1 | **Elastic collision**

We are given that the object m_1 collides with the rod with velocity v_0, and the rod is floating in free space. Given m_1, v_0, m_2, I_0, and r, we are to figure to the final velocity of m_1 after collision v_f, the velocity of m_2 after collision v_{CM}, and of course the rotation of the rod after collision ω.

We are assuming that this collision elastic.

We have, then, for conservation of linear momentum:

$$m_1 v_0 = m_1 v_f + m_2 v_{CM} \tag{1}$$

Furthermore, we understand that kinetic energy is also conserved here; therefore:

$$\frac{1}{2}m_1 {v_0}^2 + \frac{1}{2}m_1 {v_0}^2 = \left(\frac{1}{2}m_1 {v_f}^2\right) + \left(\frac{1}{2}m_1 {v_f}^2\right) + \left(\frac{1}{2}m_2 {v_{CM}}^2\right) + \left(\frac{1}{2}I_0 \omega^2\right) \tag{2}$$

$$\Rightarrow 2m_1 {v_0}^2 = \left(2m_1 {v_f}^2\right) + \left(m_2 {v_{CM}}^2\right) + \left(I_0 \omega^2\right) \tag{3}$$

as the point mass does not have any rotational inertia, and the rod is not rotating at the start.

Lastly, we understand that the angular momentum is conserved through a collision; letting the origin as the center of mass of the rod:

$$m_1 r^2 \left(\frac{v_0}{r}\right) = m_1 r^2 \left(\frac{v_f}{r}\right) + I_0 \omega \tag{4}$$

$$\Rightarrow m_1 r v_0 = m_1 r v_f + I_0 \omega \tag{5}$$

We now have a system of three equations that can be combined to solve for three unknowns v_f, v_{CM}, and ω.

Performing the actual solution digitally:

$$v_{cm} = \frac{4I_0 m_1 v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I_0 m_2} \tag{6}$$

$$v_f = \frac{(m_1 m_2 r^2 + I_0 m_1 - 2I_0 m_2)v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I_0 m_2} \tag{7}$$

and finally, we have

$$\omega = \frac{4m_1 m_2 r v_0}{m_1 m_2 r^2 + I_0 m_1 + 2I m_2} \tag{8}$$

2 | **Rigid Body Kinetic Energy**

We will start with the known expression that:

$$KE = \sum_i \frac{1}{2}m_i {v_i}^2 \tag{9}$$

Because of the fact a point v_i can be defined as a sum of the velocity from the origin plus the displace from from origin ($v_i = v_{CM} + v_i'$), we can rewrite the kinetic energy expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM} + v'_i)(V_{CM} + v'_i) \tag{10}$$

Now, we shall foil the above expression:

$$KE = \sum_i \frac{1}{2} m_i (V_{CM}^2 + 2v_{CM}v'_i + v_i'^2) \tag{11}$$

$$= \sum_i \frac{1}{2} m_i V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \tag{12}$$

$$= \frac{1}{2} M V_{CM}^2 + \sum_i m_i V_{CM} v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \tag{13}$$

$$= \frac{1}{2} M V_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \tag{14}$$

At which point, we realize that we have in the middle arrived at the definition of the center of mass in the reference frame of the center of mass—meaning that it is indeed 0 because the center of mass is at the origin of the center of mass. Moving on, then:

$$KE = \frac{1}{2} M V_{CM}^2 + V_{CM} \sum_i m_i v'_i + \sum_i \frac{1}{2} m_i v_i'^2 \tag{15}$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i v_i'^2 \tag{16}$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i (r'_i \omega)^2 \tag{17}$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \sum_i m_i r_i'^2 \omega^2 \tag{18}$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} \omega^2 \sum_i m_i r_i'^2 \tag{19}$$

$$= \frac{1}{2} M V_{CM}^2 + \frac{1}{2} I \omega^2 \; \blacksquare \tag{20}$$