HW2

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1. QUESTION

1. Prove that $\vdash_{\mathbf{Nm}} ((((\varphi \to \psi) \to \varphi) \to \varphi) \to \psi) \to \psi$.
2. Prove the De Morgan's Law in **Ni** if it holds or give a counter-model (in Kripke semantics) if it not hold.
3. Prove that $\vdash_{\mathbf{Ni}} \neg\neg\neg\varphi \to \neg\varphi$.
4. Prove that $\vdash_{\mathbf{Ni}} \neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$ and $\vdash_{\mathbf{Ni}} (\neg\neg\varphi \to \neg\neg\psi) \to \neg\neg(\varphi \to \psi)$.
5. Prove that $\vdash_{\mathbf{Ni}} \neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$.

2. ANSWER

1.[Incomplete]

$$
w\;\dfrac{u\;\dfrac{\dfrac{v\;\dfrac{\dfrac{\dfrac{[\varphi\to\psi]^u \quad [(\varphi\to\psi)\to\varphi]^v}{\varphi}\,{\to}\mathrm{E} \quad [\varphi\to\psi]^u}{\psi}\,{\to}\mathrm{E}}{((\varphi\to\psi)\to\varphi)\to\psi}\,{\to}\mathrm{I} \qquad [(((\varphi\to\psi)\to\varphi)\to\psi)\to\psi]^w}{\psi}\,{\to}\mathrm{E}}{(\varphi\to\psi)\to\psi}\,{\to}\mathrm{I}}{((((\varphi\to\psi)\to\varphi)\to\psi)\to\psi)\to(\varphi\to\psi)\to\psi}\,{\to}\mathrm{I}
$$

2. In this section we use the jargon "De Morgan's Law" in the sense of derivability in a particular proof calculus **Ni**, i.e., $\neg(\varphi \lor \psi) \vdash_{\mathbf{Ni}} \neg\varphi \land \neg\psi$ and so forth. The only one which not hold in **Ni** is $\neg(\varphi \land \psi) \vdash \neg\varphi \lor \neg\psi$. First we shall construct a counter-model $\mathcal{W} = (W, \prec, \Vdash)$ which satisfies $\neg(\varphi \land \psi)$ but not $\neg\varphi \lor \neg\psi$, then prove the other three in **Ni**.

- Consider a frame $W = \{x, y, z\}, x \prec y, x \prec z$ and $y \nprec z$. We let x does not satisfy both φ and ψ, y satisfies φ but not ψ and z satisfies ψ but not φ. Obviously, for any point $w \in W$ either $w \nVdash \varphi$ or $w \nVdash \psi$ holds, which is $w \nVdash \varphi \land \psi$. From this we can conclude that for every successor w' of any point w that $w' \nVdash \varphi \land \psi$, i.e., $\mathcal{W}$ satisfies $\neg(\varphi \land \psi)$. If we assume $\mathcal{W}$ satisfies $\neg\varphi \lor \neg\psi$, we must accept that for any point w either $\forall w' \succeq w(w' \nVdash \varphi)$ or $\forall w' \succeq w(w' \nVdash \psi)$ holds, but this leads to a contradiction hence $y, z \succeq x$ but $y \Vdash \varphi$ and $z \Vdash \psi$.

- $\neg\varphi \land \neg\psi \vdash_{\mathbf{Ni}} \neg(\varphi \lor \psi)$

$$
u\;\dfrac{v,w\;\dfrac{[\varphi\lor\psi]^u \quad \dfrac{\dfrac{\neg\varphi\land\neg\psi}{\neg\varphi}\,\land\mathrm{E}\quad[\varphi]^v}{\bot}\,{\to}\mathrm{E} \quad \dfrac{\dfrac{\neg\varphi\land\neg\psi}{\neg\psi}\,\land\mathrm{E}\quad[\psi]^w}{\bot}\,{\to}\mathrm{E}}{\bot}\,\lor\mathrm{E}}{\neg(\varphi\lor\psi)}\,{\to}\mathrm{I}
$$

- $\neg(\varphi \lor \psi) \vdash_{\mathbf{Ni}} \neg\varphi \land \neg\psi$

$$
\dfrac{u\;\dfrac{\dfrac{\dfrac{[\varphi]^u}{\varphi\lor\psi}\,\lor\mathrm{I}\quad\neg(\varphi\lor\psi)}{\bot}\,{\to}\mathrm{E}}{\neg\varphi}\,{\to}\mathrm{I}\qquad v\;\dfrac{\dfrac{\dfrac{[\psi]^v}{\varphi\lor\psi}\,\lor\mathrm{I}\quad\neg(\varphi\lor\psi)}{\bot}\,{\to}\mathrm{E}}{\neg\psi}\,{\to}\mathrm{I}}{\neg\varphi\land\neg\psi}\,\land\mathrm{I}
$$

- $\neg\varphi \lor \neg\psi \vdash_{\mathbf{Ni}} \neg(\varphi \land \psi)$

$$
w\;\dfrac{u,v\;\dfrac{\neg\varphi\lor\neg\psi\quad\dfrac{[\neg\varphi]^u\quad\dfrac{[\varphi\land\psi]^w}{\varphi}\,\land\mathrm{E}}{\bot}\,{\to}\mathrm{E}\quad\dfrac{[\neg\psi]^v\quad\dfrac{[\varphi\land\psi]^w}{\psi}\,\land\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\bot}\,\lor\mathrm{E}}{\neg(\varphi\land\psi)}\,{\to}\mathrm{I}
$$

3.

$$
\begin{prooftree}
\AxiomC{$[\varphi]^u$}
\AxiomC{$[\neg\varphi]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\neg\neg\varphi]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\varphi$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\neg\varphi \to \neg\varphi$}
\end{prooftree}
$$

4.

-

$$
\begin{prooftree}
\AxiomC{$[\varphi \to \psi]^u$}
\AxiomC{$[\varphi]^v$}
\RightLabel{$\to$E}
\BinaryInfC{ψ}
\AxiomC{$[\neg\psi]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg(\varphi \to \psi)$}
\AxiomC{$[\neg\neg(\varphi \to \psi)]^x$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg\neg\varphi]^y$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\psi$}
\LeftLabel{y}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi \to \neg\neg\psi$}
\LeftLabel{x}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$}
\end{prooftree}
$$

- We omit the implication operator $\to$ to save space.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi]^u$}
\AxiomC{$[\varphi]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\bot\mathbf{i}$}
\UnaryInfC{ψ}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\varphi\psi$}
\AxiomC{$[\neg(\varphi\psi)]^\tau$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\neg\varphi\neg\neg\psi]^\sigma$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg\psi$}
\AxiomC{$[\psi]^w$}
\LeftLabel{μ}
\RightLabel{$\to$I}
\UnaryInfC{$\varphi\psi$}
\AxiomC{$[\neg(\varphi\psi)]^\tau$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\psi$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{τ}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg(\varphi\psi)$}
\LeftLabel{σ}
\RightLabel{$\to$I}
\UnaryInfC{$(\neg\neg\varphi\neg\neg\psi)\neg\neg(\varphi\psi)$}
\end{prooftree}
$$

5.

$$
\begin{prooftree}
\AxiomC{$[\forall x\varphi(x)]^u$}
\RightLabel{$\forall$E}
\UnaryInfC{$\varphi(a)$}
\AxiomC{$[\neg\varphi(a)]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall x\varphi(x)$}
\AxiomC{$[\neg\neg\forall x\varphi(x)]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi(a)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x\neg\neg\varphi(x)$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$}
\end{prooftree}
$$

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