HW11

Y. Konpaku

May 12, 2019

1 QUESTION

1. Prove the case of (L$\forall$) that one of the active formula is principal and another is side formula in previous rule application in Proposition 3.5.5 (d.p.a. of (LC) and (RC)).

2. [Optional](*Basic Proof Theory* Ex. 3.5.11A) Prove the following simple form of *Herbrand's theorem* for **G3**[**mic**]: if Γ, Δ and φ are quantifier-free, and $\vdash_n \Gamma, \forall x\varphi \Rightarrow \Delta$, then there are $t_1, \ldots, t_m$ such that $\vdash_n \Gamma, \varphi[x/t_1], \ldots, \varphi[x/t_m] \Rightarrow \Delta$. For **G3c** we also have: if $\vdash_n \Gamma \Rightarrow \Delta, \exists x\varphi$ then for suitable $t_1, \ldots, t_m$, $\vdash_n \Gamma \Rightarrow \Delta, \varphi[x/t_1], \ldots, \varphi[x/t_m]$.

2 ANSWER

1. We prove that $\vdash_{n+1} \Gamma, \forall x\varphi, \forall x\varphi \Rightarrow \Delta$ implies $\vdash_{n+1} \Gamma, \forall x\varphi \Rightarrow \Delta$ if that holds for n-height derivation and $\vdash_{n+1} \Gamma, \forall x\varphi, \forall x\varphi \Rightarrow \Delta$ is obtained by an application of (L$\forall$), i.e., $\vdash_n \Gamma, \forall x\varphi, \varphi[x/t], \forall x\varphi \Rightarrow \Delta$ for some term t.

   Proof. First we applying induction hypothesis on $\vdash_n \Gamma, \forall x\varphi, \varphi[x/t], \forall x\varphi \Rightarrow \Delta$ to obtain that $\vdash_n \Gamma, \forall x\varphi, \varphi[x/t] \Rightarrow \Delta$. Then we applying (L$\forall$) on it again to finish the proof. □

2. *Proof.* We prove that by induction on height of the derivation. If $\vdash_0 \Gamma, \forall x\varphi \Rightarrow \Delta$ and $\vdash_0 \Gamma \Rightarrow \Delta, \exists x\varphi$, then $\forall x\varphi(\exists x\varphi)$ must be the weakening formula and we can simply delete it(or replace it by $\varphi[x/x]$ if at least one $\varphi[x/t_i]$ need to be present).

   Now assume that $\vdash_n \Gamma, \forall x\varphi \Rightarrow \Delta$ implies $\vdash_n \Gamma, \varphi[x/t_1], \ldots, \varphi[x/t_m] \Rightarrow \Delta$ and $\vdash_n \Gamma \Rightarrow \Delta, \exists x\varphi$ implies $\vdash_n \Gamma \Rightarrow \Delta, \varphi[x/t_1], \ldots, \varphi[x/t_m]$ for some $t_i (1 \leq i \leq m)$, we prove the case of $n+1$.

   If $\forall x\varphi(\exists x\varphi)$ is the principal formula, i.e., $\vdash_{n+1} \Gamma, \forall x\varphi \Rightarrow \Delta(\vdash_{n+1} \Gamma \Rightarrow \exists x\varphi, \Delta)$ is obtained by an application of (L$\forall$)((R$\exists$)), there must exist some term s s.t. $\vdash_n \Gamma, \forall x\varphi, \varphi[x/s] \Rightarrow \Delta(\vdash_n \Gamma \Rightarrow \Delta, \varphi[x/s], \exists x\varphi)$. By induction hypothesis, there exists some term t_i s.t. $\vdash_n \Gamma, \varphi[x/t_1], \ldots, \varphi[x/t_m], \varphi[x/s] \Rightarrow \Delta(\vdash_n \Gamma \Rightarrow \Delta, \varphi[x/s], \varphi[x/t_1], \ldots, \varphi[x/t_m])$, which is what we want.

   Or if it is the case that $\forall x\varphi(\exists x\varphi)$ is a weakening formula, we can do the same thing as if $n = 0$.

   Otherwise, we can simply use induction hypothesis on the premise of the last rule application then applying the corresponding rule. This rule application is always valid since Γ and Δ are both quantifier-free so this rule cannot be a quantifier rule, introduce new terms will not break anything. □