Hello, fellow person that comes across this. I have had one brief exposure with Linear Algebra following MATH 21-1 at UCSC. However, Axler is just so cool, so I am trying to learn a bit of linalg on the side to supplement my much more traditional linalg experience at the UC.

A few things of note. This whole thing is very "partial": in the sense that its contents contain many a parts of things omitted which I feel like I have a good grasp on from 21-1 such that I don't need to be reminded again; I only include things that maybe useful to me later either b/c I don't know it or I want to be reminded of it. As such, I don't think this will be helpful for most people.

1 | **1.A**

1.1 | **Things of Note**

- $\lambda \in \mathbb{F}$ is called a "scalar". I mean duh but still.

1.1.1 | **Defining a list**

A list of length n is a collection of n elements (any mathematical object?) separated by commas.

"Identical" lists are established when lists have:

- the same length
- same elements
- in the same order.

Its also called a n-tuple.

n must be a finite non-negative value. Therefore, an "infinitely long list" is not a list.

1.1.2 | **Sets vs Lists**

Lists have order and repetition. In sets, order and repetitions don't matter.

1.1.3 | $\mathbb{F}$

- A set
- Containing 2 elements $0, 1$
- Operators of "addition" and "multiplication" that satisfy the following properties

1. Properties of $\mathbb{F}$ That, with $\alpha, \beta, \lambda \in \mathbb{F}$:
    - **Commutativity** $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
    - **Associativity** $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
    - **Existence of Identities** $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$
    - **Additive Inverse** for every α, $\exists\beta$ s.t. $\alpha + \beta = 0$
    - **Multiplicative Inverse** for every $\alpha \neq 0$, $\exists\beta$ s.t. $\alpha\beta = 1$
    - **Distribution** $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.1.4 | $\mathbb{F}^n$

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \; for \; j = 1, \ldots, n\} \tag{1}$$

We say x_j is the j^{th} coordinate of $(x_1, \ldots, x_n)$.
In $\mathbb{F}^n$...

1. Addition
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \tag{2}$$
2. Scalar Multiplication
$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) \tag{3}$$
3. Zero
$$0 = (0, \ldots, 0) \tag{4}$$
4. Additive Inverse ...of $x \in \mathbb{F}^n$:
$$x + (-x) = 0 \tag{5}$$
That:
$$x = (x_1, \ldots, x_n), -x = (-x_1, \ldots, -x_n) \tag{6}$$

1.2 | In-Text Exercises

1.2.1 | Verify that $i^2 = -1$

$(0 + 1i)(0 + 1i) = (0 + 0 + 0 + ii) = -1$

1.2.2 | Defining subtraction and division

$\alpha, \beta \in \mathbb{C}$
Subtraction could be defined in that:

- Let $-\alpha$ be defined as the additive inverse of α
- Subtraction, therefore, is defined $\beta - \alpha = \beta + (-\alpha)$

Division could be defined in that:

- Let $1/\alpha$ be defined as the multiplicative inverse of α
- Subtraction, therefore, is defined $\beta/\alpha = \beta(1/\alpha)$

1.3 | **Actual Exercises**

1: Suppose $a, b \in \mathbb{R}$, $a, b \neq 0$, find $c, d \in \mathbb{R}$ s.t. $\frac{1}{(a+bi)} = c + di$

$$\frac{1}{(a+bi)} = \frac{(a-bi)}{(a+bi)(a-bi)} = \tag{7}$$
$$\Rightarrow \frac{a-bi}{a^2-(bi)^2} = c + di \tag{8}$$
$$\Rightarrow \frac{a-bi}{a^2+b^2} = c + di \tag{9}$$
$$\Rightarrow \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2} = c + di \tag{10}$$

Therefore:

$$c = \frac{a}{a^2+b^2} \tag{11}$$

$$d = \frac{-b}{a^2+b^2} \tag{12}$$

2: Show that $\frac{-1+\sqrt{3}i}{2}$ is the cube root of 1.

$$(\frac{-1+\sqrt{3}i}{2})^3 \tag{13}$$
$$\Rightarrow (\frac{-1+\sqrt{3}i}{2})(\frac{-1+\sqrt{3}i}{2})(\frac{-1+\sqrt{3}i}{2}) \tag{14}$$
$$\Rightarrow \frac{(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i)}{8} \tag{15}$$
$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \tag{16}$$
$$\Rightarrow \frac{(1-2\sqrt{3}i-3)(-1+\sqrt{3}i)}{8} \tag{17}$$
$$\Rightarrow \frac{8}{8} = 1 \tag{18}$$

3: Find two distinct square roots of i

?

4: Show that $\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in C$

Let:
$\forall a, b, c, d \in \mathbb{R}$

- $\alpha = (a + bi)$
- $\beta = (c + di)$

$$\begin{aligned}
\alpha + \beta &= (a+bi) + (c+di) &(19)\\
&= (a+c) + (b+d)i &(20)\\
&= (c+a) + (d+b)i &(21)\\
&= (c+di) + (a+bi) &(22)\\
&= \beta + \alpha \blacksquare &(23)
\end{aligned}$$

5: Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda), \forall \alpha, \beta, \lambda \in \mathbb{C}$

Let:

$\forall a, b, c, d, e, f \in \mathbb{R}$

- $\alpha = (a + bi)$
- $\beta = (c + di)$
- $\lambda = (e + fi)$

$$\begin{aligned}
(\alpha + \beta) + \lambda &= ((a+bi) + (c+di)) + (e+fi) &(24)\\
&= ((a+c) + (b+d)i) + (e+fi) &(25)\\
&= (a+c+e) + (b+d+f)i &(26)\\
&= (a + (c+e)) + (b + (d+f))i &(27)\\
&= (a+bi) + (c+e) + (d+f)i &(28)\\
&= (a+bi) + ((c+di) + (e+fi)) &(29)\\
&= \alpha + (\beta + \lambda) \blacksquare &(30)
\end{aligned}$$

2 | **1.B**

2.1 | **Things of Note**

2.1.1 | **Vector Spaces** V

A "vector"/"point" is a member of a vector space. The exact nature of scalar multiplication depends on which $\mathbb{F}$ we are working it; hence, when being precise, we say that V is a vector space "over $\mathbb{F}$".

1. Motivation
    - Addition is commutative, associative, and has identity
    - Every element has additive inverse
    - Scalar multiplication is associative
    - Addition and scalar multiplication is connected by distribution
2. Basic Operators
    - **Addition** on set V is a function that assigns $u + v \in V$ to each pair $u, v \in V$

- **Scalar Multiplication** on set V is a function that assigns $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and $v \in V$. Note that this is different than a field, because if you can't multiply two different things in and out of the field and expect it to remain. But you could multiply an element in field $\mathbb{F}$ and a vector in vector space V and expect it to stay in V.

Note also "Multiplication" is not defined as 1) there are two and 2) they behave very differently.

3. Properties For $u, v, w \in V$ and $a, b \in \mathbb{F}$.
   - **Commutativity** $u + v = v + u$
   - **Associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$.
   - **Additive Identity** $\exists 0 \in V\ s.t.\ v + 0 = v, \forall v \in V$
   - **Additive Inverse** $\forall v \in V, \exists w \in V\ s.t.\ v + w = 0$
   - **Multiplicative Identity** $1v = v$
   - **Distribution** $a(u + v) = au + av$ and $(a + b)v = av + bv$
4. Unique Additive Identity The additive identity ("zero") in a vector space must be unique. (i.e. there cannot be two distinct zeros 0 and $0'$ which both are $\in V$). This is because:

$$0 = 0 + 0' = 0' + 0 = 0' \tag{31}$$

   That — if both 0 and $0'$ are additive identities, $0 = 0'$.
5. Unique Additive Inverse Every element in a vector space has an unique additive inverse (i.e. there cannot be two distinct additive inverses of $v \in V$ w and w' which both are $\in V$).

   Suppose w and w' are both additive inverses of v, then it holds that:

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w' \tag{32}$$

   That — if both w and w' exists in V, $w = w'$.
6. Zero and Vectors $0v = 0$ for $v \in V$. $a\vec{0} = \vec{0}$ for $a \in \mathbb{F}$.

2.1.2 | $\mathbb{F}^\infty$

Wait but aren't $\mathbb{F}^n$ supposed to be made of lists, which has finite length?

I guess its just sequences of all of everything in F.

$$\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_j \in \mathbb{F}\ for\ j = 1, 2, \ldots\} \tag{33}$$

2.1.3 | $\mathbb{F}^S$

$\mathbb{F}^S$ is defined as the set of functions that maps elements in set S to $\mathbb{F}$. It is a vector space.

1. Addition Addition between $f, g \in \mathbb{F}^S$ is defined by:

$$(f + g)(x) = f(x) + g(x),\ \forall x \in S \tag{34}$$

2. Scalar Multiplication Multiplication between $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^s$, $\lambda f \in \mathbb{F}^s$ is defined as:

$$(\lambda f)(x) = \lambda f(x),\ \forall x \in S \tag{35}$$

3. $\mathbb{F}^n$ and $\mathbb{F}^\infty$ are special cases of $\mathbb{F}^s$ ...this is because a list $\{x_1, x_2, x_3, \ldots, x_n\}$ is actually a bijective mapping between $\{1, 2, 3, \ldots, n\}$ (the indexes) and the values of the list, which are all $\in \mathbb{F}$. so :tada:!

2.2 | In-Text Exercises

2.2.1 | Verify that $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

Not going to write this one out, but:

- Commutativity: via rules addition, commutation (in $\mathbb{F}$), then undoing addition
- Associativity: addition, communication, then undoing addition
- Additive Identity: addition + definition of "zero" in $\mathbb{F}^n$
- Additive Inverse: addition + additive inverse (in $\mathbb{F}$)
- Multiplicative Identity: scalar multiplication (by 1) and then identity (in $\mathbb{F}$)
- Distribution: definition of addition in $\mathbb{F}^n$, scalar multiplication, undoing definition of addition again

2.3 | Actual Exercises

1: Proof that $-(-v) = v,\ \forall v \in V$

Step	Explanation
$v = v + 0$	Additive identity
$v = v + (-v + -(-v))$	Additive inverse
$v = (v + -v) + -(-v)$	Associative property
$v = 0 + -(-v)$	Additive inverse
$v = -(-v)$ ■	Additive Identity

2: Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Proof $a = 0$ or $v = 0$.

Let $a \neq 0$. We define the multiplicative inverse of a as a^{-1}.

Step	Explanation
$v = 1v$	Multiplicative identity
$v = aa^{-1}v$	Multiplicative inverse
$v = ava^{-1}$	Commutativity
$v = 0a^{-1}$	Given
$v = 0$ ■	Number times 0

If $a = 0$, ■.

3: Suppose $v, w \in V$, explain why $\exists$ unique $x \in V$ s.t. $v + 3x = w$

Let $x = \frac{1}{3}(w - v)$; by addition and scalar multiplication, $\exists x \in V$.

Step	Explanation
$v + 3x = w$	Given
$v + 3(\frac{1}{3}(w - v)) = w$	Defined
$v + (w - v) = w$	Multiplication in $\mathbb{F}$
$v - v + w = w$	Commutativity
$w = w$ ■	Additive Inverse

Therefore, $\exists x \in V$ that satisfies the needed property.

Suppose there exists more than 1 x which satisfies this property. We call them x and x'. This would tell us the following equalities:

$v + 3x = w$, $v + 3x' = w$.

It follows from the equalities that:

$3x = w - v$, $3x' = w - v$

Then, it follows that

$3x = 3x'$

Therefore:

$$x = x' \tag{36}$$

Hence, if given that there exists $v + 3x = w$, $v + 3x' = w$, $x = x'$. Hence, there is only one unique x such that $v + 3x = w$.

3 | 1.C

Axler, in his infinite wisdom, has crammed everything that's interesting to note in Chapter 1.c.

3.1 | Things of Note

3.1.1 | Subspaces

(Woo hoo!)

A subset $U \subset V$ is called a "subspace of V" if U is also a vector space using the same addition and scalar multiplication operators.

1. Checking for Subspaces Check for three conditions:

   For $U \subset V$

   - **Additive Identity** $0 \in U$. (also could be defined as "set is nonempty", b/c if nonempty, and its closed under scalar multiplication, multiplying any element by 0 will do the trick. But often showing 0 is in it is actually simpler.)
   - **Closed Under Addition** $u, w \in U$ implies $u + w \in U$
   - **Closed Under Scalar Multiplication** $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

3.1.2 | Summing Subsets

Suppose $U_1, \ldots, U_m$ are subsets of V. The "sum" of the subsets ($U_1 + \cdots + U_m$) is the set of all possible sums of elements in $U_1, , U_m$. That is:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\} \tag{37}$$

1. Properties of the Sums of Subspaces Suppose $U_1, \ldots, U_m$ are subspaces of V. $U_1 + \cdots + U_m$ is the smallest subspace of V containing all of $U_1, \ldots, U_m$.

3.1.3 | Direct Sum

Suppose $U_1, \dots, U_m$ are subspaces of V

- Sum $U_1 + \dots + U_m$ is a direct sum if every element $u \in U_1 + \dots + U_m$ can be only written only one way as a sum $u_1 + \dots + u_m$
- Direct sum is noted as $U_1 \oplus \dots \oplus U_m$

”Sums is the union, direct sums is the disjoint union”.

1. Checking for Direct Sums Suppose $U_1, \dots, U_m$ are subspaces of V. $U_1 + \dots + U_m$ is a direct sum iff the only way to write 0 as a sum $u_1 + \dots + u_m$ is by taking each u_j equaling to 0. This could be implied from the definition of a direct sum: that there is only one way to write a 0 as $u_1 + \dots + u_m$, and being closed scalar multiplication means that you could multiply 0 to each subspace individually and they still have to add up to 0.
2. Direct Sum of Two Subspaces Suppose U, W are subspaces of V.

   $U + W$ is a direct sum iff $U \cap W = \{0\}$.

3.2 | In-Text Excercises

3.2.1 | Summing Subspaces, an Example

Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$ and $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Then:

$$U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\} \tag{38}$$

We verify this by writing out the sum.

$$\begin{aligned} U + W &= \{(x_u, x_u, y_u, y_u) + (x_w, x_w, x_w, y_w) : x_u, y_u \in U, x_w, y_w \in W\} \quad (39) \\ &= \{(x_u + x_w, x_u + x_w, y_u + x_w, y_u + y_w) : x_u, y_u \in U, x_w, y_w \in W\} \quad (40) \\ &= \{\left(\underbrace{x}_{x_u + x_w}, x, \underbrace{y}_{y_u + x_w}, \underbrace{z}_{y_u + y_w}\right) : x, y, z \in \mathbb{F}\} \quad (41) \end{aligned}$$

3.2.2 | Verify sums equal to spaces

Suppose U, W are subspaces of $\mathbb{F}^3$.

$$\begin{aligned} U &= \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \quad (42) \\ W &= \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \quad (43) \end{aligned}$$

Verify $\mathbb{F}^3 = U \oplus W$

$$U + W = \{(x, y, z) \in F^3 : x, y, z \in F\} = \mathbb{F}^3 \tag{44}$$

:tada:?

3.3 | **Actual Excercises**

1: For each of the following subsets of $\mathbb{F}^3$, determine if it is a subspace of $\mathbb{F}^3$.

$$U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} \tag{45}$$

We could see that $(0, 0, 0) \in U$.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3}), u_2 = (x_3, x_4, \frac{-x_3-2x_4}{3}).u_1, u_2 \in U$.

$u_1 + u_2 = (x_1 + x_3, x_2 + x_4, \frac{-(x_1+x_3)-2(x_2+x_4)}{3})$. Define $x_1 + x_3 = x$, $x_2 + x_4 = y$. Therefore: $u_1 + u_2 = (x, y, \frac{-x-2y}{3})$.

$x + 2y - x - 2y = 0$.

Therefore, U is closed under addition.

Let $u_1 = (x_1, x_2, \frac{-x_1-2x_2}{3})$. We scale each element by scalar factor λ.

$\lambda u_1 = (\lambda x_1, \lambda x_2, \lambda\frac{-x_1-2x_2}{3})$.

$\lambda x_1 + 2\lambda x_2 - \lambda x_1 - 2\lambda x_2 = 0$. Therefore, U is closed under scalar multiplication.

Therefore, it is a subspace of $\mathbb{F}^3$.

10: Suppose that U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Given U_1 and U_2 are both subspaces, 0 must be in both subsets hence it must be in their intersection.

Let $u_1 = (x_1, y_1, z_1)$. Let $u_2 = (x_2, y_2, z_2)$. Finally, let $u_1, u_2 \in U_1 \cap U_2$. By this last fact, we could derive the fact that $u1, u2 \in U_1, U_2$.

As U_1 is a subspace, U_1 closed under addition. Since $u_1, u_2 \in U_1$, $u_1 + u_2 \in U_1$. As U_2 is a subspace, U_2 closed under addition. Since $u_1, u_2 \in U_2$, $u_1 + u_2 \in U_2$.

As $u_1 + u_2 \in U_1, U_2$, it is additionally $u_1 + u_2 \in U_1 \cap U_2$. $U_1 \cap U_2$ is therefore closed under addition.

By the same token….

As U_1 is a subspace, U_1 is closed under scalar multiplication. Since $u_1 \in U_1$, $\lambda u_1 \in U_1$. As U_2 is a subspace, U_2 is closed under scalar multiplication. Since $u_1 \in U_2$, $\lambda u_1 \in U_2$.

Therefore, as $\lambda u_1 \in U_1, U_2$, it is additionally true that $u_1 \in U_1 \cap U_2$, it is therefore closed under multiplication.

4 | **2.A**

What's with the balancing of these chapters? Like Span and Linear Independence is squished in one, but then Bases gets a whole chapter and so does dimension.

4.1 | **Things of Note**

- Lists of vectors are usually denoted as a list without parentheses

4.1.1 | **Linear Combination**

A linear combination of a list is a sum of vectors in the form:

$$a_1v_1 + \cdots + a_mv_m \tag{46}$$

where, $a_1, \ldots, a_m \in \mathbb{F}$.

1. Verifying Linear Combinations You could note that a linear combination is actually just a linear system of equations. That:

   To check (x, y, z) is a linear combination of (v_1, v_2, v_3), (w_1, w_2, w_3), figure if there exists a pair (a_1, a_2) such that…

   $$\begin{cases} x = a_1v_1 + a_2w_1 \\ y = a_1v_2 + a_2w_2 \\ z = a_1v_3 + a_2w_3 \end{cases} \tag{47}$$

   If so, there exists the requisite scalars such that the linear combination is possible.

4.1.2 | **Linear Span**

The span is the set of all linear combinations of a list of vector. That:

$$span(v_1, \ldots, v_m) = \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbb{F}\} \tag{48}$$

The span of an empty list of vectors $()$ is defined to be $\{0\}$. Due to the fact that any subspace must be closed under scalar multiplication + addition, the span of a list of vectors in V is the smallest subspace of V containing all the vectors in that list.

1. Spanning List If $span(v_1, \ldots, v_m) = V$, we say that $v_1, \ldots, v_m$ spans V.

2. Finite-Dimensional Vector Space A vector space is ”finite-dimensional” if some list of vectors in it could span the space. i.e. there exists a list of vectors that span the space. Otherwise, it is called ”infinite dimensional”.

   Every subspace of a finite dimentional vector space is finite dimentional.

4.1.3 | **Polynomials**

We quickly recap the definition of a polynomial. A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial if $\exists a_0, \ldots, a_m \in \mathbb{F}$ s.t.

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m, \forall z \in \mathbb{F} \tag{49}$$

By the same token, $\mathcal{P}($

1. Degree of a Polynomial A polynomial $p \in \mathcal{P}(\mathbb{F})$ has degree m if $\exists a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that…

   $$p(z) = a_0 + a_1z + \cdots + a_mz^m, \forall z \in \mathbb{F} \tag{50}$$

   If p has degree m, we write $deg\ p = m$. A zero-polynomial is said to have degree $-\infty$.

2. $\mathcal{P}_m(\mathbb{F})$ For a non-negative integer m, $\mathcal{P}_m(\mathbb{F})$ is defined as the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

   You will therefore notice, then, that:

$$\mathcal{P}_m(\mathbb{F}) = span(1, z, \dots, z^m) \tag{51}$$

4.1.4 | **Linear Independence**

Linear independence exists by a only *unique* choice of scalars $a_1, \dots, a_m$ to form any given $v \in span(v_1, \dots, v_m)$. This could also be stated as that:

that the only choice $a_1, \dots, a_m \in \mathbb{F}$ that makes $a_1v_1 + \dots + a_mv_m = 0$ is the "trivial" case whereby $a_1 = \dots = a_m = 0$.

The empty set is also defined as linearly independent.

4.1.5 | **Linear Dependence**

A list of vectors in V is linearly dependent if its not linearly independent.

A list $v_1, \dots, v_m$ of vectors is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$ that's not all 0, such that $a_1v_1 + \dots + a_mv_m = 0$.

1. Linear Dependence Lemma Suppose $v_1, \dots, v_m$ is an linearly dependent list in V. Then, exists a $j \in \{1, 2, \dots, m\}$ such that:

   (a) $v_j \in span(v_1, \dots, v_{j-1})$

   (b) If the j^{th} term is removed, the span remains the same

2. Lengths of Lin. Indp List In a finite-dimentional vector space, the length of every linearly independent list is less than or equal to the length of every spanning list.

4.2 | **In-Text Excercises**

4.2.1 | $\mathcal{P}(\mathbb{F})$ **is a subspace of** $\mathbb{F}^{\mathbb{F}}$

Zero exists in the set as, for a polynomial, $(a_0, \dots, a_m) = (0, \dots, 0)$ would create a function $f : \mathbb{F} \to 0$.

Due to commutativity, we could group and factor-out input-variable z such that the sum of two polynomials become $(a_{0a} + a_{0b}) + (a_{1a} + a_{1b})z + \dots + (a_{ma} + a_{mb})z^m$, which would be another polynomial. This would be closed under addition.

Due to distribution, a scalar λ multiplied to a polynomial would just scale every value by λ resulting in $\lambda a_0 + \lambda a_1 z + \dots + \lambda a_m z^m$, which would be another polynomial. This would be closed under scalar multiplication.

Therefore, the set of polynomials in $\mathbb{F}$ is as subspace.

4.2.2 | **Show that** $\mathcal{P}(\mathbb{F})$ **in infinite-dimensional**

Any list $U \subset \mathcal{P}(\mathbb{F})$ would contain a highest-degree polynomial with degree m. Therefore, the element $z^{m+1} \in \mathcal{P}(\mathbb{F})$ would not be in the span of the list: making the list not span the entire space. Therefore, there could not be a list that spans $\mathcal{P}(\mathbb{F})$, making it infinite-dimentional.

5 | 2.B

5.1 | Things of Note

(wow that was short.)

5.1.1 | Bases

A basis of V is a list of vectors in V that's linearly independent **and** spans V. A list $v_1, \ldots, v_n$ could only be a basis if and only if every vector $v \in V$ could be written as a linear combination of $v_1, \ldots, v_n$ uniquely. The "uniquely" part makes linear independence, the "could be written" part makes span.

1. Reducing a Basis Every spanning list in a vector space could be reduced to a basis of the vector space. Basically by removing everything that makes the span linearly dependent.

   And therefore, every finite-dimentional vector space has a basis. Because you could just get a span of the space and keep reducing.

2. Building a Basis The dual of the previous condition is also true. Take a linearly independent list, keep adding vectors that still maintains the linearly independent-ness of the list, and at some point you will get a basis.

   As an correlary to this, we could claim that any subspace U of V is part of the a direct sum that adds up to V. This is because we could expand ("pad") $u_1, \ldots, u_m \in U$ with vectors $w_1, \ldots, w_n$ until we form a basis of V. We could see that $V = U + W$ because the collection of the vectors that make them up is the basis of V. And also because of that they are linearly independent.

   because they are linearly independent, there is only one way to write $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ as a linear combination, making it a direct sum.

5.2 | In-Text Exercises

5.2.1 | The list $1, z, \ldots, z^m$ is a basis for $\mathcal{P}_m(\mathbb{F})$

We proof first that the sequence spans $\mathcal{P}_m(\mathbb{F})$. This is easy to see, because, definitionally, a polynomial in $\mathcal{P}_m(\mathbb{F})$ a made of a linear combination of $1, z, \ldots, z^m$.

This list is furthermore linearly independent as, the only set of scalars $a_1, \ldots, a_{m+1}$ by which $1, z, \ldots, z^m$ could be scaled to be 0 is in the case by which all scalars are 0, making this list linearly independent.

6 | 2.C

6.1 | Things of Note

6.1.1 | Dimension!

$dim\ V$ of a finite-dimentional vector space is the length of any basis of the space. Because the length of linear independent list must be $\leq$ spanning list, there is only one length that's possible in a space such that a list is a basis (linear independent AND spanning.)

- $dim\ \mathbb{F}^n = n$ because the standard basis $\mathbb{F}^n$ as length n

- $dim\ \mathcal{P}_m(\mathbb{F}) = m + 1$ because the basis of the space $1, z, \dots, z^m$ has $m + 1$ elements

1. Subspace Dimension If V has a finite dimension + if U is a subspace of V, $dim\ U \leq dim\ V$.
2. A John McHugh Special: "Half is good enough" Linearly independent and/or spanning lists of vectors in V with length $dim\ V$ is a basis of V. So proving two (length, basis, linearly indep., spanning) proves all four.
3. Dimension of a Sum If U_1 and U_2 are finite-dimentional, then:

$$dim(U_1 + U_2) = dim\ U_1 + dim\ U_2 - dim(U_1 \cap U_2) \tag{52}$$

6.2 | In-Text Excercises

6.2.1 | Show that 1, $(x-5)^2$, $(x-5)^3$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$

Define U, as given by the question:

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\} \tag{53}$$

By thinking a little hard, we could see that $1, (x-5)^2, (x-5)^3 \in U$. A linear combination of these elements are shown:

$$a + b(x-5)^2 + c(x-5)^3 = 0 \tag{54}$$

We could see that the degree of 0 is 0. Therefore the coefficient for the x^3 term is 0. As the 3rd-degree term in the left expression is cx^3, we therefore deduct that $c = 0$. We repeat this to find $b = 0, a = 0$. Therefore, the list as prescribed is linearly independent. Furthermore, we could see that the dimension of U would have to be ≥ 3 as the length of this linearly independent list is 3. As U is a subspace of $\mathcal{P}_3$, U has a dimension ≤ 4 as the dimension of $\mathcal{P}_3$ is 4 (bases $1, z, \dots, z^3$).

We further show that, due to the fact that $U \neq \mathcal{P}_3(\mathbb{R})$ (i.e. for instance, $x^2 \in \mathcal{P}_3(\mathbb{R}), x^2 \notin U$), and if $dim\ U = 4$, extending the bases of U towards that for $\mathcal{P}_3(\mathbb{R})$ would exceed $dim\ \mathcal{P}_3(\mathbb{R})$. Therefore, $dim\ U = 3$.

Given the list as given is an linearly independent list with a length of 3 in a subspace of dimension 3, the list is a basis of U.

7 | 3.A

7.1 | Things of Note

7.1.1 | Linear Maps

A linear map form V to W is a function $T : V \to W$ which holds…

- **Commutativity**: $T(u + v) = Tu + Tv, \forall u, v \in V$
- **Homogeneity** $T(\lambda v) = \lambda(Tv), \forall \lambda \in \mathbb{F}, v \in V$

A set of all linear maps from $V \to W$ is denoted as $\mathcal{L}(V, W)$.

1. Zero Let the symbol 0, in the context of linear maps define a function that takes a element in V and maps it to the additive identity in W. That is

$$0v = 0 \tag{55}$$

2. Identity The identity map, denoted in I, is the function that maps each element to itself.

$$Iv = v, I \in \mathcal{L}(V, V) \tag{56}$$

   Note that this only makes sense if I maps from one vector space back to the same space.

3. Differentiation Define a $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that…

$$Dp = p' \tag{57}$$

   The map D is therefore the map of "differentiation." Note that, because of linear maps' homogeneity and commutativity, we could therefore see the basic rules of differentiation ($(f+g)' = f' + g'$, $(\lambda f)' = \lambda f'$).

4. Integration Define a $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ such that…

$$Tp = \int_0^1 p(x)dx \tag{58}$$

   The same reasoning as above show the basic commutativity and homogeneity of integration.

5. $\mathbb{F}^n \to \mathbb{F}^m$ Let $\mathbb{F}^n$ and $\mathbb{F}^m$ be two spaces. Let m and n be positive integers, let $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, and define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ by:

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n) \tag{59}$$

   Every linear map for $\mathbb{F}^n$ to $\mathbb{F}^m$ look like this.

7.1.2 | $\mathcal{L}(V, W)$

1. …is a vector space Suppose that $S, T \in \mathcal{L}(V, W)$

   - $(S + T)(v) = Sv + Tv$
   - $(\lambda T)(v) = \lambda(Tv)$

   Which, after some wrangling which we will do below, shows that $\mathcal{L}(V, W)$ is a vector space.

2. …has the "product" defined Suppose, now, that $T \in \mathcal{L}(V, W), M \in \mathcal{L}(W, V)$

   $(TM)(v) = T(Mv)$

   TM, therefore, is simply $T \circ M$ but when they are linear maps they are considered the "product"

   Linear map products are…

   - Associative $(T_1T_2)T_3 = T_1(T_2T_3)$
   - Identitative $TI = IT = T$, if I is the identity in the domain of T
   - Distributive $(S_1 + S_2)T = S_1TS_2T$, and visa versa

3. Takes 0 to 0 Suppose T is a linear map $T : V \to W$, then $T(0) = 0$, as defined in both spaces. So zeros stay zero.

7.1.3 | Linear Maps & Basis of the Domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$, then there exists a **unique** linear map $T : V \to W\ s.t.\ Tv_j = w_j$.

That, if a list of vectors is a basis of the domain of linear maps, there exists a unique linear map by which any n (length of the basis of domain) vectors in the codomain could be the output of the linear map applied to each basis.

7.2 | In-Text Experiences

7.2.1 | Linear Map from Basis of Domain is Unique

We first show that there is a linear map $T : V \to W$.

Define:

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n \tag{60}$$

We could see that, because $v_1, \ldots, v_n$ is a basis of V, the entirety of V is in domain for T (as inputs to T is a linear combination of all elements in V). T is furthermore a function as every element in V could be *uniquely* written (as $v_1, \ldots, v_n$ is a basis) as a linear combination. Therefore, for the expression above, every $Tv_j =$ some w_j by simply setting $c_j = 1, j_{not\ j} = 0$.

We now proof the function T is a linear map.

$$\begin{aligned}
T(u+v) &= T(a_1v_1 + \cdots + a_nv_n + c_1v_1 + \cdots + c_nv_n) && (61)\\
&= T((a_1+c_1)v_1 + \cdots + (a_n+c_n)v_n) && (62)\\
&= (a_1+c_1)w_1 + \cdots + (a_n+c_n)w_n && (63)\\
&= a_1w_1 + \cdots + a_nw_n + c_1w_1 + \cdots + c_nw_n && (64)\\
&= (a_1w_1 + \cdots + a_nw_n) + (c_1w_1 + \cdots + c_nw_n) && (65)\\
&= T(a_1v_1 + \cdots + a_nv_n) + T(c_1v_1 + \cdots + c_nv_n) && (66)\\
&= Tu + Tv && (67)
\end{aligned}$$

Hence, the function is commutative.

$$\begin{aligned}
T(\lambda v) &= T(\lambda c_1v_1 + \cdots + \lambda c_nv_n) && (68)\\
&= \lambda c_1w_1 + \cdots + \lambda c_nw_n && (69)\\
&= \lambda(c_1w_1 + \cdots + c_nw_n) && (70)\\
&= \lambda T(c_1v_1 + \cdots + c_nv_n) && (71)\\
&= \lambda Tv && (72)
\end{aligned}$$

The function is therefore homogeneous. Therefore, the function T is a linear map from V to W. Any linear map T such that $Tv_j = w_j$ would, after each element v_j be scaled by c_j ($T(C-jv_j) = c_jw_j$) (homogeneity) and every $j = 1, \ldots, n$ summed ($T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$) (additivity), we arrive at the unique linear map T as highlighted above.

Hence, T is uniquely determined upon $span(v_1, \ldots, v_n)$. As $v_1, \ldots, v_n$ is the basis of V, T is unique upon V.

7.2.2 | Verifying that $\mathcal{L}(V, W)$ is a vector space

With the operations of scalar multiplication and addition defined, we could check that $\mathcal{L}(V, W)$ is a vector space.

- **Commutativity** $(S+T)(v) = Sv + Tv = (vectors\ are\ commutative) = Tv + Sv = (T+S)(v)$
- **Associativity** per the same token above, as associative property of vectors
- **Additive Identity** $0 \in \mathcal{L}(V, W)$. $0v + Sv = 0 + Sv = Sv$
- **Additive Inverse** Scalar multiplication by -1? #TODO
- **Multiplicative Identity** Scalar multiplication by 1
- **Distribution** $\lambda((S+T)(V)) = \lambda(Sv + Tv) = \lambda Sv + \lambda Tv$

8 | 3.B

8.1 | Things of Note

8.1.1 | Null Space

For a linear map $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted as $null\ T$, is the subset of V containing vectors which T has the ability to map to the 0 in W. That:

$$null\ T = \{v \in V : Tv = 0\} \tag{73}$$

Axler's Noticings…

- Notably, if $T = 0$, then the entirety of V is its null space
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map ($Dp = p'$). Only constants are going to have a derivative of 0; therefore, the null space of D is the set of constant functions
- Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is defined as the x^2 by $(Tp)(x) = x^2p(x)$, the only polynomial s.t. $x^2p(x) = 0, \forall x \in \mathbb{R}$ is the 0; hence $null\ T = \{0\}$.

1. Null Space is a Subspace Suppose $T \in \mathcal{L}(V, W)$, then $null\ T$ is a subspace of V.

8.1.2 | Injectivity

A function $T : V \to W$ is "injective" if $Tu = Tv$ implies $u = v$. i.e. that each output corresponds to one unique input.

So, if T is "injective", $u \neq v$ implies that $Tu \neq Tv$.

This would imply, of course, that the $null\ T$ of an injective function is simply $\{0\}$ — because as each output corresponds to a unique input, and $T0 = 0$, so the null space is going to only contain 0.

8.1.3 | Range

A function T from $V \to W$ has range defined as the subset of W which takes the form Tv, that:

$$range\ T = \{Tv : v \in V\} \tag{74}$$

Based on some reasoning, you could see that the range of a function T is a subspace of W.

8.1.4 | Surjectivity

A function $T : V \to W$ is surjective if its range equals W. As in, the entirely of the output space has some input for which it is possible to map to it.

8.1.5 | Fundamental Theorem of Linear Maps

This is kind of important. Because of the above results, we arrive that:

Suppose V is a finite-dimentional vector space and $T \in \mathcal{L}(V, W)$; then, $range\ T$ is finite-dimentional and:

$$dim\ V = dim\ null\ T + dim\ range\ T \tag{75}$$

"the rank-nullity theorem". It is proven below.

8.1.6 | Map to Smaller Space is not Injective; Map to Larger Space is not Surjective

Suppose V and W are finite-dimentional s.t. $dim\ V > dim\ W$. There are therefore no linear map from V to W which is injective.

Suppose V and W are finite-dimentional s.t. $dim\ V < dim\ W$. There are therefore no linera map from V to W that is surjective.

8.1.7 | Noticings about Linear Systems

1. A homogenous linear system with more variables than equations has nonzero solutions
2. A inhomogenous linear system with more equations than variables has no solution for some constant terms

8.2 | In-Text Exercises

8.2.1 | Null Space is a Subspace

Suppose $T \in \mathcal{L}(V, W)$. It has a zero (because $T(0) = 0$), and $0+0 = 0$ and $\lambda 0 = 0$, and 0, as we established a sentence earlier, is in T. Therefore, $null\ T$ is a subspace.

8.2.2 | Proving the Fundamental Theorem of Linear Maps

Suppose V is a finite-dimentional vector space and $T \in \mathcal{L}(V, W)$.

Let $u_1, \ldots, u_m$ be a basis of $null\ T$, then $dim\ null\ T = m$. We know that the null space of T is a subset of V, we could extend its basis to that of V: that $u_1, \ldots, u_m, v_1, \ldots, v_n$. Hence, $dim\ V = m + n$.

The fact that the list $u_1, \ldots, u_m, v_1, \ldots, v_n$ represents the basis of V, we know that every $v \in V$ can be represented as:

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n,\ where\ a_j, b_j \in \mathbb{F} \tag{76}$$

Let's apply T to both ends!

$$\begin{aligned} Tv &= T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) &(77)\\ &= T(a_1u_1 + \cdots + a_mu_m) + T(b_1v_1 + \cdots + b_nv_n) \text{ commutivity} &(78)\\ &= 0 + T(b_1v_1 + \cdots + b_nv_n) \text{ null space} &(79)\\ &= b_1Tv_1 + \cdots + b_nTv_n \text{ commutivity, homogenicity} &(80)\end{aligned}$$

Given this, we now know that $b_1Tv_1 + \cdots + b_nTv_n$ is a spanning set of the range of T (as every input to T (i.e. $v \in V$) could be represented as such.) To show that it is now a basis of $range\ T$, we need to show that its linearly independent.

Suppose that there are a set of values $c_1, \ldots, c_n \in \mathbb{F}$ such that…

$$c_1Tv_1 + \cdots + c_nTv_n = 0 \tag{81}$$

Factoring out T using commutivity, we arrive at that:

$$T(c_1v_1 + \cdots + c_nv_n) = 0 \tag{82}$$

Woah! T maps this combination to 0! Therefore:

$$c_1v_1 + \cdots + c_nv_n \in null\ T \tag{83}$$

We know that, from the assumptions before, $u_1, \ldots, u_m$ is a basis of $null\ T$. Therefore, we could claim that:

$$c_1v_1 + \cdots + c_nv_n = d_1u_1 + \cdots + d_mu_m \tag{84}$$

We know that, because they were all parts of bases of V, $u_1, \ldots, u_m, v_1, \ldots, v_n$ are all linearly independent. There is only one way to write each side as an equation of each other, and $c_1 = \cdots = c_n = d_1 \cdots = d_m = 0$ is one valid way to make this equation valid, therefore — due to the linear independence — it is the only way.

Therefore, $c_1 = \cdots = c_n = d_1 \cdots = d_m = 0$. Plugging that allllll the way back to:

$$c_1Tv_1 + \cdots + c_nTv_n = 0 \tag{85}$$

We could now see that $Tv_1, \ldots, Tv_n$ is linearly independent. Therefore, it is a basis of the range of T (as we established before it spans the range of T.)

And lastly! We know that $dim\ V = m + n$. m is the length of the basis of the null space, n is the length of the basis of the range. Therefore $dim\ V = dim\ null\ T + dim\ range\ T$ ■.

Phew.

9 | **3.C**

9.1 | **Things of Note**

Matrices are an efficient method of recording a linear transformation $T : V \to W$ in terms of the basis of W.

9.1.1 | **Matrix** $A_{j,k}$

Let m and n denote positive integers. An m by n matrix A is an array of elements of $\mathbb{F}$ with m rows and n columns.

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \tag{86}$$

We could see that, there is m rows and n columns. $A_{j,k}$ denotes the entry in row j, column k.

9.1.2 | **Matrix as a Linear Map** $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The "matrix of" t w.r.t. these bases is the m by n matrix $\mathcal{M}(T)$ such that…

$$T_{v_k} = A_{1,k}w_1 + \dots + A_{m,k}w_m \tag{87}$$

So you get each basis vector v_k back from multiplying pairwise each row in the column k with the corresponding basis vector w_j, then adding it up.

If the bases used are not clear from the context, we say its a matrix $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$.

When constructing a matrix from its basis, line the domain's basis up along the columns and the range's basis up along the rows. Fill in the values column-by-column based on what scalars such that the linear combinations of the basis vectors at the range would result in the value at the column.

To figure what bases to use:

- When working with $\mathbb{F}^n$, use the standard basis $(1, \dots, 0), \dots, (0, \dots, 1)$
- When working with $\mathcal{P}_m(\mathbb{F})$, use the standard basis $1, x, x^2, \dots, x^m$

1. Adding Matrices Matrices add in the way you'd expect. $(A + C)_{j,k} = A_{j,k} + C_{j,k}$. Suppose $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.
2. Scaling Matricies Matricies scale by a scalar also in the way you'd expect. $(\lambda A)_{j,k} = \lambda A_{j,k}$. Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda\mathcal{M}(T)$.
3. OMG we defined adding and scaling First, here's a notation. $\mathbb{F}^{m,n}$ is the set of all m by n matrices with entries in $\mathbb{F}$. So it maps from a vector space with $dim\ n$ to $dim\ m$.

   Ladies and gentlemen, $\mathbb{F}^{m,n}$ is a vector space. Mind blown. Shock surprise. :tada:

   $dim\ \mathbb{F}^{m,n} = mn$

9.1.3 | Matrix Product

The product of two matrices $A : m \to n$, $C : n \to p$ is defined as:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k} \tag{88}$$

That is, the value at new matrix j, k involves taking row j from the first matrix and column k from the second matrix, multiplying them together, then adding the resulting values.

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

1. Describing rows/columns Suppose A is an m by n matrix.

   - $A_{j,\cdot}$ is row j of the matrix — a 1 by n matrix consisting of row j of A
   - $A_{\cdot,k}$ is column k of the matrix — a n by 1 matrix consisting of column k of A

2. Revising the Product Statement Therefore, $(AC)_{j,k} = A_{j,\cdot} \cdot C_{\cdot,k}$, if treating A and C instead of as column/row vectors but as matrices.

   Furthermore, $(AC)_{\cdot,k} = A \cdot C_{\cdot,k}$: that each column in the output of a matrix multiplication is the whole of matrix A multiplied upon the column $C_{\cdot,k}$.

3. Scaling is a Linear Combination Suppose A is an m by n matrix. $c = (c_1, \ldots, c_n)$ is an n by 1 (its a column but I dunno how to typeset it inline) matrix.

   Then, $Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.

9.2 | In-Text Exercises

9.2.1 | Set of Matricies from N to M are Vector Space

Its closed under addition adding does not change size, so in same space, scaling does not change size, its closed under scalar multiplication, identity is 0 where 0, commutivity and associativity comes from the fact that $\mathbb{F}$ is a field.