1 | **Problem 1**

The plane that passes through the vector $\vec{P}_o$ and is perpendicular to $\vec{n}$ is defined by:

$$\{\vec{r} : (\vec{r} - \vec{P}_o) \cdot \vec{n} = 0, \vec{P}_o \in \mathbb{R}, \vec{n} \in \mathbb{R}\}$$

This works because:

- The $(\vec{r} - \vec{P}_o)$ term is similar to when you subtract some value from x (in a cartisian plane) to shift it over to the right. In a senes, here you are subtracting the postition vector from every single vector on the plane defined by $\vec{r}$, thus shifting the plane to $\vec{P}_o$.
- Setting the dot product between the term above and $\vec{n}$ to 0, ensures that the plane is perpendicular to the normal vector $\vec{n}$

2 | **Problem 2**

First we can defined:

- $\vec{n} = (n_x, n_y, n_z)$
- $\vec{P}_o = (P_{ox}, P_{oy}, P_{oz})$
- $\vec{r} = (x, y, z)$

Then we can evaluate: $(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$:

$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$
so we see that the cartisian definition of a plane is: $xn_x + yn_y + zn_z = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$

From this we see:

- $A = n_x$
- $B = n_y$
- $C = n_z$
- $D = P_{ox}n_x + P_{oy}n_y + P_{oz}n_z$

Therefore, the normal vector is (A, B, C)

3 | **Problem 3**

First we can define:

- $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle$
- $\vec{r} = \langle x, y, z \rangle$

Then we can evaluate the equation:

$\hat{n} \cdot \vec{r} = D$
$\Rightarrow \langle \frac{n_x}{|\vec{n}|}, \frac{n_y}{|\vec{n}|}, \frac{n_z}{|\vec{n}|} \rangle \cdot \langle x, y, z \rangle = D$
$\Rightarrow \frac{n_x}{|\vec{n}|}x + \frac{n_y}{|\vec{n}|}y + \frac{n_z}{|\vec{n}|}z = D$

This is the equation for a cartisian definition of a plane, with $\hat{n}$ representing the normal vector and D representing $\frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$ which is the distance from the origin to the plane.

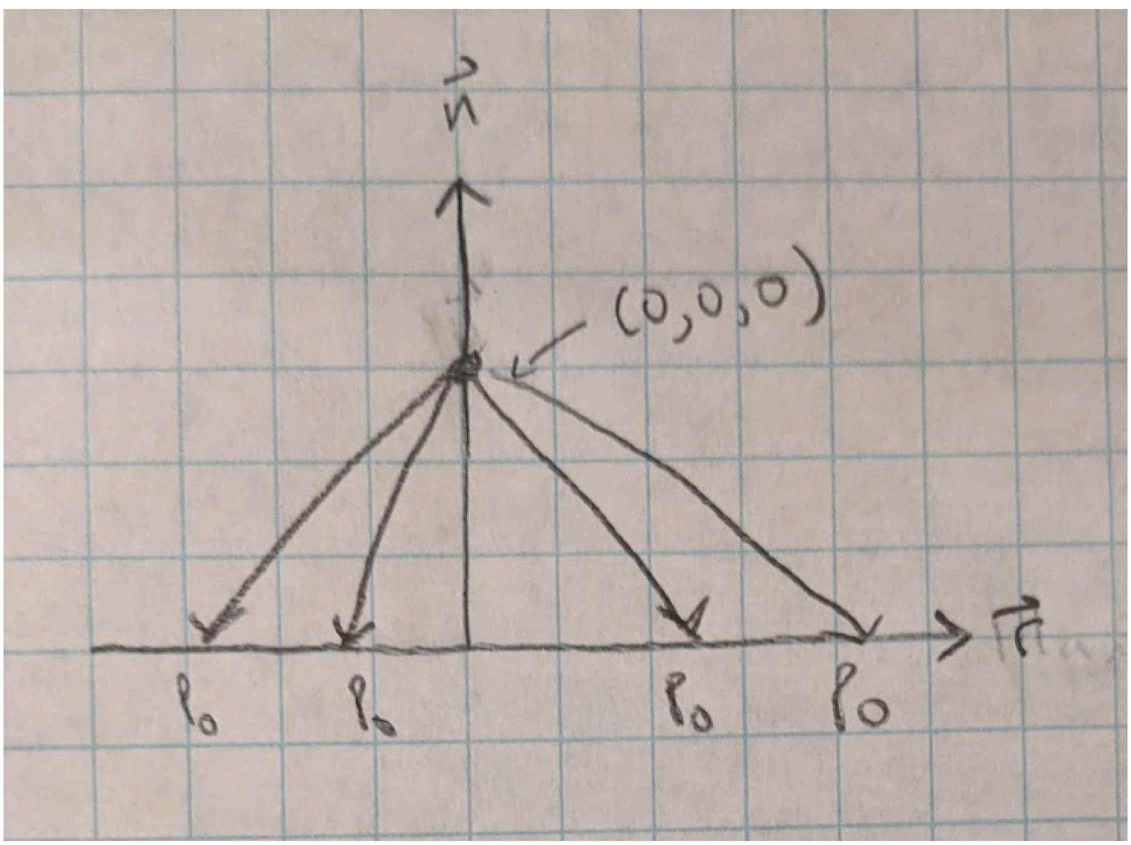
The value of D is found by starting with the orginal vector definition of a plane:

$(\vec{r} - \vec{P}_o) \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} - \vec{P}_o \cdot \vec{n} = 0$
$\Rightarrow \vec{r} \cdot \vec{n} = \vec{P}_o \cdot \vec{n}$
$\Rightarrow \frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow \vec{r} \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$\Rightarrow D = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

Looking at problem 4 we see that this is the distance from the plane to the origin.

4 | **Problem 4**

We can start with this drawing:



In the image we see that there are multiple "point $\vec{P}_o$'s" that go from point P_o to the plane. We are trying to solve for d which is the shortest distance from the plane to the point, it can also be defined as the length of the $\vec{P}_o$ that is perpendiclar to the plane. Because all of the $\vec{P}_o$'s come from the same point and $\vec{n}$ is perpendicular to the plane we can find d by finding $comp_{\hat{n}}\vec{P}_o$:

$d = comp_{\hat{n}}\vec{P}_o = 1 \cdot |\vec{P}_o| \cos(\theta)$
$= |\hat{n}||\vec{P}_o| \cos(\theta)$
$= \vec{P}_o \cdot \hat{n}$
$= \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$

From problem 2 we know that $\vec{P}_o \cdot \vec{n} = 4$ and that $\vec{n} = \langle 1, 2, 3 \rangle$ and thus:

$$d = \frac{4}{\sqrt{14}}$$

5 | Problem 5

We can do something similar to what was done in the problem above in which the $\hat{n}$ component of $\vec{P}_o$ can be used as the distance d, but because $\vec{P}_o$ is center at the origin and the plane may not pass through the origin, the postion vector of the plane, we'll call it $\vec{P}_1$, has to be subtracted from $\vec{P}_o$. To find the length of $\vec{P}_1$ we can do what we did in problem 4.

We'll break this problem into three parts: 1) finding the length of $\vec{P}_o$ parallel to $\hat{n}$ 2) findng the distance of $\vec{P}_1$ parallel to $\hat{n}$ 3) subtracting the two:

5.1 | Finding length of $\vec{P}_o$

From problem 2 we know that $\vec{n} = \langle A, B, C \rangle$. From problem 4 we know that we can find the length of $\vec{P}_o$ parallel to $\hat{n}$ by:

$\vec{P}_o \cdot \hat{n} = \frac{\vec{P}_o \cdot \vec{n}}{|\vec{n}|}$
$= \frac{\langle x_o, y_o, z_o \rangle \cdot \langle A, B, C \rangle}{|\langle A, B, C \rangle|}$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2 + B^2 + C^2}}$

5.2 | Finding the length of $\vec{P}_1$

From problem 2 we know that $\vec{n} = \langle A, B, C \rangle$. From problem 4 we know that we can find the length of $\vec{P}_1$ parallel to $\hat{n}$ by:

$$\vec{P}_1 \cdot \hat{n} = \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$$

From problem 2 we know that the dot product between the position vector and the noraml vector is D. Thus:

length of$\vec{P}_1 = \vec{P}_1 \cdot \hat{n}$
$= \frac{\vec{P}_1 \cdot \vec{n}}{|\vec{n}|}$
$= \frac{D}{|\vec{n}|}$
$= \frac{D}{|\langle A, B, C \rangle|}$
$= \frac{D}{\sqrt{A^2 + B^2 + C^2}}$

5.3 | Subtratcting the two:

$d = \vec{P}_o - \vec{P}_1$
$= \frac{Ax_o + By_o + Cz_o}{\sqrt{A^2+B^2+C^2}} - \frac{D}{\sqrt{A^2+B^2+C^2}}$
$= \frac{Ax_o + By_o + Cz_o - D}{\sqrt{A^2+B^2+C^2}}$

Lastly, you would take the absolute value of the numorator because distances are positive (and the denominator is already positive due to the squaring of A, B, and C). Thus:

$$d = \frac{|Ax_o + By_o + Cz_o - D|}{\sqrt{A^2+B^2+C^2}}$$

6 | Problem 6

First we can define:

- $\vec{A}(t) = A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}$
- $\vec{B}(t) = B_x(t)\hat{i} + B_y(t)\hat{j} + B_z(t)\hat{k}$

Thus:

$\frac{d}{dt}(\vec{A}(t) \cdot \vec{B}(t)) = \frac{d}{dt}(A_x(t)B_x(t) + A_y(t)B_y(t) + A_z(t)B_z(t))$
$= \frac{d}{dt}(A_x(t)B_x(t)) + \frac{d}{dt}(A_y(t)B_y(t)) + \frac{d}{dt}(A_z(t)B_z(t))$
$= A'_x(t)B_x(t) + A_x(t)B'_x(t) + A'_y(t)B_y(t) + A_y(t)B'_y(t) + A'_z(t)B_z(t) + A_z(t)B'_z(t)$
$= (A'_x(t)B_x(t) + A'_y(t)B_y(t) + A'_z(t)B_z(t)) + (A_x(t)B'_y(t) + A_y(t)B'_y(t) + A_z(t) + B'_z(t))$
$= \vec{A}'(t) \cdot \vec{B}(t) + \vec{A}(t) \cdot \vec{B}'(t)$
$= \frac{d\vec{A}(t)}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}(t)}{dt}$

7 | Problem 7

First we can define:

- $\vec{r} = |\vec{r}|\hat{r}$
  - this works because the unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, and so you can multiply both sides by $|\vec{r}|$

We can start by finding the derivative of $\vec{r}$ with the definition above:

$\frac{d}{dt}\vec{r}(t) = \frac{d}{dt}(|\vec{r}(t)|\hat{r}(t))$
$= (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$

We can take this equation and solve for $\frac{d}{dt}|\vec{r}(t)|$:

$\frac{d}{dt}\vec{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t))$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \hat{r}(t) = (\frac{d}{dt}|\vec{r}(t)|)\hat{r}(t) \cdot \hat{r}(t) + |\vec{r}(t)|(\frac{d}{dt}\hat{r}(t)) \cdot \hat{r}(t)$
$\Rightarrow \frac{d}{dt}\vec{r}(t) \cdot \frac{\vec{r}(t)}{|\vec{r}(t)|} = (\frac{d}{dt}|\vec{r}(t)|) \cdot 1 + |\vec{r}(t)| \cdot 0$
$\Rightarrow \frac{1}{|\vec{r}(t)|}\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}|\vec{r}(t)|$

This means the $\hat{r}$ component of the $\frac{d\vec{r}}{dt}$ is equal to the rate of change of the magnitude of $\vec{r}$.

This proof relied on the fact that the dot product between a vectors of constant length and it's derivative is zero. To prove that this is true let's first think about the group of vectors of a constant length. In other words:

$$\{\vec{r} : |\vec{r}| = c, c \in \mathbb{R}\}$$

This group of vectors would form a sphere. Now we can think about the definition of an derivative:

$$\frac{d}{dt}\vec{r}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

Because $\vec{r}(t)$ is of constant length we know that and value of $\vec{r}(t)$ will be on the circle, and any vector that has its tip on the sphere will be perpendicular to a plane tangent to the sphere (the tangent plane).

We also know that $\Delta\vec{r}(t)$ is the vector between the two tips of $\vec{r}(t + \Delta t)$ and $\vec{r}(t)$ (due to the definition of vector subtraction).

Now, let's say that $\vec{r}(t)$ is perpendicular to plane A. As Δt approches zero, $\vec{r}(t + \Delta t)$ will become closer and closer to becoming the same as $\vec{r}(t)$. Thus, $\Delta\vec{r}(t)$ will get closer and closer to being on plane A, which is perpendicular to $\vec{r}(t)$, yielding a dot product of zero.

This proof also relied on the fact that the product rule still holds between a vector function and a scalar function. Here is why that is true:

$\frac{d}{dt}(f(t) \cdot \vec{r}(t)) = \frac{d}{dt}(f(t) \cdot r_x(t), f(t) \cdot r_y(t), f(t) \cdot r_x(t))$
$= (f(t)r'_x(t) + f'(t)r_x(t), f(t)r'_y(t) + f'(t)r_y(t), f(t)r'_z(t) + f'(t)r_z(t))$
$= (f(t)r'_x(t), f(t)r'_y(t), f(t)r'_z(t)) + (f'(t)r_x(t), f'(t)r_y(t), f'(t)r_z(t))$
$= f(t)(r'_x(t), r'_y(t), r'_z(t)) + f'(t)(r_x(t), r_y(t), r_z(t))$
$= f(t)\frac{d\vec{r}(t)}{dt} + \frac{df(t)}{dt}\vec{r}(t)$

Note: for this problem, I got Albert's help with the initial proof. The proof that the dot product between a vector of constant length and its derivative was gotten from in class.

8 | **Problem 8**

If we start with the vector equation for a 3D line we get:

$$(x, y, z) = (x_o, y_o, z_o) + t(a, b, c) = (x_o + ta, y_o + tb, z_o + tc)$$

Where (x_o, y_o, z_o) is the position vector of the line (the line passes through this point), and (a, b, c) is the direction in which the line is traveling.

We can take the x, y and z components of the vector form and convert them into the parametric form of the equation of a 3D line:

$(x, y, z) = (x_o + ta, y_o + tb, z_o + tc)$
$\Rightarrow x = x_o + ta$
$\Rightarrow y = y_o + tb$
$\Rightarrow z = z_o + tc$

Lastly, we can solve each of the parameterized components for t and set them equal to each other to get the symmetric form:

$\Rightarrow t = \frac{x - x_o}{a}$
$\Rightarrow t = \frac{y - y_o}{b}$
$\Rightarrow t = \frac{z - z_o}{c}$
$\Rightarrow \frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$

With this in mind we can look at the given symmetric equation:

$\frac{x-2}{2} = \frac{y-1}{3} = 2 - z$
$\Rightarrow \frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{-1}$

From this we see that the position vector is $(2,1,2)$ and the direction of the line is the same direction as $(2,3,-1)$

Similar to finding the distance between the plane and the origin we will need the normal vector. The dot product between the normal vector and the vector that describes the directing of the line must equal zero thus:

$\vec{n} \cdot \vec{d_o} = 0$
$(n_x, n_y, n_z) \cdot (2,3,-1) = 0$
$2n_x + 3n_y - n_z = 0$
$2n_x + 3n_y = n_z$

There are many was that two vectors can be perpendicular, but in this case the normal vector also has to pass through a poin on the line described above. Thus:

$2(2+2t) + 3(1+3t) = 2 - t$
$\Rightarrow 4 + 4t + 3 + 9t = 2 - t$
$\Rightarrow 14t = -5$
$\Rightarrow t = -\frac{5}{14}$
$\Rightarrow (n_x, n_y, n_z) = (2 - \frac{10}{14}, 1 - \frac{15}{14}, 2 + \frac{5}{14}) = (\frac{18}{14}, \frac{-1}{14}, \frac{33}{14})$

Because we only need the unit vector of the normal vector (see problem 4), we can redefine it with nicer numbers by multiplying it by a constant (nicer numbers but direction is maintained):

$$\vec{n} = (18, -1, 33)$$

Therefore:

$d = \frac{\vec{P_o} \cdot \vec{n}}{|\vec{n}|}$
$= \frac{36-1+66}{\sqrt{18^2+(-1)^2+33^2}}$
$= \frac{101}{\sqrt{1414}}$

Note: for this problem I used this source to see what the different forms of the equation for a 3D line: but I proved that the vector form equaled the symmetric form myself: `https://math.stackexchange.com/questions/404440/what-is-the-equation-for-a-3d-line`.

9 | **Problem 9**

First we can rewrite the two equations of the plane as:

1. $Ax + By + Cz = D_1$
2. $A_x + B_y + C_z = D2$

Then we can find the distance between the origin and the first plane using the equation from problem 4:

We know that the normal vector is (A, B, C) and that the dot product between the normal vector and the position vector is D_1. Therefore:

$d_1 = \vec{P}_1 \cdot \hat{n}_1$
$= \frac{\vec{P}_1 \cdot \vec{n}_1}{|\vec{n}_1|}$
$= \frac{D_1}{|(A,B,C)|}$
$= \frac{D_1}{|A\hat{i}+B\hat{j}+C\hat{k}|}$

Next we can find the distance between the origin and the second plane using the equation from problem 4:

We know that the normal vector is (A, B, C) and that the dot product between the normal vector and the position vector is D_2. Therefore:

$d_2 = \vec{P}_2 \cdot \hat{n}_2$
$= \frac{\vec{P}_2 \cdot \vec{n}_2}{|\vec{n}_2|}$
$= \frac{D_2}{|(A,B,C)|}$
$= \frac{D_2}{|A\hat{i}+B\hat{j}+C\hat{k}|}$

Lastly, because the tow planes have the same normal vector, they are parallel. This means we can find the distance between the two planes by subtracting the distances between each plane and the origin:

We have to take the absolute value because distance cannot be negative:

$d = |d_1 - d_2|$
$= |\frac{D_1}{|A\hat{i}+B\hat{j}+C\hat{k}|} - \frac{D_2}{|A\hat{i}+B\hat{j}+C\hat{k}|}|$
$= \frac{|D_1 - D_2|}{|A\hat{i}+B\hat{j}+C\hat{k}|}$