Homework 1
M1522.001800 Computer Vision (2019 Spring)

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[Important] Reference to work of others. You can refer to another person's key idea in writing the source code, but in this case you must leave a reference in the writeup. If you take someone else's idea and leave a reference, you will only get 0 points in the "Implementation" part of that problem in the scoring process. However, please note that if you do not mention the reference even though you have brought in someone else's idea, you will get 0 points for all the problems in that homework.

1 Composing Filters [5 pts]

$\mathcal{G}$ and $\mathcal{E}$ are linear kernels, that is, they can be implemented with convolution and we can use the properties of convolutions(commutative, associative). Let the image be denoted by $\mathcal{I}$, then we can denote applying $\mathcal{G}$ followed by $\mathcal{E}$ on $\mathcal{I}$ by using convolution, $\mathcal{G} * \mathcal{E} * \mathcal{I}$. And by below procedures,
(assume that each kernel size are the same)

$$\begin{aligned} \mathcal{G} * (\mathcal{E} * \mathcal{I}) &= (\mathcal{G} * \mathcal{E}) * \mathcal{I} \text{ (by associative rule)} && (1) \\ &= (\mathcal{E} * \mathcal{G}) * \mathcal{I} \text{ (by commutative rule)} && (2) \\ &= \mathcal{E} * (\mathcal{G} * \mathcal{I}) \text{ (by associative rule)} && (3) \end{aligned}$$

applying order between $\mathcal{G}$ and $\mathcal{E}$ doesn't matter. they will produce the same result. On the other hand, $\mathcal{M}$ is a non-linear filter, so that we can't implement the filter with convolution, and the result will depend on applying order between $\mathcal{M}$ and $\mathcal{E}$.

2 Identity for Convolution [10 pts]

Dirac Delta function δ is a function that satisfies below equation.

$$\delta(\text{x}) = \begin{cases} \infty & \text{x} = 0 \\ 0 & \text{x} \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

Then, convolution between δ and some function f over t is denoted by

$$\begin{aligned}
(f * \delta)(t) &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau && (4)\\
&= \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau && (5)\\
&= \int_{-\infty}^{\infty} f(t)\delta(\tau)d\tau && (6)\\
&= f(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau && (7)\\
&= f(t) && (8)
\end{aligned}$$

(5) hold because of the commutative property of convolution. By definition of dirac delta function $f(t-\tau)\delta(\tau)$ term would have meaning only when $\tau = 0$. Therefore, we can change $f(t-\tau)$ to $f(t)$. Finally, by definition of dirac delta function again, the term after $f(t)$ is going to be 1.
Now we can conclude that dirac delta function is the identity of convolution.

3 Decomposing a Steerable Filter [10 pts]

Definition of gaussian kernel for one-dimension is below.

$$\mathcal{G}_{\mathbf{t}} = \frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{t^2}{2\sigma^2})$$

Then, we can decompose 2-d gaussian kernel into two successive convolution of 1-d gaussian kernel. For some function f

$$\begin{aligned}
(\mathcal{G} * f)(t,s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-x, s-y)\mathcal{G}(x,y)dxdy && (9)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-x, s-y)\frac{1}{2\pi\sigma^2}exp(-\frac{x^2+y^2}{2\sigma^2})dxdy && (10)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-x, s-y)\frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{x^2}{2\sigma^2})\frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{y^2}{2\sigma^2})dxdy && (11)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-x, s-y)\mathcal{G}_{\mathbf{x}}(x)\mathcal{G}_{\mathbf{y}}(y)dxdy && (12)\\
&= \int_{-\infty}^{\infty}\mathcal{G}_{\mathbf{y}}(y)\left(\int_{-\infty}^{\infty} f(t-x, s-y)\mathcal{G}_{\mathbf{x}}(x)dx\right)dy && (13)\\
&= (\mathcal{G}_{\mathbf{y}} * (\mathcal{G}_{\mathbf{x}} * f(t,s)) && (14)
\end{aligned}$$

Let length of range of t, s be n, so that filter size is n x n. Then, convolving with $\mathcal{G}$ in a single step takes $O(n^2)$ computational steps. On the other hand, convolving with $\mathcal{G}_{\mathbf{x}}$ and $\mathcal{G}_{\mathbf{y}}$ in two steps takes only $O(2n)$ computational steps, because 1-dimensional gaussian kernel has n filter size. In conclusion, the latter is better in terms of computational efficiency

4 Convolution theorem [10 pts]

$$
\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} G(w)exp(2\pi xw)dw \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)H(w-\tau)exp(2\pi xw)d\tau dw && \text{Definition of Inverse Fourier Transform} \\
&= \int_{-\infty}^{\infty} F(\tau)\int_{-\infty}^{\infty} H(w-\tau)exp(2\pi xw)dwd\tau && \text{Changing the order of integration} \\
&= \int_{-\infty}^{\infty} F(\tau)\int_{-\infty}^{\infty} H(v)exp(2\pi x(v+\tau))dvd\tau && \text{By } v = \text{x - } \tau \\
&= \int_{-\infty}^{\infty} F(\tau)exp(2\pi x\tau)d\tau\int_{-\infty}^{\infty} H(v)exp(2\pi xv)dv && \text{Separating terms for } \tau\text{and } v \\
&= f(x)h(x)
\end{aligned}
$$