2(\theta_0) x_f \quad (18)$$

$$\Rightarrow - \frac{1}{4(H-h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (19)$$

$$= \frac{1}{4(H-h_0)} (2x_f \frac{dx_f}{d\theta_0}) \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (20)$$

$$\Rightarrow - \frac{1}{4(H-h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (21)$$

$$= \frac{dx_f}{d\theta_0} \frac{1}{2(H-h_0)} x_f \cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0} \tan(\theta_0) \quad (22)$$

$$\Rightarrow - \frac{1}{4(H-h_0)} x_f^2 (2\cos^{-3}(\theta_0) \sin(\theta_0)) - \sec^2(\theta_0) x_f \quad (23)$$

$$= \frac{dx_f}{d\theta_0} \left(\frac{1}{2(H-h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right) \quad (24)$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{- \frac{(\cos^{-3}(\theta_0) \sin(\theta_0))}{2(H-h_0)} x_f^2 - \sec^2(\theta_0) x_f}{\left(\frac{1}{2(H-h_0)} x_f \cos^{-2}(\theta_0) + \tan(\theta_0) \right)} \quad (25)$$

3.2.3 | Optimizing for x_f for θ_0 via $\frac{dx_f}{d\theta_0}$

We now set $\frac{dx_f}{d\theta_0} = 0$ in order to figure critical points for the value of x_f .

$$\frac{dx_f}{d\theta_0} = \frac{-\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{1}{2(H-h_0)}x_f\cos^{-2}(\theta_0) + \tan(\theta_0)\right)} \quad (26)$$

$$\Rightarrow 0 = \frac{-\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f}{\left(\frac{1}{2(H-h_0)}x_f\cos^{-2}(\theta_0) + \tan(\theta_0)\right)} \quad (27)$$

$$\Rightarrow 0 = -\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 - \sec^2(\theta_0)x_f \quad (28)$$

$$\Rightarrow \sec^2(\theta_0)x_f = -\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f^2 \quad (29)$$

$$\Rightarrow \sec^2(\theta_0) = -\frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{2(H-h_0)}x_f \quad (30)$$

$$\Rightarrow 2\sec^2(\theta_0)(H-h_0) = -(\cos^{-3}(\theta_0)\sin(\theta_0))x_f \quad (31)$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{(\cos^{-3}(\theta_0)\sin(\theta_0))}{\sec^2(\theta_0)} \quad (32)$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos^3(\theta_0)\sec^2(\theta_0)} \quad (33)$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{\sin(\theta_0)}{\cos(\theta_0)} \quad (34)$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \tan(\theta_0) \quad (35)$$

$$\Rightarrow \theta_0 = \arctan\left(\frac{-2(H-h_0)}{x_f}\right) \quad (36)$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize x_f for a given solved x_f . To figure the actual statement that would optimize for both,

3.3 | Solving and optimizing for x_f

We will now return to our original expression for the final y-position ($= 0$) to create an expression for x_f .

3.3.1 | Solving for x_f

We first take the previous expression for x_f and supply the expression for θ_0 .

$$0 = \frac{-1}{4} \frac{-1}{(H - h_0)} x_f^2 \cos^{-2}(\theta_0) + x_f \tan(\theta_0) + h_0 \quad (37)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{-1}{(H - h_0)} x_f^2 \cos^{-2}(\arctan(\frac{-2(H - h_0)}{x_f})) + x_f \tan(\arctan(\frac{-2(H - h_0)}{x_f})) \quad (38)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{-1}{(H - h_0)} x_f^2 \cos^{-2}(\arctan(\frac{-2(H - h_0)}{x_f})) + x_f \frac{-2(H - h_0)}{x_f} \quad (39)$$

$$\Rightarrow 0 = \frac{-1}{4} \frac{-1}{(H - h_0)} x_f^2 ((\frac{-2(H - h_0)}{x_f})^2 + 1) + x_f \frac{-2(H - h_0)}{x_f} \quad (40)$$

$$\Rightarrow 0 = \frac{1}{4(H - h_0)} (-2(H - h_0))^2 + x_f^2) + -2(H - h_0) \quad (41)$$

$$\Rightarrow 0 = \frac{-(H - h_0)}{2} + \frac{x_f^2}{4(H - h_0)} + -2(H - h_0) \quad (42)$$

$$\Rightarrow \frac{x_f^2}{4(H - h_0)} = \frac{(H - h_0)}{2} + 2(H - h_0) \quad (43)$$

$$\Rightarrow x_f^2 = 2(H - h_0)^2 + 8(H - h_0)^2 \quad (44)$$

$$\Rightarrow x_f^2 = 10(H - h_0)^2 \quad (45)$$

3.3.2 | Finding $\frac{dx_f^2}{dh_0}$

We know that, by optimizing for x_f^2 , x_f is optimized due to the setup of the problem of the behavior of the length of line.

Hence, we take the *first* derivative, though of x_f^2 w.r.t. h_0 and with H held constant.

$$x_f^2 = 10(H - h_0)^2 \quad (46)$$

$$\Rightarrow \frac{dx_f^2}{dh_0} = \frac{d}{dh_0} 10(H - h_0)^2 \quad (47)$$

$$= 10 \frac{d}{dh_0} (H - h_0)^2 \quad (48)$$

$$= 10 \times -2(H - h_0) \quad (49)$$

$$= -20(H - h_0) \quad (50)$$

3.3.3 | Optimizing for x_f^2 via h_0 via $\frac{dx_f^2}{dh_0}$

We again set the value of $\frac{dx_f^2}{dh_0} = 0$ and solve for critical points.

$$x_f^2 = -20(H - h_0) \quad (51)$$

$$\Rightarrow 0 = -20(H - h_0) \quad (52)$$

$$(53)$$