HW4

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1. QUESTION

1. Prove that **G1m** $\vdash \Rightarrow ((((\varphi\psi)\varphi)\varphi)\psi)\psi$.
2. [*Basic Proof Theory* Ex. 3.1.3A] Prove in **G1m** that $\varphi \Rightarrow \varphi$ for arbitrary φ from prime instances $p \Rightarrow p$.
3. Prove that if **G1m** $\vdash \Gamma \Rightarrow \Delta$ then $\Delta \neq \emptyset$.
4. [Optional] Prove that if **G1**[**ic**] $\vdash \Gamma \Rightarrow \Delta, \bot$ then **G1**[**ic**] $\vdash \Gamma \Rightarrow \Delta$(in **G1i**, $\Delta = \emptyset$).

2. ANSWER

1.

\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\RightLabel{LW}
\UnaryInfC{$\varphi, (\varphi\psi)\varphi \Rightarrow \varphi$}
\RightLabel{R$\to$}
\UnaryInfC{$\varphi \Rightarrow ((\varphi\psi)\varphi)\varphi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\psi \Rightarrow \psi$}
\RightLabel{LW}
\UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
\RightLabel{L$\to$}
\BinaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi, \varphi \Rightarrow \psi$}
\RightLabel{R$\to$}
\UnaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi \Rightarrow \varphi\psi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\RightLabel{RW}
\UnaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi, \varphi \Rightarrow \varphi$}
\RightLabel{L$\to$}
\BinaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi, (\varphi\psi)\varphi \Rightarrow \varphi$}
\RightLabel{R$\to$}
\UnaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi \Rightarrow ((\varphi\psi)\varphi)\varphi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\psi \Rightarrow \psi$}
\RightLabel{LW}
\UnaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi, \psi \Rightarrow \psi$}
\RightLabel{L$\to$}
\BinaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi, (((\varphi\psi)\varphi)\varphi)\psi \Rightarrow \psi$}
\RightLabel{LC}
\UnaryInfC{$(((\varphi\psi)\varphi)\varphi)\psi \Rightarrow \psi$}
\RightLabel{R$\to$}
\UnaryInfC{$\Rightarrow ((((\varphi\psi)\varphi)\varphi)\psi)\psi$}
\end{prooftree}

2. We prove that by induction on the formation rules of the wff.

- **BASE G1m** $\vdash p \Rightarrow p$ for $p = \bot$ or p atomics holds by definition.
- **IH** By induction hypothesis **G1m** $\vdash \varphi \Rightarrow \varphi$ and **G1m** $\vdash \psi \Rightarrow \psi$ holds, we prove that we can derive $\varphi * \psi \Rightarrow \varphi * \psi$ for $* = \wedge, \vee, \to$ and $Qx\varphi \Rightarrow Qx\varphi$ for $Q = \forall, \exists$ in **G1m**.

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  \begin{prooftree}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\varphi \Rightarrow \varphi$}
  \RightLabel{L$\wedge$}
  \UnaryInfC{$\varphi \wedge \psi \Rightarrow \varphi$}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\psi \Rightarrow \psi$}
  \RightLabel{L$\wedge$}
  \UnaryInfC{$\varphi \wedge \psi \Rightarrow \psi$}
  \RightLabel{R$\wedge$}
  \BinaryInfC{$\varphi \wedge \psi \Rightarrow \varphi \wedge \psi$}
  \end{prooftree}

  –
  \begin{prooftree}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\varphi \Rightarrow \varphi$}
  \RightLabel{R$\vee$}
  \UnaryInfC{$\varphi \Rightarrow \varphi \vee \psi$}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\psi \Rightarrow \psi$}
  \RightLabel{R$\vee$}
  \UnaryInfC{$\psi \Rightarrow \varphi \vee \psi$}
  \RightLabel{L$\vee$}
  \BinaryInfC{$\varphi \vee \psi \Rightarrow \varphi \vee \psi$}
  \end{prooftree}

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  \begin{prooftree}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\varphi \Rightarrow \varphi$}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\psi \Rightarrow \psi$}
  \RightLabel{LW}
  \UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
  \RightLabel{L$\to$}
  \BinaryInfC{$\varphi, \varphi \to \psi \Rightarrow \psi$}
  \RightLabel{R$\to$}
  \UnaryInfC{$\varphi \to \psi \Rightarrow \varphi \to \psi$}
  \end{prooftree}

  –
  \begin{prooftree}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\varphi[x := x] \Rightarrow \varphi[x := x]$}
  \RightLabel{L$\forall$}
  \UnaryInfC{$\forall x\varphi \Rightarrow \varphi[x := x]$}
  \RightLabel{R$\forall$}
  \UnaryInfC{$\forall x\varphi \Rightarrow \forall x\varphi$}
  \end{prooftree}

  –
  \begin{prooftree}
  \AxiomC{}
  \RightLabel{IH}
  \UnaryInfC{$\varphi[x := x] \Rightarrow \varphi[x := x]$}
  \RightLabel{R$\exists$}
  \UnaryInfC{$\varphi[x := x] \Rightarrow \exists x\varphi$}
  \RightLabel{L$\exists$}
  \UnaryInfC{$\exists x\varphi \to \exists x\varphi$}
  \end{prooftree}

  Q.E.D

3. In this and the next question we would do induction on the formation rules of the sequent, i.e., the derivation rules of **G1**[**mic**].

- **BASE (Ax)** Obviously, the succedent must have exactly one formula so Δ can not be empty.
- **IH** Every rule except RW and L$\to$ simply hold the cardinality of the succedent in every premises, L$\to$ holds the second and RW add exactly one formula on the right hand side so their succedent are all non-empty multiset by induction hypothesis.

Q.E.D

4. We can prove that in **G1c** and **G1i** simultaneously and just let the induction hypothesis to be "if $\Gamma \Rightarrow \Delta$ is derivable in **G1**[**ic**] and $\bot \in \Delta$, then $\Gamma \Rightarrow \Theta$ is also derivable s.t. $\bot \notin \Theta \wedge \forall \varphi \neq \bot (\varphi \in \Delta \to \varphi \in \Theta)$ (denoted as $\Theta = \Delta/\bot$)".

   - **BASE (Ax, L$\bot$)** For $\bot \Rightarrow$ and $\varphi \Rightarrow \varphi (\varphi \neq \bot)$, the property holds vacuously. In the case of $\bot \Rightarrow \bot$, we always have $\bot \Rightarrow$ from L$\bot$.
   - **IH**
     - **L* for $* \neq \to$** In these cases, all formulas in the succedent are side formula and the rules just simply keep them up. We can easily eliminate the $\bot$ in conclusion by apply the induction hypothesis before the application of rules. We choose L$\wedge$ to prove, and the others are similar.
       Assume that from a derivation of $\varphi, \Gamma \Rightarrow \Delta$ we can conclude the existence of a derivation of $\varphi, \Gamma \Rightarrow \Theta$ s.t. $\Theta = \Delta/\bot$. Then we can derive $\varphi \wedge \psi, \Gamma \Rightarrow \Theta$ instead of $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ via the following application of L$\wedge$:
       $$\dfrac{\dfrac{}{\varphi, \Gamma \Rightarrow \Theta}\,\text{IH}}{\varphi \wedge \psi, \Gamma \Rightarrow \Theta}\,\text{L}\wedge$$
     - **R* for $* = \wedge, \vee, \to, \forall, \exists$** In these cases, since $\bot$ cannot match any principal formula in the conclusion so it can only fall into the side formulas. In **G1i**, there is no place for side formula in the succedent and this assertion is vacuously true. In **G1c**, applying the induction hypothesis directly may cause the corresponding rule invalid when $\bot$ is an active formula, we shall use RW to rectify that. We choose R$\wedge$ to prove, and the other cases are similar.
       From the induction hypothesis, we can obtain $\Gamma \Rightarrow \Theta, \varphi$ and $\Gamma \Rightarrow \Theta, \psi$ from $\Gamma \Rightarrow \Delta, \varphi$ and $\Gamma \Rightarrow \Delta, \psi$ s.t. $\Theta = \Delta/\bot$. Moreover, if φ or ψ is $\bot$, we can only obtain $\Gamma \Rightarrow \Theta$. We can construct a derivation of $\Gamma \Rightarrow \Theta, \varphi \wedge \psi$ from the latter by R$\wedge$:
       $$\dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow \Theta}\,\text{IH}}{\Gamma \Rightarrow \Theta, \varphi}\,\text{RW} \qquad \dfrac{}{\Gamma \Rightarrow \Theta, \psi}\,\text{IH}}{\Gamma \Rightarrow \Theta, \varphi \wedge \psi}\,\text{R}\wedge$$
       This derivation exhibited how to handle the case that one of the active formula(φ) is $\bot$, and the derivation for the other three cases are highly similar.
     - **L$\to$** This rule is slightly different in **G1c** and **G1i**. From the induction hypothesis, we can obtain $\Gamma \Rightarrow \Theta, \varphi$ from $\Gamma \Rightarrow \Delta, \varphi$ for $\varphi \neq \bot$ in **G1c**, $\Gamma \Rightarrow \Theta$ from $\Gamma \Rightarrow \Delta, \varphi$ in **G1c** and $\Gamma \Rightarrow$ from $\Gamma \Rightarrow \varphi$ in **G1i** for $\varphi = \bot$ s.t. $\Theta = \Delta/\bot$. The second premise is similar. We choose one of these cases(**G1c**, $\varphi = \bot$) to prove:
       $$\dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow \Theta}\,\text{IH}}{\Gamma \Rightarrow \Theta, \varphi}\,\text{RW} \qquad \dfrac{}{\psi, \Gamma \Rightarrow \Theta}\,\text{IH}}{\varphi \to \psi, \Gamma \Rightarrow \Theta}\,\text{L}\to$$
     - **RC** This rule only exists in **G1c**. We also let Δ to denote the side formulas in succedent and $\Theta = \Delta/\bot$. If the principal(active) formula φ is $\bot$, we can obtain $\Gamma \Rightarrow \Theta$ from $\Gamma \Rightarrow \Delta, \varphi, \varphi$ immediately. Otherwise, there exists a derivation of $\Gamma \Rightarrow \Theta, \varphi, \varphi$ from induction hypothesis and we can derive $\Gamma \Rightarrow \Theta, \varphi$ from it easily:
       $$\dfrac{\dfrac{}{\Gamma \Rightarrow \Theta, \varphi, \varphi}\,\text{IH}}{\Gamma \Rightarrow \Theta, \varphi}\,\text{RW}$$
     - **RW** If the principal(weak) formula φ is $\bot$, by using induction hypothesis on the premise $\Gamma \Rightarrow \Delta$ we can obtain $\Gamma \Rightarrow \Theta$ for $\Theta = \Delta/\bot$ immediately, and which is exactly what we want. Otherwise, we simply applying RW again:
       $$\dfrac{\dfrac{}{\Gamma \Rightarrow \Theta}\,\text{IH}}{\Gamma \Rightarrow \Theta, \varphi}\,\text{RW}$$

   In **G1i**, there is at most one formula on the right hand side so we already finished the proof. In **G1c**, we just simply applying RW finite many times to eliminate exactly one $\bot$ in the succedent and preserve the others.

   Q.E.D