HW9

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1 QUESTION

1. Prove the case of (L$\exists$) and (R$\exists$) for **G3**[**mi**] in 3.5.2(α).

2 ANSWER

1. *Proof.*

   (**R**$\exists$) For arbitrary fresh renaming $*$ free in the conclusion $\Gamma \Rightarrow \exists x\varphi$, we can find a proper renaming $**$ s.t. $\Gamma^{**} \equiv \Gamma^*$. If $*$ does not rename the bounded x in $\exists x\varphi$, we let $**$ has the same activity on φ as $*$. Then we can obtain that $(\varphi^x_t)^* \equiv (\varphi^x_t)^{**} \equiv (\varphi^{**})^x_t \equiv (\varphi^*)^x_t$ hence the effects of bounded variable renaming and term substitution are orthogonal. We can apply (R$\exists$) on $[\Gamma \Rightarrow \varphi^x_t]^{**} \equiv \Gamma^* \Rightarrow (\varphi^*)^x_t$ by induction hypothesis which concludes $\Gamma^* \Rightarrow \exists x\varphi^* \equiv [\Gamma \Rightarrow \exists x\varphi]^*$ since $\exists x\varphi^* \equiv (\exists x\varphi)^*$ for $*$ is free for x in $\exists x\varphi$.

   Otherwise, $\exists x\varphi$ will be rename to $\exists y(\varphi^*)^x_y$ s.t. $y \notin \mathrm{FV}(\varphi^*) = \mathrm{FV}(\varphi)$. Since y is free in φ we have $\varphi^x_t \equiv \varphi^{xy}_{y\,t}$. Then we can define $**$ as the same way and use the fact that substitution and renaming are commutable again: $[\Gamma \Rightarrow \varphi^x_t]^{**} \equiv \Gamma^* \Rightarrow (\varphi^*)^{xy}_{y\,t}$. If we apply (R$\exists$) on it by induction hypothesis then we can conclude $\Gamma^* \Rightarrow \exists y(\varphi^*)^x_y$, which is exactly what we want.

   (**L**$\exists$) The case that $*$ does not rename x in $\exists x\varphi$ in conclusion is very similar to (R$\exists$), now we focus on the case that $(\exists x\varphi)^* \equiv \exists z(\varphi^*)^x_z$ for some z fresh. Since z is fresh, we have $(\varphi^x_y)^* \equiv (\varphi^*)^x_y \equiv (\varphi^*)^{xz}_{zy}$. We can apply (L$\exists$) on $[\Gamma, \varphi^x_y \Rightarrow \psi]^{**} \equiv \Gamma^*, (\varphi^*)^{xz}_{zy} \Rightarrow \psi^*$ to obtain $\Gamma^*, \exists z(\varphi^*)^x_z \Rightarrow \psi^*$ because y is free for Γ, ψ and $\exists x\varphi$(this implies $y \equiv x$ or $y \notin \mathrm{FV}(\varphi) = \mathrm{FV}((\varphi^*)^x_z)$) and changing bound variables does not affect free variables.

$\square$