11 | **cubic and a line**

11.1 | **show tangency**

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 - x^3) = 8x - 3x^2\Big|_3$$
$$= 24 - 27 = -3$$
$$\frac{dy}{dx} = \frac{d}{dx}(18 - 3x) = -3$$

11.2 | **area between curves**

$$\int_3^6 18 - 3x - 4x^2 + x^3 dx \to \frac{1}{4}x^4 - \frac{1}{3}4x^3 - \frac{1}{2}3x^2 + 18x + C$$
$$= \frac{1}{4}(6)^4 - \frac{1}{3}4(6)^3 - \frac{1}{2}3(6)^2 + 18(6)$$
$$- \frac{1}{4}(3)^4 + \frac{1}{3}4(3)^3 + \frac{1}{2}3(3)^2 - 18(3)$$
$$= \boxed{\frac{261}{4}}$$

12 | **estimate area**

Right handed Riemann Sum:
$$0.5 + 4 + 10 + 13 + 10 + 0 = 37.5$$

13 | **estimate area again**

$$4(200 + 2700 + 1100 + 4000 + 200) = 32800$$

14 | **area between curves**

$$\int_0^{10} 2200e^{0.024t} dx - \int_0^{10} 1360e^{0.018t} dx = \frac{1}{0.024}2200e^{0.024t} - \frac{1}{0.018}1360e^{0.018t}$$
$$\implies \quad \frac{1}{0.024}2200e^{0.24} - \frac{1}{0.018}1360e^{0.18} - \frac{1}{0.024}2200 + \frac{1}{0.018}1360 \quad \approx 9964$$

The area represents the population over those ten years.

15 | **meaning of area**

The shaded region represents the profit made between producing 50 units and 100 units.

16 | **slicing pizza into three using parallel cuts**

The problem of placing slices is the same if we only worry about the top half of the pizza. Thus, we can choose some x for the first slice s.t.

$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt = \int_{x}^{7}\sqrt{7^2-t^2}dt$$

$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt - \int_{x}^{7}\sqrt{7^2-t^2}dt = 0$$

$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt + \int_{7}^{x}\sqrt{7^2-t^2}dt = 0$$

$$2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{-7}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) = 0$$

$$2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) = 0$$

$$2\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt + \int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt = 0$$

$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt = 0$$

$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt = 0$$

$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + \frac{49\pi}{4} = 0$$

Now, we need to use trigonometric substitution, apparently.

$$x = a\sin\theta, dx = a\cos\theta d\theta$$

Or, you could use David's method, which is just better (cut horizontally instead of vertically)

$$\int_{-7}^{7}\sqrt{49-x^2} - adx = \frac{49\pi}{3}$$

$$\int_{-7}^{7}\sqrt{49-x^2}dx - \int_{-7}^{7}adx = \frac{49\pi}{3}$$

$$\frac{49\pi}{2} - \int_{-7}^{7}adx = \frac{49\pi}{3}$$

$$\frac{49\pi}{2} - (7a - -7a) = \frac{49\pi}{3}$$

$$\frac{49\pi}{6} = 14a$$

$$a = \frac{49\pi}{84} = \frac{7\pi}{12}$$

Since a is the upper half of the center portion, the width of each slice is $2a = \frac{7\pi}{6}$

17 | **tractrix**

17.1 | **derivative**

At any moment, if the boat is at (x, y) and the puller is at $(0, h)$, then the velocity of the boat is in the direction

$$\frac{\Delta y}{\Delta x}$$

Where $\Delta x = -x$ and Δy can be found using the Pythagorean Theorem

$$L^2 = \Delta y^2 + x^2$$
$$\implies \quad \Delta y = \sqrt{L^2 - x^2}$$

Thus, the boat is moving in the direction

$$\frac{\sqrt{L^2 - x^2}}{-x}$$

17.2 | **integral**

$$\int \frac{\sqrt{L^2 - x^2}}{-x} dx = -\int \frac{1}{x} \sqrt{L^2 - x^2} dx$$

Let $x = L \sin\theta, dx = L \cos\theta d\theta$

$$= -\int \frac{1}{L \sin\theta} \sqrt{L^2 - L^2 \sin^2\theta} dx$$
$$= -\int \frac{1}{L \sin\theta} L \sqrt{1 - \sin^2\theta} dx$$
$$= -\int \frac{1}{L \sin\theta} L \sqrt{\cos^2\theta} dx$$
$$= -\int \frac{L \cos\theta}{L \sin\theta} L \cos\theta d\theta$$
$$= -\int L \frac{\cos^2\theta}{\sin\theta} d\theta$$
$$= -L \int \frac{1}{\sin\theta} d\theta + L \int \sin\theta d\theta$$
$$= L \ln|\csc\theta + \cot\theta| - L \cos\theta + C$$
$$= L \ln\left|\frac{L}{x} + \frac{\sqrt{L^2 - x^2}}{x}\right| - \sqrt{L^2 - x^2} + C$$

18 | **TODO water displacement**

Plan: find a function $f(r)$ which represents the amount of water displaced for any radius, then take the derivative and find roots.