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Homework Assignment 5

Алгебра и геометрия, 2 семестр

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Задача 1.
Определить тип поверхности в зависимости от параметра k
$x^2 + z^2 + 4xy - 2xz - 4yz - 4x - 8y + 4z + k = 0$

Решение:
$(a_{11} = 1; a_{22} = 0; a_{33} = 1; a_{12} = 2; a_{13} = -1; a_{23} = -2; a_{14} = -2; a_{24} = -4; a_{34} = 2; a_{44} = k)$

$$J_3 = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & -2 \\ -1 & -2 & -1 \end{vmatrix} = 0$$

$$J_2 = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix} = -8$$

$$J_4 = \begin{vmatrix} 1 & 2 & -1 & -2 \\ 2 & 0 & -2 & -4 \\ -1 & -2 & 1 & 2 \\ -2 & -4 & 2 & k \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & -2 \\ 2 & 0 & -2 & -4 \\ 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & k \end{vmatrix} = 0$$

$$\tilde{J}_3 = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -4 \\ -1 & -4 & k \end{vmatrix} + \begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -2 & 2 & k \end{vmatrix} + \begin{vmatrix} 0 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & k \end{vmatrix} = 32 - 8k$$

$J_1 = 2$

$J(\lambda) = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = -\lambda^3 + 2\lambda^2 + 8\lambda = -\lambda(\lambda+2)(\lambda-4)$
$$\begin{cases} \lambda = 0 \\ \lambda = -2 \\ \lambda = 4 \end{cases}$$
-2x$^2 + 4y^2 + \frac{32-8k}{-8} = 0$
$x^2 - 4y^2 + \frac{32-8k}{16} = 0$

$k = 4 : x^2 - 4y^2 = 0 => \tilde{J}_3 = 0$ (Пара пересекающихся плоскостей)
$k! = 4 => \tilde{J}_3! = 0$ (Гиперболический цилиндр)

Задача 2.
Определить тип линии, центр (при наличии), фокусы и директрисы:
$$\begin{cases} x^2 - y^2 - z^2 = 1, \\ x - y = 3. \end{cases}$$

Решение:

$$\begin{cases} x^2 - y^2 - z^2 = 1 \\ x - y = 3 \end{cases}$$

вектор нормали n = (1, -1, 0)

$\Rightarrow e_3' = \{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\}$

$](\cdot)A(3,0,0)$

$](\cdot)B(0,-3,0)$

= (3, 3, 0) $\Rightarrow e_1' = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\}$

$$e_2' = \frac{1}{2} \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \frac{1}{2} * 2k = k$$

$\Rightarrow e_2' = \{0, 0, 1\}$

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} * \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$]A = (3,0,0)$ центр

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} * \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{cases} \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z' + 3 \\ \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z' \\ y' \end{cases}$$

подставим в исходную систему

$$\begin{cases} z' = 0 \\ x^2 - y^2 - z^2 = 1 \end{cases} \Rightarrow \begin{cases} z' = 0 \\ (\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z' + 3)^2 - (\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z')^2 - (y')^2 = 1 \end{cases}$$

$$\begin{cases} z' = 0 \\ (\frac{1}{\sqrt{2}}x' + 3)^2 - (\frac{1}{\sqrt{2}}x')^2 - (y')^2 = 1 \end{cases} \Rightarrow \begin{cases} z' = 0 \\ \frac{x'^2}{2} + \frac{6x'}{\sqrt{2}} + 9 - \frac{x'^2}{2} - y'^2 = 1 \end{cases}$$

$$\begin{cases} z' = 0 \\ y'^2 = \frac{6x'}{\sqrt{2}} - 8 \end{cases} \Rightarrow \begin{cases} z' = 0 \\ y'^2 = 6\sqrt{2}(x' - \frac{4\sqrt{2}}{3}) \end{cases} \Rightarrow \text{парабола}$$

найдем фокус и директрису

$$\begin{cases} z' = 0 \\ y'^2 = 6\sqrt{2}(x' - \frac{4\sqrt{2}}{3}) \end{cases}$$

$$\begin{cases} k = 0 \\ h = 4\sqrt{2}_{\overline{3}} \\ p = 3\sqrt{2}_{\overline{2}} \end{cases}$$

F = ($\frac{17\sqrt{2}}{6}$, 0,0)

d = $\frac{4\sqrt{2}}{3}$ - $\frac{3\sqrt{2}}{2} = -\frac{\sqrt{2}}{6}$

Вершина: ($\frac{4\sqrt{2}}{3}$,0, 0)

вернемся к Oxyz:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} x-3 \\ y \\ z \end{pmatrix}$$

$$\text{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x-3) + \frac{1}{\sqrt{2}}y \\ z \\ \frac{1}{\sqrt{2}}(x-3) - \frac{1}{\sqrt{2}}y \end{pmatrix}$$

подставим x', y', z':

$$\begin{cases} y = x-3 \\ z^2 = 6\sqrt{2}(\frac{1}{\sqrt{2}}(x-3) + \frac{1}{\sqrt{2}}y - \frac{4\sqrt{2}}{3}) \end{cases} \Rightarrow \begin{cases} y = x-3 \\ z^2 = 6\sqrt{2}(\frac{1}{\sqrt{2}}(x-3) + \frac{1}{\sqrt{2}}(x-3) - \frac{4\sqrt{2}}{3}) \end{cases}$$

$$\Rightarrow \begin{cases} y = x-3 \\ z^2 = 6\sqrt{2}(\frac{2}{\sqrt{2}}x - \frac{6}{\sqrt{2}} - \frac{4\sqrt{2}}{3}) \end{cases} \Rightarrow \begin{cases} y = x-3 \\ z^2 = 6\sqrt{2}(\frac{2}{\sqrt{2}}x - \frac{10}{3\sqrt{2}}) \end{cases}$$

$$\begin{cases} k = 0 \\ h = \frac{10}{3\sqrt{2}} \\ p = \frac{3\sqrt{2}}{2} \end{cases}$$

вершина: $(\frac{10}{3\sqrt{2}}, 0, 0)$

d $= \frac{10}{3\sqrt{2}} - \frac{3\sqrt{2}}{2} = \frac{1}{3\sqrt{2}}$

F $= (\frac{19}{3\sqrt{2}}$, 0,0)

Задача 3.

Определить тип линии, центр (при наличии), фокусы и директрисы:

$$\begin{cases} x^2 - y^2 + z^2 = 1, \\ x + 2y = 1. \end{cases}$$

Решение:

$$\begin{cases} x^2 - y^2 + z^2 = 1 \\ x + 2y = 1 \end{cases}$$

вектор нормали n = {1, 2, 0}

$\Rightarrow e_3' = \{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\}$

$(\cdot)A(1,0,0)$

$(\cdot)B(-1,1,0)$

$\vec{AB} = \{-2, 1, 0\}$.

e$_1' = \{-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\}$

$e_2' = e_1' \times e_3'$

$e_2' = \{0, 0, 1\}$

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$]A(1,0,0) -$ центр

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = -\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z' + 1 \\ y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}z' \\ z = y' \end{cases} \Rightarrow \begin{cases} z' = 0 \\ \left(1 - \frac{2}{\sqrt{5}}x'\right)^2 - \left(\frac{1}{\sqrt{5}}x'\right)^2 + y'^2 = 1 \end{cases}$$

$$\left(1 - \frac{2}{\sqrt{5}}x'\right)^2 - \left(\frac{1}{\sqrt{5}}x'\right)^2 + y'^2 = 1$$

$$\frac{-4\sqrt{5}}{5}x' + \frac{3}{5}x'^2 + y'^2 = 0$$

$$\left(x' - \frac{2\sqrt{5}}{3}\right)^2 + \frac{5}{3}y'^2 = \frac{20}{9}$$

$$\frac{\left(x' - \frac{2\sqrt{5}}{3}\right)^2}{\left(\frac{2\sqrt{5}}{3}\right)^2} + \frac{y'^2}{\left(\frac{2}{\sqrt{3}}\right)^2} = 1$$

$$\begin{cases} a = \frac{2\sqrt{5}}{3} \\ b = \frac{2}{\sqrt{3}} \end{cases} \Rightarrow c = \frac{2\sqrt{2}}{3}; \quad \varepsilon = \frac{\sqrt{10}}{5}$$

центр: $\left(\frac{2\sqrt{5}}{3}; 0; 0\right)$

фокусы:$F_1'\left(\frac{2\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}; 0; 0\right); F_2'\left(\frac{2\sqrt{5}}{3} - \frac{2\sqrt{2}}{3}; 0; 0\right)$

директрисы:$x' = \pm\frac{a}{\varepsilon} + h \Rightarrow \begin{cases} x' = \frac{2\sqrt{5}}{3} + \frac{5\sqrt{2}}{3} \\ x' = \frac{2\sqrt{5}}{3} - \frac{5\sqrt{2}}{3} \end{cases}$

перейдем в старую Oxyz:

$$\begin{cases} x = 1 - \frac{2}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{3} = -\frac{1}{3} \\ y = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{3} = \frac{2}{3} \\ z = 0 \end{cases} \Rightarrow \text{центр } \left(-\frac{1}{3}; \frac{2}{3}; 0\right)$$

фокусы:

$$F_1 \begin{cases} x = 1 - \frac{2}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\right) \\ y = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\right) \\ z = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{3} - \frac{4\sqrt{10}}{15} \\ y = \frac{2}{3} + \frac{2\sqrt{10}}{15} \\ z = 0 \end{cases}$$

$$F_2 \begin{cases} x = 1 - \frac{2}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} - \frac{2\sqrt{2}}{3}\right) \\ y = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} - \frac{2\sqrt{2}}{3}\right) \\ z = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{3} + \frac{4\sqrt{10}}{15} \\ y = \frac{2}{3} - \frac{2\sqrt{10}}{15} \\ z = 0 \end{cases}$$

$F_1\left(-\frac{1}{3} + \frac{4\sqrt{10}}{15}; \frac{2}{3} - \frac{2\sqrt{10}}{15}; 0\right)$ и $F_2\left(-\frac{1}{3} - \frac{4\sqrt{10}}{15}; \frac{2}{3} + \frac{2\sqrt{10}}{15}; 0\right)$

директрисы:

$$d_1 \begin{cases} x_1 = 1 - \frac{2}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} + \frac{5\sqrt{2}}{3}\right) = -\frac{1}{3} - \frac{10\sqrt{10}}{15} \\ y_1 = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} + \frac{5\sqrt{2}}{3}\right) = \frac{2}{3} + \frac{\sqrt{10}}{3} \end{cases}$$

$$d_2 \begin{cases} x_1 = 1 - \frac{2}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} - \frac{5\sqrt{2}}{3}\right) = -\frac{1}{3} + \frac{10\sqrt{10}}{15} \\ y_1 = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{3} - \frac{5\sqrt{2}}{3}\right) = \frac{2}{3} - \frac{\sqrt{10}}{3} \end{cases}$$

центр: $\begin{cases} x = 1 - \frac{2}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{3} = -\frac{1}{3} \\ y = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{3} = \frac{2}{3} \\ z = 0 \end{cases} \Rightarrow \left(-\frac{1}{3}; \frac{2}{3}; 0\right)$
