STUDIES IN PROOF THEORY

WILFRIED BUCHHOLZ - KURT SCHÜTTE

PROOF THEORY OF IMPREDICATIVE SUBSYSTEMS OF ANALYSIS

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INTRODUCTION

The definition of a set of natural numbers is called impredicative if it refers to the completed totality of all sets of natural numbers. The least upper bound principle of classical analysis provides an important example of the necessary use of such an impredicative definition. So the full theory of real numbers (analysis) cannot be obtained by strict predicativeness. But for certain subsystems of analysis (as e.g. the system (Δ^1_1-CA)) a predicative interpretation can be given. This was first done by S. FEFERMAN [8] after he and K. SCHÜTTE [24] independently from each other had determined the limiting ordinal Γ_0 of predicativity.

The first constructive consistency proof for an essentially impredicative subsystem of analysis (the so called Π^1_1-analysis) was given by G. TAKEUTI [29]. In [25] proof-theoretical investigations of such impredicative systems are carried out following the work of Takeuti. After the publication of [25] essential progress in the proof-theoretical investigations of relatively strong subsystems of analysis was achieved mainly by W. BUCHHOLZ [2]-[4], S. FEFERMAN [9], H. FRIEDMAN [10], J.Y. GIRARD [11], [12], G. JÄGER [13]-[18], W. POHLERS [19]-[23], W. SIEG [27] and W.W. TAIT [28]. There the methods of Takeuti were successfully replaced by more perspicuous techniques which also made it possible to determine the precise proof-theoretical ordinals of the systems considered.

The present book provides a uniform proof-theoretical treatment of several essentially impredicative subsystems of analysis, beginning with simple Π^1_1-analysis and ending with the system (Δ^1_2-CA) + (BR) of Δ^1_2-comprehension and bar-rule. For the proof-theoretical analysis of these systems a certain constructive system of ordinal notations suffices which we call $T(\Omega)$. But for the proof-theoretical treatment of stronger subsystems of analysis or set theory, as carried out by G. JÄGER and W. POHLERS [18], a much stronger system of ordinal notations is needed. In this book we do not present these stronger proof-theoretical studies but only give a reference to them in § 36.

In chapter I the ordinal notation system $T(\Omega)$ is developed.

In chapters II and III upper bounds for the provable ordinals of the various impredicative subsystems of analysis are established. Chapter IV contains formalized well ordering proofs for certain segments of $T(\Omega)$ which are used to show that the upper bounds of chapters II and III are best possible and thus are in fact the proof-theoretical ordinals of the respective formal systems.

CHAPTER I

ORDINALS

In this chapter we develop a constructive notation system $T(\Omega)$ for ordinals which we will use in chapters II and III for the proof theoretical treatment of several impredicative subsystems of classical analysis. The notation system $T(\Omega)$ is closely related to the notation system $\overline{\theta}(\Omega)$ which is described in [25] as a subsystem of $\overline{\theta}(\{g\})$ in [1]. $T(\Omega)$ is based on collapsing functions ψ_σ which are more convenient for proof theoretical investigations than the basic functions θ_α of $\overline{\theta}(\Omega)$. (There is a close connection between the ordinals $\psi_\sigma(\alpha)$ and $\theta_\alpha(\Omega_\sigma)$).

§ 1. – Fundamentals

We begin by considering the ordinals in a nonconstructive way as they are determined for instance in classical set theory *ZFC* of Zermelo and Fraenkel with the axiom of choice.

By the small Greek letters $\alpha, \beta, \gamma, \delta, \eta, \xi, \sigma, \tau$ (also subscripts) we always denote ordinals, and by i, k, m, n we denote ordinals $< \omega$ or natural numbers (including 0). $\alpha + \beta$ and ω^α may be defined in the usual way. $\alpha \cdot n$ and $\omega_n(\alpha)$ are defined in the following inductive way:

$$\alpha \cdot 0 := 0, \quad \alpha \cdot (n+1) := \alpha \cdot n + \alpha,$$

$$\omega_0(\alpha) := \alpha, \quad \omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}.$$

Let E be the class of ε-numbers, i.e. $\alpha \in E \Leftrightarrow \omega^\alpha = \alpha$.

DEFINITION. $\gamma =_{NF} \omega^\alpha + \beta$ (γ has the *normal form* $\omega^\alpha + \beta$) means that $\gamma = \omega^\alpha + \beta$ holds with $\alpha < \gamma$ and $\beta < \gamma$.

Obviously, we have:

LEMMA 1.1.

a) If $\gamma =_{NF} \omega^\alpha + \beta$, then α and β are uniquely determined by γ.

b) For γ there are α, β with $\gamma =_{NF} \omega^\alpha + \beta$ if and only if $0 < \gamma \notin E$.

LEMMA 1.2. For $\gamma =_{NF} \omega^\alpha + \beta$ and $\delta \in E$ we have

a) $\gamma < \delta \Leftrightarrow \alpha < \delta$

b) $\delta < \gamma \Leftrightarrow \delta \leq \alpha$

DEFINITION of Ω_σ.

$\Omega_0 := 0$. For $\sigma > 0$ let Ω_σ be the lest ordinal of cardinality $\aleph_\sigma$.

Then we have

LEMMA 1.3.

a) $\sigma \leq \Omega_\sigma$

b) $\sigma > 0 \Rightarrow \Omega_\sigma \in E$

c) For any ordinal α there is a uniquely determined ordinal σ such that $\Omega_\sigma \leq \alpha < \Omega_{\sigma+1}$.

For a set S of ordinals we make the following definitions:

$S < \alpha$ means that $\xi < \alpha$ holds for all $\xi \in S$.

$\sup S := \min\{\eta : \xi \leq \eta$ for all $\xi \in S\}$.

§ 2. - The Functions ψ_σ

By induction on α we define $\psi_\sigma(\alpha)$ as the smallest ordinal which does not belong to the set $C_\sigma(\alpha)$ of ordinals, where $C_\sigma(\alpha)$ is the closure of the set of all ordinals $\leq \Omega_\sigma$ with respect to the mappings $\xi, \eta \mapsto \omega^\xi + \eta$, $\gamma \mapsto \Omega_\gamma$ and $\tau, \beta \mapsto \psi_\tau(\beta)$ for $\beta < \alpha$.

INDUCTIVE DEFINITION of sets $C_\sigma^n(\alpha)$ and $C_\sigma(\alpha)$ of ordinals and the ordinal $\psi_\sigma(\alpha)$ (by induction on α with a subsidiary induction on n).

(C1) $\gamma \leq \Omega_\sigma \Rightarrow \gamma \in C_\sigma^n(\alpha)$

(C2) $\Omega_\sigma < \gamma =_{NF} \omega^\xi + \eta$, $\{\xi, \eta\} \subseteq C_\sigma^n(\alpha) \Rightarrow \gamma \in C_\sigma^{n+1}(\alpha)$

(C3) $\sigma < \gamma \in C_\sigma^n(\alpha) \Rightarrow \Omega_\gamma \in C_\sigma^{n+1}(\alpha)$

(C4) $\sigma \leq \tau, \beta < \alpha$, $\{\tau, \beta\} \subseteq C_\sigma^n(\alpha) \Rightarrow \psi_\tau(\beta) \in C_\sigma^{n+2}(\alpha)$

(C5) $C_\sigma(\alpha) := \bigcup\{C_\sigma^n(\alpha) : n < \omega\}$

(C6) $\psi_\sigma(\alpha) := \min\{\eta : \eta \notin C_\sigma(\alpha)\}$

It follows that $C_\sigma^m(\alpha) \subseteq C_\sigma^n(\alpha)$ holds for $m < n$.

For brevity we write $\psi\sigma\alpha$ instead of $\psi_\sigma(\alpha)$.

LEMMA 2.1. $\sigma \leq \tau$, $\alpha \leq \beta \Rightarrow C_\sigma(\alpha) \subseteq C_\tau(\beta)$, $\psi\sigma\alpha \leq \psi\tau\beta$.

This follows immediately from the definitions.

LEMMA 2.2. $\Omega_\sigma < \psi\sigma\alpha < \Omega_{\sigma+1}$.

Proof. $\Omega_\sigma < \psi\sigma\alpha$ according to (C1), (C5) and (C6). By induction on n we see that $C_\sigma^n(\alpha)$ has cardinality $< \aleph_{\sigma+1}$. Then also $C_\sigma(\alpha)$ has cardinality $< \aleph_{\sigma+1}$, hence $\psi\sigma\alpha < \Omega_{\sigma+1}$ according to (C6).

LEMMA 2.3. $\psi\sigma\alpha \in E$.

Proof. Suppose $\psi\sigma\alpha \notin E$. Then we have $\Omega_\sigma < \psi\sigma\alpha =_{NF} \omega^\xi + \eta$ by Lemmata 2.2 and 1.1, hence ξ, $\eta \in C_\sigma(\alpha)$ according to (C6) and $\psi\sigma\alpha \in C_\sigma(\alpha)$ by (C2) in contradiction to (C6).

LEMMA 2.4.

a) $\gamma =_{NF} \omega^\xi + \eta \in C_\sigma^n(\alpha) \Rightarrow \xi$, $\eta \in C_\sigma(\alpha)$

b) $\Omega_\gamma \in C_\sigma^n(\alpha) \Rightarrow \gamma \in C_\sigma(\alpha)$

c) $\Omega_\tau \leq \gamma < \Omega_{\tau+1}$, $\gamma \in C_\sigma^n(\alpha) \Rightarrow \tau \in C_\sigma(\alpha)$

Proofs by induction on n.

LEMMA 2.5. $\gamma \in C_\sigma^n(\alpha)$, $\gamma < \Omega_{\sigma+1} \Rightarrow \gamma < \psi\sigma\alpha$.

Proof by induction on n.

1. $\gamma \in C_\sigma^n(\alpha)$ by (C1) or (C3). Then we have $\gamma \leq \Omega_\sigma < \psi\sigma\alpha$ by Lemma 2.2.

2. $\gamma \in C_\sigma^n(\alpha)$ by (C2). Then we have $\gamma =_{NF} \omega^\xi + \eta$ with $\xi < \psi\sigma\alpha$ by I.H. (Induction Hypothesis). Since $\psi\sigma\alpha \in E$, it follows that $\gamma < \psi\sigma\alpha$ holds.

3. $\gamma \in C_\sigma^n(\alpha)$ by (C4). Then we have $\gamma = \psi\tau\beta$ with $\beta < \alpha$. $\gamma < \Omega_{\sigma+1}$ implies $\tau \leq \sigma$ by Lemma 2.2, hence $\gamma \leq \psi\sigma\alpha$ by Lemma 2.1. Since $\psi\sigma\alpha \notin C_\sigma(\alpha)$, it follows that $\gamma < \psi\sigma\alpha$ holds.

COROLLARY 2.5. $\beta < \gamma < \Omega_{\sigma+1}$, $\gamma \in C_\sigma(\alpha) \Rightarrow \beta \in C_\sigma(\alpha)$.

Proof. The assumptions imply $\beta < \gamma < \psi\sigma\alpha$ by Lemma 2.5, hence $\beta \in C_\sigma(\alpha)$ by (C6).

LEMMA 2.6.

a) $\alpha < \beta$, $\alpha \in C_\sigma(\beta) \Rightarrow \psi\sigma\alpha < \psi\sigma\beta$

b) $\alpha \in C_\sigma(\alpha)$, $\beta \in C_\sigma(\beta)$, $\psi\sigma\alpha = \psi\sigma\beta \Rightarrow \alpha = \beta$.

Proof. a) $\sigma \leq \Omega_\sigma$ implies $\sigma \in C_\sigma(\beta)$ by $(C1)$. Therefore, the assumptions imply $\psi\sigma\alpha \in C_\sigma(\beta)$ by $(C4)$, hence $\psi\sigma\alpha < \psi\sigma\beta$ by Lemmata 2.2 and 2.5.

b) Suppose $\alpha < \beta$. Then the assumptions imply $\alpha \in C_\sigma(\beta)$ by Lemma 2.1 and $\psi\sigma\alpha < \psi\sigma\beta$ by a). In the same way also the assumption $\beta < \alpha$ yields a contradiction to the assumption $\psi\sigma\alpha = \psi\sigma\beta$.

LEMMA 2.7. If $\alpha < \beta$ and there is no $\delta \in C_\sigma(\alpha)$ such that $\alpha \leq \delta < \beta$, then $\gamma \in C_\sigma^n(\beta)$ implies $\gamma \in C_\sigma(\alpha)$.

Proof by induction on n.

1. $\gamma \in C_\sigma^n(\beta)$ by $(C1)$. Then we have also $\gamma \in C_\sigma(\alpha)$ by $(C1)$.

2. $\gamma \in C_\sigma^n(\beta)$ by $(C2)$ or $(C3)$. Then $\gamma \in C_\sigma(\alpha)$ follows from the I.H.

3. $\gamma \in C_\sigma^n(\beta)$ by $(C4)$. Then we have $\gamma = \psi\tau\delta$ with $\sigma \leq \tau, \delta < \beta$ and by I.H. $\tau, \delta \in C_\sigma(\alpha)$. Therefore by the assumption it is not the case that $\alpha \leq \delta < \beta$, hence $\delta < \beta$ implies $\delta < \alpha$. Then we have $\gamma = \psi\tau\delta \in C_\sigma(\alpha)$ by $(C4)$.

LEMMA 2.8. If $\beta = \min\{\xi : \alpha \leq \xi \in C_\sigma(\alpha)\}$, then $C_\sigma(\alpha) = C_\sigma(\beta)$, hence $\psi\sigma\alpha = \psi\sigma\beta$ and $\beta \in C_\sigma(\beta)$.

Proof. The assumption implies $C_\sigma(\alpha) \subseteq C_\sigma(\beta)$ by Lemma 2.1 and $C_\sigma(\beta) \subseteq C_\sigma(\alpha)$ by Lemma 2.7.

LEMMA 2.9. $\gamma \in C_\sigma^n(\alpha) \Rightarrow \gamma + 1 \in C_\sigma^{n+1}(\alpha)$.

Proof by induction on n. We have the following five cases.

1. $\gamma < \Omega_\sigma$. Then also $\gamma + 1 < \Omega_\sigma$, hence $\gamma + 1 \in C_\sigma^{n+1}(\alpha)$ by $(C1)$.

2. $\gamma = \Omega_\sigma = 0$. Then we have $0 \in C_\sigma^n(\alpha)$ by $(C1)$ and $\gamma + 1 =_{NF} \omega^0 + 0 \in C_\sigma^{n+1}(\alpha)$ by $(C2)$.

3. $\gamma = \Omega_\sigma > 0$. Then we have $\gamma, 1 \in C_\sigma^n(\alpha)$ by $(C1)$ and $\gamma + 1 =_{NF} \omega^\gamma + 1 \in C_\sigma^{n+1}(\alpha)$ by $(C2)$.

4. $\Omega_\sigma < \gamma =_{NF} \omega^\xi + \eta \in C_\sigma^n(\alpha)$. Then we have $n > 0$ and $\xi, \eta \in C_\sigma^{n-1}(\alpha)$, hence $\eta + 1 \in C_\sigma^n(\alpha)$ by I.H. and $\gamma + 1 =_{NF} \omega^\xi + \eta + 1 \in C_\sigma^{n+1}(\alpha)$ by $(C2)$.

5. $\Omega_\sigma < \gamma \in E$. Then we have $n > 0$, $0 \in C_\sigma^{n-1}(\alpha)$ by $(C1)$, $1 =_{NF} \omega^0 + 0 \in C_\sigma^n(\alpha)$ by $(C1)$ or $(C2)$ and $\gamma + 1 =_{NF} \omega^\gamma + 1 \in C_\sigma^{n+1}(\alpha)$ by $(C2)$.

LEMMA 2.10. If $\gamma \in C_\sigma^n(\alpha)$, $\sigma \leq \tau$ and $\beta \leq \alpha$, then there exists $\delta = \min\{\xi : \gamma \leq \xi \in C_\tau(\beta)\} \in C_\sigma^n(\alpha)$.

Proof by induction on n.

1. $\gamma \in C_\sigma^n(\alpha)$ by $(C1)$. Then we have also $\gamma \in C_\tau(\beta)$ according to $\sigma \leq \tau$. The assertion follows for $\delta = \gamma$.

2. $\gamma \in C_\sigma^n(\alpha)$ by $(C2)$. Then we have $n > 0$, $\gamma =_{NF} \omega^{\gamma_1} + \gamma_2$ and $\gamma_1, \gamma_2 \in C_\sigma^{n-1}(\alpha)$. By I.H. we have $\delta_i = \min\{\xi : \gamma_i \leq \xi \in C_\tau(\beta)\} \in C_\sigma^{n-1}(\alpha)$ $(i = 1, 2)$.

2.1. $\gamma_1 \in C_\tau(\beta)$. Then there is an $m < \omega$ such that $\gamma_2 \leq \delta_2 < \omega^{\gamma_1} \cdot m \in C_\tau(\beta)$. The assertion follows for $\delta =_{NF} \omega^{\gamma_1} + \delta_2$.

2.2. $\gamma_1 \notin C_\tau(\beta)$. Then we have $\gamma < \omega^{\delta_1}$. The assertion follows for $\delta = \omega^{\delta_1}$.

3. $\gamma \in C_\sigma^n(\alpha)$ by $(C3)$. Then we have $n > 0$, $\gamma = \Omega_{\gamma_0}$ and $\gamma_0 \in C_\sigma^{n-1}(\alpha)$. By I.H. we have $\delta_0 = \min\{\xi : \gamma_0 \leq \xi \in C_\tau(\beta)\} \in C_\sigma^{n-1}(\alpha)$. The assertion follows for $\delta = \Omega_{\delta_0}$.

4. $\gamma \in C_\sigma^n(\alpha)$ by $(C4)$. Then we have $n \geq 2$, $\gamma = \psi\gamma_1\gamma_2$ and $\gamma_1, \gamma_2 \in C_\sigma^{n-2}(\alpha)$. By I.H. we have $\delta_i = \min\{\xi : \gamma_i \leq \xi \in C_\tau(\beta)\} \in C_\sigma^{n-2}(\alpha)$ $(i = 1, 2)$.

4.1. $\gamma_1 \in C_\tau(\beta)$ and $\delta_2 < \beta$. Then we have also $\delta_2 < \alpha$, and the assertion holds for $\delta = \psi\gamma_1\delta_2$.

4.2. $\gamma_1 \in C_\tau(\beta)$ and $\beta \leq \delta_2$. Then we have $\gamma_1 + 1 \in C_\sigma^{n-1}(\alpha)$ by Lemma 2.9, and the assertion holds for $\delta = \Omega_{\gamma_1+1}$.

4.3. $\gamma_1 \notin C_\tau(\beta)$. Then we have also $\Omega_{\gamma_1} \notin C_\tau(\beta)$, and the assertion holds for $\delta = \Omega_{\delta_1}$.

LEMMA 2.11. If $\gamma \in C_\sigma^{n+2}(\alpha)$ holds by $(C4)$, then there are uniquely determined $\tau \geq \sigma$ and $\beta < \alpha$ such that $\gamma = \psi\tau\beta$, $\beta \in C_\tau(\beta)$ and $\tau, \beta \in C_\sigma^n(\alpha)$.

Proof. By the assumption we have $\tau \geq \sigma$ and β_0 such that $\gamma = \psi\tau\beta_0$ and $\tau, \beta_0 \in C_\sigma^n(\alpha)$. By Lemma 2.10 there exists $\beta = \min\{\xi : \beta_0 \leq \xi \in C_\tau(\beta_0)\} \in C_\sigma^n(\alpha)$. Then by Lemma 2.8 $C_\tau(\beta_0) = C_\tau(\beta)$, $\gamma = \psi\tau\beta$ and $\beta \in C_\tau(\beta)$. If $\beta = \beta_0$, we have $\beta < \alpha$. Otherwise, $\beta_0 \notin C_\tau(\beta_0) = C_\tau(\beta)$. Then $\sigma \leq \tau$ implies $\beta_0 \notin C_\sigma(\beta)$. Therefore, $\beta_0 \in C_\sigma(\alpha)$ implies also in this case $\beta < \alpha$. The uniqueness of τ and β follows from Lemmata 2.2. and 2.6 b).

§ 3. – The Notation System $T(\Omega)$

INDUCTIVE DEFINITION of the set $T(\Omega)$ of ordinals and the *degree* $dg(\gamma)$ of $\gamma \in T(\Omega)$.

$(T1)$ $0 \in T(\Omega)$, $dg(0) := 0$.

$(T2)$ If $\gamma =_{NF} \omega^\alpha + \beta$ and $\alpha, \beta \in T(\Omega)$, then $\gamma \in T(\Omega)$ and $dg(\gamma) := \max\{dg(\alpha), dg(\beta)\} + 1$.

$(T3)$ If $0 < \sigma \in T(\Omega)$, then $\Omega_\sigma \in T(\Omega)$ and $dg(\Omega_\sigma) := dg(\sigma) + 1$.

$(T4)$ If $\sigma, \alpha \in T(\Omega)$ and $\alpha \in C_\sigma(\alpha)$, then $\psi_\sigma(\alpha) \in T(\Omega)$ and $dg(\psi_\sigma(\alpha)) := \max\{dg(\sigma), dg(\alpha)\} + 1$.

As before, we write $\psi\sigma\alpha$ instead of $\psi_\sigma(\alpha)$.

DEFINITION. $\Lambda := \min\{\xi : 0 < \xi = \Omega_\xi\}$.
(This ordinal Λ exists in classical set theory).

LEMMA 3.1. $\gamma \in T(\Omega) \Rightarrow \gamma < \Lambda$.
Proof by induction on $dg(\gamma)$.

REMARK. The elements of the set $T(\Omega)$ may be considered as terms which are composed by the symbols $0, +, \omega, \Omega, \psi$ and parentheses according to the inductive definition of $T(\Omega)$.

We call these terms *ordinal terms*. They denote ordinals in the sense of § 2. By Lemmata 2.2, 2.3, 2.6 b) and 3.1, any two distinct ordinal terms denote distinct ordinals. Therefore, the degree $dg(\gamma)$ of an ordinal $\gamma \in T(\Omega)$ is uniquely determined. We have $1 :=_{NF} \omega^0 + 0 \in T(\Omega)$, $\omega :=_{NF} \omega^1 + 0 \in T(\Omega)$ and $\varepsilon_0 := \psi 00 \in T(\Omega)$.

LEMMA 3.2. $\gamma \in T(\Omega) \Rightarrow \gamma \in C_0(\Lambda)$.
Proof by induction on $dg(\gamma)$ using Lemma 3.1.

LEMMA 3.3. $\gamma \in C_0^n(\alpha) \Rightarrow \gamma \in T(\Omega)$.
Proof by induction on n using Lemma 2.11.

THEOREM 3.4. The set of those ordinals of $T(\Omega)$ which are $< \Omega_1$ is exactly the segment of all ordinals $< \psi 0 \Lambda$.

Proof. By Lemmata 3.2 and 3.3 we have $T(\Omega) = C_0(\Lambda)$. The Theorem follows by Lemma 2.5.

INDUCTIVE DEFINITION of the set $G_\sigma \gamma \subseteq T(\Omega)$ for $\sigma, \gamma \in T(\Omega)$.
1. $G_\sigma 0 := \emptyset$ (empty).
2. If $\gamma =_{NF} \omega^\xi + \eta$, then $G_\sigma \gamma := G_\sigma \xi \cup G_\sigma \eta$.
3. $G_\sigma \Omega_\gamma := G_\sigma \gamma$.
4. If $\beta \in C_\tau(\beta)$ and $\tau < \sigma$, then $G_\sigma \psi \tau \beta := \emptyset$.
5. If $\beta \in C_\tau(\beta)$ and $\sigma \leq \tau$, then $G_\sigma \psi \tau \beta := \{\beta\} \cup G_\sigma \tau \cup G_\sigma \beta$.

LEMMA 3.5. $\gamma \in C_\sigma(\alpha) \Leftrightarrow G_\sigma \gamma < \alpha$ for $\gamma, \sigma \in T(\Omega)$.
Proof by induction on $dg(\gamma)$ using Lemmata 2.2, 2.4 and 2.11.

COROLLARY 3.5. $\psi_\sigma(\alpha)$ is an ordinal term of $T(\Omega)$ if and only if σ and α are ordinal terms of $T(\Omega)$ and $G_\sigma \alpha < \alpha$ holds.

Proof by Lemma 3.5, since an ordinal term $\psi_\sigma(\alpha)$ has to satisfy the condition $\alpha \in C_\sigma(\alpha)$.

THEOREM 3.6.

a) For any term composed by the symbols of $T(\Omega)$ it is decidable whether it is an ordinal term of $T(\Omega)$.

b) For any two distinct ordinal terms α and β of $T(\Omega)$ it is decidable whether $\alpha < \beta$ or $\beta < \alpha$ holds.

Proof by induction on the lengths of the terms using Lemmata 2.2, 2.3, 2.6 a) and Corollary 3.5.

In the following the letters $\alpha, \beta, \gamma, \delta, \eta, \mu, \nu, \xi, \pi, \sigma, \tau$ (also with subscript) always denote ordinals of $T(\Omega)$.

§ 4. – Majorization

For proof theoretical investigations we need a majorizing relation of ordinals which is invariant with respect to some collapsing procedures.

DEFINITIONS.

1. $\alpha \lhd_\mu \beta$ means that $\alpha < \beta$ and for all $\delta, \eta, \pi : \alpha \leq \delta \leq \min\{\beta, \eta\}$, $\delta, \mu \in C_\pi(\eta) \Rightarrow \alpha \in C_\pi(\eta)$.
2. $\alpha \lhd \beta$ (α is *majorized* by β) means that $\alpha \lhd_0 \beta$ holds.
3. $\alpha \trianglelefteq \beta$ means that $\alpha \lhd \beta$ or $\alpha = \beta$ holds.

LEMMA 4.1.

a) $\alpha \lhd \beta \Rightarrow \alpha \lhd_\mu \beta$
b) $\alpha < \beta \Rightarrow \alpha \lhd_\alpha \beta$
c) $\alpha < \beta < \gamma$, $\alpha \lhd_\mu \gamma \Rightarrow \alpha \lhd_\mu \beta$
d) $\alpha < \varepsilon_0$, $\alpha < \beta \Rightarrow \alpha \lhd \beta$
e) $0 < \beta < \varepsilon_0 \Rightarrow \alpha \lhd \alpha + \beta$
f) $\alpha < \beta < \Omega_1 \Rightarrow \alpha \lhd \beta$
g) $\alpha \lhd \beta \Rightarrow \alpha + 1 \trianglelefteq \beta$

Proofs are immediate by the definitions.

LEMMA 4.2. $\alpha \lhd_\mu \beta$, $\beta \lhd_\mu \gamma \Rightarrow \alpha \lhd_\mu \gamma$.

Proof. By the assumptions we have $\alpha < \gamma$. Suppose $\alpha \leq \delta \leq \min\{\gamma, \eta\}$ and $\delta, \mu \in C_\pi(\eta)$. If $\delta \leq \beta$ we obtain $\alpha \in C_\pi(\eta)$ according to $\alpha \lhd_\mu \beta$. If $\beta < \delta$ we obtain $\beta < \eta$ and $\beta \in C_\pi(\eta)$ according to $\beta \lhd_\mu \gamma$ and furthermore $\alpha \in C_\pi(\eta)$ according to $\alpha \lhd_\mu \beta$.

LEMMA 4.3. $\alpha \lhd_\mu \beta,\ \beta < \omega^{\gamma+1} \Rightarrow \omega^\gamma + \alpha \lhd_\mu \omega^\gamma + \beta$.

Proof. By the assumptions we have $\omega^\gamma + \alpha < \omega^\gamma + \beta$. Suppose $\omega^\gamma + \alpha \leq \delta \leq \min\{\omega^\gamma + \beta,\ \eta\}$ and $\delta,\ \mu \in C_\pi(\eta)$. Then we have $\delta = \omega^\gamma + \delta_o$, $\alpha \leq \delta_o \leq \min\{\beta, \eta\}$ and $\gamma, \delta_o \in C_\pi(\eta)$. According to $\alpha \lhd_\mu \beta$ we obtain $\alpha \in C_\pi(\eta)$ and furthermore $\omega^\gamma + \alpha \in C_\pi(\eta)$.

COROLLARY 4.3. $\omega^\alpha \cdot n \lhd \omega^\alpha \cdot (n+1)$.

Proof by induction on n using $0 \lhd \omega^\alpha$ and Lemma 4.3.

LEMMA 4.4. $\alpha \lhd_\mu \beta \Rightarrow \omega^\alpha \cdot n \lhd_\mu \omega^\beta$.

Proof. By the assumption we have $\omega^\alpha \cdot n < \omega^\beta$. The assertion holds for $n = 0$ according to Lemma 4.1 d). Now suppose $n > 0$, $\omega^\alpha \cdot n \leq \delta \leq \min\{\omega^\beta, \eta\}$ and $\delta,\ \mu \in C_\pi(\eta)$. Then we have $\delta = \omega^{\delta_1} + \delta_2$, $\delta_2 < \delta$, $\alpha \leq \delta_1 \leq \min\{\beta, \eta\}$ and $\delta_1 \in C_\pi(\eta)$. According to $\alpha \lhd_\mu \beta$ we obtain $\alpha \in C_\pi(\eta)$ which implies $\omega^\alpha \cdot n \in C_\pi(\eta)$.

COROLLARY 4.4. If $\beta = 0$ or $\beta \in E$, then $\omega_n(\beta + 1) \lhd \omega_{n+1}(\beta + 1)$.

Proof. If $\beta \in E$, then $\beta \lhd \beta + 1$ implies $\beta = \omega^\beta \lhd \omega^{\beta+1}$ by Lemma 4.4. Then we obtain $\beta + 1 \lhd \omega^{\beta+1}$ which also holds for $\beta = 0$. The assertion follows by induction on n using Lemma 4.4.

LEMMA 4.5. $\alpha \lhd_\mu \beta \Rightarrow \Omega_\alpha \lhd_\mu \Omega_\beta$.

Proof. By the assumption we have $\Omega_\alpha < \Omega_\beta$. Suppose $\Omega_\alpha \leq \delta \leq \min\{\Omega_\beta, \eta\}$ and $\delta,\ \mu \in C_\pi(\eta)$. Then we have $\Omega_\sigma \leq \delta < \Omega_{\sigma+1}$ such that $\alpha \leq \sigma \leq \min\{\beta, \eta\}$ and $\sigma \in C_\pi(\eta)$. According to $\alpha \lhd_\mu \beta$ we obtain $\alpha \in C_\pi(\eta)$ which implies $\Omega_\alpha \in C_\pi(\eta)$.

COROLLARY 4.5. $\Omega_\sigma < \alpha \leq \Omega_{\sigma+1} \Rightarrow \Omega_\sigma \lhd \alpha$.

Proof. $\sigma \lhd \sigma + 1$ implies $\Omega_\sigma \lhd \Omega_{\sigma+1}$ by Lemma 4.5. and $\Omega_\sigma \lhd \alpha$ by Lemma 4.1 c).

LEMMA 4.6. $\alpha \lhd_\mu \beta$, $\mu \in C_\sigma(\alpha)$ and $\beta \in C_\sigma(\beta)$ implies

a) $\alpha \in C_\sigma(\alpha)$

b) $\psi\sigma\alpha \lhd_\mu \psi\sigma\beta$.

Proof. a) By Lemma 2.8 there is a $\gamma = \min\{\xi : \alpha \leq \xi \in C_\sigma(\alpha)\}$ such that $\gamma \in C_\sigma(\gamma) = C_\sigma(\alpha)$. Then we have $\gamma, \mu \in C_\sigma(\gamma)$. Since $\beta \in C_\sigma(\beta)$, we have $\alpha \leq \gamma = \min\{\beta, \gamma\}$. According to $\alpha \lhd_\mu \beta$ we obtain $\alpha \in C_\sigma(\gamma)$, hence $\alpha \in C_\sigma(\alpha)$.

b) $\alpha < \beta$ and $\alpha \in C_\sigma(\alpha)$ implies $\psi\sigma\alpha < \psi\sigma\beta$ by Lemmata 2.1 and 2.6 a). Suppose $\psi\sigma\alpha \leq \delta \leq \min\{\psi\sigma\beta, \eta\}$ and $\delta,\ \mu \in C_\pi(\eta)$. We prove $\psi\sigma\alpha \in C_\pi(\eta)$ by induction on $dg(\delta)$. Since $\Omega_\sigma < \delta < \Omega_{\sigma+1}$, we have only the following three cases.

1. $\delta < \Omega_\pi$. Then we have $\psi\sigma\alpha < \Omega_\pi$, hence $\psi\sigma\alpha \in C_\pi(\eta)$.

2. $\Omega_\pi < \delta =_{NF} \omega^{\delta_1} + \delta_2$, $\psi\sigma\alpha \leq \delta_1 < \min\{\psi\sigma\beta, \eta\}$ and $\delta_1 \in C_\pi(\eta)$. Since $dg(\delta_1) < dg(\delta)$, we obtain $\psi\sigma\alpha \in C_\pi(\eta)$ by the I.H.

3. $\Omega_\pi < \delta = \psi\sigma\delta_o$, $\delta_o \in C_\sigma(\delta_o)$, $\alpha \leq \delta_o \leq \beta$, σ, $\delta_o \in C_\pi(\eta)$ and $\delta_o < \eta$. According to $\alpha \lhd_\mu \beta$ we obtain $\alpha \in C_\pi(\eta)$. Together with $\sigma \in C_\pi(\eta)$ and $\alpha \leq \delta_o < \eta$ we obtain $\psi\sigma\alpha \in C_\pi(\eta)$.

COROLLARY 4.6. $\alpha = \alpha_o + 1 \in C_\sigma(\alpha)$ implies $\alpha_o \in C_\sigma(\alpha)$ and $\psi\sigma\alpha_o \lhd \psi\sigma\alpha$.

Proof by Lemma 4.6 for $\mu = 0$ using $\alpha_o \lhd \alpha$.

§ 5. - Fundamental Functions and Fundamental Sequences

In proof theoretical investigations we shall use some fundamental functions to control transfinite inferences.

DEFINITION. A function $f : \mathrm{dom}(f) \to T(\Omega)$ with the *domain* $\mathrm{dom}(f) \subseteq T(\Omega)$ is said to be a *fundamental function* if the following holds.

($F1$) If $\beta \in \mathrm{dom}(f)$ and $\alpha < \beta$, then $\alpha \in \mathrm{dom}(f)$ and $f(\alpha) \lhd_\alpha f(\beta)$.

($F2$) If $\beta \in \mathrm{dom}(f)$ and $f(0) \leq \delta < f(\beta)$, then there is an $\alpha < \beta$ such that $f(\alpha) \leq \delta < f(\alpha + 1)$ and $f(\alpha) \lhd f(\alpha + 1)$.

($F3$) If $\alpha \in \mathrm{dom}(f)$ and $f(\alpha) \in C_\pi(\eta)$, then $\alpha \in C_\pi(\eta)$.

LEMMA 5.1. If f is a fundamental function and $\alpha \in \mathrm{dom}(f)$, then $\alpha \leq f(\alpha)$.

Proof by induction on α using ($F1$).

DEFINITION. Let Id_β be the identity function with domain $\mathrm{dom}(Id_\beta) := \{\alpha \in T(\Omega) : \alpha \leq \beta\}$ and $Id_\beta(\alpha) := \alpha$ for all $\alpha \in \mathrm{dom}(Id_\beta)$.

LEMMA 5.2. Id_β is a fundamental function.

The proof is trivial.

DEFINITIONS with respect to a fundamental function f.

1. Let $\omega^\gamma + f$ be the function with domain $\mathrm{dom}(\omega^\gamma + f) := \{\alpha \in \mathrm{dom}(f) : f(\alpha) < \omega^{\gamma+1}\}$ and $(\omega^\gamma + f)(\alpha) := \omega^\gamma + f(\alpha)$ for all $\alpha \in \mathrm{dom}(\omega^\gamma + f)$.

2. Let ω^f be the function with domain $\mathrm{dom}(\omega^f) := \mathrm{dom}(f)$ and $(\omega^f)(\alpha) := \omega^{f(\alpha)}$ for all $\alpha \in \mathrm{dom}(\omega^f)$.

3. Let Ω_f be the function with domain $\mathrm{dom}(\Omega_f) := \mathrm{dom}(f)$ and $(\Omega_f)(\alpha) := \Omega_{f(\alpha)}$ for all $\alpha \in \mathrm{dom}(\Omega_f)$.

4. Let $\psi\sigma f$ be the function with domain $\mathrm{dom}(\psi\sigma f) := \{\alpha \in \mathrm{dom}(f) : \alpha < \Omega_{\sigma+1},\ f(\alpha) \in C_\sigma(f(\alpha))\}$ and $(\psi\sigma f)(\alpha) := \psi\sigma(f(\alpha))$ for all $\alpha \in \mathrm{dom}(\psi\sigma f)$.

LEMMA 5.3. If f is a fundamental function, then also $\omega^\gamma + f$, ω^f, Ω_f and $\psi\sigma f$ are fundamental functions.

Proof. The assertion for $\omega^\gamma + f$, ω^f and Ω_f follows from Lemmata 4.3, 4.4 and 4.5. Now we prove the assertion for $\psi\sigma f$.

1. Suppose $\beta \in \mathrm{dom}(\psi\sigma f)$ and $\alpha < \beta$. Then we have $\alpha, \beta \in \mathrm{dom}(f)$, $f(\alpha) \lhd_\alpha f(\beta)$, $f(\beta) \in C_\sigma(f(\beta))$ and $\beta < \Omega_{\sigma+1}$. By Lemma 2.8 there is a $\gamma = \min\{\xi : f(\alpha) \le \xi \in C_\sigma(f(\alpha))\}$ such that $\gamma \in C_\sigma(\gamma) = C_\sigma(f(\alpha))$. Since $f(\beta) \in C_\sigma(f(\beta))$, we have $f(\alpha) \le \gamma \le f(\beta)$. If $\gamma = f(\beta)$, we obtain $\beta \in C_\sigma(\gamma) = C_\sigma(f(\alpha))$ and $\alpha \in C_\sigma(f(\alpha))$, since $\alpha < \beta < \Omega_{\sigma+1}$. Otherwise there is a $\mu < \beta$ such that $f(\mu) \le \gamma < f(\mu+1)$, $\alpha \le \mu$ and $f(\mu) \lhd f(\mu+1)$. Then $\gamma \in C_\sigma(\gamma)$ implies $f(\mu) \in C_\sigma(\gamma)$ and $\mu \in C_\sigma(\gamma) = C_\sigma(f(\alpha))$. Since $\alpha \le \mu < \Omega_{\sigma+1}$, we obtain $\alpha \in C_\sigma(f(\alpha))$. In any case we have $\alpha \in C_\sigma(f(\alpha))$. Together with $f(\alpha) \lhd_\alpha f(\beta)$ and $f(\beta) \in C_\sigma(f(\beta))$ we obtain $f(\alpha) \in C_\sigma(f(\alpha))$ by Lemma 4.6 a). Then we have $\alpha \in \mathrm{dom}(\psi\sigma f)$. By Lemma 4.6 b) we obtain $(\psi\sigma f)(\alpha) \lhd_\alpha (\psi\sigma f)(\beta)$. Hence $(F1)$ holds for $\psi\sigma f$.

2. Suppose $\beta \in \mathrm{dom}(\psi\sigma f)$ and $(\psi\sigma f)(0) \le \delta < (\psi\sigma f)(\beta)$. We prove by induction on $dg(\delta)$ that there is an $\alpha < \beta$ such that $(\psi\sigma f)(\alpha) \le \delta < (\psi\sigma f)(\alpha+1)$ and $(\psi\sigma f)(\alpha) \lhd (\psi\sigma f)(\alpha+1)$. If $\delta =_{NF} \omega^{\delta_1} + \delta_2$ the assertion follows from the I.H. Otherwise we have $\delta = \psi\sigma\delta_0$, $\delta_0 \in C_\sigma(\delta_0)$ and $f(0) \le \delta_0 < f(\beta)$. Then there is an $\alpha < \beta$ such that $f(\alpha) \le \delta_0 < f(\alpha+1)$ and $f(\alpha) \lhd f(\alpha+1)$, which implies $(\psi\sigma f)(\alpha) \le \delta < (\psi\sigma f)(\alpha+1)$ and by Lemma 4.6 b) $(\psi\sigma f)(\alpha) \lhd (\psi\sigma f)(\alpha+1)$. Hence $(F2)$ holds for $\psi\sigma f$.

3. Suppose $\alpha \in \mathrm{dom}(\psi\sigma f)$. If $\sigma < \pi$, we have $\alpha \in C_\pi(\eta)$, since $\alpha < \Omega_{\sigma+1}$. If $\sigma \ge \pi$ and $(\psi\sigma f)(\alpha) \in C_\pi(\eta)$, we obtain $f(\alpha) \in C_\pi(\eta)$ and $\alpha \in C_\pi(\eta)$. Hence also $(F3)$ holds for $\psi\sigma f$.

LEMMA 5.4. If f is a fundamental function with $\alpha, \Omega_{\tau+1} \in \mathrm{dom}(f)$, $\alpha < \beta = \psi\tau(f(\alpha))$ and $f(\alpha) \lhd f(\Omega_{\tau+1})$, then also $f(\beta) \lhd f(\Omega_{\tau+1})$.

Proof. We have $f(\beta) \lhd_\beta f(\Omega_{\tau+1})$. Suppose $f(\beta) \le \delta \le \min\{f(\Omega_{\tau+1}), \eta\}$ and $\delta \in C_\pi(\eta)$. Then $f(\alpha) < f(\beta)$ and $f(\alpha) \lhd f(\Omega_{\tau+1})$ implies $f(\alpha) \in C_\pi(\eta)$ and $f(\alpha) < \eta$. If $\delta = f(\Omega_{\tau+1})$ we obtain $\Omega_{\tau+1} \in C_\pi(\eta)$ and $\tau \in C_\pi(\eta)$. Otherwise there is a $\mu < \Omega_{\tau+1}$ such that $f(\mu) \le \delta < f(\mu+1)$ and $f(\mu) \lhd f(\mu+1)$. Then $\delta \in C_\pi(\eta)$ implies $f(\mu) \in C_\pi(\eta)$ and $\mu \in C_\pi(\eta)$, which implies $\tau \in C_\pi(\eta)$, since $\beta = \psi\tau(f(\alpha)) \le \mu < \Omega_{\tau+1}$. In any case we obtain $\tau, f(\alpha) \in C_\pi(\eta)$ and $f(\alpha) < \eta$, which implies $\beta = \psi\tau(f(\alpha)) \in C_\pi(\eta)$. According to $f(\beta) \lhd_\beta f(\Omega_{\tau+1})$ we obtain $f(\beta) \in C_\pi(\eta)$. Hence we have $f(\beta) \lhd f(\Omega_{\tau+1})$.

COROLLARY 5.4. If f is a fundamental function with $\Omega_{\tau+1} \in \mathrm{dom}(f)$, then $f(\psi\tau(f(0))) \lhd f(\Omega_{\tau+1})$.

Proof by Lemma 5.4 for $\alpha = 0$ using $f(0) \lhd f(\Omega_{\tau+1})$.

LEMMA 5.5. If f is a fundamental function with $\Omega_{\tau+1} \in \mathrm{dom}(f)$, $\sigma \le \tau$, $f(\Omega_{\tau+1}) \in C_\sigma(f(\Omega_{\tau+1}))$ and g is the function with domain $\mathrm{dom}(g) = \{\alpha : \alpha \le \omega\}$, $g(n) = \psi\sigma(f(\beta_n))$ and $g(\omega) = \psi\sigma(f(\Omega_{\tau+1}))$ where $\beta_0 = 0$ and $\beta_{n+1} = \psi\tau(f(\beta_n))$, then also g is a fundamental function.

Proof. 1. At first we prove

$$\beta_n < \beta_{n+1} \quad \text{and} \quad f(\beta_n) \lhd f(\Omega_{\tau+1}) \tag{1}$$

by induction on n. The assertion (1) holds for $n = 0$, since $\beta_0 = 0$.

Now we prove it for $n+1$ under the assumption that it holds for n. $f(\beta_n) \lhd f(\Omega_{\tau+1})$ and $f(\Omega_{\tau+1}) \in C_\sigma(f(\Omega_{\tau+1})) \subseteq C_\tau(f(\Omega_{\tau+1}))$ implies $f(\beta_n) \in C_\tau(f(\beta_n))$ by Lemma 4.6 a). Using $\beta_n < \beta_{n+1}$ we obtain τ, $f(\beta_n) \in C_\tau(f(\beta_{n+1}))$ and $f(\beta_n) < f(\beta_{n+1})$ which implies $\beta_{n+1} = \psi\tau(f(\beta_n)) \in C_\tau(f(\beta_{n+1}))$ and $\beta_{n+1} < \psi\tau(f(\beta_{n+1})) = \beta_{n+2}$.

From $\beta_n < \beta_{n+1}$ and $f(\beta_n) \lhd f(\Omega_{\tau+1})$ we also obtain $f(\beta_{n+1}) \lhd f(\Omega_{\tau+1})$ by Lemma 5.4, which completes the inductive proof of (1).

$f(\beta_n) \lhd f(\Omega_{\tau+1})$ and $f(\Omega_{\tau+1}) \in C_\sigma(f(\Omega_{\tau+1}))$ implies $f(\beta_n) \in C_\sigma(f(\beta_n))$ by Lemma 4.6 a). Together with $f(\beta_n) < f(\beta_{n+1})$ we obtain

$$g(n) = \psi\sigma(f(\beta_n)) < \psi\sigma(f(\beta_{n+1})) = g(n+1) \tag{2}$$

by Lemmata 2.1 and 2.6 a). By Lemma 4.6 b) we obtain

$$g(n) = \psi\sigma(f(\beta_n)) \lhd \psi\sigma(f(\Omega_{\tau+1})) = g(\omega) \tag{3}$$

The condition $(F1)$ for g follows from (2) and (3).

2. To obtain also $(F2)$ for g, we prove by induction on γ:

(4) For $\gamma < \psi\tau(f(\Omega_{\tau+1}))$ there is an n such that $\beta_n \leq \gamma < \beta_{n+1}$.

If $\gamma < \beta_1$, we have $\beta_0 \leq \gamma < \beta_1$. Now suppose $\beta_1 = \psi\tau(f(0)) \leq \gamma < \psi\tau(f(\Omega_{\tau+1}))$. If $\gamma =_{NF} \omega^\xi + \eta$, the assertion follows from the I.H. Otherwise we have $\gamma = \psi\tau\gamma_0$, $\gamma_0 \in C_\tau(\gamma_0)$ and $f(0) \leq \gamma_0 < f(\Omega_{\tau+1})$. Then there is an $\alpha < \Omega_{\tau+1}$ such that $f(\alpha) \leq \gamma_0 < f(\alpha+1)$ and $f(\alpha) \lhd f(\alpha+1)$. In this case $\gamma_0 \in C_\tau(\gamma_0)$ implies $f(\alpha) \in C_\tau(\gamma_0)$, $\alpha \in C_\tau(\gamma_0)$ and $\alpha < \psi\tau\gamma_0 = \gamma$. Hence by the I.H. there is an n such that $\beta_n \leq \alpha < \beta_{n+1}$. It follows that $f(\beta_n) \leq \gamma_0 < f(\beta_{n+1})$ and $\beta_{n+1} = \psi\tau(f(\beta_n)) \leq \gamma < \psi\tau(f(\beta_{n+1})) = \beta_{n+2}$, which completes the proof of (4). Now we prove by induction on $dg(\delta)$:

(5) If $g(0) \leq \delta < g(\omega)$, then there is an n such that $g(n) \leq \delta < g(n+1)$.

If $\delta =_{NF} \omega^\xi + \eta$, the assertion follows from the I.H. Otherwise we have $\delta = \psi\sigma\delta_0$, $\delta_0 \in C_\sigma(\delta_0)$ and $f(0) \leq \delta_0 < f(\Omega_{\tau+1})$. Then there is an $\mu < \Omega_{\tau+1}$ such that $f(\mu) \leq \delta_0 < f(\mu+1)$ and $f(\mu) \lhd f(\mu+1)$. In this case $\delta_0 \in C_\sigma(\delta_0)$ implies $f(\mu) \in C_\sigma(\delta_0)$, $\mu \in C_\sigma(\delta_0) \subseteq C_\tau(f(\Omega_{\tau+1}))$ and $\mu < \psi\tau(f(\Omega_{\tau+1}))$. Hence by (4) there is an n such that $\beta_n \leq \mu < \beta_{n+1}$. It follows that $f(\beta_n) \leq \delta_0 < f(\beta_{n+1})$ and $g(n) = \psi\sigma(f(\beta_n)) \leq \delta < \psi\sigma(f(\beta_{n+1})) = g(n+1)$, which completes the proof of (5).

The condition $(F2)$ for g follows from (2), (3) and (5). The condition $(F3)$ is trivial for g.

Let L be the set of limit numbers of $T(\Omega)$. For any $\gamma \in L$ we define a distinguished *fundamental sequence* $\gamma[\nu]$ $(\nu < \mathrm{tp}(\gamma))$ of *type* $\mathrm{tp}(\gamma)$ in the following way.

INDUCTIVE DEFINITION of $\mathrm{tp}(\gamma)$ and $\gamma[\nu]$ for $\gamma \in L$ and $\nu < \mathrm{tp}(\gamma)$.

$(L1)$ $\gamma = \omega^{\alpha+1}$:

$\mathrm{tp}(\gamma) := \omega$, $\gamma[n] := \omega^\alpha \cdot n$.

$(L2)$ $\gamma =_{NF} \omega^\alpha + \beta$, $\beta \in L$:

$\mathrm{tp}(\gamma) := \mathrm{tp}(\beta)$, $\gamma[\nu] := \omega^\alpha + \beta[\nu]$.

$(L3)$ $\gamma = \omega^\alpha > \alpha$, $\alpha \in L$:

$\mathrm{tp}(\gamma) := \mathrm{tp}(\alpha)$, $\gamma[\nu] := \omega^{\alpha[\nu]}$.

$(L4)$ $\gamma = \Omega_{\sigma+1}$:

$\mathrm{tp}(\gamma) := \gamma$, $\gamma[\nu] := \nu$.

$(L5)$ $\gamma = \Omega_\alpha$, $\alpha \in L$:

$\mathrm{tp}(\gamma) := \mathrm{tp}(\alpha)$, $\gamma[\nu] := \Omega_{\alpha[\nu]}$.

$(L6)$ $\gamma = \psi\sigma\alpha$, $\alpha \in C_\sigma(\alpha)$, $\alpha \notin L$:

$\mathrm{tp}(\gamma) := \omega$, $\gamma[n] := \omega_n(\beta+1)$ where $\beta := \Omega_\sigma$ if $\alpha = 0$ and $\beta := \psi\sigma\alpha_0$ if $\alpha = \alpha_0 + 1$.

$(L7)$ $\gamma = \psi\sigma\alpha$, $\alpha \in C_\sigma(\alpha)$, $\alpha \in L$, $\mathrm{tp}(\alpha) < \Omega_{\sigma+1}$:

$\mathrm{tp}(\gamma) := \mathrm{tp}(\alpha)$, $\gamma[\nu] := \psi\sigma(\alpha[\nu])$.

$(L8)$ $\gamma = \psi\sigma\alpha$, $\alpha \in C_\sigma(\alpha)$, $\alpha \in L$, $\mathrm{tp}(\alpha) = \Omega_{\tau+1} \geq \Omega_{\sigma+1}$:

$\mathrm{tp}(\gamma) := \omega$, $\gamma[n] := \psi\sigma(\alpha[\beta_n])$ where $\beta_0 := 0$ and $\beta_{n+1} := \psi\tau(\alpha[\beta_n])$.

LEMMA 5.6. For $\gamma \in L$:

a) $\mathrm{tp}(\gamma)$ is either ω or an ordinal $\Omega_{\sigma+1}$.

b) $\mathrm{tp}(\gamma) \leq \gamma$.

c) $\gamma \in C_\pi(\eta) \Rightarrow \mathrm{tp}(\gamma) \in C_\pi(\eta)$.

Proof by induction on $\mathrm{dg}(\gamma)$.

DEFINITION. For $\gamma \in L$ let F_γ be the function with domain $\mathrm{dom}(F_\gamma) := \{\alpha \in T(\Omega) : \alpha \leq \mathrm{tp}(\gamma)\}$, $F_\gamma(\alpha) := \gamma[\alpha]$ for $\alpha < \mathrm{tp}(\gamma)$ and $F_\gamma(\mathrm{tp}(\gamma)) := \gamma$.

THEOREM 5.7. For $\gamma \in L$ the function F_γ is a fundamental function.

Proof by induction on $\mathrm{dg}(\gamma)$. The assertion follows in the case $(L1)$ from Corollary 4.3 and Lemma 4.4, is trivial in the case $(L4)$ and follows in the case $(L6)$ from Corollary 4.4 and Corollary 4.6, since in this case γ is the least ε-number greater than β. In the remaining cases $(L2)$, $(L3)$, $(L5)$, $(L7)$ and $(L8)$ the assertion follows from the I.H. by Lemmata 5.3 and 5.5.

COROLLARY 5.7. $\gamma \in L \Rightarrow \gamma = \sup\{\gamma[\nu] : \nu < \mathrm{tp}(\gamma)\}$.

Proof by Theorem 5.7 using the condition $(F2)$ of fundamental functions.

§ 6. - Sets of Coefficients

Only for the well ordering proofs in Chapter IV we need the coefficient sets which we define in this section.

INDUCTIVE DEFINITION of the set $K_\sigma \gamma$ of σ-coefficients of an ordinal $\gamma \in T(\Omega)$ (by induction on $dg(\gamma)$).

1. $K_\sigma 0 := \emptyset$ (empty)
2. If $\gamma =_{NF} \omega^\xi + \eta$, then $K_\sigma \gamma := K_\sigma \xi \cup K_\sigma \eta$.
3. If $\tau > 0$, then

$$K_\sigma \Omega_\tau := \begin{cases} \{\Omega_\tau\} & \text{for } \tau \leq \sigma \\ K_\sigma \tau & \text{for } \sigma < \tau \end{cases}$$

4. If $\beta \in C_\tau(\beta)$, then

$$K_\sigma \psi \tau \beta := \begin{cases} \{\psi \tau \beta\} & \text{for } \tau \leq \sigma \\ K_\sigma \tau \cup K_\sigma \beta & \text{for } \sigma < \tau \end{cases}$$

LEMMA 6.1. $K_\sigma \gamma$ is a finite set of ε-numbers of $T(\Omega)$ which are $< \Omega_{\sigma+1}$.
Proof by induction on $dg(\gamma)$.

LEMMA 6.2. $\gamma \in C_\sigma(\alpha) \Rightarrow K_\sigma \gamma \subseteq C_\sigma(\alpha)$.
Proof by induction on $dg(\gamma)$.

LEMMA 6.3. $\alpha \in C_\sigma(\alpha) \Rightarrow K_\sigma \alpha < \psi \sigma \alpha$.
Proof. $\alpha \in C_\sigma(\alpha)$ implies $K_\sigma \alpha \subseteq C_\sigma(\alpha)$ by Lemma 6.2. The assertion follows by Lemmata 6.1 and 2.5.

§ 7. - The Ordinal Γ_0

The ordinal Γ_0 is the least upper bound of predicatively provable well orderings (see for instance [8] or [25]). We determine in this section the ordinal term of $T(\Omega)$ which denotes the ordinal Γ_0 (only to show that Γ_0 is essentially smaller than the proof theoretical ordinals of the subsystems of analysis which we investigate in Chapters II and III).
The ordinals $\Omega_1 \cdot \gamma$ and $\Omega_1^\sigma \cdot \gamma$ may be defined in the usual way.

They have the following recursive characterization:

1. $\Omega_1 \cdot 0 = \Omega_1^\sigma \cdot 0 = 0$
2. $\gamma =_{NF} \omega^\alpha + \beta \Rightarrow \Omega_1 \cdot \gamma = \omega^{\Omega_1 + \alpha} + \Omega_1 \cdot \beta$
3. $\gamma \in E \Rightarrow \Omega_1 \cdot \gamma = \omega^{\Omega_1 + \gamma}$
4. $\gamma =_{NF} \omega^\alpha + \beta \Rightarrow \Omega_1^\sigma \cdot \gamma = \omega^{\Omega_1 \cdot \sigma + \alpha} + \Omega_1^\sigma \cdot \beta$
5. $\gamma \in E \Rightarrow \Omega_1^\sigma \cdot \gamma = \omega^{\Omega_1 \cdot \sigma + \gamma}$

Any ordinal $\gamma > 0$ has a *Cantor normal form* to the *base* Ω_1:

$$\gamma = \Omega_1^{\alpha_1} \cdot \beta_1 + \ldots + \Omega_1^{\alpha_n} \cdot \beta_n \qquad (n \geq 1)$$

where $\alpha_1 > \ldots > \alpha_n$ and $0 < \beta_i < \Omega_1$ $(i = 1, \ldots, n)$.

LEMMA 7.1.

a) $\Omega_1 \cdot \gamma \in C_0(\eta) \Leftrightarrow \gamma \in C_0(\eta)$
b) $\Omega_1^\sigma \cdot \gamma \in C_0(\eta)$, $0 < \gamma < \Omega_1 \Rightarrow \sigma \in C_0(\eta)$
c) $\Omega_1^\sigma \cdot \gamma \in C_0(\eta) \Rightarrow \gamma \in C_0(\eta)$
d) $\sigma, \gamma \in C_0(\eta) \Rightarrow \Omega_1^\sigma \cdot \gamma \in C_0(\eta)$

Proof by induction on $dg(\gamma)$ using the above recursive characterization.

LEMMA 7.2. $\alpha < \psi 0 (\Omega_1^{\Omega_1}) \Rightarrow \alpha \in C_0(\Omega_1^\alpha)$.

Proof. $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ implies $\alpha \in C_0(\Omega_1^{\Omega_1})$. If $C_0(\Omega_1^\alpha) = C_0(\Omega_1^{\Omega_1})$, if follows that $\alpha \in C_0(\Omega_1^\alpha)$. Otherwise there is by Lemma 2.7 a $\delta \in C_0(\Omega_1^\alpha)$ such that $\Omega_1^\alpha \leq \delta < \Omega_1^{\Omega_1}$. Then we have $\delta = \Omega_1^{\alpha_1} \cdot \beta_1 + \delta_0$ where $\alpha \leq \alpha_1 < \Omega_1$, $0 < \beta_1 < \Omega_1$ and $\delta_0 < \Omega_1^{\alpha_1}$. $\delta \in C_0(\Omega_1^\alpha)$ implies $\Omega_1^{\alpha_1} \cdot \beta_1 \in C_0(\Omega_1^\alpha)$ and by Lemma 7.1 b) $\alpha_1 \in C_0(\Omega_1^\alpha)$. Since $\alpha \leq \alpha_1 < \Omega_1$, it follows by Corollary 2.5 that $\alpha \in C_0(\Omega_1^\alpha)$.

LEMMA 7.3. If $\psi 0 (\Omega_1^{\alpha+1}) \leq \beta < \Omega_1$, then there is a uniquely determined $\eta > 0$ such that $\psi 0 (\Omega_1^{\alpha+1} \cdot \eta) \leq \beta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$ and $\Omega_1^{\alpha+1} \cdot \eta \in C_0(\Omega_1^{\alpha+1} \cdot \eta)$.

Proof by induction on dg (β). There are only the following two cases for β.

1. $\beta =_{NF} \omega^{\beta_1} + \beta_2$. Then the assumption implies $\psi 0 (\Omega_1^{\alpha+1}) \leq \beta_1 < \Omega_1$. Therefore there is by the I.H. an $\eta > 0$ such that $\psi 0 (\Omega_1^{\alpha+1} \cdot \eta) \leq \beta_1 < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$ and $\Omega_1^{\alpha+1} \cdot \eta \in C_0(\Omega_1^{\alpha+1} \cdot \eta)$. It follows that $\psi 0 (\Omega_1^{\alpha+1} \cdot \eta) < \beta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$.

2. $\beta = \psi 0 \beta_o$, $\beta_o \in C_o(\beta_o)$ and $\Omega_1^{\alpha+1} \leq \beta_o$. In this case there is an $\eta > 0$ such that $\Omega_1^{\alpha+1} \cdot \eta \leq \beta_o < \Omega_1^{\alpha+1} \cdot (\eta + 1)$. It follows that $\psi 0 (\Omega_1^{\alpha+1} \cdot \eta) \leq \beta$ and according to $\beta_o \in C_o(\beta_o)$ also $\beta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$. For $\delta := \min \{\xi : \Omega_1^{\alpha+1} \cdot \eta \leq \xi \in C_o(\Omega_1^{\alpha+1} \cdot \eta)\}$ we have $\delta = \Omega_1^{\alpha+1} \cdot \eta + {} + \delta_o \leq \beta_o$, $\delta_o < \Omega_1^{\alpha+1}$ and $\delta \in C_o(\Omega_1^{\alpha+1} \cdot \eta)$. It follows that $\Omega_1^{\alpha+1} \cdot \eta \in C_o(\Omega_1^{\alpha+1} \cdot \eta)$.

The uniqueness of η is obvious.

LEMMA 7.4. If $\gamma = \Omega_1^{\alpha+1} \cdot \eta + \Omega_1^{\alpha} \cdot \delta$ and $\delta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$, then $\delta \in C_o(\gamma)$.

Proof. The assumption $\delta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$ implies $\delta \in C_o(\Omega_1^{\alpha+1} \cdot \cdot (\eta + 1))$. If $C_o(\gamma) = C_o(\Omega_1^{\alpha+1} \cdot (\eta + 1))$, we have $\delta \in C_o(\gamma)$. Otherwise there is by Lemma 2.7 a $\beta \in C_o(\gamma)$ such that $\gamma \leq \beta < \Omega_1^{\alpha+1} \cdot (\eta + 1)$. In this case we have $\beta = \Omega_1^{\alpha+1} \cdot \eta + \Omega_1^{\alpha} \cdot \beta_1 + \beta_2$, $\delta \leq \beta_1 < \Omega_1$ and $\beta_2 < \Omega_1^{\alpha}$. $\beta \in C_o(\gamma)$ implies Ω_1^{α}, $\beta_1 \in C_o(\gamma)$ and by Lemma 7.1 c) $\beta_1 \in C_o(\gamma)$. Since $\delta \leq \beta_1 < \Omega_1$, it follows that $\delta \in C_o(\gamma)$.

COROLLARY 7.4. $\beta < \psi 0 (\Omega_1^{\alpha+1}) \Rightarrow \beta \in C_o(\Omega_1^{\alpha} \cdot \beta)$.

Proof. This is the special case of Lemma 7.4 for $\eta = 0$ and $\delta = \beta$.

DEFINITION of $[\alpha, \beta]$ for $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and $\beta < \Omega_1$.

1. If $\beta < \psi 0 \Omega_1$, then $[0, \beta] := \beta$.

2. If $\alpha > 0$ and $\beta < \psi 0 (\Omega_1^{\alpha+1})$, then $[\alpha, \beta] := \Omega_1^{\alpha} \cdot (1 + \beta)$.

3. If $\beta = \psi 0 (\Omega_1^{\alpha+1} \cdot \eta) + \delta$, $\eta > 0$ and $\delta < \psi 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$, then $[\alpha, \beta] := \Omega_1^{\alpha+1} \cdot \eta + \Omega_1^{\alpha} \cdot \delta$.

(According to Lemma 7.3, this is a complete and unambiguous definition of $[\alpha, \beta]$ for all $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and all $\beta < \Omega_1$).

LEMMA 7.5. For $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and $\beta < \Omega_1$:

a) $[\alpha, \beta] \in C_o([\alpha, \beta])$

b) $\beta_o < \beta \Rightarrow [\alpha, \beta_o] < [\alpha, \beta]$

c) $\alpha < \psi 0 [\alpha, \beta]$

d) $[\alpha, \psi 0 (\Omega_1^{\Omega_1})] = \Omega_1^{\Omega_1}$

e) $\beta = \psi 0 [\alpha, \beta]$ if there is an $\eta > 0$ such that $\beta = \psi 0 (\Omega_1^{\alpha+1} \cdot \eta)$; otherwise $\beta < \psi 0 [\alpha, \beta]$.

Proof. a) follows from Lemmata 7.1 d), 7.2, 7.3, 7.4 and Corollary 7.4.

b) is obvious.

c) $\alpha \in C_o([\alpha, \beta])$ by Lemma 7.2, hence $\alpha < \psi 0 [\alpha, \beta]$ by Lemma 2.5.

d) $\Omega_1^{\Omega_1} = \Omega_1^{\alpha+1} \cdot \Omega_1^{\Omega_1}$, since $\alpha + 1 < \Omega_1$. It follows by the definition of $[\alpha, \beta]$ that $[\alpha, \psi 0 (\Omega_1^{\Omega_1})] = \Omega_1^{\alpha+1} \cdot \Omega_1^{\Omega_1} = \Omega_1^{\Omega_1}$.

e) We have the following two cases.

1. $\beta < \psi 0 (\Omega_1^{\alpha+1})$, $\Omega_1^{\alpha} \cdot \beta \leq [\alpha, \beta]$. In this case we have by Corollary 7.4 $\beta \in C_o(\Omega_1^{\alpha} \cdot \beta) \subseteq C_o([\alpha, \beta])$, hence $\beta < \psi 0 [\alpha, \beta]$.

2. $\beta = \psi 0 (\Omega_1^{\alpha+1} \cdot \eta) + \delta$, $\eta > 0$, $\delta < \gamma 0 (\Omega_1^{\alpha+1} \cdot (\eta + 1))$, $[\alpha, \beta] = {} = \Omega_1^{\alpha+1} \cdot \eta + \Omega_1^{\alpha} \cdot \delta$. In this case we have $\beta = \psi 0 [\alpha, \beta]$ if $\delta = 0$, otherwise $\beta < \psi 0 [\alpha, \beta]$.

DEFINITION of $\varphi \alpha \beta$ for $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and $\beta < \Omega_1$. $\varphi 0 \beta := \omega^{\beta}$, $\varphi (1 + \alpha) \beta := \psi 0 [\alpha, \beta]$.

LEMMA 7.6. For $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and $\beta < \Omega_1$:

a) $\beta_o < \beta \Rightarrow \varphi \alpha \beta_o < \varphi \alpha \beta$

b) $\alpha_o < \alpha \Rightarrow \varphi \alpha_o (\varphi \alpha \beta) = \varphi \alpha \beta$

c) If $\varphi \xi \beta = \beta$ for all $\xi < 1 + \alpha$, then there is a $\beta_o < \Omega_1$ such that $\beta = \varphi (1 + \alpha) \beta_o$.

d) $\alpha < \varphi \alpha \beta$

e) $\varphi \alpha (\psi 0 (\Omega_1^{\Omega_1})) = \psi 0 (\Omega_1^{\Omega_1})$.

Proof. a) Holds for $\alpha = 0$ and follows for $\alpha > 0$ from Lemma 7.5 a) and b).

b) Holds for $\alpha_o = 0$, since $\varphi \alpha \beta$ $(\alpha > 0)$ is an ε-number. If $\alpha_o > 0$, we have $\alpha_o = 1 + \alpha_1$, $\alpha = 1 + \alpha_2$, $\alpha_1 < \alpha_2$, $\varphi \alpha \beta = \psi 0 [\alpha_2, \beta]$ and $[\alpha_2, \beta] = {} = \Omega_1^{\alpha_2+1} \cdot \eta + \Omega_1^{\alpha_2} \cdot \delta > 0$. Since $\alpha_1 + 1 \leq \alpha_2$, there is an $\eta_1 > 0$ such that $[\alpha_2, \beta] = \Omega_1^{\alpha_1+1} \cdot \eta_1$, hence $\varphi \alpha \beta = \psi 0 (\Omega_1^{\alpha_1+1} \cdot \eta_1)$. It follows by Lemma 7.5 e) that $\varphi \alpha_o (\varphi \alpha \beta) = \psi 0 [\alpha_1, \varphi \alpha \beta] = \varphi \alpha \beta$.

c) The assumption $\varphi 0 \beta = \beta < \Omega_1$ implies that β is an ε-number $< \Omega_1$. Therefore we have $\beta = \psi 0 \xi$, where $\xi \in C_o(\xi)$.

1. $\alpha = 0$. If $\xi < \Omega_1$, then $\xi \in C_o(\xi)$ implies $\xi < \psi 0 \Omega_1$. In this case we have $[0, \xi] = \xi$, hence $\beta = \psi 0 [0, \xi] = \varphi 1 \xi$. Otherwise there are $\eta > 0$ and $\delta < \Omega_1$ such that $\xi = \Omega_1 \cdot \eta + \delta$. In this case we have $\beta_o = \psi 0 (\Omega_1 \cdot \eta) + \delta$ such that $\beta = \psi 0 [0, \beta_o] = \varphi 1 \beta_o$.

2. $\alpha > 0$. Then we have for all $\sigma < \alpha$

$$\psi 0 \xi = \beta = \varphi (1 + \sigma) \beta = \psi 0 [\sigma, \beta]$$

It follows by Lemma 7.5 e) that for each $\sigma < \alpha$ there is an $\eta_\sigma > 0$ such that $\xi = \Omega_1^{\sigma+1} \cdot \eta_\sigma$. By considering the Cantor normal form for ξ to the base Ω_1 we see that there are η and $\delta < \Omega_1$ such that $\xi = \Omega_1^{\alpha+1} \cdot \eta + \Omega_1^\alpha \cdot \delta > 0$. If $\eta = 0$, we have $\delta = 1 + \beta_o$ and $\beta = \psi 0 [\alpha, \beta_o] = \varphi (1 + \alpha) \beta_o$. If $\eta > 0$, we have $\beta_o := \psi 0 (\Omega_1^{\alpha+1} \cdot \eta) + \delta$ such that $\beta = \psi 0 [\alpha, \beta_o] = \varphi (1 + \alpha) \beta_o$.

d) and e) hold for $\alpha = 0$ and follow for $\alpha > 0$ from Lemma 7.5 c) and d).

LEMMA 7.7. If $0 < \gamma < \psi 0 (\Omega_1^{\Omega_1})$, then there is an $\alpha < \gamma$ such that $\gamma \leq \varphi \alpha 0$.

Proof. We have only the following two cases for γ.

1. $\gamma =_{NF} \omega^{\gamma_1} + \gamma_2$. If $\gamma_1 = \gamma_2 = 0$, we have $\gamma = \varphi 0 0$. If $\gamma_1 = 0$ and $\gamma_2 > 0$, we have $1 < \gamma < \omega < \varphi 1 0$. If $\gamma_1 > 0$, we have $\gamma_1 + 1 \leq \varphi \gamma_1 0$ by Lemma 7.6 c) and $\gamma_1 < \gamma < \omega^{\gamma_1 + 1} \leq \omega^{\varphi \gamma_1 0} = \varphi \gamma_1 0$.

2. $\gamma = \psi 0 \gamma_o$, $\gamma_o \in C_o(\gamma_o)$, $\gamma_o < \Omega_1^{\Omega_1}$. If $\gamma_o = 0$, we have $1 < \gamma = \varphi 1 0$. Otherwise we have $\gamma_o = \Omega_1^{\alpha_1} \cdot \beta_1 + \delta$, $\alpha_1 < \Omega_1$, $0 < \beta_1 < \Omega_1$, $\delta < \Omega_1^{\alpha_1}$. In this case $\gamma_o \in C_o(\gamma_o)$ implies $\alpha_1 \in C_o(\gamma_o)$ and $\alpha_1 < \psi 0 \gamma_o = \gamma$. It follows that $1 + \alpha_1 + 1 < \gamma < \psi 0 (\Omega_1^{\alpha_1 + 1}) = \psi 0 [\alpha_1 + 1, 0] = \varphi (1 + \alpha_1 + 1) 0$. Lemma 7.6 a), b) and c) shows that the ordinals $\varphi \alpha \beta$ have the usual properties such that we can define Γ_o in the following way.

DEFINITION. Let Γ_o be the least ordinal $\gamma > 0$ such that $\varphi \alpha \beta < \gamma$ for all $\alpha < \gamma$ and $\beta < \gamma$.

THEOREM 7.8. $\Gamma_o = \psi 0 (\Omega_1^{\Omega_1})$.

Proof. For $\alpha < \psi 0 (\Omega_1^{\Omega_1})$ and $\beta < \psi 0 (\Omega_1^{\Omega_1})$ we have by Lemma 7.6 a) and e) $\varphi \alpha \beta < \varphi \alpha (\psi 0 (\Omega_1^{\Omega_1})) = \psi 0 (\Omega_1^{\Omega_1})$. Therefore Γ_o exists and $\Gamma_o \leq \psi 0 (\Omega_1^{\Omega_1})$. But $\psi 0 (\Omega_1^{\Omega_1}) \leq \Gamma_o$ by Lemma 7.7, hence $\Gamma_o = \psi 0 (\Omega_1^{\Omega_1})$.

REMARK. In our notation system $T(\Omega)$ we have:

$$1 =_{NF} \omega^o + 0,\ \Omega_1 \cdot 2 =_{NF} \omega^{\Omega_1} + \Omega_1,\ \Omega_1^2 =_{NF} \omega^{\Omega_1 \cdot 2} + 0,\ \Omega_1^{\Omega_1} =_{NF} \omega^{\Omega_1^2} + 0.$$

It follows that $\Omega_1^{\Omega_1} \in C_o(0) \subseteq C_o(\Omega_1^{\Omega_1}) \subseteq T(\Omega)$.

CHAPTER II

SUBSYSTEMS OF ANALYSIS WITH Π_1^1-COMPREHENSION

In this chapter and in chapter III we consider formal and semiformal systems which contain recursive number theory with quantification over number variables and predicate variables. In these systems the real numbers are definable by certain predicates (representing sets of rational numbers) and universal and existential statements about real numbers are formalizable. Our systems are to be understood as proper subsystems of the classical theory of real numbers, since they do not contain the general comprehension axiom but only special cases of comprehension. We carry out the proof theoretical treatment by using the ordinals of our notation system $T(\Omega)$ to determine the proof theoretical ordinals of some impredicative subsystems of analysis.

§ 8. - The Language of a Formal System A_2 of Second Order Arithmetic

As *primitive symbols* we use

1. Denumerable infinitely many free and bound number variables and predicate variables. (All predicate variables are to be 1-place).

2. Symbols for n-place recursive functions and n-place recursive predicates ($n \geq 1$).

3. The symbols 0, $'$, $\perp$, $\rightarrow$ and $\forall$.

4. Parentheses and comma.

By an n-place *nominal form* ($n \geq 1$) we mean a non-empty finite string of symbols which contains no other symbols than primitive symbols of our language and the *nominal symbols* $*_1, \ldots, *_n$.

If $\mathcal{C}$ is an n-place nominal form ($n \geq 1$) and $r_1, \ldots, r_n$ are non-empty finite strings of symbols then $\mathcal{C} [r_1, \ldots, r_n]$ denotes the result of replacing the nominal symbols $*_1, \ldots, *_n$ in $\mathcal{C}$ by $r_1, \ldots, r_n$ respectively.

By capital script letters we always denote nominal forms.

INDUCTIVE DEFINITION of *terms*.

1. The symbol 0 is a term.
2. Every free number variable is a term.
3. If t is a term then so is t' (denoting the *successor* of t).
4. If f is a symbol for an n-place recursive function ($n \geq 1$) and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term.

Terms built up according to 1. and 3. only are called *numerals*. The numerals $0, 0', 0'', \ldots$ denote the natural numbers $0, 1, 2, \ldots$ A term is called *numerical* if it contains no variable. Every numerical term t has a uniquely determined *value* which is a natural number calculable according to the meaning of the symbols occurring in t.

The *prime formulas* are

1. $\perp$ (*falsum*).
2. $U(t)$ where U is a free predicate variable and t is a term.
3. $p(t_1, \ldots, t_n)$ where p is a symbol for an n-place recursive predicate ($n \geq 1$) and $t_1, \ldots, t_n$ are terms.

A prime formula is called *constant* if it contains no variable. Every constant prime formula P is decidably either *true* or *false* according to the meaning of the symbols occurring in P.

INDUCTIVE DEFINITION of *formulas*.

1. Every prime formula is a formula.
2. If A and B are formulas then so is $(A \to B)$.
3. If a is a free number variable, $\mathfrak{A}[a]$ is a formula and x is a bound number variable which does not occur in $\mathfrak{A}$ then $\forall x \mathfrak{A}[x]$ is a formula.
4. If U is a free predicate variable, $\mathfrak{F}[U]$ is a formula and X is a bound predicate variable which does not occur in $\mathfrak{F}$ then $\forall X \mathfrak{F}[X]$ is a formula.

By the *length* of a formula we mean the number of symbols $\to$ and $\forall$ occurring in the formula.

A formula is said to be *arithmetical* if it contains no bound predicate variable.

Two formulas are said to be *equivalent* (in a strong sense) if they are formulas $\mathfrak{C}[s_1, \ldots, s_n]$ and $\mathfrak{C}[t_1, \ldots, t_n]$ where s_i and t_i ($i = 1, \ldots, n$) are numerical terms of equal values.

As *syntactical variables* we use

a, b, c, d	for free number variables,
x, y, z	for bound number variables,
U, V, W	for free predicate variables,
X, Y, Z	for bound predicate variables,
s, t	for terms,
A, B, C, D, E, F, G	for formulas,
$\mathfrak{A}, \mathfrak{B}$	for nominal forms such that $\mathfrak{A}[t]$, $\mathfrak{B}[t]$ are formulas,
$\mathfrak{F}, \mathfrak{G}$	for nominal forms such that $\mathfrak{F}[U], \mathfrak{G}[U]$ are formulas.

We shall also use these syntactical variables with subscripts.

Negation, disjunction, conjunction, bijunction and *existential quantifiers* are defined in the usual way as follows.

$$\neg A := (A \to \perp)$$
$$(A \vee B) := ((A \to \perp) \to B)$$
$$(A \wedge B) := ((A \to (B \to \perp)) \to \perp)$$
$$(A \leftrightarrow B) := ((A \to B) \wedge (B \to A))$$
$$\exists x \mathfrak{A}[x] := \neg \forall x \neg \mathfrak{A}[x]$$
$$\exists X \mathfrak{F}[X] := \neg \forall X \neg \mathfrak{F}[X]$$

For brevity we omit parentheses in formulas where misunderstanding is not possible.

If $\mathfrak{F}[U]$ and $\mathfrak{A}[t]$ are formulas where U does not occur in $\mathfrak{F}$, then $\mathfrak{F}[\mathfrak{A}]$ denotes the result of replacing each occurrence of $U(.)$ in $\mathfrak{F}[U]$ by $\mathfrak{A}[.]$. This expression $\mathfrak{F}[\mathfrak{A}]$ is a formula if the bound variables in $\mathfrak{A}$ are chosen in an appropriate way in particular if $\mathfrak{F}$ and $\mathfrak{A}$ have no bound variable in common.

As in [25] we define *positive parts* and *negative parts* of a formula F in the following inductive way.

1. F is a positive part of F.
2. If $(A \to B)$ is a positive part of F, then A is a negative part of F and B is a positive part of F.
3. If $(A \to \perp)$ is a negative part of F, then A is a positive part of F.

These parts of a formula F have the properties that the truth (falsehood) of a positive (negative) part of F implies the truth of the formula F with respect to the usual semantics.

To denote a positive part or a negative part of a formula, we define *P-forms* (*positive forms*) and *N-forms* (*negative forms*) as 1-place nominal forms in the following inductive way.

1. $*_1$ is a P-form.

2. If $\mathfrak{P}$ is a P-form and A is a formula, then $\mathfrak{P}\ [(*_1 \to A)]$ is an N-form and $\mathfrak{P}\ [(A \to *_1)]$ is a P-form.

3. In $\mathfrak{N}$ is an N-form then $\mathfrak{N}\ [(*_1 \to \perp)]$ is a P-form.

According to these definitions, a formula A is a positive part (negative part) of a formula F if and only if there is a P-form $\mathfrak{P}$ (an N-form $\mathfrak{N}$) such that F is the formula $\mathfrak{P}\ [A]$ ($\mathfrak{N}\ [A]$).

By an *NP*-form we mean a 2-place nominal form $\mathfrak{Q}$ such that $\mathfrak{Q}[*_1, A]$ is an N-form and $\mathfrak{Q}[A, *_1]$ is a P-form for any formula A.

As *syntactical variables* we use (also with subscripts) $\mathfrak{P}$ for P-forms, $\mathfrak{N}$ for N-forms and $\mathfrak{Q}$ for *NP*-forms.

A positive part of a formula is called *minimal* if it is not of the form $(A \to B)$. A negative part of a formula is called to be *minimal* if it is not of the form $(A \to \perp)$.

$F \overset{s}{\vdash} G$ (*G follows structurally from F*) means that every minimal positive part of F which is not a false constant prime formula also occurs as a positive part of G and every minimal negative part of F which is not a true constant prime formula also occurs as a negative part of G. (Obviously in this case F implies G with respect to the usual semantics).

INDUCTIVE DEFINITION of the set $PV(F)$ of free predicate variables which occur in the scope of a predicate quantifier of F.

1. If F is a prime formula then $PV(F)$ is empty.

2. $PV(A \to B) := PV(A) \cup PV(B)$

3. $PV(\forall x\, \mathfrak{A}[x]) := PV(\mathfrak{A}[0])$

4. $PV(\forall X \mathfrak{F}\ [X])$ is the set of free predicate variables occurring in $\mathfrak{F}$.

INDUCTIVE DEFINITION of *weak formulas*.

1. Every prime formula is a weak formula.

2. $(A \to B)$ is a weak formula if A and B are weak formulas.

3. $\forall x\, \mathfrak{A}[x]$ is a weak formula if $\mathfrak{A}[0]$ is a weak formula.

4. $\forall X \mathfrak{F}\ [X]$ is a weak formula if $\mathfrak{F}\ [U]$ is a weak formula and $U \notin PV(\mathfrak{F}\ [U])$ holds for a free predicate variable U which does not occur in $\mathfrak{F}$.

These weak formulas correspond to G. TAKEUTI'S [30] *isolated formulas*. Among them there are in particular the Π^1_1-formulas $\forall X \mathfrak{F}\ [X]$ where $\mathfrak{F}$ does not contain predicate quantifiers.

The comprehension axiom-schema for weak formulas is essentially equivalent to the comprehension axiom-schema for Π^1_1-formulas.

Therefore we shall call the comprehension for weak formulas also Π^1_1-comprehension.

INDUCTIVE DEFINITION of the *weak* and *strong* predicate quantifiers in a formula.

1. A prime formula does not contain predicate quantifiers.

2. A predicate quantifier in $(A \to B)$ is a weak (strong) quantifier if the corresponding quantifier in A or B is a weak (strong) quantifier.

3. A predicate quantifier in $\forall x\, \mathfrak{A}[x]$ is a weak (strong) quantifier if the corresponding quantifier in $\mathfrak{A}[0]$ is a weak (strong) quantifier.

4. The predicate quantifier $\forall X$ in $\forall X \mathfrak{F}\ [X]$ is a weak quantifier if $\forall X \mathfrak{F}\ [X]$ is a weak formula. Otherwise it is a strong quantifier. Any other predicate quantifier in $\forall X \mathfrak{F}\ [X]$ is a weak (strong) quantifier if the corresponding quantifier in $\mathfrak{F}\ [U]$ is a weak (strong) quantifier.

A formula is called to be a *strong formula* if it contains a strong predicate quantifier.

§ 9. – Axioms and Basic Inferences of the Formal System A_2

Axioms:

($A x$ 1) $\mathfrak{P}\ [A]$ if A is a true constant prime formula.

($A x$ 2) $\mathfrak{N}\ [A]$ if A is a false constant prime formula.

($A x$ 3) $\mathfrak{Q}[A, B]$ if A and B are equivalent prime formulas.

($A x$ 4) $\mathfrak{C}\ [a_1, \ldots, a_n]$ $(n \geq 1)$ if for every sequence of numerals $m_1, \ldots, m_n$ $\mathfrak{C}\ [m_1, \ldots, m_n]$ is one of the axioms ($A x$ 1) – ($A x$ 3).

($A x$ 5) $\forall x (\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \forall x\, \mathfrak{A}[x])$.
(Complete induction)

REMARK. It is not in general decidable whether a formula is an ($A x$ 4) axiom. A formula therefore may only be used as an ($A x$ 4) axiom if there is a general procedure which shows that the formula does satisfy the conditions for such an axiom.

Structural inferences:

(S 0) $F \vdash G$
if $F \overset{s}{\models} G$ holds.

Principal inferences:

(S 1) $\mathfrak{N}[\neg A], \mathfrak{N}[B] \vdash \mathfrak{N}[(A \to B)]$
if B is not the formula $\perp$.

(S 2.0) $\mathfrak{F}[\mathfrak{A}[a]] \vdash \mathfrak{F}[\forall x \mathfrak{A}[x]]$
if a does not occur in the conclusion.

(S 2.1) $\mathfrak{F}[\mathfrak{F}[U]] \vdash \mathfrak{F}[\forall X \mathfrak{F}[X]]$
if U does not occur in the conclusion.

(S 3.0) $\mathfrak{A}[t] \to \mathfrak{N}[\forall x \mathfrak{A}[x]] \vdash \mathfrak{N}[\forall x \mathfrak{A}[x]]$

(S 3.1) $\mathfrak{F}[U] \to \mathfrak{N}[\forall X \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X \mathfrak{F}[X]]$

The indicated positive part or negative part in the conclusion of a principal inference is said to be the *principal part* of the given principal inference.

Cuts:

(cut) $A \vee F, A \to F \vdash F$

The formula denoted by A in the premises of a cut is called the *cut formula* of the given cut.

The *basic inferences* of the formal system A_2 are the structural inferences, the principal inferences and the cuts.

By additional axiom schemas and additional basic inference rules we shall extend A_2 to some impredicative subsystems of analysis.

§ 10. – Π^1_1-Comprehension and Bar-Induction

As axiom schema of Π^1_1-*Comprehension* we use

$(\Pi^1_1\text{-}CA)$ $\exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$
for weak formulas $\mathfrak{A}[t]$.

By an *arithmetical relation* we mean a 2-place nominal form $\mathfrak{R}$ such that $\mathfrak{R}[s,t]$ is an arithmetical formula. For an arithmetical relation $\mathfrak{R}$ and a formula $\mathfrak{A}[t]$ we define

$\mathrm{Prog}[\mathfrak{R}, \mathfrak{A}] := \forall y (\forall x (\mathfrak{R}[x,y] \to \mathfrak{A}[x]) \to \mathfrak{A}[y])$
($\mathfrak{A}$ is a *progressive predicate with respect to* $\mathfrak{R}$)

$Wf[\mathfrak{R}] := \forall X (\mathrm{Prog}[\mathfrak{R}, X] \to \forall y X(y))$
($\mathfrak{R}$ is *well founded*)

The *Bar-Induction* is formulated by the axiom schema

$(BI_{\mathfrak{R}})$ $Wf[\mathfrak{R}] \to (\mathrm{Prog}[\mathfrak{R}, \mathfrak{A}] \to \forall y \mathfrak{A}[y])$
for arithmetical relations $\mathfrak{R}$ and arbitrary formulas $\mathfrak{A}[t]$.

This is a special case of the axiom schema

(BI) $\forall X \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$
for arithmetical formulas $\mathfrak{F}[U]$ and arbitrary formulas $\mathfrak{A}[t]$.

REMARK. If we write $\mathfrak{F}[\mathfrak{A}]$ we always suppose that the bound variables in $\mathfrak{A}$ are chosen in such a way that $\mathfrak{F}[\mathfrak{A}]$ is a formula. (See the explanation of $\mathfrak{F}[\mathfrak{A}]$ on page 31).

In fact, $(BI_{\mathfrak{R}})$ and (BI) are equivalent in A_2. Therefore we shall use (BI) as axiom schema of *Bar-Induction*.

The corresponding *Bar-Induction Rule* is the inference rule

$$\forall X \mathfrak{F}[X] \vdash \mathfrak{F}[\mathfrak{A}]$$

which is equivalent to the inference rule

(BR) $\mathfrak{F}[U] \vdash \mathfrak{F}[\mathfrak{A}]$
for arithmetical formulas $\mathfrak{F}[U]$ where U does not occur in $\mathfrak{F}$.

A generalization of (BI) is the axiom schema

$(\Pi^1_1\text{-BI})$ $\forall X \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$
for weak formulas $\forall X \mathfrak{F}[X]$ and arbitrary formulas $\mathfrak{A}[t]$.

THEOREM 10. In the formal system A_2, the axiom schema $(\Pi^1_1\text{-}CA)$ follows from the axiom schema $(\Pi^1_1\text{-}BI)$.

Proof. Let $\mathfrak{A}[t]$ be a weak formula. Then we have

$$\forall X \neg \forall y (X(y) \leftrightarrow \mathfrak{A}[y]) \to \neg \forall y (\mathfrak{A}[y] \leftrightarrow \mathfrak{A}[y])$$

by $(\Pi^1_1\text{-}BI)$. The formula

(1) $$\forall y(\mathfrak{A}[y] \leftrightarrow \mathfrak{A}[y]) \to \exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

follows by a structural inference. The formula

(2) $$\forall y(\mathfrak{A}[y] \leftrightarrow \mathfrak{A}[y]) \vee \exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

is derivable in A_2. From (1) and (2) we obtain

$$(\Pi^1_1\text{-}CA) \qquad \exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

by a cut.

§ 11. - The Formal Systems PA, $\overline{PA}$ and PB

Let PA be the formal system A_2 with the additional axiom schema $(\Pi^1_1\text{-}CA)$.

Let $\overline{PA}$ be the formal system A_2 with the additional axiom schema $(\Pi^1_1\text{-}CA)$ and the additional basic inference rule (BR).

Let PB be the formal system A_2 with the additional axiom schema $(\Pi^1_1\text{-}BI)$.

Inductive Definition of $PA_k \overset{n}{\vdash} F$.

1. If F is an axiom of PA then $PA_k \overset{n}{\vdash} F$ holds for all natural numbers k and n.

2. If $PA_k \overset{n}{\vdash} F_i$ holds for each premise F_i of a basic inference of PA then $PA_k \overset{n+1}{\vdash} F$ holds for the conclusion F of that inference.

3. If $PA_k \overset{n}{\vdash} F_0$ holds for the premise F_0 of an inference (BR) then $PA_{k+1} \overset{n}{\vdash} F$ holds for the conclusion F of that inference.

According to this definition, $PA_k \overset{n}{\vdash} F$ implies $PA_i \overset{m}{\vdash} F$ for all $i \geq k$ and $m \geq n$.

A formula F is said to be *derivable* in PA if there a natural number n such that $PA_0 \overset{n}{\vdash} F$ holds.

A formula F is said to be *derivable* in $\overline{PA}$ if there are natural numbers k and n such that $PA_k \overset{n}{\vdash} F$ holds.

Inductive Definition on $PB \overset{n}{\vdash} F$.

1. If F is an axiom of PB then $PB \overset{n}{\vdash} F$ holds for any natural number n.

2. If $PB \overset{n}{\vdash} F_i$ holds for each premise F_i of a basic inference of PB then $PB \overset{n+1}{\vdash} F$ holds for the conclusion F of that inference.

According to this definition, $PB \overset{n}{\vdash} F$ implies $PB \overset{m}{\vdash} F$ for all $m \geq n$.

A formula F is said to be *derivable* in PB if there is a natural number n such that $PB \overset{n}{\vdash} F$ holds.

§ 12. - The Ramified System PB^*

The *formulas* of the system PB^* are the results of the following replacements in formulas of the system A_2:

1. Every free number variable is replaced by a numeral.

2. Every free predicate variable U is replaced by U^n where n is a natural number.

3. Every strong predicate quantifier $\forall X$ is replaced by $\forall X^\omega$. (The weak predicate quantifiers remain unchanged).

According to this definition, for every formula F of the system PB^* there is a *corresponding formula* of the system A_2 which results from F by cancelling all upper indices of free predicate variables and strong predicate quantifiers. These corresponding formulas contain no free number variables. For a formula F of the system PB^*, we define $PV(F) := PV(F')$ where F' is the corresponding formula of the system A_2. A formula of the system PB^* is said to be a *prime formula*, an *arithmetical formula*, a *weak formula* or a *strong formula* if the corresponding formula of the system A_2 is such a formula.

P-forms, *N-forms*, *NP-forms* and $F \overset{s}{\vdash} G$ are defined for PB^* in the same way as for A_2. We also use in PB^* the same *syntactical variables* as in A_2.

Inductive Definition of the *grade* $gr(F)$ of a formula F in PB^*.

1. $gr(F) := 0$ if F is a prime formula or a formula $\forall X \mathfrak{F}[X]$.
2. $gr(A \to B) := \max\{gr(A), gr(B)\} + 1$
3. $gr(\forall x\, \mathfrak{A}[x]) := gr(\mathfrak{A}[0]) + 1$
4. $gr(\forall X^\omega\, \mathfrak{F}[X]) := gr(\mathfrak{F}[U^0]) + 1$.

Inductive Definition of the *stage* $st(F)$ of a formula F in PB^*.

1. $st(F) = st(\neg F) := 0$ if F is a constant prime formula.
2. $st(U^n(t)) = st(\neg U^n(t)) := n$
3. $st(A \to B) := \max\{st(\neg A), st(B)\}$,
   $st(\neg (A \to B]) := \max\{st(A), st(\neg B)\}$.

4. $st(\forall x\, \mathfrak{A}[x]) := st(\mathfrak{A}[0])$, $st(\neg \forall x\, \mathfrak{A}[x]) := st(\neg \mathfrak{A}[0])$.
5. $st(\forall X \mathfrak{F}[X]) := st(\mathfrak{F}[U^0])$, $st(\neg \forall X \mathfrak{F}[X]) := st(\mathfrak{F}[U^0]) + 1$.
6. $st(\forall X^\omega \mathfrak{F}[X]) = st(\neg \forall X^\omega \mathfrak{F}[X] := \omega$.

According to this definition, every weak formula has a stage $< \omega$, and every strong formula has the stage ω.

LEMMA 12.1. (*Stage Lemma*)

a) For every *P*-form $\mathfrak{G}$ and every *N*-form $\mathfrak{N}$ we have

$$st(\mathfrak{G}[A]) = \max\{st(A),\ st(\mathfrak{G}[\perp])\}$$
$$st(\mathfrak{N}[A]) = \max\{st(\neg A),\ st(\mathfrak{N}[\perp])\}$$

b) If U^n appears in a formula $\mathfrak{F}[U^n]$ and $U \notin PV(\mathfrak{F}[U^n])$, then $st(\mathfrak{F}[U^n]) = \max\{n, st(\mathfrak{F}[U^0])\}$.

c) If $st(\forall X \mathfrak{F}[X]) = n$ then also $st(\mathfrak{F}[U^n]) = n$.

Proof of a) by induction of the lengths of the nominal forms and $\mathfrak{N}$. Proof of b) by induction of the length of $\mathfrak{F}$.

c) follows from b).

Axioms of PB^*:

$(Ax\,1)$ and $(Ax\,2)$ like the corresponding axioms of A_2 in § 9.

$(Ax\,3^*)$ $\mathfrak{Q}[A, B]$ if A and B are equivalent formulas of grade 0.

Principal inferences of PB^*:

$(S\,1)$ and $(S\,3.0)$ like the corresponding inferences of A_2 in § 9.

$(S\,2.0^*)$ $\mathfrak{G}[\mathfrak{A}[n]]$ for every numeral $n \vdash \mathfrak{G}[\forall x\, \mathfrak{A}[x]]$

$(S\,2.1^*)$ $\mathfrak{G}_0[\mathfrak{F}[U^n]] \vdash \mathfrak{G}[\forall X \mathfrak{F}[X]]$
if $st(\forall X \mathfrak{F}[X]) = n$, U does not occur in the conclusion and $\mathfrak{G}_0[\mathfrak{F}[U^n]]$ is the result of replacing every positive part $\forall X \mathfrak{F}[X]$ in $\mathfrak{G}[\mathfrak{F}[U^n]]$ by $\perp$.

$(S\,2.2^*)$ $\mathfrak{G}[\mathfrak{F}[U^n]]$ for every natural number $n \vdash \mathfrak{G}[\forall X^\omega \mathfrak{F}[X]]$
if U does not occur in the conclusion.

$(S\,3.1^*)$ $\mathfrak{F}[U^n] \to \mathfrak{N}[\forall X^\omega \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X^\omega \mathfrak{F}[X]]$

(The inferences $(S\,2.0^*)$ and $(S\,2.2^*)$ have infinitely many premises).

The *principal parts* of the principal inferences and the *cuts* are defined as before in § 9. The *grade of a cut* is the grade of its cut formula.

REMARK. The system PB^* has no principal inference with a negative principal part $\forall X \mathfrak{F}[X]$. To derive formulas with a negative part $\forall X \mathfrak{F}[X]$ we shall use the Ω_{n+1}-rule which is described in the following.

INDUCTIVE DEFINITION of $PB^* \left|\frac{\gamma}{m}\right. F$ for $\gamma \in T(\Omega)$ and $m < \omega$.

1. If F is an axiom of PB^* then $PB^* \left|\frac{\gamma}{m}\right. F$ holds for all $\gamma \in T(\Omega)$ and $m < \omega$.

2. If $PB^* \left|\frac{\beta}{m}\right. F_i$ and $\beta \lhd \gamma$ holds for every premise F_i of a principal inference or a cut of grade $< m$, then $PB^* \left|\frac{\gamma}{m}\right. F$ holds for the conclusion F of that inference.

3. (Ω_{n+1}*-rule*) $PB^* \left|\frac{\gamma}{m}\right. F$ holds under the following assumptions:

a) $\forall X \mathfrak{F}[X]$ is a formula of stage n.

b) f is a fundamental function (according to § 5) such that $\Omega_{n+1} \in \mathrm{dom}(f)$ and $f(\Omega_{n+1}) \trianglelefteq \gamma$.

c) $PB^* \left|\frac{f(0)}{m}\right. \forall X \mathfrak{F}[X] \vee F$.

d) $PB^* \left|\frac{\alpha}{0}\right. \mathfrak{G}[\forall X \mathfrak{F}[X]] \Rightarrow PB^* \left|\frac{f(\alpha)}{m}\right. \mathfrak{G}[F]$ for every $\alpha < \Omega_{n+1}$ $(\alpha \in T(\Omega))$ and every P-form $\mathfrak{G}$ such that $\mathfrak{G}[\forall X \mathfrak{F}[X]]$ is a formula of stage n.

REMARK. The Ω_{n+1}-rule contains a hidden cut

$$\forall X \mathfrak{F}[X] \vee F, \quad \forall X \mathfrak{F}[X] \to F \vdash F$$

according to the meaning of the assumptions c) and d), but formally this hidden cut is not a cut. Such a formulation of the Ω_{n+1}-rule will be useful for our proof theoretical treatment.

LEMMA 12.2. $PB^* \left|\frac{\alpha}{m}\right. F$, $\alpha \trianglelefteq \beta$, $m \le n \Rightarrow PB^* \left|\frac{\beta}{n}\right. F$.

This follows immediately from the inductive definition.

LEMMA 12.3. (*Replacement rules*)

a) $PB^* \left|\frac{\gamma}{m}\right. F \Rightarrow PB^* \left|\frac{\gamma}{m}\right. G$
if F and G are equivalent formulas.

b) $PB^* \left|\frac{\gamma}{m}\right. \mathfrak{F}[U^n] \Rightarrow PB^* \left|\frac{\gamma}{m}\right. \mathfrak{F}[V^n]$
if U does not occur in $\mathfrak{F}$.

Proofs by induction on γ.

LEMMA 12.4. (*Inversion rules*).

a) $PB^* \vdash^{\gamma}_{m} \mathfrak{M}[(A \to B)] \Rightarrow PB^* \vdash^{\gamma}_{m} \mathfrak{M}[\neg A]$

$PB^* \vdash^{\gamma}_{m} \mathfrak{M}[(A \to B)] \Rightarrow PB^* \vdash^{\gamma}_{m} \mathfrak{M}[B]$

b) $PB^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall x\, \mathfrak{A}[x]] \Rightarrow PB^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{A}[t]]$

c) $PB^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall X^{\omega}\, \mathfrak{F}\ [X]] \Rightarrow PB^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{F}\ [U^n]]$

Proof by induction on γ using Lemma 12.3.

LEMMA 12.5. (*Structural rule*) $PB^* \vdash^{\gamma}_{m} F \Rightarrow PB^* \vdash^{\gamma}_{m} G$ if $F \vdash^{s} G$ holds.

Proof by induction on γ. We have the following five cases.

1. F is an axiom. Then also G is an axiom, hence the assertion is trivial.

2. $PB^* \vdash^{\gamma}_{m} F$ is derived by an inference (S 1). Then F, G are formulas $\mathfrak{M}_{\mathrm{I}}[(A \to B)]$, $\mathfrak{M}_{\mathrm{II}}[(A \to B)]$ where B is not the formula $\perp$, and we have $\beta \lhd \gamma$ such that $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{I}}[\neg A]$ and $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{I}}[B]$. Let $\mathfrak{M}_1[\neg A]$, $\mathfrak{M}_2[B]$ be the results of replacing every negative part $(A \to B)$ in $\mathfrak{M}_{\mathrm{I}}[\neg A]$, $\mathfrak{M}_{\mathrm{I}}[B]$ by $\neg A, B$ respectively. Then we have $\mathfrak{M}_1[\neg A] \vdash^{s} \mathfrak{M}_{\mathrm{II}}[\neg A]$ and $\mathfrak{M}_2[B] \vdash^{s} \mathfrak{M}_{\mathrm{II}}[B]$. We obtain $PB^* \vdash^{\beta}_{m} \mathfrak{M}_1[\neg A]$, $PB^* \vdash^{\beta}_{m} \mathfrak{M}_2[B]$ from $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{I}}[\neg A]$, $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{I}}[B]$ by the inversion rule a) and $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{II}}[\neg A]$, $PB^* \vdash^{\beta}_{m} \mathfrak{M}_{\mathrm{II}}[B]$ by the I.H. (Induction Hypothesis). The assertion $PB^* \vdash^{\gamma}_{m} \mathfrak{M}_{\mathrm{II}}[(A \to B)]$ follows by an inference (S 1).

3. $PB^* \vdash^{\gamma}_{m} F$ is derived by an inference (S 2.0*). Then F, G are formulas $\mathfrak{S}_{\mathrm{I}}[\forall x\, \mathfrak{A}[x]]$, $\mathfrak{S}_{\mathrm{II}}[\forall x\, \mathfrak{A}[x]]$, and we have $\beta \lhd \gamma$ such that $PB^* \vdash^{\beta}_{m} \mathfrak{S}_{\mathrm{I}}[\mathfrak{A}[n]]$ holds for all n. Let $\mathfrak{S}_n[\mathfrak{A}[n]]$ be the result of replacing every positive part $\forall x\, \mathfrak{A}[x]$ in $\mathfrak{S}_{\mathrm{I}}[\mathfrak{A}[n]]$ by $\mathfrak{A}[n]$. Then we have $\mathfrak{S}_n[\mathfrak{A}[n]] \vdash^{s} \mathfrak{S}_{\mathrm{II}}[\mathfrak{A}[n]]$. We obtain $PB^* \vdash^{\beta}_{m} \mathfrak{S}_n[\mathfrak{A}[n]]$ from $PB^* \vdash^{\beta}_{m} \mathfrak{S}_{\mathrm{I}}[\mathfrak{A}[n]]$ by the inversion rule b) and $PB^* \vdash^{\beta}_{m} \mathfrak{S}_{\mathrm{II}}[\mathfrak{A}[n]]$ by the I.H. The assertion $PB^* \vdash^{\gamma}_{m} \mathfrak{S}_{\mathrm{II}}[\forall x\, \mathfrak{A}[x]]$ follows by an inference (S 2.0*).

4. $PB^* \vdash^{\gamma}_{m} F$ is derived by an inference (S 2.2*). Then the assertion follows from the inversion rule c) and the I.H. in the same way as in the 3. case.

5. $PB^* \vdash^{\gamma}_{m} F$ is derived by another principal inference or by a cut of grade $< m$ or by the Ω_{n+1}-rule. Then the assertion follows immediately from the I.H.

LEMMA 12.6.

$PB^* \vdash^{2m}_{0} \mathfrak{Q}[F, F]$ for every *NP*-form $\mathfrak{Q}$ if $m \geq gr(F)$.

Proof by induction on $gr(F)$. We have the following five cases.

1. $gr(F) = 0$. Then $\mathfrak{Q}[F, F]$ is an axiom, hence the assertion holds.

2. F is a formula $(A \to \perp)$. Then the assertion follows from the I.H., since $gr(A) < gr(F)$ and $\mathfrak{Q}[F, F]$ is a formula $\mathfrak{Q}_0[A, A]$.

3. F is a formula $(A \to B)$ where B is not the formula $\perp$. Then we have $m > 0$ and by I.H.

$$PB^* \vdash^{2m-1}_{0} \mathfrak{Q}[\neg A, (A \to B)], \quad PB^* \vdash^{2m-1}_{0} \mathfrak{Q}[B, (A \to B)]$$

The assertion follows by an inference (S 1).

4. F is a formula $\forall x\, \mathfrak{A}[x]$. Then we have $m > 0$ and by I.H.

$$PB^* \vdash^{2m-2}_{0} \mathfrak{A}[n] \to \mathfrak{Q}[\forall x\, \mathfrak{A}[x], \mathfrak{A}[n]]$$

for every numeral n. From these formulas we obtain

$$PB^* \vdash^{2m-1}_{0} \mathfrak{Q}[\forall x\, \mathfrak{A}[x], \mathfrak{A}[n]]$$

by inferences (S 3.0). The assertion follows by an inference (S 2.0*).

5. F is a formula $\forall X^{\omega}\, \mathfrak{F}\ [X]$. Then the assertion follows from the I.H. by inferences (S 3.1*) and (S 2.2*) in the same way as in the 4. case.

LEMMA 12.7.

a) $PB^* \vdash^{2m+2n}_{0} \forall x(\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \mathfrak{A}[n])$ for every numeral n if $gr(\mathfrak{A}[t]) = m$.

b) $PB^* \vdash^{\omega+1}_{0} \forall x(\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \forall x\, \mathfrak{A}[x])$.

Proof of a) by induction on n. Let F_n be the formula

$$\forall x(\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \mathfrak{A}[n]).$$

The assertion holds for $n = 0$ by Lemma 12.6. Now we prove the assertion for $n + 1 = n'$ under the assumption that it holds for n. From this assumption we obtain

$$PB^* \vdash^{2m+2n}_{0} (\mathfrak{A}[n] \to \perp) \to F_{n'} \tag{1}$$

by the structural rule. Furthermore we have

$$PB^* \vdash^{2m+2n}_{0} \mathfrak{A}[n'] \to F_{n'} \tag{2}$$

by Lemma 12.6. From (1) and (2) we obtain

$$PB^* \vdash^{2m+2n+1}_{0} (\mathfrak{A}[n] \to \mathfrak{A}[n']) \to F_{n'}$$

by an inference (S 1). The assertion

$$PB^* \vdash_0^{2m+2n+2} F_{n'}$$

follows by an inference (S 3.0).

Proof of b). Since $2m + 2n \lhd \omega$, we have according to a)

$$PB^* \vdash_0^{\omega} \forall x (\mathfrak{A}[x] \to \mathfrak{A}[x'])) \to (\mathfrak{A}[0] \to \mathfrak{A}[n])$$

for every numeral n. The assertion b) follows by an inference (S 2.0*).

LEMMA 12.8. If $\forall X \mathfrak{F}[X]$ and $\mathfrak{S}[\forall X \mathfrak{F}[X]]$ are formulas of stage n and $\alpha < \Omega_{n+1}$, then

$$PB^* \vdash_0^{\alpha} \mathfrak{S}[\forall X \mathfrak{F}[X]] \Rightarrow PB^* \vdash_0^{\alpha} \mathfrak{S}[\mathfrak{F}[U^n]]$$

Proof by induction on α. We have the following three cases.

1. $\mathfrak{S}[\forall X \mathfrak{F}[X]]$ is an axiom. Then also $\mathfrak{S}[\mathfrak{F}[U^n]]$ is an axiom since the formula $\mathfrak{S}[\forall X \mathfrak{F}[X]]$ of stage n does not contain a negative part which is equivalent to the formula $\forall X \mathfrak{F}[X]$ of stage n.

2. $PB^* \vdash_0^{\alpha} \mathfrak{S}[\forall X \mathfrak{F}[X]]$ is derived by a principal inference. If the indicated positive part $\forall X \mathfrak{F}[X]$ of the formula $\mathfrak{S}[\forall X \mathfrak{F}[X]]$ is the principal part of this inference, then the assertion follows from the premise by Lemmata 12.2, 12.3b) and 12.5. Otherwise the assertion follows from the I.H. and L. 12.3 b).

3. $PB^* \vdash_0^{\alpha} \mathfrak{S}[\forall X \mathfrak{F}[X]]$ is derived by an Ω_{m+1}-rule. Then $\alpha < \Omega_{n+1}$ implies $m < n$. It follows that no formula of stage m contains the formula $\forall X \mathfrak{F}[X]$ of stage n. Therefore, the assertion follows from the I.H.

LEMMA 12.9. Let $\mathfrak{S}[\mathfrak{F}[U^n]]$ be a weak formula where U does not occur in $\mathfrak{S}$ or $\mathfrak{F}$ and $U \notin PV(\mathfrak{F}[U^n])$. Let $\mathfrak{A}[t]$ be an arbitrary formula and $\alpha < \Omega_{n+1}$. Then we have

$$PB^* \vdash_0^{\alpha} \mathfrak{S}[\mathfrak{F}[U^n]] \Rightarrow PB^* \vdash_0^{\Omega_{n+1}+\alpha} \mathfrak{S}[\mathfrak{F}[\mathfrak{A}]]$$

Proof by induction on α. We have the following three cases.

1. $\mathfrak{S}[\mathfrak{F}[U^n]]$ is an axiom. If also $\mathfrak{S}[\mathfrak{F}[\mathfrak{A}]]$ is an axiom, the assertion holds. Otherwise $\mathfrak{S}[\mathfrak{F}[\mathfrak{A}]]$ is a formula $\mathfrak{Q}[\mathfrak{A}[s], \mathfrak{A}[t]]$ where s and t are numerical terms of equal value. In this case the assertion follows from Lemmata 12.2, 12.3 a) and 12.6, since $2m \lhd \Omega_{n+1} + \alpha$.

2. $PB^* \vdash_0^{\alpha} \mathfrak{S}[\mathfrak{F}[U^n]]$ is derived by a principal inference. Then the assertion follows from the I.H. by Lemma 4.3, since $\mathfrak{S}[\mathfrak{F}[U^n]]$ is a weak formula and $U \notin PV(\mathfrak{F}[U^n])$.

3. $PB^* \vdash_0^{\alpha} \mathfrak{S}[\mathfrak{F}[U^n]]$ is derived by an Ω_{m+1}-rule. Then $\alpha < \Omega_{n+1}$ implies $m < n$. It follows that no formula of stage m contains U^n. Therefore the assertion follows by the I.H. and Lemma 5.3.

LEMMA 12.10. $PB^* \vdash_0^{\Omega_{n+1} \cdot 2} \forall X \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$ for every formula $\forall X \mathfrak{F}[X]$ of stage n and every arbitrary formula $\mathfrak{A}[t]$.

Proof. Let f be the fundamental function $F_{\Omega_{n+1} \cdot 2}$, i.e.

$$\operatorname{dom}(f) = \{\alpha \in T(\Omega) : \alpha \leq \Omega_{n+1}\} \text{ and } f(\alpha) = \Omega_{n+1} + \alpha \text{ for all } \alpha \in \operatorname{dom}(f)$$

according to § 5. Then we have as an (Ax 3*) axiom

(1) $$PB^* \vdash_0^{f(0)} \forall X \mathfrak{F}[X] \vee (\forall X \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}])$$

For $\alpha < \Omega_{n+1}$ and every formula $\mathfrak{S}[\forall X \mathfrak{F}[X]]$ of stage n, we have by Lemmata 12.8 and 12.9

$$PB^* \vdash_0^{\alpha} \mathfrak{S}[\forall X \mathfrak{F}[X]] \Rightarrow PB^* \vdash_0^{f(\alpha)} \mathfrak{S}[\mathfrak{F}[\mathfrak{A}]]$$

If follows by the structural rule that

(2) $$PB^* \vdash_0^{\alpha} \mathfrak{S}[\forall X \mathfrak{F}[X]] \Rightarrow PB^* \vdash_0^{f(\alpha)} \mathfrak{S}[\forall X \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]]$$

The assertion follows from (1) and (2) by the rule Ω_{n+1}-rule.

§ 13. – The Reduction Procedure of PB^*

LEMMA 13.1. If C is a formula of grade m which is not of the form $(A \to B)$ and $\delta \leq \omega^{\gamma}$, then we have

$$PB^* \vdash_m^{\omega^{\gamma}} \mathfrak{S}[C], \quad PB^* \vdash_m^{\delta} C \to F \Rightarrow PB^* \vdash_m^{\omega^{\gamma}+\delta} \mathfrak{S}[F]$$

Proof by induction on δ. Suppose

(1) $$PB^* \vdash_m^{\omega^{\gamma}} \mathfrak{S}[C]$$

(2) $$PB^* \vdash_m^{\delta} C \to F$$

1. Let $C \to F$ be an axiom. Then we have the following three cases.

1.1. F is an axiom. Then also $\mathfrak{F}\ [F]$ is an axiom, and the assertion holds.

1.2. C is a false constant prime formula. Then we have $\mathfrak{F}\ [C] \vdash^{s} \mathfrak{F}\ [F]$. In this case, the assertion follows from (1) by the structural rule, since $\omega^{\gamma} \trianglelefteq \omega^{\gamma} + \delta$ holds.

1.3. F is a formula $\mathfrak{F}_0 [C_0]$ where C and C_0 are equivalent formulas of grade 0. Then we have $\mathfrak{F}\ [C_0] \vdash^{s} \mathfrak{F}\ [F]$. In this case the assertion follows from (1) by the replacement rule a) and the structural rule.

2. Let $PB^* \vdash^{\delta}_{m} C \to F$ be derived by a principal inference or a cut of grade $< m$. Then we have the following three cases.

2.1. C is a formula $\forall x\, \mathfrak{A}[x]$ and there is $\delta_0 \lhd \delta$ such that

$$PB^* \vdash^{\delta_0}_{m} \mathfrak{A}[t] \to (\forall x\, \mathfrak{A}[x] \to F) \tag{3}$$

holds. Then from (1) and (3) we obtain

$$PB^* \vdash^{\omega^{\gamma}+\delta_0}_{m} \mathfrak{A}[t] \to \mathfrak{F}\ [F] \tag{4}$$

by the structural rule and the I.H. From (1) we obtain

$$PB^* \vdash^{\omega^{\gamma}}_{m} \mathfrak{A}[t] \lor \mathfrak{F}\ [F] \tag{5}$$

by the inversion rule b) and the structural rule. From (4) and (5) the assertion follows by a cut with cut formula $\mathfrak{A}[t]$ of grade $< m$.

2.2. C is a formula $\forall X^{\omega} \mathfrak{F}\ [X]$ and there is $\delta_0 \lhd \delta$ such that

$$PB^* \vdash^{\delta_0}_{m} \mathfrak{F}\ [U^n] \to (\forall X^{\omega} \mathfrak{F}\ [X] \to F)$$

holds. Then the assertion follows by the inversion rule c) corresponding to 2.1.

2.3. $C \to F$ is the conclusion of a principal inference with principal part in F or the conclusion of a cut of grade $< m$. Then the assertion follows from I.H. and the structural rule.

3. Let $PB^* \vdash^{\delta}_{m} C \to F$ be derived by an Ω_{n+1}-rule with a fundamental function f. Then the assertion follows from the I.H. by the Ω_{n+1}-rule with the fundamental function $\omega^{\gamma} + f$.

LEMMA 13.2. $PB^* \vdash^{\gamma}_{m+1} F \Rightarrow PB^* \vdash^{\omega^{\gamma}}_{m} F$

Proof by induction on γ.

1. Let F be an axiom. Then the assertion is trivial.

2. Let $PB^* \vdash^{\gamma}_{m+1} F$ be derived by a principal inference or a cut of grade $< m$. Then the assertion follows from the I.H. by Lemma 4.4.

3. Let $PB^* \vdash^{\gamma}_{m+1} F$ be derived by a cut of grade m. Then we have $\beta \lhd \gamma$ and a formula C of grade m such that

$$PB^* \vdash^{\beta}_{m+1} C \lor F, \qquad PB^* \vdash^{\beta}_{m+1} C \to F$$

hold. By the I.H. we obtain

$$PB^* \vdash^{\omega^{\beta}}_{m} C \lor F, \qquad PB^* \vdash^{\omega^{\beta}}_{m} C \to F$$

3.1. Suppose C is not of the form $(A \to B)$. Then the assertion follows from Lemma 13.1 and the structural rule, since $\omega^{\beta} + \omega^{\beta} \lhd \omega^{\gamma}$ by Lemma 4.4.

3.2. Suppose C is a formula $(A \to B)$. Then we have

$$PB^* \vdash^{\omega^{\beta}}_{m} (A \to B) \lor F \tag{1}$$

$$PB^* \vdash^{\omega^{\beta}}_{m} (A \to B) \to F \tag{2}$$

From (2) we obtain

$$PB^* \vdash^{\omega^{\beta}}_{m} A \lor F \tag{3}$$

$$PB^* \vdash^{\omega^{\beta}}_{m} B \to F \tag{4}$$

by the inversion rule a). From (3) and (1) we obtain

$$PB^* \vdash^{\omega^{\beta}}_{m} A \lor (B \lor F), \qquad PB^* \vdash^{\omega^{\beta}}_{m} A \to (B \lor F)$$

by the structural rule and

$$PB^* \vdash^{\omega^{\beta}\cdot 2}_{m} B \lor F \tag{5}$$

by a cut with cut formula A of grade $< m$. The assertion follows from (5) and (4) by a cut with cut formula B of grade $< m$.

4. Let $PB^* \vdash^{\gamma}_{m+1} F$ be derived by an Ω_{n+1}-rule with a fundamental function f. Then the assertion follows from the I.H. by the Ω_{n+1}-rule with the fundamental function ω^{f}.

THEOREM 13.3. (*Cut Elimination Theorem*) $PB^* \vdash^{\gamma}_{m} F \Rightarrow PB^* \vdash^{\omega_m(\gamma)}_{0} F$.

Proof. This follows from Lemma 13.2 by induction on m.

THEOREM 13.4. (*Collapsing Theorem*) If F is a weak formula of stage $\leq n$ and $\gamma \in C_n(\gamma)$, then we have

$$PB^* \vdash^{\gamma}_{0} F \Rightarrow PB^* \vdash^{\psi n\gamma}_{0} F$$

Proof by induction on γ.

1. Let F be an axiom. Then the assertion is trivial.

2. Let $PB^* \vdash^{\gamma}_{0} F$ be derived by a principal inference. Then also the premises of that inference are weak formulas of stages $\leq n$. Therefore the assertion follows from the I.H. by Lemma 4.6.

3. Let $PB^* \vdash^{\gamma}_{0} F$ be derived by an Ω_{m+1}-rule with respect to a formula $\forall X \mathfrak{F}[X]$ of stage m and a fundamental function f. Then we have $\Omega_{m+1} \in \mathrm{dom}(f)$ and $f(\Omega_{m+1}) \trianglelefteq \gamma \in C_n(\gamma)$, hence $f(\Omega_{m+1}) \in C_n(f(\Omega_{m+1}))$ and $\psi n(f(\Omega_{m+1})) \trianglelefteq \psi n\gamma$ by Lemma 4.6.

3.1. $m < n$. Then $\Omega_{m+1} \leq \Omega_n$ and $f(\Omega_{m+1}) \in C_n(f(\Omega_{m+1}))$ implies $\Omega_{m+1} \in \mathrm{dom}(\psi nf)$. Therefore the assertion follows from the I.H. by the Ω_{m+1}-rule with the fundamental function ψnf.

3.2. $n \leq m$. According to the Ω_{m+1}-rule we have

$$(1) \qquad PB^* \vdash^{f(0)}_{0} \forall X \mathfrak{F}[X] \vee F$$

In this case $\forall X \mathfrak{F}[X] \vee F$ is a weak formula of stage m. $f(0) \lhd f(\Omega_{m+1}) \in C_n(f(\Omega_{m+1}))$ implies $f(0) \in C_n(f(0)) \subseteq C_m(f(0))$ by Lemma 4.6. Therefore (1) and $f(0) \lhd \gamma$ implies

$$(2) \qquad PB^* \vdash^{\psi m(f(0))}_{0} \forall X \mathfrak{F}[X] \vee F$$

by the I.H. Here $\forall X \mathfrak{F}[X] \vee F$ is a weak formula $\mathfrak{G}[\forall X \mathfrak{F}[X]]$ of stage m. Furthermore we have $\psi m(f(0)) < \Omega_{m+1}$ and $\mathfrak{G}[F] \vdash^{s} F$. Therefore from (2) we obtain

$$(3) \qquad PB^* \vdash^{f(\alpha)}_{0} F \quad \text{for} \quad \alpha := \psi m(f(0))$$

by the premises of the Ω_{m+1}-rule and the structural rule.

We have $f(\alpha) \lhd f(\Omega_{m+1})$ by Corollary 5.4. Together with $f(\Omega_{m+1}) \trianglelefteq \gamma \in C_n(\gamma)$ we obtain $f(\alpha) \in C_n(f(\alpha))$ and $\psi n(f(\alpha)) \lhd \psi n\gamma$ by Lemma 4.6. Therefore the assertion $PB^* \vdash^{\psi n\gamma}_{0} F$ follows from (3) by the I.H.

§ 14. – Interpretation of *PB* in *PB**

A formula F^* of the system PB^* is said to be an *interpretation* of a formula F of the system PB if F^* is the result of replacing every free number variable in F by a numeral, every free predicate variable U in F by U^n and every strong predicate quantifier $\forall X$ in F by $\forall X^\omega$.

THEOREM 14.1 (*Interpretation Theorem*) If $PB \vdash^{k} F$ holds, then there is an $m < \omega$ such that $PB^* \vdash^{\Omega_\omega + 2k}_{m} F^*$ holds for every interpretation F^* of F.

Proof by induction on k.

1. Let F be an axiom of PB. If F is one of the axioms $(Ax\,1)$-$(Ax\,4)$, then F^* is an axiom of PB^*, hence the assertion holds. If F is an $(Ax\,5)$ axiom, the assertion follows from Lemma 12.7 b). If F is a $(\Pi^1_1$-$BI)$ axiom, the assertion follows from Lemma 12.10.

2. Let $PB \vdash^{k} F$ be derived by a structural inference. Then the assertion follows from the I.H. by the structural rule of PB^*.

3. Let $PB \vdash^{k} F$ be derived by a cut or a principal inference, the principal part of which is not a weak formula $\forall X \mathfrak{F}[X]$. Then the assertion follows from the I.H. by a corresponding inference of PB^*.

4. Let $PB \vdash^{k} F$ be derived by an inference $(S\,2.1)$, the principal part of which is a weak formula. Then F^* is a formula $\mathfrak{G}[\forall X \mathfrak{F}[X]]$ and $k > 0$. By the I.H. we have

$$(1) \qquad PB^* \vdash^{\Omega_\omega + 2k-2}_{m} \mathfrak{G}[\mathfrak{F}[U^n]]$$

where U does not occur in F^* and n may be chosen such that $st(\forall X \mathfrak{F}[X]) = n$. Let $\mathfrak{G}_0[\mathfrak{F}[U^n]]$ be the result of replacing every positive part $\forall X \mathfrak{F}[X]$ in $\mathfrak{G}[\mathfrak{F}[U^n]]$ by $\perp$. Then we have

$$\mathfrak{G}[\mathfrak{F}[U^n]] \vdash^{s} \forall X \mathfrak{F}[X] \vee \mathfrak{G}_0[\mathfrak{F}[U^n]]$$

therefore by the structural rule it follows from (1) that

$$(2) \qquad PB^* \vdash^{\Omega_\omega + 2k-2}_{m} \forall X \mathfrak{F}[X] \vee \mathfrak{G}_0[\mathfrak{F}[U^n]]$$

Furthermore it follows by the structural rule from Lemma 12.10 that

$$(3) \qquad PB^* \vdash^{\Omega_\omega + 2k-2}_{m} \forall X \mathfrak{F}[X] \rightarrow \mathfrak{G}_0[\mathfrak{F}[U^n]]$$

We can suppose $m > 0 = gr(\forall X \mathfrak{F}[X])$. Then we obtain

$$(4) \qquad PB^* \vdash^{\Omega_\omega + 2k-1}_{m} \mathfrak{G}_0[\mathfrak{F}[U^n]]$$

from (2) and (3) by a cut. The assertion follows from (4) by an inference $(S\,2.1^*)$.

5. Let $PB \overset{k}{\vdash} F$ be derived by an inference (S 3.1), the principal part of which is a weak formula. Then F^* is a formula $\mathfrak{N}[\forall X \mathfrak{F}[X]]$ and $k > 0$. By the I.H. we have

$$(5) \qquad PB^* \vdash_{m}^{\Omega_\omega + 2k-2} \mathfrak{F}[U^n] \to \mathfrak{N}[\forall X \mathfrak{F}[X]]$$

From Lemma 12.10 it follows by the structural rule that

$$(6) \qquad PB^* \vdash_{m}^{\Omega_\omega + 2k-2} \mathfrak{F}[U^n] \vee \mathfrak{N}[\forall X \mathfrak{F}[X]]$$

We can suppose $m > st(\mathfrak{F}[U^n)])$. Then the assertion follows from (5) and (6) by a cut.

DEFINITIONS. 1. By the *zero-interpretation* of a formula F of A_2 we mean the result of replacing every free number variable in F by the numeral 0, every free predicate variable U in F by U° and every strong predicate quantifier $\forall X$ in F by $\forall X^\omega$.

2. We define the *stage* $st(F)$ of a formula F of A_2 by $st(F) := st(F^\circ)$ where F° is the zero-interpretation of F.

THEOREM 14.2. (*Upper Bound Theorem* of PB) For every in PB derivable formula F of stage 0 there is an ordinal $\alpha < \psi 0 (\psi \omega 0)$ such that $PB^* \vdash_{0}^{\alpha} F^\circ$ holds for the zero-interpretation F° of F. ($\psi \omega 0 = \varepsilon_{\Omega_\omega + 1}$ is the least ε-number $> \Omega_\omega$).

Proof. From the assumption it follows by the Interpretation Theorem 14.1 that there are k, $m < \omega$ such that

$$PB^* \vdash_{m}^{\gamma} F^\circ$$

holds for $\gamma := \Omega_\omega + 2k$. By the Cut Elimination Theorem 13.3 it follows that

$$PB^* \vdash_{0}^{\beta} F^\circ$$

holds for $\beta := \omega_m(\gamma) < \psi \omega 0$. Obviously we have $\beta \in C_0(0) \subseteq C_0(\beta)$. Therefore it follows by the Collapsing Theorem 13.4 that

$$PB^* \vdash_{0}^{\alpha} F^\circ$$

holds for $\alpha := \psi 0 \beta < \psi 0 (\psi \omega 0)$.

REMARK. The result of Theorem 14.2 first was proved by W. POHLERS [20] with respect to another ordinal notation system.

§ 15. – The Semiformal System PA'

The *formulas* of PA' are those formulas of A_2 which contain no free number variable.

The *grade* $gr(F)$ of a formula F of PA' is defined corresponding to the grade in PB^* such that $gr(F) = gr(F^*)$ holds for every interpretation F^* of F.

Axioms of PA':

(Ax 1), (Ax 2), (Ax 3) corresponding to these axioms of A_2 (see § 2).

Principal inferences of PA':

(S 1), (S 2.1), (S 3.0) corresponding to these inferences of A_2.

(S 2.0') $\mathfrak{F}[\mathfrak{A}[n]]$ for every numeral $n \vdash \mathfrak{F}[\forall x \mathfrak{A}[x]]$

(S 3.1') $\mathfrak{F}[\mathfrak{A}] \to \mathfrak{N}[\forall X \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X \mathfrak{F}[X]]$
if $\forall X \mathfrak{F}[X]$ and $\mathfrak{A}[t]$ are weak formulas. (Then also $\mathfrak{F}[\mathfrak{A}]$ is a weak formula).

(S 3.2') $\mathfrak{F}[U] \to \mathfrak{N}[\forall X \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X \mathfrak{F}[X]]$
if $\forall X \mathfrak{F}[X]$ is a strong formula.

Cuts of PA' corresponding to the cuts of A_2.

The system PA' is called *semiformal* because it has inferences with infinitely many premises.

INDUCTIVE DEFINITION of $PA' \vdash_{m}^{\gamma} F$ for $\gamma < \Omega_1$ ($\gamma \in T(\Omega)$) and $m < \omega$.

1. If F is an axiom of PA' then $PA' \vdash_{m}^{\gamma} F$ holds for all $\gamma < \Omega_1$ and $m < \omega$.

2. If $PA' \vdash_{m}^{\beta} F_i$ and $\beta < \gamma < \Omega_1$ holds for every premise F_i of a principal inference of PA' or a cut of grade $< m$, then $PA' \vdash_{m}^{\gamma} F$ holds for the conclusion F of that inference.

LEMMA 15.1. $PA' \vdash_{m}^{\alpha} F$, $\alpha \leq \beta < \Omega_1$, $m \leq n < \omega \Rightarrow PA' \vdash_{n}^{\beta} F$. This follows immediately from the inductive definition.

LEMMA 15.2. (*Replacement rules*)

a) $PA' \vdash_{m}^{\gamma} F \Rightarrow PA' \vdash_{m}^{\gamma} G$

if F and G are equivalent formulas.

b) $PA' \vdash_{m}^{\gamma} \mathfrak{F}[U] \Rightarrow PA' \vdash_{m}^{\gamma} \mathfrak{F}[V]$

if U does not occur in $\mathfrak{F}$.

Proof by induction on γ.

LEMMA 15.3. (*Inversion rules*).

a) $PA' \vdash^{\gamma}_{m} \mathfrak{N}[(A \to B)] \Rightarrow PA' \vdash^{\gamma}_{m} \mathfrak{N}[\neg A]$
$PA' \vdash^{\gamma}_{m} \mathfrak{N}[(A \to B)] \Rightarrow PA' \vdash^{\gamma}_{m} \mathfrak{N}[B]$

b) $PA^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall x\, \mathfrak{A}[x]] \Rightarrow PA' \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{A}[t]]$

c) $PA' \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall X \mathfrak{F}[X]] \Rightarrow PA' \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{F}[U]]$

Proof by induction on γ using the replacement rules.

LEMMA 15.4. (*Structural rule*) $PA' \vdash^{\gamma}_{m} F \Rightarrow PA' \vdash^{\gamma}_{m} G$ if $F \vdash^{s} G$ holds.
Proof corresponding to the proof of Lemma 12.5.

LEMMA 15.5. $PA' \vdash^{2m}_{0} \mathfrak{Q}[F, F]$ for every *NP*-form $\mathfrak{Q}$ if F is a formula of length $\leq m$. (The length of F is the number of symbols $\to$ and $\forall$ occurring in F).

Proof by induction on the length of F corresponding to the proof of Lemma 12.6.

LEMMA 15.6. For every formula $\mathfrak{A}[t]$ we have

$$PA' \vdash^{\omega+1}_{0} \forall x (\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \forall x\, \mathfrak{A}[x])$$

Proof corresponding the proof of Lemma 12.7.

LEMMA 15.7. For every weak formula $\mathfrak{A}[t]$ we have

$$PA' \vdash^{\omega}_{0} \exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

Proof. Let m be the length of the formula $\mathfrak{A}[n]$. Then by Lemma 15.5. we have

(1) $$PA' \vdash^{2m}_{0} \neg\, (\mathfrak{A}[n] \to \mathfrak{A}[n]) \to \bot$$

for every numeral n. From (1) by an inference (S 1) we obtain

(2) $$PA' \vdash^{2m+1}_{0} (\mathfrak{A}[n] \leftrightarrow \mathfrak{A}[n])$$

since $(\mathfrak{A}[n] \leftrightarrow \mathfrak{A}[n])$ is the formula

$$((\mathfrak{A}[n] \to \mathfrak{A}[n]) \to \neg\, (\mathfrak{A}[n] \to \mathfrak{A}[n])) \to \bot$$

From (2) we obtain by an inference (S 2.0′) and by the structural rule

(3) $$PA' \vdash^{2m+2}_{0} \neg \forall y (\mathfrak{A}[y] \leftrightarrow \mathfrak{A}[y]) \to \exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

The assertion follows from (3) by an inference (S 3.1′).

LEMMA 15.8. If C is a formula of grade $m + 1$ which is not of the form $(A \to B)$, then we have

$$PA' \vdash^{\gamma}_{m+1} \mathfrak{S}\ [C],\ PA' \vdash^{\delta}_{m+1} C \to F \Rightarrow PA' \vdash^{\gamma+\delta}_{m+1} \mathfrak{S}\ [F]$$

Proof by induction on δ.

1. Let $C \to F$ be an axiom. Then also F and $\mathfrak{S}\ [F]$ are axioms since $gr[C] > 0$. Hence the assertion holds.

2. Let $PA' \vdash^{\delta}_{m+1} C \to F$ be derived by a cut of grade $< m + 1$ or by a principal inference, the principal part of which is in F. Then the assertion follows from the I.H. and the structural rule.

3. Let $PA' \vdash^{\delta}_{m+1} C \to F$ be derived by a principal inference with principal part C. This can only be an inference (S 3.0) or (S 3.2′) since $gr(C) > 0$. In these cases we have a proof corresponding to that of Lemma 13.1.

LEMMA 15.9.

$$PA' \vdash^{\gamma}_{m+2} F \Rightarrow PA' \vdash^{\omega^{\gamma}}_{m+1} F$$

Proof corresponding to the proof of Lemma 13.2 using Lemma 15.8.

THEOREM 15.10. (*Cut Reduction Theorem*)

$$PA' \vdash^{\gamma}_{m+1} F \Rightarrow PA' \vdash^{\omega_m(\gamma)}_{1} F$$

Proof. This follows from Lemma 15.9 by induction on m.

§ 16. – Interpretation of PA' in PB^*

To prove the Interpretation Theorem for PA' in PB^* we have to use some Lemmata about majorization of ordinals and about derivability in PB^*.

LEMMA 16.1. $\beta < \gamma < \Omega_1 \Rightarrow \Omega_\omega \cdot \omega^\beta \lhd \Omega_\omega \cdot \omega^\gamma$. (We have $\Omega_\omega \cdot \omega^\beta = \omega^{\Omega_\omega + \beta}$).

Proof by the Lemmata 4.1 f), 4.3 and 4.4.

LEMMA 16.2. $PB^* \vdash_0^{\delta} F,\ \delta < \omega^{\gamma+1} \Rightarrow PA' \vdash_0^{\omega^{\gamma}+\delta} F$.

Proof by induction on δ using Lemma 4.3 and Lemma 5.3.

LEMMA 16.3. $\alpha \in C_0(\alpha),\ \alpha < \beta \Rightarrow \alpha \lhd \beta$.

Proof. Suppose $\alpha \leq \delta \leq \min\{\beta, \eta\}$ and $\delta \in C_\tau(\eta)$. Then $\alpha \in C_0(\alpha)$ implies $\alpha \in C_\tau(\eta)$, hence $\alpha \lhd \beta$ according to the definition of $\lhd$.

LEMMA 16.4. $\gamma < \psi 0 (\Omega_\omega \cdot \Omega_1) \Rightarrow \gamma \in C_0(\Omega_\omega \cdot \omega^\gamma)$.

Proof. $\gamma < \psi 0 (\Omega_\omega \cdot \Omega_1)$ implies $\gamma \in C_0(\Omega_\omega \cdot \Omega_1)$. If $C_0(\Omega_\omega \cdot \omega^\gamma) = C_0(\Omega_\omega \cdot \Omega_1)$, we have $\gamma \in C_0(\Omega_\omega \cdot \omega^\gamma)$. Otherwise by Lemma 2.8 there is an ordinal δ such that $\Omega_\omega \cdot \omega^\gamma \leq \delta < \Omega_\omega \cdot \Omega_1$ and $\delta \in C_0(\delta) = C_0(\Omega_\omega \cdot \omega^\gamma)$. In this case we have $\delta = \Omega_\omega \cdot \delta_1 + \delta_2$, $\omega^\gamma \leq \delta_1 < \Omega_1$ and $\delta_2 < \Omega_1$. Then $\delta \in C_0(\Omega_\omega \cdot \omega^\gamma)$ implies $\delta_1 \in C_0(\Omega_\omega \cdot \omega^\gamma)$. It follows by Corollary 2.5 that $\gamma \in C_0(\Omega_\omega \cdot \omega^\gamma)$ holds, since $\gamma \leq \delta_1 < \Omega_1$.

LEMMA 16.5. Suppose $PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} A \vee F$ and $PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} A \rightarrow F$ where A and F are weak formulas and $\gamma < \psi 0 (\Omega_\omega \cdot \Omega_1)$. Then there is an ordinal $\delta \lhd \Omega_\omega \cdot \omega^{\gamma+1}$ such that $PB^* \vdash_0^{\delta} F$ holds.

Proof. Let n be the maximum of the stages of $A \vee F$ and $A \rightarrow F$. Then from the assumptions by the Collapsing Theorem 13.4 we obtain

$$PB^* \vdash_0^{\psi n(\Omega_\omega + \omega^\gamma)} A \vee F, \qquad PB^* \vdash_0^{\psi n(\Omega_\omega + \omega^\gamma)} A \rightarrow F$$

It follows by a cut that

$$PB^* \vdash_m^{\psi n(\Omega_\omega + \omega^\gamma)+1} F$$

holds for $m := gr(A) + 1$. Then by the Cut Elimination Theorem we obtain

$$PB^* \vdash_0^{\omega_m(\psi n(\Omega_\omega \cdot \omega^\gamma)+1)} F$$

and by Lemma 16.2 we obtain

$$PB^* \vdash_0^{\delta} F$$

for $\delta := \Omega_\omega \cdot \omega^\gamma + \omega_m(\psi n (\Omega_\omega \cdot \omega^\gamma) + 1)$. By Lemma 16.4 we have $\gamma \in C_0(\Omega_\omega \cdot \omega^\gamma) \subseteq C_0(\delta)$. It follows that also $\delta \in C_0(\delta)$ holds.

Therefore by Lemma 16.3 we have $\delta \lhd \Omega_\omega \cdot \omega^{\gamma+1}$, since $\delta < \Omega_\omega \cdot \omega^{\gamma+1}$.

THEOREM 16.6. (*Interpretation Theorem*).

If $\gamma < \psi 0 (\Omega_\omega \cdot \Omega_1)$ and $PA' \vdash_1^{\gamma} F$ holds for a weak formula F, then $PB^* \vdash_0^{\Omega_\omega \cdot \omega^{\gamma+1}} F^*$ holds for every interpretation F^* of F.

Proof by induction on γ.

1. Let F be an axiom of PA'. Then F^* is an axiom of PB^*, hence the assertion holds.

2. Let $PA' \vdash_1^{\gamma} F$ be derived by an inference $(S\,1)$, $(S\,2.1')$ or $(S\,3.0)$. Then we obtain the assertion from the I.H. using a corresponding inference of PB^* and Lemma 16.1.

3. Let $PA' \vdash_1^{\gamma} F$ be derived by an inference $(S\,2.1)$. Then F^* is a formula $\mathfrak{G}\,[\forall X\, \mathfrak{F}\,[X]]$ and by the I.H. and Lemma 16.1 we have

$$PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} \mathfrak{G}\,[\mathfrak{F}\,[U^n]] \tag{1}$$

where U does not occur in F^* and n may be chosen such that $n = st(\forall X\, \mathfrak{F}\,[X])$. Let $\mathfrak{G}_0[\mathfrak{F}\,[U^n]]$ be the result of replacing every positive part $\forall X \mathfrak{F}\,[X]$ in $\mathfrak{G}\,[\mathfrak{F}\,[U^n]]$ by $\perp$. Then we have

$$\mathfrak{G}\,[\mathfrak{F}\,[U^n]] \vdash^{s} \forall X\, \mathfrak{F}\,[X] \vee \mathfrak{G}_0[\mathfrak{F}\,[U^n]]$$

Therefore by the structural rule from (1) we obtain

$$PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} \forall X\, \mathfrak{F}\,[X] \vee \mathfrak{G}_0[\mathfrak{F}\,[U^n]] \tag{2}$$

Furthermore from Lemma 12.10 and the structural rule we obtain

$$PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} \forall X\, \mathfrak{F}\,[X] \rightarrow \mathfrak{G}_0[\mathfrak{F}\,[U^n]] \tag{3}$$

By Lemma 16.5 it follows from (2) and (3) that there is an ordinal $\delta \lhd \Omega_\omega \cdot \omega^{\gamma+1}$ such that

$$PB^* \vdash_0^{\delta} \mathfrak{G}_0[\mathfrak{F}\,[U^n]] \tag{4}$$

holds. The assertion follows from (4) by an inference $(S\,2.1^*)$.

4. Let $PA' \vdash_1^{\gamma} F$ be derived by an inference $(S\,3.1')$. Then F^* is a formula $\mathfrak{N}\,[\forall x\, \mathfrak{F}\,[X]]$ and by the I.H. and Lemma 16.1 we have

$$PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} \mathfrak{F}\,[\mathfrak{A}] \rightarrow \mathfrak{N}\,[\forall X\, \mathfrak{F}\,[X]] \tag{5}$$

From Lemma 12.10 and the structural rule we obtain

$$PB^* \vdash_0^{\Omega_\omega \cdot \omega^\gamma} \mathfrak{F}\,[\mathfrak{A}] \vee \mathfrak{N}\,[\forall X\, \mathfrak{F}\,[X]] \tag{6}$$

The assertion follows from (5) and (6) by Lemma 16.5.

5. Let $PA' \vdash^{\gamma}_{1} F$ be derived by a cut of grade 0. Then the cut formula is also a weak formula. Therefore the assertion follows from the I.H. by Lemmata 16.1 and 16.5.

§ 17. - Embedding PA and $\overline{PA}$ in PA'

LEMMA 17.1. If $PB^* \vdash^{\alpha}_{0} F^{\circ}$ holds for the zero-interpretation F° of a formula F of PA' where $\alpha < \Omega_1$ and $st(F) = 0$, then also $PA' \vdash^{\alpha}_{0} F$ holds.

Proof by induction on α. $PB^* \vdash^{\alpha}_{0} F^{\circ}$ cannot be derived by a cut and it cannot be derived by an Ω_{n+1}-rule since $\alpha < \Omega_1$.

1. Let F° be an axiom of PB^*. Then F is an axiom of PA' since $st(F) = 0$.

2. Let $PB^* \vdash^{\alpha}_{0} F^{\circ}$ be derived by a principal inference of PB^*. This only can be an inference (S 1), (S 2.0*), (S 2.1*) or (S 3.0). In all these cases the assertion follows from the I.H. and the structural rule of PA'.

LEMMA 17.2. If $PA' \vdash^{\alpha}_{0} \mathfrak{F}[U]$ holds for an arithmetical formula $\mathfrak{F}[U]$ where $\alpha < \Omega_1$ and U does not occur in $\mathfrak{F}$, then $PA' \vdash^{\omega+\alpha}_{0} \mathfrak{F}[\mathcal{A}]$ holds for an arbitrary formula $\mathcal{A}[t]$ of PA'.

Proof by induction on α.

1. Let $\mathfrak{F}[U]$ be an axiom of PA'. If also $\mathfrak{F}[\mathcal{A}]$ is an axiom, the assertion holds. Otherwise $\mathfrak{F}[\mathcal{A}]$ is a formula $\mathcal{Q}[\mathcal{A}[s], \mathcal{A}[t]]$ where s and t are numerical terms of equal values. Then the assertion follows from Lemma 15.5. and the replacement rule a).

2. If $\mathfrak{F}[U]$ is not an axiom, $PA' \vdash^{\alpha}_{0} \mathfrak{F}[U]$ can only be derived by an inference (S 1), (S 2.0′) or (S 3.0). In all these cases the assertion follows from the I.H.

INDUCTIVE DEFINITION of ordinals μ_n and ν_n.

$$\nu_0 := \varepsilon_0 = \psi\, 00, \qquad \nu_{n+1} := \psi\, 0\, (\Omega_\omega \cdot \nu_n + 1),$$

$$\mu_0 := \omega + 1, \qquad \mu_{n+1} := \psi\, 0\, (\Omega_\omega \cdot \nu_n).$$

LEMMA 17.3.

a) $\Omega_\omega \cdot \nu_n \in C_0(\Omega_\omega \cdot \nu_n)$

b) $\mu_n < \nu_n < \mu_{n+1} < \psi\, 0\, (\Omega_\omega \cdot \Omega_1)$

c) ν_n is the least ε-number $> \mu_n$.

Proof of a) by induction on n using Lemma 16.4. b) and c) follows from a) and the definitions.

DEFINITION. By a *numerical substitute* of a formula F of A_2 we mean the result of replacing every free number variable in F by a numeral.

THEOREM 17.4. (*Embedding Theorem*).

a) If $PA_k \vdash^{n} F$ holds, then there is an $m < \omega$ such that $PA' \vdash^{\mu_k+n}_{m} F'$ holds for every numerical substitute F' of F.

b) If $PA_k \vdash^{n} F$ holds for a formula F of stage 0, then there is an ordinal $\alpha < \mu_{k+1}$ such that $PA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

We prove a) and b) simultaneously by induction on $k + n$. Suppose $PA_k \vdash^{n} F$ and let F' be a numerical substitute of F.

1. Proof of a).

1.1. Let F be an axiom of PA. If F is one of the axioms (Ax 1)-(Ax 4), then F' is an axiom of PA'. If F is an axiom (Ax 5) or (Π^1_1-CA), then the assertion follows from Lemmata 15.6. and 15.7.

1.2. Let $PA_k \vdash^{n} F$ be derived by a structural inference. Then the assertion follows from the I.H. by the structural rule of PA'.

1.3. Let $PA_k \vdash^{n} F$ be derived by a principal inference of PA or by a cut. Then the assertion follows from the I.H. by a corresponding inference of PA'.

1.4. Let $PA_k \vdash^{n} F$ be derived by an inference (BR). Then we have $k > 0$ and $PA_{k-1} \vdash^{n} F_0$ for the premise F_0 of that inference. Let F'_0 be the numerical substitute of F_0 such that also $F'_0 \vdash F'$ is an inference (BR). By the I.H. for b) there is an $\alpha < \mu_k$ such that $PA' \vdash^{\alpha}_{0} F'_0$ holds for the arithmetical formula F'_0. It follows by Lemma 17.2. that $PA' \vdash^{\omega+\alpha}_{0} F'$ holds. Then also $PA' \vdash^{\mu_k+n}_{0} F'$ holds, since $\omega + \alpha < \mu_k + n$.

2. Proof of b). Let F be a formula of stage 0. It is already proved that there is an $m < \omega$ such that $PA' \vdash^{\mu_k+n}_{0} F'$ holds. It follows by the Cut Reduction Theorem 15.10 that $PA' \vdash^{\gamma}_{1} F'$ for $\gamma := \omega_m(\mu_k + n) < \nu_k$. It follows by the Interpretation Theorem 16.6. that $PB^* \vdash^{\Omega_\omega \cdot \omega^{\gamma+1}}_{0} F^{\circ}$ holds for the zero-interpretation F° of F'. From Lemma 16.4 we obtain $\Omega_\omega \cdot \omega^{\gamma+1} \in C_0(\Omega_\omega \cdot \omega^{\gamma+1})$. It follows by the Collapsing Theorem 13.4 that $PB^* \vdash^{\alpha}_{0} F^{\circ}$ holds for

$$\alpha = \psi\, 0\, (\Omega_\omega \cdot \omega^{\gamma+1}) < \psi\, 0\, (\Omega_\omega \cdot \nu_k) = \nu_{k+1}.$$

By Lemma 17.1 we obtain $PA' \vdash^{\alpha}_{0} F'$.

THEOREM 17.5. (*Upper Bound Theorem* for PA and $\overline{PA}$)

a) If a formula F of stage 0 is derivable in PA, then there is an ordinal $\alpha < \psi\, 0\, (\Omega_\omega \cdot \varepsilon_0)$ such that $PA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

b) If a formula F of stage 0 is derivable in $\overline{PA}$, then there is an ordinal $\alpha < \psi 0 (\Omega_\omega \cdot \Omega_1)$ such that $PA' \left|\frac{\alpha}{0}\right. F'$ holds for every numerical substitute F' of F.

Proof. This follows immediately from Theorem 17.4. b).

REMARK. The result of Theorem 17.5 a) was first proved by W. BUCHHOLZ [2] with respect to another ordinal notation system.

CHAPTER III

SUBSYSTEMS OF ANALYSIS WITH Π^1_2-SEPARATION

Stronger subsystems of analysis than the systems of chapter II are systems with Δ^1_2-comprehension. Such systems were first investigated by S. FEFERMAN [9] and H. FRIEDMAN [10]. Later on the proof theoretical ordinals of such systems were determined by G. TAKEUTI and M. YASUGI [31] and by W. POHLERS [20]. In this chapter we use the formally stronger Π^1_2-separation instead of Δ^1_2-comprehension because we can carry out the proof theoretical treatment for Π^1_2-separation by a generalization of the Buchholz $\Omega_{\sigma+1}$-rule.

§ 18. – Π^1_2-Separation and Δ^1_2-Comprehension

INDUCTIVE DEFINITION of Π^1_2-formulas and Σ^1_2-formulas in A_2.

1. Every prime formula and every weak formula $\forall X \mathfrak{F}[X]$ is both a Π^1_2-formula and a Σ^1_2-formula.

2. A formula $(A \to B)$ is a Π^1_2-formula (Σ^1_2-formula) if A is a Σ^1_2-formula (Π^1_2-formula) and B is a Π^1_2-formula (Σ^1_2-formula).

3. A formula $\forall x \mathfrak{A}[x]$ is a Π^1_2-formula (Σ^1_2-formula) if $\mathfrak{A}[0]$ is a Π^1_2-formula (Σ^1_2-formula).

4. A strong formula $\forall X \mathfrak{F}[X]$ is a Π^1_2-formula if $\mathfrak{F}[U]$ is a Π^1_2-formula. It is not a Σ^1_2-formula.

LEMMA 18.1. A formula of A_2 is both a Π^1_2-formula and a Σ^1_2-formula if and only if it is a weak formula.

Proof by induction on the length of the formula.

REMARK. The above defined Π^1_2-formulas and Σ^1_2-formulas are more general than the usual Π^1_2-formulas $\forall X \exists Y \mathfrak{C}[X, Y]$ and Σ^1_2-formulas $\exists X \forall Y \mathfrak{C}[X, Y]$ where $\mathfrak{C}[U, V]$ is an arithmetical formula.

But the axioms and inference rules in this chapter with respect to our Π^1_2-formulas and Σ^1_2-formulas are not essentially stronger than the corresponding axioms and inference rules with respect to the usual Π^1_2-formulas and Σ^1_2-formulas.

We may consider the set theoretical meaning of a nominal form $\mathfrak{A}$ and a predicate variable U as follows: The nominal form $\mathfrak{A}$ denotes the *class* $\{x : \mathfrak{A}[x]\}$ of natural numbers and the predicate variable U denotes the *set* $\{x : U[x]\}$ of natural numbers. The complement of this set may be denoted by $\bar{U} := \{x : \neg U(x)\}$. A class $\mathfrak{A}$ is said to be a Π^1_2-class (Σ^1_2-class) if $\mathfrak{A}[t]$ is a Π^1_2-formula (Σ^1_2-formula).

The axiom schema of Π^1_2-separation is the statement: Any two disjoint Π^1_2-classes $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be separated by a set X such that $\mathfrak{A}_1 \subseteq X$ and $\mathfrak{A}_2 \subseteq \bar{X}$.

The complement of a Π^1_2-class is a Σ^1_2-class. Therefore the axiom schema of Π^1_2-separation is equivalent to the statement:

If $\mathfrak{A} \subseteq \mathfrak{B}$ holds for a Π^1_2-class $\mathfrak{A}$ and a Σ^1_2-class $\mathfrak{B}$, then there is a set X such that $\mathfrak{A} \subseteq X$ and $X \subseteq \mathfrak{B}$.

We use in the language of A_2 the *abbreviations*

$$\mathfrak{A} = \mathfrak{B} := \forall y(\mathfrak{A}[y] \leftrightarrow \mathfrak{B}[y])$$

$$\mathfrak{A} \subseteq \mathfrak{B} := \forall y(\mathfrak{A}[y] \rightarrow \mathfrak{B}[y])$$

$$(\mathfrak{A} \subseteq U \subseteq \mathfrak{B}) := \forall y(\mathfrak{A}[y] \rightarrow U(y)) \wedge \forall y(U(y) \rightarrow \mathfrak{B}[y])$$

Then the axiom schema of Π^1_2-*Separation* is formalized by

(Π^1_2-*SA*) $\quad \mathfrak{A} \subseteq \mathfrak{B} \rightarrow \exists X(\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$
for a Π^1_2-formula $\mathfrak{A}[t]$ and a Σ^1_2-formula $\mathfrak{B}[t]$.

The corresponding Π^1_2-*Separation Rule* is the inference rule

(Π^1_2-*SR*) $\quad \mathfrak{A} \subseteq \mathfrak{B} \vdash \exists X(\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$
for a Π^1_2-formula $\mathfrak{A}[t]$ and a Σ^1_2-formula $\mathfrak{B}[t]$.

The axiom schema of Δ^1_2-*Comprehension* is formalized by

(Δ^1_2-*CA*) $\quad \mathfrak{A} = \mathfrak{B} \rightarrow \exists X \forall y(X(y) \leftrightarrow \mathfrak{A}[y])$
for a Π^1_2-formula $\mathfrak{A}[t]$ and a Σ^1_2-formula $\mathfrak{B}[t]$.

The corresponding Δ^1_2-*Comprehension Rule* is the inference rule

(Δ^1_2-*CR*) $\quad \mathfrak{A} = \mathfrak{B} \vdash \exists X \forall y(X(y) \leftrightarrow \mathfrak{A}[y])$.

LEMMA 18.2.

a) (Π^1_2-*SA*) implies (Δ^1_2-*CA*).

b) (Π^1_2-*SR*) implies (Δ^1_2-*CR*).

c) (Δ^1_2-*CR*) implies (Π^1_1-*CA*).

The proofs are trivial.

REMARK. According to an unpublished proof by R. MANSFIELD (Δ^1_2-*CA*) implies the axiom of choice (Σ^1_2-*AC*) for Σ^1_2-formulas which implies (Π^1_2-*SA*). Hence (Π^1_2-*SA*) and (Δ^1_2-*CA*) are equivalent in A_2. But in this book we only prove that the subsystems of analysis with (Π^1_2-*SA*) and with (Δ^1_2-*CA*) have the same proof theoretical ordinal.

§ 19. - The Formal Systems SR, SA and $\overline{SA}$

Let SR be the formal system A_2 with the additional axiom schema (Π^1_1-*BI*) and the additional basic inference rule (Π^1_2-*SR*).

Let SA be the formal system A_2 with the additional axiom schema (Π^1_2-*SA*).

Let $\overline{SA}$ be the formal system A_2 with the additional axiom schema (Π^1_2-*SA*) and the additional basic inference rule (*BR*).

INDUCTIVE DEFINITION of $SR \overset{n}{\vdash} F$.

1. If F is an axiom of SR then $SR \overset{n}{\vdash} F$ holds for every natural number n.

2. If $SR \overset{n}{\vdash} F_i$ holds for every premise F_i of a basic inference of SR then $SR \overset{n+1}{\vdash} F$ holds for the conclusion F of that inference.

According to this definition $SR \overset{n}{\vdash} F$ implies $SR \overset{m}{\vdash} F$ for all $m > n$.

A formula F is said to be *derivable* in SR if there is a natural number n such that $SR \overset{n}{\vdash} F$ holds.

INDUCTIVE DEFINITION of $SA_k \overset{n}{\vdash} F$.

1. If F is an axiom of SA then $SA_k \overset{n}{\vdash} F$ holds for all natural numbers k and n.

2. If $SA_k \overset{n}{\vdash} F_i$ holds for every premise F_i of a basic inference of SA then $SA_k \overset{n+1}{\vdash} F$ holds for the conclusion F of that inference.

3. If $SA_k \overset{n}{\vdash} F_0$ holds for the premise F_0 of an inference (*BR*) then $SA_{k+1} \overset{n}{\vdash} F$ holds for the conclusion F of that inference.

According to this definition $SA_k \overset{n}{\vdash}$ implies $SA_i \overset{m}{\vdash} F$ for all $i \geq k$ and $m \geq n$.

A formula F is said to be *derivable* in SA if there is a natural number n such that $SA_0 \overset{n}{\vdash} F$ holds.

A formula F is said to be *derivable* in $\overline{SA}$ if there are natural numbers k and n such that $SA_k \overset{n}{\vdash} F$ holds.

§ 20. - The Language of a Ramified System SR^*

By an SR^**-interpretation* of a formula F of A_2 we mean the result of replacing every free number variable in F by a numeral, every free predicate variable U in F by a U^α where $\alpha < \omega^\omega$ and every strong predicate quantifier $\forall X$ in F by $\forall X^\beta$ where β is a limit ordinal $< \omega^\omega$. The *basic formulas* of SR^* are the SR^*-interpretations of formulas of A_2.

According to this definition, for every basic formula F of SR^* there is a *corresponding formula* of A_2 which results from F by cancelling all upper indices of free predicate variables and strong predicate quantifiers. For a basic formula F of SR^* we define $PV(F) := PV(F')$ where F' is the corresponding formula of A_2. A basic formula of SR^* is said to be a *prime formula*, an *arithmetical formula*, a *weak formula* or a *strong formula* , a Π^1_2*-formula* or a Σ^1_2*-formula* if the corresponding formula of A_2 is such a formula.

INDUCTIVE DEFINITION of the *formulas* of SR^*.

1. Every basic formula of SR^* is a formula of SR^*.

2. If $\mathfrak{F}[U^\circ]$ is a strong basic Π^1_2-formula of SR^* where the free predicate variable U and the bound predicate variable X do not occur in $\mathfrak{F}$ and $U \notin PV(\mathfrak{F}[U^\circ])$, then $\forall X' \mathfrak{F}[X]$ is a formula of SR^*. This formula is a Π^1_2-formula and is not a Σ^1_2-formula. $PV(\forall X' \mathfrak{F}[X])$ is the set of free predicate variables occurring in $\mathfrak{F}$.

3. If A and B are formulas of SR^* then $(A \to B)$ is a formula of SR^* and $PV(A \to B) := PV(A) \cup PV(B)$. This formula $(A \to B)$ is a Π^1_2-formula (Σ^1_2-formula) if A is a Σ^1_2-formula (Π^1_2-formula) and B is a Π^1_2-formula (Σ^1_2-formula).

REMARK. According to this definition, a quantifier $\forall X'$ cannot appear in the scope of another quantifier in a formula of SR^* and it cannot appear in a basic formula of SR^*. From strong basic Π^1_2-formulas $\mathfrak{F}[U^\circ]$ where U and X do not occur in $\mathfrak{F}$ and $U \notin PV(\mathfrak{F}[U^\circ])$ we can construct two different kinds of universal formulas: $\forall X^\beta \mathfrak{F}[X]$ and $\forall X' \mathfrak{F}[X]$. We shall use the formulas $\forall X^\beta \mathfrak{F}[X]$ as interpretations of the corresponding formulas of SR and we shall use $\forall X' \mathfrak{F}[X]$ in the proof of the Interpretation Theorem 22.2 with respect to an inference (Π^1_2-SR).

Therefore the deduction procedure in SR^* has to be different with respect to $\forall X^\beta \mathfrak{F}[X]$ and with respect to $\forall X' \mathfrak{F}[X]$.

P-forms, N-forms, NP-forms and $F \overset{s}{\vdash} G$ are defined in SR^* in the same way as in A_2. We also use in SR^* the same *syntactical variables* as in A_2.

INDUCTIVE DEFINITION of the *grade* $gr(F)$ of a formula F in SR^*.

1. $gr(F) := 0$ if F is a prime formula or a formula $\forall X \mathfrak{F}[X]$ or a formula $\forall X' \mathfrak{F}[X]$.
2. $gr(A \to B) := \max\{gr(A), gr(B)\} + 1$
3. $gr(\forall x\, \mathfrak{A}[x]) := gr(\mathfrak{A}[0]) + 1$
4. $gr(\forall X^\beta \mathfrak{F}[X]) := gr(\mathfrak{F}[U^\circ]) + 1$.

INDUCTIVE DEFINITION of the *stage* $st(F)$ of a formula F in SR^*.

1. $st(F) = st(\neg F) := 0$ if F is a constant prime formula.
2. $st(U^\alpha(t)) = st(\neg U^\alpha(t)) := \alpha$
3. $st(A \to B) := \max\{st(\neg A), st(B)\}$,
   $st(\neg(A \to B)) := \max\{st(A), st(\neg B)\}$.
4. $st(\forall x\, \mathfrak{A}[x]) := st(\mathfrak{A}[0])$, $st(\neg \forall x\, \mathfrak{A}[x]) := st(\neg \mathfrak{A}[0])$.
5. $st(\forall X \mathfrak{F}[X]) := st(\mathfrak{F}[U^\circ])$, $st(\neg \forall X \mathfrak{F}[X]) := st(\mathfrak{F}[U^\circ]) + 1$.
6. $st(\forall X^\beta \mathfrak{F}[X]) = st(\neg \forall X^\beta \mathfrak{F}[X]) := \max\{\beta, st(\mathfrak{F}[U^\circ])\}$.
7. $st(\forall X' \mathfrak{F}[X]) = st(\neg \forall X' \mathfrak{F}[X]) := st(\mathfrak{F}[U^\circ])$.

LEMMA 20.1. (*Stage Lemma*)

a) For every P-form $\mathfrak{P}$ and every N-form $\mathfrak{N}$ we have $st(\mathfrak{P}[A]) = \max\{st(A), st(\mathfrak{P}[\bot])\}$, $st(\mathfrak{N}[A]) = \max\{st(\neg A), st(\mathfrak{N}[\bot])\}$.

b) If U^α appears in a formula $\mathfrak{F}[U^\alpha]$ and $U \notin PV(\mathfrak{F}[U^\alpha])$, then $st(\mathfrak{F}[U^\alpha]) = \max\{\alpha, st(\mathfrak{F}[U^\circ])\}$.

c) $st(\mathfrak{F}[U^\alpha]) \le \max\{\alpha + \omega, st(\mathfrak{F}[U^\circ])\}$ for every formula $\mathfrak{F}[U^\alpha]$.

d) If $st(\forall X \mathfrak{F}[X]) = \sigma$, then also $st(\mathfrak{F}[U^\circ]) = \sigma$.

e) If $st(\forall X' \mathfrak{F}[X]) = \sigma$, then also $st(\mathfrak{F}[U^\circ]) = \sigma$.

f) $st(\mathfrak{F}[U^\alpha]) \le st(\forall X^\beta \mathfrak{F}[X])$ for $\alpha < \beta$.

Proof of a) by induction on the lengths of the nominal forms $\mathfrak{P}$ and $\mathfrak{N}$. Proof of b) and c) by induction on the length of $\mathfrak{F}$.

d) and e) follow from b), f) follows from c).

INDUCTIVE DEFINITION of the *positively* and *negatively occurring* predicate quantifiers in a formula of SR^*.

1. In a prime formula no predicate quantifier occurs.

2. A predicate quantifier occurs positively (negatively) in a formula $(A \to B)$ if it occurs negatively (positively) in A or positively (negatively) in B.

3. A predicate quantifier occurs positively (negatively) in a formula $\forall x \mathfrak{A}[x]$ if it occurs positively (negatively) in $\mathfrak{A}[0]$.

4. If F is a formula $\forall X \mathfrak{F}[X]$, $\forall X^\beta \mathfrak{F}[X]$ or $\forall X' \mathfrak{F}[X]$, then the indicated predicate quantifier occurs positively in F. Any other predicate quantifiers occurs positively (negatively) in F if it occurs positively (negatively) in $\mathfrak{F}[U^\sigma]$.

LEMMA 20.2. A strong predicate quantifier $\forall X^\beta \mathfrak{F}[X]$ or $\forall X' \mathfrak{F}[X]$ does not occur negatively (positively) in a Π^1_2-formula (Σ^1_2-formula) of SR^*.

Proof by induction on the length of the formula.

DEFINITION of $F \overset{w}{\vdash} G$ for formulas F, G of SR^*.

$F \overset{w}{\vdash} G$ (G follows by weakening of predicate quantifiers from F) means that either F and G are equivalent formulas or G is equivalent to the result of replacing some positively or negatively occurring predicate quantifiers $\forall X_i^{\beta_i}$ in F by $\forall X_i^{\gamma_i}$ where γ_i is a limit ordinal $< \omega^\omega$ and $\gamma_i < \beta_i$ or $\gamma_i > \beta_i$ respectively holds.

Obviously, we have

LEMMA 20.3. $F \overset{w}{\vdash} G \Rightarrow \mathfrak{S}[F] \overset{w}{\vdash} \mathfrak{S}[G]$.

§ 21. – Deductions in SR^*

All formulas in this section are formulas of SR^*.

Axioms of SR^*:

$(Ax\,1)$, $(Ax\,2)$ corresponding to these axioms of A_2.

$(Ax\,3^*)$ $\mathfrak{Q}[A, B]$ if A and B are equivalent formulas of grade 0.

Principal inferences of SR^*:

$(S\,1)$, $(S\,3.0)$ corresponding to these inferences of A_2.

$(S\,2.0^*)$ $\mathfrak{S}[\mathfrak{A}[n]]$ for every numeral $n \vdash \mathfrak{S}[\forall x \mathfrak{A}[x]]$

$(S\,2.1^*)$ $\mathfrak{S}_0[\mathfrak{F}[U^\sigma]] \vdash \mathfrak{S}[A]$
if A is a formula $\forall X \mathfrak{F}[X]$ or $\forall X' \mathfrak{F}[X]$ of stage σ, U does not occur in the conclusion and $\mathfrak{S}_0[\mathfrak{F}[U^\sigma]]$ is the result of replacing every positive part A in $\mathfrak{S}[\mathfrak{F}[U^\sigma]]$ by $\perp$.

$(S\,2.2^*)$ $\mathfrak{S}[\mathfrak{F}[U^\alpha]]$ for all $\alpha < \beta \vdash \mathfrak{S}[\forall X^\beta \mathfrak{F}[X]]$
if U does not occur in the conclusion.

$(S\,3.1^*)$ $\mathfrak{F}[U^\alpha] \to \mathfrak{N}[\forall X^\beta \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X^\beta \mathfrak{F}[X]]$
for $\alpha < \beta$.

Cuts of SR^* corresponding to the cuts of PB^*. The grade of a cut is the grade of its cut formula.

REMARK. As in PB^* there is no principal inference in SR^* with a negative principal part $\forall X \mathfrak{F}[X]$ or $\forall X' \mathfrak{F}[X]$.

INDUCTIVE DEFINITION of $SR^* \overset{\gamma}{\underset{m}{\vdash}} F$ for $\gamma \in T(\Omega)$ and $m < \omega$.

1. If F is an axiom of SR^* then $SR^* \overset{\gamma}{\underset{m}{\vdash}} F$ holds for all $\gamma \in T(\Omega)$ and $m < \omega$.

2. If $SR^* \overset{\beta}{\underset{m}{\vdash}} F_i$ and $\beta \lhd \gamma$ holds for every premise F_i of a principal inference of SR^* or a cut of grade $< m$, then $SR^* \overset{\gamma}{\underset{m}{\vdash}} F$ holds for the conclusion F of that inference.

3. ($\Omega_{\sigma+1}$-*rule*) $SR^* \overset{\gamma}{\underset{m}{\vdash}} F$ holds under the following assumptions:

a) A is a formula $\forall X \mathfrak{F}[X]$ or $\forall X' \mathfrak{F}[X]$ of stage σ.

b) f is a fundamental function such that $\Omega_{\sigma+1} \in \mathrm{dom}(f)$ and $f(\Omega_{\sigma+1}) \trianglelefteq \gamma$.

c) $SR^* \overset{f(0)}{\underset{m}{\vdash}} A \vee F$

d) $SR^* \overset{\alpha}{\underset{0}{\vdash}} \mathfrak{S}[A] \Rightarrow SR^* \overset{f(\alpha)}{\underset{m}{\vdash}} \mathfrak{S}[F]$

for all $\alpha < \Omega_{\sigma+1}$ ($\alpha \in T(\Omega)$) and every P-form $\mathfrak{S}$ such that $\mathfrak{S}[A]$ is a Π^1_2-formula of stage σ.

LEMMA 21.1. $SR^* \overset{\alpha}{\underset{m}{\vdash}} F$, $\alpha \trianglelefteq \beta$, $m \leq n \Rightarrow SR^* \overset{\beta}{\underset{n}{\vdash}} F$.

This follows immediately from the inductive definition.

LEMMA 21.2. (*Replacement rules*)

a) $SR^* \overset{\gamma}{\underset{m}{\vdash}} F \Rightarrow SR^* \overset{\gamma}{\underset{m}{\vdash}} G$

if F and G are equivalent formulas.

b) $SR^* \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{S}[U^\alpha] \Rightarrow SR^* \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{S}[V^\alpha]$

if U does not occur in $\mathfrak{S}$.

c) $SR^* \overset{\gamma}{\underset{m}{\vdash}} F \Rightarrow SR^* \overset{\gamma}{\underset{m}{\vdash}} G$

for basic formulas F, G if $F \overset{w}{\vdash} G$ holds.

LEMMA 21.3. (*Inversion rules*)

a) $SR^* \vdash^{\gamma}_{m} \mathfrak{N}[(A \to B)] \Rightarrow SR^* \vdash^{\gamma}_{m} \mathfrak{N}[\neg A]$

$SR^* \vdash^{\gamma}_{m} \mathfrak{N}[(A \to B)] \Rightarrow SR^* \vdash^{\gamma}_{m} \mathfrak{N}[B]$

b) $SR^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall x\, \mathfrak{A}[x]] \Rightarrow SR^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{A}[t]]$

c) $SR^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\forall X^{\beta}\, \mathfrak{F}[X]],\ \alpha < \beta \Rightarrow SR^* \vdash^{\gamma}_{m} \mathfrak{S}\ [\mathfrak{F}[U^{\alpha}]]$

Proof by induction on γ using the replacement rule a) and b).

LEMMA 21.4. (*Structural rule*)

$$SR^* \vdash^{\gamma}_{m} F, \quad F \vdash^{s} G \Rightarrow SR^* \vdash^{\gamma}_{m} G$$

Proof by induction on γ using the inversion rules corresponding to the proof of Lemma 12.5.

LEMMA 21.5. $SR^* \vdash^{2m}_{0} \mathfrak{Q}[F, F]$ for every *NP*-form $\mathfrak{Q}$ if $m \geq gr(F)$.
Proof by induction on $gr(F)$ corresponding to the proof of Lemma 12.6.

LEMMA 21.6. For every formula $\mathfrak{A}[t]$:

$$SR^* \vdash^{\omega+1}_{0} \forall x (\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \forall x\, \mathfrak{A}[x])$$

Proof corresponding to the proof of Lemma 12.7.

LEMMA 21.7. If A is a formula $\forall X\, \mathfrak{F}[X]$ or $\forall X'\, \mathfrak{F}[X]$ of stage σ, $\mathfrak{S}\ [A]$ a Π^1_2-formula of stage σ and $\alpha < \Omega_{\sigma+1}$, then

$$SR^* \vdash^{\alpha}_{0} \mathfrak{S}\ [A] \Rightarrow SR^* \vdash^{\alpha}_{0} \mathfrak{S}\ [\mathfrak{F}[U^{\sigma}]]$$

Proof by induction on α corresponding to the proof of Lemma 12.8.

LEMMA 21.8. Let $\mathfrak{S}\ [\mathfrak{F}[U^{\sigma}]]$ be a basic Π^1_2-formula where U does not occur in $\mathfrak{S}$ or $\mathfrak{F}$ and $U \notin PV(\mathfrak{F}[U^{\sigma}])$. Let $\mathfrak{A}[t]$ be an arbitrary basic formula and $\alpha < \Omega_{\sigma+1}$. Then we have

$$SR^* \vdash^{\alpha}_{0} \mathfrak{S}\ [\mathfrak{F}[U^{\sigma}]] \Rightarrow SR^* \vdash^{\Omega_{\sigma+1}+\alpha}_{0} \mathfrak{S}\ [\mathfrak{F}[\mathfrak{A}]]$$

Proof by induction on α corresponding to the proof of Lemma 12.9.

LEMMA 21.9. If A is a formula $\forall X \mathfrak{F}[X]$ or $\forall X'\, \mathfrak{F}[X]$ of stage σ and $\mathfrak{A}[t]$ is an arbitrary basic formula, then we obtain

$$SR^* \vdash^{\Omega_{\sigma+1}\cdot 2}_{0} A \to \mathfrak{F}[\mathfrak{A}]$$

Proof by $\Omega_{\sigma+1}$-rule using the Lemmata 21.7 and 21.8 corresponding to the proof of Lemma 12.10.

THEOREM 21.10. (*Cut Elimination Theorem*) $SR^* \vdash^{\gamma}_{m} F \Rightarrow SR^* \vdash^{\omega_m(\gamma)}_{0} F$.
Proof corresponding to the proof of Theorem 13.3.

LEMMA 21.11. Let $\mathfrak{F}[U^{\sigma}]$ be a strong basic Π^1_2-formula of stage $\sigma < \beta < \omega^{\omega}$ where U and X do not occur in $\mathfrak{F}$, $U \notin PV(\mathfrak{F}[U^{\sigma}])$ and β is a limit ordinal. Let $\mathfrak{A}[t]$ be an arbitrary basic formula. Then we obtain

$$SR^* \vdash^{\Omega_{\sigma+2}}_{0} \forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$$

Proof. For $m := gr(\mathfrak{F}[U^{\sigma}])$ by Lemma 21.5 we have

$$SR^* \vdash^{2m}_{0} \mathfrak{F}[U^{\sigma}] \to (\forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[U^{\sigma}])$$

By an inference $(S\,3.1^*)$ we obtain

$$SR^* \vdash^{2m+1}_{0} \forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[U^{\sigma}]$$

By an inference $(S\,2.1^*)$ we obtain

$$SR^* \vdash^{2m+2}_{0} \forall X^{\beta}\, \mathfrak{F}[X] \to \forall X'\, \mathfrak{F}[X]$$

By Lemmata 21.1 and 21.4 we also have

(1) $$SR^* \vdash^{\Omega_{\sigma+1}\cdot 2}_{0} \forall X'\, \mathfrak{F}[X] \vee (\forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}])$$

From Lemmata 21.9 and 21.4 we obtain

(2) $$SR^* \vdash^{\Omega_{\sigma+1}\cdot 2}_{0} \forall X'\, \mathfrak{F}[X] \to (\forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}])$$

From (1) and (2) by a cut of grade 0 we obtain

(3) $$SR^* \vdash^{\Omega_{\sigma+2}}_{1} \forall X^{\beta}\, \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$$

since $\Omega_{\sigma+1} \cdot 2 \lhd \Omega_{\sigma+2}$. From (3) by the Cut Elimination Theorem 21.10 we obtain the assertion since $\omega_1(\Omega_{\sigma+2}) = \Omega_{\sigma+2}$.

THEOREM 21.12. (*Collapsing Theorem*) If F is a Π^1_2-formula of stage $\leq \sigma$ and $\gamma \in C_{\sigma}(\gamma)$, then we have

$$SR^* \vdash^{\gamma}_{0} F \Rightarrow SR^* \vdash^{\psi_{\sigma}\gamma}_{0} F$$

Proof by induction on γ corresponding to the proof of Theorem 13.4.

§ 22. - Interpretation of SR in SR^*

A basic formula F^β of SR^* is said to be a β-*interpretation* of a formula F of SR if β is a limit ordinal $< \omega^\omega$ and F^β is the result of replacing every free number variable in F by a numeral, every free predicate variable U in F by a U^α $(\alpha < \beta)$ and every strong predicate quantifier $\forall X$ in F by $\forall X^\beta$.

LEMMA 22.1. For every β-interpretation F^β of a formula F of SR we have

a) $st(F^\beta) < \beta$ if F is a weak formula,

b) $st(F^\beta) = \beta$ if F is a strong formula.

Proof by induction on the length of F.

THEOREM 22.2. (*Interpretation Theorem*) If $SR \vdash^{n} F$ and $0 < \beta = \omega^{n+1} \cdot \delta < \omega^\omega$, then

$$SR^* \vdash^{\Omega_{\beta+n}}_{0} F^\beta$$

for every β-interpretation F^β of F.

Proof by induction on n.

1. Let F be an axiom of SR. If F is one of the axioms $(Ax\,1)$-$(Ax\,4)$, then F^β is an axiom of SR^* and the assertion holds. If F is an axiom $(Ax\,5)$, then the assertion follows from Lemma 21.6. If F is an axiom $(\Pi^1_1\text{-}BI)$, then by Lemmata 22.1 and 21.9 there is a $\sigma < \beta$ such that $SR^* \vdash^{\Omega_{\sigma+1}\cdot 2}_{0} F^\beta$ holds. By Lemma 21.1 we obtain the assertion, since $\Omega_{\sigma+1} \cdot 2 \lhd \Omega_{\beta+n}$.

2. Let $SR \vdash^{n} F$ be derived by a structural inference. Then the assertion follows from the I.H. by the structural rule of SR^*.

3. Let $SR \vdash^{n} F$ be derived by a principal inference $(S\,1)$, $(S\,2.0)$, $(S\,3.0)$, $(S\,2.1)$ or $(S\,3.1)$, the principal part of which is not a weak formula $\forall X \mathfrak{F}[X]$. Then the assertion follows from the I.H. by a corresponding inference of SR^*.

4. Let $SR \vdash^{n} F$ be derived by an inference $(S\,2.1)$, the principal part of which is a weak formula. Then F^β is a formula $\mathfrak{G}\;[\forall X\,\mathfrak{F}[X]]$ and $n > 0$. By the I.H. we have

$$(1) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \mathfrak{G}\;[\mathfrak{F}[U^\sigma]]$$

where U does not occur in F^β and $\sigma := st(\forall X\,\mathfrak{F}[X]) < \beta$. Let $\mathfrak{G}_0[\mathfrak{F}[U^\sigma]]$ be the result of replacing every positive part $\forall X \mathfrak{F}[X]$ in $\mathfrak{G}\;[\mathfrak{F}[U^\sigma]]$ by $\perp$. Then we have

$$\mathfrak{G}\;[\mathfrak{F}[U^\sigma]] \vdash^{s} \forall X \mathfrak{F}[X] \vee \mathfrak{G}_0[\mathfrak{F}[U^\sigma]]$$

Therefore from (1) by the structural rule we obtain

$$(2) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \forall X \mathfrak{F}[X] \vee \mathfrak{G}_0[\mathfrak{F}[U^\sigma]]$$

From Lemmata 21.9, 21.4 and 21.1 we obtain

$$(3) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \forall X \mathfrak{F}[X] \to \mathfrak{G}_0[\mathfrak{F}[U^\sigma]]$$

From (2) and (3) by a cut of grade 0 and by the Cut Elimination Theorem 21.10 we obtain

$$SR^* \vdash^{\omega_1(\Omega_{\beta+n-1}+1)}_{0} \mathfrak{G}_0[\mathfrak{F}[U^\sigma]]$$

The assertion follows by an inference $(S\,2.1^*)$.

5. Let $SR \vdash^{n} F$ be derived by an inference $(S\,3.1)$, the principal part of which is a weak formula. Then F^β is a formula $\mathfrak{N}\,[\forall X \mathfrak{F}[X]]$ and $n > 0$. By the I.H. we have

$$(4) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \mathfrak{F}[U^\alpha] \to \mathfrak{N}\,[\forall X \mathfrak{F}[X]], \text{ for any } \alpha < \beta.$$

From Lemmata 21.9, 21.4 and 21.1 we obtain

$$(5) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \mathfrak{F}[U^\alpha] \vee \mathfrak{N}\,[\forall X \mathfrak{F}[X]]$$

From (4) and (5) the assertion follows by a cut and by the Cut Elimination Theorem 21.10, since $\omega_m(\Omega_{\beta+n-1}) \lhd \Omega_{\beta+n}$.

6. Let $SR \vdash^{n} F$ be derived by a cut. Then the assertion follows from the I.H. by a cut and by the Cut Elimination Theorem 21.10.

7. Let $SR \vdash^{n} F$ be derived by an inference $(\Pi^1_2\text{-}SR)$ and let F be a weak formula. Then F^β is a formula $\exists X(\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$ and $n > 0$. By the I.H. we have

$$(6) \qquad SR^* \vdash^{\Omega_{\beta+n-1}}_{0} \mathfrak{A} \subseteq \mathfrak{B}$$

By Lemma 22.1 a) we have $st(F^\beta) < \beta$. Then also

$$\sigma := st(\forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})) < \beta$$

and by Lemma 21.9 we have

$$SR^* \vdash^{\Omega_{\sigma+1}\cdot 2}_{0} \forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B}) \to \neg (\mathfrak{A} \subseteq \mathfrak{A} \wedge \mathfrak{A} \subseteq \mathfrak{B})$$

By the structural rule we obtain

(7) $$SR^* \left|\frac{\Omega_{\sigma+1} \cdot 2}{0}\right. \mathfrak{A} \subseteq \mathfrak{A} \to (\mathfrak{A} \subseteq \mathfrak{B} \to \exists X (\mathfrak{A} \subseteq X \subseteq \mathfrak{B}))$$

From Lemma 21.5 by an inference (S 2.0*) we obtain

(8) $$SR^* \left|\frac{\omega}{0}\right. \mathfrak{A} \subseteq \mathfrak{A}$$

The assertion follows from (6), (7) and (8) by cuts (after using the structural rule) and by the Cut Elimination Theorem 21.10.

8. Let $SR \left|\frac{n}{}\right. F$ be derived by an inference (Π^1_2-SR) and let F be a strong formula. Then F^β is a formula $\exists X^\beta (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$ and $n > 0$. Since all upper indices of free predicate variables in $\mathfrak{A}$ and $\mathfrak{B}$ are $< \beta = \omega^{n+1} \cdot \delta$, there is a $\sigma = \omega^n \cdot \delta_0 < \beta$ such that $\sigma > 0$ and all upper indices of free predicate variables in $\mathfrak{A}$ and $\mathfrak{B}$ are $< \sigma$. Let $\mathfrak{A}'$, $\mathfrak{B}$ be the results of replacing every predicate quantifier $\forall X^\beta$ in $\mathfrak{A}$, $\mathfrak{B}$ by $\forall X^\sigma$. Then $\mathfrak{A}' \subseteq \mathfrak{B}$ is a σ-interpretation of the premise of the given inference (Π^1_2-SR). Therefore by the I.H. we have

(9) $$SR^* \left|\frac{\Omega_{\sigma+n-1}}{0}\right. \mathfrak{A}' \subseteq \mathfrak{B}$$

By Lemma 22.1 b) the formula $\neg (\mathfrak{A}' \subseteq U^\sigma \subseteq \mathfrak{B}')$ has stage σ. Therefore by Lemma 22.11 we have

$$SR^* \left|\frac{\Omega_{\sigma+2}}{0}\right. \forall X^\beta \neg (\mathfrak{A}' \subseteq X \subseteq \mathfrak{B}) \to \neg (\mathfrak{A}' \subseteq \mathfrak{A}' \wedge \mathfrak{A}' \subseteq \mathfrak{B})$$

By the structural rule we obtain

(10) $$SR^* \left|\frac{\Omega_{\sigma+2}}{0}\right. \mathfrak{A}' \subseteq \mathfrak{A}' \to (\mathfrak{A}' \subseteq \mathfrak{B} \to \exists X^\beta (\mathfrak{A}' \subseteq X \subseteq \mathfrak{B}))$$

From Lemma 21.5 by an inference (S 2.0*) we obtain

(11) $$SR^* \left|\frac{\omega}{0}\right. \mathfrak{A}' \subseteq \mathfrak{A}'$$

From (9), (10) and (11) by cuts (after using the structural rule) and by the Cut Elimination Theorem 21.10 we obtain

(12) $$SR^* \left|\frac{\Omega_{\beta+n}}{0}\right. \exists X^\beta (\mathfrak{A}' \subseteq X \subseteq \mathfrak{B})$$

All predicate quantifiers $\forall X^\sigma_i$ in the formula $\exists X^\beta (\mathfrak{A}' \subseteq X \subseteq \mathfrak{B})$ are negatively occurring in it (according to Lemma 20.2). Therefore by the replacement rule c) we obtain the assertion

$$SR^* \left|\frac{\Omega_{\beta+n}}{0}\right. \exists X^\beta (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$$

DEFINITION. By the *zero-interpretation* of a weak formula F of SR we mean the result of replacing every free number variable in F by the numeral 0 and every free predicate variable U in F by U^0. By the *stage* of a weak formula of SR we mean the stage of its zero-interpretation.

THEOREM 22.3. (*Upper Bound Theorem* for SR) For every in SR derivable weak formula F of stage 0 there is an ordinal $\alpha < \psi\, 0\, (\Omega_{\omega^\omega})$ such that $SR^* \left|\frac{\alpha}{0}\right. F^0$ holds for the zero-interpretation F^0 of F.

Proof. By the assumption there is an n such that $SR \left|\frac{n}{}\right. F$ holds. Let F^0 be the zero-interpretation of F, $\beta := \omega^{n+1}$ and $\gamma := \Omega_{\beta+n}$.

Then by the Interpretation Theorem 22.2 we have $SR^* \left|\frac{\gamma}{0}\right. F^0$. Obviously $\gamma \in C_0(\gamma)$. Therefore by the Collapsing Theorem 21.12 we obtain $SR^* \left|\frac{\alpha}{0}\right. F^0$ for $\alpha := \psi\, 0\, \gamma < \psi\, 0\, (\Omega_{\omega^\omega})$.

REMARK. The result of Theorem 22.3 was first proved by G. TAKEUTI and M. YASUGI [31] with respect to ordinal diagrams. An earlier proof theoretical treatment of (Δ^1_2-CR) analysis was developed by S. FEFERMAN [9].

§ 23. – The Semiformal System SA'

The *formulas* of the system SA' are those formulas of A_2 which do not contain free number variables.

INDUCTIVE DEFINITION of the *isolated* and *complex* predicate quantifiers in a formula of SA'.

1. A prime formula contains no predicate quantifiers.

2. A predicate quantifier in a formula ($A \to B$) is isolated (complex) if it is isolated (complex) in A or B.

3. A predicate quantifier is isolated (complex) in a formula $\forall x\, \mathfrak{A}[x]$ if it is isolated (complex) in $\mathfrak{A}[t]$.

4. The predicate quantifier $\forall X$ ahead of a formula $\forall X \mathfrak{F}[X]$ is isolated if $\mathfrak{F}[U]$ is a Π^1_2-formula and $U \notin PV(\mathfrak{F}[U])$ holds for a free predicate variable U which does not occur in $\mathfrak{F}$. Otherwise it is a complex quantifier. Any other predicate quantifier in the formula $\forall X \mathfrak{F}[X]$ is isolated (complex) if it is isolated (complex) in $\mathfrak{F}[U]$.

According to this definition we have:

a) There are as well weak formulas as strong formulas $\forall X \mathfrak{F}[X]$ with an isolated predicate quantifier $\forall X$.

b) A formula $\forall X \mathfrak{F}[X]$ with an isolated predicate quantifier $\forall X$ is a Π^1_2-formula. It is a Σ^1_2-formula if and only if it is a weak formula.

c) Every formula $\forall X \mathfrak{F}[X]$ with a complex predicate quantifier $\forall X$ is a strong formula.

d) A formula $\forall X \mathfrak{F}[X]$ with a complex predicate quantifier $\forall X$ is a Π^1_2-formula if and only if $\mathfrak{F}[U]$ is a Π^1_2-formula. It is not a Σ^1_2-formula.

INDUCTIVE DEFINITION of the *degree* $dg(F)$ of a formula F of SA'.

1. $dg(F) := 0$ if F is a prime formula or a formula $\forall X \mathfrak{F}[X]$ with an isolated predicate quantifier $\forall X$.
2. $dg(A \to B) := \max\{dg(A), dg(B)\} + 1$.
3. $dg(\forall x \mathfrak{A}[x]) := dg(\mathfrak{A}[0]) + 1$.
4. $dg(\forall X \mathfrak{F}[X]) := dg(\mathfrak{F}[U]) + 1$ if $\forall X$ is a complex predicate quantifier.

According to this definition, a formula of degree 0 does not contain negatively occurring complex predicate quantifiers.

Axioms of SA':

$(Ax\,1)$, $(Ax\,2)$, $(Ax\,3)$ corresponding to these axioms of A_2.

Principal inferences of SA':

$(S\,1)$, $(S\,2.1)$, $(S\,3.0)$, $(S\,3.1)$ corresponding to these inferences of A_2.

$(S\,2.0')$ $\quad \mathfrak{G}[\mathfrak{A}[n]]$ for every numeral $n \vdash \mathfrak{G}[\forall x \mathfrak{A}[x]]$

$(S\,3.2')$ $\quad \mathfrak{A} \subseteq \mathfrak{B} \vee \mathfrak{N}[\forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})] \vdash \mathfrak{N}[\forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})]$ if $\mathfrak{A}[t]$ is a Π^1_2-formula and $\mathfrak{B}[t]$ is a Σ^1_2-formula. (In this case $\forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$ is a formula with an isolated predicate quantifier $\forall X$).

Cuts of SA' corresponding to the cuts of A_2. The *degree* of a cut is the degree of its cut formula.

INDUCTIVE DEFINITION of $SA' \overset{\gamma}{\underset{m}{\vdash}} F$ for $\gamma < \Omega_1$ $(\gamma \in T(\Omega))$ and $m < \omega$.

1. If F is an axiom of SA' then $SA' \overset{\gamma}{\underset{m}{\vdash}} F$ holds for all $\gamma < \Omega_1$ and $m < \omega$.
2. If $SA' \overset{\beta}{\underset{m}{\vdash}} F_i$ and $\beta < \gamma$ holds for every premise F_i of a principal inference of SA' or a cut of degree $< m$, then $SA' \overset{\gamma}{\underset{m}{\vdash}} F$ holds for the conclusion F of that inference.

LEMMA 23.1. $SA' \overset{\alpha}{\underset{m}{\vdash}} F$, $\alpha \leq \beta < \Omega_1$, $m \leq n < \omega \Rightarrow SA' \overset{\beta}{\underset{n}{\vdash}} F$.

This follows immediately from the inductive definition.

LEMMA 23.2. (*Replacement rules*)

a) $SA' \overset{\gamma}{\underset{m}{\vdash}} F \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} G$

if F and G are equivalent formulas.

b) $SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{F}[U] \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{F}[V]$

if U does not occur in F.

Proof by induction on γ.

LEMMA 23.3. (*Inversion rules*)

a) $SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{N}[(A \to B)] \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{N}[\neg A]$

$SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{N}[(A \to B)] \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{N}[B]$

b) $SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{G}[\forall x \mathfrak{A}[x]] \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{G}[\mathfrak{A}[t]]$

c) $SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{G}[\forall X \mathfrak{F}[X]] \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} \mathfrak{G}[\mathfrak{F}[U]]$

Proof by induction on γ using the replacement rules.

LEMMA 23.4. (*Structural rule*)

$$SA' \overset{\gamma}{\underset{m}{\vdash}} F,\ F \overset{s}{\vdash} G \Rightarrow SA' \overset{\gamma}{\underset{m}{\vdash}} G$$

Proof by induction on γ using the inversion rules corresponding to the proof of Lemma 12.5.

LEMMA 23.5. $SA' \overset{2m}{\underset{0}{\vdash}} \mathfrak{Q}[F, F]$ for every *NP*-form $\mathfrak{Q}$ if F is a formula of length $\leq m$.

Proof by induction on the length of F corresponding to the proof of Lemma 12.6.

LEMMA 23.6. For every formula $\mathfrak{A}[t]$:

$$SA' \overset{\omega+1}{\underset{0}{\vdash}} \forall x (\mathfrak{A}[x] \to \mathfrak{A}[x']) \to (\mathfrak{A}[0] \to \forall x \mathfrak{A}[x])$$

Proof corresponding to the proof of Lemma 12.7.

LEMMA 23.7. For every Π^1_2-formula $\mathfrak{A}[t]$ and every Σ^1_2-formula $\mathfrak{B}[t]$ we obtain

$$SA' \overset{\omega}{\underset{0}{\vdash}} \mathfrak{A} \subseteq \mathfrak{B} \to \neg \forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})$$

Proof. By Lemma 23.5 there is a $k < \omega$ such that

$$SA' \overset{k}{\underset{0}{\vdash}} \mathfrak{A} \subseteq \mathfrak{B} \vee (\mathfrak{A} \subseteq \mathfrak{B} \to \neg \forall X \neg (\mathfrak{A} \subseteq X \subseteq \mathfrak{B}))$$

holds. The assertion follows by an inference $(S\,3.2')$.

THEOREM 23.8 (*Cut Reduction Theorem*)

$$SA' \vdash^{\gamma}_{m+1} F \Rightarrow SA' \vdash^{\omega_m(\gamma)}_{1} F$$

Proof corresponding to the proof of Theorem 15.10.

§ 24. – The Ramified System SA^*

By an *SA^*-interpretation* of a formula F of SA' we mean the result of replacing every free predicate variable U in F by a U^α ($\alpha < \Omega_1$, $\alpha \in T(\Omega)$) and every complex predicate quantifier $\forall X$ in F by $\forall X^\beta$ where β is a limit ordinal $< \Omega_1$ of $T(\Omega)$ (the isolated predicate quantifiers in F remain unchanged). The *formulas* of SA^* are the SA^*-interpretations of the formulas of SA'.

According to this definition, for every formula F of SA^* there is a *corresponding formula* of SA' which results from F by cancelling all upper indices of free predicate variables and complex predicate quantifiers. For a formula F of SA^* we define $dg(F) := dg(F')$ and $PV(F) := PV(F')$ where F' is the corresponding formula of SA'. A formula of SA^* is said to be a *prime formula*, an *arithmetical formula*, a *weak formula* or a *strong formula*, a *Π^1_2-formula* or *Σ^1_2-formula* if the corresponding formula of SA' is such a formula.

P-forms, *N-forms*, *NP-forms* and $F \vdash^{s} G$ are defined in SA^* in the same way as before in A_2. We also use the same *syntactical variables*.

The *positively* and *negatively occurring* predicate quantifiers in a formula of SA^* and $F \vdash^{w} G$ for formulas F, G of SA^* are defined as before in § 20.

INDUCTIVE DEFINITION of the *level* $lv(F)$ of a formula F of SA^*.

1. $lv(F) = lv(\neg F) := 0$ if F is a constant prime formula.
2. $lv(U^\alpha(t)) = lv(\neg U^\alpha(t)) := \alpha$.
3. $lv(A \to B) := \max\{lv(\neg A), lv(B)\}$,
   $lv(\neg(A \to B)) := \max\{lv(A), lv(\neg B)\}$.
4. $lv(\forall x\, \mathfrak{A}[x]) := lv(\mathfrak{A}[0])$, $lv(\neg \forall x\, \mathfrak{A}[x]) := lv(\neg \mathfrak{A}[0])$.
5. $lv(\forall X\, \mathfrak{F}[X]) := lv(\mathfrak{F}[U^0])$, $lv(\neg \forall X\, \mathfrak{F}[X]) := lv(\mathfrak{F}[U^0]) + 1$.
6. $lv(\forall X^\beta\, \mathfrak{F}[X]) = lv(\neg \forall X^\beta\, \mathfrak{F}[X]) := \max\{\beta, lv(\mathfrak{F}[U^0])\}$.

LEMMA 24.1. (*Level Lemma*)

a) For every P-form $\mathfrak{G}$ and every N-form $\mathfrak{N}$ we have
$lv(\mathfrak{G}[A]) = \max\{lv(A), lv(\mathfrak{G}[\bot])\}$,
$lv(\mathfrak{N}[A]) = \max\{lv(\neg A), lv(\mathfrak{N}[\bot])\}$.

b) If U^α appears in a formula $\mathfrak{F}[U^\alpha]$ and $U \notin PV[U^\alpha]$), then $lv(\mathfrak{F}[U^\alpha]) = \max\{\alpha, lv(\mathfrak{F}[U^0])\}$.

c) $lv(\mathfrak{F}[U^\alpha]) \leq \max\{\alpha + \omega, lv(\mathfrak{F}[U^0])\}$ for every formula $\mathfrak{F}[U^\alpha]$.

d) If $lv(\forall X\, \mathfrak{F}[X]) = \sigma$, then also $lv(\mathfrak{F}[U^0]) = \sigma$.

e) $lv(\mathfrak{F}[U^\alpha]) \leq lv(\forall X^\beta\, \mathfrak{F}[X])$ for $\alpha < \beta$.

Proof corresponding to the proof of the Stage Lemma 20.1.

Axioms of SA^*:

($Ax\,1$), ($Ax\,2$) corresponding to these axioms of A_2.

($Ax\,3^*$) $\mathfrak{A}[A, B]$ if $dg(A) = dg(B) = 0$ and $A \vdash^{w} B$ holds.

Principal inferences of SA^*:

($S\,1$), ($S\,3.0$) corresponding to these inferences of A_2.

($S\,2.0^*$) $\mathfrak{G}[\mathfrak{A}[n]]$ for every numeral $n \vdash \mathfrak{G}[\forall x\, \mathfrak{A}[x]]$.

($S\,2.1^*$) $\mathfrak{G}_0[\mathfrak{F}[U^\alpha]]$ for all $\alpha \leq lv(\forall X\, \mathfrak{F}[X]) \vdash \mathfrak{G}[\forall X\, \mathfrak{F}[X]]$ if U does not occur in the conclusion and $\mathfrak{G}_0[\mathfrak{F}[U^\alpha]]$ is the result of replacing every positive part $\forall X\, \mathfrak{F}[X]$ in $\mathfrak{G}[\mathfrak{F}[U^\alpha]]$ by $\bot$.

($S\,2.2^*$) $\mathfrak{G}[\mathfrak{F}[U^\alpha]]$ for all $\alpha < \beta \vdash \mathfrak{G}[\forall X^\beta\, \mathfrak{F}[X]]$ if U does not occur in the conclusion.

($S\,3.1^*$) $\mathfrak{F}[U^\alpha] \to \mathfrak{N}[\forall X^\beta\, \mathfrak{F}[X]] \vdash \mathfrak{N}[\forall X^\beta\, \mathfrak{F}[X]]$ for $\alpha < \beta$.

Cuts of SA^* corresponding to the cuts of A_2. The *degree* of a cut is the degree of its cut formula.

INDUCTIVE DEFINITION of $SA^* \vdash^{\gamma}_{m} F$ for $\gamma \in T(\Omega)$ and $m < \omega$.

1. If F is an axiom of SA^*, then $SA^* \vdash^{\gamma}_{m}$ holds for all $\gamma \in T(\Omega)$ and $m < \omega$.

2. If $SA^* \vdash^{\beta_i}_{m} F_i$ and $\beta_i \lhd \gamma$ holds for every premise F_i of a principal inference of SA^* or a cut of degree $< m$, then $SA^* \vdash^{\gamma}_{m} F$ holds for the conclusion F of that inference.

3. ($\Omega_{\sigma+1}$-*rule*) $SA^* \vdash^{\gamma}_{m} F$ holds under the following assumptions:

a) $\forall X\, \mathfrak{F}[X]$ is a formula of level σ.

b) f is a fundamental function such that $\Omega_{\sigma+1} \in \mathrm{dom}(f)$ and $f(\Omega_{\sigma+1}) \trianglelefteq \gamma$.

c) $SA^* \vdash^{f(0)}_{m} \forall X\, \mathfrak{F}[X] \vee F$.

d) $SA^* \vdash^{\alpha}_{0} \mathfrak{G}[\forall X\, \mathfrak{F}[X]] \Rightarrow SA^* \vdash^{f(\alpha)}_{m} \mathfrak{G}[F]$.

for all $\alpha < \Omega_{\sigma+1}$ ($\alpha \in T(\Omega)$) and every *P-form* $\mathfrak{G}$ such that $\mathfrak{G}[\forall X\, \mathfrak{F}[X]]$ is a Π^1_2-formula of level σ.

LEMMA 24.2. $SA^* \vdash_m^\alpha F$, $\alpha \trianglelefteq \beta \in T(\Omega)$, $m \leq n < \omega \Rightarrow SA^* \vdash_n^\beta F$. This follows immediately from the inductive definition.

LEMMA 24.3. (*Replacement rules*)

a) $SA^* \vdash_m^\gamma F \Rightarrow SA^* \vdash_m^\gamma G$

if $F \vdash^w G$ holds.

b) $SA^* \vdash_m^\gamma \mathfrak{F}[U^\alpha] \Rightarrow SA^* \vdash_m^\gamma \mathfrak{F}[V^\alpha]$

if U does not occur in $\mathfrak{F}$.

Proof by induction on γ.

LEMMA 24.4. (*Inversion rules*)

a) $SA^* \vdash_m^\gamma \mathfrak{N}[(A \to B)] \Rightarrow SA^* \vdash_m^\gamma \mathfrak{N}[\neg A]$

$SA^* \vdash_m^\gamma \mathfrak{N}[(A \to B)] \Rightarrow SA^* \vdash_m^\gamma \mathfrak{N}[B]$.

b) $SA^* \vdash_m^\gamma \mathfrak{S}\ [\forall x\, \mathfrak{A}[x]] \Rightarrow SA^* \vdash_m^\gamma \mathfrak{S}\ [\mathfrak{A}[t]]$.

c) $SA^* \vdash_m^\gamma \mathfrak{S}\ [\forall X^\beta\, \mathfrak{F}[X]]$, $\alpha < \beta \Rightarrow SA^* \vdash_m^\gamma \mathfrak{S}\ [\mathfrak{F}[U^\alpha]]$.

Proof by induction on γ using the replacement rules.

LEMMA 24.5. (*Structural rule*)

$$SA^* \vdash_m^\gamma F,\ F \vdash^s G \Rightarrow SA^* \vdash_m^\gamma G$$

Proof by induction on γ using the inversion rules corresponding to the proof of Lemma 12.5.

LEMMA 24.6.

$SA^* \vdash_0^{2m} \mathfrak{Q}[F,F]$ for every *NP*-form $\mathfrak{Q}$ if $m \geq dg(F)$.

Proof by induction on $dg(F)$ corresponding to the proof of Lemma 12.6.

LEMMA 24.7. If $\forall X \mathfrak{F}[X]$ is a formula of level σ and $\mathfrak{A}[t]$ is an arbitrary formula, then we have

$$SA^* \vdash_0^{\Omega_{\sigma+1}\cdot 2} \forall X\, \mathfrak{F}[X] \to \mathfrak{F}[\mathfrak{A}]$$

Proof by the $\Omega_{\sigma+1}$-rule corresponding to the proof of Lemma 12.10.

THEOREM 24.8. (*Cut Elimination Theorem*) $SA^* \vdash_m^\gamma F \Rightarrow SA^* \vdash_0^{\omega_m(\gamma)} F$.

Proof corresponding to the proof of Theorem 13.3.

THEOREM 24.9. (*Collapsing Theorem*) If F is a Π^1_2-formula of level $\leq \sigma$ and $\gamma \in C_\sigma(\gamma)$, then we have

$$SA^* \vdash_0^\gamma F \Rightarrow SA^* \vdash_0^{\psi_\sigma\gamma} F$$

Proof by induction on γ corresponding to the proof of Theorem 13.4.

§ 25. – Interpretation of SA' in SA^*

A formula F^β of SA^* is said to be a β-*interpretation* of a formula F of SA' if β is a limit ordinal $< \Omega_1$ of $T(\Omega)$ and F^β is the result of the following replacements in F:

1. Every free predicate variable U in F has to be replaced by a U^α where α is an ordinal $< \beta$.

2. Every complex predicate quantifier $\forall X$ positively occurring in F has to be replaced by $\forall X^\eta$ where η is a limit ordinal $< \beta$.

3. Every complex predicate quantifier $\forall X$ negatively occurring in F has to be replaced by $\forall X^\beta$.

LEMMA 25.1. For every β-interpretation F^β of a formula F of SA' we have:

a) $lv(\forall X\, \mathfrak{F}[X]) < \beta$ if $\forall X\, \mathfrak{F}[X]$ is a positive part of F^β.

b) $lv(\forall X\, \mathfrak{F}[X]) \leq \beta$ if $\forall X\, \mathfrak{F}[X]$ is a negative part of F^β.

Proof. This holds because there are no negatively occurring complex predicate quantifiers in a formula $\forall X \mathfrak{F}[X]$.

THEOREM 25.2. (*Interpretation Theorem*)

If $SA' \vdash_1^\gamma F$ and $0 < \beta = \omega^{1+\gamma}\cdot\delta < \Omega_1$, then we obtain

$$SA^* \vdash_0^{\Omega_{\beta+\gamma+3}} F^\beta$$

for every β-interpretation F^β of F.

Proof by induction on γ.

1. Let F be an axiom of SA'. Then F^β is an axiom of SA^*, hence the assertion holds.

2. Let $SA' \mid\!\frac{\gamma}{1} F$ be derived by a principal inference of SA', the principal part of which is not of degree 0. Then the assertion follows from the I.H. by a corresponding inference of SA^*.

3. Let $SA' \mid\!\frac{\gamma}{1} F$ be derived by an inference (S 2.1), the principal part of which is of degree 0. Then F^β is a formula $\mathfrak{G}\ [\forall X\, \mathfrak{F}\, [X]]$ and by Lemma 25.1 a) we have $lv\, (\forall X\, \mathfrak{F}\, [X]) < \beta$.

Therefore by the I.H. we have

(1) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \mathfrak{G}\ [\mathfrak{F}\ [U^\alpha]] \quad \text{for all} \quad \alpha \leq lv\, (\forall X\, \mathfrak{F}\, [X])$$

Let $\mathfrak{G}_0\, [\mathfrak{F}\ [U^\alpha]]$ be the result of replacing every positive part $\forall X\, \mathfrak{F}\, [X]$ in $\mathfrak{G}\ [\mathfrak{F}\ [U^\alpha]]$ by $\perp$. Then we have

$$\mathfrak{G}\ [\mathfrak{F}\ [U^\alpha]] \mid\!\frac{s}{}\ \forall X\, \mathfrak{F}\, [X] \vee \mathfrak{G}_0\, [\mathfrak{F}\ [U^\alpha]]$$

Therefore from (1) by the structural rule we obtain

(2) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \forall X\, \mathfrak{F}\, [X] \vee \mathfrak{G}_0\, [\mathfrak{F}\ [U^\alpha]]$$

From Lemmata 24.7, 24.5, and 24.2 we obtain

(3) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \forall X\, \mathfrak{F}\, [X] \rightarrow \mathfrak{G}_0\, [\mathfrak{F}\ [U^\alpha]]$$

By a cut of degree 0 from (2) and (3) we obtain

$$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}+1}{1} \mathfrak{G}_0\, [\mathfrak{F}\ [U^\alpha]] \quad \text{for all} \quad \alpha \leq lv\, (\forall X\, \mathfrak{F}\, [X])$$

By an inference (S 2.1*) we obtain

$$SA^* \mid\!\frac{\Omega_{\beta+\gamma+3}}{1} \mathfrak{G}\ [\forall X\, \mathfrak{F}\, [X]]$$

By the Cut Elimination Theorem 24.8 we obtain the assertion since $\omega_1\, (\Omega_{\beta+\gamma+3}) = \Omega_{\beta+\gamma+3}$.

4. Let $SA' \mid\!\frac{\gamma}{1} F$ be derived by an inference (S 3.1), the principal part of which is a formula of degree 0. Then F^β is a formula $\mathfrak{N}[\forall X\, \mathfrak{F}\, [X]]$, and by the I.H. we have

(4) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \mathfrak{F}\ [U^\alpha] \rightarrow \mathfrak{N}\ [\forall X\, \mathfrak{F}\, [X]]$$

By Lemma 25.1 b) we have $\sigma := lv\, (\forall X\, \mathfrak{F}\, [X]) \leq \beta$. From Lemmata 24.7, 24.5 and 24.2 we obtain

(5) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \mathfrak{F}\ [U^\alpha] \vee \mathfrak{N}\ [\forall X\, \mathfrak{F}\, [X]]$$

since $\Omega_{\sigma+1} \cdot 2 \lhd \Omega_{\beta+\gamma+2}$. The assertion follows from (4) and (5) by a cut and the Cut Elimination Theorem 24.8.

5. Let $SA' \mid\!\frac{\gamma}{1} F$ be derived by an inference (S 3.2′). Then F^β is a formula $\mathfrak{N}\ [\forall X \neg\, (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})]$ and by the I.H. we have

(6) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \mathfrak{A} \subseteq \mathfrak{B} \vee \mathfrak{N}\ [\forall X \neg\, (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})]$$

From Lemmata 24.7, 24.5 and 24.2 we obtain

(7) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} \mathfrak{A} \subseteq \mathfrak{A} \rightarrow (\mathfrak{A} \subseteq \mathfrak{B} \rightarrow \mathfrak{N}\ [\forall X \neg\, (\mathfrak{A} \subseteq X \subseteq \mathfrak{B})])$$

From Lemma 24.6 by an inference (S 2.0*) we obtain

(8) $$SA^* \mid\!\frac{\omega}{0} \mathfrak{A} \subseteq \mathfrak{A}$$

From (6), (7) and (8) the assertion follows by cuts and by the Cut Elimination Theorem 24.8.

6. Let $SA' \mid\!\frac{\gamma}{1} F$ be derived by a cut of degree 0. Then we have $\gamma_0 < \gamma$ and a formula A of degree 0 such that

$$SA' \mid\!\frac{\gamma_0}{1} A \vee F, \qquad SA' \mid\!\frac{\gamma_0}{1} A \rightarrow F$$

hold. Since all upper indices of free predicate variables and positively occurring complex predicate quantifiers in F^β are $< \beta = \omega^{1+\gamma} \cdot \delta$, there is an ordinal $\sigma = \omega^{1+\gamma_0} \cdot \delta_0 < \beta$ such that $\sigma > 0$ and all upper indices of free predicate variables and positively occurring complex predicate quantifiers in F^β are $< \sigma$.

Let F^σ be the result of replacing all negatively occurring $\forall X^\beta$ in F^β by $\forall X^\sigma$. Then there is a σ-interpretation $A^* \rightarrow F^\sigma$ of $A \rightarrow F$ and a β-interpretation $A^* \vee F^\beta$ of $A \vee F$ where all complex predicate quantifiers in A^* have the upper index σ. From the I.H. and Lemma 24.2 we obtain

(9) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} A^* \rightarrow F^\sigma$$

(10) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} A^* \vee F^\beta$$

By the replacement rule a) from (9) we obtain

(11) $$SA^* \mid\!\frac{\Omega_{\beta+\gamma+2}}{0} A^* \rightarrow F^\beta$$

The assertion follows from (10) and (11) by a cut and the Cut Elimination Theorem 24.8.

§ 26. - Embedding SA and $\overline{SA}$ in SA'

By the *zero-interpretation* of a weak formula F of A_2 we mean the result of replacing every free number variable in F by the numeral 0 and every free predicate variable U in F by U°.

For a weak formula F of A_2 we define the *level* $lv(F) := lv(F^\circ)$ where F° is the zero-interpretation of F.

LEMMA 26.1. If $SA^* \vdash^{\alpha}_{0} F^\circ$ holds for the zero-interpretation F° of a weak formula F of SA' where $\alpha < \Omega_1$ and $lv(F) = 0$, then also $SA' \vdash^{\alpha}_{0} F$ holds.

Proof by induction on α corresponding to the proof of Lemma 17.1.

LEMMA 26.2. If $SA' \vdash^{\alpha}_{0} \mathfrak{F}[U]$ holds for an arithmetical formula $\mathfrak{F}[U]$ where $\alpha < \Omega_1$ and U does not occur in $\mathfrak{F}$, then $SA' \vdash^{\omega+\alpha}_{0} \mathfrak{F}[\mathfrak{A}]$ holds for an arbitrary formula $\mathfrak{A}[t]$ of SA'.

Proof by induction on α corresponding to the proof of Lemma 17.2.

LEMMA 26.3. $\gamma < \psi 0 (\Omega_{\Omega_1}) \Rightarrow \gamma \in C_0(\Omega_\gamma)$.

Proof. $\gamma < \psi 0 (\Omega_{\Omega_1})$ implies $\gamma \in C_0(\Omega_{\Omega_1})$. If $C_0(\Omega_\gamma) = C_0(\Omega_{\Omega_1})$, we have $\gamma \in C_0(\Omega_\gamma)$. Otherwise by Lemma 2.8 there is an ordinal δ such that $\Omega_\gamma \leq \delta < \Omega_{\Omega_1}$ and $\delta \in C_0(\Omega_\gamma)$. In this case we have $\Omega_\sigma \leq \delta < \Omega_{\sigma+1}$ and $\gamma \leq \sigma < \Omega_1$. Then $\delta \in C_0(\Omega_\gamma)$ implies $\sigma \in C_0(\Omega_\gamma)$ and $\gamma \in C_0(\Omega_\gamma)$ by Corollary 2.5.

INDUCTIVE DEFINITION of ordinals $\sigma(n)$ and $\tau(n)$.

$$\tau(0) := \varepsilon_0, \qquad \tau(n+1) := \psi 0 (\Omega_{\tau(n)} + 1)$$
$$\sigma(0) := \omega + 1, \qquad \sigma(n+1) := \psi 0 (\Omega_{\tau(n)})$$

LEMMA 26.4.

a) $\Omega_{\tau(n)} \in C_0(\Omega_{\tau(n)})$

b) $\sigma(n) < \tau(n) < \sigma(n+1) < \psi 0 (\Omega_{\Omega_1})$

c) $\tau(n)$ is the least ε-number $> \sigma(n)$.

Proof of a) by induction on n using Lemma 26.4. b) and c) follow from a) and the definitions.

A *numerical substitute* of a formula F of A_2 is the result of replacing every free number variable in F by a numeral.

THEOREM 26.5. (*Embedding Theorem*)

a) If $SA_k \vdash^{n} F$ holds, then there is an $m < \omega$ such that $SA' \vdash^{\sigma(k)+n}_{m} F'$ holds for every numerical substitute F' of F.

b) If $SA_k \vdash^{n}$ holds for a formula F of level 0, then there is an ordinal $\alpha < \sigma(k+1)$ such that $SA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

We prove a) and b) simultaneously by induction on $k+n$. Suppose $SA_k \vdash^{n} F$ and let F' be a numerical substitute of F.

1. Proof of a).

1.1. Let F be an axiom of SA. If F is one of the axioms (Ax 1)-(Ax 4) then F' is an axiom of SA' and the assertion a) holds. If F is an axiom (Ax 5) or (Π^1_2-SA) then the assertion a) follows from Lemmata 23.6 and 23.7.

1.2. Let $SA_k \vdash^{n} F$ be derived by a structural inference. Then the assertion follows from the I.H. by the structural rule of SA'.

1.3. Let $SA_k \vdash^{n} F$ be derived by a principal inference of SA. Then the assertion follows from the I.H. by a corresponding inference of SA'.

1.4. Let $SA_k \vdash^{n} F$ be derived by an inference (BR). Then we have $k > 0$ and $SA_{k-1} \vdash^{n} F_0$ for the premise F_0 of that inference. Let F'_0 be the numerical substitute of F_0 such that also $F'_0 \vdash F'$ is an inference (BR). By the I.H. for b) there is an $\alpha < \sigma(k)$ such that $SA' \vdash^{\alpha}_{0} F'_0$ holds for the arithmetical formula F'_0. It follows by Lemma 26.2 that $SA' \vdash^{\omega+\alpha}_{m} F'$ holds. Then also $SA' \vdash^{\sigma(k)+n}_{m} F'$ holds, since $\omega + \alpha < \sigma(k) + n$.

2. Proof of b). Let F be a weak formula of level 0. It is already proved that there is an $m < \omega$ such that $SA' \vdash^{\sigma(k)+n}_{m} F'$. It follows by the Cut Reduction Theorem 23.8 that $SA' \vdash^{\gamma}_{1} F'$ holds for $\gamma := \omega_m(\sigma(k)+n) < \tau(k)$. Let F° be the zero-interpretation of F' and $\sigma := \omega^{1+\gamma} + \gamma + 3 < \tau(k)$. Then by the Interpretation Theorem 25.2 we obtain $SA^* \vdash^{\Omega_\sigma}_{0} F^\circ$. By Lemma 26.3 we have $\Omega_\sigma \in C_0(\Omega_\sigma)$. Therefore by the Collapsing Theorem 24.9 we obtain $SA^* \vdash^{\alpha}_{0} F^\circ$ for $\alpha := \psi 0 (\Omega_\sigma) < \psi 0 (\Omega_{\tau(k)}) = \sigma(k+1)$. By Lemma 26.1 we obtain $SA' \vdash^{\alpha}_{0} F'$.

THEOREM 26.6. (*Upper Bound Theorem* for SA and $\overline{SA}$).

a) For every in SA derivable weak formula F of level 0 there is an ordinal $\alpha < \psi 0 (\Omega_{\varepsilon_0})$ such that $SA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

b) For every in $\overline{SA}$ derivable weak formula F of level 0 there is an ordinal $\alpha < \psi 0 (\Omega_{\Omega_1})$ such that $SA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

Proof. This follows immediately from Theorem 26.5 b).

REMARK. The result of Theorem 26.6 a) was first proved by G. TAKEUTI and M. YASUGI [31] with respect to ordinal diagrams.

Before H. FRIEDMAN [10] had established a reduction of Δ^1_2-CA to a system of iterated Π^1_1-comprehensions. The first purely proof theoretic argument for Friedman's result was given by FEFERMAN and SIEG [9a].

CHAPTER IV

WELL ORDERING PROOFS

Within the subsystems of analysis considered in chapters II and III we carry out well ordering proofs for various segments of $T(\Omega)$. From this we conclude that the ordinals which are established as upper bounds in chapters II and III are in fact the proof-theoretical ordinals of the respective formal systems.

§ 27. – Formalizing $T(\Omega)$ in Second Order Arithmetic

Since $T(\Omega)$ is a constructive system of ordinal notations with a decidable ordering relation, we can assume that we have a bijective mapping $\alpha \mapsto \overline{\alpha}$ from the set $T(\Omega)$ of ordinal terms onto a recursive set $\overline{T(\Omega)}$ of positive natural numbers with the following properties.

1. There is a 1-place recursive predicate T such that $T(n)$ is true if and only if $n \in \overline{T(\Omega)}$ holds.

2. There are 2-place recursive predicates $\prec$ and $\preccurlyeq$ such that $\prec(m, n)$ is true if and only if there are ordinals $\alpha, \beta \in T(\Omega)$ such that $\overline{\alpha} = m, \overline{\beta} = n$ and $\alpha < \beta$ holds and $\preccurlyeq(m, n)$ is true if and only if there are ordinals α, $\beta \in T(\Omega)$ such that $\overline{\alpha} = m$, $\overline{\beta} = n$ and $\alpha \leq \beta$ holds.

3. There is a 2-place recursive predicate C such that $C(m, n)$ is true if and only if there are ordinals $\sigma, \alpha \in T(\Omega)$ such that $\overline{\alpha} = m$, $\overline{\alpha} = n$ and $\alpha \in C_\sigma(\alpha)$ holds.

4. There is a 3-place recursive predicate K such that $K(k, m, n)$ is true if and only if there are ordinals $\alpha, \beta, \gamma \in T(\Omega)$ such that $\overline{\sigma} = k$, $\overline{\beta} = m$, $\overline{\gamma} = n$ and $\gamma \in K_\alpha \beta$ holds.

5. There is a 1-place recursive function N such that the value of $N(n)$ is $\overline{n}$ (for $n < \omega$).

6. There is a 1-place recursive function S such that the value of $S(n)$ is $\bar{\sigma}$ if there are ordinals α, $\sigma \in T(\Omega)$ such that $\bar{\alpha} = n$ and $\Omega_\sigma \leq \alpha < \Omega_{\sigma+1}$ holds. Otherwise the value of $S(n)$ is 0.

7. There is a 1-place recursive function dg such that the value of $dg(n)$ is the degree $dg(\alpha)$ if there is an ordinal $\alpha \in T(\Omega)$ such that $\bar{\alpha} = n$ holds. Otherwise the value of $dg(n)$ is 0.

8. For any other fundamental n-place $T(\Omega)$-function f of Chapter I there is an n-place recursive function f_T such that the value of $f_T(m_1, \ldots, m_n)$ is $\bar{\beta}$ if there are ordinals $\alpha_1, \ldots, \alpha_n, \beta \in T(\Omega)$ such that $\bar{\alpha}_1 = m_1, \ldots, \bar{\alpha}_n = m_n$ and $f(\alpha_1, \ldots, \alpha_n) = \beta$ holds. Otherwise the value of $f_T(m_1, \ldots, m_n)$ is 0.

We use the above denoted symbols of recursive predicates and recursive functions in our formal language of second order arithmetic and write

$$s \prec t, \quad s \preccurlyeq t, \quad u \in K_s t, \quad Nt, \quad St$$

instead of

$$\prec(s, t), \quad \preccurlyeq(s, t), \quad K(s, t, u), \quad N(t), \quad S(t)$$

where s, t, u are arbitrary terms of our formal language.

Instead of f_T we shall use in our formal language the corresponding notations of $T(\Omega)$ such that for instance $s + t$, ω^s, Ω_s and $\psi\, st$ are to be considered as terms of our formal language with respect to the assumed mapping of ordinals.

We also write α instead of $\bar{\alpha}$ in terms and formulas of our formal language such that for instance $\omega + t$, $\omega_n(\alpha)$ are to be considered as terms and $\mathfrak{A}[\varepsilon_0]$ is to be considered as a formula of our language.

Furthermore we use the following *abbreviations*.

$\mathfrak{A} = \mathfrak{B} :\equiv \forall x\, (\mathfrak{A}[x] \leftrightarrow \mathfrak{B}[x])$

$\mathfrak{A} \subseteq \mathfrak{B} :\equiv \forall x\, (\mathfrak{A}[x] \to \mathfrak{B}[x])$

$K_s t \subseteq \mathfrak{A} :\equiv \forall x\, (x \in K_s t \to \mathfrak{A}[x])$

$(\mathfrak{A} \cup \mathfrak{B})[t] :\equiv \mathfrak{A}[t] \vee \mathfrak{B}[t]$

$(\mathfrak{A} \cap \mathfrak{B})[t] :\equiv \mathfrak{A}[t] \wedge \mathfrak{B}[t]$

$(\mathfrak{A})_s[t] :\equiv \mathfrak{A}[t] \wedge St \preccurlyeq s$

$t \preccurlyeq \mathfrak{A} :\equiv \exists x\, (t \preccurlyeq x \wedge \mathfrak{A}[x])$

$\forall x \prec t\, \mathfrak{A}[x] :\equiv \forall x\, (x \prec t \to \mathfrak{A}[x])$

$Pr[\mathfrak{B}] :\equiv \forall y\, (\forall x \prec y\, \mathfrak{B}[x] \wedge T(y) \to \mathfrak{B}[y])$
($\mathfrak{B}$ is *progressive* with respect to $\prec$)

$Prg[\mathfrak{A}, \mathfrak{B}] :\equiv \forall y\, (\forall x \prec y\, (\mathfrak{A}[x] \to \mathfrak{B}[x]) \wedge \mathfrak{A}[y] \wedge T(y) \to \mathfrak{B}[y])$

$Ac[\mathfrak{A}, t] :\equiv \mathfrak{A}[t] \wedge T(t) \wedge \forall Y\, (Prg[\mathfrak{A}, Y] \to \forall z \prec t\, (\mathfrak{A}[z] \to Y(z)))$
(t is *accessible* in $\mathfrak{A}$ with respect to $\prec$)

$Wo[\mathfrak{A}] :\equiv \forall x\, (\mathfrak{A}[x] \to Ac[\mathfrak{A}, x])$
($\mathfrak{A}$ is *well ordered* with respect to $\prec$)

The following Lemmata in this section indicate formulas which are derivable in the formal system A_2.

LEMMA 27.1.

a) $\forall x\, ((\mathfrak{A}[x] \to \mathfrak{B}[x]) \leftrightarrow \mathfrak{C}[x]) \to (Prg[\mathfrak{A}, \mathfrak{B}] \leftrightarrow Pr[\mathfrak{C}])$

b) $T(s) \wedge Prg[\mathfrak{A}, \mathfrak{B}] \to Prg[(\mathfrak{A})_s, \mathfrak{B}]$

c) $Ac[\mathfrak{A}, t] \leftrightarrow Ac[(\mathfrak{A})_{St}, t]$

d) $\mathfrak{A} \subseteq \mathfrak{B} \wedge Wo[\mathfrak{B}] \to Wo[\mathfrak{A}]$

e) $Wo[\mathfrak{A}] \wedge \mathfrak{A}[t] \to T[t] \wedge \forall Y\, (Prg(\mathfrak{A}, Y] \to Y(t))$

Proof immediately by the definitions.

LEMMA 27.2. For $\mathfrak{B}[t] :\equiv Ac[\mathfrak{A}, t]$ we have

a) $\mathfrak{B} \subseteq \mathfrak{A} \wedge \mathfrak{B} \subseteq T$

b) $\mathfrak{B}[t] \to \forall x \prec t\, (\mathfrak{A}[x] \to \mathfrak{B}[x])$

c) $Wo[\mathfrak{B}]$

Proof immediately by the definitions.

LEMMA 27.3. $Wo[\mathfrak{A}] \wedge Wo[\mathfrak{B}] \to Wo[\mathfrak{A} \cup \mathfrak{B}]$.

Proof. We have

$$\begin{cases} Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \forall x \prec a\, (\mathfrak{A}[x] \to U(x)) \wedge b \prec a \\ \to (\forall x \prec b\, (\mathfrak{B}[x] \to U(x)) \wedge \mathfrak{B}[b] \to U(b)) \end{cases}$$

Let $\mathfrak{B}_a[b]$ be the formula $\mathfrak{B}[b] \wedge b \prec a$. Then we obtain

$$Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \forall x \prec a\, (\mathfrak{A}[x] \to U(x)) \to Prg[\mathfrak{B}_a, U]$$

and by Lemma 27.1 e)

$$(1) \quad \begin{cases} Wo[\mathfrak{B}] \wedge Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \forall x \prec a\, (\mathfrak{A}[x] \to U(x)) \\ \to \forall x \prec a\, (\mathfrak{B}[x] \to U(x)) \end{cases}$$

We also have

$$(2)\quad \begin{cases} Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \forall x \prec a(\mathfrak{A}[x] \to U(x)) \wedge \forall x \prec a(\mathfrak{B}[x] \to U(x)) \\ \to (\mathfrak{A}[a] \wedge T(a) \to U(a)) \end{cases}$$

From (1) and (2) we obtain

$$Wo[\mathfrak{B}] \wedge Prg(\mathfrak{A} \cup \mathfrak{B}\, U] \to Prg[\mathfrak{A}, U]$$

and by Lemma 27.1 e)

$$(3)\qquad Wo[\mathfrak{A}] \wedge Wo[\mathfrak{B}] \wedge Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \mathfrak{A}[a] \to U(a)$$

In the same way we obtain

$$(4)\qquad Wo[\mathfrak{A}] \wedge Wo[\mathfrak{B}] \wedge Prg[\mathfrak{A} \cup \mathfrak{B}, U] \wedge \mathfrak{B}[a] \to U(a)$$

From (3) and (4) we obtain

$$Wo[\mathfrak{A}] \wedge Wo[\mathfrak{B}] \wedge Prg[\mathfrak{A} \cup \mathfrak{B}\, U] \to \forall x((\mathfrak{A} \cup \mathfrak{B})[x] \to U(x))$$

which yields the assertion

$$Wo[\mathfrak{A}] \wedge Wo[\mathfrak{B}] \to Wo[\mathfrak{A} \cup \mathfrak{B}]$$

LEMMA 27.4. For any formula $\mathfrak{A}[t]$ there is a formula $\mathfrak{A}'[t]$ such that the following formulas are derivable:

a) $\mathfrak{A}'[t] \to \forall x \prec \omega^t\, \mathfrak{A}[x]$

b) $Pr[\mathfrak{A}] \to Pr[\mathfrak{A}']$

Proof. Let $\mathfrak{A}'[t]$ be the formula

$$\forall y(\forall x \prec y\, \mathfrak{A}[x] \to \forall x \prec y + \omega^t\, \mathfrak{A}[x])$$

Then the formula a) is derivable. To prove b) we use the abbreviation

$$\mathfrak{C}[a, b, c] := Pr[\mathfrak{A}] \wedge \forall z \prec a\, \mathfrak{A}'[z] \wedge \forall x \prec b\, \mathfrak{A}[x] \wedge c \prec \omega^a$$

and prove by induction on $dg(c)$

$$(1)\qquad \mathfrak{C}[a, b, c] \to \mathfrak{A}[b + c]$$

If $c = NO$, then $\mathfrak{A}[b + c]$ follows from $Pr[\mathfrak{A}] \wedge \forall x \prec b\, \mathfrak{A}[x]$.

Otherwise we have $c = \omega^u + v$, $u \prec a$, $v \prec c$ and $dg(v) < dg(c)$. In this case $\forall z \prec a\, \mathfrak{A}'[z] \wedge \forall x \prec b\, \mathfrak{A}[x]$ implies $\forall x \prec b + \omega^u\, \mathfrak{A}[x]$. Hence we obtain

$$\mathfrak{C}[a, b, c] \to \mathfrak{C}[a, b + \omega^u, v]$$

Since $dg(v) < dg(c)$, by the I.H. we obtain

$$\mathfrak{C}[a, b, c] \to \mathfrak{A}[b + \omega^u + v]$$

which completes the proof of (1). From (1) we obtain

$$(2)\qquad Pr[\mathfrak{A}] \wedge \forall z \prec a\, \mathfrak{A}'[z] \wedge \forall x \prec b\, \mathfrak{A}[x] \to \forall x \prec b + \omega^a\, \mathfrak{A}[x]$$

From (2) we obtain

$$Pr[\mathfrak{A}] \wedge \forall z \prec a\, \mathfrak{A}'[z] \to \mathfrak{A}'[a]$$

which yields the assertion b).

LEMMA 27.5. $Pr[\mathfrak{A}] \to \mathfrak{A}[\omega_n(0)]$.

Proof by induction on n. Obviously, the assertion holds for $n = 0$. Now we prove the assertion for $n + 1$ under the assumption that it holds for n. By Lemma 27.4 there is a formula $\mathfrak{A}'[t]$ such that we have

$$(1)\qquad \mathfrak{A}'[\omega_n(0)] \to \forall x \prec \omega_{n+1}(0)\, \mathfrak{A}[x]$$

$$(2)\qquad Pr[\mathfrak{A}] \to Pr[\mathfrak{A}']$$

By our assumption we have

$$(3)\qquad Pr[\mathfrak{A}'] \to \mathfrak{A}'[\omega_n(0)]$$

From (1), (2) and (3) we obtain

$$Pr[\mathfrak{A}] \to \forall x \prec \omega_{n+1}(0)\, \mathfrak{A}[x]$$

which implies the assertion

$$Pr[\mathfrak{A}] \to \mathfrak{A}[\omega_{n+1}(0)]$$

for $n + 1$.

DEFINITIONS.

$$M_s^{\mathfrak{A}}[t] := St \preccurlyeq s \wedge \forall x \prec \Omega_s (\mathfrak{A}[x] \to K_{Sx} t \subseteq \mathfrak{A})$$

$$W_s^{\mathfrak{A}}[t] := Ac[M_s^{\mathfrak{A}}, t]$$

LEMMA 27.6.

a) $M_{NO}^{\mathfrak{A}} = (T)_{NO}$

b) $M_s^{\mathfrak{A}}[t] \wedge s \preccurlyeq u \to K_u t \subseteq M_s^{\mathfrak{A}}$

c) $v \prec \omega^u + v \to (M_s^{\mathfrak{A}}[\omega^u + v] \leftrightarrow M_s^{\mathfrak{A}}[u] \wedge M_s^{\mathfrak{A}}[v])$

d) $M_s^{\mathfrak{A}}[\Omega_s] \leftrightarrow M_s^{\mathfrak{A}}[s]$

e) $St = s \wedge M_s^{\mathfrak{A}}[t] \to M_s^{\mathfrak{A}}[\Omega_s]$

f) $\forall x \prec \Omega_s (\mathfrak{A}[x] \leftrightarrow \mathfrak{B}[x]) \to M_s^{\mathfrak{A}} = M_s^{\mathfrak{B}}$

Proof immediately by the definitions.

LEMMA 27.7.

a) $W_s^{\mathfrak{A}} \subseteq M_s^{\mathfrak{A}} \wedge M_s^{\mathfrak{A}} \subseteq T$

b) $T(s) \to W_s^{\mathfrak{A}}[NO]$

c) $W_s^{\mathfrak{A}}[t] \leftrightarrow \forall x \prec t (M_s^{\mathfrak{A}}[x] \to W_s^{\mathfrak{A}}[x]) \wedge M_s^{\mathfrak{A}}[t]$

d) $St = s \wedge W_s^{\mathfrak{A}}[t] \to W_s^{\mathfrak{A}}[\Omega_s] \wedge W_s^{\mathfrak{A}}[s]$

e) $Wo[W_s^{\mathfrak{A}}]$

f) $\forall x \prec \Omega_s (\mathfrak{A}[x] \leftrightarrow \mathfrak{B}[x]) \to W_s^{\mathfrak{A}} = W_s^{\mathfrak{B}}$

Proof immediately by the definitions.

§ 28. - Distinguished Predicates

Similarly to the definition of distinguished sets (ausgezeichnete Mengen) in [1] we define

$D[\mathfrak{A}] := \mathfrak{A} \subseteq T \wedge \forall x (x \preccurlyeq \mathfrak{A} \to (\mathfrak{A})_{Sx} = W_{Sx}^{\mathfrak{A}})$
($\mathfrak{A}$ is a *distinguished predicate*)

$\mathfrak{W}[t] := \exists X (D[X] \wedge X(t))$
($\mathfrak{W}$ turns out be the maximal distinguished predicate)

$W_s^U[t]$ and $D[U]$ are weak formulas, $\mathfrak{W}[t]$ is a Σ_2^1-formula.

The Lemmata in this section indicate formulas which are derivable in the formal system A_2 with the additional axiom schema of arithmetical comprehension

$$\exists X \forall y (X(y) \leftrightarrow \mathfrak{A}[y])$$

for any arithmetical formula $\mathfrak{A}[t]$.

LEMMA 28.1. $D[\mathfrak{A}] \to Wo[\mathfrak{A}]$.

Proof. $D[\mathfrak{A}] \wedge \mathfrak{A}[a]$ implies

1) $(\mathfrak{A})_{Sa} = W_{Sa}^{\mathfrak{A}}$	by the definition of $D[\mathfrak{A}]$,
2) $W_{Sa}^{\mathfrak{A}}[a]$	by 1),
3) $Ac[W_{Sa}^{\mathfrak{A}}, a]$	by 2) and Lemma 27.7 e),
4) $Ac[\mathfrak{A}, a]$	by 1), 3) and Lemma 27.1 c).

Hence we obtain

$$D[\mathfrak{A}] \to \forall x (\mathfrak{A}[x] \to Ac[\mathfrak{A}, x])$$

which yields the assertion $D[\mathfrak{A}] \to Wo[\mathfrak{A}]$.

LEMMA 28.2.

$$D[U] \wedge D[V] \wedge t \preccurlyeq U \wedge t \preccurlyeq V \to (U)_{St} = (V)_{St}.$$

Proof.

$$D[U] \wedge D[V] \wedge t \preccurlyeq U \wedge t \preccurlyeq V \wedge \forall x \prec a ((U \cup V)[x] \to (U \cap V)[x]) \wedge Sa \preccurlyeq St$$

implies

1) $\forall x \prec a (U(x) \leftrightarrow V(x))$	
2) $(U)_{Sa} = W_{Sa}^U \wedge (V)_{Sa} = W_{Sa}^V$	by the definition of $D[U]$ and $D[V]$,
3) $W_{Sa}^U = W_{Sa}^V$	by 1) and Lemma 27.7 f),
4) $(U)_{Sa} = (V)_{Sa}$	by 2) and 3),
5) $(U \cup V)[a] \to (U \cap V)[a]$	by 4).

Hence we obtain

(1) $D[U] \wedge D[V] \wedge t \preccurlyeq U \wedge t \preccurlyeq V \to Prg[(U \cup V)_{St}, U \cap V]$

By Lemmata 28.1, 27.3 and 27.1 d) we obtain

(2) $$D[U] \wedge D[V] \to Wo[(U \cup V)_{St}]$$

From (1) and (2) by arithmetical comprehension we obtain

$$D[U] \wedge D[V] \wedge t \preccurlyeq U \wedge t \preccurlyeq V \to \forall x((U \cup V)_{St}[x] \to (U \cap V)[x])$$

which implies the assertion

$$D[U] \wedge D[V] \wedge t \preccurlyeq U \wedge t \preccurlyeq V \to (U)_{St} = (V)_{St}$$

COROLLARY 28.2. $D[U] \wedge t \preccurlyeq U \to (U)_{St} = (\mathfrak{W})_{St}$

Proof. By Lemma 28.2 we have

$$D[U] \wedge t \preccurlyeq U \wedge Sa \preccurlyeq St \wedge D[V] \wedge V(a) \to U(a)$$

which by the definition of $\mathfrak{W}[a]$ implies

(1) $$D[U] \wedge t \preccurlyeq U \wedge Sa \preccurlyeq St \wedge \mathfrak{W}[a] \to U(a)$$

We also have

(2) $$D[U] \wedge U(a) \to \mathfrak{W}[a]$$

The assertion

$$D[U] \wedge t \preccurlyeq U \to (U)_{St} = (\mathfrak{W})_{St}$$

follows from (1) and (2).

LEMMA 28.3. $D[\mathfrak{W}]$.

Proof. $D[U] \wedge a \preccurlyeq U$ implies

1) $(U)_{Sa} = W^U_{Sa}$ by the definition of $D[U]$,
2) $(U)_{Sa} = (\mathfrak{W})_{Sa}$ by Corollary 28.2,
3) $W^U_{Sa} = W^{\mathfrak{W}}_{Sa}$ by Lemma 27.7 f) and 2),
4) $(\mathfrak{W})_{Sa} = W^{\mathfrak{W}}_{Sa}$ by 1), 2), 3).

Hence we obtain

$$D[U] \wedge a \preccurlyeq U \to (\mathfrak{W})_{Sa} = W^{\mathfrak{W}}_{Sa}$$

which implies

(1) $$\forall x(x \preccurlyeq \mathfrak{W} \to (\mathfrak{W})_{Sx} = W^{\mathfrak{W}}_{Sx})$$

We also have

(2) $$\mathfrak{W} \subseteq T$$

The assertion $D[\mathfrak{W}]$ follows from (1) and (2).

LEMMA 28.4. $D[\mathfrak{A}] \wedge \mathfrak{A}[t] \to \mathfrak{A}[\Omega_{St}] \wedge \mathfrak{A}[St]$.

Proof. $D[\mathfrak{A}] \wedge \mathfrak{A}[t]$ implies

1) $(\mathfrak{A})_{St} = W^{\mathfrak{A}}_{St}$ by the definition of $D[\mathfrak{A}]$,
2) $W^{\mathfrak{A}}_{St}[t]$ by 1),
3) $W^{\mathfrak{A}}_{St}[\Omega_{St}] \wedge W^{\mathfrak{A}}_{St}[St]$ by 2) and Lemma 27.7 d),
4) $\mathfrak{A}[\Omega_{St}] \wedge \mathfrak{A}[St]$ by 1) and 3).

LEMMA 28.5.

$$D[\mathfrak{A}] \wedge \mathfrak{A}[\Omega_{St}] \wedge \forall x \prec t(M^{\mathfrak{A}}_{St}[x] \to \mathfrak{A}[x]) \wedge M^{\mathfrak{A}}_{St}[t] \to \mathfrak{A}[t]$$

Proof.

$$D[\mathfrak{A}] \wedge \mathfrak{A}[\Omega_{St}] \wedge \forall x \prec t(M^{\mathfrak{A}}_{St}[x] \to \mathfrak{A}[x]) \wedge M^{\mathfrak{A}}_{St}[t]$$

implies

1) $(\mathfrak{A})_{St} = W^{\mathfrak{A}}_{St}$ by the definition of $D[\mathfrak{A}]$,
2) $\forall x \prec t(M^{\mathfrak{A}}_{St}[x] \to W^{\mathfrak{A}}_{St}[x])$ by 1),
3) $W^{\mathfrak{A}}_{St}[t]$ by 2) and $M^{\mathfrak{A}}_{St}[t]$ and Lemma 27.7 c),
4) $\mathfrak{A}[t]$ by 1) and 3).

LEMMA 28.6. $D[U] \wedge U(s) \wedge U(t) \to U(\omega^s + t)$.

Proof. We can assume $t \prec \omega^s + t$, since otherwise the assertion is trivial. We use the following abbreviations.

$$\mathfrak{A}_a[b] := b \prec \omega^a + b \to U(\omega^a + b)$$
$$\mathfrak{B}[a] := \forall y(U(y) \to \mathfrak{A}_a[y])$$
$$\mathfrak{C}_1[a] := D[U] \wedge \forall x \prec a(U(x) \to \mathfrak{B}[x]) \wedge U(a)$$
$$\mathfrak{C}_2[a, b] := \mathfrak{C}_1[a] \wedge \forall y \prec b(U(y) \to \mathfrak{A}_a[y]) \wedge U(b) \wedge b \prec \omega^a + b$$
$$\mathfrak{C}_3[a, b, c] := \mathfrak{C}_2[a, b] \wedge c \prec \omega^a + b \wedge M^U_{Sa}[c]$$

By induction on $dg(c)$ we prove

(1) $$\mathfrak{C}_3[a, b, c] \to U(c)$$

We have the following three cases.

1. $c \preccurlyeq a$. Then $U(c)$ follows from $U(a) \wedge M^U_{Sa}[c] \wedge D[U]$ by Lemma 27.7 c).

2. $a \prec c \prec \omega^a$. Then we have $c = \omega^{c_1} + c_2$, $c_1 \prec a$, $c_2 \prec c$ and $dg(c_i) < dg(c)$. In this case $\mathfrak{C}_3[a, b, c]$ implies $\mathfrak{C}_3[a, b, c_i]$ and by the I.H. $U(c_i)$ for $i = 1, 2$. Together with $\forall x \prec a(U(x) \to \mathfrak{B}[x])$ we obtain $\mathfrak{B}[c_1]$ and $U[c]$.

3. $c = \omega^a + b_0$, $b_0 \prec b$ and $dg(b_0) < dg(c)$. In this case $\mathfrak{C}_3[a, b, c]$ implies $\mathfrak{C}_3[a, b, b_0]$ and by the I.H. $U(b_0)$. Together with $\forall y \prec b(U(y) \to \mathfrak{A}_a[y])$ we obtain $U(c)$.

From (1) we obtain

(2) $$\mathfrak{C}_2[a, b] \to \forall x \prec \omega^a + b(M^U_{Sa}[x] \to U(x))$$

By Lemma 27.7 a) and d) we also have

(3) $$\mathfrak{C}_3[a, b] \to S(\omega^a + b) = Sa \wedge U(\Omega_{Sa}) \wedge M^U_{Sa}[\omega^a + b]$$

From (2) and (3) by Lemma 28.5 we obtain

$$\mathfrak{C}_2[a, b] \to U(\omega^a + b)$$

By the definition of $\mathfrak{C}_2[a, b]$ we obtain

$$\mathfrak{C}_1[a] \to Prg[U, \mathfrak{A}_a]$$

By Lemma 28.1 and arithmetical comprehension we obtain

$$\mathfrak{C}_1[a] \to \forall y(U(y) \to \mathfrak{A}_a[y])$$

By the definitions of $\mathfrak{B}$ and $\mathfrak{C}_1[a]$ we obtain

$$D[U] \to Prg[U, \mathfrak{B}]$$

By Lemma 28.1 and arithmetical comprehension we obtain

$$D[U] \wedge U(s) \to \mathfrak{B}[s]$$

which implies the assertion

$$D[U] \wedge U(s) \wedge U(t) \to U(\omega^s + t)$$

COROLLARY 28.6. $\mathfrak{W}[s] \wedge \mathfrak{W}[t] \to \mathfrak{W}[\omega^s + t]$.

Proof. We can assume $t \prec \omega^s + t$. Then by Lemma 28.2 we have

$$D[U] \wedge U(s) \wedge D(V) \wedge V(t) \to U(s) \wedge U(t)$$

By Lemma 28.6 we obtain

$$D[U] \wedge U(s) \wedge D[V] \wedge V(t) \to U(\omega^s + t)$$

which implies the assertion

$$\mathfrak{W}[s] \wedge \mathfrak{W}[t] \to \mathfrak{W}[\omega^s + t]$$

LEMMA 28.7.

a) $D[\mathfrak{A}] \wedge \exists x \mathfrak{A}[x] \to \mathfrak{A}[NO]$

b) $D[U] \wedge \exists x U(x) \wedge St \preccurlyeq s \wedge K_s t \subseteq U \to U(t)$

c) $\exists x \mathfrak{W}[x] \wedge St \preccurlyeq s \wedge K_s t \subseteq \mathfrak{W} \to \mathfrak{W}[t]$

Proof of a). $D[\mathfrak{A}] \wedge \mathfrak{A}[a]$ implies

1) $(\mathfrak{A})_{Sa} = W^{\mathfrak{A}}_{Sa}$ by the definition of $D[\mathfrak{A}]$,

2) $\mathfrak{A}[NO]$ by 1), since we have $W^{\mathfrak{A}}_{Sa}[NO]$ by Lemma 27.7 b).

Proof of b) by induction on $dg(t)$. We have the following three cases.

1. $t = NO$. Then we obtain $U(t)$ by a).

2. $t = \omega^{t_1} + t_2$, $dg(t_i) < dg(t)$ $(i = 1, 2)$. In this case $K_s t \subseteq U$ implies $K_s t_i \subseteq U$ and by the I.H. $U(t_i)$ for $i = 1, 2$. By Lemma 28.6 we obtain $U(t)$.

3. $t = \bar{\alpha}$ where $\alpha \in E$. Then the assertion is trivial, since in this case $K_\sigma \alpha = \{\alpha\}$ holds for $\sigma \geq S\alpha$.

Proof of c) corresponding to the proof of b) by a) and Corollary 28.6.

LEMMA 28.8. $D[U] \wedge U(\Omega_s) \wedge U(t) \wedge C(s, t) \to U(\psi s t)$.

Proof. We use the following abbreviations.

$$\mathfrak{A}[b] := \forall x(U(\Omega_x) \wedge C(x, b) \to U(\psi x b))$$
$$\mathfrak{C}_1[b] := D[U] \wedge \forall y \prec b(U(y) \to \mathfrak{A}[y]) \wedge U(b)$$
$$\mathfrak{C}_2[a, b] := \mathfrak{C}_1[b] \wedge U(\Omega_a) \wedge C(a, b)$$
$$\mathfrak{C}_3[a, b, c] := \mathfrak{C}_2[a, b] \wedge c \prec \psi a b \wedge M^U_a[c]$$

By induction on $dg(c)$ we prove

(1) $$\mathfrak{C}_3[a, b, c] \to U(c)$$

We have the following three cases.

1. $c \preccurlyeq \Omega_a$. Then $U(c)$ follows from $U(\Omega_a) \wedge M_a^U[c]$.

2. $\Omega_a \prec c = \omega^{c_1} + c_2$, $dg(c_i) < dg(c)$ $(i = 1, 2)$. In this case $\mathfrak{C}_3[a, b, c]$ implies $\mathfrak{C}_3[a, b, c_i]$ and by the I.H. $U(c_i)$ for $i = 1, 2$. By Lemma 28.6 we obtain $U(c)$.

3. $c = \psi a b_0$, $C(a, b_0)$, $b_0 \prec b$ and $dg(b_0) < dg(c)$. In this case we prove

(1′) $\mathfrak{C}_3[a, b, c] \wedge \forall x \prec d(U(x) \to K_{Sx} b_0 \subseteq U) \wedge U(d) \to K_{Sd} b_0 \subseteq U$

If $Sd \prec a$, then $K_{Sd} b_0 \subseteq U$ follows from $M_a^U[\psi a b_0] \wedge U(d)$. Now suppose $a \preccurlyeq Sd \wedge e \in K_{Sd} b_0$. Then $C(a, b_0) \wedge C(a, b)$ implies $C(Sd, b_0) \wedge C(Sd, b)$ and by Lemma 6.3 $e \prec \psi(Sd) b_0 \prec \psi(Sd) b$. $U(d) \wedge \forall x \prec d(U(x) \to K_{Sx} b_0 \subseteq U)$ and $e \in K_{Sd} b_0$ implies $U(\Omega_{Sd}) \wedge M_{Sd}^U[e]$. Hence we obtain $\mathfrak{C}_3[Sd, b, e]$. Since $dg(e) \leq dg(b_0) < dg(c)$, by the I.H. we obtain $U(e)$ which completes the proof of (1′).

Let $\mathfrak{B}[d]$ be the formula $K_{Sd} b_0 \subseteq U$. Then from (1′) we obtain

$$\mathfrak{C}_3[a, b, c] \to Prg[U, \mathfrak{B}]$$

By Lemma 28.1 and arithmetical comprehension we obtain

$$\mathfrak{C}_3[a, b, c] \to M_{Sb}^U[b_0]$$

$U(b) \wedge M_{Sb}^U[b_0] \wedge b_0 \prec b$ implies $U(b_0)$. Together with

$$\forall y \prec b(U(y) \to \mathfrak{A}[y]) \wedge U(\Omega_a) \wedge C(a, b_0)$$

we obtain $\mathfrak{A}[b_0]$ and $U(\psi a b_0)$ which completes the proof of (1).

From (1) we obtain

(2) $$\mathfrak{C}_2[a, b] \to \forall x \prec \psi a b(M_a^U[x] \to U(x))$$

We also have

(3) $$\mathfrak{C}_2[a, b] \to U(\Omega_a) \wedge M_a^U[\psi a b]$$

From (2) and (3) by Lemma 28.5 we obtain

$$\mathfrak{C}_2[a, b] \to U(\psi a b)$$

By the definition of $\mathfrak{C}_2[a, b]$ we obtain

$$\mathfrak{C}_1[b] \to \forall x(U(\Omega_x) \wedge C(x, b) \to U(\psi x b))$$

By the definition of $\mathfrak{A}$ and $\mathfrak{C}_1[b]$ we obtain

$$D[U] \to Prg[U, \mathfrak{A}]$$

By Lemma 28.1 and arithmetical comprehension we obtain

$$D[U] \wedge U(t) \to \mathfrak{A}[t]$$

which implies the assertion

$$D[U] \wedge U(\Omega_s) \wedge U(t) \wedge C(s, t) \to U(\psi s t)$$

COROLLARY 28.8. $\mathfrak{W}[\Omega_s] \wedge \mathfrak{W}[t] \wedge C(s, t) \to \mathfrak{W}[\psi s t]$

Proof. By Lemma 28.2 we have

$$D[U] \wedge U(\Omega_s) \wedge D[V] \wedge V(t) \to (U(\Omega_s) \wedge U(t)) \vee (V(\Omega_s) \wedge V(t))$$

By Lemma 28.8 we obtain

$$D[U] \wedge U(\Omega_s) \wedge D[V] \wedge V(t) \wedge C(s, t) \to U(\psi s t) \vee V(\psi s t)$$

which implies the assertion

$$\mathfrak{W}[\Omega_s] \wedge \mathfrak{W}[t] \wedge C(s, t) \Rightarrow \mathfrak{W}[\psi s t]$$

LEMMA 28.9. $D[\mathfrak{A}] \wedge \mathfrak{A}[a] \wedge \mathfrak{A}[b] \to K_{Sb} a \subseteq \mathfrak{A}$.

Proof. $D[\mathfrak{A}] \wedge \mathfrak{A}[a] \wedge \mathfrak{A}[b]$ implies

1) $(\mathfrak{A})_{Sa} = W_{Sa}^{\mathfrak{A}}$	by the definition of $D[\mathfrak{A}]$
2) $W_{Sa}^{\mathfrak{A}}[a]$	by 1),
3) $M_{Sa}^{\mathfrak{A}}[a]$	by 2),
4) $Sb \prec Sa \to K_{Sb} a \subseteq \mathfrak{A}$	by 3) and $\mathfrak{A}[b]$,
5) $Sa \preccurlyeq Sb \to K_{Sb} a \subseteq M_{Sa}^{\mathfrak{A}}$	by 3) and Lemma 27.6 b),
6) $Sa \preccurlyeq Sb \to K_{Sb} a \subseteq W_{Sa}^{\mathfrak{A}}$	by 2), 5) and Lemma 27.7 c),
7) $Sa \preccurlyeq Sb \to K_{Sb} a \subseteq \mathfrak{A}$	by 1) and 6),
8) $K_{Sb} a \subseteq \mathfrak{A}$	by 4) and 7).

COROLLARY 28.9. $D[\mathfrak{A}] \to (\mathfrak{A})_s \subseteq M_s^{\mathfrak{A}}$.

Proof. By Lemma 28.9 we have

$$D[\mathfrak{A}] \wedge \mathfrak{A}[a] \to \forall x(\mathfrak{A}[x] \to K_{Sx} a \subseteq \mathfrak{A})$$

which implies the assertion.

LEMMA 28.10.

$$D[\mathfrak{A}] \wedge \forall x \prec \Omega_s (M_s^{\mathfrak{A}}[x] \to \mathfrak{A}[x]) \to \forall x \prec \Omega_s (\mathfrak{A}[x] \leftrightarrow W_s^{\mathfrak{A}}[x])$$

Proof. $D[\mathfrak{A}]$ implies $W_s^{\mathfrak{A}} \subseteq M_s^{\mathfrak{A}}$ and by Corollary 28.9 $(\mathfrak{A})_s \subseteq M_s^{\mathfrak{A}}$. Hence

$$D[\mathfrak{A}] \wedge \forall x \prec \Omega_s (M_s^{\mathfrak{A}}[x] \to \mathfrak{A}[x])$$

implies

$$\forall x \prec \Omega_s ((W_s^{\mathfrak{A}}[x] \to \mathfrak{A}[x]) \wedge (\mathfrak{A}[x] \to M_s^{\mathfrak{A}}[x]))$$

which by Lemma 28.1 and the definition of $W_s^{\mathfrak{A}}$ implies

$$\forall x \prec \Omega_s (\mathfrak{A}[x] \leftrightarrow W_s^{\mathfrak{A}}[x])$$

LEMMA 28.11. $D[U] \wedge W_s^U[\Omega_s] \wedge a \prec s \to (W_s^U)_a = W_a^U$

Proof. First we prove

(1) $D[U] \wedge W_a^U[b] \wedge \forall x \prec c (U(x) \to K_{Sx} b \subseteq U) \wedge U(c) \to K_{Sc}\, b \subseteq U$

We have the following three cases.

1. $Sc \prec a$. Then $K_{Sc}\, b \subseteq U$ follows from $W_a^U[b] \wedge U(c)$.

2. $Sc = a$. Then $D[U] \wedge W_a^U[b] \wedge U(c)$ implies $(U)_a = W_a^U$ and $U(b)$, hence by Lemma 28.9 $K_{Sc}\, b \subseteq U$.

3. $a \prec Sc$. Then $D[U] \wedge U(c)$ implies

1) $(U)_{Sc} = W_{Sc}^U$	
2) $W_{Sc}^U[c]$	by 1),
$\forall x \prec c (U(x) \to K_{Sx}\, b \subseteq U)$ implies	
3) $M_{Sc}^U[b]$	
4) $W_{Sc}^U[b]$	by 2) and 3), since $b \prec c$,
5) $U(b)$	by 1) and 4),
6) $K_{Sc}\, b \subseteq U$.	by 5) and Lemma 28.9.

Let $\mathfrak{A}[c]$ be the formula $K_{Sc}\, b \subseteq U$. Then from (1) we obtain

$$D[U] \wedge W_a^U[b] \to Prg[U, \mathfrak{A}]$$

By Lemma 28.1 and arithmetical comprehension we obtain

$$D[U] \wedge W_a^U[b] \to \forall x (U(x) \to K_{Sx}\, b \subseteq U)$$

which implies

(2) $$D[U] \wedge a \prec s \wedge W_a^U[b] \to M_s^U[b]$$

We also have

(3) $$Sb \prec s \wedge W_s^U[\Omega_s] \wedge M_s^U[b] \to W_s^U[b]$$

From (2) and (3) we obtain

(4) $$D[U] \wedge W_s^U[\Omega_s] \wedge a \prec s \to W_a^U \subseteq (W_s^U)_a$$

We also have

(5) $$a \prec s \to (W_s^U)_a \subseteq M_a^U \wedge Wo[(W_s^U)_a]$$

From (4) and (5) by the definition of W_a^U we obtain the assertion

$$D[U] \wedge W_s^U[\Omega_s] \wedge a \prec s \to (W_s^U)_a = W_a^U$$

LEMMA 28.12.

$$D[U] \wedge \forall x \prec \Omega_s (M_s^U[x] \to U(x)) \wedge M_s^U[\Omega_s] \to D[W_s^U] \wedge W_s^U[\Omega_s]$$

Proof. Let $\mathfrak{A}[t]$ be the formula $W_s^U[t]$.

$$D[U] \wedge \forall x \prec \Omega_s (M_s^U[x] \to U(x)) \wedge M_s^U[\Omega_s]\cdot$$

implies

1) $\forall x \prec \Omega_s (U(x) \leftrightarrow \mathfrak{A}[x])$	by Lemma 28.10,
2) $\forall x \prec \Omega_s (M_s^U[x] \to W_s^U[x])$	by 1),
3) $W_s^U[\Omega_s]$	by 2) and $M_s^U[\Omega_s]$,
4) $a \prec s \to (W_s^U)_a = W_a^U$	by 3) and Lemma 28.11,
5) $a \prec s \to (\mathfrak{A})_a = W_a^{\mathfrak{A}}$	by 1), 4) and the definition of $\mathfrak{A}$,
6) $(\mathfrak{A})_s = W_s^{\mathfrak{A}}$	by 1),
7) $D[\mathfrak{A}]$	by 5) and 6) since $\mathfrak{A} \subseteq T$ and $t \preccurlyeq \mathfrak{A} \to St \preccurlyeq s$ holds,
8) $D[W_s^U] \wedge W_s^U[\Omega_s]$	by 3) and 7).

LEMMA 28.13. $D[U] \wedge \Omega_{Sa} \preccurlyeq U \wedge a \prec \Omega_s \wedge M_s^U[a] \to U(a)$

Proof. First we prove

(1) $$D[U] \wedge a \prec \Omega_s \wedge M_s^U[a] \wedge \forall x \prec c\,(U(x) \to K_{Sx}\, a \subseteq U) \wedge U(c) \to K_{Sc}\, a \subseteq U$$

We have the following two cases

1. $Sc \prec s$. Then $K_{Sc}\, a \subseteq U$ follows from $M_s^U[a] \wedge U(c)$.
2. $s \preccurlyeq Sc$. Then $D[U] \wedge U(c)$ implies

1) $(U)_{Sc} = W_{Sc}^U$

2) $W_{Sc}^U[c]$ by 1),

$\forall x \prec c\,(U(x) \to K_{Sx}\, a \subseteq U)$ implies

3) $M_{Sc}^U[a]$

4) $W_{Sc}^U[a]$ by 2) and 3), since $a \prec c$,

5) $U(a)$ by 1) and 4),

6) $K_{Sc}\, a \subseteq U$ by 5) and Lemma 28.9.

Let $\mathfrak{A}[c]$ be the formula $K_{Sc}\, a \subseteq U$. Then from (1) we obtain

$$D[U] \wedge a \prec \Omega_s \wedge M_s^U[a] \to Prg[U, \mathfrak{A}]$$

By Lemma 28.1 and arithmetical comprehension we obtain

(2) $$D[U] \wedge a \prec \Omega_s \wedge M_s^U[a] \wedge U(b) \to K_{Sb}\, a \subseteq U$$

By Lemma 28.7 b) we have

(3) $$D[U] \wedge \Omega_{Sa} \preccurlyeq b \wedge U(b) \wedge K_{Sb}\, a \subseteq U \to U(a)$$

From (2) and (3) we obtain the assertion

$$D[U] \wedge \Omega_{Sa} \preccurlyeq U \wedge a \prec \Omega_s \wedge M_s^U[a] \to U(a)$$

LEMMA 28.14.

$$D[U] \wedge \forall x \prec s\,(\Omega_x \preccurlyeq U) \wedge M_s^U[\Omega_s] \to D[W_s^U] \wedge W_s^U[\Omega_s]$$

Proof.

$$D[U] \wedge \forall x \prec s\,(\Omega_x \preccurlyeq U) \wedge M_s^U[\Omega_s]$$

implies

1) $\forall x \prec \Omega_s\,(M_s^U[x] \to U(x))$ by Lemma 28.13,

2) $D[W_s^U] \wedge W_s^U[\Omega_s]$ by 1) and Lemma 28.12.

LEMMA 28.15. $\mathfrak{W}[t] \wedge St = NO \to Ac[T, t]$.

Proof. $\mathfrak{W}[t] \wedge St = NO$ implies

1) $(\mathfrak{W})_{NO} = W_{NO}^{\mathfrak{W}}$ by Lemma 28.3,

2) $W_{NO}^{\mathfrak{W}}[t]$ by 1),

3) $Ac[M_{NO}^{\mathfrak{W}}, t]$ by 2),

4) $Ac[T, t]$ by 3) and Lemmata 27.6 a) and 27.1 c).

REMARK. According to Lemma 28.15 we obtain a well ordering proof for the segment of all ordinals $< \alpha < \Omega_1$ of $T(\Omega)$ by proving $\mathfrak{W}[\alpha]$.

§ 29. – Well Ordering Proof with $(\Pi_1^1\text{-}CA)$

The Lemmata in this section indicate formulas which are derivable in the formal system A_2 with the additional axiom schema $(\Pi_1^1\text{-}CA)$ of Π_1^1-comprehension.

LEMMA 29.1.

a) $D[U] \wedge \forall x \prec \Omega_s\,(M_s^U[x] \to U(x)) \wedge M_s^U[\Omega_s] \to \mathfrak{W}[\Omega_s]$

b) $D[U] \wedge \forall x \prec s\,(\Omega_x \preccurlyeq U) \wedge M_s^U[\Omega_s] \to \mathfrak{W}[\Omega_s]$

Proof by Lemmata 28.12 and 28.14 using $(\Pi_1^1\text{-}CA)$ with respect to the weak formula $W_s^U[t]$.

LEMMA 29.2. $\forall x\, \mathfrak{W}[\Omega_{Nx}]$.

Proof. By the definition of W_{NO}^U we have

$$D[W_{NO}^U] \wedge W_{NO}^U[\Omega_{NO}]$$

From this formula by $(\Pi_1^1\text{-}CA)$ we obtain

(1) $$\mathfrak{W}[\Omega_{NO}]$$

We also have

$$D[U] \wedge U(\Omega_{Na}) \rightarrow \forall x \prec N(a')(\Omega_x \preccurlyeq U) \wedge M^U_{N(a')}[\Omega_{N(a')}]$$

By Lemma 29.1 b) we obtain

$$D[U] \wedge U(\Omega_{Na}) \rightarrow \mathfrak{W}[\Omega_{N(a')}]$$

which implies

$$(2) \qquad \mathfrak{W}[\Omega_{Na}] \rightarrow \mathfrak{W}[\Omega_{N(a')}]$$

The assertion $\forall x\, \mathfrak{W}[\Omega_{Nx}]$ follows from (1) and (2) by complete induction.

LEMMA 29.3. For any arithmetical formula $\mathfrak{A}_U[t]$ we have

$$Prg[(\mathfrak{W})_{Ns}, \mathfrak{A}_{(\mathfrak{W})_{Ns}}] \wedge (\mathfrak{W})_{Ns}[t] \rightarrow \mathfrak{A}_{(\mathfrak{W})_{Ns}}[t]$$

Proof. By Lemmata 28.1 and 27.1 e) and arithmetical comprehension we have

$$(1) \qquad D[U] \wedge Prg[(U)_{Ns}, \mathfrak{A}_{(U)_{Ns}}] \wedge (U)_{Ns}[t] \rightarrow \mathfrak{A}_{(U)_{Ns}}[t]$$

By Corollary 28.2 we have

$$(2) \qquad D[U] \wedge U(\Omega_{Ns}) \rightarrow (U)_{Ns} = (\mathfrak{W})_{Ns}$$

From (1) and (2) we obtain

$$D[U] \wedge U(\Omega_{Ns}) \wedge Prg[(\mathfrak{W})_{Ns}, \mathfrak{A}_{(\mathfrak{W})_{Ns}}] \wedge (\mathfrak{W})_{Ns}[t] \rightarrow \mathfrak{A}_{(\mathfrak{W})_{Ns}}[t]$$

and furthermore

$$(3) \qquad \mathfrak{W}[\Omega_{Ns}] \wedge Prg[(\mathfrak{W})_{Ns}, \mathfrak{A}_{(\mathfrak{W})_{Ns}}] \wedge (\mathfrak{W})_{Ns}[t] \rightarrow \mathfrak{A}_{(\mathfrak{W})_{Ns}}[t]$$

By Lemma 29.2 we have

$$(4) \qquad \mathfrak{W}[\Omega_{Ns}]$$

The assertion follows from (3) and (4).

In the following we use the abbreviation

$$\pi := \psi 0 (\Omega_\omega \cdot \Omega_1)$$

LEMMA 29.4.

a) $t \prec \pi \wedge T(s) \rightarrow C(s, \Omega_\omega \cdot t)$

b) $t \prec \pi \wedge \mathfrak{M}[t] \wedge s \prec \omega \rightarrow M^{\mathfrak{M}}_s[\Omega_\omega \cdot t]$

Proof. a) follows from Lemma 16.4. b) follows from Lemmata 28.3 and 28.9, since we have $K_a(\Omega_\omega \cdot t) = K_a t$ for $a \prec \omega$ and $t \prec \Omega_1$.

DEFINITION

$$\mathfrak{H}[t] := \begin{cases} \forall x \forall y (t \prec \pi \wedge \mathfrak{M}[t] \wedge (\mathfrak{M})_{Nx}[y] \\ \rightarrow \forall z \preccurlyeq Nx (C(z, \Omega_\omega \cdot t + y) \rightarrow \mathfrak{M}[\psi z (\Omega_\omega \cdot t + y)])) \end{cases}$$

LEMMA 29.5. $Pr[\mathfrak{H}]$.

Proof. We use the following abbreviations.

$$\mathfrak{A}_{a,c}[b] := \forall z \preccurlyeq Nc(C(z, \Omega_\omega \cdot a + b) \rightarrow \mathfrak{M}[\psi z (\Omega_\omega \cdot a + b)])$$

$$\mathfrak{C}_1[a] := \forall u \prec a\, \mathfrak{H}[u] \wedge a \prec \pi \prec \mathfrak{M}[a]$$

$$\mathfrak{C}_2[a, b, c] := \mathfrak{C}_1[a] \wedge \forall y \prec b(\mathfrak{M}[y] \rightarrow \mathfrak{A}_{a,c}[y]) \wedge \mathfrak{M}[b] \wedge Sb \preccurlyeq Nc$$

$$\mathfrak{C}_3[a, b, c, d] := \mathfrak{C}_2[a, b, c] \wedge d \preccurlyeq Nc \wedge C(d, \Omega_\omega \cdot a + b)$$

$$\mathfrak{C}_4[a, b, c, d, e] := \mathfrak{C}_3[a, b, c, d] \wedge e \prec \psi d(\Omega_\omega \cdot a + b) \wedge M^{\mathfrak{M}}_d[e]$$

By induction on $dg(e)$ we prove

$$(1) \qquad \mathfrak{C}_4[a, b, c, d, e] \rightarrow \mathfrak{M}[e]$$

We have the following four cases.

1. $e \preccurlyeq \Omega_d$. Then $\mathfrak{M}[e]$ follows from $M^{\mathfrak{M}}_d[e]$ and $\mathfrak{M}[\Omega_d]$ which by Lemma 29.2 follows from $d \preccurlyeq Nc$.

2. $\Omega_d \prec e = \omega^{e_1} + e_2$, $dg(e_i) < dg(e)$ $(i = 1, 2)$. In this case $\mathfrak{C}_4[a, b, c, d, e]$ implies $\mathfrak{C}_4[a, b, c, d, e_i]$ and by the I.H. $\mathfrak{M}[e_i]$ for $i = 1, 2$. By Corollary 28.6 we obtain $\mathfrak{M}[e]$.

3. $e = \psi d(\Omega_\omega \cdot a + b_0)$, $C(d, \Omega_\omega \cdot a + b_0)$, $b_0 \prec b$, $dg(b_0) < dg(e)$. In this case we prove

$$(1.1) \qquad \mathfrak{C}_4[a, b, c, d, e] \wedge u < c \wedge M^{\mathfrak{M}}_{Nu}[b_0] \rightarrow K_{Nu} b_0 \subseteq \mathfrak{M}$$

If $Nu \prec d$, then $K_{Nu}\, b_0 \subseteq \mathfrak{W}$ follows from $M_d^{\mathfrak{W}}[e]$, since by Lemma 29.2 we have $\mathfrak{W}[\Omega_{Nu}]$. Now suppose $d \preccurlyeq Nu$ and $v \in K_{Nu}\, b_0$. Then

$$C(d, \Omega_\omega \cdot a + b_0) \wedge C(d, \Omega_\omega \cdot a + b)$$

implies

$$C(Nu, \Omega_\omega \cdot a + b_0) \wedge C(Nu, \Omega_\omega \cdot a + b)$$

and by Lemma 6.3

$$v \prec \psi(Nu)(\Omega_\omega \cdot a + b_0) \prec \psi(Nu)(\Omega_\omega \cdot a + b),$$

and $M_{Nu}^{\mathfrak{W}}[b_0]$ implies $M_{Nu}^{\mathfrak{W}}[v]$.

Hence we obtain $\mathfrak{C}_4[a, b, c, Nu, v]$. Since $dg(v) \leq dg(b_0) < dg(e)$, by the I.H. we obtain $\mathfrak{W}[v]$ which completes the proof of (1.1).

From (1.1) by complete induction we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to K_{Sb_0}\, b_0 \subseteq \mathfrak{W}$$

By Lemma 28.7 c) we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to \mathfrak{W}[b_0]$$

$\forall y \prec b(\mathfrak{W}[y] \to \mathcal{A}_{a,c}[y])$, $\mathfrak{W}[b_0]$, $b_0 \prec b$ and $C(d, \Omega_\omega \cdot a + b_0)$ implies $\mathfrak{W}[e]$. Hence we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to \mathfrak{W}[e]$$

4. $e = \psi d(\Omega_\omega \cdot a_0 + b_0)$, $C(d, \Omega_\omega \cdot a_0 + b_0)$, $a_0 \prec a$, $b_0 \prec \Omega_\omega$. In this case we prove

$$\mathfrak{C}_4[a, b, c, d, e] \wedge M_{Nu}^{\mathfrak{W}}[b_0] \to K_{Nu}\, b_0 \subseteq \mathfrak{W} \tag{1.2}$$

If $Nu \prec d$, then $K_{Nu}\, b_0 \subseteq \mathfrak{W}$ follows from $M_d^{\mathfrak{W}}[e]$. Now suppose $d \preccurlyeq Nu$ and $v \in K_{Nu}\, b_0$. Then $C(d, \Omega_\omega \cdot a_0 + b_0)$ implies $C(Nu, \Omega_\omega \cdot a_0 + b_0)$. By Lemma 29.4 a) we also have $C(Nu, \Omega_\omega \cdot a)$. By Lemma 6.3 we obtain $v \prec \psi(Nu)(\Omega_\omega \cdot a_0 + b_0) \prec \psi(Nu)(\Omega_\omega \cdot a)$. $M_{Nu}^{\mathfrak{W}}[b_0]$ implies $M_{Nu}^{\mathfrak{W}}[v]$. Hence we obtain $\mathfrak{C}_4[a, NO, u, Nu, v]$.

Since $dg(v) \leq dg(b_0) < dg(e)$, by the I.H. we obtain $\mathfrak{W}[v]$ which completes the proof of (1.2).

From (1.2) by complete induction we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to K_{Sb_0}\, b_0 \subseteq \mathfrak{W}$$

By Lemma 28.7 c) we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to \mathfrak{W}[b_0] \tag{1.3}$$

$\mathfrak{C}_1[a] \wedge a_0 \prec a$ implies $\mathcal{H}[a_0]$. $\mathfrak{W}[a] \wedge Sa = NO \wedge a_0 \prec a$ implies $\mathfrak{W}[a_0]$. Hence we have

$$\mathfrak{C}_4[a, b, c, d, e] \to \mathcal{H}[a_0] \wedge \mathfrak{W}[a_0] \tag{1.4}$$

From (1.3) and (1.4) we obtain

$$\mathfrak{C}_4[a, b, c, d, e] \to \mathfrak{W}[e]$$

which completes the proof of (1).

From (1) we obtain

$$\mathfrak{C}_3[a, b, c, d] \to \forall x \prec \psi d(\Omega_\omega \cdot a + b)(M_d^{\mathfrak{W}}[x] \to \mathfrak{W}[x]) \tag{2}$$

By Lemmata 29.2 and 29.4 b) we also obtain

$$\mathfrak{C}_3[a, b, c, d] \to \mathfrak{W}[\Omega_d] \wedge M_d^{\mathfrak{W}}[\psi d(\Omega_\omega \cdot a + b)] \tag{3}$$

From (2) and (3) by Lemmata 28.3 and 28.5 we obtain

$$\mathfrak{C}_3[a, b, c, d] \to \mathfrak{W}[\psi d(\Omega_\omega \cdot a + b)]$$

By the definition of $\mathfrak{C}_3[a, b, c, d]$ we obtain

$$\mathfrak{C}_1[a] \to Prg[(\mathfrak{W})_{Nc}, \mathcal{A}_{a,c}]$$

By Lemma 29.3 we obtain

$$\mathfrak{C}_1[a] \wedge (\mathfrak{W})_{Nc}[b] \to \mathcal{A}_{a,c}[b]$$

which implies the assertion $Pr[\mathcal{H}]$.

THEOREM 29.6. (*Lower Bound Theorem* for (Π_1^1-CA)). For any ordinal $\alpha < \psi 0(\Omega_\omega \cdot \varepsilon_0)$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schema (Π_1^1-CA).

Proof. For $\alpha < \psi 0(\Omega_\omega \cdot \varepsilon_0)$ there is an n such that $\alpha < \psi 0(\Omega_\omega \cdot \omega_n(0))$. By Lemmata 29.5 and 27.5 we have

$$\mathcal{H}[\omega_n(0)] \tag{1}$$

We also have

(2) $$\omega_n(0) \prec \pi \wedge \mathfrak{W}[\omega_n(0)] \wedge C(0, \Omega_\omega \cdot \omega_n(0))$$

From (1) and (2) we obtain $\mathfrak{W}[\psi 0 (\Omega_\omega \cdot \omega_n(0)]$. Then also $\mathfrak{W}[\alpha]$ is derivable.

REMARK. The result of Theorem 29.6 was first proved by W. BUCHHOLZ [4] with respect to another system of ordinal notations.

§ 30. - Well Ordering Proof with (Π^1_1-CA) and (BR)

LEMMA 30.1. If formulas $\mathfrak{W}[t] \wedge St = NO$ and $Pr[\mathcal{A}]$ are derivable in the formal system A_2 with the additional basic inference rule (BR), then the formula $\mathcal{A}[t]$ is derivable in that system.

Proof. From $\mathfrak{W}[t] \wedge St = NO$ by Lemma 28.15 we obtain $Ac[T, t]$, hence

(1) $$Prg[T, U] \to U(t)$$

Since (1) is an arithmetical formula, by (BR) we obtain

(2) $$Prg[T, \mathcal{A}] \to \mathcal{A}[t]$$

Since $Prg[T, \mathcal{A}]$ is equivalent to $Pr[\mathcal{A}]$, from (2) and $Pr[\mathcal{A}]$ we obtain $\mathcal{A}[t]$.

LEMMA 30.2. $\psi 0 (\Omega_\omega \cdot \Omega_1) = \sup\{\pi[n] : n < \omega\}$ where $\pi[0] = \psi 0 \Omega_\omega$ and $\pi[n+1] = \psi 0 (\Omega_\omega \cdot \pi[n])$.

Proof. For $\pi = \psi 0 (\Omega_\omega \cdot \Omega_1)$ we have $\pi = \psi 0 \alpha$, $\alpha \in C_0(\alpha)$ and $\alpha = \Omega_\omega \cdot \Omega_1 = \omega^{\Omega_\omega + \Omega_1}$. By the definition of $tp(\gamma)$ and $\gamma[\nu]$ for $\gamma \in L$ on page 22 we obtain $tp(\alpha) = \Omega_1$, $\alpha[\nu] = \omega^{\Omega_\omega + \nu} = \Omega_\omega \cdot \omega^\nu$, $\mathrm{tp}[\pi] = \omega$ and $\pi[n] = \psi 0 (\alpha[\beta_n])$ where $\beta_0 = 0$ and $\beta_{n+1} = \psi 0 (\alpha[\beta_n]) = \pi[n]$. We obtain $\pi[0] = \psi 0 \Omega_\omega$ and $\pi[n+1] = \psi 0 (\Omega_\omega \cdot \pi[n])$. By Corollary 5.7 we have

$$\psi 0 (\Omega_\omega \cdot \Omega_1) = \sup\{\pi[n] : n < \omega\}.$$

THEOREM 30.3. (*Lower Bound Theorem* for (Π^1_1-CA) + (BR)). For any ordinal $\alpha < \psi 0 (\Omega_\omega \cdot \Omega_1)$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schema (Π^1_1-CA) and the additional basic inference rule (BR).

Proof. By Lemma 30.2 we only have to prove that $\mathfrak{W}[\pi[n]]$ is derivable for any natural number n. We prove it by induction on n. Since $\pi[0] < \psi 0 (\Omega_\omega \cdot \varepsilon_0)$, by Theorem 29.6 $\mathfrak{W}[\pi[0]]$ is derivable. Now suppose that $\mathfrak{W}[\pi[n]]$ is derivable. Since $S(\pi[n]) = NO$, by Lemmata 29.5 and 30.1 $\mathcal{H}[\pi[n]]$ is derivable which implies $\mathfrak{W}[\pi[n+1]]$.

§ 31. - Well Ordering Proof with (Π^1_1-CA) and (BI)

The Lemmata in this section indicate formulas which are derivable in the formal system A_2 with the additional axiom schemata (Π^1_1-CA) and (BI).

LEMMA 31.1. $Prg[(\mathfrak{W})_{St}, \mathcal{A}] \wedge \mathfrak{W}[t] \to \mathcal{A}[t]$.

Proof. By Lemma 28.1 we have

$$D[U] \wedge U(t) \to \forall X (Prg[(U)_{St}, X] \to X(t))$$

By an application of (BI) we obtain

(1) $$D[U] \wedge U(t) \to (Prg[(U)_{St}, \mathcal{A}] \to \mathcal{A}[t])$$

By Corollary 28.2 we have

(2) $$D[U] \wedge U(t) \to (U)_{St} = (\mathfrak{W})_{St}$$

From (1) and (2) we obtain

$$D[U] \wedge U(t) \to (Prg[(\mathfrak{W})_{St}, \mathcal{A}] \to \mathcal{A}[t])$$

which implies the assertion.

DEFINITION. $\mathfrak{W}_\psi[t] := \forall x (C(Nx, t) \to \mathfrak{W}[\psi(Nx)t])$.

LEMMA 31.2. $Prg[M^{\mathfrak{W}}_\omega, \mathfrak{W}_\psi]$.

Proof. We use the following abbreviations.

$$\mathcal{A}[a] := \forall y \prec a (M^{\mathfrak{W}}_\omega[y] \to \mathfrak{W}_\psi[y]) \wedge M^{\mathfrak{W}}_\omega[a]$$
$$\mathcal{C}[a, b, c] := \mathcal{A}[a] \wedge C(Nb, a) \wedge c \prec \psi(Nb)a \wedge M^{\mathfrak{W}}_{Nb}[c]$$

By induction on $dg(c)$ we prove

$$\mathfrak{C}[a, b, c] \to \mathfrak{W}[c] \tag{1}$$

We have the following three cases.

1. $c \preccurlyeq \Omega_{Nb}$. Then $\mathfrak{W}[c]$ follows from $M_{Nb}^{\mathfrak{W}}[c]$ and $\mathfrak{W}[\Omega_{Nb}]$ which we have by Lemma 29.2.

2. $\Omega_{Nb} \prec c = \omega^{c_1} + c_2$, $dg(c_i) < dg(c)$. Then $\mathfrak{C}[a, b, c]$ implies $\mathfrak{C}[a, b, c_i]$ and by the I.H. $\mathfrak{W}[c_i]$ for $i = 1, 2$. By Corollary 28.6 we obtain $\mathfrak{W}[c]$.

3. $c = \psi(Nb)\,a_0$, $C(Nb, a_0)$, $a_0 \prec a$ and $dg(a_0) < dg(c)$.

In this case we prove

$$\mathfrak{C}[a, b, c] \wedge M_{Nd}^{\mathfrak{W}}[a_0] \to K_{Nd}\,a_0 \subseteq \mathfrak{W} \tag{1.1}$$

If $d < b$, then $K_{Nd}\,a_0 \subseteq \mathfrak{W}$ follows from $M_{Nb}^{\mathfrak{W}}[a_0]$. Now suppose $b \leq d \wedge e \in K_{Nd}\,a_0$. Then $C(Nb, a_0) \wedge C(Nb, a)$ implies $C(Nd, a_0) \wedge C(Nd, a)$ and by Lemma 6.3 $e \prec \psi(Nd)\,a_0 \prec \psi(Nd)\,a$.

$M_{Nd}^{\mathfrak{W}}[a_0]$ implies $M_{Nd}^{\mathfrak{W}}[e]$. Hence we obtain $\mathfrak{C}[a, d, e]$. Since $dg(e) \leq dg(a_0) < dg(c)$, by the I.H. we obtain $\mathfrak{W}[e]$ which completes the proof of (1.1).

From (1.1) by complete induction we obtain

$$\mathfrak{C}[a, b, c] \to M_{\omega}^{\mathfrak{W}}[a_0] \tag{1.2}$$

$\mathfrak{A}[a] \wedge a_0 \prec a \wedge M_{\omega}^{\mathfrak{W}}[a_0]$ implies $\mathfrak{W}_{\psi}[a_0]$. Together with $C(Nb, a_0)$ we obtain $\mathfrak{W}[c]$. Hence (1.2) implies

$$\mathfrak{C}[a, b, c] \to \mathfrak{W}[c]$$

which completes the proof of (1).

From (1) we obtain

$$\mathfrak{A}[a] \wedge C(Nb, a) \to \forall x \prec \psi(Nb)\,a\,(M_{Nb}^{\mathfrak{W}}[x] \to \mathfrak{W}[x]) \tag{2}$$

We also have

$$\mathfrak{A}[a] \wedge C(Nb, a) \to \mathfrak{W}[\Omega_{Nb}] \wedge M_{Nb}^{\mathfrak{W}}[\psi(Nb)\,a] \tag{3}$$

From (2) and (3) by Lemma 28.5 we obtain

$$\mathfrak{A}[a] \wedge C(Nb, a) \to \mathfrak{W}[\psi(Nb)\,a]$$

which yields the assertion

$$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{W}_{\psi}]$$

LEMMA 31.3. $Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \to \mathfrak{A}[\Omega_{\omega} + 1]$.

Proof. For $a \prec \Omega_{\omega}$ we have $\mathfrak{W}[\Omega_{Sa}]$ by Lemma 29.2. Then $a \prec \Omega_{\omega} \wedge Sb \preccurlyeq Sa \wedge M_{\omega}^{\mathfrak{W}}[b]$ implies $K_{Sa}\,b \subseteq \mathfrak{W}$ and by Lemma 28.7 c) $\mathfrak{W}[b]$. Hence we obtain

$$a \prec \Omega_{\omega} \to (M_{\omega}^{\mathfrak{W}})_{Sa} \subseteq (\mathfrak{W})_{Sa}$$

Since by Corollary 28.9 we also have $(\mathfrak{W})_{Sa} \subseteq (M_{\omega}^{\mathfrak{W}})_{Sa}$, we obtain

$$a \prec \Omega_{\omega} \to (M_{\omega}^{\mathfrak{W}})_{Sa} = (\mathfrak{W})_{Sa} \tag{1}$$

$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \wedge a \prec \Omega_{\omega}$ implies

1) $Prg\,[(M_{\omega}^{\mathfrak{W}})_{Sa}, \mathfrak{A}]$
2) $Prg\,[(\mathfrak{W})_{Sa}, \mathfrak{A}]$ by 1) and (1),
3) $\mathfrak{W}[a] \to \mathfrak{A}[a]$ by 2) and Lemma 31.1,
4) $M_{\omega}^{\mathfrak{W}}[a] \to \mathfrak{A}[a]$ by 3) and (1).

Hence we obtain

$$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \to \forall x \prec \Omega_{\omega}\,(M_{\omega}^{\mathfrak{W}}[x] \to \mathfrak{A}[x]) \tag{2}$$

From (2) we obtain

$$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \to (M_{\omega}^{\mathfrak{W}}[\Omega_{\omega}] \to \mathfrak{A}[\Omega_{\omega}])$$

and

$$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \to (M_{\omega}^{\mathfrak{W}}[\Omega_{\omega} + 1] \to \mathfrak{A}[\Omega_{\omega} + 1]) \tag{3}$$

Since $K_n(\Omega_{\omega} + 1)$ is empty for all $n < \omega$, we have

$$M_{\omega}^{\mathfrak{W}}[\Omega_{\omega} + 1] \tag{4}$$

From (3) and (4) we obtain the assertion

$$Prg\,[M_{\omega}^{\mathfrak{W}}, \mathfrak{A}] \to \mathfrak{A}[\Omega_{\omega} + 1]$$

LEMMA 31.4. For any formula $\mathfrak{A}[t]$ there is a formula $\mathfrak{A}'[t]$ such that the following formulas are derivable:

a) $\mathfrak{A}'[t] \to \forall x \prec \omega^t (M_\omega^{\mathfrak{W}}[x] \to \mathfrak{A}[x])$

b) $Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \to Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}']$

Proof. We define

$$\mathfrak{B}[t] := M_\omega^{\mathfrak{W}}[t] \to \mathfrak{A}[t]$$
$$\mathfrak{A}'[t] := \forall y (\forall x \prec y\, \mathfrak{B}[x] \to \forall x \prec y + \omega^t\, \mathfrak{B}[x])$$

Then the formula a) is derivable. To prove the derivability of b) we use the abbreviation

$$\mathfrak{C}[a, b, c] := \begin{cases} Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \wedge \forall z \prec a (M_\omega^{\mathfrak{W}}[z] \to \mathfrak{A}'[z]) \\ \wedge \forall x \prec b\, \mathfrak{B}[x] \wedge c \prec \omega^a \wedge M_\omega^{\mathfrak{W}}[b + c] \end{cases}$$

and prove by induction on $dg(c)$

(1) $$\mathfrak{C}[a, b, c] \to \mathfrak{A}[b + c]$$

We have the following two cases.

1. $b + c = b$. Then $Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \wedge \forall x \prec b\, \mathfrak{B}[x]$ implies $\mathfrak{B}[b]$. Together with $M_\omega^{\mathfrak{W}}[b]$ we obtain $\mathfrak{A}[b]$.

2. $c = \omega^{c_1} + c_2$, $c_1 \prec a$ and $dg(c_2) < dg(c)$. Then

$$\forall z \prec a (M_\omega^{\mathfrak{W}}[z] \to \mathfrak{A}'[z])$$

implies $M_\omega^{\mathfrak{W}}[c_1] \to \mathfrak{A}'[c_1]$. $M_\omega^{\mathfrak{W}}[b + c]$ implies $M_\omega^{\mathfrak{W}}[c_1]$. Hence we obtain $\mathfrak{A}'[c_1]$. Together with $\forall x \prec b\, \mathfrak{B}[x]$ we obtain $\forall x \prec b + \omega^{c_1}\, \mathfrak{B}[x]$. So we obtain $\mathfrak{C}[a, b + \omega^{c_1}, c_2]$. Since $dg(c_2) < dg(c)$, by the I.H. we obtain $\mathfrak{A}[b + \omega^{c_1} + c_2]$ which completes the proof of (1).

From (1) we obtain

(2) $$\begin{cases} Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \wedge \forall z \prec a (M_\omega^{\mathfrak{W}}[z] \to \mathfrak{A}'[z]) \wedge \forall x \prec b\, \mathfrak{B}[x] \\ \to \forall x \prec b + \omega^a\, \mathfrak{B}[x] \end{cases}$$

From (2) we obtain

$$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \wedge \forall z \prec a (M_\omega^{\mathfrak{W}}[z] \to \mathfrak{A}'[z]) \to \mathfrak{A}'[a]$$

which implies the assertion b).

LEMMA 31.5. $Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \to \mathfrak{A}[\omega_n(\Omega_\omega + 1)]$.

Proof by induction on n. By Lemma 31.3 the assertion holds for $n = 0$. Now we prove the assertion for $n + 1$ under the assumption that it holds for n. By Lemma 31.4 there is a formula $\mathfrak{A}'[t]$ such that we have

(1) $$\mathfrak{A}'[\omega_n(\Omega_\omega + 1)] \to \forall x \prec \omega_{n+1}(\Omega_\omega + 1)(M_\omega^{\mathfrak{W}}[x] \to \mathfrak{A}[x])$$

(2) $$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \to Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}']$$

By our assumption we have

(3) $$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}'] \to \mathfrak{A}'[\omega_n(\Omega_\omega + 1)]$$

From (1), (2) and (3) we obtain

$$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \to \forall x \prec \omega_{n+1}(\Omega_\omega + 1)(M_\omega^{\mathfrak{W}}[x] \to \mathfrak{A}[x])$$

which implies

$$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \wedge M_\omega^{\mathfrak{W}}[\omega_{n+1}(\Omega_\omega + 1)] \to \mathfrak{A}[\omega_{n+1}(\Omega_\omega + 1)]$$

Since we also have $M_\omega^{\mathfrak{W}}[\omega_{n+1}(\Omega_\omega + 1)]$ we obtain the assertion

$$Prg[M_\omega^{\mathfrak{W}}, \mathfrak{A}] \to \mathfrak{A}[\omega_{n+1}(\Omega_\omega + 1)]$$

for $n + 1$.

THEOREM 31.6. (*Lower Bound Theorem* for $(\Pi_1^1\text{-}CA) + (BI)$). For any ordinal $\alpha < \psi 0(\psi \omega 0)$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schemata $(\Pi_1^1\text{-}CA)$ and (BI).

Proof. $\psi \omega 0$ is the least ε-number $> \Omega_\omega$. Hence for $\alpha < \psi 0(\psi \omega 0)$ there is an n such that $\alpha < \psi 0(\omega_n(\Omega_\omega + 1))$.

By Lemmata 31.2 and 31.5 we have $\mathfrak{W}_\psi[\omega_n(\Omega_\omega + 1)]$. Since we also have $C(0, \omega_n(\Omega_\omega + 1))$, we obtain $\mathfrak{W}[\psi 0(\omega_n(\Omega_\omega + 1))]$. Then also $\mathfrak{W}[\alpha]$ is derivable.

REMARK. The result of Theorem 31.6 was first proved by W. BUCHHOLZ and W. POHLERS [6] with respect to another ordinal notation system.

§ 32. - Well Ordering Proof with $(\Delta^1_2\text{-}CR)$

The Lemmata in this section indicate formulas which are derivable in the formal system A_2 with the additional basic inference rule $(\Delta^1_2\text{-}CR)$. Since $(\Delta^1_2\text{-}CR)$ implies $(\Pi^1_1\text{-}CA)$ we here can use the Lemmata of § 29.

DEFINITION. $\vartheta[t] := \forall X(D[X] \wedge \Omega_{St} \preccurlyeq X \to X(t))$
$\vartheta[t]$ is a Π^1_2-formula.

LEMMA 32.1. $T(s) \wedge \forall x \prec s(\Omega_x \preccurlyeq \mathfrak{M}) \to \forall x \prec \Omega_s(\vartheta[x] \leftrightarrow \mathfrak{M}[x])$.

Proof. By Lemma 28.2 we have

$$D[U] \wedge U(a) \to (D[V] \wedge \Omega_{Sa} \preccurlyeq V \to V(a))$$

which implies

(1) $$\mathfrak{M}[a] \to \vartheta[a]$$

By the definition of ϑ we obtain

$$D[U] \wedge \Omega_{Sa} \preccurlyeq U \to (\vartheta[a] \to U(a))$$

which implies

(2) $$\Omega_{Sa} \preccurlyeq \mathfrak{M} \to (\vartheta[a] \to \mathfrak{M}[a])$$

The assertion of Lemma 32.1 follows from (1) and (2).

In the following we write $\Omega(t)$ instead of Ω_t.

LEMMA 32.2. $D[U] \wedge U(\Omega(t)) \to \mathfrak{M}[\Omega(t+\omega^n)]$.

Proof by induction on n. By Lemma 29.1 b) we have

(1) $$D[U] \wedge U(\Omega(t)) \wedge M^U_{t+1}[\Omega(t+1)] \to \mathfrak{M}[\Omega(t+1)]$$

We also have

(2) $$D[U] \wedge U(\Omega(t)) \to M^U_{t+1}[\Omega(t+1)]$$

The assertion for $n = 0$ follows from (1) and (2).

Now we prove the assertion for $n+1$ under the assumption that it holds for n. By this assumption we have

$$D[U] \wedge U(\Omega(t+\omega^n \cdot m)) \to \mathfrak{M}[\Omega(t+\omega^n \cdot m')]$$

which implies

(3) $$\mathfrak{W}[\Omega(t+\omega^n \cdot m)] \to \mathfrak{W}[\Omega(t+\omega^n \cdot m')]$$

We also have

(4) $$D[U] \wedge U(\Omega(t)) \to \mathfrak{W}[\Omega(t)]$$

From (3) and (4) by complete induction we obtain

$$D[U] \wedge U(\Omega(t)) \to \forall x \prec \omega\, \mathfrak{W}[\Omega(t+\omega^n \cdot x)]$$

which implies

(5) $$D[U] \wedge U(\Omega(t)) \to \forall x \prec t+\omega^{n+1}(\Omega_x \preccurlyeq \mathfrak{W})$$

By Lemma 32.1 we obtain

(6) $$D[U] \wedge U(\Omega(t)) \to \forall x \prec \Omega(t+\omega^{n+1})(\vartheta[x] \leftrightarrow \mathfrak{W}[x])$$

Let $\mathfrak{A}[a]$ be the Π^1_2-formula

$$D[U] \wedge U(\Omega(t)) \wedge a \prec \Omega(t+\omega^{n+1}) \to \vartheta[a]$$

and let $\mathfrak{B}[a]$ be the Σ^1_2-formula

$$D[U] \wedge U(\Omega(t)) \wedge a \prec \Omega(t+\omega^{n+1}) \to \mathfrak{W}[a]$$

Then from (6) we obtain

$$\forall x(\mathfrak{A}[x] \leftrightarrow \mathfrak{B}[x])$$

By an application of $(\Delta^1_2\text{-}CR)$ we obtain

$$\exists Y \forall x(Y(x) \leftrightarrow \mathfrak{B}[x])$$

which implies

(7) $$D[U] \wedge U(\Omega(t)) \to \exists Y \forall x \prec \Omega(t+\omega^{n+1})(Y(x) \leftrightarrow \mathfrak{W}[x])$$

From (4), (5), (6), (7) and Lemma 28.3 we obtain

(8) $$D[U] \wedge U(\Omega(t)) \to \exists Y(D[Y] \wedge \forall x \prec t+\omega^{n+1}(\Omega_x \preccurlyeq Y) \wedge Y(\Omega(t)))$$

Obviously, we have

$$D[U] \wedge U(\Omega(t)) \to M^{U}_{t+\omega^{n+1}}[\Omega(t+\omega^{n+1})]$$

Therefore from Lemma 29.1 b) we obtain

(9) $$D[U] \wedge \forall x \prec t+\omega^{n+1}(\Omega_x \preccurlyeq U) \wedge U(\Omega(t)) \to \mathfrak{W}[\Omega(t+\omega^{n+1})]$$

From (8) and (9) we obtain the assertion

$$D[U] \wedge U(\Omega(t)) \to \mathfrak{W}[\Omega(t+\omega^{n+1})]$$

for $n+1$.

COROLLARY 32.2. $\mathfrak{W}[\Omega(\omega^n)]$.

Proof. By Lemma 32.2 we have

$$D[U] \wedge U(NO) \to \mathfrak{W}[\Omega(\omega^n)]$$

which implies

$$\mathfrak{W}[NO] \to \mathfrak{W}[\Omega(\omega^n)]$$

Since we have $\mathfrak{W}[NO]$ by Lemma 29.2, we obtain $\mathfrak{W}[\Omega(\omega^n)]$.

THEOREM 32.3. (*Lower Bound Theorem* for $(\Delta^1_2\text{-}CR)$). For any ordinal $\alpha < \psi 0(\Omega(\omega^\omega))$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional basic inference rule $(\Delta^1_2\text{-}CR)$.

Proof. For $\alpha < \psi 0(\Omega(\omega^\omega))$ there is an n such that $\alpha < \psi 0(\Omega(\omega^n))$. By Corollary 32.2 the formula $\mathfrak{W}[\Omega(\omega^n)]$ is derivable. Since we also have $\mathfrak{W}[NO] \wedge C(0, \Omega(\omega^n))$, by Corollary 28.8 we obtain $\mathfrak{W}[\psi 0(\Omega(\omega^n))]$. Then also $\mathfrak{W}[\alpha]$ is derivable.

REMARK. The result of Theorem 32.3 was first proved by G. TAKEUTI and M. YASUGI [31] with respect to ordinal diagrams.

§ 33. – Well Ordering Proofs with $(\Delta^1_2\text{-}CA)$ and with $(\Delta^1_2\text{-}CA)+(BR)$

DEFINITION. $\mathfrak{W}_\Omega[t] := \mathfrak{W}[\Omega_t]$.

LEMMA 33.1. The formula $Pr[\mathfrak{W}_\Omega]$ is derivable in the formal system A_2 with the additional axiom schema $(\Delta^1_2\text{-}CA)$.

Proof. $\forall x \prec a\, \mathfrak{W}[\Omega_x] \wedge b \prec a$ implies $\mathfrak{W}[\Omega_b] \wedge \mathfrak{W}[\Omega_{Sb}]$, hence by Lemma 28.9 $K_{Sb}[\Omega_b] = K_{Sb}[b] \subseteq \mathfrak{W}$ and by Lemma 28.7 c) $\mathfrak{W}[b]$. Hence we obtain

(1) $$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \forall x \prec a\, \mathfrak{W}[x]$$

If $Sa = NO$ we have $\mathfrak{W}[\Omega_{Sa}]$ by Lemma 29.2. Otherwise we have $Sa \prec a$ and therefore in any case

(2) $$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \mathfrak{W}[\Omega_{Sa}]$$

From (1) and (2) by Lemma 28.5 we obtain

(3) $$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \mathfrak{W}[a]$$

By Lemma 32.1 we have

$$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \forall x \prec \Omega_a(\vartheta[x] \leftrightarrow \mathfrak{W}[x])$$

Using $(\Delta^1_2\text{-}CA)$ we obtain

(4) $$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \exists Y \forall x \prec \Omega_a(Y(x) \leftrightarrow \mathfrak{W}[x])$$

From (3), (4) and Lemma 28.3 we obtain

(5) $$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \exists Y(D[Y] \wedge \forall x \prec a(\Omega_x \preccurlyeq Y) \wedge Y(a))$$

Obviously, we have

$$D[U] \wedge U(a) \to M^{U}_{a}[\Omega_a]$$

Therefore from Lemma 29.1 b) we obtain

(6) $$D[U] \wedge \forall x \prec a(\Omega_x \preccurlyeq U) \wedge U(a) \to \mathfrak{W}[\Omega_a]$$

From (5) and (6) we obtain

$$\forall x \prec a\, \mathfrak{W}[\Omega_x] \to \mathfrak{W}[\Omega_a]$$

which yields the assertion $Pr[\mathfrak{W}_\Omega]$.

THEOREM 33.2. (*Lower Bound Theorem* for $(\Delta^1_2\text{-}CA)$). For any ordinal $\alpha < \psi 0(\Omega_{\varepsilon_0})$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schema $(\Delta^1_2\text{-}CA)$.

Proof. For $\alpha < \psi 0 (\Omega_{\varepsilon_0})$ there is an n such that $\alpha < \psi 0 (\Omega_{\omega_n(0)})$. By Lemmata 27.5 and 33.1 the formula $\mathfrak{W} [\Omega_{\omega_n(0)}]$ is derivable. Since we also have $\mathfrak{W} [NO] \wedge C(0, \Omega_{\omega_n(0)})$, by Corollary 28.8 we obtain $\mathfrak{W} [\psi 0 (\Omega_{\omega_n(0)})]$. Then also $\mathfrak{W} [\alpha]$ is derivable.

REMARK. The result of Theorem 33.2 was first proved by G. TAKEUTI and M. YASUGI [31] with respect to ordinal diagrams.

LEMMA 33.3. $\psi 0 (\Omega_{\Omega_1}) = \sup \{\gamma [n] : n < \omega\}$ where $\gamma [0] = \psi 00$ and $\gamma [n+1] = \psi 0 (\Omega_{\gamma[n]})$.

Proof. For $\gamma = \psi 0 (\Omega_{\Omega_1})$ by the definition of $tp (\gamma)$ and $\gamma [\nu]$ for $\gamma \in L$ on page 22 we have $\gamma = \psi 0 \alpha$, $\alpha = \Omega_{\Omega_1}$, $\alpha \in C_0 (\alpha)$ $tp (\alpha) = \Omega_1$, $\alpha [\nu] = \Omega_\nu$, $tp (\gamma) = \omega$ and $\gamma [n] = \psi 0 (\alpha [\beta_n])$ where $\beta_0 = 0$ and $\beta_{n+1} = \psi 0 (\alpha [\beta_n]) = \gamma [n]$. We obtain $\gamma [0] = \psi 00$ and $\gamma [n+1] = \psi 0 (\Omega_{\gamma[n]})$. By Corollary 5.7 we have $\psi 0 (\Omega_{\Omega_1}) = \sup \{\gamma [n] : n < \omega\}$.

THEOREM 33.4. (*Lower Bound Theorem* for $(\Delta^1_2\text{-}CA) + (BR)$). For any ordinal $\alpha < \psi 0 (\Omega_{\Omega_1})$ the formula $\mathfrak{W}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schema $(\Delta^1_2\text{-}CA)$ and the additional basic inference rule (BR).

Proof. By Lemma 33.3 we only have to prove the derivability of $\mathfrak{W} [\gamma [n]]$ for any natural number n. We prove it by induction on n. By Theorem 33.2 we have $\mathfrak{W} [\gamma [0]]$. Now suppose $\mathfrak{W} [\gamma [n]]$. Since $S (\gamma [n]) = NO$, by Lemmata 30.1 and 33.1 we obtain $\mathfrak{W} [\Omega_{\gamma[n]}]$. Using Corollary 28.8 we obtain $\mathfrak{W} [\gamma [n+1]]$.

§ 34. - Well Ordering Proof with $(\Delta^1_2\text{-}CA)$ and (BI)

LEMMA 34.1.

$$M^{\mathfrak{W}}_s [t] \leftrightarrow St \preccurlyeq s \wedge \forall y \prec \Omega_s \ \forall X (D [X] \wedge X(y) \rightarrow K_{Sy} t \subseteq X)$$

Proof. By Corollary 28.2 we have

$$D [U] \wedge U (a) \rightarrow (U)_{Sa} = (\mathfrak{W})_{Sa}$$

which implies

$$(\mathfrak{W} [a] \rightarrow K_{Sa} t \subseteq \mathfrak{W}) \rightarrow (D [U] \wedge U (a) \rightarrow K_{Sa} t \subseteq U)$$

and

(1) $\quad (\mathfrak{W} [a] \rightarrow K_{Sa} t \subseteq \mathfrak{W}) \rightarrow \forall X (D [X] \wedge X (a) \rightarrow K_{Sa} t \subseteq X)$

We also have

$$\forall X (D [X] \wedge X (a) \rightarrow K_{Sa} t \subseteq X) \rightarrow (D [U] \wedge U (a) \rightarrow K_{Sa} t \subseteq U)$$

which implies

(2) $\quad \forall X (D [X] \wedge X (a) \rightarrow K_{Sa} t \subseteq X) \rightarrow (\mathfrak{W} [a] \rightarrow K_{Sa} t \subseteq \mathfrak{W})$

The assertion follows from (1) and (2) by the definition of $M^{\mathfrak{W}}_s [t]$.

LEMMA 34.2. The formula $Prg [\mathfrak{W}, \mathfrak{W}_\Omega]$ is derivable in the formal system A_2 with the additional axiom schema $(\Delta^1_2\text{-}CA)$.

Proof. By induction on $dg (b)$ we prove

(1) $\quad \forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \wedge b \prec \Omega_a \wedge M^{\mathfrak{W}}_a [b] \rightarrow \mathfrak{W} [b]$

If $b = NO$, we have $\mathfrak{W} [b]$ by Lemma 29.2. Now suppose $b \neq NO$. Then we have $dg (Sb) < dg (b)$. $M^{\mathfrak{W}}_a [b]$ implies $M^{\mathfrak{W}}_a [Sb]$. By the I.H. we obtain $\mathfrak{W} [Sb]$. Then $\forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \wedge Sb \prec a$ implies $\mathfrak{W} [\Omega_{Sb}]$. Then $M^{\mathfrak{W}}_a [b] \wedge Sb \prec a$ implies $K_{Sb} b \subseteq \mathfrak{W}$ and by Lemma 28.7 c) $\mathfrak{W} [b]$, which completes the proof of (1).

From (1) we obtain

$$\forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \rightarrow \forall x \prec \Omega_a (M^{\mathfrak{W}}_a [x] \rightarrow \mathfrak{W} [x])$$

Together with Corollary 28.9 we obtain

(2) $\quad \forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \rightarrow \forall x \prec \Omega_a (M^{\mathfrak{W}}_a [x] \leftrightarrow \mathfrak{W} [x])$

$\mathfrak{W} [t]$ is a Σ^1_2-formula and by Lemma 34.1 $M^{\mathfrak{W}}_a [t]$ is equivalent to a Π^1_2-formula. Therefore by an application of $(\Delta^1_2\text{-}CA)$ we obtain

(3) $\quad \forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \rightarrow \exists Y \forall x \prec \Omega_a (Y (x) \leftrightarrow \mathfrak{W} [x])$

From (2) and (3) we obtain

(4) $$\begin{cases} \forall x \prec a (\mathfrak{W} [x] \rightarrow \mathfrak{W} [\Omega_x]) \wedge \mathfrak{W} [a] \\ \rightarrow \exists Y (D [Y] \wedge \forall x \prec \Omega_a (M^Y_a [x] \rightarrow Y(x)) \wedge Y (a)) \end{cases}$$

Since we have

$$D [U] \wedge U (a) \rightarrow M^U_a [\Omega_a]$$

from Lemma 29.1 a) we obtain

(5) $$D[U] \wedge \forall x \prec \Omega_a (M^U_a[x] \to U(x)) \wedge U(a) \to \mathfrak{M}[\Omega_a]$$

From (4) and (5) we obtain

$$\forall x \prec a (\mathfrak{M}[x] \to \mathfrak{M}[\Omega_x]) \wedge \mathfrak{M}[a] \to \mathfrak{M}[\Omega_a]$$

which yields the assertion $Prg[\mathfrak{M}, \mathfrak{M}_\Omega]$.

LEMMA 34.3. The formula

$$\mathfrak{M}[t] \to \mathfrak{M}[\Omega_t]$$

is derivable in the formal system A_2 with the additional axiom schemata $(\Delta^1_2\text{-}CA)$ and (BI).

Proof by Lemmata 31.1 and 34.2.

THEOREM 34.4. For any ordinal $\alpha \in T(\Omega)$ the formula $\mathfrak{M}[\alpha]$ is derivable in the formal system A_2 with the additional axiom schemata $(\Delta^1_2\text{-}CA)$ and (BI).

Proof by induction on $dg(\alpha)$ using Lemma 29.2 (for $\alpha = 0$), Corollary 28.6, Corollary 28.8 and Lemma 34.3.

§ 35. - The Proof Theoretical Ordinals

Let S be the formal system A_2 with some additional axiom schemata or basic inference rules. The *proof theoretical ordinal* of S is defined to be the least ordinal β such that there is no recursive well ordering of order type β whose well foundedness is provable in S. We denote the proof theoretical ordinal of S by $|S|$.

An ordinal β_1 is said to be a *lower bound* of the system S if β_1 is an ordinal $< \Omega_1$ of $T(\Omega)$ such that for any ordinal $\alpha < \beta_1$ the formula $\mathfrak{M}[\alpha]$ is derivable in the system S.

An ordinal β_2 is said to be an *upper bound* of the system S if β_2 is an ε-number $< \Omega_1$ of $T(\Omega)$ and for any in S derivable formula F of stage 0 there is an ordinal $\alpha < \beta_2$ such that $PA' \vdash^{\alpha}_{0} F'$ holds for every numerical substitute F' of F.

LEMMA 35.1. If β_1 is a lower bound of S, then $\beta_1 \leq |S|$.

Proof immediately by the definitions.

To prove that also $|S| \leq \beta_2$ holds for every upper bound β_2 of S we consider the following

ASSUMPTIONS. Let β_2 be an upper bound of S and let R be a recursive well ordering of natural numbers of order type β_2 with the following properties:

1. There is a 1-place recursive predicate F_R such that $F_R(n)$ is true if and only if n is in the field of R.

2. There is a 2-place recursive predicate $<_R$ such that $<_R(m, n)$ is true if and only if m, n are in the field of R and m precedes n in the order relation R. We shall write $s <_R t$ instead of $<_R(s, t)$.

If n is the field of R we denote by $|n|_R$ the order type of $\{x : x <_R n\}$ with respect to R. We shall write α instead of n if $|n|_R = \alpha$ holds. Let $Pr_R[U]$ be the formula

$$\forall y (\forall x (x <_R y \to U(x)) \to (F_R(y) \to U(y)))$$

We call a formula A a *distinguished* $U(\gamma)$*-formula* if $\gamma < \beta_2$ and A is an arithmetical formula of the following kind:

1. $U(\gamma)$ is a positive part of A. Any other minimal positive part of A is a formula $U(\delta_i)$ where $\gamma \leq \delta_i < \beta_2$.

2. $Pr_R[U]$ is a negative part of A. Any other minimal negative part of A is a true constant prime formula or a formula $U(n_i)$ which is not equivalent to a positive part of A.

Under our assumptions we prove

LEMMA 35.2. If $PA' \vdash^{\alpha}_{0} A$ holds for a distinguished $U(\gamma)$-formula A, then $\gamma < \omega \cdot \alpha$.

Proof by induction on α. The formula A is not an axiom of PA'. $PA' \vdash^{\alpha}_{0} A$ is derivable only by an inference (S 3.0) with principal part $Pr_R[U]$. Therefore there is an $\alpha_0 < \alpha$ and a numerical term t such that we have

$$PA' \vdash^{\alpha_0}_{0} ((\forall x (x <_R t \to U(x)) \to (F_R(t) \to U(t))) \to A$$

By applications of the replacement rule a) and the inversion rule a) we obtain

(1) $$PA' \vdash^{\alpha_0}_{0} U(n) \to A$$

(2) $$PA' \vdash^{\alpha_0}_{0} \forall x (x <_R n \to U(x)) \vee . A$$

We have the following two cases.

1. $U(n)$ is not equivalent to a positive part of A. Then $U(n) \to A$ is a distinguished $U(\gamma)$-formula and from (1) by the I.H. we obtain $\gamma < \omega \cdot \alpha_0 < \omega < \alpha$.

2. $U(n)$ is equivalent to a positive part of A. Then we have $\gamma \leq |n|_R < \beta_2$. Suppose $\omega \cdot \alpha \leq \gamma$. Then there are infinitely many ordinals δ such that $\omega \cdot \alpha_0 < \delta < \omega \cdot \alpha \leq \gamma$. Therefore we can choose an m such that $|m|_R = \gamma_0$, $\omega \cdot \alpha_0 < \gamma_0 < \gamma$ and $U(m)$ is not equivalent to a negative part of A. From (2) by the inversion rule b) we obtain

$$PA' \overset{\alpha_0}{\underset{0}{\vdash}} (m <_R n \to U(m)) \vee A$$

where $m <_R n$ is a true constant prime formula. Then

$$(m <_R n \to U(m)) \vee A$$

is a distinguished $U(\gamma_0)$-formula. By the I.H. we obtain $\gamma_0 < \omega \cdot \alpha_0$ in contradiction to the assumed choice of m. Hence we obtain $\gamma < \omega \cdot \alpha$.

COROLLARY 35.2. The well ordering of R is not provable in S.

Proof. Suppose that the well ordering of R is provable in S. Then the formula

$$F := \forall X (Pr_R[X] \to \forall y (F_R(y) \to X(y)))$$

is derivable in S. Since F is a formula of stage 0 and β_2 is an upper bound of S it follows that there is an $\alpha < \beta_2$ such that we have

$$PA' \overset{\alpha}{\underset{0}{\vdash}} F$$

By the inversion rules b) and c) we obtain

$$PA' \overset{\alpha}{\underset{0}{\vdash}} Pr_R[U] \to (F_R(\gamma) \to U(\gamma))$$

for any ordinal $\gamma < \beta_2$.

$$Pr_R[U] \to (F_R(\gamma) \to U(\gamma))$$

is a distinguished $U(\gamma)$-formula for any $\gamma < \beta_2$. Therefore by Lemma 35.2 we obtain $\gamma < \omega \cdot \alpha$ for all $\gamma < \beta_2$, hence $\beta_2 \leq \omega \cdot \alpha$.

Since β_2 is an ε-number we have $\beta_2 = \omega \cdot \beta_2$. It follows that $\beta_2 \leq \alpha$ holds in contradiction to our assumptions.

LEMMA 35.3. If β_2 is an upper bound of S, then $|S| \leq \beta_2$.

Proof. This follows from Corollary 35.2.

LEMMA 35.4. For any ordinal $\alpha < \Omega_1$ of $T(\Omega)$ we have:

a) If $SR' \overset{\alpha}{\underset{0}{\vdash}} F^0$ holds for the zero-interpretation F^0 of a formula F of stage 0, then $PA' \overset{\alpha}{\underset{0}{\vdash}} F'$ holds for every numerical substitute F' of F.

b) If $SA' \overset{\alpha}{\underset{0}{\vdash}}$ holds for a formula F of level 0, then also $PA' \overset{\alpha}{\underset{0}{\vdash}}$ holds.

Proof by induction on α.

NOTATIONS. If C_1 and C_2 denote axiom schemata or inference rules, we denote by $|C_1|$ the proof theoretical ordinal of the formal system A_2 with the additional axiom schema or basic inference rule C_1 and we denote by $|C_1 + C_2|$ the proof theoretical ordinal of the formal system A_2 with the additional axiom schemata or basic inference rules C_1 and C_2.

By Lemmata 35.1 and 35.3 we obtain the following proof theoretical ordinals.

THEOREM 35.5.

a) $|(\Pi^1_1\text{-}CA)| = \psi\, 0\, (\Omega_\omega \cdot \varepsilon_0)$

b) $|(\Pi^1_1\text{-}CA) + (BR)| = \psi\, 0\, (\Omega_\omega \cdot \Omega_1)$

c) $|(\Pi^1_1\text{-}CA) + (BI)| = |(\Pi^1_1\text{-}BI)| = \psi\, 0\, (\psi\, \omega\, 0)$

d) $|(\Delta^1_2\text{-}CR)| = |(\Pi^1_2\text{-}SR)| = |(\Pi^1_2\text{-}SR) + (\Pi^1_1\text{-}BI)| = \psi\, 0\, (\Omega_{\omega^\omega})$

e) $|(\Delta^1_2\text{-}CA)| = |(\Pi^1_2\text{-}SA)| = \psi\, 0\, (\Omega_{\varepsilon_0})$

f) $|(\Delta^1_2\text{-}CA) + (BR)| = |(\Pi^1_2\text{-}SA) + (BR)| = \psi\, 0\, (\Omega_{\Omega_1})$

Proof.

a) Follows from Theorem 29.6 and Theorem 17.5 a).

b) Follows from Theorem 30.3 and Theorem 17.5 b).

c) We have

$$\psi\, 0\, (\psi\, \omega\, 0) \leq |(\Pi^1_1\text{-}CA) + (BI)|$$

by Theorem 31.6,

$$|(\Pi^1_1\text{-}CA) + (BI)| \leq |(\Pi^1_1\text{-}BI)|$$

by Theorem 10 and

$$|(\Pi^1_1\text{-}BI)| \leq \psi\, 0\, (\psi\, \omega\, 0)$$

by Theorem 14.2 and Lemma 17.1.

d) We have

$$\psi\, 0\, (\Omega_{\omega^\omega}) \leq |(\Delta^1_2\text{-}CR)|$$

by Theorem 32.2,

$$|(\Delta^1_2\text{-}CR)| \leq |(\Pi^1_2\text{-}SR)|$$

by Lemma 18.2 b) and

$$|(\Pi^1_2\text{-}SR) + (\Pi^1_1\text{-}BI)| \leq \psi\, 0\, (\Omega_{\omega^\omega})$$

by Theorem 22.2 and Lemma 35.4 a).

e) We have

$$\psi\, 0\, (\Omega_{\varepsilon_0}) \leq |(\Delta^1_2\text{-}CA)|$$

by Theorem 33.2,

$$|(\Delta^1_2\text{-}CA)| \leq |(\Pi^1_2\text{-}SA)|$$

by Lemma 18.2 a) and

$$|(\Pi^1_2\text{-}SA)| \leq \psi\, 0\, (\Omega_{\varepsilon_0})$$

by Theorem 26.6 a) and Lemma 35.4 b).

f) Follows from Theorem 33.4, Lemma 18.2 a), Theorem 26.6 b) and Lemma 35.4 b).

REMARK. All these subsystems of analysis are essentially *impredicative*, since their proof theoretical ordinals are essentially greater than Γ_0 (see § 7).

§ 36. - Stronger Subsystems of Analysis and Set Theory

Due to G. JÄGER [13]-[15], the proof-theoretical ordinals described in § 35 are also the limiting numbers of certain subsystems of set theory into which the respective subsystems of analysis can be imbedded.

Up to now the strongest subsystem of analysis which has been proof-theoretically analyzed is the system $(\Delta^1_2\text{-}CA) + (BI)$. For this system and a corresponding system of set theory the proof-theoretical ordinal has been characterized by G. JÄGER and W. POHLERS [18]. From the well ordering proof of § 34 it follows that this ordinal cannot be represented in the notation system $T(\Omega)$.

For this an essentially stronger system of notations is needed which makes use of the first (recursively) inaccessible ordinal as e.g. the notation system $T(I)$ developed in [7].

In [18] the upper bound theorem for $(\Delta^1_2\text{-}CA) + (BI)$ and the corresponding subsystem of set theory is proved by Pohlers' method of local predicativity. A different proof of the upper bound theorem for the corresponding system $(\Pi^1_2\text{-}SA) + (BI)$ is given in [26]. This proof is based on a generalization of Buchholz' $\Omega_{\sigma+1}$-rule.

The extensive proof-theoretical investigations of G. Jäger and W. Pohlers will be presented in a forthcoming volume of the Springer series «Ergebnisse der Mathematik und ihrer Grenzgebiete».

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