Srednicki QFT: Chapter 34

Left- and Right-Handed Spinor Fields

Cadabra2 expressions and numerical verification

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Contents

1 Lorentz Group Representations

The Lorentz algebra in four dimensions is isomorphic to $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ via the non-Hermitian combinations

$$N_i \equiv \tfrac{1}{2}(J_i - iK_i), \qquad N_i^\dagger \equiv \tfrac{1}{2}(J_i + iK_i), \tag{1}$$

satisfying $[N_i, N_j] = i\varepsilon_{ijk}N_k$, $[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk}N_k^\dagger$, $[N_i, N_j^\dagger] = 0$.

Irreducible representations are therefore labelled by two numbers $n, n' \in \{0, \frac{1}{2}, 1, \ldots\}$:

Srednicki label	Physics label	Dimensions	Field	Index type
$(1,1)$	$(0,0)$	1	scalar $\phi(x)$	none
$(2,1)$	$(\frac{1}{2},0)$	2	left-handed Weyl ψ_a	undotted
$(1,2)$	$(0,\frac{1}{2})$	2	right-handed Weyl $\psi^\dagger_{\dot a}$	dotted
$(2,2)$	$(\frac{1}{2},\frac{1}{2})$	4	vector A^μ	spacetime

Convention note. Srednicki labels representations by their *dimensions* $(2n{+}1, 2n'{+}1)$. The physics literature often uses the spins directly, writing $(\frac{1}{2}, 0)$ for what Srednicki calls $(2, 1)$. Both label the same object.

2 Left-Handed Spinor Field ψ_a

A left-handed Weyl field $\psi_a(x)$ lives in the $(2, 1)$ representation. Under a finite Lorentz transformation Λ:

$$U(\Lambda)^{-1}\,\psi_a(x)\,U(\Lambda) = L_a{}^b(\Lambda)\,\psi_b(\Lambda^{-1}x). \tag{34.1}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$:

$$L_a{}^b(1{+}\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\,\delta\omega_{\mu\nu}\,(S_{\rm L}^{\mu\nu})_a{}^b. \tag{34.3}$$

In Cadabra2 notation, the left-handed field and its generator are:

$$\psi_{\alpha}, \qquad (S^{\mu\nu}{}_{L})\,{}_{\alpha}{}^{\beta}$$

3 Generators $S_{\rm L}^{\mu\nu}$ in the $(2, 1)$ Representation

3.1 Commutation relations

The six 2×2 generator matrices $S_{\rm L}^{\mu\nu}$ (antisymmetric: $S_{\rm L}^{\mu\nu} = -S_{\rm L}^{\nu\mu}$) satisfy the Lorentz algebra (eq. 34.4):

$$[S_{\rm L}^{\mu\nu}, S_{\rm L}^{\rho\sigma}] = i\Big(g^{\nu\rho}S_{\rm L}^{\mu\sigma} - g^{\mu\rho}S_{\rm L}^{\nu\sigma} - g^{\nu\sigma}S_{\rm L}^{\mu\rho} + g^{\mu\sigma}S_{\rm L}^{\nu\rho}\Big), \tag{34.4}$$

with metric $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

3.2 Explicit matrices

Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{34.8}$$

Spatial rotation generators (eq. 34.9).

$$(S_{\mathrm{L}}^{ij})_a{}^b = \tfrac{1}{2}\varepsilon^{ijk}\sigma_k \tag{34.9}$$

$$S_{\mathrm{L}}^{12} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad S_{\mathrm{L}}^{13} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{L}}^{23} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Boost generators (eq. 34.10).

$$(S_{\mathrm{L}}^{k0})_a{}^b = \tfrac{i}{2}\sigma_k \tag{34.10}$$

$$S_{\mathrm{L}}^{10} = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{L}}^{20} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{L}}^{30} = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

Physical interpretation. The i factor in the boost generators means boosts are *not unitary*: the Lorentz group is non-compact. Rotations (S^{ij}) are Hermitian; boosts (S^{k0}) are anti-Hermitian.

Numerical verification. ✓ All $\binom{6}{2} = 15$ commutator pairs satisfy eq. (34.4). Max error: $0.0e+00$.

4 Right-Handed Spinor Field $\psi^\dagger_{\dot a}$

4.1 Why hermitian conjugation flips the representation

For ψ_a in $(2,1)$: N_i acts as $\frac{1}{2}\sigma_i$ (spin-$\frac{1}{2}$), $N_i^\dagger$ acts trivially (spin-0). Taking † swaps $N_i \leftrightarrow N_i^\dagger$. Therefore $(\psi_a)^\dagger \equiv \psi^\dagger_{\dot a}$ lives in $(1,2)$.

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot a}(x). \tag{34.11}$$

In Cadabra2: $\psi^\dagger$

4.2 Right-handed generators and the dotted-index rule

The dotted index $\dot{a}$ signals membership in $(1,2)$. The generators satisfy (eq. 34.17):

$$\boxed{(S_{\mathrm{R}}^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -\left[(S_{\mathrm{L}}^{\mu\nu})_{a}{}^{b}\right]^{*}} \tag{34.17}$$

This has a physical consequence:

- **Rotation generators** (S^{ij}): real parts unchanged, imaginary parts flip sign. Since σ_1, σ_3 are real and σ_2 is purely imaginary, $S_{\mathrm{R}}^{ij} = -[S_{\mathrm{L}}^{ij}]^*$ differs from S_{L}^{ij} by the sign of σ_2 components.
- **Boost generators** (S^{k0}): the i flips sign, so $S_{\mathrm{R}}^{k0} = -[S_{\mathrm{L}}^{k0}]^* = -\frac{i}{2}\sigma_k$. Boosts are reversed — consistent with parity $L \leftrightarrow R$.

Explicit $S_{\mathrm{R}}^{\mu\nu}$ matrices.

$$S_{\mathrm{R}}^{12} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad S_{\mathrm{R}}^{13} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{R}}^{23} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$$S_{\mathrm{R}}^{10} = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{R}}^{20} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \qquad S_{\mathrm{R}}^{30} = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}$$

5 The ε Symbol — SL(2,$\mathbb{C}$) Metric

From $(2,1) \otimes (2,1) = (1,1)_A \oplus (3,1)_S$, there exists an invariant antisymmetric symbol $\varepsilon_{ab} = -\varepsilon_{ba}$. In Cadabra2: $\epsilon_{\alpha\beta}$.

5.1 Normalization (Srednicki convention, eq. 34.22)

$$\varepsilon^{12} = \varepsilon_{21} = +1, \qquad \varepsilon^{21} = \varepsilon_{12} = -1. \tag{2}$$

$$\varepsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Completeness (eq. 34.23):

$$\varepsilon_{ab}\,\varepsilon^{bc} = \delta_a{}^c. \tag{3}$$

5.2 Raising and lowering

$$\psi^a = \varepsilon^{ab}\,\psi_b \qquad \text{(Cadabra2: } \epsilon^{\alpha\beta}\psi_\beta) \tag{4}$$

$$\psi_a = \varepsilon_{ab}\,\psi^b \tag{5}$$

Sign trap (eq. 34.27):

$$\psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b. \tag{6}$$

The contraction $\psi^a\chi_a = -\psi_a\chi^a$ carries an essential minus sign. The same $\varepsilon_{\dot{a}\dot{b}}$ structure holds for dotted indices.

Numerical verification. ✓ $\varepsilon_{ab}\,\varepsilon^{bc} = \delta_a{}^c$ and invariance under SL(2,ℂ): max error $0.0e+00$.

6 Lorentz-Invariant Spinor Products

6.1 Left-handed ("angle bracket")

$$\langle\psi\chi\rangle \equiv \varepsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \psi^\alpha\chi_\alpha \tag{35.21 preview}$$

Cadabra2: $\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta$.

Antisymmetry (Grassmann + ε antisymmetric): $\langle\psi\chi\rangle = -\langle\chi\psi\rangle$.

6.2 Right-handed ("square bracket")

$$[\psi^\dagger] \equiv \varepsilon_{\dot\alpha\dot\beta}\psi^{\dagger\dot\alpha\dot\beta} \tag{7}$$

Cadabra2: $\epsilon\bar\psi\,\bar\chi$.

7 The σ^μ Symbol — Vector/Spinor Dictionary

A field $A_{a\dot a}$ in $(2,2)$ maps to a 4-vector via (eq. 34.28):

$$A_{a\dot a} = \sigma^\mu_{a\dot a}\,A_\mu, \qquad \sigma^\mu_{a\dot a} = (I,\,\vec\sigma). \tag{34.30}$$

$$\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and $\bar\sigma^\mu_{\dot a a} = (I, -\vec\sigma)$:

$$\bar\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \bar\sigma^1 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \quad \bar\sigma^2 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \quad \bar\sigma^3 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Key identity.

$$\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) \equiv \sigma^\mu_{a\dot a}\,\bar\sigma^{\nu\,\dot a a} = 2\,g^{\mu\nu}. \tag{8}$$

✓ Verified numerically: max error $0.0e+00$.

8 Summary

Object	Expression
Left-handed field	ψ_α (undotted index)
Right-handed field	$\psi^\dagger_{\dot\alpha} = (\psi_\alpha)^\dagger$ (dotted index)
Rotation generator	$(S_L^{ij})_a{}^b = \frac{1}{2}\varepsilon^{ijk}\sigma_k$
Boost generator	$(S_L^{k0})_a{}^b = \frac{i}{2}\sigma_k$
R-generators	$S_R^{\mu\nu} = -[S_L^{\mu\nu}]^*$
Raise index	$\psi^a = \varepsilon^{ab}\psi_b$
Lower index	$\psi_a = \varepsilon_{ab}\psi^b$
Sign identity	$\psi^a\chi_a = -\psi_a\chi^a$
Angle bracket	$\langle\psi\chi\rangle = \varepsilon^{\alpha\beta}\psi_\alpha\chi_\beta$
Square bracket	$[\psi^\dagger] = \varepsilon_{\dot\alpha\dot\beta}\psi^{\dagger\dot\alpha\dot\beta}$
Vector dictionary	$\sigma^\mu_{a\dot a} = (I, \vec\sigma)$, $\bar\sigma^\mu_{\dot a a} = (I, -\vec\sigma)$
Trace identity	$\mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = 2g^{\mu\nu}$

Next: Chapter 35 develops the index-free dot/bar notation, derives the σ-algebra identities, and constructs the Weyl Lagrangian.