HW2

Y. Konpaku

1. QUESTION

1. Prove that $\vdash_{\mathbf{Nm}} ((((\varphi \to \psi) \to \varphi) \to \varphi) \to \psi) \to \psi$.
2. Prove the De Morgan's Law in **Ni** if it holds or give a counter-model (in Kripke semantics) if it not hold.
3. Prove that $\vdash_{\mathbf{Ni}} \neg\neg\neg\varphi \to \neg\varphi$.
4. Prove that $\vdash_{\mathbf{Ni}} \neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$ and $\vdash_{\mathbf{Ni}} (\neg\neg\varphi \to \neg\neg\psi) \to \neg\neg(\varphi \to \psi)$.
5. Prove that $\vdash_{\mathbf{Ni}} \neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$.

2. ANSWER

2. In this section we use the jargon "De Morgan's Law" in the sense of derivability in a particular proof calculus **Ni**, i.e., $\neg(\varphi \lor \psi) \vdash_{\mathbf{Ni}} \neg\varphi \land \neg\psi$ and so forth. The only one which not hold in **Ni** is $\neg(\varphi \land \psi) \vdash \neg\varphi \lor \neg\psi$. First we shall construct a counter-model $\mathcal{W} = (W, \prec, \Vdash)$ which satisfies $\neg(\varphi \land \psi)$ but not $\neg\varphi \lor \neg\psi$, then prove the other three in **Ni**.

- Consider a frame $W = \{x, y, z\}, x \prec y, x \prec z$ and $y \nprec z$. We let x does not satisfy both φ and ψ, y satisfies φ but not ψ and z satisfies ψ but not φ. Obviously, for any point $w \in W$ either $w \nVdash \varphi$ or $w \nVdash \psi$ holds, which is $w \nVdash \varphi \land \psi$. From this we can conclude that for every successor w' of any point w that $w' \nVdash \varphi \land \psi$, i.e., $\mathcal{W}$ satisfies $\neg(\varphi \land \psi)$. If we assume $\mathcal{W}$ satisfies $\neg\varphi \lor \neg\psi$, we must accept that for any point w either $\forall w' \succeq w(w' \nVdash \varphi)$ or $\forall w' \succeq w(w' \nVdash \psi)$ holds, but this leads to a contradiction hence $y, z \succeq x$ but $y \Vdash \varphi$ and $z \Vdash \psi$.

- $\neg\varphi \land \neg\psi \vdash_{\mathbf{Ni}} \neg(\varphi \lor \psi)$

$$
u\ \cfrac{v, w\ \cfrac{[\varphi \lor \psi]^u \qquad \cfrac{\cfrac{\cfrac{\neg\varphi \land \neg\psi}{\neg\varphi}\land\mathrm{E} \qquad [\varphi]^v}{\bot}\to\mathrm{E} \qquad \cfrac{\cfrac{\neg\varphi \land \neg\psi}{\neg\psi}\land\mathrm{E} \qquad [\psi]^w}{\bot}\to\mathrm{E}}{\bot}\lor\mathrm{E}}{\neg(\varphi \lor \psi)}\to\mathrm{I}
$$

- $\neg(\varphi \lor \psi) \vdash_{\mathbf{Ni}} \neg\varphi \land \neg\psi$

$$\dfrac{u\ \dfrac{\dfrac{\dfrac{[\varphi]^u}{\varphi \lor \psi}\,\lor\mathrm{I} \qquad \neg(\varphi \lor \psi)}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I} \qquad v\ \dfrac{\dfrac{\dfrac{[\psi]^v}{\varphi \lor \psi}\,\lor\mathrm{I} \qquad \neg(\varphi \lor \psi)}{\bot}\,\to\mathrm{E}}{\neg\psi}\,\to\mathrm{I}}{\neg\varphi \land \neg\psi}\,\land\mathrm{I}$$

- $\neg\varphi \lor \neg\psi \vdash_{\mathbf{Ni}} \neg(\varphi \land \psi)$

$$w\ \dfrac{u, v\ \dfrac{\neg\varphi \lor \neg\psi \qquad \dfrac{[\neg\varphi]^u \qquad \dfrac{[\varphi \land \psi]^w}{\varphi}\,\land\mathrm{E}}{\bot}\,\to\mathrm{E} \qquad \dfrac{[\neg\psi]^v \qquad \dfrac{[\varphi \land \psi]^w}{\psi}\,\land\mathrm{E}}{\bot}\,\to\mathrm{E}}{\bot}\,\lor\mathrm{E}}{\neg(\varphi \land \psi)}\,\to\mathrm{I}$$

3.

$$w\ \dfrac{u\ \dfrac{\dfrac{v\ \dfrac{\dfrac{[\varphi]^u \qquad [\neg\varphi]^v}{\bot}\,\to\mathrm{E}}{\neg\neg\varphi}\,\to\mathrm{I} \qquad [\neg\neg\neg\varphi]^w}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I}}{\neg\neg\neg\varphi \to \neg\varphi}\,\to\mathrm{I}$$

4.

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$$x\ \dfrac{y\ \dfrac{w\ \dfrac{\dfrac{v\ \dfrac{\dfrac{u\ \dfrac{\dfrac{\dfrac{[\varphi \to \psi]^u \qquad [\varphi]^v}{\psi}\,\to\mathrm{E} \qquad [\neg\psi]^w}{\bot}\,\to\mathrm{E}}{\neg(\varphi \to \psi)}\,\to\mathrm{I} \qquad [\neg\neg(\varphi \to \psi)]^x}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I} \qquad [\neg\neg\varphi]^y}{\bot}\,\to\mathrm{E}}{\neg\neg\psi}\,\to\mathrm{I}}{\neg\neg\varphi \to \neg\neg\psi}\,\to\mathrm{I}}{\neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)}\,\to\mathrm{I}$$

- We omit the implication operator $\to$ to save space.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi]^u$}
\AxiomC{$[\varphi]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\bot\mathbf{i}$}
\UnaryInfC{ψ}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\varphi\psi$}
\AxiomC{$[\neg(\varphi\psi)]^\tau$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\neg\varphi\neg\neg\psi]^\sigma$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg\neg\psi$}
\AxiomC{$[\psi]^w$}
\LeftLabel{μ}
\RightLabel{$\to$I}
\UnaryInfC{$\varphi\psi$}
\AxiomC{$[\neg(\varphi\psi)]^\tau$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\psi$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{τ}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg(\varphi\psi)$}
\LeftLabel{σ}
\RightLabel{$\to$I}
\UnaryInfC{$(\neg\neg\varphi\neg\neg\psi)\neg\neg(\varphi\psi)$}
\end{prooftree}
$$

5.

$$
\begin{prooftree}
\AxiomC{$[\forall x\varphi(x)]^u$}
\RightLabel{$\forall$E}
\UnaryInfC{$\varphi(a)$}
\AxiomC{$[\neg\varphi(a)]^v$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\forall x\varphi(x)$}
\AxiomC{$[\neg\neg\forall x\varphi(x)]^w$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\varphi(a)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x\neg\neg\varphi(x)$}
\LeftLabel{w}
\RightLabel{$\to$I}
\UnaryInfC{$\neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$}
\end{prooftree}
$$