1 | **Deriving Velocity in Polar Coordinates**

In this problem, we aim to determine functions f_1, f_2, that represents the velocity and acceleration vectors respectively of a circularly movable reference frame.

1.1 | **Defining First Principles**

We begin by defining $\vec{r}$, the vector representing the radius of the rotating frame, as follows:

$$\vec{r} = r\cos(\theta)\hat{i} + r\sin\theta\hat{j} \tag{1}$$

We define θ as the angle up from the horizontal at which $\vec{r}$ is located, and therefore the vector $\vec{r}$ is simply the magnitude thereof r projected upon that angle $(\cos\theta, \sin\theta)$ into a vector.

Furthermore, we define a unit vector in the direction of $\vec{r}$ as $\hat{r}$. That is:

$$\hat{r} = \cos(\theta)\hat{i} + \sin\theta\hat{j} \tag{2}$$

Lastly, we define a vector $\hat{\theta}$, a unit vector orthogonal to $\vec{r}$. It is defined as such as the direction of $\vec{\theta}$ would, at any given instance, be perpendicular to the direction of $\vec{r}$ and parallel to the direction to its movement.

$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{3}$$

1.2 | **Determining changes in direction**

We now create definitions for changes in "direction" — the changes present in $\hat{r}$ and $\hat{\theta}$ — which we will leverage later.

$$\begin{align}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{4}\\
&= -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} \tag{5}\\
&= \dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j}) \tag{6}\\
&= \dot{\theta}\hat{\theta} \tag{7}
\end{align}$$

Hence, the change in the direction of $\hat{r}$, aptly and intuitively, could be modeled by the change in the angle θ times the direction of θ.

$$\begin{align}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j}) \tag{8}\\
&= -\dot{\theta}\cos\theta\hat{i} - \dot{\theta}\sin\theta\hat{j} \tag{9}\\
&= -\dot{\theta}(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{10}\\
&= -\dot{\theta}\hat{r} \tag{11}
\end{align}$$

We now note that, indeed, the change in the direction of θ is modeled by the direction at which $\vec{r}$ exists, and the angle of θ as θ must be orthogonal to $\vec{r}$.

1.3 | Solving for $f_1 = \vec{v}$

We now begin to solve for a function $f_1(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, r) = \vec{v}$. We know that the velocity of the frame as a whole could be modeled by the following expression:

$$\vec{v} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt} \tag{12}$$
$$= r\dot{\theta}\hat{\theta} + \hat{r}\dot{r} \tag{13}$$

The derivation of this expression simply follows variable substitution to result in the expression modeling velocity: that the velocity of the point is determined by the radius-scaled velocity in the angle of the object (the velocity that's tangent to the radius), plus the velocity of the radius itself (how fast it grows, the radial velocity.)

1.4 | Solving for $f_2 = \vec{a}$.

We here wish to determine a function $f_2(\hat{r}, \hat{\theta}, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}, r) = \vec{a} = \frac{d}{dt}\vec{v}$

Deriving the value of this expression, therefore, simply acts as a matter of taking the derivative of the top-derived expression and performing variable substitution for $\dot{\hat{r}}$ and $\dot{\hat{\theta}}$ as determined above as needed.

$$\vec{a} = \frac{d}{dt}(r\dot{\theta}\hat{\theta} + \hat{r}\dot{r}) \tag{14}$$
$$= ((\frac{d}{dt}r)\dot{\theta}\hat{\theta} + ((\frac{d}{dt}\dot{\theta})\hat{\theta} + (\frac{d}{dt}\hat{\theta})\dot{\theta})r) + ((\frac{d}{dt}\hat{r})\dot{r} + (\frac{d}{dt}\dot{r})\hat{r}) \tag{15}$$
$$=((\dot{r})\dot{\theta}\hat{\theta} + ((\ddot{\theta})\hat{\theta} + (-\dot{\theta}\hat{i})\dot{\theta})r) + ((\dot{\theta}\hat{\theta})\dot{r} + (\ddot{r})\hat{r}) \tag{16}$$
$$=(\dot{r}\dot{\theta}\hat{\theta} + \ddot{\theta}\hat{\theta}r - \dot{\theta}^2\hat{r}r) + (\dot{\theta}\hat{\theta}\dot{r} + \ddot{r}\hat{r}) \tag{17}$$
$$=\hat{\theta}(2\dot{r}\dot{\theta} + \ddot{\theta}r) + \hat{r}(\ddot{r} - \dot{\theta}^2 r) \tag{18}$$

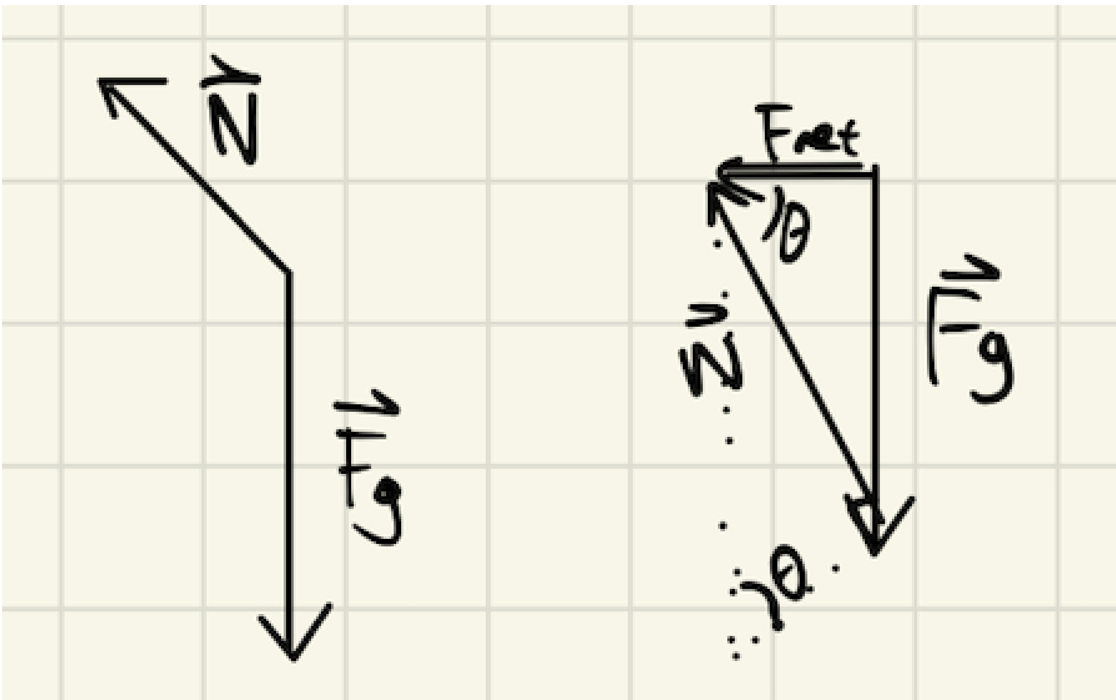
We could now figure that acceleration is actually four terms, scaled by their respective directions ($\hat{\theta}$ and $\hat{r}$). That is:

- $2\dot{r}\dot{\theta}$ is a term representing the coriolis force: a fictitious force pushing objects as either of radius change or spin speed changes.
- $\ddot{\theta}r$ is a term that models the tangential acceleration: how quickly does the object accelerates at a direction tangent to the spin. The "tangential acceleration." We could see this if $\ddot{r} = \dot{\theta} = 0$.
- $\ddot{r}$ is a term that models the acceleration of the radius, as it evidently shows. The "radial acceleration". We could see this if $\dot{\theta} = \ddot{\theta} = 0$.
- $\dot{\theta}^2 r$ is a term that models the inward acceleration that maintains the shape of the circle. It is $\dot{\theta}^2 r = \omega^2 r$, the "centripetal acceleration." We could see this if $\dot{r} = \ddot{r} = \ddot{\theta} = 0$

For a point to stay on the circle, you have to have components of acceleration INTO the circle as well to prevent the side spin from making it fly off. Hence, "tangential acceleration" + "centripetal acceleration."

2 | Mass in a Cone

For this problem, we deduct the fact, given the uniform velocity v_0 and the uniform path in which the particle exists, that the object is indeed in uniform circular motion. As such, there is only one main source of acceleration — that of centripetal acceleration. Furthermore, it is only influenced by forces normal and gravitational. This situation could therefore be modeled in the manner shown below:



Via this diagram, we could therefore determine that the net force inwards is $F_{net} = ma = \frac{F_g}{\tan\theta} = \frac{mg}{\tan\theta}$.
Therefore, we could determine that the value of acceleration inwards is

$$ma = \frac{mg}{\tan\theta} \tag{19}$$
$$\Rightarrow a = \frac{g}{\tan\theta} \tag{20}$$

As this problem deals exclusively with uniform circular motion, we additionally set $a = \frac{{v_0}^2}{r}$.
Therefore, to solve for r, we simply equate the two expression by a and solve.

$$\frac{g}{\tan\theta} = \frac{{v_0}^2}{r} \tag{21}$$
$$\Rightarrow r = \frac{{v_0}^2\tan\theta}{rg} \tag{22}$$

3 | Whirling Block

In this case of the whiling block, we could treat the behavior of the block as the behavior of the *radial* component of the spin: as the force of tension acting upon the block B could necessarily translate to forces acting in the radial-direction upon mass m.

We therefore are concerned with the following expression:

$$a_r = \ddot{r} - \omega^2 r \tag{23}$$

Therefore, a_r the acceleration of m in the radial direction (and therefore, the acceleration of B) is a variable that could be modeled by $\ddot{r}$, the radial acceleration (which, because the position of B directly affects the radius of the circular payload per the tension of the string, is also the acceleration by gravity of B) and ω given.

Substituting the above statements and deductions in symbolic form into the previous expression:

$$a_r = g + \omega^2 R \tag{24}$$

4 | **Loop The Loop**

Analysing the situation of the problem, we could first make a simplifying deduction: that the loop-de-loop's completion without loss of contact can be simplified to the successful counteraction of the force of gravity at the *top* of the loop-de-loop. This is due to the fact that, at any other angle at which the object could be at, a (x-direction) component of the acceleration of gravity would be in the direction along the track: which would not cause lost of contact. It is only at the top of the loop that all of the acceleration of gravity is applied directly towards the direction that would cause the object to loose contact.

Hence, as long as the velocity at entry is high enough such that the whole of gravitation force is counteracted, the object will stay on the track. This is also equivalent so saying that we wish to solve for where our normal force to be exactly 0 (point where "contact" is lost, that, based on the expression for uniform circular motion, $mg + N = ma = \frac{mv^2}{R}, N = 0$.

We therefore proceed to solve for v^2 (as, for the second part of this problem, v^2 would be more easily treated) based on this expression.

$$mg = \frac{mv^2}{R} \tag{25}$$
$$\Rightarrow g = \frac{v^2}{R} \tag{26}$$
$$\Rightarrow v^2 = gR \tag{27}$$

Finally, to figure the height at which the needed v^2 could be achieved, we leverage energetic expressions.

The potential energy at the height of the ramp would equal $mg\Delta h$. As the entire height we are descend before entry of the loop-de-loop is h_0, we deduct that the potential energy at the top of the ramp is mgh_0.

We wish to convert all of the potential energy to kinetic energy at the bottom of the loop. We assume that, given the object is a "block", there are no rotational kinetic energy. That:

$$mgh_0 = \frac{1}{2}mv^2 \tag{28}$$

Performing variable substitution upon the derived value needed of v^2 and solving for h_0:

$$mgh_0 = \frac{1}{2}mv^2 \tag{29}$$

$$\Rightarrow mgh_0 = \frac{1}{2}mgR \tag{30}$$

$$\Rightarrow h_0 = \frac{1}{2}R \tag{31}$$

4.1 | TODO wat. no. this is wrong.

5 | Tropical Storm

As the wind closer to the equator, it would veer further off to the Westernly direction (the ”left”) under the earth’s rotation from west-to-east (”counterclockwise”). As the earth is rotating, the wind’s inertia prevents it from rotating as fast as with the Earth, and hence it would ”lag” behind the rotation of the earth from the reference frame of the universe. As the earth is rotating away from the west, the winds closer to the equator would ”lag” further behind in the Westernly direction. In the reference frame of the Earth, this results in a phantom coriolis force towards the Westernly direction that increases in magnitude as the wind blows closer to the equator.

The same is true in the opposite direction. As the wind blows farther away from the equator, by the same logic, it would — in the reference point of Earth — experience a force that seems to push it in the Easternly direction because its source location (closer to the equator) is traveling faster than its destination location (farter from equator). Hence