HW5

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1. QUESTION

1. We say two calculi $\mathbf{C}$ and $\mathbf{D}$ are equivalent, denoted as $\mathbf{C} \simeq \mathbf{D}$, iff they derive the same set of strings. Prove that $\mathbf{G1s4} \simeq \mathbf{G1s4'}$ where $\mathbf{G1s4'} := \mathbf{G1t} \oplus (\mathrm{R}\Box) = \mathbf{G1s4} \oplus (\mathrm{K}\Box)$ and $\mathbf{G1s4} \simeq \mathbf{G1s4''}$ where $\mathbf{G1s4''} := \mathbf{G1k4} \oplus (\mathrm{L}\Box)$ if it holds or give a counterexample if not.

2. Prove that $\mathbf{G1k4} \vdash \Rightarrow \Box\neg(\varphi \to \Box\varphi) \to \Box\bot$ and give a countermodel of $\neg\Box\neg(\varphi \to \Box\varphi) = \Diamond(\varphi \to \Box\varphi)$ in $\mathbf{K4}$.

2. ANSWER

1. **Proof.** We simply show that $(\mathrm{K}\Box)$ is derivable from $(\mathrm{L}\Box)$ and $(\mathrm{R}\Box)$ to prove $\mathbf{G1s4} \simeq \mathbf{G1s4'}$. Similarly, we show that $(4\Box)$ is derivable from $(\mathrm{L}\Box)$ and $(\mathrm{R}\Box)$ then $(\mathrm{R}\Box)$ from $(\mathrm{L}\Box)$ and $(4\Box)$ to prove $\mathbf{G1s4} \simeq \mathbf{G1s4''}$.

   - For any sequent with the form $\Theta \Rightarrow \eta$, we can derive $\Box\Theta \Rightarrow \Box\eta$ by following derivation, where $\mathrm{L}\Box^{|\Theta|}$ means applying $(\mathrm{L}\Box)$ on every formula in Θ. This operation can always be finished since Θ must be finitely many. Other similar notations below have the same meaning.

   $$\dfrac{\dfrac{\Theta \Rightarrow \eta}{\Box\Theta \Rightarrow \eta}\,\mathrm{L}\Box^{|\Theta|}}{\Box\Theta \Rightarrow \Box\eta}\,\mathrm{R}\Box$$

   - Similarly, we derive $(4\Box)$ by applying following rules on any sequent with the form $\Theta, \Box\Theta \Rightarrow \eta$.

   $$\dfrac{\dfrac{\dfrac{\Theta, \Box\Theta \Rightarrow \eta}{\Box\Theta, \Box\Theta \Rightarrow \eta}\,\mathrm{L}\Box^{|\Theta|}}{\Box\Theta \Rightarrow \eta}\,\mathrm{LC}^{|\Box\Theta|}}{\Box\Theta \Rightarrow \Box\eta}\,\mathrm{R}\Box$$

   - We only need $(4\Box)$ to derive $(\mathrm{R}\Box)$.

   $$\dfrac{\dfrac{\Box\Theta \Rightarrow \eta}{\Theta, \Box\Theta \Rightarrow \eta}\,\mathrm{LW}^{|\Theta|}}{\Box\Theta \Rightarrow \Box\eta}\,4\Box$$

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   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\mathrm{Ax}}{\varphi \Rightarrow \Box\varphi, \varphi}\,\mathrm{RW}}{\Rightarrow \varphi \to \Box\varphi, \varphi}\,\mathrm{R}{\to} \quad \dfrac{\dfrac{}{\bot \Rightarrow}\,\mathrm{L}\bot}{\bot \Rightarrow \varphi}\,\mathrm{RW}}{\neg(\varphi \to \Box\varphi) \Rightarrow \varphi}\,\mathrm{L}{\to}}{\Box\neg(\varphi \to \Box\varphi), \neg(\varphi \to \Box\varphi) \Rightarrow \varphi}\,\mathrm{LW}}{\Box\neg(\varphi \to \Box\varphi) \Rightarrow \Box\varphi}\,4\Box}{\Box\neg(\varphi \to \Box\varphi), \varphi \Rightarrow \Box\varphi}\,\mathrm{LW}}{\Box\neg(\varphi \to \Box\varphi) \Rightarrow \varphi \to \Box\varphi}\,\mathrm{R}{\to}}{\Box\neg(\varphi \to \Box\varphi) \Rightarrow \varphi \to \Box\varphi, \bot}\,\mathrm{RW} \quad \dfrac{\dfrac{}{\bot \Rightarrow \bot}\,\mathrm{Ax}}{\bot, \Box\neg(\varphi \to \Box\varphi) \Rightarrow \bot}\,\mathrm{LW}}{\Box\neg(\varphi \to \Box\varphi), \neg(\varphi \to \Box\varphi) \Rightarrow \bot}\,\mathrm{L}{\to}}{\Box\neg(\varphi \to \Box\varphi) \Rightarrow \Box\bot}\,4\Box}{\Rightarrow \Box\neg(\varphi \to \Box\varphi) \to \Box\bot}\,\mathrm{R}{\to}$$

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- Consider a model contains only one point w and an empty relation $\prec$, i.e., $w \nprec w$. Since w has no successor, $\Box\neg(\varphi \to \Box\varphi)$ holds vacuously and its negation is false in this model.