HW7

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1 QUESTION

1. (*Basic Proof Theory* Ex. 3.3.3B)

   (a) For N-systems, let us write $\vdash \Gamma \Rightarrow \varphi$ if φ can be deduced from assumptions in Γ. Let $\mathbf{Nc}'$ be $\mathbf{Nc}$ with $\perp_{\mathbf{c}}$ replaced by the Peirce rule (P), defined in 2.5.2. Show $\mathbf{Nc} \vdash \Gamma \Rightarrow \varphi$ iff $\mathbf{Nc}' \vdash \Gamma \Rightarrow \varphi$.

   In the following two pats of the exercise we extend the equivalence between N-systems and G-systems to fragments $\mathcal{X}$ s.t. $\{\rightarrow\} \subset \mathcal{X} \subset \{\rightarrow, \vee, \wedge, \forall, \exists\}$. Let $\mathbf{G}$ be $\mathcal{X}$-$\mathbf{G2c} \oplus$ (Cut).

   (b) If $\mathcal{S} \equiv \Gamma \Rightarrow \varphi, \Delta$, then a sequent $\mathcal{S}^* \equiv \Gamma, \Delta \rightarrow \varphi \Rightarrow \varphi$ is call a 1-equivalent of $\mathcal{S}$; $\Delta \rightarrow \varphi$ abbreviates $\psi_1 \rightarrow \varphi, \psi_2 \rightarrow \varphi, \dots, \psi_n \rightarrow \varphi$ for $\Delta \equiv \psi_1, \psi_2, \dots, \psi_n$. Show that any two 1-equivalents $\mathcal{S}^*$, $\mathcal{S}^{**}$ of a sequent $\mathcal{S}$ are provably equivalent in $\mathbf{G}$.

   (c) Show that $\mathbf{G} \vdash \Gamma \Rightarrow \varphi$ iff $\mathcal{X}$-$\mathbf{Nc}' \vdash \Gamma \Rightarrow \varphi$. *Hint.* For the proof from left to right, show by induction on the depth of deductions in $\mathbf{G}$, that whenever $\mathbf{G} \vdash \mathcal{S}$, then for some 1-equivalent $\mathcal{S}^*$ of $\mathcal{S}$, $\mathbf{Nc}' \vdash \mathcal{S}^*$.

2. (*Basic Proof Theory* Ex. 3.3.3C) Prove equivalence of $\mathbf{G1i}$ with the Hilbert system $\mathbf{Hi}$ directly, that is to say, not via the equivalence of $\mathbf{G1i}$ with natural deduction.

2 ANSWER

1. (a) We can prove that (P) is derivable from $\perp_{\mathbf{c}}$ directly, but it seems that we cannot derive $\perp_{\mathbf{c}}$ from (P) without $\perp_{\mathbf{i}}$ (incomplete, need a proof).

   Proof.
   - $\mathbf{Nc} \vdash \Gamma \overset{\delta}{\Rightarrow} \varphi$ implies $\mathbf{Nc}' \vdash \Gamma \overset{\delta'}{\Rightarrow} \varphi$: for any sub-derivation δ_1 end with $\perp_{\mathbf{c}}$ in δ, we can convert it into a derivation δ_2 which use (P) and $\perp_{\mathbf{i}}$ instead of $\perp_{\mathbf{c}}$ and then formulate a new derivation δ' in $\mathbf{Nc}'$ by replacing every occurrence of δ_1 in δ by δ_2.

   $$\begin{array}{c} [\neg\varphi]^u \\ \vdots\,\pi \\ u\,\dfrac{\perp}{\varphi}\,\perp_{\mathbf{c}} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} [\neg\varphi]^u \\ \vdots\,\pi \\ \dfrac{\perp}{\varphi}\,\perp_{\mathbf{i}} \\ u\,\dfrac{\varphi}{\varphi}\,\mathrm{P} \end{array}$$

   - $\mathbf{Nc}' \vdash \Gamma \overset{\delta}{\Rightarrow} \varphi$ implies $\mathbf{Nc} \vdash \Gamma \overset{\delta'}{\Rightarrow} \varphi$: similar.

   $$\begin{array}{c} [\varphi \rightarrow \psi]^u \\ \vdots\,\pi \\ u\,\dfrac{\varphi}{\varphi}\,\mathrm{P} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{[\varphi]^u \qquad [\neg\varphi]^v}{\dfrac{\perp}{\psi}\,\perp_{\mathbf{c}}}\rightarrow\!\mathrm{E} \\ u\,\dfrac{\qquad}{\varphi \rightarrow \psi}\rightarrow\!\mathrm{I} \\ \vdots\,\pi \\ \dfrac{\varphi \qquad\qquad [\neg\varphi]^v}{v\,\dfrac{\perp}{\varphi}\,\perp_{\mathbf{c}}}\rightarrow\!\mathrm{E} \end{array}$$

   $\square$

   (b) We prove this by showing that for any sequent $\mathcal{S}$, $\mathbf{G} \vdash \mathcal{S}$ iff $\mathbf{G} \vdash \mathcal{S}^*$ for arbitrary $\mathcal{S}^*$ which is 1-equivalent to $\mathcal{S}$. Note that there is at least one formula on the succedent in any sequent hence we are in a fragment which does not contain $\perp$ so (L$\perp$).

   Proof. Let $\mathcal{S} \equiv \Gamma \Rightarrow \Delta$ and $\Delta \equiv \varphi_0, \varphi_1, \dots, \varphi_n (n \geq 0)$, we denote $\Delta \setminus \varphi_i$ as Θ_i. We can prove that by induction on the cardinality of Δ. Hence $\mathcal{S} \equiv \mathcal{S}^*$ for $|\Delta| = 0$, now we show the successive case.

- If $\mathbf{G} \vdash \Gamma \Rightarrow \Delta$ implies $\mathbf{G} \vdash \Gamma, \Theta_i \to \varphi_i \Rightarrow \varphi_i$ for arbitrary Γ, Δ and $0 \le i \le |\Delta|$, then $\mathbf{G} \vdash \Gamma \overset{\delta}{\Rightarrow} \varphi, \Delta$ implies $\mathbf{G} \vdash \Gamma, \Delta \to \varphi \overset{\pi}{\Rightarrow} \varphi$ for arbitrary φ.

$$\dfrac{\dfrac{\overset{\vdots\,\delta}{\Gamma \Rightarrow \varphi, \Delta}}{\Gamma \Rightarrow \varphi, \varphi_i, \Theta_i}\equiv \qquad \dfrac{}{\varphi, \Gamma \Rightarrow \varphi, \Theta_i}\text{Ax}}{\Gamma, \varphi_i \to \varphi \Rightarrow \varphi, \Theta_i}\text{L}{\to} \quad \xrightarrow{\text{by IH}} \quad \dfrac{\overset{\vdots\,\pi}{\Gamma, \varphi_i \to \varphi, \Theta_i \to \varphi \Rightarrow \varphi}}{\Gamma, \Delta \to \varphi \Rightarrow \varphi}\equiv$$

- If $\mathbf{G} \vdash \Gamma, \Theta_i \to \varphi_i \Rightarrow \varphi_i, \Lambda$ implies $\mathbf{G} \vdash \Gamma \Rightarrow \Delta, \Lambda$ for arbitrary Γ, Δ, Λ and $0 \le i \le |\Delta|$, then $\mathbf{G} \vdash \Gamma, \Delta \to \varphi \overset{\delta}{\Rightarrow} \varphi, \Lambda$ implies $\mathbf{G} \vdash \Gamma \overset{\pi}{\Rightarrow} \varphi, \Delta, \Lambda$ for arbitrary φ.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi_i \Rightarrow \varphi_i, \varphi}\text{Ax}}{\Gamma \Rightarrow \varphi_i, \varphi_i \to \varphi}\text{R}{\to} \qquad \dfrac{\overset{\vdots\,\delta}{\Gamma, \Delta \to \varphi \Rightarrow \varphi, \Lambda}}{\Gamma, \varphi_i \to \varphi, \Theta_i \to \varphi \Rightarrow \varphi, \Lambda}\equiv}{\Gamma, \Gamma, \Theta_i \to \varphi \Rightarrow \varphi, \varphi_i, \Lambda}\text{Cut}}{\Gamma, \Theta_i \to \varphi \Rightarrow \varphi, \varphi_i, \Lambda}\text{LC}^{|\Gamma|} \quad \xrightarrow{\text{by IH}} \quad \dfrac{\overset{\vdots\,\pi}{\Gamma \Rightarrow \varphi, \Theta_i, \varphi_i, \Lambda}}{\Gamma \Rightarrow \varphi, \Delta, \Lambda}\equiv$$

$\square$

(c) We use $\Gamma \vdash \Delta$ to denote there is a $\mathcal{X}$-$\mathbf{Nc}'$ derivation of φ from Γ, i.e., $\mathcal{X}$-$\mathbf{Nc}' \vdash \Gamma \Rightarrow \varphi$ for *every*(not some!) $\varphi \in \Delta$.

Lemma 1 (Weakening)**.** *If* $\Gamma \vdash \Delta$, *then* $\Gamma, \Theta \vdash \Delta$. $\square$

Lemma 2 (Contraction)**.** *If* $\Gamma, \Gamma, \Theta \vdash \Delta$, *then* $\Gamma, \Theta \vdash \Delta$. $\square$

Lemma 3 (Cut)**.** *If* $\Gamma \overset{\delta}{\vdash} \Delta, \Sigma$ *and* $\Delta, \Theta \overset{\pi}{\vdash} \Pi$, *then* $\Gamma, \Theta \vdash \Pi$.

Proof. Just put δ_φ on every occurrence of φ for every $\varphi \in \Delta$ in π. $\square$

Lemma 4 (Chain)**.** $\Delta \to \varphi, \varphi \to \psi \vdash \Delta \to \psi$.

Proof. We use Δ in derivation to indicate that for every $\chi \in \Delta$ there exists a corresponding derivation.

$$\dfrac{u\ \dfrac{\dfrac{[\Delta]^u \qquad \dfrac{}{\Delta \to \varphi}\text{Asm}}{\varphi}{\to}\text{E} \qquad \dfrac{}{\varphi \to \psi}\text{Asm}}{\psi}{\to}\text{E}}{\Delta \to \psi}{\to}\text{I}$$

$\square$

Lemma 5 (Flip)**.** *For any* $\Gamma, \Delta \equiv \varphi_1, \ldots, \varphi_n$ *and* $\Theta_i \equiv \Delta \setminus \varphi_i$, $\Gamma, \Theta_i \to \varphi_i \vdash \varphi_i$ *iff* $\Gamma, \Theta_j \to \varphi_j \vdash \varphi_j$ *for every* $1 \le i, j \le n$.

Proof. We prove that $\Gamma, \Theta_i \to \varphi_i \overset{\delta}{\vdash} \varphi_i$ implies $\Gamma, \Theta_j \to \varphi_j \vdash \varphi_j$ w.l.o.g. Note that $\varphi_i \to \varphi_j \in \Theta_j \to \varphi_j$ and $\Theta_j \to \varphi_j, \varphi_j \to \varphi_i \vdash \Delta \to \varphi_i$ from Lemma 4 and Lemma 1.

$$u\ \dfrac{\dfrac{\overset{\vdots\,\delta}{\varphi_i} \qquad \dfrac{}{\varphi_i \to \varphi_j}\text{Asm}}{\varphi_j}{\to}\text{E}}{\varphi_j}\text{P} \qquad \text{where } \dfrac{\dfrac{}{\Gamma, \Theta_j \to \varphi_j}\text{Asm} \qquad [\varphi_j \to \varphi_i]^u}{\vdots\,\delta}$$

$\square$

- $\mathbf{G} \vdash \Gamma \Rightarrow \varphi$ implies $\Gamma \vdash \varphi$:

  Proof. We prove the property $P(n)$ that for any sequent $\Gamma \Rightarrow \Delta$, if $\mathbf{G} \vdash_n \Gamma \Rightarrow \Delta$ then for some $\Gamma, \Theta_i \to \varphi_i \Rightarrow \varphi$ 1-equivalent to $\Gamma \Rightarrow \Delta$ s.t. $\Gamma, \Theta_i \to \varphi_i \vdash \varphi$ for any $n \in \omega$ by induction on the height of proofs in $\mathbf{G}$. Note that there is no need to consider the empty succedent for the same reason which we explains in (b).

  BASE (Ax) There is only one zero-premise rule (Ax) in a $\bot$-free fragment $\mathbf{G}$. For any sequent $\Gamma, \varphi \Rightarrow \varphi, \Delta$ introduced by (Ax), we can prove $\Gamma, \Delta \to \varphi, \varphi \vdash \varphi$ by simply write φ down as a proof.

  IH Assume that for every $\mathbf{G} \vdash_n \Gamma \Rightarrow \Delta$ there exist a 1-equivalent sequent $\Gamma, \Theta_i \to \varphi_i \Rightarrow \varphi_i$ s.t. $\Gamma, \Theta_i \to \varphi_i \vdash \varphi$, we prove that for every sequent with a proof which height is $n+1$ the property $P(n+1)$ holds.

(LC) and (RC) The property P holds iff the diagram commute.

$$\begin{tikzcd}
\varphi,\varphi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] \arrow[d, "\text{LC}"] & \varphi,\varphi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, "\text{by IH}"] & \varphi,\varphi,\Gamma,\Theta_i \to \varphi_i \vdash \varphi_i \arrow[d, "\text{Lemma 2}"] \\
\varphi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] & \varphi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, dashed, "P"] & \varphi,\Gamma,\Theta_i \to \varphi_i \vdash \varphi_i
\end{tikzcd}$$

The case of (RC) can be similar to (LC) since we have the freedom to choose which formula in the succedent left, or we may need to prove $\vdash \varphi \to \varphi$ if we choose to preserve φ on the right hand side.

(L$\wedge$) Similar to (RC), just use Lemma 3 and $\varphi_0 \wedge \varphi_1 \vdash \varphi_i$ for $i = 0, 1$.

(L$\vee$) Use (1).

$$\begin{tikzcd}
\varphi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] \arrow[d, "\text{L}\vee"] & \varphi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, "\text{by IH}"] & \varphi,\Gamma,\Theta_i \to \varphi_i \overset{\delta}{\vdash} \varphi_i \arrow[d, "(1)"] \\
\varphi\vee\psi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] & \varphi\vee\psi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, dashed, "P"] & \varphi\vee\psi,\Gamma,\Theta_i \to \varphi_i \vdash \varphi_i \\
\psi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] \arrow[u, "\text{L}\vee"] & \psi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, "\text{by IH}"] & \psi,\Gamma,\Theta_i \to \varphi_i \overset{\pi}{\vdash} \varphi_i \arrow[u, "(1)"]
\end{tikzcd}$$

$$u,v\;\dfrac{\dfrac{}{\varphi\vee\psi}\,\text{Asm} \qquad \begin{matrix}\dfrac{[\varphi]^u \quad \dfrac{}{\Gamma,\Theta_i\to\varphi_i}\,\text{Asm}}{} \\ \vdots\;\delta \\ \varphi_i\end{matrix} \qquad \begin{matrix}\dfrac{[\psi]^v \quad \dfrac{}{\Gamma,\Theta_i\to\varphi_i}\,\text{Asm}}{} \\ \vdots\;\pi \\ \varphi_i\end{matrix}}{\varphi_i}\,\vee\text{E} \tag{1}$$

(L$\to$) Same as (RC), there are two kinds of possible 1-equivalent sequent of the left premise, we choose $\Gamma, \Delta \to \varphi \Rightarrow \varphi$ to simplify the proof.

$$\begin{tikzcd}
\Gamma \Rightarrow \Delta,\varphi \arrow[r, leftrightarrow, "\text{1-equivalent}"] \arrow[d, "\text{L}\to"] & \Gamma,\Delta \to \varphi \Rightarrow \varphi \arrow[r, "\text{by IH}"] & \Gamma,\Delta \to \varphi \overset{\delta}{\vdash} \varphi \arrow[d, "(2)"] \\
\varphi\to\psi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] & \varphi\to\psi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, dashed, "P"] & \varphi\to\psi,\Gamma,\Theta_i \to \varphi_i \vdash \varphi_i \\
\psi,\Gamma \Rightarrow \Delta \arrow[r, leftrightarrow, "\text{1-equivalent}"] \arrow[u, "\text{L}\to"] & \psi,\Gamma,\Theta_i \to \varphi_i \Rightarrow \varphi_i \arrow[r, "\text{by IH}"] & \psi,\Gamma,\Theta_i \to \varphi_i \overset{\pi}{\vdash} \varphi_i \arrow[u, "(2)"]
\end{tikzcd}$$

$$u\;\dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{}{\Gamma,\Theta_i\to\varphi_i}\,\text{Asm} \quad [\varphi_i\to\varphi]^u}{} \\ \vdots\;\delta \\ \varphi\end{matrix} \qquad \dfrac{}{\varphi\to\psi}\,\text{Asm}}{\psi}\,\to\text{E} \qquad \dfrac{}{\Gamma,\Theta_i\to\varphi_i}\,\text{Asm}}{} \\ \vdots\;\pi \\ \varphi_i\end{matrix}}{\varphi_i}\,\text{P} \tag{2}$$

(L$\forall$) Similar to (L$\wedge$), use Lemma 3 and $\forall x\varphi \vdash \varphi[x/t]$ for arbitrary term t.

(L$\exists$) Similar to (L$\vee$), this derivation is valid because $y \notin \mathrm{FV}(\exists x\varphi)$ implies $y \notin \mathrm{FV}(\varphi)$ or $y \equiv x$.

$$u\;\dfrac{\dfrac{}{\exists x\varphi}\,\text{Asm} \qquad \begin{matrix}\dfrac{[\varphi[x/y]]^u \quad \dfrac{}{\Gamma,\Theta_i\to\varphi_i}\,\text{Asm}}{} \\ \vdots \\ \varphi_i\end{matrix}}{\varphi_i}\,\exists\text{E}$$

(**R**$\wedge$) We choose $\Gamma, \Delta \to \varphi \Rightarrow \varphi$ and $\Gamma, \Delta \to \psi \Rightarrow \psi$. Similar to Lemma 4, we first prove that $\Delta \to \varphi_0 \wedge \varphi_1 \vdash \Delta \to \varphi_i$ for $i = 0, 1$.

$$
\begin{prooftree}
\AxiomC{$[\Delta]^u$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Delta \to \varphi_0 \wedge \varphi_1$}
\RightLabel{$\to$E}
\BinaryInfC{$\varphi_0 \wedge \varphi_1$}
\RightLabel{$\wedge$E}
\UnaryInfC{φ_i}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\Delta \to \varphi_i$}
\end{prooftree}
$$

Then by Lemma 3 the following derivation is valid:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Gamma, \Delta \to \varphi \wedge \psi$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{φ}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Gamma, \Delta \to \varphi \wedge \psi$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\RightLabel{$\wedge$I}
\BinaryInfC{$\varphi \wedge \psi$}
\end{prooftree}
$$

(**R**$\vee$) We choose $\Gamma, \Delta \to \varphi \Rightarrow \varphi$ to proof. Here the occurrence of Δ also means some $\chi \in \Delta$, but we "instantiate" and merge them in the conclusion $\Delta \to \varphi$ inside the derivation, i.e., for every $\chi \in \Delta$, there exists a corresponding sub-derivation of $\chi \to \varphi$.

$$
\begin{prooftree}
\AxiomC{$[\Delta]^u$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Delta \to \varphi \vee \psi$}
\RightLabel{$\to$E}
\BinaryInfC{$\varphi \vee \psi$}
\AxiomC{$[\varphi]^{v_1}$}
\AxiomC{$[\psi]^{v_2}$}
\RightLabel{$\vee$I}
\UnaryInfC{$\varphi \vee \psi$}
\AxiomC{$[\varphi \vee \psi \to \varphi]^w$}
\RightLabel{$\to$E}
\BinaryInfC{φ}
\LeftLabel{v_1, v_2}
\RightLabel{$\vee$E}
\TrinaryInfC{φ}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\Delta \to \varphi$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{Γ}
\BinaryInfC{}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{φ}
\RightLabel{$\vee$I}
\UnaryInfC{$\varphi \vee \psi$}
\LeftLabel{w}
\RightLabel{P}
\UnaryInfC{$\varphi \vee \psi$}
\end{prooftree}
$$

(**R**$\to$) Similar to (R$\vee$), we choose $\varphi, \Gamma, \Delta \to \psi \Rightarrow \psi$.

$$
\begin{prooftree}
\AxiomC{$[\varphi]^u$}
\AxiomC{$[\varphi]^u$}
\AxiomC{$[\Delta]^v$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Delta \to \varphi \to \psi$}
\RightLabel{$\to$E}
\BinaryInfC{$\varphi \to \psi$}
\RightLabel{$\to$E}
\BinaryInfC{ψ}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\Delta \to \psi$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{Γ}
\TrinaryInfC{}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\varphi \to \psi$}
\end{prooftree}
$$

(**R**$\forall$) Similar. Note that $y \notin \mathrm{FV}(\forall x \varphi)$ implies $y \notin \mathrm{FV}(\varphi)$ or $y \equiv x$.

$$
\begin{prooftree}
\AxiomC{$[\Delta]^u$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{$\Delta \to \forall x \varphi$}
\RightLabel{$\to$E}
\BinaryInfC{$\forall x \varphi$}
\RightLabel{$\forall$E}
\UnaryInfC{$\varphi[x/y]$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\Delta \to \varphi[x/y]$}
\AxiomC{}
\RightLabel{Asm}
\UnaryInfC{Γ}
\BinaryInfC{}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\varphi[x/y]$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \varphi$}
\end{prooftree}
$$

(**R**$\exists$) Similar to (R$\vee$), we choose $\Gamma, \Delta \to \varphi[x/t] \Rightarrow \varphi[x/t]$ to proof and take a fresh variable y s.t. $y \notin \mathrm{FV}(\Gamma, \Delta)$, $\mathrm{FV}(\varphi)$ and $\mathrm{FV}(t)$ so the existential elimination rule ($\exists$E) can be applied safely. We can always take fresh variables in every branch correspond to some $\chi \in \Delta$ hence Δ is finite.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\Delta]^u \qquad \dfrac{}{\Delta \to \exists x\varphi}\,\text{Asm}}{\exists x\varphi}\,{\to}\text{E} \qquad \dfrac{\dfrac{[\varphi[x/y]]^v}{\exists x\varphi}\,\exists\text{I} \qquad [\exists x\varphi \to \varphi[x/t]]^w}{\varphi[x/t]}\,{\to}\text{E}}{\varphi[x/t]}\,v\;\exists\text{E}}{\Delta \to \varphi[x/t]}\,u\;{\to}\text{I} \qquad \dfrac{}{\Gamma}\,\text{Asm}}{\begin{array}{c}\vdots\\ \dfrac{\dfrac{\varphi[x/t]}{\exists x\varphi}\,\exists\text{I}}{\exists x\varphi}\,w\;\text{P}\end{array}}
$$

(**Cut**) Let the left premise be $\Gamma \Rightarrow \Delta, \varphi$ and the right premise be $\varphi, \Theta \Rightarrow \Lambda, \psi$, we prove that $\Gamma, \Delta \to \varphi \overset{\delta}{\vdash} \varphi$ and $\varphi, \Theta, \Lambda \to \psi \overset{\pi}{\vdash} \psi$ implies $\Gamma, \Theta, \Delta \to \psi, \Lambda \to \psi \vdash \psi$.

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, \Delta \to \psi}\,\text{Asm} \qquad [\psi \to \varphi]^u}{\begin{array}{c}\vdots\,\delta\\ \varphi\end{array}} \qquad \dfrac{}{\Theta, \Lambda \to \psi}\,\text{Asm}}{\begin{array}{c}\vdots\,\pi\\ \dfrac{\psi}{\psi}\,u\;\text{P}\end{array}}
$$

$\square$

- $\Gamma \vdash \varphi$ implies $\mathbf{G} \vdash \Gamma \Rightarrow \varphi$:

  Proof. We prove the $\mathcal{X}$-fragment of $\mathbf{GNc}'$, which is a sequent style natural deduction proof calculus exhibited in class replaces $\bot_\mathbf{c}$ with $\bot_\mathbf{i}$ and (P), is equivalent to $\mathbf{G}$ by induction, where (P) defined as follows:

  $$\dfrac{\Gamma, [\varphi \to \psi] \vdash \varphi}{\Gamma \vdash \varphi}\,\text{P}$$

  Note that $[\varphi]$ means one or more occurrence of φ in the antecedent and the previous definition of $\vdash$ still make sense here.

  BASE (Asm) $\varphi \vdash \varphi$ implies $\varphi \Rightarrow \varphi$ by (Ax).

  IH We can use (LW) and (RW) freely since they are admissible in $\mathbf{G2c}$ so $\mathbf{G}$.

  ($\wedge$**I**)

  $$\dfrac{\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \Gamma \Rightarrow \varphi\end{array}}{\Gamma, \Delta \Rightarrow \varphi}\,\text{LW}^{|\Delta|} \qquad \dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \Delta \Rightarrow \psi\end{array}}{\Gamma, \Delta \Rightarrow \psi}\,\text{LW}^{|\Gamma|}}{\Gamma, \Delta \Rightarrow \varphi \wedge \psi}\,\text{R}\wedge$$

  ($\wedge$**E**) We choose one case to prove, the other one is almost same.

  $$\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \Gamma \Rightarrow \varphi \wedge \psi\end{array} \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi \wedge \psi \Rightarrow \varphi}\,\text{L}\wedge}{\Gamma \Rightarrow \varphi}\,\text{Cut}$$

  ($\vee$**I**) Same as ($\wedge$E).

  $$\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \Gamma \Rightarrow \varphi\end{array}}{\Gamma \Rightarrow \varphi \vee \psi}\,\text{R}\vee$$

  ($\vee$**E**)

  $$\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \Gamma \Rightarrow \varphi \vee \psi\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ [\varphi], \Delta \Rightarrow \chi\end{array}}{\varphi, \Delta \Rightarrow \chi}\,\text{LC}^{|[\varphi]|}}{\varphi, \Delta, \Theta \Rightarrow \chi}\,\text{LW}^{|\Theta|} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ [\psi], \Theta \Rightarrow \chi\end{array}}{\psi, \Theta \Rightarrow \chi}\,\text{LC}^{|[\psi]|}}{\psi, \Delta, \Theta \Rightarrow \chi}\,\text{LW}^{|\Delta|}}{\varphi \vee \psi, \Delta, \Theta \Rightarrow \chi}\,\text{L}\vee}{\Gamma, \Delta, \Theta \Rightarrow \chi}\,\text{Cut}$$

(→**I**)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$[\varphi], \Gamma \Rightarrow \psi$}
\RightLabel{LC$^{|[\varphi]|}$}
\UnaryInfC{$\varphi, \Gamma \Rightarrow \psi$}
\RightLabel{R$\to$}
\UnaryInfC{$\Gamma \Rightarrow \varphi \to \psi$}
\end{prooftree}$$

(→**E**)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi \to \psi$}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Delta \Rightarrow \varphi$}
\RightLabel{RW}
\UnaryInfC{$\Delta \Rightarrow \varphi, \psi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\psi, \Delta \Rightarrow \psi$}
\RightLabel{L$\to$}
\BinaryInfC{$\varphi \to \psi, \Delta \Rightarrow \psi$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma, \Delta \Rightarrow \psi$}
\end{prooftree}$$

(∀**I**) Note that $y \equiv x$ or $y \notin \mathrm{FV}(\varphi)$ implies $y \notin \mathrm{FV}(\forall x\varphi)$ and $y \notin \mathrm{FV}(\exists x\varphi)$, this makes the derivation here and for (∃E) below valid.

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi[x/y]$}
\RightLabel{R$\forall$}
\UnaryInfC{$\Gamma \Rightarrow \forall x\varphi$}
\end{prooftree}$$

(∀**E**)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \forall x\varphi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\varphi[x/t] \Rightarrow \varphi[x/t]$}
\RightLabel{L$\forall$}
\UnaryInfC{$\forall x\varphi \Rightarrow \varphi[x/t]$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma \Rightarrow \varphi[x/t]$}
\end{prooftree}$$

(∃**I**)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi[x/t]$}
\RightLabel{R$\exists$}
\UnaryInfC{$\Gamma \Rightarrow \exists x\varphi$}
\end{prooftree}$$

(∃**E**)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \exists x\varphi$}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$[\varphi[x/y]], \Delta \Rightarrow \psi$}
\RightLabel{LC$^{|[\varphi[x/y]]|}$}
\UnaryInfC{$\varphi[x/y], \Delta \Rightarrow \psi$}
\RightLabel{L$\exists$}
\UnaryInfC{$\exists x\varphi, \Delta \Rightarrow \psi$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma, \Delta \Rightarrow \psi$}
\end{prooftree}$$

(P)

$$\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\varphi \Rightarrow \psi, \varphi$}
\RightLabel{R$\to$}
\UnaryInfC{$\Rightarrow \varphi \to \psi, \varphi$}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma, [\varphi \to \psi] \Rightarrow \varphi$}
\RightLabel{LC$^{|[\varphi\to\psi]|}$}
\UnaryInfC{$\Gamma, \varphi \to \psi \Rightarrow \varphi$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma \Rightarrow \varphi, \varphi$}
\RightLabel{RW}
\UnaryInfC{$\Gamma \Rightarrow \varphi$}
\end{prooftree}$$

(+)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi$}
\RightLabel{LW}
\UnaryInfC{$\psi, \Gamma \Rightarrow \varphi$}
\end{prooftree}$$

(−)

$$\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma, \varphi, \varphi \Rightarrow \psi$}
\RightLabel{LC}
\UnaryInfC{$\Gamma, \varphi \Rightarrow \psi$}
\end{prooftree}$$

□

2. We also prove that the sequent version of **Hi** is equivalent to **G1i** and let $\Gamma \vdash \varphi$ to denote $\mathbf{GHi} \vdash \Gamma \Rightarrow \varphi$. We may omit the implication operator to save space, and add any formula in the antecedent freely by (+). We will use these conclusions already known:

$$\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\,\text{DT} \quad \dfrac{\Gamma \vdash \varphi \to \psi}{\Gamma, \varphi \vdash \psi}\,\text{DT}^{-1}$$

Moreover, we regard α-equivalent formulas as same one hence we already prove they are equivalent(up to provability) in **H** systems.

- $\Gamma \vdash \varphi$ implies $\mathbf{G1i} \vdash \Gamma \Rightarrow \varphi$: To prove that, we only need to show that all axioms in **Hi** is derivable in **G1i** since all these rules are highly similar to the corresponding rules in **Ni** which we already proved in previous question((Cut) is admissible in **G1i**).

  –
  $$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow}\,\text{L}\bot}{\bot \Rightarrow \varphi}\,\text{RW}}{\Rightarrow \bot\varphi}\,\text{R}$$

  –
  $$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi, \psi \Rightarrow \varphi}\,\text{LW}}{\varphi \Rightarrow \psi\varphi}\,\text{R}}{\Rightarrow \varphi\psi\varphi}\,\text{R}$$

  –
  $$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi\psi\chi, \varphi \Rightarrow \varphi}\,\text{LW} \quad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi, \psi \Rightarrow \varphi}\,\text{LW} \quad \dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Ax}}{\varphi, \psi \Rightarrow \psi}\,\text{LW} \quad \dfrac{\dfrac{}{\chi \Rightarrow \chi}\,\text{Ax}}{\varphi, \psi, \chi \Rightarrow \chi}\,\text{LW}}{\varphi, \psi, \psi\chi \Rightarrow \chi}\,\text{L}}{\varphi\psi\chi, \varphi, \psi \Rightarrow \chi}\,\text{L}}{\varphi\psi\chi, \varphi\psi, \varphi \Rightarrow \chi}\,\text{L}}{(\varphi\psi\chi)(\varphi\psi)(\varphi\chi)}\,\text{R}^3$$

  –
  $$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi \Rightarrow \varphi \vee \psi}\,\text{R}\vee}{\Rightarrow \varphi(\varphi \vee \psi)}\,\text{R} \quad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi \wedge \psi \Rightarrow \varphi}\,\text{L}\wedge}{\Rightarrow (\varphi \wedge \psi)\varphi}\,\text{R}$$

  –
  $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi, \psi\chi \Rightarrow \varphi}\,\text{LW} \quad \dfrac{\dfrac{}{\chi \Rightarrow \chi}\,\text{Ax}}{\varphi, \psi\chi, \chi \Rightarrow \chi}\,\text{LW}}{\varphi, \varphi\chi, \psi\chi \Rightarrow \chi}\,\text{L} \quad \dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Ax}}{\psi, \varphi\chi \Rightarrow \psi}\,\text{LW} \quad \dfrac{\dfrac{}{\chi \Rightarrow \chi}\,\text{Ax}}{\psi, \varphi\chi, \chi \Rightarrow \chi}\,\text{LW}}{\psi, \varphi\chi, \psi\chi \Rightarrow \chi}\,\text{L}}{\varphi\chi, \psi\chi, \varphi \vee \psi \Rightarrow \chi}\,\text{L}\vee}{\Rightarrow (\varphi\chi)(\psi\chi)(\varphi \vee \psi)\chi}\,\text{R}^3$$

  –
  $$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\varphi, \psi \Rightarrow \varphi}\,\text{LW} \quad \dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Ax}}{\varphi, \psi \Rightarrow \psi}\,\text{LW}}{\varphi, \psi \Rightarrow \varphi \wedge \psi}\,\text{R}\wedge}{\Rightarrow \varphi\psi(\varphi \wedge \psi)}\,\text{R}^2$$

  –
  $$\dfrac{\dfrac{\dfrac{}{\varphi^x_t \Rightarrow \varphi^x_t}\,\text{Ax}}{\forall x\varphi \Rightarrow \varphi^x_t}\,\text{L}\forall}{\Rightarrow \forall x\varphi \to \varphi^x_t}\,\text{R} \quad \dfrac{\dfrac{\dfrac{}{\varphi^x_t \Rightarrow \varphi^x_t}\,\text{Ax}}{\varphi^x_t \Rightarrow \exists x\varphi}\,\text{R}\exists}{\Rightarrow \varphi^x_t \to \exists x\varphi}\,\text{R}$$

  –
  $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Ax} \quad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax}}{\psi, \varphi \Rightarrow \varphi}\,\text{LW}}{\psi\varphi, \psi \Rightarrow \varphi}\,\text{L}}{\forall x\psi\varphi, \psi \Rightarrow \varphi^{xy}_{yx} \equiv \varphi}\,\text{L}\forall}{\forall x\psi\varphi, \psi \Rightarrow \forall y\varphi^x_y}\,\text{R}\forall}{\Rightarrow \forall x\psi\varphi \to \psi \to \forall y\varphi^x_y}\,\text{R}^2$$

–

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{Ax} \qquad \dfrac{\dfrac{}{\psi \Rightarrow \psi}\,\text{Ax}}{\varphi, \psi \Rightarrow \psi}\,\text{LW}}{\varphi\psi, \varphi \Rightarrow \psi}\,\text{L}}{\forall x\varphi\psi, \varphi^{xy}_{yx} \equiv \varphi \Rightarrow \psi}\,\text{L}\forall}{\forall x\varphi\psi, \exists y\varphi^x_y \Rightarrow \psi}\,\text{L}\exists}{\Rightarrow \forall x\varphi\psi \to \exists y\varphi^x_y \to \psi}\,\text{R}^2$$

- **G1i** $\vdash \Gamma \Rightarrow \varphi$ implies $\Gamma \vdash \varphi$: We only consider the sequents in **G1i** which has exactly one formula in the succedent, this can be done by put arbitrary formula in an empty succedent.

  (Ax),(LW) and (LC) Equivalent to (Asm), (+) and (−) respectively.

  (**L**$\bot$)

$$\dfrac{\dfrac{}{\vdash \bot\varphi}\,\text{Ax}}{\bot \vdash \varphi}\,\text{DT}^{-1}$$

  (**L**$\wedge$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\varphi \wedge \psi)\varphi}\,\text{Ax}}{\Gamma, \varphi \wedge \psi \vdash \varphi}\,\text{DT}^{-1} \qquad \dfrac{\dfrac{\overset{\vdots\,\text{IH}}{\Gamma, \varphi \vdash \chi}}{\Gamma \vdash \varphi\chi}\,\text{DT}^{-1}}{\Gamma, \varphi \wedge \psi \vdash \varphi\chi}\,(+)}{\Gamma, \varphi \wedge \psi \vdash \chi}\,\text{MP}$$

  (**R**$\wedge$)

$$\dfrac{\overset{\vdots\,\text{IH}}{\Gamma \vdash \psi} \qquad \dfrac{\overset{\vdots\,\text{IH}}{\Gamma \vdash \varphi} \qquad \dfrac{}{\Gamma \vdash \varphi\psi(\varphi \wedge \psi)}\,\text{Ax}}{\Gamma \vdash \psi(\varphi \wedge \psi)}\,\text{MP}}{\Gamma \vdash \varphi \wedge \psi}\,\text{MP}$$

  (**L**$\vee$)

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\text{IH}}{\Gamma, \psi \vdash \chi}}{\Gamma \vdash \psi\chi}\,\text{DT} \qquad \dfrac{\dfrac{\overset{\vdots\,\text{IH}}{\Gamma, \varphi \vdash \chi}}{\Gamma \vdash \varphi\chi}\,\text{DT} \qquad \dfrac{}{\Gamma \vdash (\varphi\chi)(\psi\chi)(\varphi \vee \psi)\chi}\,\text{Ax}}{\Gamma \vdash (\psi\chi)(\varphi \vee \psi)\chi}\,\text{MP}}{\Gamma \vdash (\varphi \vee \psi)\chi}\,\text{MP}}{\Gamma, \varphi \vee \psi \vdash \chi}\,\text{DT}^{-1}$$

  (**R**$\vee$)

$$\dfrac{\overset{\vdots\,\text{IH}}{\Gamma \vdash \varphi} \qquad \dfrac{}{\Gamma \vdash \varphi(\varphi \vee \psi)}\,\text{Ax}}{\Gamma \vdash \varphi \vee \psi}\,\text{MP}$$

  (**L**$\to$)

$$\dfrac{\dfrac{\overset{\vdots\,\text{IH}}{\Gamma \vdash \varphi}}{\Gamma, \varphi\psi \vdash \varphi}\,(+) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\text{IH}}{\psi, \Gamma \vdash \chi}}{\varphi, \psi, \Gamma \vdash \chi}\,(+)}{\Gamma \vdash \varphi\psi\chi}\,\text{DT}^2 \qquad \dfrac{}{\Gamma \vdash (\varphi\psi\chi)(\varphi\psi)(\varphi\chi)}\,\text{Ax}}{\Gamma \vdash (\varphi\psi)(\varphi\chi)}\,\text{MP}}{\Gamma, \varphi\psi \vdash \varphi\chi}\,\text{DT}^{-1}}{\Gamma, \varphi\psi \vdash \chi}\,\text{MP}$$

  (**R**$\to$)

$$\dfrac{\overset{\vdots\,\text{IH}}{\Gamma, \varphi \vdash \psi}}{\Gamma \vdash \varphi\psi}\,\text{DT}$$

(**L**∀)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x\varphi \to \varphi_t^x}\ \text{Ax}}{\Gamma, \forall x\varphi \vdash \varphi_t^x}\ \text{DT}^{-1} \qquad \dfrac{\dfrac{\dfrac{\vdots\ \text{IH}}{\Gamma, \varphi_t^x \vdash \psi}}{\Gamma \vdash \varphi_t^x\psi}\ \text{DT}}{\Gamma, \forall x\varphi \vdash \varphi_t^x\psi}\ (+)}{\Gamma, \forall x\varphi \vdash \psi}\ \text{MP}$$

(**R**∀)

$$\dfrac{\dfrac{\vdots\ \text{IH}}{\Gamma \vdash \varphi_y^x}}{\Gamma \vdash \forall x\varphi^{xy}_{yx} \equiv \forall x\varphi}\ (\forall\text{I})$$

(**L**∃)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \text{IH}}{\Gamma, \varphi_y^x \vdash \psi}}{\Gamma \vdash \varphi_y^x\psi}\ \text{DT}}{\Gamma \vdash \forall x(\varphi_y^x\psi)_x^y \overset{y\notin \text{FV}(\psi)}{\equiv} \forall x\varphi\psi}\ (\forall\text{I}) \qquad \dfrac{}{\Gamma \vdash \forall x\varphi\psi \to \exists y\varphi_y^x \to \psi}\ \text{Ax}}{\Gamma \vdash \exists y\varphi_y^x \to \psi \overset{y\notin \text{FV}(\exists x\varphi)}{\equiv} \exists x\varphi \to \psi}\ \text{MP}}{\Gamma, \exists x\varphi \vdash \psi}\ \text{DT}^{-1}$$

(**R**∃)

$$\dfrac{\dfrac{\vdots\ \text{IH}}{\Gamma \vdash \varphi_t^x} \qquad \dfrac{}{\Gamma \vdash \varphi_t^x \to \exists x\varphi}\ \text{Ax}}{\Gamma \vdash \exists x\varphi}\ \text{MP}$$