Homework 2
M1522.001800 Computer Vision (2019 Spring)
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1 Theory Questions [30 pts]

1.1 Question : Combining Light Sources (5 points)

if we assume that each source(s_1, s_2) illuminates the Lambertian surface non-simultaneously, then each image intensity(surface radiance) will be like below.
(ρ is albedo, $\vec{s_1}$ and $\vec{s_2}$ are unit vectors which demonstrate the direction of sources)

$$L_1 = \frac{\rho}{\pi}(I_1\vec{n} \cdot \vec{s_1}) \tag{1}$$
$$L_2 = \frac{\rho}{\pi}(I_2\vec{n} \cdot \vec{s_2}) \tag{2}$$

And by adding two image intensity, we can get image intensity which we can see as it is caused by one source(s_3)

$$L = \frac{\rho}{\pi}(I_1\vec{n} \cdot \vec{s_1}) + \frac{\rho}{\pi}(I_2\vec{n} \cdot \vec{s_2}) \tag{3}$$
$$= \frac{\rho}{\pi}\vec{n} \cdot (I_1\vec{s_1} + I_2\vec{s_2}) \tag{4}$$
$$= \frac{\rho}{\pi}\vec{n} \cdot (I_3\vec{s_3}) \tag{5}$$

So, it is obvious that the direction of unit vector $\vec{s_3}$ has the direction just as that of $I_1\vec{s_1} + I_2\vec{s_2}$ and the intensity of s_3 is same as $|I_1\vec{s_1} + I_2\vec{s_2}|$

1.2 Question : Solid Angle (5 points)

solid angle subtended at the center of icosahedron by its all faces is the area of sphere itself which is 4π. And, solid angle subtended by each face is same as the others. So, the solid angle subtended by one of its faces is 4π divided by 20.
answer : $\frac{\pi}{5}$

1.3 Question : Lambertian Albedo (5 points)

By properties of Lambertian BRDF, below two inequalites holds.
(here, π denotes ϕ, ϕ denotes azimuthal angle of the output direction, and θ_o denotes polar angle of the output direction)

$$\text{(positivity)} \qquad f_r(\widehat{v_i}, \widehat{v_o}) \geq 0$$

$$\text{(conservation of energy)} \qquad \int_{hemisphere} f_r(\widehat{v_i}, \widehat{v_o}) cos(\theta_o) d\omega_o \leq 1$$

Using first property, it is certain that albedo is greater than 0 by the definition of BRDF on Lambertain surface.

$$f_r(\widehat{v_i}, \widehat{v_o}) = \frac{\rho}{\pi} \geq 0 \tag{6}$$

$$\rho \geq 0 \tag{7}$$

And using second property, we can derive albedo is less than 1 or equal to 1.

$$\int_{hemisphere} f_r(\widehat{v_i}, \widehat{v_o}) cos(\theta_o) d\omega_o = \int_{hemisphere} \frac{\rho}{\pi} cos(\theta_o) d\omega_o \tag{8}$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{\rho}{\pi} cos(\theta_o) sin(\theta_o) d\theta_o d\phi \tag{9}$$

$$= \int_0^{2\pi} \frac{\rho}{\pi} \int_0^{\frac{\pi}{2}} \frac{sin(2\theta_o)}{2} d\theta_o d\phi \tag{10}$$

$$= \int_0^{2\pi} \frac{\rho}{2\pi} d\phi \tag{11}$$

$$= \rho \leq 1 \tag{12}$$

For equation between (8) and (9), we can change vector notation(w) to spherical system notation(θ, ϕ). differential w (dA in image below) is represented by multiplying $rd\theta$ with $rsin\theta d\phi$. Then we can say $dw = sin\theta d\theta d\phi \because$ r = 1

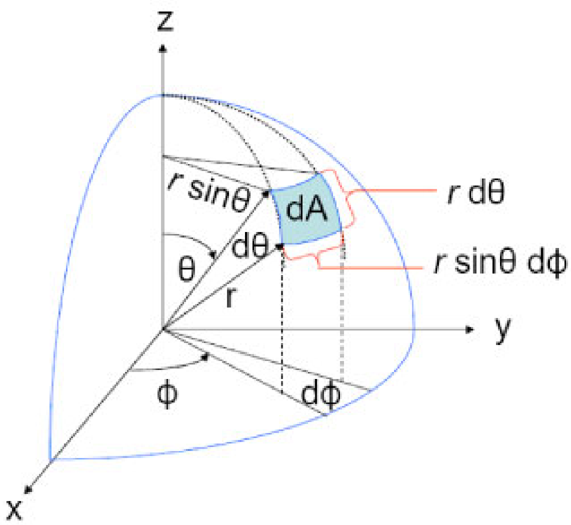

Figure 1: differential relation between vector and spherical coordinate system

Finally, combining two facts((7) and (12)), it is proved that $0 \leq \rho \leq 1$.

1.4 Question : Computing Normals of a Sphere (10 points)

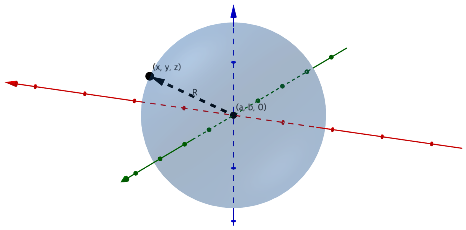

Figure 2: the projected sphere

To find normal vector of the sphere at point (u, v), I drew the original sphere before projection with center at (a, b, 0) as this sphere formula : $(x-a)^2 + (y-b)^2 + z^2 = R^2$. Then it is certain that a arbitrary point (x, y, z) subtracted by origin (a, b, 0) will be the normal vector of the sphere at point (x, y) in image coordinates. Therefore, for the point (u, v), the normal vector is follow.

$$\begin{aligned}
\vec{n} &= (u, v, z) - (a, b, 0) \quad where \quad z = \pm\sqrt{R^2 - (u-a)^2 - (v-b)^2} && (13) \\
&= (u-a, u-b, \pm\sqrt{R^2 - (u-a)^2 - (v-b)^2}) && (14)
\end{aligned}$$

1.5 Question : Computing Scene Normals (5 points)

The intensity(I) of each image pixel, which means surface radiance, can be described as below formula.
(ρ : albedo, B : brightness of source, $\vec{n}$: normal unit vector of surface, $\vec{s}$: source unit vector)

$$I = \frac{\rho}{\pi} B \vec{n} \cdot \vec{s}$$

if three equations for each intensity are given, these equations form a single matrix equation.

$$I = \begin{bmatrix} I1 \\ I2 \\ I3 \end{bmatrix} = \begin{bmatrix} \vec{s1}^T \\ \vec{s2}^T \\ \vec{s3}^T \end{bmatrix} \vec{N} = S\vec{N}$$

$$where \quad \vec{N} = \frac{\rho}{\pi}\vec{n}$$

Then, multiplying each side by S^{-1}, you can get $\vec{N}$. For $\vec{n}$ is unit vector, $\left|\vec{N}\right|$ is going to be $\frac{\rho}{\pi}$. So the normal vector of the surface is $\frac{\vec{N}}{|\vec{N}|}$ and albedo is $\pi\left|\vec{N}\right|$.

if equations is more than three, then multiplying each side by Moore-Penrose pseudo inverse $(S^T S)^{-1} S^T$, it is possible to get least squares solution.

$$I = S\vec{N} \tag{15}$$

$$S^T I = S^T S\vec{N} \tag{16}$$

$$\vec{N} = (S^T S)^{-1} S^T I \tag{17}$$