1 | Setup

The ball launcher problem involves an energetic optimization to figure, given the situation as shown in the above image, the parameters h_0 and θ_0 that would best create a maximum launch distance x_f.

In this problem, we will define the axis such that the "lower-left" corner of the wood block (corner sharing x value of the starting position of the marble, but on the "ground") as $(-w, 0)$, where w is the width of the wooden block. Therefore, we derive the x-value of the location of the launch of the projectile as $x = 0$. We define the direction towards with the marble is launching as positive-x, so as the marble rolls, its position's x value increases. We will define the location of the marble before starting as positive y, and as the marble decreases in height, its position's y value decreases.

We define the start of the experiment as time t_0, the moment the marble leaves the track and travels as a projectile as t_1, and the end — in the moment when the marble hits the ground — as t_f. We will call the marble m_0.

2 | Figuring the Velocity at t_1

In order to expedite the process of derivation, we will leverage an energetic argument instead of that of kinematics for figuring the velocity at launch. The change-in-height that m_0 experiences before t_1 is $\Delta h = H - h_0$. Therefore, the potential energy expenditure is $\Delta PE_{grav} = mg\Delta h = m_0 g(H - h_0)$. Assuming that the marble starts out with 0 kinetic energy, we deduct that, at the moment of it finishing its descent, it will possess kinetic energy $KE = 0 + m_0 g(H - h_0) = m_0 g(H - h_0)$.

For this derivation, for now, we ignore $KE_{rotational}$, hence, we could roughly deduct the statement that $KE_{translational} \approx m_0 g(H - h_0)$.

Creating this statement, we could deduct a statement that we could leverage to solve for the velocity at t_1 named $\vec{v_0}$.

$$m_0 g(H - h_0) = \frac{1}{2} m_0 \vec{v_0}^2 \tag{1}$$

$$g(H - h_0) = \frac{1}{2} \vec{v_0}^2 \tag{2}$$

$$2g(H - h_0) = \vec{v_0}^2 \tag{3}$$

$$\vec{v_0} = \sqrt{2g(H - h_0)} \tag{4}$$

This velocity vector could be easily split into its two constituent parts. Namely:

$$\begin{cases} \vec{v_{0x}} = \sqrt{2g(H - h_0)} cos(\theta_0) \\ \vec{v_{0y}} = \sqrt{2g(H - h_0)} sin(\theta_0) \end{cases}$$

3 | Figuring the Maximum Possible Travel Distance

Here, we devise an function for x_f w.r.t. $\vec{v_{0y}}$, $\vec{v_{0x}}$, h_0, m_0.

3.1 | Setup for Kinematics

We first will leverage the parametric equations for position in kinematics in order to ultimately result in a function for x_f.

$$\begin{cases} x(t) = \frac{1}{2}a_{0x}t^2 + v_{0x}t + x_0 \\ y(t) = \frac{1}{2}a_{0y}t^2 + v_{0y}t + y_0 \end{cases}$$

Given the situation of our problem, we could modify the pair as follows:

$$\begin{cases} x(t) = v_{0x}t \\ y(t) = \frac{-1}{2}gt^2 + v_{0y}t + h_0 \end{cases}$$

as…

- there are no acceleration in the x-direction at the point of launch
- the only acceleration in the y-direction is that due to gravity
- the start x-position of the marble at launch is, as defined above, $x = 0$
- the start y-position of the marble at launch is, as defined above, $y = h_0$

3.2 | Solving and optimizing for $\frac{dx_f}{d\theta_0}$

We need to maximize $\frac{dx_f}{d\theta_0}$ as one out of two components to optimize for. Once we figure that value, we then supply the corresponding maximum value then optimize again for $\frac{h_0}{d\theta_0}$. The position equations above could be leveraged to figure a value for x_f.

3.2.1 | Setup for Solution

We first create a set of equations modeling the location of the marble at t_f.

$$\begin{cases} x(t_f) = x_f = v_{0x}t_f = t_f\sqrt{2g(H-h_0)}cos(\theta_0) \\ y(t_f) = 0 = \frac{-1}{2}g{t_f}^2 + v_{0y}t_f + h_0 = \frac{-1}{2}g{t_f}^2 + t_f\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \end{cases}$$

We first solve for t_f, and supply it to the first equation.

$$t_f = \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)} \tag{5}$$

Finally, we substitute the definition of t_f into $y(t_f)$.

$$y(t_f) = 0 = \frac{-1}{2}g\frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}^2 + \frac{x_f}{\sqrt{2g(H-h_0)}cos(\theta_0)}\sqrt{2g(H-h_0)}sin(\theta_0) + h_0 \tag{6}$$

We will now proceed to simplify the expression further

$$\begin{aligned} 0 =& \frac{-1}{4}\frac{-{x_f}^2}{(H-h_0)cos^2(\theta_0)} + x_f tan(\theta_0) + h_0 &(7)\\ =& \frac{-1}{4}\frac{-{x_f}^2}{(H-h_0)}cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 &(8)\\ =& \frac{-1}{4}\frac{-1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 &(9) \end{aligned}$$

3.2.2 | Finding $\frac{dx_f}{d\theta_0}$

We leverage implicit differentiation to figure a value for $\frac{dx_f}{d\theta_0}$. We set x_f as a differentiable function, and h_0 and H as both constants.

$$0 = \frac{-1}{4}\frac{-1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 \tag{10}$$

$$\Rightarrow \frac{d}{d\theta_0}0 = \frac{d}{d\theta_0}(\frac{1}{4}\frac{1}{(H-h_0)}{x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0) \tag{11}$$

$$\Rightarrow 0 = \frac{1}{4}\frac{1}{(H-h_0)}\frac{d}{d\theta_0}{x_f}^2 cos^{-2}(\theta_0) + \frac{d}{d\theta_0}x_f tan(\theta_0) + \frac{d}{d\theta_0}h_0 \tag{12}$$

$$\Rightarrow 0 = \frac{1}{4}\frac{1}{(H-h_0)}((\frac{d}{d\theta_0}{x_f}^2)cos^{-2}(\theta_0) + {x_f}^2(\frac{d}{d\theta_0}cos^{-2}(\theta_0)))+ \tag{13}$$

$$((\frac{d}{d\theta_0}x_f)tan(\theta_0) + (\frac{d}{d\theta_0}tan(\theta_0))x_f) + 0 \tag{14}$$

$$\Rightarrow 0 = \frac{1}{4(H-h_0)}((2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + {x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)))+ \tag{15}$$

$$(\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f) \tag{16}$$

$$\Rightarrow 0 = \frac{1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0))+ \tag{17}$$

$$\frac{dx_f}{d\theta_0}tan(\theta_0) + sec^2(\theta_0)x_f \tag{18}$$

$$\Rightarrow -\frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{19}$$

$$= \frac{1}{4(H-h_0)}(2x_f\frac{dx_f}{d\theta_0})cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{20}$$

$$\Rightarrow -\frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{21}$$

$$= \frac{dx_f}{d\theta_0}\frac{1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + \frac{dx_f}{d\theta_0}tan(\theta_0) \tag{22}$$

$$\Rightarrow -\frac{1}{4(H-h_0)}{x_f}^2(2cos^{-3}(\theta_0)sin(\theta_0)) - sec^2(\theta_0)x_f \tag{23}$$

$$= \frac{dx_f}{d\theta_0}(\frac{1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0)) \tag{24}$$

$$\Rightarrow \frac{dx_f}{d\theta_0} = \frac{-\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{25}$$

3.2.3 | Optimizing for x_f for θ_0 via $\frac{dx_f}{d\theta_0}$

We now set $\frac{dx_f}{d\theta_0} = 0$ in order to figure critical points for the value of x_f.

$$\frac{dx_f}{d\theta_0} = \frac{-\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{26}$$

$$\Rightarrow 0 = \frac{-\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f}{(\frac{1}{2(H-h_0)}x_f cos^{-2}(\theta_0) + tan(\theta_0))} \tag{27}$$

$$\Rightarrow 0 = -\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 - sec^2(\theta_0)x_f \tag{28}$$

$$\Rightarrow sec^2(\theta_0)x_f = -\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}{x_f}^2 \tag{29}$$

$$\Rightarrow sec^2(\theta_0) = -\frac{(cos^{-3}(\theta_0)sin(\theta_0))}{2(H-h_0)}x_f \tag{30}$$

$$\Rightarrow 2sec^2(\theta_0)(H-h_0) = -(cos^{-3}(\theta_0)sin(\theta_0))x_f \tag{31}$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{(cos^{-3}(\theta_0)sin(\theta_0))}{sec^2(\theta_0)} \tag{32}$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{sin(\theta_0)}{cos^3(\theta_0)sec^2(\theta_0)} \tag{33}$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = \frac{sin(\theta_0)}{cos(\theta_0)} \tag{34}$$

$$\Rightarrow \frac{-2(H-h_0)}{x_f} = tan(\theta_0) \tag{35}$$

$$\Rightarrow \theta_0 = arctan(\frac{-2(H-h_0)}{x_f}) \tag{36}$$

As there is one critical point per the range, and that there must be at least one maximum point, we determine that the derived expression will maximize x_f for a given solved x_f. To figure the actual statement that would optimize for both,

3.3 | Solving and optimizing for x_f

We will now return to our original expression for the final y-position ($= 0$) to create an expression for x_f.

3.3.1 | Solving for x_f

We first take the previous expression for x_f and supply the expression for θ_0.

$$
\begin{aligned}
0 =& \frac{-1}{4} \frac{-1}{(H-h_0)} {x_f}^2 cos^{-2}(\theta_0) + x_f tan(\theta_0) + h_0 && (37)\\
\Rightarrow 0 =& \frac{-1}{4} \frac{-1}{(H-h_0)} {x_f}^2 cos^{-2}(arctan(\frac{-2(H-h_0)}{x_f})) + x_f tan(arctan(\frac{-2(H-h_0)}{x_f}) && (38)\\
\Rightarrow 0 =& \frac{-1}{4} \frac{-1}{(H-h_0)} {x_f}^2 cos^{-2}(arctan(\frac{-2(H-h_0)}{x_f})) + x_f \frac{-2(H-h_0)}{x_f} && (39)\\
\Rightarrow 0 =& \frac{-1}{4} \frac{-1}{(H-h_0)} {x_f}^2 ((\frac{-2(H-h_0)}{x_f})^2 + 1) + x_f \frac{-2(H-h_0)}{x_f} && (40)\\
\Rightarrow 0 =& \frac{1}{4(H-h_0)} (-2(H-h_0))^2 + {x_f}^2) + -2(H-h_0) && (41)\\
\Rightarrow 0 =& \frac{-(H-h_0)}{2} + \frac{{x_f}^2}{4(H-h_0)} + -2(H-h_0) && (42)\\
\Rightarrow \frac{{x_f}^2}{4(H-h_0)} =& \frac{(H-h_0)}{2} + 2(H-h_0) && (43)\\
\Rightarrow {x_f}^2 =& 2(H-h_0)^2 + 8(H-h_0)^2 && (44)\\
\Rightarrow {x_f}^2 =& 10(H-h_0)^2 && (45)
\end{aligned}
$$

3.3.2 | Optimizing for ${x_f}^2$ via h_0 via $\frac{d{x_f}^2}{dh_0}$

We know that, by optimizing for ${x_f}^2$, x_f is optimized due to the setup of the problem of the behavior of the length of line.

Hence, we take the *first* derivative, though of ${x_f}^2$ w.r.t. h_0 and with H held constant.

$$
\begin{aligned}
{x_f}^2 =& 10(H-h_0)^2 && (46)\\
\Rightarrow \frac{d{x_f}^2}{dh_0} =& \frac{d}{dh_0} 10(H-h_0)^2 && (47)\\
=& 10 \frac{d}{dh_0} (H-h_0)^2 && (48)\\
=& 10 \times -2(H-h_0) && (49)\\
& && (50)
\end{aligned}
$$