HW3

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1. QUESTION

1. [*Basic Proof Theory* Lemma 2.3.2] For φ negative, $\mathbf{M} \vdash \varphi \leftrightarrow \neg\neg\varphi$.
2. [*Basic Proof Theory* Ex. 2.3.6A] Show that in **Ni** all instances of $\bot_\mathbf{i}$ are derivable from the instances of $\bot_\mathbf{i}$ with atomic conclusion. Show that in **Nc**, for the language without $\vee, \exists$, all instances of $\bot_\mathbf{c}$ are derivable from instances $\bot_\mathbf{c}$ with atomic conclusion.

2. ANSWER

1. We use induction on the formation rules of negative formulas. Note that for any formula φ that $\mathbf{M} \vdash \varphi \to \neg\neg\varphi$ always holds, so we only need to prove $\mathbf{M} \vdash \neg\neg\varphi \to \varphi$ for φ negative. We shall choose **Nm** as the formulation of minimal logic to prove this result for it equivalent to **Hm**.

   - **BASE ($\bot$, $\neg p$ for p atomic)**
     - $\bot$

$$
\begin{prooftree}
\AxiomC{$[\bot]^u$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\bot$}
\AxiomC{$[\neg\neg\bot]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\bot \to \bot$}
\end{prooftree}
$$

     - $\neg p$ Same as HW2.3

   - **IND HYPS ($\varphi \wedge \psi, \varphi \to \psi, \forall x\varphi$ for φ, ψ negative)** By induction hypothesis $\mathbf{M} \vdash \neg\neg\varphi \to \varphi$ and $\mathbf{M} \vdash \neg\neg\psi \to \psi$ holds, and we will show that these connectives and quantifiers preserve this property.
     - We merge the two similar branches of the proof tree to save space.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi, \neg\psi]^{u,v}$}
\AxiomC{$[\varphi \wedge \psi]^{\tau,\sigma}$}
\RightLabel{$\wedge$E}
\UnaryInfC{φ, ψ}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{τ, σ}
\RightLabel{$\to$I}
\UnaryInfC{$\neg(\varphi \wedge \psi)$}
\AxiomC{$[\neg\neg(\varphi \wedge \psi)]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u, v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi, \neg\neg\psi$}
\AxiomC{}
\RightLabel{Ind Hyps}
\UnaryInfC{$\neg\neg\varphi \to \varphi, \neg\neg\psi \to \psi$}
\RightLabel{$\to$E}
\BinaryInfC{φ, ψ}
\RightLabel{$\wedge$I}
\UnaryInfC{$\varphi \wedge \psi$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg(\varphi \wedge \psi) \to (\varphi \wedge \psi)$}
\end{prooftree}
$$

     - By HW2.4 we only need to prove that $\mathbf{M} \vdash (\neg\neg\varphi \to \neg\neg\psi) \to \varphi \to \psi$.

$$
\begin{prooftree}
\AxiomC{$[\varphi]^u$}
\AxiomC{$[\neg\varphi]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\neg\varphi \to \neg\neg\psi]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg\psi$}
\AxiomC{}
\RightLabel{Ind Hyps}
\UnaryInfC{$\neg\neg\psi \to \psi$}
\RightLabel{$\to$E}
\BinaryInfC{ψ}
\LeftLabel{u}
\RightLabel{$\to$E}
\UnaryInfC{$\varphi \to \psi$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$(\neg\neg\varphi \to \neg\neg\psi) \to \varphi \to \psi$}
\end{prooftree}
$$

     - Similarly, we only prove $\mathbf{M} \vdash \forall x\neg\neg\varphi \to \forall x\varphi$ by HW2.5.

$$
\begin{prooftree}
\AxiomC{$[\forall x\neg\neg\varphi]^u$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{}
\RightLabel{Ind Hyps}
\UnaryInfC{$\neg\neg\varphi \to \varphi$}
\RightLabel{$\to$E}
\BinaryInfC{φ}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x\varphi$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\forall x\neg\neg\varphi \to \forall x\varphi$}
\end{prooftree}
$$

   Q.E.D

2. We prove that by means of induction on the conclusion. The variant of $\bot_{\mathbf{i}}$ and $\bot_{\mathbf{c}}$ which restrict to atomic conclusions are denoted as $\bot_{\mathbf{i}}^*$ and $\bot_{\mathbf{c}}^*$.

   - **BASE** Hold by $\bot_{\mathbf{i}}^*$ and $\bot_{\mathbf{c}}^*$.
   - **IND HYPS**($\wedge, \vee, \to, \forall, \exists$) By induction hypothesis there exists a derivation of φ and ψ from $\bot$ by using $\bot_{\mathbf{i}}^*$ and from a derivation from $\neg\varphi$ and $\neg\psi$ to $\bot$ by using $\bot_{\mathbf{c}}^*$, and we also denote them as $\bot_{\mathbf{i}}$ and $\bot_{\mathbf{c}}$. Moreover, we denote the derivation from a negative assumption to the absurdity in $\bot_{\mathbf{c}}$ which need to be filled by the user of the rule as Π. Note that the variable substitution only affects on atomic formulas so the induction hypothesis is valid on any Gentzen-style subformula $\varphi[x := a]$ of $\forall x\varphi$ and $\exists x\varphi$.
     - $\varphi \wedge \psi$
       - $\bot_{\mathbf{i}}$

$$\dfrac{\dfrac{\bot}{\varphi, \psi}\,\bot_{\mathbf{i}}}{\varphi \wedge \psi}\,\wedge\mathrm{I}$$

       - $\bot_{\mathbf{c}}$

$$\tau,\sigma\;\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^{\tau,\sigma}}{\varphi, \psi}\,\wedge\mathrm{E} \qquad [\neg\varphi, \neg\psi]^{u,v}}{\bot}\,\to\mathrm{E}}{\neg(\varphi \wedge \psi)}\,\to\mathrm{I}$$
$$\Pi$$
$$\dfrac{u, v\;\dfrac{\bot}{\varphi, \psi}\,\bot_{\mathbf{c}}}{\varphi \wedge \psi}\,\wedge\mathrm{I}$$

     - $\varphi \vee \psi$

$$\dfrac{\dfrac{\bot}{\varphi}\,\bot_{\mathbf{i}}}{\varphi \vee \psi}\,\vee\mathrm{I}$$

     - $\varphi \to \psi$
       - $\bot_{\mathbf{i}}$

$$u\;\dfrac{\dfrac{\bot}{\psi}\,\bot_{\mathbf{i}}}{\varphi \to \psi}\,\to\mathrm{I}$$

       - $\bot_{\mathbf{c}}$

$$v\;\dfrac{\dfrac{\dfrac{[\varphi]^u \qquad [\varphi \to \psi]^v}{\psi}\,\to\mathrm{E} \qquad [\neg\psi]^w}{\bot}\,\to\mathrm{E}}{\neg(\varphi \to \psi)}\,\to\mathrm{I}$$
$$\Pi$$
$$u\;\dfrac{w\;\dfrac{\bot}{\psi}\,\bot_{\mathbf{c}}}{\varphi \to \psi}\,\to\mathrm{I}$$

     - $\forall x\varphi$ We may need to take a fresh variable a which is no free occurrence in the assumptions and φ to avoid clash. Hence the proof tree is finite and the variables are at least countably many, we can always take such a variable to finish our proof.
       - $\bot_{\mathbf{i}}$

$$\dfrac{\dfrac{\bot}{\varphi[x := a]}\,\bot_{\mathbf{i}}}{\forall x\varphi}\,\forall\mathrm{I}$$

       - $\bot_{\mathbf{c}}$

$$u\;\dfrac{\dfrac{\dfrac{[\forall x\varphi]^u}{\varphi[x := a]}\,\forall\mathrm{E} \qquad [\neg\varphi[x := a]]^v}{\bot}\,\to\mathrm{E}}{\neg\forall x\varphi}\,\to\mathrm{I}$$
$$\Pi$$
$$v\;\dfrac{\dfrac{\bot}{\varphi[x := a]}\,\bot_{\mathbf{c}}}{\forall x\varphi}\,\forall\mathrm{I}$$

- $\exists x\varphi$

$$\dfrac{\dfrac{\perp}{\varphi[x := t]}\,\perp_{\mathbf{i}}}{\exists x\varphi}\,\exists\mathrm{I}$$

Q.E.D

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