> Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

If $T = 0$, then $0^2 = 0$ and 0 is self-adjoint. Thus, let $T \neq 0$.

In 7.1, Axler asserts that V is finite-dimensional.

By the complex spectral theorem, T has a diagonal matrix w.r.t. an orthonormal basis of V.

Let these entries equal $d_1, \ldots, d_n$. T^k will have on it's diagonal $d_1^k, \ldots, d_n^k$. For each d_i, $d_i^8 = d_i^9$. The only values in $\mathbb{F}$ that satisfy this are zero and one; thus every d_i must be a zero or a one.

Thus, $T = T^2$ and T is self-adjoint.

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$$TT^* = T^*T$$

First, we will show that $T^2 = T$. Suppose T is invertible. Then,

$$\begin{aligned} T^9 &= T^8 \\ T^9T^{-7} &= T^8T^{-7} \\ T^2 &= T \end{aligned}$$

Suppose T is not invertible and not equal to zero. Then, T has some zero entries on it's diagonal and some non-zero entries on it's diagonal.