STUDIES IN PROOF THEORY
LECTURE NOTES

GERHARD JÄGER

THEORIES FOR ADMISSIBLE SETS
A UNIFYING APPROACH TO PROOF THEORY

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C O N T E N T S

INTRODUCTION.

In a series of papers between 1878 and 1897 Cantor created the theory of infinite sets. Although his ideas were first met with scepticism and distrust, the last decade of the previous century proved their importance for mathematics.

Cantor's conception of set, however, was too naive, and several antinomies were discovered, the most famous one due to Russell. Fortunately all these antinomies were of logical-foundational character and did not directly affect the mathematical consequences of set theory. In order to rescue set theory for mathematics, many logicians therefore started to think more closely about the nature of sets and the principles of set existence and set generation.

Various systems of set theory emerged from this work which all avoided the known antinomies and were strong enough to formalize abstract mathematical reasoning. Today Zermelo-Fraenkel set theory ZF with or without the axiom of choice is the generally accepted formalism and can be considered as the 'official' framework for doing mathematics.

The situation was less clear at the beginning of this century. Many systems of set theory were available, all without apparent inconsistencies but also without guarantee that inconsistencies would never appear. Several mathematicians therefore thought that to develop mathematics in these systems was to build mathematics on uncertain foundations.

As a reaction, Brouwer and his school tried to banish the critical principles and to allow only that part of mathematical reasoning which could be justified constructively. Hilbert, on the other hand, was convinced of the importance of the nonconstructive, abstract framework provided by set theory. He wished to justify the use of abstract techniques by showing that their use never leads to contradictions.

His program can be sketched as follows:

(i) Axiomatize and formalize all of mathematics.

(ii) Show that applications of the axioms and rules of inference never lead to contradictions.

In (ii) one has only to manipulate with finite proof trees, and so Hilbert had hoped that this step could be carried through with purely finitistic methods accepted by everybody. To realize this program proof procedures had to be investigated. On this account, Hilbert called the new discipline 'Proof Theory'.

In 1930 Gödel showed that Hilbert's program does not work. His second incompleteness theorem says that a consistent theory Th which contains a reasonable amount of number theory cannot prove its own consistency.

Although Gödel had devastated Hilbert's program, proof theory was not dead. The decisive new idea came from Gentzen [27]. In this paper he gave a consistency proof for number theory which was finite modulo transfinite induction along a well-ordering of order-type ε_0. He also showed that well-orderings of smaller order-types do not suffice. Hence ε_0

can be considered as the ordinal which characterizes that part of number theory which goes beyond finitary reasoning or, in other words, ε_0 characterizes the infinitary content of number theory. For the proof Gentzen uses the technique of cut elimination, which became one of the standard instruments in proof theory. Gentzen's methods and ideas were later taken up, modified and extended by other logicians in order to study various formal systems.

Although Zermelo-Fraenkel set theory is the established framework for mathematics and (practically) nobody doubts its consistency, there are good reasons for considerng weaker theories:

1. ZF is a theory which presupposes a Platonistic concept of set. It seems impossible to give a philosophical justification for the set existence axioms of ZF which is not based on a Platonistic world view.

2. More serious from the point of view of mathematics is the fact that ZF heavily violates the principle of parsimony. Most theorems of ordinary mathematics can already be proved in theories much weaker than ZF.

3. ZF is a formalism developed for studying large ordinals and cardinals and does not pay much attention to sets of low degree of constructibility. ZF has very few natural models, and therefore these models only give very limited information about the theorems of ZF.

Proof theory nowadays is concerned with the investigation of the proof possibilities of mathematical systems. One aspect of this research is the ordinal analysis of formal theories. The proof-theoretic ordinal $|Th|$ of a theory Th is the least ordinal α, such that the consistency of Th can be proved in Peano arithmetic PA plus the scheme of transfinite induction along a primitive recursive standard well-ordering of order-type α. In section 2.3 we present a different definition of proof-theoretic ordinal, but both notions agree in all relevant cases. Experience has shown that the proof-theoretic ordinal $|Th|$ is a very good measure for the strength of Th, so that we call $|Th|$ the proof-theoretic strength of Th.

Historically logicians first carried through the ordinal analysis of subsystems of second order arithmetic A_2 as defined in section 0.3. Roughly speaking, A_2 can be considered as the subsystem of ZF where we speak about natural numbers and sets of natural numbers only. It makes sense to study subsystems of A_2, since an old observation, probably established by Hilbert, Bernays and Weyl, says, that most of ordinary mathematics can be carried through in subsystems of A_2. This, however, can be accomplished only by using a heavy coding machinery.

Later the interest shifted to theories for iterated inductive definitions and their relations to subsystems of A_2. We do not quote special results here but refer to the textbooks of Schütte [47] and Takeuti [58] and to the recent Lecture Notes of Buchholz-Feferman-Pohlers-Sieg [4].

Feferman's preface in [4] contains a comprehensive description of the development during the last 20 years. Important contributions to this field of logic have been made by (in alphabetical order): Buchholz, Feferman, Friedman, Howard, Kreisel, Pohlers, Schütte, Sieg, Tait, Takeuti, Zucker et. al.

The first significant proof-theoretic results obtained for set theory are to be found in Feferman [11], where he introduces predicatively reducible subsystems of ZF. Friedman [23] is concerned with intuitionistic set theories and their importance for constructive mathematics, a track we do not follow here. In both cases the proof-theoretic analysis is carried out by reducing the set theories to subsystems of A_2. In order to do this, a lot of coding is necessary.

In this Habilitationsschrift we will present a unifying approch to the proof theory of theories in strength between Peano arithmetic PA and the theory KPi defined below. The central notion is that of admissible set; the central method is the systematic use of theories for (iterated) admissible sets.

In the beginning of the sixties, several attempts were made to develop recursion theories on more general domains than the natural numbers. In this connection we mention the metarecursion theory of Kreisel-Sacks [39], Takeuti's recursion theory on the ordinals [56,57] and Kripke's definition of α-recursive functions by means of an equation calculus [40]. Platek [42] then introduced admissible sets as natural domains for such abstract recursion theories. In doing this he also bridged the gap between recursion theory and set theory. Later the work of Barwise showed the importance of admissible sets for the logic of infinitary languages (cf. [1]). Here we will employ theories for admissible sets in proof-theoretic research.

N denotes the set of natural numbers. Over the natural numbers as urelements we fix the largest possible universe of sets V_N defined by the following recursion on the ordinals:

$$
\begin{aligned}
V_N(0) &:= \emptyset ;\\
V_N(\alpha+1) &:= \mathrm{Pow}(N \cup V_N(\alpha)) ;\\
V_N(\lambda) &:= \cup\{V_N(\xi) : \xi < \lambda\}, \text{ if } \lambda \text{ is a limit;}\\
V_N &:= \cup\{V_N(\xi) : \mathrm{Ord}(\xi)\} .
\end{aligned}
$$

The use of the natural numbers as urelements is theoretically superfluous, but has several technical advantages as should become clear later. Very often we will only consider the subhierarchy $(L_\alpha : \mathrm{Ord}(\alpha))$ of the sets constructible over the natural numbers.

Let L_1 be the usual first order language of Peano arithmetic. The theories for admissible sets are formulated in the extended language $L_* = L_1(\in, S, N)$ in which L_1 is augmented by: a membership relation symbol $\in$; a unary relation symbol S to express that an object is a set and not an urelement; and a set constant N for the set of the natural numbers as urelements. A Δ_0 formula is an L_* formula in which all quantifiers are bounded, i.e. of the form $(\forall x \in a)$ or $(\exists x \in a)$.

A transitive set d in V_N is called admissible above N if it contains N and satisfies the following Kripke-Platek axioms:

(Pair) $\exists z(a \in z \ \& \ b \in z)$;

(Transitive Hull) $\exists z(a \subset z \ \& \ z \text{ transitive})$;

(Δ_0 Separation) $\exists z(z = \{x \in a : A(x)\})$;

(Δ_0 Collection) $(\forall x \in a)\exists y A(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)A(x,y)$

for all Δ_0 formulas A. The ordinal α is admissible above N if L_α is admissible above N.

The situation of an admissible universe above N is described by the theory KPu, where we have Peano arithmetic as the theory for the urelements, the Kripke-Platek axioms as the only set existence axioms and the following principles of induction:

(IND_N) $A(0) \ \& \ (\forall x \in N)(A(x) \rightarrow A(x+1)) \rightarrow (\forall x \in N)A(x)$;

($IND_\in$) $\forall x((\forall y \in x)A(y) \rightarrow A(x)) \rightarrow \forall x A(x)$

for arbitrary L_* formulas A.

KPu^r is taken to be KPu with the schemes (IND_N) and ($IND_\in$) replaced by the axioms

(I_N) $\forall a[0 \in a \ \& \ (\forall x \in N)(x \in a \rightarrow x+1 \in a) \rightarrow (\forall x \in N)(x \in a)]$;

($I_\in$) $\forall a[a \neq \emptyset \rightarrow (\exists x \in a)(\forall y \in a)(y \notin x)]$.

KPu^o is taken to be KPu with (IND_N) restricted to (I_N) and $(IND_\in)$ omitted completely.

In order to speak about universes which are limits of admissible sets, we replace in KPu the scheme of Δ_0 collection by the limit axiom

(Lim) $\forall x \exists y (x \in y$ & y admissible)

and call this theory KPl. KPi finally is KPu + (Lim) and formalizes recursively inaccessible universes, i.e. universes which are admissible limits of admissibles. The restricted theories KPl^r, KPl^o, KPi^r and KPi^o are defined according to KPu^r and KPu^o.

KPi is the strongest system we will consider. It corresponds in proof-theoretic strength to the second order theory $(\Delta^1_2\text{-CA}) + (BI)$ and to Feferman's theory T_0 for explicit mathematics; for proofs cf. Feferman [15], Jäger [32] and Jäger-Pohlers [35].

It is well established that large parts of ordinary mathematics can be developed in (subsystems of) KPi, $(\Delta^1_2\text{-CA}) + (BI)$ and T_0. Subsystems of second order arithmetic have been preferred for this research originally (and still are), but there are strong reasons, to study theories in more flexible languages too.

In set theories it is possible to treat problems which in (subsystems of) A_2 cannot even be formulated. Typical examples are propositions of the form $\forall x(x$ uncountable group $\rightarrow A(x))$ where sets of higher cardinalities are involved. If we consider this statement in a set theory like KPi, we have to be aware of its relativized meaning. The existence of an uncountable group is consistent with KPi but not provable in KPi. Nevertheless it is interesting to determine the proof-theoretic strength which is required for the proof of A(x), provided that x is an uncountable group.

In the case of countable mathematics, it is not clear yet, how much one gains by working in subsystems of KPi rather than $(\Delta^1_2\text{-CA}) + (\text{BI})$. It is reasonable to hope that proofs in subsystems of KPi sometimes are more perspicuous since coding can be avoided.

The main reason, however, for introducing theories for admissible sets is the observation that they present themselves as a very uniform and powerful proof-theoretic tool:

1. Although systems for admissible sets are relatively weak subsystems of ZF, they are strong enough to develop a fair amount of definability theory. Together with the expressive power of L_*, these results then make it easy to embed other systems into theories for admissible sets.

2. Theories for admissible sets are very close to initial segments of the constructible hierarchy. The relevant closure properties of these initial segments can be simulated in a system of ramified set theory which has been completely analyzed in Jäger [29] and Jäger-Pohlers [35].

To sum up, it can be said that theories for admissible sets are important for proof-theoretic research since they help to discover and exploit connections between proof theory and definability theory.

Let Th be an arbitrary theory. A typical proof-theoretic analysis of Th using theories for admissible sets proceeds as follows:

(i) Find the least α such that L_α induces a model of Th (e.g. if $Th \subset \mathcal{L}_2$, then $L_\alpha \cap Pow(N) \models Th$).

(ii) Find a theory for admissible sets Th_0 which formalizes L_α.

(iii) By restricting the induction principles in Th_0, find the weakest subtheory Th_1 of Th_0 such that

$$Th \vdash A \implies Th_1 \vdash A$$

for all sentences A of the theory Th.

(iv) Embed Th_1 into ramified set theory; calculate $|Th_1|$.

(v) In all interesting cases we have $|Th| = |Th_1|$.

A variety of examples will be presented which support the thesis that this procedure gives sharp proof-theoretic bounds. Observe that the strength of a theory Th then is determined by the following two parameters:

(a) the number of admissibles

(b) the amount of induction

needed to interpret Th in a theory of admissible sets.

The following three case-studies are for the specialists; missing definitions and proofs will be given later.

(a) $L_{\Omega_\omega} \cap Pow(N)$ is the least standard model of ATR_0. KPl is a formalization of L_{Ω_ω} in terms of iterated admissible sets. ATR_0 is contained in the subtheory KPl^0 of KPl.
$\Gamma_0 = |KPl^0| = |ATR_0|$.

(b) $L_{\Omega_\omega} \cap Pow(N)$ also is the least standard model of $(\Pi^1_1\text{-CA})$. For the embedding, however, we need more induction. $(\Pi^1_1\text{-CA})$ is contained in $KPl^r + (IND_N)$.
$\overline{\theta}(\Omega_\omega \cdot \varepsilon_0)0 = |KPl^r + (IND_N)| = |(\Pi^1_1\text{-CA})|$.

(c) $L_{i_0} \cap Pow(N)$ is the least standard model of $(\Delta^1_2\text{-CA}) + (BI)$. KPi is a formalization of L_{i_0}. $(\Delta^1_2\text{-CA}) + (BI)$ is contained in KPi but in no subtheory obtained by restricting induction.
$\overline{\theta}^0(\overline{\theta}^1\varepsilon_{i_0+1}0)0 = |KPi| = |(\Delta^1_2\text{-CA}) + (BI)|$.

Since (IND_N) and $(IND_\in)$ are true in all sets L_α, these sets are not suited for distinguishing between a theory Th and its restricted versions Th^r and Th^0. To extract more information from the sets L_α, we will consider partial models. L_α is called a Π_2 model of the theory Th if $L_\alpha \models A$ for every Π_2 sentence provable in Th.

We end this introduction with a plan of our Habilitationsschrift.

Roughly speaking, one can distinguish between predicative and impredicative proof theory. We will carry out a complete proof-theoretic analysis of predicative theories for admissible sets and will establish their connections to related systems in different languages. The approach here is very uniform and avoids special ad hoc methods. In the case of impredicative theories for admissible sets, we are interested in their exact correspondence to subsystems of A_2. Together with Buchholz-Feferman-Pohlers-Sieg [4], Pohlers [44], Jäger [31] and Jäger-Pohlers [35], this gives a clear picture of a certain amount of impredicative mathematics.

In section 0 we collect the basic syntactic notions which will be needed later. Some basic theories are introduced.

Section 1 reviews several well-known results of the definability theory of admissible sets. We put special emphasis on the real part RP(d) of an admissible set d and the companion A_X of a set X of subsets of N.

Proof theory starts in section 2, where we introduce the concepts of admissible cover Th^c and admissible extension Th^e of a theory Th. If Th is formulated in the language L_* and has intended model M, then Th^e can be regarded as formalization of the least admissible set which contains M as element. We prove, for example:

(a) $Th^e \vdash A^M \iff Th \vdash A$

(b) $Th^e + (I_N) \vdash A^M \iff Th + (IND_N) \vdash A$

(c) $Th^e \vdash^{<\alpha} A^M \implies Th \vdash^{<\phi\alpha 0} A$

for all sentences A of the language of Th and all ε-numbers α.

Th^c is the corresponding notion for theories Th in L_1. Theorems 2.1 - 2.4 give a complete proof-theoretic characterization of Th^c and Th^e, which will be used in the following two sections.

In section 3 we consider the theory KPu and its restrictions. We study their connections to subsystems of A_2 and calculate their proof-theoretic ordinals and minimal Π_2 models.

Section 4 treats theories for iterated admissible sets without foundation. The strongest results of this section are:

(a) $|KPi^o| = |KPl^o| = \Gamma_o$

(b) $KPi^o \vdash \forall x(x \text{ admissible} \rightarrow A^x) \quad \Longrightarrow \quad L_{\Gamma_o} \models A$

for every Π_2 sentence A of L_*. By proving the axiom β, comparability of well-orderings and the principle of arithmetic transfinite recursion in KPl^o, we can compare KPi^o and KPl^o with Friedman's ATR_o and Simpson's ATR_o^S defined in [21,22] and [52]. Another group of results of this section considers subtheories of KPi^o with finitely many universes and their relations to the iterated fixed point theories of Feferman [16].

In section 5 we use results of Simpson [51,52] to develop the theory of hyperarithmetic sets in the second order theory ATR_o. It follows that the class of all hyperarithmetic representation trees is, provably in ATR_o, a model of KPu^r.

It is the purpose of section 6 to underline the importance of KPi^0 for theories of strength Γ_0. We start with a short description of predicative mathematics and the Feferman-Schütte characterization of predicativity by the ordinal Γ_0. Then we survey several theories of proof-theoretic ordinal Γ_0 which have been introduced by Feferman, Friedman, Martin-Löf, Schütte, Simpson et al. for completely different philosophical and mathematical reasons. It is very striking, and supports the unifying power of KPi^0, to observe that each of these theories can be embedded into KPi^0 in a very natural way. So far we have not found a theory of strength Γ_0 where this is not true. Therefore we may regard KPi^0 as the 'strongest' theory of proof-theoretic strength Γ_0.

Sections 7 and 8 turn to impredicative theories for iterated admissible sets and to subsystems of second order arithmetic with Π^1_1 and Δ^1_2 comprehension. We concentrate on interpreting impredicative subsystems of KPi into $(\Delta^1_2\text{-CA}) + (\text{BI})$ and vice versa. As corollary we obtain that the addition of the axiom 'all sets are countable' does not increase the proof-theoretic strength of these theories.

In the appendix we use ramified set theory in order to give the detailed proofs of the main results of section 2.

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§0. PRELIMINARIES.

The essential purpose of this section is to present the languages and proof systems of first order arithmetic, second order arithmetic and set theory with urelements. Apart from that, we introduce some basic axioms and describe principles of induction on the natural numbers and $\in$-induction. All the notions we use are more or less standard; for first and second order arithmetic see any textbook on mathematical logic (e.g. Shoenfield [48]), for set theory with urelements refer to Barwise [1].

We disregard standard conventions in some insignificant cases only. Formulas are built up from atomic formulas and negated atomic formulas; the negation of an arbitrary formula is defined in the obvious way via De Morgan's laws and the law of double negation. In the setup of the proof systems we follow Tait [54,55] instead of using the more familiar Hilbert or Gentzen calculus. These modifications have some technical advantages but are not essential.

0.1. The languages L_1, L_2 and L_*.

The vocabulary of L_1 consists of constants 0,1,2,3,... for all natural numbers, countably many number variables u, v,w,x,y,z,..., a 1-ary relation symbol R, two 2-ary relation symbols Eq, Sc and two 3-ary relation symbols Pl, Ti. Here Eq stands for the primitive recursive equality relation, and

Sc, Pl, Ti stand for the graphs of the primitive recursive successor, plus and times function, respectively. L_1 terms are the number constants and (free) number variables; we denote them by i,j,k,l,m,n.

The atomic formulas are all those of the form $J(k_1,\dots,k_n)$ where J is an n-ary relation symbol. Instead of Eq(m,n) we write $m =_N n$. The formulas A of L_1 and their negations $\neg A$ are defined inductively as follows.

<u>Inductive definition of the L_1 formulas</u>.

1. If A is an atomic formula, then A and $\neg A$ are L_1 formulas of length 0.
2. If A and B are L_1 formulas of lengths m and n, then (A & B) and (A v B) are L_1 formulas of length max(m,n)+1.
3. If A(u) is an L_1 formula of length n, then $\forall x A(x)$ and $\exists x A(x)$ are L_1 formulas of length n+1.

<u>Inductive definition of the negation $\neg A$ of an L_1 formula A</u>.

1. $\neg\neg A := A$, if A is an atomic formula.
2. $\neg(A \& B) := (\neg A \vee \neg B)$; $\neg(A \vee B) := (\neg A \& \neg B)$.
3. $\neg\forall x A(x) := \exists x \neg A(x)$; $\neg\exists x A(x) := \forall x \neg A(x)$.

If we denote the length of a formula A by $|A|$, then we have $|A| = |\neg A|$.

<u>Definition of TRUE</u>. By TRUE we denote the collection of all atomic formulas and negated atomic formulas which are true under their intended interpretation. That is, we have for all number constants m,n,k:

$$\left.\begin{array}{l}(\neg)\ Eq(m,n)\\(\neg)\ Sc(m,n)\\(\neg)\ Pl(m,n,k)\\(\neg)\ Ti(m,n,k)\end{array}\right\}\begin{array}{c}\text{belongs to}\\\text{TRUE if}\end{array}\left\{\begin{array}{l}n \text{ is (not) equal to } m;\\n \text{ is (not) equal to } m+1;\\k \text{ is (not) equal to } m+n;\\k \text{ is (not) equal to } m\cdot n.\end{array}\right.$$

The second order language L_2 is an extension of L_1 obtained by adjoining countably many set variables U,V,W,X, Y,Z,... and the membership relation symbol $\in$. The atomic formulas of L_2 are the atomic formulas of L_1 and all $(k \in X)$ for arbitrary k and X.

The L_2 formulas form the smallest collection containing all atomic formulas and negated atomic formulas of L_2 closed under &, v, numerical quantification $\forall x$, $\exists x$ and the following set quantification: If A(U) is an L_2 formula, then so are $\forall XA(X)$ and $\exists XA(X)$, where the length is increased by 1. The negation $\neg A$ of an L_2 formula A is defined as the negation of an L_1 formula with the additional $\neg\forall XA(X) := \exists X\neg A(X)$ and $\neg\exists XA(X) := \forall X\neg A(X)$.

A formula A of L_2 is called <u>arithmetic</u> or Π^1_0, if no set quantifiers occur in it; A may contain free set variables. Now suppose that A is an L_2 formula of the form $Q_1X_1\ldots Q_nX_nB(X_1,\ldots,X_n)$, where $B(U_1,\ldots,U_n)$ is arithmetic and $Q_1\ldots Q_n$ is an alternating sequence of quantifiers, $n \geq 1$. If Q_1 is existential (universal), then A is said to be a Σ^1_n (Π^1_n) formula. Obviously A is Σ^1_n iff $\neg A$ is Π^1_n.

All of the set theories which we will consider comprise two different kinds of objects: urelements and sets. They are formulated in the extended language L_* in which L_1 is augmented by: a membership relation symbol $\in$, a unary relation symbol S, used to express that an object is a set, and a set constant N for the set of the natural numbers. If an object is not a set, we call it an urelement. By the ontological axioms we will determine later that the urelements are precisely the natural numbers. The terms of L_* are the number constants, the constant N and the free variables; they are denoted by a,b,c,d,.... Since L_* is not so familiar, we list the basic definitions.

Inductive definition of the Δ_0 formulas of L_*.

1. If J is an n-ary relation symbol of L_*, then $J(a_1,\dots,a_n)$ and $\neg J(a_1,\dots,a_n)$ are Δ_0 formulas of length 0.
2. If A and B are Δ_0 formulas of lengths m and n, then $(A \,\&\, B)$ and $(A \vee B)$ are Δ_0 formulas of length max(m,n)+1.
3. If A(u) is a Δ_0 formula of length n, then $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are Δ_0 formulas of length n+2.

Inductive definition of the L_* formulas.

1. Each Δ_0 formula of length n is an L_* formula of length n.
2. If A and B are L_* formulas of length m and n, then $(A \,\&\, B)$ and $(A \vee B)$ are L_* formulas of length max(m,n)+1.
3. If A(u) is an L_* formula of length n, then $\forall x A(x)$ and $\exists x A(x)$ are L_* formulas of length n+1 and $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are L_* formulas of length n+2.

Inductive definition of the Σ and Π formulas of L_*.

1. Each Δ_0 formula is a Σ and Π formula.

2. If A and B are Σ (Π) formulas, then (A & B) and (A v B) are Σ (Π) formulas.

3. If A(u) is a Σ (Π) formula, then $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are Σ (Π) formulas.

4. If A(u) is a Σ formula, then $\exists x A(x)$ is a Σ formula.

5. If A(u) is a Π formula, then $\forall x A(x)$ is a Π formula.

Definition of the Σ_n and Π_n formulas of L_*, $n \geq 1$.

If $A(u_1,\ldots,u_n)$ is a Δ_0 formula, then $\exists x_1 \forall x_2 \ldots Q_n x_n A(x_1,\ldots,x_n)$ is a Σ_n formula and $\forall x_1 \exists x_2 \ldots Q_n x_n A(x_1,\ldots,x_n)$ a Π_n formula.

The negation $\neg A$ of an L_* formula A is defined as the negation of an L_1 formula with the extra clauses: $\neg(\forall x \in a)A(x) := (\exists x \in a)\neg A(x)$ and $\neg(\exists x \in a)A(x) := (\forall x \in a)\neg A(x)$. Then we have: A is Δ_0 iff $\neg A$ is Δ_0; A is Σ iff $\neg A$ is Π; A is Σ_n iff $\neg A$ is Π_n. If A is an L_* formula, then A^a is the result of replacing each unrestricted quantifier $\forall x(\ldots)$ and $\exists x(\ldots)$ in A by $(\forall x \in a)(\ldots)$ and $(\exists x \in a)(\ldots)$, respectively. To increase readability, we sometimes drop or insert brackets.

Definition.

1. $a \subset b \;:\Longleftrightarrow\; (\forall x \in a)(x \in b)$.

2. $a = b \;:\Longleftrightarrow\; \begin{cases} (\neg S(a) \;\&\; \neg S(b) \;\&\; a =_N b) \;\vee \\ (S(a) \;\&\; S(b) \;\&\; a \subset b \;\&\; b \subset a) \;. \end{cases}$

3. $\mathrm{Tran}(a) \;:\Longleftrightarrow\; (\forall x \in a)(\forall y \in x)(y \in a)$.

$$4.\ a = \{x \in b : A(x)\} \ :\Longleftrightarrow \begin{cases} S(a)\ \&\ (\forall x \in a)(x \in b\ \&\ A(x))\ \& \\ (\forall x \in b)(\neg A(x) \vee x \in a)\ . \end{cases}$$

In each language L_1, L_2 and L_* we use some standard abbreviations. So $A \rightarrow B$ and $A \leftrightarrow B$ stand for $\neg A \vee B$ and $(A \rightarrow B)\ \&\ (B \rightarrow A)$, respectively. Underlined letters denote finite sequences of terms or variables; e.g. $\underline{a} = a_1,\ldots,a_n$, $\underline{k} = k_1,\ldots,k_n$, $\underline{U} = U_1,\ldots,U_n$.

Convention. The notation $A[\underline{u}]$ $(A[\underline{U},\underline{v}])$ is used to indicate that all free variables of A come from the list $\underline{u}$ $(\underline{U},\underline{v})$; $A(\underline{u})$ $(A(\underline{U},\underline{v}))$ may contain other free variables besides $\underline{u}$ $(\underline{U},\underline{v})$.

A sentence is a formula without free variables. A formula A is called R-positive, if all appearances of R in A are positive, i.e. A contains no subformula $\neg R(a)$. We often write $A(R^+)$ for an R-positive formula A.

0.2. The proof systems Z_1, Z_2 and Z_*.

For each language L_1, L_2 and L_* we define Tait-style proof systems Z_1, Z_2 and Z_* (cf. [54,55]). The rules of inference are formulated for finite sets of formulas Γ, Λ, Γ_1, Λ_1, ..., which should be interpreted disjunctively. We write (for example) Γ, Λ, A, B for the union of Γ, Λ and $\{A,B\}$.

Basic rules of Z_1, Z_2 and Z_*.

(B) $\Gamma , \neg A , A$, if A is atomic;

(T) Γ , A , if A is in TRUE.

Normal rules of Z_1.

$$(\&)\quad \frac{\Gamma , A_0 \qquad \Gamma , A_1}{\Gamma , A_0 \,\&\, A_1} \qquad\qquad (\vee)\quad \frac{\Gamma , A_i \quad \text{for } i = 0 \text{ or } i = 1}{\Gamma , A_0 \vee A_1}$$

$$(\forall x)\quad \frac{\Gamma , A(u)}{\Gamma , \forall x A(x)} \qquad\qquad (\exists x)\quad \frac{\Gamma , A(k)}{\Gamma , \exists x A(x)}$$

where in $(\forall x)$ the free variable u is not to occur in the conclusion.

Normal rules of Z_2.

(&), (v), $(\forall x)$, $(\exists x)$ as before;

$$(\forall X)\quad \frac{\Gamma , A(U)}{\Gamma , \forall X A(X)} \qquad\qquad (\exists X)\quad \frac{\Gamma , A(U)}{\Gamma , \exists X A(X)}$$

where in $(\forall X)$ the free variable U is not to occur in the conclusion.

Normal rules of Z_*.

(&), (v) as before;

$$(b\forall)\ \frac{\Gamma,\ u \in a \to A(u)}{\Gamma,\ (\forall x \in a)A(x)} \qquad (b\exists)\ \frac{\Gamma,\ b \in a\ \&\ A(b)}{\Gamma,\ (\exists x \in a)A(x)}$$

$$(\forall)\ \frac{\Gamma,\ A(u)}{\Gamma,\ \forall x A(x)} \qquad (\exists)\ \frac{\Gamma,\ A(b)}{\Gamma,\ \exists x A(x)}$$

where in $(b\forall)$ and $(\forall)$ the free variable u is not to occur in the conclusion.

<u>Cut rule of Z_1, Z_2 and Z_*.</u>

$$(cut)\ \frac{\Gamma,\ A \qquad \Gamma,\ \neg A}{\Gamma}$$

The degree of this cut is the length $|A| = |\neg A|$ of its cut formulas.

Let L be one of the languages L_1, L_2 or L_*, Γ a finite set of L formulas and Z the corresponding proof system. Derivability of Γ in Z, $Z \vdash \Gamma$, is defined inductively.

<u>Inductive definition of $Z \vdash^n_m \Gamma$.</u>

1. If Γ is a basic rule of Z, then $Z \vdash^n_m \Gamma$ for all natural numbers m and n.
2. If $Z \vdash^{n_i}_m \Gamma_i$ and $n_i < n$ for every premise Γ_i of a normal rule or a cut of degree $< m$, then we have $Z \vdash^n_m \Gamma$ for the conclusion Γ of that inference.

$Z \vdash \Gamma$ $(Z \vdash_0 \Gamma)$ means $Z \vdash^n_m \Gamma$ $(Z \vdash^n_0 \Gamma)$ for some m and n (some n). If $Z \vdash_0 \Gamma$, then this derivation is called cut-free since it does not use the cut rule. Cut elimination for the proof systems Z is the usual cut elimination for propositional logic; cf. e.g. [54].

Theorem 0.1 (Cut elimination).

(a) $Z \vdash^{n}_{m+1} \Gamma \implies Z \vdash^{2^n}_{m} \Gamma$;

(b) $Z \vdash \Gamma \implies Z \vdash_{0} \Gamma$.

A theory Th in L is specified by a set Ax(Th) of L formulas, the axioms of Th.

$$\mathrm{Th} \vdash^{n}_{m} \Gamma$$

means that there exists a finite set $\{A_1,\dots,A_k\}$ of universal closures of axioms of Th such that

$$Z \vdash^{n}_{m} \{\neg A_1,\dots,\neg A_k\}, \Gamma \quad .$$

$\mathrm{Th} \vdash \Gamma$ is defined analogously. Put in other terms (if L is L_1 or L_*), a finite set Λ of L formulas is called a Th-axiom set, if each element of Λ is the negation of a Th axiom or has the form $\exists x_1 \dots \exists x_k \neg A[\underline{a}, x_1,\dots,x_n]$, where $A[\underline{u}, v_1,\dots,v_k]$ is an axiom of Th. Then $\mathrm{Th} \vdash \Gamma$ if $Z \vdash \Lambda,\Gamma$ for some Th-axiom set Λ.

In general we will work rather informally in a given theory. If we write 'Lemma n.k (Th)' or 'Theorem n.k (Th)', this means that the statement of Lemma n.k or Theorem n.k is provable in the theory Th.

Let Th_1 and Th_2 be theories in L. We write $\mathrm{Th}_1 \subset \mathrm{Th}_2$ if every theorem of Th_1 is a theorem of Th_2. We write $\mathrm{Th}_1 \equiv \mathrm{Th}_2$ if $\mathrm{Th}_1 \subset \mathrm{Th}_2$ and $\mathrm{Th}_2 \subset \mathrm{Th}_1$. If C is a class of L formulas, then $\mathrm{Th} + C$ is the extension of Th with the formulas in C as additional axioms.

0.3. The theories PA, EA and ES.

The axioms of Peano arithmetic PA are the axioms for the successor function, the recursive definitions of plus and times, the equality axioms and the scheme of complete induction on the natural numbers.

1. Successor.

$\neg Sc(k,0)$;

$k \neq_N 0 \rightarrow \exists x Sc(x,k)$;

$\exists x Sc(k,x)$;

$Sc(n,n+1)$ for every number constant n;

$Sc(k,i) \,\&\, Sc(k,j) \rightarrow i =_N j$;

$Sc(i,k) \,\&\, Sc(j,k) \rightarrow i =_N j$.

2. Plus.

$Pl(k,m,i) \,\&\, Pl(k,m,j) \rightarrow i =_N j$;

$\exists x Pl(k,m,x)$;

$Pl(k,0,k)$;

$Pl(k,m,n) \,\&\, Sc(m,i) \,\&\, Sc(n,j) \rightarrow Pl(k,i,j)$.

3. Times.

$Ti(k,m,i) \,\&\, Ti(k,m,j) \rightarrow i =_N j$;

$\exists x Ti(k,m,x)$;

$Ti(k,0,0)$;

$Ti(k,m,n) \,\&\, Sc(m,i) \,\&\, Pl(n,k,j) \rightarrow Ti(k,i,j)$.

4. Equality.

$k =_N k$;

$k =_N m \rightarrow (A(k) \rightarrow A(m))$ for all atomic formulas.

5. Complete induction.

(IND_N) $A(0)$ & $\forall x \forall y(A(x)$ & $Sc(x,y) \rightarrow A(y)) \rightarrow \forall x A(x)$

for arbitrary formulas.

Elementary analysis EA is one of our two basic systems of second order arithmetic. In addition to the axioms of PA with the schemes of equality and complete induction formulated for L_2 formulas, it contains the axiom scheme $(\Pi^1_0\text{-CA})$ of arithmetic comprehension

$(\Pi^1_0\text{-CA})$ $\exists X \forall x(x \in X \leftrightarrow A(x))$

for any arithmetic formula $A(u)$. EA_0 is the subsystem of EA where the scheme of complete induction (IND_N) is replaced by the induction axiom

(I_N) $\forall X[0 \in X$ & $\forall x \forall y(x \in X$ & $Sc(x,y) \rightarrow y \in X) \rightarrow \forall x(x \in X)]$.

If C is a collection of L_2 formulas, we write C (C_0) for the theory $EA + C$ $(EA_0 + C)$. To obtain an increasing sequence of second order theories we may consider the following schemes of Π^1_n and Δ^1_n comprehension

$(\Pi^1_n\text{-CA})$ $\exists X \forall x(x \in X \leftrightarrow A(x))$

$(\Delta^1_n\text{-CA})$ $\forall x(A(x) \leftrightarrow B(x)) \rightarrow \exists X \forall x(x \in X \leftrightarrow A(x))$

for all Π^1_n formulas $A(u)$ and Σ^1_n formulas $B(v)$. Then we have:

$$(\Pi^1_0\text{-CA}) \subset (\Delta^1_1\text{-CA}) \subset (\Pi^1_1\text{-CA}) \subset (\Delta^1_2\text{-CA}) \subset \dots$$

Finally, full classical analysis A_2 is EA plus unlimited comprehension

(CA) $\exists X \forall x (x \in X \leftrightarrow A(x))$

for arbitrary L_2 formulas A(u). In the presence of (CA), the axiom of induction (I_N) is equivalent to the scheme (IND_N).

We now present three versions of elementary set theory: ES, ES^r and ES^o. These theories have the same set existence axioms but differ in the assumptions they make about foundation and complete induction on the natural numbers. ES, ES^r and ES^o are formulated in L_*; the axioms of ES are divided into the following five groups.

1. Ontological axioms. These axioms can be considered as implicit definitions of the predicate S and the set constant N:

$S(a) \leftrightarrow a \notin N$;

$a \in b \rightarrow S(b)$;

$J(\underline{a}) \rightarrow \underline{a} \in N$ for every relation symbol J of L_1.

2. Number-theoretic axioms. For every axiom $A[\underline{u}]$ of PA:

$\underline{a} \in N \rightarrow A^N[\underline{a}]$.

3. Equality axioms.

$a = a$;

$a = b \rightarrow (A(a) \rightarrow A(b))$ for every atomic formula of L_*.

4. Set existence axioms.

(Pair) $\exists z(a \in z \ \& \ b \in z)$;

(Transitive Hull) $\exists z(a \subset z \ \& \ \mathrm{Tran}(z))$;

(Δ_0-Sep) $\exists z(z = \{x \in a : A(x)\})$ for all Δ_0 formulas $A(u)$.

This scheme is called Δ_0 separation.

5. Axioms of induction. These axioms provide complete induction on the natural numbers (IND_N) and the usual $\in$-induction ($\mathrm{IND}_\in$); both for arbitrary L_* formulas:

(IND_N) $A(0) \ \& \ (\forall x, y \in N)(A(x) \ \& \ Sc(x,y) \rightarrow A(y)) \rightarrow (\forall x \in N)A(x)$.

($\mathrm{IND}_\in$) $\forall x((\forall y \in x)A(y) \rightarrow A(x)) \rightarrow \forall x A(x)$.

ES^r is taken to be the theory ES with (IND_N) and ($\mathrm{IND}_\in$) replaced by the axioms (I_N) and ($I_\in$):

(I_N) $0 \in a \ \& \ (\forall x, y \in N)(x \in a \ \& \ Sc(x,y) \rightarrow y \in a) \rightarrow (\forall x \in N)(x \in a)$;

($I_\in$) $(\exists x \in a)(0 = 0) \rightarrow (\exists x \in a)(\forall y \in a)(y \notin x)$.

ES^o is taken to be ES with (IND_N) restricted to (I_N) and ($\mathrm{IND}_\in$) omitted completely.

In general, if C is a class of L_* formulas, then (C-IND_N) and (C-$\mathrm{IND}_\in$) denote the schemes (IND_N) and ($\mathrm{IND}_\in$) restricted to the formulas in C. Over ES^o, (I_N) and ($I_\in$) are equivalent to (Δ_0-IND_N) and (Δ_0-$\mathrm{IND}_\in$).

The significance of the Σ formulas comes from the principle of Σ persistency. It is logically valid; here we state it for ES^0.

<u>Σ persistency</u>. For every Σ formula A we have

$$ES^0 \vdash a \subset b \ \& \ A^a \rightarrow (A^b \ \& \ A) .$$

0.4. <u>Embedding of L_2 into L_*</u>.

With each L_2 formula $A[\underline{U},\underline{v}]$ we associate an L_* formula $A^*[\underline{u},\underline{v}]$ defined by

$$\underline{v} \in N \ \& \ \underline{u} \in Pow(N) \rightarrow A'[\underline{u},\underline{v}] .$$

$u \in Pow(N)$ is an abbreviation of the L_* formula $S(u) \ \& \ u \subset N$ and A' results from A by replacing all numerical quantifiers $\forall x(\ldots)$, $\exists x(\ldots)$ in A by $(\forall x \in N)(\ldots)$, $(\exists x \in N)(\ldots)$ and all set quantifiers $\forall Y(\ldots)$, $\exists Y(\ldots)$ in A by $\forall y(y \in Pow(N) \rightarrow \ldots)$, $\exists y(y \in Pow(N) \ \& \ \ldots)$.

In this sense L_2 can be considered as a sublanguage of L_*. When working in the context of set theories, we often identify the L_2 formula $A[\underline{U},\underline{v}]$ and its translation $A^*[\underline{u},\underline{v}]$. Clearly we have for every L_2 formula A:

$$EA_0 \vdash A \implies ES^0 \vdash A ;$$

$$EA \vdash A \implies ES^0 + (IND_N) \vdash A .$$

0.5. Extensions of L_2 and L_*.

For the actual work in L_2 and L_*, it is often very convenient to use formulations with function symbols for primitive recursive functions (at least). So assume that L is L_2 or L_* and Th a theory in L. The language L(PR) results from L by adding a function constant for every primitive recursive function and extending the syntax accordingly. Th(PR) is Th formulated in L(PR) plus the defining equations for the primitive recursive functions. The following remark is then obvious.

Remark. Let Th be a theory in L, $EA_0 \subset Th$. For every formula A of L(PR) there exists a formula B of L with the same free variables such that:

(i) $EA_0(PR) \vdash A \leftrightarrow B$.

(ii) If A is a Π^1_n (Σ^1_n, Δ_0, Π, Σ) formula of L(PR), then B is a Π^1_n (Σ^1_n, Δ_0, Π, Σ) formula of L.

(iii) $Th(PR) \vdash A \iff Th \vdash B$.

Therefore we can identify Th(PR) and Th in all relevant cases.

If Th is a set theory in L_*, we have to consider the following definitional extension as well. Suppose that A describes in Th a set-theoretic function on the universe,

$$Th \vdash \forall \underline{x} \exists ! y A[\underline{x}, y] .$$

We then adjoin a new function symbol F_A and the axiom

$$(F_A) \qquad \forall\underline{x}\forall y(F_A(\underline{x}) = y \leftrightarrow A[\underline{x},y]) .$$

A function symbol F_A is called a Σ function symbol if its definition clause A is a Σ formula.

A formula $B(F_A(\underline{a}))$ is considered as an abbreviation for $\exists x(A[\underline{a},x] \;\&\; B(x))$. One has to be careful, however, since the Δ_0 formulas are not closed under quantification restricted by one of these function symbols. If $B(u)$ is Δ_0, then $(\exists x \in F_A(\underline{a}))B(x)$ stands for $\exists y(A[\underline{a},y] \;\&\; (\exists x \in y)B(x))$ and is in general not a Δ_0 formula.

<u>Convention</u>. Strictly speaking we use $=_N$ and $<_N$ as relation symbols for the primitive recursive equality and less relation on the natural numbers. In L_1 and L_2, however, we often drop the subscript N and write $=$ and $<$ instead of $=_N$ and $<_N$, respectively.

§1. ADMISSIBLE SETS.

In this section we turn to the central object of our interest, admissible sets. The following reviews some of their definability theory and serves as motivation for the proof-theoretic work in the subsequent sections. The reader who is interested in a detailed exposition of the set-theoretic, model-theoretic or recursion theoretic aspects of admissible sets and structures may consult Barwise's monograph [1]. Here we confine ourselves to relevant notions and consequences.

1.1. Structures for L_1, L_2 and L_*.

A structure for L_1 is an ordered 6-tuple

$$M = (|M|, \mathrm{Eq}^M, \mathrm{Sc}^M, \mathrm{Pl}^M, \mathrm{Ti}^M, \mathrm{R}^M)$$

with

(i) $|M|$ is a set which contains the natural numbers;
(ii) Eq^M and Sc^M are 2-ary relations on $|M|$;
(iii) Pl^M and Ti^M are 3-ary relations on $|M|$;
(iv) R^M is a 1-ary relation on $|M|$.

In the L_1 structure M, $|M|$ is the range of the number variables; Eq^M, Sc^M, Pl^M, Ti^M and R^M are interpretations of the relation symbols Eq, Sc, Pl, Ti and R, respectively.

A structure for L_2 is a pair $(M;S)$ where
(i) M is a structure for L_1;
(ii) S is a non-empty collection of subsets of $|M|$;
(iii) $|M| \cap S = \emptyset$.

A <u>structure for L_*</u> is a triple $(M;a,E)$ where

(i) M is a structure for L_1;

(ii) a is a set such that $|M| \in a$;

(iii) $E \subset a \times (a-|M|)$.

In the L_2 structure $(M;S)$, S is the range of the set variables of L_2. In the L_* structure $(M;a,E)$, a is the range of the variables of L_*, $|M|$ is the interpretation of the set constant N, $a-|M|$ is the interpretation of the relation symbol S and E is the interpretation of the membership relation $\in$.

If L is one of the languages L_1, L_2 or L_* and K is an L structure, then the notion of truth in K, $K \models A$ for L formulas A, is defined as usual; cf. e.g. Shoenfield [48]. K is called a model of the formal system Th if each theorem of Th is true in K. Observe that the ontological axioms are true in all L_* structures.

1.2. <u>Standard structures</u>.

Fortunately, we do not have to consider arbitrary structures but can concentrate on so called standard structures.

A <u>standard structure for L_1</u> is a 6-tuple

$$(N,Q) = (N,\ Eq^N,\ Sc^N,\ Pl^N,\ Ti^N,\ Q)$$

where N is the set of natural numbers, $Eq^N = \{(x,x) : x \in N\}$

$Sc^{N} = \{(x,x+1) : x \in N\}$, $Pl^{N} = \{(x,y,x+y) : x,y \in N\}$, $Ti^{N} = \{(x,y,x \cdot y) : x,y \in N\}$ and Q is a subset of N that interprets R.

The <u>standard structures for L_2</u> are all structures of the form $(N,Q;S)$ where S is a collection of subsets of N with $Q \in S$. Very often we identify $(N,\emptyset;S)$ with S and speak of the L_2 structure S.

Before saying more about L_* structures, we fix the largest possible universe of sets over the urelement structure N once and for all. We write Ord(a) if a is an ordinal and use small Greek letters to denote ordinals; the class of all ordinals is called ON. By recursion on ON we define:

$$
\begin{aligned}
V_N(0) &:= \emptyset\ ; \\
V_N(\alpha+1) &:= \mathrm{Pow}(N \cup V_N(\alpha))\ ; \\
V_N(\lambda) &:= \bigcup\{V_N(\xi) : \xi < \lambda\}\ , \quad \text{if } \lambda \text{ is a limit;} \\
V_N &:= \bigcup\{V_N(\xi) : \mathrm{Ord}(\xi)\}
\end{aligned}
$$

where Pow(a) is the power set of a. By letting $V_N(0) = \emptyset$ rather than $V_N(0) = N$, we achieve that V_N does not contain natural numbers. V_N is the collection of all sets over the natural numbers as urelements. The standard structure N, the universe of sets V_N and the membership relation $\in$ on $(N \cup V_N) \times V_N$ constitute our set-theoretic world.

<u>Standard structures for L_*</u> are 4-tuples $(N,Q;a,\in)$ where a is a transitive set in V_N which contains N and $Q \subset N$ as

elements. If a is such a set, we often write $(Q;a)$ for the structure $(N,Q;a,\in)$ and a for $(\emptyset;a)$; it is always clear from the context whether we mean the set or the structure.

Definition. The ordinal of a, denoted by $o(a)$, is the least ordinal not in a. We define for $a \in V_N$:

$$o(a) := \min\{\xi : \xi \notin a\} .$$

1.3. Constructible sets.

Most important for us are standard structures which are determined by initial segments of the constructible hierarchy à la Gödel (cf. [1,48]). Suppose that a is an element of V_N. $L_\alpha(a)$, the collection of sets constructible from a in α steps, is generated from a by iterating the operation $b \to \mathrm{Def}(b)$ α times;

$$\mathrm{Def}(b) := \{x \subset b : x \text{ is definable over } (N,\emptyset;b,\in) \text{ by some } L_* \text{ formula with parameters from } b\} .$$

Let $TC(a)$ be the transitive closure of a, i.e. the smallest transitive set $x \supset a$, and define:

$$L_0(a) := N \cup \{N,a\} \cup TC(a) ;$$

$$L_{\alpha+1}(a) := \mathrm{Def}(L_\alpha(a)) ;$$

$$L_\lambda(a) := \bigcup\{L_\xi(a) : \xi < \lambda\} , \text{ if } \lambda \text{ is limit;}$$

$$L(a) := \bigcup\{L_\xi(a) : \mathrm{Ord}(\xi)\} .$$

N and a are elements of $L_\alpha(a)$; the ordinals of $L_\alpha(a)$ are at least the ordinals less than α; $L_\alpha(a)$ is a transitive set in V_N. We write L_α and L for $L_\alpha(\emptyset)$ and $L(\emptyset)$, respectively.

Caution. Our definition of $L_\alpha(a)$ slightly departs from the usual definition of the constructible hierarchy since we construct sets over the ground structure N; each set $L_\alpha(a)$ is infinite for example.

1.4. Admissible sets.

The most important notion of this Habilitationsschrift is the notion of admissible set or, more precisely, admissible set above the natural numbers as urelements. Arbitrary admissible sets with or without urelements are treated for example in Barwise [1]; here we can concentrate on those admissibles which are characterized as the standard models of the theory KPu.

Definition. KPu is the theory ES plus the scheme of Δ_0 collection

$$(\Delta_0\text{-Col}) \quad (\forall x \in a)\exists y A(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)A(x,y)$$

for all Δ_0 formulas A.

Definition.

(a) A set $a \in V_N$ is called an admissible set above N if a is transitive and $a \models \text{KPu}$.

(b) An ordinal α is called admissible above N if L_α is an admissible set above N.

Remark and convention. In the last definition we used the phrase 'above *N*' in order to emphasize the importance of the ground structure *N*. Since N is an element of L_ω, we can easily see that L_ω is not a model of Δ_0 collection, and so ω is not an admissible ordinal above *N*. The admissible ordinals as defined in the literature (cf. e.g. [1]) are exactly the admissible ordinals above *N* and the ordinal ω. We will use 'admissible' to mean 'admissible above *N*'. There is no great harm in doing this since each admissible set above *N* is admissible in the usual sense.

Before we give some examples of admissible sets, we need a definition. A set a is called hereditarily countable if there exists a 1-1 function f with domain TC(a) and range a subset of N, i.e. $f : TC(a) \xrightarrow{1-1} N$.

Definition. $HC := \{a \in V_N : a \text{ is hereditarily countable}\}$.

Lemma 1.1.
(a) HC is admissible;
(b) If κ is a cardinal greater than ω, then $V_N(\kappa)$ and L_κ are admissibles.

This lemma is obvious for HC and regular cardinals but requires some argument for singular cardinals. As a consequence we have that each element in V_N is contained in an admissible set. The following definition and theorem characterize the smallest admissible set a^+ which contains a.

For the proofs of Lemma 1.1 and Theorem 1.1 see [1].

Definition (Next admissible). For $a, \alpha \in V_N$:

(a) $a^+ := \bigcap\{x \in V_N : a \in x \;\&\; x \text{ is admissible}\}$;

(b) $\alpha^+ := \min\{\xi \in V_N : \alpha < \xi \;\&\; \xi \text{ is admissible}\}$.

Theorem 1.1.

(a) a^+ is the smallest admissible set containing a;

(b) $a^+ = L_\alpha(a)$ for $\alpha = o(a^+)$;

(c) $\alpha^+ = o(L_\alpha{}^+)$.

Definition.

(a) We define by recursion on the ordinals:

$$\Omega_0 := \omega ;$$

$$\Omega_{\alpha+1} := \Omega_\alpha{}^+ ;$$

$$\Omega_\lambda := \sup\{\Omega_\xi : \xi < \lambda\} , \text{ if } \lambda \text{ is limit.}$$

(b) $i_o := \min\{\Omega_\xi : \Omega_\xi \text{ is admissible} \;\&\; \xi \text{ is limit}\}$.

Ω is the function which enumerates the admissible ordinals and their limits. The least fixed-point of Ω is a limit of admissibles but not admissible. The least admissible fixed-point of Ω is the first recursively inaccessible ordinal i_o. If λ is a limit ordinal, then L_{Ω_λ} is a model of the axiom $\forall x \exists y(x \in y \;\&$ y is admissible). L_{i_o} is the least standard model of KPu + $\forall x \exists y(x \in y \;\&\; y$ is admissible).

Next admissible sets have some interesting applications in characterizing Π^1_1 and Δ^1_1 subsets of N. The following result emerges from the work of several logicians and allows different generalizations. The version here is a special case of a theorem in [1].

<u>Definition</u>. Suppose that P and Q are subsets ot N and a is an admissible set.

(a) P is Π^1_1 in Q if there exists a Π^1_1 formula $A[u,\underline{v},X]$ of L_2 and $\underline{n} \in N$ such that for all $m \in N$:

$$m \in P \iff Pow(N) \models A[m,\underline{n},Q] ;$$

P is Δ^1_1 in Q if P and N-P are Π^1_1 in Q.

(b) P is Σ_1 on a if there exists a Σ_1 formuls $A[u,\underline{v}]$ of L_* and $\underline{c} \in a$ such that for all $b \in a$:

$$b \in P \iff a \models A[b,\underline{c}] ;$$

P is Δ_1 on a if P and N-P are Σ_1 on a.

<u>Theorem 1.2</u>. If P and Q are subsets of N, then we have

(a) P is Π^1_1 in Q iff P is Σ_1 on Q^+;

(b) P is Δ^1_1 in Q iff P is Δ_1 on Q^+;

(c) P is Δ^1_1 in Q iff $P \in Q^+$.

1.5. <u>Positive inductive definitions</u>.

Let Q be a subset of N and $A[R^+,X,x]$ an R-positive arithmetic formula of L_1. Then A and Q induce a monotone operator Γ^Q_A from Pow(N) to Pow(N):

$$\Gamma^Q_A : Pow(N) \rightarrow Pow(N) ; \qquad \Gamma^Q_A(X) := \{x \in N : N \models A[X,Q,x]\} .$$

Γ_A^Q is monotone (i.e. $X \subset Y$ implies $\Gamma_A^Q(X) \subset \Gamma_A^Q(Y)$), and so the least fixed point of Γ_A^Q can be constructed by the following recursion on the ordinals:

$$I_A^\alpha(Q) \quad := \{x \in \mathbb{N} : \mathcal{N} \models A[I_A^{<\alpha}(Q), Q, x]\} ;$$

$$I_A^{<\alpha}(Q) := \cup\{I_A^\xi(Q) : \xi < \alpha\} ;$$

$$I_A(Q) \quad := \cup\{I_A^\xi(Q) : \mathrm{Ord}(\xi)\} .$$

By a cardinality argument there exists an ordinal γ such that $I_A^\gamma(Q) = I_A^{<\gamma}(Q)$. The least γ with this property is called the closure ordinal of Γ_A^Q and denoted by $|\Gamma_A^Q|$. If n is an element of $I_A(Q)$, then we define its inductive norm $|n|_A^Q$ by

$$|n|_A^Q := \min\{\xi : n \in I_A^\xi(Q)\}$$

and obtain $|\Gamma_A^Q| = \sup\{|n|_A^Q+1 : n \in I_A(Q)\}$.

Typical examples of such inductive definitions are Kleene's $\mathcal{O}$ and its relativized versions $\mathcal{O}^Q$. Let $\{e\}$ be an enumeration of all partial recursive functions and $\{e\}^Q$ an enumeration of all partial functions recursive in Q. Tot(e) and $\mathrm{Tot}^Q(e)$ express that $\{e\}$ and $\{e\}^Q$ are total, respectively. Let $<_{\mathcal{O}}^Q$ be the smallest transitive relation such that

(i) $k <_{\mathcal{O}}^Q 2^k$ for all k;

(ii) $\{e\}^Q(k) <_{\mathcal{O}}^Q 3\cdot 5^e$ for all e,k with $k \in \mathrm{dom}(\{e\}^Q)$.

Clearly $k <_{\mathcal{O}}^Q m$ is definable by an arithmetic formula $B[Q,k,m]$.

Definition.

(a) $\mathcal{CO}(R,Q,n) \;:\Longleftrightarrow\; n = 0 \lor \exists x(n = 2^x \;\&\; R(x)) \lor \exists e[n = 3\cdot 5^e \;\&$
$\forall y(\{e\}^Q(y) <_{\mathcal{O}}^Q \{e\}^Q(y+1) \;\&\; R(\{e\}^Q(y)))]$;

(b) $\mathcal{O}^Q(n) \;:\Longleftrightarrow\; \forall Z[\forall x(\mathcal{CO}(Z,Q,x) \to x \in Z) \to n \in Z]$;

(c) $\omega_1^Q := |\Gamma_{\mathcal{CO}}^Q|$.

$\mathcal{O}^Q$ defines the least fixed point of the monotone operator $\Gamma_{\mathcal{CO}}^Q$. Further, $|\Gamma_A^Q| \leq \omega_1^Q$ for all R-positive arithmetic formulas $A[R^+,X,x]$ since $\mathcal{O}^Q$ is complete Π_1^1 in Q (cf. [45]). If Q is the empty set, we write $\mathcal{O}$ and ω_1^{ck} instead of $\mathcal{O}^Q$ and ω_1^Q, respectively; ck stands for Church-Kleene.

The following theorem indicates the close connections between positive inductive definitions and admissible ordinals. The first part is folklore in recursion theory, proved e.g. in [1]. The second part is a famous result of Sacks and Friedman-Jensen [24].

Theorem 1.3.
(a) $o(Q^+) = \omega_1^Q$ for every subset Q of N.
(b) For every countable admissible ordinal α $(>\omega)$ there exists a subset Q of N such that $\alpha = \omega_1^Q$.

1.6. The real part of an admissible set.

It is common practice in mathematical logic to identify the real numbers with the subsets of N. The real part of an admissible set a is therefore defined as the set of all reals in a.

Definition (Real part). For $a \in V_N$:

$$RP(a) := \{x \in a : x \subset N\} .$$

The problem is to characterize those sets of reals which are real parts of admissible sets. However, before going into these matters, we would like to emphasize Lemma 1.2 below. It follows from Lemma 3.2 and Corollary 7.1 and deals with three simple but important types of real parts.

Lemma 1.2.

(a) For α admissible: $RP(L_\alpha) \models (\Delta^1_1\text{-CA})$.

(b) For α limit of admissibles: $RP(L_\alpha) \models (\Pi^1_1\text{-CA})$.

(c) For α recursively inaccessible: $RP(L_\alpha) \models (\Delta^1_2\text{-CA})$.

By Σ^1_n choice (Σ^1_n-AC) is meant the axiom scheme

$$(\Sigma^1_n\text{-AC}) \quad \forall x \exists X A(x,X) \rightarrow \exists Z \forall x A(x,(Z)_x)$$

for any Σ^1_n formula A of L_2. Here we use $\langle .,. \rangle$ as a standard pairing function on N and write $m \in (Z)_n$ for $\langle m,n \rangle \in Z$. It is well known that (Δ^1_n-CA) is a consequence of (Σ^1_n-AC).

Definition. Let $Q(m,n)$ be a formula of L_2, possibly with parameters. If we wish to regard Q as a binary relation rather than a formula, we write $m\,Q\,n$ for $Q(m,n)$. Then we define for any L_2 formula A(x) the following L_2 formulas:

(a) $FD(Q,m) :\Longleftrightarrow \exists x(m\,Q\,x \vee x\,Q\,m)$ (field);

(b) $LO(\mathcal{Q}) :\Longleftrightarrow \begin{cases} \forall x \neg(x\,\mathcal{Q}\,x) \\ \forall x,y,z(x\,\mathcal{Q}\,y \,\&\, y\,\mathcal{Q}\,z \to x\,\mathcal{Q}\,z) \\ \forall x,y[FD(\mathcal{Q},x) \,\&\, FD(\mathcal{Q},y) \to (x\,\mathcal{Q}\,y \vee x=y \vee y\,\mathcal{Q}\,x)] \end{cases}$

(linear ordering);

(c) $PROG(\mathcal{Q},A) :\Longleftrightarrow \forall x[\forall y(y\,\mathcal{Q}\,x \to A(y)) \to A(x)]$ (progressiveness);

(d) $TI(\mathcal{Q},A) :\Longleftrightarrow PROG(\mathcal{Q},A) \to \forall x A(x)$ (transfinite induction);

(e) $WF(\mathcal{Q}) :\Longleftrightarrow \forall X TI(\mathcal{Q},X)$ (well-foundedness);

(f) $WO(\mathcal{Q}) :\Longleftrightarrow LO(\mathcal{Q}) \,\&\, WF(\mathcal{Q})$ (well-ordering).

If $\mathcal{Q}(m,n)$ is the formula $\langle m,n\rangle \in X$ we write $FD(X,m)$, $LO(X)$, $PROG(X,A)$, $TI(X,A)$, $WF(X)$ and $WO(X)$ instead of $FD(\mathcal{Q},m)$, $LO(\mathcal{Q})$, $Prog(\mathcal{Q},A)$, $TI(\mathcal{Q},A)$, $WF(\mathcal{Q})$ and $WO(\mathcal{Q})$, respectively.

Definition. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are sets of reals, $\mathcal{X} \subset \mathcal{Y}$. $\mathcal{X}$ is called Π^1_1 over $\mathcal{Y}$, if there exist a Π^1_1 formula $A[X,\underline{Y}]$ and $\underline{Q} \in \mathcal{Y}$ such that for all $P \in \mathcal{Y}$:

$$P \in \mathcal{X} \quad \Longleftrightarrow \quad \mathcal{Y} \models A[P,\underline{Q}] \; .$$

Definition.

(a) $\mathcal{WO} := \{X \subset N : Pow(N) \models WO(X)\}$.

(b) A set of reals $\mathcal{X}$ is Π^1_1 strong if $\mathcal{X} \cap \mathcal{WO}$ is Π^1_1 over $\mathcal{X}$.

$WO(X)$ is equivalent to a Π^1_1 formula, and so $\mathcal{WO}$ is Π^1_1 over $Pow(N)$. If $\mathcal{X}$ is a β-model, then $\mathcal{X}$ is Π^1_1 strong. According to Harrington the converse is not true. S. Friedman proved the following theorem (cf. [26]):

Theorem 1.4. X is the real part of an admissible set iff

(i) X is Π^1_1 strong and $X \models (\Sigma^1_2\text{-AC})$ or

(ii) X is not Π^1_1 strong and $X \models (\Sigma^1_1\text{-AC})$.

1.7. The companion of a set of reals.

If a is an element of V_N, then RP(a) is a set of reals. Now we start with a set of reals X and construct the companion set A_X in V_N. The basic idea is to represent the elements of A_X as well-founded trees in X. We need some additional considerations, since we work over the ground structure N.

The notation for trees is standard and essentially taken from Shoenfield [48]: A finite sequence $k_1,\ldots,k_n$ of natural numbers is primitive recursively coded by the sequence number $\langle k_1,\ldots,k_n\rangle$; the set of sequence numbers is designated by SEQ; $s,t,s_1,t_1,\ldots$ range over elements of SEQ; $\langle\rangle$ is the number of the empty sequence; lh(s) gives the length of the sequence (coded by) s; $s = \langle (s)_0,\ldots,(s)_{lh(s)-1}\rangle$; $s*t$ is the concatenation of s and t, hence $\langle x_1,\ldots,x_n\rangle * \langle y_1,\ldots,y_m\rangle = \langle x_1,\ldots,x_n,y_1,\ldots,y_m\rangle$; $s \subseteq t$ $(s \subset t)$ holds if s is an initial (proper) segment of t. Functions are defined by sets X satisfying $\forall x \exists ! y (\langle x,y\rangle \in X)$, and we use function variables f,g,h to range over such sets; $\overline{f}(k)$ is the sequence number $\langle f(0),\ldots,f(k-1)\rangle$, f is a bijection from X to Y if $\forall x(x \in X \to f(x) \in Y)$, $\forall x(x \notin X \to f(x) = 0)$, $\forall y \exists x(y \in Y \to x \in X \;\&\; f(x) = y)$ and $\forall x,y(x \in X \;\&\; y \in X \;\&\; f(x) = f(y) \to x = y)$.

Definition.

(a) A non-empty subset P of SEQ is called a tree if we have for all s and t: $s \subset t \ \& \ t \in P \rightarrow s \in P$.

(b) If P is a tree and $s \in P$, then P_s denotes the tree $\{t : s*t \in P\}$.

(c) A tree P is well-founded if it has no infinite descending branch, i.e. $\forall f \exists x (\overline{f}(x) \notin P)$.

We now wish to describe the interpretation of hereditarily countable sets as trees. The distinction between urelements and sets is reflected by a different use of odd and even numbers: Codes for urelements are composed of odd numbers; the structure of an element of HC modulo urelements is described by a tree in the even numbers.

Definition.

(a) Each natural number k is represented by the N-tree

$$[k] := \{<>, <2k+1>\} .$$

(b) A tree T is called a special tree if for all t, k and m:

(i) $t*<2k+1> \in T \ \& \ m \neq 2k+1 \rightarrow t*<m> \notin T$;

(ii) $t*<k> \in T \rightarrow (\forall i < lh(t))[(t)_i \text{ is even}]$.

Definition (Equivalence of special trees).

(a) A function f is an isomorphism between the special trees S and T, in symbols $f : S \simeq T$, if f is a bijection from S to T such that:

(i) f is order preserving, i.e. $s \subset t \leftrightarrow f(s) \subset f(t)$;

(ii) $s*\langle 2k+1\rangle \in S \rightarrow f(s*\langle 2k+1\rangle) = f(s)*\langle 2k+1\rangle$;

(iii) $f(s)*\langle 2k+1\rangle \in T \rightarrow s*\langle 2k+1\rangle \in S$.

(b) $S \simeq T :\Longleftrightarrow \exists f(f : S \simeq T)$.

$\simeq$ is an equivalence relation on the class of all special trees; $f : S \simeq T$ is an arithmetic formula of L_2 and therefore $S \simeq T$ a Σ^1_1 formula. The following lemma is obvious.

<u>Lemma 1.3 (EA_0)</u>.
(a) If S and T are special trees, $f : S \simeq T$ and $s \in S$, then we have $S_s \simeq T_{f(s)}$.
(b) If T is a special tree and $k \in N$ such that $T \simeq [k]$, then $T = [k]$.

<u>Definition</u>. A tree T is a representation tree if

(i) T is a well-founded special tree,

(ii) $\forall s,x,y[s*\langle 2x\rangle \in T \;\&\; s*\langle 2y\rangle \in T \;\&\; T_{s*\langle 2x\rangle} \simeq T_{s*\langle 2y\rangle} \rightarrow x = y]$.

We write REP(T) to express that T is a representation tree. Clearly REP(T) is logically equivalent to a Π^1_1 formula.

The previous definition is similar to Simpson's definition of 'suitable tree' in [52]. For many purposes well-founded special trees would be enough, but adding (ii) has the effect that there exists exactly one isomorphism between two isomorphic representation trees.

Lemma 1.4 (EA_0). If T is a representation tree and $t \in T$, then T_t is a representation tree as well.

Lemma 1.5 (EA_0). If we have $f : S \simeq T$ and $g : S \simeq T$ for two representation trees S and T, then $f = g$.

Both lemmas follow directly from the definition of representation trees; in the second case we prove $f(s) = g(s)$ for all $s \in S$ by induction on $lh(s)$.

Definition (Membership). For special trees S and T:

$$S \tilde{\in} T :\Longleftrightarrow REP(S) \;\&\; REP(T) \;\&\; \exists x(\langle 2x\rangle \in T \;\&\; S \simeq T_{\langle 2x\rangle}) .$$

Lemma 1.6 ($(\Sigma_1^1\text{-}AC)_0$). Let T and T' be two representation trees.
(a) If T and T' are N-trees, then $T \simeq T'$ iff $T = T'$.
(b) If T is an N-tree and T' is not an N-tree, then $T \not\simeq T'$.
(c) If T and T' are not N-trees, then

$$T \simeq T' \leftrightarrow \forall S(S \tilde{\in} T \leftrightarrow S \tilde{\in} T') .$$

(d) $S \tilde{\in} T \rightarrow \exists! f \exists! x(\langle 2x\rangle \in T \;\&\; f : S \simeq T_{\langle 2x\rangle})$.

Proof. (a) and (b) follow from Lemma 1.3(b); the implication from left to right of (c) is obvious by the definition of $\tilde{\in}$. Now we prove the converse direction of (c). Since T and T' are representation trees, we obtain from the assumption

(1) $(\forall\langle 2x\rangle \in T)(\exists!\langle 2y\rangle \in T')(T_{\langle 2x\rangle} \simeq T'_{\langle 2y\rangle})$,

(2) $(\forall\langle 2y\rangle \in T')(\exists!\langle 2x\rangle \in T)(T_{\langle 2x\rangle} \simeq T'_{\langle 2y\rangle})$.

From (1) it follows that the Σ^1_1 formula

$$\exists f(\langle 2k\rangle \in T \ \& \ \langle 2m\rangle \in T' \ \& \ f : T_{\langle 2k\rangle} \simeq T'_{\langle 2m\rangle})$$

is equivalent to the Π^1_1 formula

$$\forall f[\langle 2k\rangle \in T \ \& \ \forall x(\langle 2x\rangle \in T' \ \& \ f : T_{\langle 2k\rangle} \simeq T'_{\langle 2x\rangle} \rightarrow x = m)]$$

By (Δ^1_1-CA) there exists a function h with

$$h(k) = m \leftrightarrow \begin{cases} (\langle 2k\rangle \notin T \ \& \ m = 0) \vee \\ (\langle 2k\rangle \in T \ \& \ \langle 2m\rangle \in T' \ \& \ T_{\langle 2k\rangle} \simeq T'_{\langle 2m\rangle}) . \end{cases}$$

Thus we have

$$\forall x \exists f(\langle 2x\rangle \in T \rightarrow f : T_{\langle 2x\rangle} \simeq T'_{\langle 2h(x)\rangle}) .$$

Hence we can use (Σ^1_1-AC) in order to find a function g with

$$\forall x(\langle 2x\rangle \in T \rightarrow (g)_x : T_{\langle 2x\rangle} \simeq T'_{\langle 2h(x)\rangle}) .$$

Finally, by arithmetic comprehension, there exists the following function f:

$$f(m) := \begin{cases} \langle 2h(k)\rangle * (g)_k(s) , & \text{if } m = \langle 2k\rangle * s \in T ; \\ \langle\rangle & , \text{if } m = \langle\rangle ; \\ 0 & , \text{if } m \notin T . \end{cases}$$

It is easy to check that $f : T \simeq T'$.

(d) follows from Lemma 1.4, Lemma 1.5 and the definition of representation trees. ./.

The idea is now to identify the representation tree T with the hereditarily countable set $|T|$. $|T|$ is defined by recursion on the well-founded tree T.

Definition.

(a) $|[k]| := k$ for all $k \in N$.

(b) If T is a representation tree but not an N-tree, then

$$|T| := \{|T_{<2x>}| : <2x> \in T\} .$$

Lemma 1.7. Suppose that S and T are representation trees. Then we have:

(a) $|S| = |T| \iff S = T$;

(b) $|S| \in |T| \iff S \tilde{\in} T$.

Proof. Let d(P) be the depth of the well-founded tree P. Using Lemma 1.6, we prove (a) by a straightforward induction on max(d(S),d(T)); (b) follows immediately from (a). ./.

Remarks.

1. It is an easy exercise to show that

$$HC = \{|T| : Pow(N) \models REP(T)\} .$$

2. Comparable representations of hereditarily countable sets as subsets of N have been considered by Feferman [11], H. Friedman [23], S. Friedman [26] and Simpson [52]. The version here has some advantages which will become clear in section 5.

Definition (Companion). Suppose that X is a set of reals. The companion A_X of X is the following element of V_N:

$$A_X := \{|T| : T \in X \ \& \ Pow(N) \models REP(T)\} .$$

Lemma 1.8. If X is a model of EA, then $RP(A_X) = X$.

Proof. For $Q \in X$ we define

$$T_Q := \{<>\} \cup \{<2x> : x \in Q\} \cup \{<2x,2x+1> : x \in Q\} .$$

Then T_Q is a representation tree, $T_Q \in X$ and $|T_Q| = Q$. Hence $Q \in RP(A_X)$.

Now suppose that $T \in X$ is a representation tree and $|T| \in Pow(N)$. Clearly the set

$$Q_T := \{y : \exists x(<x,2y+1> \in T)\}$$

is an element of X and $Q_T = |T|$. This implies $|T| \in X$. ./.

Lemma 1.9. If a is an admissible set or a union of admissible sets, then $A_{RP(a)} \subset a$.

Proof. Suppose first that a is admissible and T a representation tree in a. By induction on T we show $(\forall t \in T)(|T_t| \in a)$ which yields $|T| \in a$ by Δ_0 collection.

If a is a union of admissibles, then the assertion follows from the previous case since $A_{RP(\cup b)} = \cup\{A_{RP(a)} : a \in b\}$ . ./.

The converse direction is not true in general. Let card(b) denote the cardinality of b and suppose that a is an admissible such that $card(a) > card(Pow(N))$. It follows $card(A_{RP(a)}) \leq card(RP(a)) \leq card(Pow(N)) < card(a)$. But then $A_{RP(a)}$ is a proper subset of a.

Definition. An admissible set a is called locally countable if

$$a \models \forall x(x \text{ is countable}) .$$

Lemma 1.10. If a is a locally countable admissible set or a union of locally countable admissible sets, then $a \subset A_{RP(a)}$.

Proof. Choose an arbitrary transitive set b in a. Then there exists a function $f \in a$ which maps b 1-1 into a subset of N. Without loss of generality we may assume that f(x) is a multiple of 4 for every $x \in b - N$. We define a tree T^c for every $c \in b$. If $c \in N$, then $T^c := [c]$; otherwise $T^c :=$

$$\{<f(x_1),\ldots,f(x_n)> : n \in N \ \&\ x_n \notin N \ \&\ x_n \in \ldots \in x_1 \in c\} \cup$$
$$\{<f(x_1),\ldots,f(x_n),4x+2> : n \in N \ \&\ x \in N \ \&\ x \in x_n \in \ldots \in x_1 \in c\} \cup$$
$$\{<f(x_1),\ldots,f(x_n),4x+2,2x+1> : n \in N \ \&\ x \in N \ \&\ x \in x_n \in \ldots \in x_1 \in c\}.$$

It is easy to check that each T^c is a representation tree in RP(a) which satisfies $|T^c| = c$. This implies $a \subset A_{RP(a)}$ since each element of a is element of a transitive set in a. ./.

Theorem 1.5.

(a) If $X \subset Pow(N)$ is a model of EA, then

$$RP(A_X) = X.$$

(b) If a is a locally countable admissible set or a union of locally countable admissible sets, then

$$A_{RP(a)} = a.$$

Proof. Immediate from Lemma 1.8 - 1.10. ./.

§2. THE ADMISSIBLE COVER AND EXTENSION OF A THEORY.

After a brief discussion of admissible sets in the previous section, we now turn to theories with these sets as intended models. We begin with two general concepts for describing an admissible universe above a given underlying theory: For a theory Th in L_1 we define its admissible cover Th^c and for a theory Th in L_* its admissible extension Th^e.

The notions of admissible cover and extension are very general and interesting by their own right. We will apply them in §3 and §4 for the proof-theoretic analysis of predicative theories for admissible sets. Proofs, however, are postponed to the appendix, since they require the introduction of systems of ramified set theory which we do not need elsewhere.

2.1. The admissible cover Th^c of Th.

Let PA^- be Peano arithmetic PA minus the scheme of complete induction, and assume that Th is a theory in L_1 which contains PA^-. Th^c is formulated in the language L_* and consists of the following axioms.

1. Ontological and equality axioms. As in ES.

2. Th-axioms. For every axiom $A[\underline{u}]$ of Th:

$(\forall \underline{x} \in N) A^N[\underline{x}]$.

3. Kripke-Platek axioms.

(Pair) $\exists z(a \in z \ \& \ b \in z)$;

(Transitive Hull) $\exists z(a \subset z \ \& \ \mathrm{Tran}(z))$;

(Δ_0-Sep) $\exists z(z = \{x \in a : A(x)\})$ for all Δ_0 formulas $A(u)$;

(Δ_0-Col) $(\forall x \in a)\exists y A(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)A(x,y)$ for all Δ_0 formulas $A(u,v)$.

If (N,Q) is a model of Th, then $(N,Q;Q^+,\in)$ is the least standard model of Th^c that contains Q as element. In this model (IND_N) and $(\mathrm{IND}_\in)$ are trivially satisfied. Observe, however, that Th^c is weak with respect to induction. We do not have $\in$-induction in Th^c and only as much induction on N as can be lifted from Th. For example, if Th is PA, Th^c contains induction on N for all formulas of the form A^N, with A an L_1 formula, but not for arbitrary L_* formulas.

Theorem 2.1. Suppose that Th is a theory in L_1, $\mathrm{PA}^- \subset \mathrm{Th}$. Then we have for any sentence A of L_1:

(a) $\mathrm{Th}^c \vdash A^N \iff \mathrm{Th} \vdash A$;

(b) $\mathrm{Th}^c + (I_N) \vdash A^N \iff \mathrm{Th} + (\mathrm{IND}_N) \vdash A$;

(c) $\mathrm{Th}^c + (I_N) + (I_\in) \vdash A^N \iff \mathrm{Th} + (\mathrm{IND}_N) \vdash A$.

The implications from right to left are obvious; for the converse directions see the appendix. In 2.4 we will consider $Th^c + (IND_N)$; a general treatment of $Th^c + (IND_N) + (IND_\in)$ in this vein is not possible. In the special case $Th = PA$, we have $Th^c + (IND_N) + (IND_\in) = KPu$, and this theory is studied in §3.

2.2. The admissible extension Th^e of Th.

The definition of the admissible extension is similar to the definition of the admissible cover. Now we suppose that Th is a set theory in the language L_* or an extension $L_*(\underline{e})$ of L_* by finitely many set constants $\underline{e} = e_1, \ldots, e_n$. We also assume that Th contains ES^o. Since Th already describes a universe of sets, we do not form a set-theoretic cover of Th but formalize an admissible end extension of Th.

In order to define Th^e, we first extend the language $L_*(\underline{e})$ by a new set constant M to the language $L_*(\underline{e}, M)$. Th^e then is the theory in $L_*(\underline{e}, M)$ that consists of the following axioms.

1. Ontological and equality axioms. As in ES.

2. M-axioms.

$Tran(M)$; $b \in M$ for all constants b of Th.

3. Th-axioms. For every axiom $A[\underline{u}]$ of Th:

$(\forall \underline{x} \in M) A^M[\underline{x}]$.

4. Kripke-Platek axioms. As in Th^c.

We would like to emphasize that the existence of an infinite descending sequence of sets outside M is consistent with Th^e. However, if a is a model of Th, then a^+ is a standard model of Th^e that contains a.

<u>Theorem 2.2</u>. Suppose that Th is a theory in $L_*(\underset{\sim}{e})$, $ES^0 \subset Th$. Then we have for any sentence A of $L_*(\underset{\sim}{e})$:

(a) $Th^e \vdash A^M \iff Th \vdash A$;

(b) $Th^e + (I_N) \vdash A^M \iff Th + (IND_N) \vdash A$;

(c) $Th^e + (I_N) + (I_\in) \vdash A^M \iff Th + (IND_N) + (IND_\in) \vdash A$.

The implications from right to left are obvious again; the converse directions will be proved in the appendix. In 2.4 we analyze $Th^e + (IND_N)$.

2.3. <u>Ordinals and theories</u>.

In order to establish conservation results for the theories $Th^c + (IND_N)$ and $Th^e + (IND_N)$ similar to those of Theorem 2.1 and Theorem 2.2 , we introduce infinitary versions Th^c_∞ and Th^e_∞. Now we provide the necessary tools from ordinal theory.

For each ordinal α we define a function $\phi\alpha$ from ordinals to ordinals by the following recursion: $\phi 0 \xi$ is ω^ξ; for $\alpha > 0$, $\phi\alpha\xi$ is the ξth simultaneous fixed point of all functions $\phi\beta$ with $\beta < \alpha$. The least α such that $\phi\alpha 0 = \alpha$ is called the least strongly critical ordinal and normally denoted by Γ_0. We write ε_α instead of $\phi 1\alpha$ and call these ordinals ε-numbers. For further details and information cf. Feferman [9] and Schütte [47]. The following definition and its immediate consequences will be needed later.

Definition. For $n < \omega$ we define:

$$\phi(0|\alpha) := \alpha ; \quad \phi(n+1|\alpha) := \phi(\phi(n|\alpha))0 .$$

Lemma 2.1. For $m,n < \omega$ and $\gamma < \Gamma_0$:

(a) $\phi(m|\phi(n|\alpha)) = \phi(m+n|\alpha)$;

(b) $\Gamma_0 = \sup\{\phi(i|\gamma) : i < \omega\}$.

Definition. Let Th be a theory formulated in (an extension of) L_1, L_2 or L_*.

(a) We say that the ordinal α is provable in Th, if there exists a primitive recursive well-ordering $\mathcal{Q}$ of order-type α such that $\mathrm{Th} \vdash \mathrm{TI}(\mathcal{Q},R)$.

(b) The proof-theoretic ordinal of Th, denoted by $|\mathrm{Th}|$, is the least ordinal not provable in Th.

(c) We say that Th is a theory of predicative strength, if $|\mathrm{Th}| \leqq \Gamma_0$.

Remarks.

1. In practice we have: Two theories Th_1 and Th_2 prove the same arithmetic statements, possibly with parameters, iff $|\mathrm{Th}_1| = |\mathrm{Th}_2|$.

2. The notion of predicativity will be discussed in more detail in section 6.

2.4. Infinitary systems.

In this subsection we study the admissible cover and extension in presence of the ω-rule. As a consequence of this, we obtain a characterization of $Th^c + (IND_N)$ and $Th^e + (IND_\in)$.

In the infinitary proof system Z_1^∞ we use the language L_1 but now exclude the free number variables.

Basic rules of Z_1^∞. As in Z_1.

Normal rules of Z_1^∞.

(&), (v) as in Z_1;

$$(\forall n^\infty)\ \frac{\Gamma, A(n) \quad \text{for all number constants } n}{\Gamma, \forall x A(x)}$$

$$(\exists n^\infty)\ \frac{\Gamma, A(n) \quad \text{for some number constant } n}{\Gamma, \exists x A(x)}$$

Cut rule of Z_1^∞. As in Z_1.

The rule $(\forall n^\infty)$ sometimes is called ω-rule. It is a rule with infinitely many premises, and so we will have infinite proof trees in general.

Z^{∞}_{*} is the infinitary proof system for $L_{*}(\underline{e})$. Here we allow free variables, but make sure by an additional basic rule that they do not denote natural numbers.

<u>Basic rules of Z^{∞}_{*}.</u>

(B), (T) as in Z_{*};

(N) $\Gamma, n \in N$, if n is a number constant;

(S) $\Gamma, a \notin N$, if a is not a number constant.

<u>Normal rules of Z^{∞}_{*}.</u>

(&), (v) as in Z_{*};

$$(b\forall^{\infty}) \quad \frac{\Gamma, b \in a \rightarrow A(b) \quad \text{for all } b}{\Gamma, (\forall x \in a)A(x)}$$

$$(b\exists^{\infty}) \quad \frac{\Gamma, b \in a \,\&\, A(b) \quad \text{for some } b}{\Gamma, (\exists x \in a)A(x)}$$

$$(\forall^{\infty}) \quad \frac{\Gamma, A(b) \quad \text{for all } b}{\Gamma, \forall x A(x)}$$

$$(\exists^{\infty}) \quad \frac{\Gamma, A(b) \quad \text{for some } b}{\Gamma, \exists x A(x)}$$

<u>Cut rule of Z^{∞}_{*}.</u> As in Z_{*}

Let Z^{∞} denote one of the proof systems Z^{∞}_{1} or Z^{∞}_{*}.

Inductive definition of $Z^{\infty} \vdash_{\beta}^{\alpha} \Gamma$.

1. If Γ is a basic rule of Z^{∞}, then we have $Z^{\infty} \vdash_{\beta}^{\alpha} \Gamma$ for all ordinals α and β.

2. If $Z^{\infty} \vdash_{\beta}^{\alpha_i} \Gamma_i$ and $\alpha_i < \alpha$ for every premise Γ_i of a normal rule or a cut of degree $< \beta$, then we have $Z^{\infty} \vdash_{\beta}^{\alpha} \Gamma$ for the conclusion Γ of that inference.

We write $Z^{\infty} \vdash^{\alpha} \Gamma$ for $Z^{\infty} \vdash_{0}^{\alpha} \Gamma$; $Z^{\infty} \vdash^{<\alpha} \Gamma$ means $Z^{\infty} \vdash^{\beta} \Gamma$ for some $\beta < \alpha$. Note that each cut is of finite degree.

Lemma 2.2 (Cut elimination).

(a) $Z^{\infty} \vdash_{n+1}^{\alpha} \Gamma \implies Z^{\infty} \vdash_{n}^{\omega^{\alpha}} \Gamma$;

(b) $Z^{\infty} \vdash_{\omega}^{\alpha} \Gamma \implies Z^{\infty} \vdash^{\varepsilon_{\alpha}} \Gamma$.

Proofs of this cut elimination property and of the following results can be found in Schütte [47] and Tait [54].

Lemma 2.3. If $\mathcal{Q}$ is a primitive recursive well-ordering such that

$$Z_1^{\infty} \vdash^{<\alpha} TI(\mathcal{Q},R),$$

then the order-type of $\mathcal{Q}$ is less than ω^{α}.

Lemma 2.4. For every formula $A(u)$ of $L_*(\underline{e})$:

$$Z_*^{\infty} \vdash^{<\omega+\omega} A(0) \ \& \ (\forall x,y \in N)(A(x) \ \& \ Sc(x,y) \rightarrow A(y)) \rightarrow (\forall x \in N)A(x).$$

Lemma 2.5. If $\mathcal{Q}$ is a primitive recursive well-ordering of order-type α, then we have

$$Z_*^\infty \vdash^{\omega\alpha+\omega} TI(\mathcal{Q},A)$$

for every formula $A(u)$ of $L_*(\underline{e})$.

Let L be one of the languages L_1 or $L_*(\underline{e})$ and Z^∞ the corresponding infinitary proof system. If Th is a theory in L, then we write

$$Th_\infty \vdash^\alpha_\beta \Gamma ,$$

if there exists a finite set $\{A_1,\ldots,A_n\}$ of universal closures of axioms of Th such that

$$Z^\infty \vdash^\alpha_\beta \{\neg A_1,\ldots,\neg A_n\} , \Gamma .$$

$Th_\infty \vdash^{<\alpha} \Gamma$ is defined analogously. Theorem 2.3 and Theorem 2.4 below are proved in the appendix.

Theorem 2.3. Suppose that Th is a theory in L_1, $PA^- \subset Th$. Then we have for any L_1 sentence A and ε-number α:

$$Th^c + (I_\in)_\infty \vdash^{<\alpha} A^N \implies Th_\infty \vdash^{<\phi\alpha 0} A .$$

Corollary 2.1. If Th is a theory in L_1 with $PA^- \subset Th$ and A an L_1 sentence, then

$$Th^c + (IND_N) + (I_\in) \vdash A^N \implies Th_\infty \vdash^{<\phi\varepsilon_0 0} A .$$

Proof. By a standard embedding argument, using Lemma 2.4, we obtain from the assumption that $Th^c + (I_\in) \vdash_n^{\omega+m} A^N$ for some $m, n < \omega$. Cut elimination and Theorem 2.3 imply the assertion ./.

<u>Theorem 2.4</u>. Suppose that Th is a theory in $L_*(\underline{e})$, $ES^o \subset Th$. Then we have for any $L_*(\underline{e})$ sentence A and ε-number α:

$$Th^e_\infty \vdash^{<\alpha} A^M \implies Th_\infty \vdash^{<\phi\alpha 0} A .$$

<u>Corollary 2.2</u>. If Th is a theory in $L_*(\underline{e})$ with $ES^o \subset Th$ and A an $L_*(\underline{e})$ sentence, then

$$Th^e + (IND_N) \vdash A^M \implies Th_\infty \vdash^{<\phi\varepsilon_0 0} A .$$

The proof of this corollary corresponds to the proof of Corollary 2.1.

§3. THE THEORY KPu AND ITS RESTRICTIONS.

We will study set theories with minimal standard models of the form L_α, where α is admissible, limit of admissibles or admissible limit of admissibles. The situation of an admissible universe is formalized by the axioms of Kripke-Platek set theory. Of course, they do not provide a complete axiomatization of L_{Ω_1} but still reflect central properties of this set. Our theory KPu corresponds to Barwise's theory KPU^+ in [1] with PA as additional theory for the urelements. KPu^r and KPu^o are subtheories of KPu obtained by restricting induction.

Definition.

(a) KPu is the theory ES with the additional axiom scheme of Δ_o collection

$$(\Delta_o\text{-Col}) \quad (\forall x \in a)\exists y A(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)A(x,y)$$

for all Δ_o formulas $A(u,v)$.

(b) KPu^r is ES^r plus (Δ_o-Col) and KPu^o is ES^o plus (Δ_o-Col).

At this stage one sees the disadvantage of restricting attention to standard structures: Since (IND_N) and ($IND_\in$) are true in each standard model, these models cannot be used to distinguish between KPu, KPu^r and KPu^o. To retain the naturalness of standard structures and to get more proof-theoretic information out of the sets L_α, we introduce the following concept.

Definition. Let Th be a theory in $\mathcal{L}_*$. L_α is a Π_n model (Σ_n model) of Th if $L_\alpha \models A$ for each Π_n (Σ_n) sentence provable in Th.

We are especially interested in Π_2 models and will see that natural theories with different proof-theoretic ordinals have different minimal Π_2 models.

3.1. KPu^o and KPu^r.

The theories KPu^o and KPu^r are well known from the previous section; KPu^o is $PA^c + (I_N)$ and KPu^r is $PA^c + (I_N) + (I_\in)$. Hence Theorem 2.1 implies that KPu^o and KPu^r are conservative extensions of PA in the following sense.

Theorem 3.1. We have for every sentence A of $\mathcal{L}_1$:

(a) $KPu^o \vdash A^N \iff PA \vdash A$;

(b) $KPu^r \vdash A^N \iff PA \vdash A$.

In spite of their proof-theoretic weakness, KPu^o and KPu^r have some important closure properties with respect to set existence. Careful reading of Barwise's book [1] reveals how much of set theory can be developed without using (IND_N) and $(IND_\in)$ for arbitrary formulas. Here we just mention Σ reflection and Δ separation, which will be needed later.

Lemma 3.1 (Σ reflection). KPu^o proves for every Σ formula A:

$$A \leftrightarrow \exists x A^x .$$

Lemma 3.2 (Δ separation). If $A(u)$ is a Π formula and $B(u)$ a Σ formula of L_*, then KPu^0 proves

$$(\forall x \in a)(A(x) \leftrightarrow B(x)) \rightarrow \exists z(z = \{x \in a : A(x)\}) .$$

As a corollary we obtain from Lemma 3.2 that the second order theory $(\Delta^1_1\text{-CA})_0$ is contained in KPu^0. Remember that we identify an L_2 sentence A with its translation into L_*.

Corollary 3.1. If A is an L_2 sentence, then

$$(\Delta^1_1\text{-CA})_0 \vdash A \quad \Longrightarrow \quad KPu^0 \vdash A .$$

Some additional conventions. $Fun(f)$ expresses that f is a function; $dom(f)$ denotes its domain and $rng(f)$ its range. $f(x)$ is the value of f at x if $Fun(f)$ and $x \in dom(f)$. $Fun_{1-1}(f)$ means that f is a 1-1 function. It is easy to see that the formulas $Fun(f)$, $dom(f) = a$, $rng(f) = b$, $f(x) = y$ and $Fun_{1-1}(f)$ are Δ_0 (cf. [1]).

We now adjoin the axiom of countability (C). It is obvious that (C) implies the usual axiom of choice. We use it in order to prove Σ replacement in $KPu^0 + (C)$.

(C) $\quad \forall x \exists f(Fun_{1-1}(f) \;\&\; dom(f) = x \;\&\; rng(f) \subset N)$

Lemma 3.3 (Σ replacement). $KPu^0 + (C)$ proves for every Δ_0 formula $A(u,v)$:

$$(\forall x \in a)\exists y A(x,y) \rightarrow \exists f[Fun(f) \;\&\; dom(f) = a \;\&\; (\forall x \in a)A(x,f(x))] .$$

Proof. We work in $KPu^{o} + (C)$ and assume $(\forall x \in a)\exists y A(x,y)$. By Δ_o collection and (C) we find a set b and a 1-1 function g such that

$$(\forall x \in a)(\exists y \in b)A(x,y) \ \& \ dom(g) = b \ \& \ rng(g) \subset N .$$

Now by Δ_o separation we define the function

$$f := \{(x,y) \in a \times b : A(x,y) \ \& \ (\forall z \in b)(g(z) <_N g(y) \rightarrow \neg A(x,z))\} ,$$

where $<_N$ is the primitive recursive less relation on N. By (I_N) we can easily check that f satisfies our lemma ./.

Remark. Lemma 3.3 implies that the second order theory $(\Sigma^1_1\text{-}AC)_o$ is contained in $KPu^{o} + (C)$.

It could be shown that the inclusion of (C) does not increase the proof-theoretic strength of theories for admissible sets. We will not follow this track now but rather turn to the minimal Π_2 model of KPu^{o} and KPu^{r}. In §7 and §8, however, we will come back to the axiom (C).

If A is a formula of L_*, then $A^{[\alpha]}$ denotes the formula which is obtained from A by replacing all unrestricted universal quantifiers $\forall x(\ldots)$ by $(\forall x \in L_\alpha)(\ldots)$. If Γ is a set of L_* formulas $\{A_1,\ldots,A_n\}$, then we write $\Gamma^{[\alpha]}$ for the formula $A_1^{[\alpha]} \vee \ldots \vee A_n^{[\alpha]}$.

Lemma 3.4. Assume that $\Lambda[\underline{u}]$ is a KPu^{r}-axiom set and $\Gamma[\underline{u}]$ a set of L_* formulas. If

$$Z_* \vdash_0^n \Lambda[\underline{u}], \Gamma[\underline{u}],$$

then we have, for all β, $\underline{a} \in L_\beta$ and $\gamma \geqq \beta + 3^n$,

$$L_\gamma \models \Gamma^{[\beta]}[\underline{a}].$$

Proof by induction on n. If the main formula of the last inference does not belong to $\Lambda[\underline{u}]$ or is not the negation of an axiom of KPu^r, then the assertion follows from the I.H. (induction hypothesis) immediately. Suppose therefore that the main formula $B[\underline{u}]$ of the last inference belongs to $\Lambda[\underline{u}]$ and is the negation of a KPu^r axiom. We fix β, $\underline{a} \in L_\beta$ and $\gamma \geqq \beta + 3^n$ and make the following observations:

(i) Each axiom of KPu^r is a Δ_0 formula, a Σ_1 formula or an instance of $(\Delta_0\text{-Col})$.

(ii) If $C[\underline{u}]$ is a Δ_0 or Σ_1 axiom of KPu^r, then $L_{\beta+1} \models C[\underline{a}]$.

We now distinguish several cases:

1. If $\neg B[\underline{u}]$ is a KPu^r axiom and $B[\underline{u}]$ a Δ_0 formula of the form $B_0[\underline{u}] \,\&\, B_1[\underline{u}]$, then there are $n_0, n_1 < n$ such that

(1) $Z_* \vdash_0^{n_i} \Lambda[\underline{u}], B_i[\underline{u}], \Gamma[\underline{u}]$

for $i = 0,1$. By the I.H. we obtain for $i = 0,1$ that

(2) $L_\gamma \models B_i[\underline{a}] \vee \Gamma^{[\beta]}[\underline{a}]$.

Hence $L_\gamma \models \Gamma^{[\beta]}[\underline{a}]$ by (ii).

2. If $\neg B[\underline{u}]$ is a KPu^r axiom and $B[\underline{u}]$a Δ_0 formula but not a conjunction, then the proof is similar.

3. If $\neg B[\underline{u}]$ is a KPu^r axiom and $B[\underline{u}]$ is of the form $\forall z B_0[\underline{u}, z]$, then B_0 is a Δ_0 formula, and there exist an $n_0 < n$ and a

variable v different from $\underline{u}$ such that

(3) $Z_* \vdash_0^{n_o} \Lambda[\underline{u}], B_o[\underline{u}, v], \Gamma[\underline{u}]$.

The I.H. gives for every $b \in L_{\beta+1}$ that

(4) $L_\gamma \models B_o[\underline{a}, b] \vee \Gamma^{[\beta+1]}[\underline{a}]$

since $\gamma \geq \beta+3^n \geq (\beta+1)+3^{n_o}$. This implies

(5) $L_\gamma \models \Gamma^{[\beta]}[\underline{a}]$ or $L_{\beta+1} \models \forall z B_o[\underline{a}, z]$.

From (5) and (ii) the conclusion of the lemma is immediate.

4. Now consider the case that $\neg B[\underline{u}]$ is an instance of (Δ_o-Col). Then we have a Δ_o formula $C[\underline{u}, v, w]$, a term c and $n_o, n_1 < n$ such that

(6) $Z_* \vdash_0^{n_o} \Lambda[\underline{u}], (\forall x \in c)\exists y C[\underline{u}, x, y], \Gamma[\underline{u}]$,

(7) $Z_* \vdash_0^{n_1} \Lambda[\underline{u}], \forall z(\exists x \in c)(\forall y \in z)\neg C[\underline{u}, x, y], \Gamma[\underline{u}]$.

By the I.H. we obtain from (6) for $\delta := \beta+3^{n_o}$

(8) $L_\delta \models (\forall x \in d)\exists y C[\underline{a}, x, y] \vee \Gamma^{[\beta]}[\underline{a}]$

Here d is the term c, if c is not a free variable; d is the number constant 0, if c is a free variable but does not belong to the list $\underline{u}$; if c is from the list $\underline{u}$, then d is the corresponding term in the list $\underline{a}$. From (8) one obtains

(9) $L_\gamma \models (\forall x \in d)(\exists y \in L_\delta)C[\underline{a}, x, y] \vee \Gamma^{[\beta]}[\underline{a}]$.

Since $(\delta+1)+3^{n_1} = \beta+3^{n_o}+1+3^{n_1} \leq \beta+3^n \leq \gamma$, the I.H. allows us to conclude from (7)

(10) $L_\gamma \models (\forall z \in L_{\delta+1})(\exists x \in d)(\forall y \in z)\neg C[\underline{a},x,y] \vee \Gamma^{[\delta+1]}[\underline{a}]$

The desired result follows from (9) and (10) since $L_\delta \in L_{\delta+1}$ and $\beta \leq \delta+1$. ./.

<u>Theorem 3.1</u>. L_ω is the minimal Π_2 model of KPu^0 and KPu^r.

Proof. Suppose that A is the Π_2 sentence $\forall x \exists y B[x,y]$ and $KPu^r \vdash A$. Then there exist $m,n < \omega$ and a KPu^r-axiom set $\Lambda[\underline{u},v]$ such that $Z_* \vdash^m_n \Lambda[\underline{u},v], \exists y B[v,y]$. Cut elimination in Z_* yields $Z_* \vdash^k_0 \Lambda[\underline{u},v], \exists y B[v,y]$ for some $k < \omega$. Now choose an arbitrary $a \in L_\omega$. Then there exists an $i < \omega$ such that $a \in L_i$. Lemma 3.4 gives $L_\omega \models \exists y B[a,y]$. This implies $L_\omega \models A$ since a was an arbitrary element of L_ω. Thus we have: L_ω is a Π_2 model of KPu^r and KPu^0. The minimality is obvious. ./.

<u>Remark</u>. L_ω is not the smallest standard structure which is a model of all Π_2 sentences provable in KPu^r. Let $D(N)$ be the collection of L_1 definable subsets of N; i.e. $a \in D(N)$ iff there exists an L_1 formula $A[u]$ such that $a = \{x \in N : N \models A[x]\}$. Then we define by recursion on $n < \omega$:

$$D^*_0 := N \cup D(N) ;$$

$$D^*_{n+1} := D^*_n \cup \{\{x,y\} : x,y \in D^*_n\} \cup \{x \cup y : x,y \in D^*_n\} ;$$

$$D^* := \cup\{D^*_n : n < \omega\} .$$

D^* is certainly contained in every Σ_1 and Π_2 model of KPu^0. One can show that D^* is a model of all Π_2 sentences provable in KPu^r (cf. Jäger [33]).

3.2. $KPu^{o} + (IND_N)$ and $KPu^{r} + (IND_N)$.

Theorem 3.1 tells us that the existence of ω is not provable in KPu^{r}. Now we make use of (IND_N) to lift the ordering on N to the ordinals. Unfortunately, we must be careful with the notion of ordinal in theories without $(I_{\in})$.

Definition.

(a) $\mathrm{Connex}(a) \; :\Longleftrightarrow \; (\forall x \in a)(\forall y \in a)(x \in y \vee x = y \vee y \in x)$;

(b) $\mathrm{Ord}_o(a) \; :\Longleftrightarrow \; \mathrm{Tran}(a) \;\&\; \mathrm{Connex}(a)$;

(c) $\mathrm{Ord}(a) \; :\Longleftrightarrow \; \begin{cases} \mathrm{Ord}_o(a) \;\& \\ \forall x[x \subset a \;\&\; x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z \notin y)] \end{cases}$;

(d) $\mathrm{Lim}_o(a) \; :\Longleftrightarrow \; \mathrm{Ord}_o(a) \;\&\; a \neq \emptyset \;\&\; (\forall x \in a)(\exists y \in a)(x \in y)$;

(e) $\mathrm{Lim}(a) \; :\Longleftrightarrow \; \mathrm{Ord}(a) \;\&\; \mathrm{Lim}_o(a)$.

The Δ_o formula $\mathrm{Ord}_o(a)$ expresses that a is an ordinal in a weak sense; $\in$ is not necessarily well-founded on a. The Π formula Ord(a) is the usual definition of ordinal. In the presence of $(I_{\in})$, both notions are equivalent, and Ord(a) then can be considered as a Δ_o predicate. In formal set theories small Greek letters range over the predicate Ord; $\alpha < \beta$ means $\alpha \in \beta$ as usual.

Theorem 3.2 (Existence of ω).

$$KPu^{o} + (\Sigma_1\text{-}IND_N) \vdash \exists! \xi(\mathrm{Lim}(\xi) \;\&\; (\forall \eta < \xi)\neg \mathrm{Lim}(\eta)) .$$

This uniquely determined ordinal is called ω.

Proof. We write $A(x,f)$ for the Δ_0 formula

$$x \in N \ \& \ \mathrm{Fun}(f) \ \& \ \mathrm{dom}(f) = \{y \in N : y \leq_N x\} \ \& \ f(0) = \emptyset \ \&$$
$$(\forall y \in N)(y <_N x \rightarrow f(y+1) = f(y) \cup \{f(y)\})$$

and prove by $(\Sigma_1\text{-IND}_N)$:

(1) $(\forall x \in N)\exists f A(x,f)$,

(2) $A(x,f) \ \& \ A(x,g) \rightarrow f = g$.

By the axioms $(\Delta_0\text{-Col})$ and (Transitive Hull) we can find a transitive set a such that $(\forall x \in N)(\exists f \in a)A(x,f)$. Let

$$b = \{z \in a : (\exists x \in N)(\exists f \in a)(A(x,f) \ \& \ f(x) = z)\}$$

by Δ_0 separation. It is easily seen that b is the uniquely determined witness for our theorem. ./.

Adding full induction on N to KPu^0 and KPu^r gives theories which are proof-theoretically equivalent to $(\Delta^1_1\text{-CA})$. One direction follows from Lemma 3.2:

Theorem 3.3. If A is a sentence of L_2, then

$$(\Delta^1_1\text{-CA}) \vdash A \implies \mathrm{KPu}^0 + (\mathrm{IND}_N) \vdash A .$$

To determine upper bounds for $\mathrm{KPu}^0 + (\mathrm{IND}_N)$ and $\mathrm{KPu}^r + (\mathrm{IND}_N)$ we work with the infinitary system KPu^r_∞. The proof of the following lemma is like the proof of Lemma 3.4 with derivations of length $< \omega$ replaced by derivation of arbitrary length.

<u>Lemma 3.5</u>. Assume that $\Lambda[\underline{u}]$ is a KPu^{r}-axiom set and $\Gamma[\underline{u}]$ a set of L_* formulas. If

$$Z_*^\infty \vdash^\alpha \Lambda[\underline{u}]\,,\,\Gamma[\underline{u}]\quad ,$$

then we have, for all β, $\underline{a} \in L_\beta$ and $\gamma \geq \beta+3^\alpha$,

$$L_\gamma \models \Gamma^{[\beta]}[\underline{a}]\quad .$$

<u>Theorem 3.4</u>. L_{ε_0} is the minimal Σ_1 and Π_2 model of $KPu^0 + (IND_N)$ and $KPu^r + (IND_N)$.

Proof. Let A be a Π_2 sentence provable in $KPu^r + (IND_N)$. By Lemma 2.4 we find $m,n < \omega$ and a KPu^r-axiom set $\Lambda[\underline{u}]$ such that $Z_*^\infty \vdash_n^{\omega+m} \Lambda[\underline{u}]\,,\,A$. By cut elimination in Z_*^∞ there exists an $\alpha < \varepsilon_0$ such that $Z_*^\infty \vdash^\alpha \Lambda[\underline{u}]\,,\,A$. Lemma 3.5 then implies $L_{\varepsilon_0} \models A^{[\beta]}$ for all $\beta < \varepsilon_0$, and consequently $L_{\varepsilon_0} \models A$. This proves that L_{ε_0} is a Π_2 model of $KPu^r + (IND_N)$.
It remains to show that no L_γ, with $\gamma < \varepsilon_0$, is a Σ_1 model of $KPu^0 + (IND_N)$. Let α be an ordinal less than ε_0. Then there exists a primitive recursive well-ordering $\mathcal{Q}$ of order-type α such that $KPu^0 + (IND_N) \vdash TI(\mathcal{Q},A)$ for all L_* formulas A. Without loss of generality we may assume that N is the field of $\mathcal{Q}$. By using transfinite induction along $\mathcal{Q}$, the well-ordering $\mathcal{Q}$ is lifted to the sets (cf. Theorem 3.2).

$$(*)\ KPu^0 + (IND_N) \vdash \exists f[Fun(f)\ \&\ dom(f) = N\ \&$$
$$(\forall x \in N)(f(x) = \{f(y) : y \in N\ \&\ y\,\mathcal{Q}\,x\})]\ .$$

We omit a detailed proof of $(*)$ since a similar and more general case will be considered in (the proof of) Theorem 4.6

at full length. We may now conclude that

$$L_\gamma \models \exists f[\mathrm{Fun}(f) \ \&\ \mathrm{dom}(f) = N \ \&$$
$$(\forall x \in N)(f(x) = \{f(y) : y \in N \ \&\ y \mathcal{Q} x\})]$$

for all Σ_1 models L_γ of $KPu^0 + (IND_N)$. This implies $\alpha \leq \gamma$. The proof is now complete. ./.

Theorem 3.5.

(a) If A is a sentence of $\mathcal{L}_1$, then

$$KPu^r + (IND_N) \vdash A^N \implies PA_\infty \vdash^{<\phi\varepsilon_0 0} A \quad .$$

(b) $|KPu^r + (IND_N)| = |KPu^0 + (IND_N)| = |(\Delta^1_1\text{-}CA)| = \phi\varepsilon_0 0$.

Proof. (a) We observe that $KPu^r + (IND_N)$ is $PA^c + (IND_N) + (I_\in)$ and apply Corollary 2.1.

(b) Since $(\Delta^1_1\text{-}CA) \subset KPu^0 + (IND_N) \subset KPu^r + (IND_N)$ and $\phi\varepsilon_0 0 \leq |(\Delta^1_1\text{-}CA)|$ by [8], it remains to show that $|KPu^r + (IND_N)| \leq \phi\varepsilon_0 0$. So assume that $\mathcal{Q}$ is a primitive recursive well-ordering on N and $KPu^r + (IND_N) \vdash TI(\mathcal{Q},R)$. Then (a) implies $PA_\infty \vdash^{<\phi\varepsilon_0 0} TI(\mathcal{Q},R)$, which is equivalent to $Z^\infty_1 \vdash^{<\phi\varepsilon_0 0} TI(\mathcal{Q},R)$. By Lemma 2.3 we therefore obtain that the order-type of $\mathcal{Q}$ is smaller than $\phi\varepsilon_0 0$. ./.

We finish this subsection with some remarks concerning the second recursion theorem.

Lemma 3.6 (Second recursion theorem). Let $A[R^+,\underline{a},b]$ be an R-positive Σ formula of $\mathcal{L}_*$. Then there exists a Σ formula

$B[\underline{a},b]$ of L_* (without R) such that the following is provable in $KPu^0 + (IND_N)$ for all $\underline{a}$, b:

$$B[\underline{a},b] \leftrightarrow A[B[\underline{a},.],\underline{a},b] .$$

The proof of this lemma can be taken from Barwise [1] with some obvious modifications. We are interested in the second recursion theorem in connection with inductive definitions.

Corollary 3.2. Let $A[R^+,\underline{a},b]$ be an R-positive Σ formula of L_*. Then there exists a Σ formula $P^A[\underline{a},b]$ of L_* (without R) such that the following is provable in $KPu^0 + (IND_N)$ for all $\underline{a}$, b:

$$P^A[\underline{a},b] \leftrightarrow b \in N \; \& \; A[P^A[\underline{a},.],\underline{a},b] .$$

If $A[R^+,x]$ is an R-positive L_1 formula, then this corollary says that the Σ definable class

$$P^A := \{x \in N : L_{\Omega_1} \models P^A[x]\}$$

is a fixed point of the operator

$$\Gamma_A : Pow(N) \to Pow(N) ; \qquad \Gamma_A(X) := \{x \in N : \mathcal{N} \models A[X,x]\} .$$

Since we do not have $(IND_\in)$ in $KPu^0 + (IND_N)$, we cannot expect to determine whether P^A is the smallest fixed point of the operator Γ_A.

3.3. $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$.

An often helpful theorem of Kripke-Platek set theory is the Σ recursion theorem. We will use it, for example, to construct fixed points of certain inductive definitions. The Σ recursion theorem is conceptually clearer than the second recursion theorem: The latter implicitly defines a new Σ predicate whereas the former explicitly constructs a new Σ function. The usual proof of Σ recursion in [1] can be formalized in $KPu^0 + (\Sigma_1\text{-}IND_\in)$.

Lemma 3.7 (Σ recursion theorem). Let Th be a theory in L_* which contains $KPu^0 + (\Sigma_1\text{-}IND_\in)$ and assume that G is an n+2-ary Σ function symbol of Th. Then there exists an n+1-ary Σ function symbol F of Th such that the following is a theorem of Th: For all $\underline{a}$ and all ordinals α:

$$F(\underline{a},\alpha) = G(\underline{a},\alpha,\cup\{F(\underline{a},\xi) : \xi < \alpha\}) .$$

Corollary 3.3. If Th contains $KPu^0 + (\Sigma_1\text{-}IND_\in)$ and $A[R^+,b]$ is an R-positive Δ_0 formula, then there exists a unary Σ function symbol I_A of Th such that Th proves for all α:

$$I_A^\alpha = \{x \in N : A[\cup\{I_A^\xi : \xi < \alpha\},x]\} .$$

With Σ recursion we define the usual operations on the ordinals like $\alpha+\beta$, $\alpha\cdot\beta$, α^β, etc. In $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$, which proves the existence of ω, we therefore have a Σ function symbol F with the property

$$F(\alpha) = \omega^{\bigcup\{F(\xi)\,:\,\xi<\alpha\}}\,.$$

Hence $F(\omega) = \omega^{F(\omega)}$, and so the ordinal $\varepsilon_0 = F(\omega)$ is an element of each Σ_1 model of $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$. Actually, all ordinals $< \phi\varepsilon_0 0$ are contained in the Σ_1 models of this theory.

The scheme of Σ_1 induction $(\Sigma_1\text{-}IND_\in)$ is equivalent to

$$\forall a[\forall x((\forall y \in x)\exists zA(y,z) \to \exists zA(x,z)) \to \exists zA(a,z)]$$

for all Δ_0 formulas $A(u,v)$, and it is this latter formulation of $(\Sigma_1\text{-}IND_\in)$ which we will use from now on. This has some technical advantages for the proof of Lemma 3.9 below. The following lemma takes care of the critical case in the proof of Lemma 3.9.

Lemma 3.8. Assume that $A[\underline{u},v,w]$ is a Δ_0 formula, α an ordinal, β a limit ordinal and $\underline{a},b \in L_\beta$. Then there exists a limit ordinal γ with the property $\beta \leq \gamma < \phi(\alpha+1)(\beta+\beta)$ such that

$$L_{\phi\alpha(\gamma+\gamma)} \not\models (\forall x \in L_\gamma)[(\forall y \in x)(\exists z \in L_\gamma)A[\underline{a},y,z] \to \exists zA[\underline{a},x,z]]$$

or

$$L_{\phi\alpha(\gamma+\gamma)} \not\models (\forall z \in L_\gamma)\neg A[\underline{a},b\ z]\,.$$

Proof. We suppress the parameters $\underline{a}$ and write $A(v,w)$ instead of $A[\underline{a},v,w]$. Assume that

(1) $L_{\phi\alpha(\gamma+\gamma)} \models (\forall x \in L_\gamma)[(\forall y \in x)(\exists z \in L_\gamma)A(y,z) \to \exists zA(x,z)]$

for all limit ordinals γ with the property $\beta \leq \gamma < \phi(\alpha+1)(\beta+\beta)$.

Depending on α and β, we define for all $\xi < \beta$:

$$\sigma_0 := \phi\alpha(\beta+\beta);$$

$$\sigma_{\xi+1} := \phi\alpha(\sigma_\xi+\sigma_\xi);$$

$$\sigma_\lambda := \sup\{\sigma_\xi : \xi < \lambda\}, \text{ if } \lambda \text{ is limit.}$$

It is easy to check that $\sigma_\xi < \sigma_\eta$ for $\xi < \eta$, $\xi+1 \leqq \sigma_\xi$ and $\beta < \sigma_\xi \leqq \phi(\alpha+1)(\beta+1+\xi) < \phi(\alpha+1)(\beta+\beta)$. Now we prove by induction on $\xi < \beta$:

(2) $L_{\sigma_{1+\xi}} \models (\forall x \in L_\xi)\exists z A(x,z)$.

For $\xi = 0$ this is obvious from (1) since all elements of L_0 are urelements, the empty set or the set N. If ξ is a limit ordinal, then (2) follows from the I.H. Now assume $\xi = \eta+1$. Then we have

(3) $L_{\sigma_{1+\eta}} \models (\forall x \in L_\eta)\exists z A(x,z)$

by the I.H. For $\delta = \sigma_{1+\eta}$ we obtain from (1)

(4) $L_{\phi\alpha(\delta+\delta)} \models (\forall y \in c)(\exists z \in L_\delta)A(y,z) \rightarrow \exists z A(c,z)$

for all $c \in L_{\eta+1} \subset L_\delta$. The premise of this implication is satisfied by (3), and so we have $L_{\sigma_{1+\xi}} \models \exists z A(c,z)$ for all $c \in L_\xi$. This completes the proof of (2).

Since $b \in L_\beta$ and β limit, there exists a $\xi < \beta$ such that $b \in L_\xi$. Then (2) gives for $\gamma_0 = \sigma_{1+\xi}$

(5) $L_{\gamma_0} \models \exists z A(b,z)$.

Hence $\beta < \gamma_0 < \phi(\alpha+1)(\beta+\beta)$ and

(6) $L_{\phi\alpha(\gamma_0+\gamma_0)} \not\models (\forall z \in L_{\gamma_0})\neg A(b,z)$.

The proof is thereby completed. ./.

Lemma 3.9. Assume that $\Lambda[\underline{u}]$ is a $KPu^0 + (\Sigma_1\text{-IND}_\in)$-axiom set and $\Gamma[\underline{u}]$ a set of L_* formulas. If

$$Z_*^\infty \vdash^\alpha \Lambda[\underline{u}], \Gamma[\underline{u}] \quad ,$$

then we have, for all limit ordinals β, all $\underline{a} \in L_\beta$ and all $\gamma \geq \phi\alpha(\beta+\beta)$,

$$L_\gamma \models \Gamma^{[\beta]}[\underline{a}] \; .$$

Proof by induction on α. The proof of this lemma is like that of Lemma 3.4 with $\beta+3^n$ replaced by $\phi\alpha(\beta+\beta)$ and one additional case since now we also have to treat $(\Sigma_1\text{-IND}_\in)$: Suppose that the main formula $B[\underline{u}]$ of the last inference belongs to $\Lambda[\underline{u}]$ and has the form

$$\exists a[\forall x((\forall y \in x)\exists z A[\underline{u},y,z] \rightarrow \exists z A[\underline{u},x,z]) \,\&\, \forall z \neg A[\underline{u},a,z]]$$

where $A[\underline{u},v,w]$ is Δ_0. Let us write $C[\underline{u},a]$ for the formula within the outer square brackets. Then there exist an $\alpha_0 < \alpha$ and a term d such that

(1) $Z_*^\infty \vdash^{\alpha_0} \Lambda[\underline{u}], C[\underline{u},d], \Gamma[\underline{u}]$

where d is a number constant, the set constant N, an element of the list $\underline{u}$ or a new free variable. Now choose a limit ordinal β and $\underline{a} \in L_\beta$. From the I.H. we conclude that

(2) $L_{\phi\alpha_0(\sigma+\sigma)} \models C^{[\sigma]}[\underline{a},b] \vee \Gamma^{[\sigma]}[\underline{a}]$

for some $b \in L_\beta$ and all limit ordinals $\sigma \geq \beta$. In view of Lemma 3.8 we can choose a δ such that $\beta \leq \delta < \phi(\alpha_0+1)(\beta+\beta)$ and

(3) $L_{\phi\alpha_0(\delta+\delta)} \not\models C^{[\delta]}[\underline{a},b]$.

It follows from (2) and (3) that

(4) $L_{\phi\alpha_0(\delta+\delta)} \models \Gamma^{[\delta]}[\underline{a}]$.

This implies the conclusion since $\beta \leq \delta$ and $\phi\alpha_0(\delta+\delta) < \phi(\alpha_0+1)(\beta+\beta) \leq \phi\alpha(\beta+\beta)$. ./.

<u>Theorem 3.6</u>. $L_{\phi\varepsilon_0 0}$ is a Π_2 model of $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$.

Proof. This theorem follows from Lemma 3.9 in the same way as Theorem 3.4 follows from Lemma 3.5. ./.

<u>Remarks</u>.

1. $L_{\phi\varepsilon_0 0}$ actually is the minimal Σ_1 and Π_2 model of $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$; we omit details.
2. Theorem 3.6 does not show that the proof-theoretic ordinal of $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$ is smaller or equal $\phi\varepsilon_0 0$. From Theorem 3.5 we can conclude that $\phi\varepsilon_0 0 \leq |KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)|$; we conjecture that $\phi\varepsilon_0 0$ is the proof-theoretic ordinal of this theory.
3. Cantini [5] considers a similar theory. The approach there, however, is very indirect and uses a series of intermediate reductions.

3.4. Inductive definitions in Kripke-Platek set theory.

Let $L_1(\mathrm{IND})$ be the language L_1 extended by a unary relation symbol P_A for every R-positive formula $A[R^+,u]$ of L_1. The axioms of ID_1 are the axioms of PA with the scheme of complete induction for arbitrary formulas of $L_1(\mathrm{IND})$ and the following

$(P_A.1)$ $\forall x(A[P_A,x] \rightarrow P_A(x))$,

$(P_A.2)$ $\forall x(A[B,x] \rightarrow B(x)) \rightarrow \forall x(P_A(x) \rightarrow B(x))$

for all formulas $B(u)$ of $L_1(\mathrm{IND})$. ID_1^* is the subtheory of ID_1 where $(P_A.2)$ is restricted to formulas $B(u)$ which contain only positive occurrences of the new relation symbols. $\widehat{ID}_1$ is obtained from ID_1 by omitting the second axioms $(P_A.2)$ and by replacing $(P_A.1)$ by the fixed point axiom

$$\forall x(A[P_A,x] \leftrightarrow P_A(x)) .$$

It is clear that $\widehat{ID}_1$ is contained in ID_1^*. The relation symbol P_A represents: in $\widehat{ID}_1$ an arbitrary fixed point of the operator Γ_A; in ID_1^* the least fixed point with respect to all classes definable by formulas which are positive in the fixed point constants; in ID_1 the least fixed point with respect to all classes definable in $L_1(\mathrm{IND})$.
For further information on ID_1, ID_1^* and $\widehat{ID}_1$ see [4,16].

$L_1(\mathrm{IND})$ is embedded into L_* by taking the usual translation of L_1 or L_2 into L_* and finding a suitable interpretation of the relation symbols P_A.

<u>Theorem 3.7</u>. $\widehat{ID}_1$ is contained in $KPu^0 + (IND_N)$.

Proof. Take an arbitrary R-positive L_1 formula $A[R^+, u]$. By Corollary 3.2 there exists a Σ formula $P^A[u]$ of L_* such that

$$KPu^0 + (IND_N) \vdash P^A[b] \leftrightarrow b \in N \ \& \ A[P^A, b] .$$

If we choose the Σ formula $P^A[u]$ as the interpretation of the relation symbol P_A, then $KPu^0 + (IND_N)$ obviously proves (the translation of) each theorem of $\widehat{ID}_1$. ./.

<u>Remark</u>. Aczel [unpublished notes] gives the first proof-theoretic analysis of $\widehat{ID}_1$. It is shown there that $\widehat{ID}_1$ can be reduced to $(\Sigma^1_1\text{-AC})$ and vice versa; an outline of Aczel's proof is given in Feferman [16].

We use the Σ recursion theorem instead of the second recursion theorem in order to deal with ID_1^* and ID_1. Take an R-positive L_1 formula $A[R^+, u]$ and suppose that Th is one of the theories $KPu^0 + (\Sigma_1\text{-}IND_\in)$ or KPu. By Corollary 3.3 we find a Σ function symbol I_A of Th such that Th proves

$$I_A^\alpha = \{x \in N : A[\cup\{I_A^\xi : \xi < \alpha\}, x]\}$$

for all α. Then define the Σ formula $I^A(u)$ by

$$I^A(u) \ :\Longleftrightarrow \ \exists\xi(u \in I_A^\xi) .$$

<u>Lemma 3.10</u>. Choose $A[R^+,u]$ and I^A as above.

(a) $KPu^0 + (\Sigma_1\text{-}IND_\in)$ proves that I^A is the least fixed point of the operator Γ_A which is definable by a Σ formula; i.e. $KPu^0 + (\Sigma_1\text{-}IND_\in)$ proves

(i) $a \in N \ \&\ A[I^A,a] \rightarrow I^A(a)$;

(ii) $(\forall x \in N)(A[B,x] \rightarrow B(x)) \rightarrow (\forall x \in N)(I^A(x) \rightarrow B(x))$

for every Σ formula $B(u)$ of L_*.

(b) $KPu^0 + (IND_\in)$ proves (ii) for all L_* formulas $B(u)$.

Proof. (a) We work in $KPu^0 + (\Sigma_1\text{-}IND_\in)$.

(i) Assume $a \in N \ \&\ A[I^A,a]$. In view of Σ reflection we may assume that there exists an ordinal α such that $A[(\exists \xi < \alpha)(. \in I_A),a]$. Thus we have $a \in I_A^\alpha$ by the definition of I_A. This implies $I^A(a)$.

(ii) is proved by $(\Sigma_1\text{-}IND_\in)$. Assume:

(1) $(\forall x \in N)(A[B,x] \rightarrow B(x))$,

(2) $(\forall \beta < \alpha)(\forall x \in N)(x \in I_A^\beta \rightarrow B(x))$

and choose an arbitrary $a \in I_A^\alpha$. Then $a \in N$ and $A[\cup\{I_A^\beta : \beta < \alpha\},a]$ according to the definition of I_A. By the positivity of A and (2) we obtain $A[B,a]$ and so $B(a)$ by (1). Hence

(3) $(\forall x \in N)(x \in I_A^\alpha \rightarrow B(x))$.

The Σ formula (3) is equivalent to a Σ_1 formula by Σ reflection. Therefore $(\Sigma_1\text{-}IND_\in)$ proves the claim.

(b) The same argument works for arbitrary L_* formulas if we use $(IND_\in)$ instead of $(\Sigma_1\text{-}IND_\in)$. ./.

<u>Theorem 3.8</u>. ID_1^* is contained in $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in)$.

Proof. We interpret each formula $P_A(u)$ of $L_1(IND)$ as the corresponding Σ formula $I^A(u)$. Then each $L_1(IND)$ formula B which is positive in the relation symbols P_A translates into a Σ formula of L_*. By Lemma 3.10(b) we immediately obtain the assertion. ./.

<u>Corollary 3.4</u>. Suppose that $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in) \vdash I^A(k)$ or $ID_1^* \vdash P_A(k)$ for some number k. Then $|k|_A < \phi\varepsilon_0 0$.

Proof. $ID_1^* \vdash P_A(k)$ implies $KPu^0 + (IND_N) + (\Sigma_1\text{-}IND_\in) \vdash I^A(k)$ by Theorem 3.8. Now we apply Theorem 3.6 and obtain $L_{\phi\varepsilon_0 0} \models \exists\xi(k \in I_A^\xi)$. Hence there exists an $\alpha < \phi\varepsilon_0 0$ such that $k \in I_A^\alpha$, i.e. $|k|_A < \phi\varepsilon_0 0$. ./.

Similar results for ID_1^* have been obtained by Friedman [unpublished notes] and Cantini [6]. Remark, however, that this corollary does not imply $|ID_1^*| = \phi\varepsilon_0 0$. Suppose that $\mathcal{Q}$ is a primitive recursive well-ordering and $A[R,u]$ the formula $\forall y(y \mathcal{Q} u \rightarrow R(y))$. Then $ID_1^* \vdash \forall x P_A(x)$ implies $ID_1^* \vdash TI(\mathcal{Q},R)$. It is not clear whether the converse direction holds as well since $(P_A.2)$ is restricted to formulas positive in the fixed point constants. In the case of ID_1 we have: $ID_1 \vdash \forall x P_A(x)$ is equivalent to $ID_1 \vdash TI(\mathcal{Q},R)$.

<u>Theorem 3.9</u>. ID_1 is contained in KPu.

The proof of this theorem is like the proof of Theorem 3.8 with Lemma 3.10(b) instead of Lemma 3.10(a).

3.5. KPu.

The previous theorem implies $|ID_1| \leq |KPu|$. The proof-theoretic treatment of KPu from above requires impredicative methods which will not be discussed here. Therefore we only state the relevant results and refer to Jäger [30] for proofs. The proof-theoretic ordinal of ID_1 and KPu is called the Howard ordinal and denoted by $\overline{\theta}\varepsilon_{\Omega_1+1}0$ in the notation system of [4].

Theorem 3.10. Let α be the Howard ordinal $\overline{\theta}\varepsilon_{\Omega_1+1}0$.
(a) $|KPu| = |ID_1| = \alpha$.
(b) L_α is the least Σ_1 and Π_2 model of KPu.

There is an additional system strictly between KPu^r + (IND_N) and KPu which has been omitted so far. In section 6 we will consider the theory KPu^0 + (BR) which is interesting in connection with predicative mathematics.

§4. THEORIES FOR ITERATED ADMISSIBLE SETS WITHOUT FOUNDATION.

In this section we begin the systematic analysis of theories for iterated admissible sets, concentrating on the case that $\in$-induction is not available. We are particularly interested in the theories ES^{o} and KPu^{o} strengthened by the limit axiom

(Lim) $\forall x \exists y(x \in y$ & y admissible$)$.

These theories have proof-theoretic ordinal Γ_o and so are examples of predicatively reducible theories which have fairly strong set existence axioms at the price of being weak with respect to induction principles. As preparation for studying (Lim) we first introduce the theories $KPu^{o} + (\mathcal{U}_n)$, $n < \omega$, of finitely many admissible universes.

Let L_{Ad} be the language L_* extended by a new unary relation symbol Ad to express admissibility. The meaning of Ad is defined implicitly by the Ad-axioms.

<u>Ad-axioms</u>.

(Ad1) $Ad(d) \rightarrow N \in d$ & $Tran(d)$;

(Ad2) $Ad(d) \rightarrow (\forall x,y \in d)(\exists z \in d)(x \in z$ & $y \in z)$;

(Ad3) $Ad(d) \rightarrow (\forall x \in d)(\exists z \in d)(x \subset z$ & $Tran(z))$;

(Ad4) $Ad(d)$ & $a,\underline{b} \in d \rightarrow (\exists z \in d)(z = \{x \in a : A[\underline{b},x]\})$;

(Ad5) $Ad(d)$ & $a,\underline{b} \in d$ & $(\forall x \in a)(\exists y \in d)A[\underline{b},x,y] \rightarrow$
$(\exists z \in d)(\forall x \in a)(\exists y \in z)A[\underline{b},x,y]$;

where the formula A in (Ad4) and (Ad5) is a Δ_o formula of L_*.

These axioms express that each object satisfying Ad is a transitive set which contains the set of natural numbers and reflects the set existence axioms of Kripke-Platek set theory. Whenever we formulate a set theory in the language L_{Ad}, we assume that the Ad-axioms are included in the list of ontological axioms. The Ad-axioms do not claim the existence of a set d with Ad(d) and so the theory Th formulated in L_{Ad} is a conservative extension of Th formulated in L_*.

The following lemma is proved by straightforward induction on the length of derivations in $KPu^0 + (IND_N)$ or KPu, respectively; unrestricted induction in $KPu^0 + (IND_N)$ and KPu reduces to Δ_0 induction in ES^0 and ES^r.

<u>Lemma 4.1</u>. For every L_* formula $A[\underline{u}]$ we have:

(a) $KPu^0 + (IND_N) \vdash A[\underline{a}] \implies ES^0 \vdash Ad(d) \,\&\, \underline{a} \in d \to A^d[\underline{a}]$;

(b) $KPu \vdash A[\underline{a}] \implies ES^r \vdash Ad(d) \,\&\, \underline{a} \in d \to A^d[\underline{a}]$.

4.1. <u>$KPu^0 + (U_n)$</u>.

Extend L_{Ad} to the language $L_{Ad}(d_0,\dots,d_{n-1})$ by adding n new set constants $d_0,\dots,d_{n-1}$, $n < \omega$. (U_n) is the following axiom

(U_n) $Ad(d_0) \,\&\, \dots \,\&\, Ad(d_{n-1}) \,\&\, d_0 \in d_1 \,\&\, \dots \,\&\, d_{n-2} \in d_{n-1}$.

Together with the Ad-axioms this means that $d_0,\dots,d_{n-1}$ is an increasing sequence of admissible sets. The intended models of the theories $KPu^0+(\mathcal{U}_n)$ and $KPu+(\mathcal{U}_n)$ are the structures $L_{\Omega_{n+1}}$, where $L_{\Omega_1},\ldots,L_{\Omega_n}$ are interpretations for the constants $d_0,\ldots,d_{n-1}$. Every $KPu^0+(\mathcal{U}_n)$ is a definitorial extension of a theory $Th_{(n)}$ in the language $L_*(d_0,\ldots,d_{n-1})$. To simplify notation, we identify $Th_{(n)}$ with $KPu^0+(\mathcal{U}_n)$.

Theorem 4.1. $KPu^0+(\mathcal{U}_{n+1})$ is a conservative extension of $KPu^0+(IND_N)+(\mathcal{U}_n)$ in the following sense; we have for every sentence A of $L_*(d_0,\ldots,d_{n-1})$:

$$KPu^0+(\mathcal{U}_{n+1}) \vdash A^{d_n} \iff KPu^0+(IND_N)+(\mathcal{U}_n) \vdash A .$$

Proof. Let Th be the admissible extension of $KPu^0+(\mathcal{U}_n)$. Then $KPu^0+(\mathcal{U}_{n+1})$ corresponds to $Th+(I_N)$ and the conclusion follows from Theorem 2.2(b). ./.

In order to calculate the proof-theoretic ordinal of $KPu^0+(\mathcal{U}_n)$, we also consider the connections between the infinitary versions $KPu^0+(\mathcal{U}_{n+1})_\infty$ and $KPu^0+(\mathcal{U}_n)_\infty$.

Lemma 4.2. Suppose that α is an ε-number, A a sentence of L_1 and B a sentence of $L_*(d_0,\ldots,d_{n-1})$. Then we have:

(a) $KPu^0_\infty \vdash^{<\alpha} A^N \implies PA_\infty \vdash^{<\phi\alpha 0} A$;

(b) $KPu^0+(\mathcal{U}_{n+1})_\infty \vdash^{<\alpha} B^{d_n} \implies KPu^0+(\mathcal{U}_n)_\infty \vdash^{<\phi\alpha 0} B$.

Proof. Let Th_0 and Th_1 be the admissible cover and extension of PA and $KPu^0+(\mathcal{U}_n)$, respectively. Then KPu^0 is $Th_0+(I_N)$ and $KPu^0+(\mathcal{U}_{n+1})$ corresponds to $Th_1+(I_N)$. By Lemma 2.4 we have

$$Th + (I_N)_\infty \vdash^{<\beta} C \implies Th_\infty \vdash^{<\beta} C$$

for every ε-number α and every sentence C of the language of Th, where Th is Th_0 or Th_1. Therefore the lemma under consideration is a consequence of Theorem 2.3 and Theorem 2.4. ./.

<u>Theorem 4.2</u>. We have for every $\mathcal{L}_*$ sentence A and $\mathcal{L}_1$ sentence B:

(a) $KPu^0 + (\mathcal{U}_{n+1}) \vdash A^{d_0} \implies KPu^0_\infty \vdash^{<\phi(n|\varepsilon_0)} A$;

(b) $KPu^0 + (\mathcal{U}_n) \vdash B^N \implies PA_\infty \vdash^{<\phi(n|\varepsilon_0)} B$.

Proof. (a) From the assumption we obtain by Theorem 4.1 that $KPu^0 + (IND_N) + (\mathcal{U}_n) \vdash A^{d_0}$ or $KPu^0 + (IND_N) \vdash A$ depending on whether $n > 0$ or $n = 0$. This implies by Lemma 2.4 and Lemma 2.2 that $KPu^0 + (\mathcal{U}_n)_\infty \vdash^{<\varepsilon_0} A^{d_0}$ or $KPu^0_\infty \vdash^{<\varepsilon_0} A$. The conclusion follows by Lemma 4.2(b) and induction on n. (b) follows from (a) by means of Lemma 4.2(a). ./.

<u>Corollary 4.1 (Π_2 boundedness)</u>. Suppose that A is a Π_2 sentence of $\mathcal{L}_*$. If $KPu^0 + (\mathcal{U}_{n+1})$ proves A^{d_0}, then $L_{\phi(n|\varepsilon_0)}$ is a model of A.

Proof. $KPu^0 + (\mathcal{U}_{n+1}) \vdash A^{d_0}$ implies $KPu^0_\infty \vdash^{<\phi(n|\varepsilon_0)} A$ and the assertion follows from Lemma 3.5. ./.

<u>Corollary 4.2</u>. $|KPu^0 + (\mathcal{U}_n)| \leq \phi(n|\varepsilon_0)$.

Proof. Suppose that $\mathcal{Q}$ is a primitive recursive well-ordering and $KPu^0 + (\mathcal{U}_n) \vdash TI(\mathcal{Q},R)$. By Theorem 4.2(b) we obtain $PA_\infty \vdash^{<\alpha} TI(\mathcal{Q},R)$ for $\alpha = \phi(n|\varepsilon_0)$. Hence the order-type of $\mathcal{Q}$ is smaller than $\phi(n|\varepsilon_0)$ by Lemma 2.3. ./.

Our theories of finitely many universes $KPu^0 + (\mathcal{U}_n)$ are closely related to Feferman's iterated inductive fixed point theories $\hat{ID}_n$ (cf. [16]). The language $\mathcal{L}_1^{(n)}$ of $\hat{ID}_n$ is defined by induction on $n < \omega$:

(i) $\mathcal{L}_1^{(0)}$ is $\mathcal{L}_1$;

(ii) for $n > 0$, $\mathcal{L}_1^{(n)}$ is the extension of $\mathcal{L}_1^{(n-1)}$ by a unary relation symbol P_A for each R-positive formula $A[R^+,u]$ of $\mathcal{L}_1^{(n-1)}$. We call P_A a fixed point of stage n with definition clause A.

The axioms of $\hat{ID}_n$ are the axioms of PA with the scheme of complete induction for all $\mathcal{L}_1^{(n)}$ formulas and the fixed point axioms

$$\forall x(A[P_A,x] \leftrightarrow P_A(x))$$

for all fixed points P_A of stages $\leq n$.

<u>Theorem 4.3</u>. Let $n \geq 1$. $\hat{ID}_n$ is contained in $KPu^0 + (IND_N) + (\mathcal{U}_{n-1})$ and in $KPu^0 + (\mathcal{U}_n)$.

Proof. By Theorem 4.1 it is enough to show that $KPu^0 + (IND_N) + (\mathcal{U}_{n-1})$ contains $\hat{ID}_n$. In translating $\mathcal{L}_1^{(n)}$ into $\mathcal{L}_{Ad}(d_0,\ldots,d_{n-2})$ we have to take care of the fixed points P_A. By induction on $m \leq n$ we define an interpretation of every fixed point P of stage $m \leq n$ such that:

(i) P is interpreted by a set $\hat{P} \in d_m$ if $m+1 < n$;

(ii) P is interpreted by a set $\hat{P}$ if $m+1 = n$;

(iii) P is interpreted by a Σ formula $\hat{P}(x)$ if $m = n$.

Let P be a fixed point of stage $m \leqq n$ with R-positive definition clause $A[R^+,P_1,\dots,P_k,u]$ where $P_1,\dots,P_k$ are the fixed points of stages $< m$ in A. Then $A^N[R^+,\hat{P}_1,\dots,\hat{P}_k,u]$ is an R-positive Δ_0 formula with parameters $\hat{P}_1,\dots,\hat{P}_k$.

1. If $m+1 < n$, then $P_1,\dots,P_k$ are elements of d_{m-1} and we can apply Corollary 3.2 relativized to d_{m-1}. Therefore we find a Σ formula $Q(u) = Q[P_1,\dots,P_k,u]$ such that for all b

$$Q^{d_{m-1}}(b) \leftrightarrow b \in N \ \& \ A^N[Q^{d_{m-1}},\hat{P}_1,\dots,\hat{P}_k,b] .$$

By Δ_0 separation in d_m there is a set $\hat{P} := \{x \in N : Q^{d_{m-1}}(x)\}$ in d_m which can be used as an interpretation of P.

2. $m+1 = n$. By the same argument as before we now find a set $\hat{P}$ in the universe of $KPu^0 + (IND_N) + (\mathcal{U}_{n-1})$ to interpret P.

3. If $m = n$, then Corollary 3.2 gives a Σ formula $Q(u) = Q[\hat{P}_1,\dots,\hat{P}_k,u]$ such that

$$Q(b) \leftrightarrow b \in N \ \& \ A^N[Q,\hat{P}_1,\dots,\hat{P}_k,b] .$$

The formula $Q(u)$ can be taken as the interpretation of P. In all three cases the interpretation of P satisfies the (translation of the) fixed point axiom of P. The rest of the proof is obvious. ./.

Theorem 4.4.

(a) For $n > 0$ we have:

$$|KPu^0 + (\mathcal{U}_n)| = |KPu^0 + (IND_N) + (\mathcal{U}_{n-1})| = \phi(n|\varepsilon_0) .$$

(b) $|\bigcup_{n<\omega} KPu^0 + (\mathcal{U}_n)| = |\bigcup_{n<\omega} \hat{ID}_n| = \Gamma_0 .$

Proof. Part (a) follows from Corollary 4.2, Theorem 4.1, Theorem 4.3 and Feferman's results in [16].
(b) is a consequence of (a) and Lemma 2.1. ./.

Remark. The idea of adding universes to a given theory first appears in Martin-Löf [41] where he considers constructive theories of types. In [16] Feferman introduces the theories $\hat{ID}_n$ and subtheories with finitely many universes of his system of explicit mathematics, in order to prove Hancock's conjecture.

4.2. The axiom (Lim).

The axiom (Lim) appears to be very natural in the context of iterating admissibility. It states that every set a is contained in an admissible set d and has the effect that all of its models are limits of admissible sets.

(Lim) $\forall x \exists y (x \in y \,\&\, Ad(y))$

The theories KPi^0 and KPl^0 are formulated in the language L_{Ad}; KPi^0 is $KPu^0 + (Lim)$ and KPl^0 is $ES^0 + (Lim)$. The letters 'i' and 'l' refer to 'inaccessible' and 'limit' according to the models of KPi^0 and KPl^0 in the constructible hierarchy. If L_α is a model of KPi^0, then α has to be admissible, since $L_\alpha \models KPu^0$, and a limit of admissibles, since $L_\alpha \models (Lim)$, and therefore an admissible limit of admissibles. Ordinals with this property are called recursively inaccessible ordinals.

The following results will show that KPi^o and KPl^o are proof-theoretically equivalent although they differ significantly with respect to their models. This is due to the limited principles of induction available in KPi^o and KPl^o. We will see later that the theory KPi, i.e. $KPi^o + (IND_N) + (IND_\in)$, is much stronger than KPl.

4.3. Reduction of KPi^o.

We show that KPi^o can be reduced locally to theories with finitely many universes. For every Σ formula A provable in KPi^o there exists an $n < \omega$ such that A is already provable in $KPu^o + (\mathcal{U}_n)$.

Definition. Suppose $m < n < k < \omega$.

(a) Let A be a formula of $\mathcal{L}_{Ad}$. $A^{(m,n)}$ denotes an arbitrary formula of $\mathcal{L}_{Ad}(d_0,\ldots,d_{k-1})$ which is obtained from A by

(i) replacing each unrestricted universal quantifier $\forall x(\ldots)$ by $(\forall x \in d_i)(\ldots)$ for some $i \leq m$;

(ii) replacing each unrestricted existential quantifier $\exists x(\ldots)$ by $(\exists x \in d_j)(\ldots)$ for some j, $n \leq j < k$.

(b) If $\Gamma = \{A_1,\ldots,A_t\}$ is a set of $\mathcal{L}_{Ad}$ formulas, then $\Gamma^{(m,n)}$ denotes an arbitrary set $\{A_1^{(m,n)},\ldots,A_t^{(m,n)}\}$ of formulas of $\mathcal{L}_{Ad}(d_0,\ldots,d_{k-1})$.

Lemma 4.3. Assume that $\Lambda[\underline{u}]$ is a KPi^o-axiom set and $\Gamma[\underline{u}]$ a set of $\mathcal{L}_{Ad}$ formulas. If

$$Z_* \vdash_0^n \Lambda[\underline{u}]\,,\,\Gamma[\underline{u}]\,,$$

then we have, for all $i \leq m$, $k > s \geq m+3^n$ and all terms $\underline{a}$,

$$KPu^o + (\mathcal{U}_k) \vdash \underline{a} \notin d_i \text{ , } \Gamma^{(m,s)}[\underline{a}] \text{ .}$$

Proof by induction on n. We follow the pattern of the proof of Lemma 3.4 with one difference: Instead of interpreting the set Γ in an initial segment of the constructible hierarchy, we work in the theory $KPu^o + (\mathcal{U}_k)$. As in the proof of Lemma 3.4 we concentrate on the case that the main formula $B[\underline{u}]$ of the last inference belongs to $\Lambda[\underline{u}]$ and is the negation of an axiom of KPi^o. Let $i \leq m$ and $k > s \geq m+3^n$.

1. If $\neg B[\underline{u}]$ is an ontological, number-theoretic, equality or induction axiom of KPi^o, then it is a Δ_o formula and an axiom of KPu^o as well. So by the I.H. there are no problems.

2. Suppose that $\neg B[\underline{u}]$ is the axiom (Lim). Then $B[\underline{u}]$ is the formula $\exists x \forall y (x \notin y \vee \neg Ad(y))$ and there exist a term c and an $n_o < n$ such that

$$(1) \quad Z_* \vdash_0^{n_o} \Lambda[\underline{u}] \text{ , } \forall y (c \notin y \vee \neg Ad(y)) \text{ , } \Gamma[\underline{u}] \text{ .}$$

We have $i \leq m < m+1 < m+1+3^{n_o} \leq m+3^n \leq s < k$ and obtain by the I.H.

$$(2) \quad KPu^o + (\mathcal{U}_k) \vdash \underline{a}, c^* \notin d_i \text{ , } (\forall y \in d_{m+1})(c^* \notin y \vee \neg Ad(y)) \text{ , } \Gamma^{(m,s)}[\underline{a}] \text{ ,}$$

where c^* is defined as follows: c^* is c, if c is not a free variable; c^* is the number constant 0, if c is a free variable but does not belong to the list $\underline{u}$; if c is from the list $\underline{u}$, then c^* is the corresponding term in the list $\underline{a}$. The Ad-axioms now yield

$$(3) \quad KPu^o + (\mathcal{U}_k) \vdash \underline{a} \notin d_i \text{ , } c^* \in d_i \;\&\; (\exists y \in d_{m+1})(c^* \in y \;\&\; Ad(y))$$

and the conclusion follows from (2) and (3).

3. Now take $\neg B[\underline{u}]$ to be a set-theoretic axiom of KPi^o different from (Δ_o-Col); e.g. $\neg B[\underline{u}]$ is $\exists z(c \subset z \,\&\, Tran(z))$ where c is a constant of L_{Ad} or a free variable from the list $\underline{u}$. There exists $n_o < n$ such that

$$(4)\quad Z_* \vdash_0^{n_o} \Lambda[\underline{u}]\,,\ \neg(c \subset v \,\&\, Tran(v))\,,\ \Gamma[\underline{u}]$$

for a new free variable v. By the I.H. we obtain

$$(5)\quad KPu^o + (\mathcal{U}_k) \vdash \underline{a}, w \notin d_i\,,\ \neg(c^* \subset w \,\&\, Tran(w))\,,\ \Gamma^{(m,s)}[\underline{a}]$$

where c^* is defined as before and w is a free variable not from the list $\underline{a}$. Hence

$$(6)\quad KPu^o + (\mathcal{U}_k) \vdash \underline{a} \notin d_i\,,\ (\forall z \in d_i)\neg(c^* \subset z \,\&\, Tran(z))\,,\ \Gamma^{(m,s)}[\underline{a}]\,.$$

Together with the Ad-axioms for d_i this gives the conclusion.

4. Assume finally that $\neg B[\underline{u}]$ is an instance of (Δ_o-Col). Then we have a Δ_o formula $C[\underline{u},v,w]$, a term c and $n_o, n_1 < n$ such that

$$(7)\quad Z_* \vdash_0^{n_o} \Lambda[\underline{u}]\,,\ (\forall x \in c)\exists y C[\underline{u},x,y]\,,\ \Gamma[\underline{u}]$$

$$(8)\quad Z_* \vdash_0^{n_1} \Lambda[\underline{u}]\,,\ \forall z(\exists x \in c)(\forall y \in z)\neg C[\underline{u},x,y]\,,\ \Gamma[\underline{u}]$$

By the I.H. we obtain from (7) for $s_o := m+3^{n_o}$

$$(9)\quad KPu^o + (\mathcal{U}_k) \vdash$$
$$\underline{a}, c^* \notin d_i\,,\ (\forall x \in c^*)(\exists y \in d_{s_o})C[\underline{a},x,y]\,,\ \Gamma^{(m,s)}[\underline{a}]$$

where c^* is defined as before. We have $(s_o+1)+3^{n_1} = m+3^{n_o}+1+3^{n_1} \leq m+3^n \leq s < k$. Now we apply the I.H. to (8) with m replaced by s_o+1.

(10) $KPu^o + (\mathcal{U}_k) \vdash$

$$\underline{a}, c^* \notin d_i \text{ , } (\forall z \in d_{s_o+1})(\exists x \in c^*)(\forall y \in z)\neg C[\underline{a},x,y] \text{ , } \Gamma^{(m,s)}[\underline{a}] \text{ .}$$

The conclusion follows from (9) and (10) since $d_{s_o} \in d_{s_o+1}$ and $(\underline{a} \in d_i \to c^* \in d_i)$. This completes the proof of Lemma 4.3. ./.

<u>Theorem 4.5</u>. Suppose that A is a sentence provable in KPi^o. Then there exists an $n < \omega$ such that

$$KPu^o + (\mathcal{U}_{k+1}) \vdash A^{(m,k)}$$

for all m and $k = m+n$.

This theorem follows immediately from Lemma 4.3. It tells us that KPi^o is reducible to $\bigcup_{n<\omega} KPu^o + (\mathcal{U}_n)$. The following corollaries are especially interesting.

<u>Corollary 4.3</u>.

(a) If A is a Σ sentence of $\mathcal{L}_{Ad}$ provable in KPi^o, then there exists an $n < \omega$ such that A is provable in $KPu^o + (\mathcal{U}_n)$.

(b) Suppose A is a sentence of $\mathcal{L}_*$ and suppose further that

$$KPi^o \vdash \forall x(Ad(x) \to A^x) \text{ .}$$

Then there exists an $n < \omega$ such that

$$KPu^o + (\mathcal{U}_n) \vdash A^{d_0} \text{ .}$$

<u>Corollary 4.4</u>. $|KPi^o| = |KPl^o| = \Gamma_o$.

Proof. By Theorem 4.4 and Corollary 4.3. ./.

Corollary 4.5 (Π_2 boundedness). Suppose that A is a Π_2 sentence of L_*. If KPi^0 proves

$$\forall x(Ad(x) \rightarrow A^x),$$

then L_{Γ_0} is a model of A.

Proof. Suppose that KPi^0 proves $\forall x(Ad(x) \rightarrow A^x)$. Then there exists an $n < \omega$ such that

$$KPu^0 + (\mathcal{U}_m) \vdash Ad(d_0) \rightarrow A^{d_0}$$

for all $m \geq n$. With Corollary 4.1 we conclude $L_{\Gamma_0} \models A$. ./.

4.4. KPl^0 and ATR_0.

Although we do not have $\in$-induction in KPl^0 and KPi^0, these systems are strong enough to develop a reasonable theory of well-orderings. As examples we prove the axiom β and comparability of well-orderings. Then we turn to the connections between KPl^0, KPi^0 and Friedman's theory ATR_0.

Definition.

(a) A binary relation r on a set a is well-founded if every non-empty subset of a has an r-minimal element.

$$Wf(a,r) \;:\Longleftrightarrow\; \begin{cases} r \subset a \times a \quad \& \\ \forall x[x \neq \emptyset \;\&\; x \subset a \rightarrow (\exists y \in x)(\forall z \in x)(\langle z,y\rangle \notin r)]. \end{cases}$$

(b) The axiom β asserts that every well-founded relation r on a has a collapsing function; it is the universal closure of the following formula

$$Wf(a,r) \rightarrow$$
$$\exists f[Fun(f) \;\&\; dom(f) = a \;\&\; (\forall x \in a)(f(x) = \{f(y) : (y,x) \in r\})] \;.$$

Under (Δ_0-Sep) the well-foundedness of a relation r on a is equivalent to the principle of Δ_0 transfinite induction

$$(\forall x \in a)[(\forall y \in a)((y,x) \in r \rightarrow A(y)) \rightarrow A(x)] \rightarrow (\forall x \in a)A(x)$$

for all Δ_0 formulas A(u) of L_{Ad}.

Theorem 4.6. KPl^0 proves the axiom β. Actually, if r is a well-founded relation on a and (a,r) belongs to the admissible set d, then the collapsing function f for (a,r) belongs to d as well.

Proof. Assume Wf(a,r) for some a and r. By (Lim) we find a set d with $a,r \in d$ and Ad(d). Now we define A(u,f) to be the Δ_0 formula

$$u \in a \;\&\; Fun(f) \;\&\; dom(f) = \{y \in a : (y,u) \in r \vee y = u\} \;\&$$
$$(\forall y \in dom(f))(f(y) = \{f(z) : (z,y) \in r\})$$

and work in the universe d. By Δ_0 induction on (a,r) one proves

(1) $b \in a \;\&\; A(b,f) \;\&\; A(b,g) \rightarrow f = g$;

(2) $b \in a \rightarrow (\exists f \in d)A(b,f)$.

The proof of (1) is obvious. For the proof of (2) choose $b \in a$ and suppose

(3) $(\forall x \in a)(\exists f \in d)[(x,b) \in r \rightarrow A(x,f)]$.

Since all parameters of this formula belong to d, we may use Δ_0 collection in d to find a set $c \in d$ such that

(4) $(\forall x \in a)(\exists f \in c)[(x,b) \in r \rightarrow A(x,f)]$.

Then

$$g := \bigcup\{f \in c : (\exists x \in a)[(x,b) \in r \ \& \ A(x,f)]\}$$

is an element of d by Δ_0 separation in d.

$$h := g \cup \{(b,\{g(x) : (x,b) \in r\})\}$$

is also in d and has the property A(b,h). This proves (2). By (Δ_0-Sep) we define the set

$$f := \{(x,y) \in a \times d : (\exists f \in d)(A(x,f) \ \& \ f(x) = y)\}$$

which is the collapsing function for (a,r). In fact, by Δ_0 collection applied to (2), we can see that $f \in d$, too. ./.

Remark. The existence of a collapsing function for (a,r) does not imply that r is well-founded on a as long as we work in theories without $(I_\in)$. In the presence of $(I_\in)$, however, we have for every binary relation r on a that

$$\exists f(f \text{ collapsing function for } (a,r)) \rightarrow Wf(a,r)$$

and therefore the axiom β has the effect of making the Π predicate Wf(a,r) a Δ predicate.

Definition.

(a) We write Lo(a,r) if $r \subset a \times a$ is a linear ordering of a.

(b) r is a well-ordering of a if r is a well-founded linear ordering of a;

$$\mathrm{Wo}(a,r) :\Longleftrightarrow \mathrm{Lo}(a,r) \ \& \ \mathrm{Wf}(a,r) .$$

(c) If r and s are linear orderings of a and b, respectively, then we write

$$f : (a,r) \leq (b,s)$$

to express that f is an order-preserving map of (a,r) onto an initial segment of (b,s). It is the conjunction of the following statements:

(i) $\mathrm{Lo}(a,r) \ \& \ \mathrm{Lo}(b,s) \ \& \ \mathrm{Fun}(f) \ \& \ \mathrm{dom}(f) = a \ \& \ \mathrm{rng}(f) \subset b$;

(ii) $(\forall x,y \in a)[(x,y) \in r \rightarrow (f(x),f(y)) \in s]$;

(iii) $(\forall x \in a)(\forall y \in b)[(y,f(x)) \in s \rightarrow (\exists z \in a)((z,x) \in r \ \& \ y = f(z))]$.

(d) Comparability of well-orderings is the sentence expressing that any two well-orderings are comparable; it is the (universal closure of the) following formula:

$$\mathrm{Wo}(a,r) \ \& \ \mathrm{Wo}(b,s) \rightarrow \exists f(f : (a,r) \leq (b,s) \vee f : (b,s) \leq (a,r)]$$

<u>Theorem 4.7</u>. KPl^0 proves comparability of well-orderings.

Proof. Assume $\mathrm{Wo}(a,r)$ and $\mathrm{Wo}(b,s)$. By the axiom β, which is provable in KPl^0, there exist collapsing functions f for (a,r) and g for (b,s). By induction on (a,r) we can prove that for every $c \in a$ one of the following three conditions holds:

(1) $(\exists x \in b)(f(c) = g(x))$,

(2) $f(c) = rng(g)$,

(3) $(\exists x \in a)((x,c) \in r \ \&\ f(x) = rng(g))$.

This implies one of the following alternatives:

(4) $(\forall x \in a)(\exists y \in b)(f(x) = g(y))$,

(5) $(\exists x \in a)(f(x) = rng(g))$.

In the case of (4) we have $h : (a,r) \leq (b,s)$ for h defined by $h := \{(x,y) \in a \times b : f(x) = g(y)\}$.
In the case of (5) we have $h' : (b,s) \leq (a,r)$ for h' defined by $h' := \{(x,y) \in b \times a : g(x) = f(y)\}$ . ./.

Remark. An analysis of the previous proof shows that comparability of well-orderings is provable in the theory ES^0 + (Axiom β).

In the rest of this subsection we discuss the connections between KPl^0 and the theory ATR_0 introduced by Friedman in [21,22]. His work was later incorporated and expanded in Friedman-McAloon-Simpson [25] and Simpson [51,52]. The work of Steel also contributed to the discovery of ATR_0 as a natural subsystem of second order arithmetic.

For any arithmetic formula $B[X,Y,u]$ of L_2 and any binary relation Q on N we introduce the L_2 formula $H_B(X,Y,Q)$, defined by

$$H_B(X,Y,Q) \ :\Longleftrightarrow\ \forall x \forall y(\langle x,y\rangle \in Y \leftrightarrow B[X,(Y)^y_Q,x]) ,$$

where $(Y)^y_{\mathcal{Q}}$ is the set $\{\langle x,z\rangle \in Y : z\,\mathcal{Q}\,y\}$. $H_B(X,Y,\mathcal{Q})$ means that Y codes the B-hierarchy obtained by iterating the operation $Q \to \{x : B[X,Q,x]\}$ along $\mathcal{Q}$; X acts as a set parameter in this process.

The theory ATR_o is formulated in L_2 and consists of elementary analysis with restricted induction EA_o plus the scheme (ATR) of arithmetic transfinite recursion

$$(ATR)\quad \forall Z[WO(Z) \to \forall X \exists Y H_B(X,Y,Z)]$$

for every arithmetic formula $B[X,Y,u]$. (ATR) says that every well-ordering has a B-hierarchy for all B in Π^1_o.

<u>Theorem 4.8</u>. ATR_o is contained in KPl^o; i.e.

$$ATR_o \vdash A \implies KPl^o \vdash A$$

for every sentence A of L_2 (which is identified with its translation into L_{Ad}).

Proof. Only the principle (ATR) requires any special consideration. Fix an arbitrary arithmetic formula $B[U,V,u]$, a set of natural numbers X and a binary relation coded by the set Z. By (Lim) there exists an admissible set d which contains X and Z. Now assume $WO(Z)$ so that we have transfinite induction along Z for all Δ_o formulas. Define $H(k,U)$ to be the Δ_o formula

$$H_B(X,U,Z|k) \quad \text{where} \quad Z|k := \{\langle x,y\rangle \in Z : \langle y,k\rangle \in Z \vee y = k\}.$$

As in the proof of the axiom β, induction on Z gives:

(1) $H(k,U)$ & $H(k,V) \rightarrow (\forall x \in N)(\langle x,k\rangle \in Z \rightarrow (U)_x = (V)_x)$;

(2) $k \in N \rightarrow (\exists Y \in d)H(k,Y)$.

Hence we obtain a B-hierarchy U along Z by defining

$$U := \{\langle x,y\rangle \in N \times N : (\exists Y \in d)(H(y,Y) \ \& \ x \in (Y)_y)\} . \qquad ./.$$

Remark. It is open whether KPl^0 or KPi^0 can be embedded into ATR_0. We can prove with recursion-theoretic methods that each theory $KPu^0 + (\mathcal{U}_n)$ is contained in ATR_0. In view of Theorem 4.5 this gives the proof-theoretic equivalence of KPi^0, KPl^0 and ATR_0.

Here we show the slightly weaker result that $\hat{ID}_n$ is contained in ATR_0 for $n < \omega$. In the proof we make use of the fact that ATR_0 proves the existence of countable ω-models of $(\Sigma^1_1\text{-AC})$ (due to Friedman and Simpson [22,51]) and of Aczel's interpretation of $\hat{ID}_1$ in $(\Sigma^1_1\text{-AC})$ as described in [16]. In that paper the following lemma is proved.

Lemma 4.4. For every R-positive arithmetic formula $A[R^+,\underline{X},u]$ we can find a Σ^1_1 formula $\hat{Q}[\underline{X},u]$ such that $(\Sigma^1_1\text{-AC})$ proves:

$$\hat{Q}[\underline{X},u] \leftrightarrow A[\hat{Q}[\underline{X},.],u] .$$

Theorem 4.9. $\hat{ID}_n$ is contained in ATR_0 for every $n < \omega$.

Proof. Suppose that $n < \omega$ is given. By induction on $m \leq n$ we show that each fixed point P of stage m can be interpreted by a set $\hat{P}$ in ATR_0. Then each $L_1^{(n)}$ formula translates into an arithmetic formula of L_2 and the scheme of complete induction in $\hat{ID}_n$ reduces to restricted induction in ATR_o. Suppose that P is a fixed point of stage $m \leq n$ with R-positive definition clause $A[R^+, P_1, \ldots P_k, u]$ where $P_1, \ldots, P_k$ are fixed points of stages $< m$. By the I.H. there exist sets $\hat{P}_1, \ldots, \hat{P}_k$ in ATR_o which interpret $P_1, \ldots, P_k$. By Friedman [22] and Simpson [51] ATR_o proves the existence of a set M such that $\hat{P}_1, \ldots, \hat{P}_k \in \mathcal{M} := \{(M)_x : x \in N\}$ and

$$ATR_o \vdash B^{\mathcal{M}}$$

for each sentence B provable in $(\Sigma^1_1\text{-AC})$. Here $B^{\mathcal{M}}$ is the arithmetic formula obtained from B by replacing all quantifiers $\forall X(\ldots X \ldots)$ and $\exists Y(\ldots Y \ldots)$ in B by $\forall x(\ldots (M)_x \ldots)$ and $\exists y(\ldots (M)_y \ldots)$, respectively. By Lemma 4.4 there exists a Σ^1_1 formula $\hat{Q}[\hat{P}_1, \ldots, \hat{P}_k, u]$ such that ATR_o proves

$$\hat{Q}^{\mathcal{M}}[\hat{P}_1, \ldots, \hat{P}_k, u] \leftrightarrow A[\hat{Q}^{\mathcal{M}}[\hat{P}_1, \ldots, \hat{P}_k, .], \hat{P}_1, \ldots, \hat{P}_k, u] \; .$$

We define $\hat{P} := \{x : \hat{Q}^{\mathcal{M}}[\hat{P}_1, \ldots, \hat{P}_k, x]\}$ by arithmetic comprehension and have the interpretation of the fixed point P. The rest of the embedding theorem is obvious. ./.

Corollary 4.6. $|KPi^o| = |KPl^o| = |ATR_o| = \Gamma_o$.

Proof. By Theorem 4.4, Corollary 4.4, Theorem 4.8 and Theorem 4.9. ./.

Remark. For an alternative proof of $|ATR_o| = \Gamma_o$, using Gödel's second incompleteness theorem, see Friedman-McAloon-Simpson [25].

4.5. The class of well-founded sets in KPi^0.

Sets are not necessarily well-founded in KPi^0. Therefore we introduce the class Hwf of all hereditarily well-founded sets and study the closure properties of Hwf in the theory KPi^0.

Definition. A set a is called hereditarily well-founded if there exists a transitive set $b \supset a$ which is well-founded with respect to $\in$;

$$Hwf(a) :\Longleftrightarrow \exists z(Tran(z) \ \& \ a \subset z \ \& \ Wf(z, \in | z \times z)).$$

If TC(a) is the transitive closure of a, then a is hereditarily well-founded provided that TC(a) is well-founded. In theories with restricted induction it is not always clear whether the transitive closure exists, and so we have choosen this slightly unusual definition.

Let Th_0 and Th_1 be theories in the language L_{Ad}. Each L_{Ad} formula K[u] defines a class $K = \{x : K[x]\}$ in V_N. We say that K is an inner model of Th_0 in Th_1 if

(i) $Th_1 \vdash \forall x \forall y(x \in y \ \& \ K[y] \rightarrow K[x])$,

(ii) $Th_1 \vdash A^K$

for all sentences A provable in Th_0. A^K is the formula that we obtain from A if we replace all unrestricted quantifiers $\forall x(\ldots)$ and $\exists x(\ldots)$ by $\forall x(K[x] \rightarrow \ldots)$ and $\exists x(K[x] \ \& \ \ldots)$, respectively.

Lemma 4.5.

(a) $KPi^0 \vdash a \in b \;\&\; Hwf(b) \rightarrow Hwf(a)$.

(b) $KPi^0 \vdash Hwf(a_1) \;\&\ldots\&\; Hwf(a_n) \rightarrow A^{Hwf}[a_1,\ldots,a_n]$

if $A[u_1,\ldots,u_n]$ is an instance of (Pair), (Transitive Hull), (Δ_0-Sep), (I_N), ($I_\in$) or the axiom β.

(c) There exists an instance $A[u_1,\ldots,u_n]$ of (Δ_0-Col) such that

$$KPi^0 \nvdash Hwf(a_1) \;\&\ldots\&\; Hwf(a_n) \rightarrow A^{Hwf}[a_1,\ldots,a_n] .$$

Proof. (a) and (b) follow from the definition of Hwf.

(c) Now assume that

$$KPi^0 \vdash Hwf(a_1) \;\&\ldots\&\; Hwf(a_n) \rightarrow A^{Hwf}[a_1,\ldots,a_n]$$

for all instances of (Δ_0-Col). Then Hwf is an inner model of KPu^r + (Axiom β) in KPi^0. Consequently $|KPu^r + (\text{Axiom } \beta)| \leq |KPi^0| \leq \Gamma_0$. By Theorem 8.4 this is a contradiction. ./.

Theorem 4.10.

(a) Hwf is an inner model of ES^r + (Axiom β) in KPi^0 but not an inner model of KPu^r.

(b) For every sentence A of L_2 we have:

$$ES^r + (\text{Axiom } \beta) \vdash A \implies KPi^0 \vdash A .$$

Proof. By Lemma 4.5. ./.

§5. HYPERARITHMETIC SETS IN ATR_0.

The essential purpose of this section is to provide the necessary tools for interpreting theories for admissible sets in subsystems of analysis. We take up the notions of section 1.6 and combine them with ideas from the theory of hyperarithmetic sets à la Kleene, Spector et al. (cf. [45]). Admissible sets, for example, are coded by collections of (relativized) hyperarithmetic representation trees.

We begin with some basic definitions. The set $P \in Pow(N)$ is recursive in the set $Q \in Pow(N)$ if there exists a total function $\{e\}^Q$, recursive in Q, such that $P = (e;Q)$ where we define $(e;Q) := \{x : \{e\}^Q(x) = 1\}$. The jump of Q is the set

$$j(Q) := \{e : \exists x(\{e\}^Q(e) = x)\} .$$

The iteration of the jump operation along the ordinals is given by transfinite recursion as follows:

$$H_0 := \emptyset ;$$

$$H_{\alpha+1} := j(H_\alpha) ;$$

$$H_\lambda := \text{disjoint union of } H_\beta,\ \beta < \lambda, \text{ if } \lambda \text{ is limit.}$$

In order to carry through this definition and its relativized versions in subsystems of analysis, we represent (initial segments of the) ordinals by elements of the sets $\mathcal{O}$ and $\mathcal{O}^Q$ defined in section 1.5.

From the work of Friedman and Simpson we know that many important theorems on hyperarithmetic sets can be pushed through in ATR_0, and several of their results will be used in this section. The following two lemmas go back to Friedman [20,22] and Simpson [51].

Lemma 5.1. ATR_0 proves the Σ^1_1 axiom of choice; i.e.

$$ATR_0 \vdash \forall x \exists X A(x,X) \rightarrow \exists Z \forall x A(x,(Z)_x)$$

for every Σ^1_1 formula $A(u,X)$ of L_2.

Lemma 5.2 (Numerical Π^1_1 uniformization). For every Π^1_1 formula $A[\underline{X},\underline{u},v]$ there exists a Π^1_1 formula $A'[\underline{X},\underline{u},v]$ such that ATR_0 proves:

(a) $\forall y(A'[\underline{X},\underline{u},y] \rightarrow A[\underline{X},\underline{u},y])$;

(b) $\exists y A[\underline{X},\underline{u},y] \rightarrow \exists! y A'[\underline{X},\underline{u},y]$.

Definition. Let $H(X,Y,e)$ be the arithmetic formula expressing that for all $k <^X_{\mathcal{O}} e$ or $k = e$:

(i) $(Y)_k = X$, if $k = 0$;

(ii) $(Y)_k = j((Y)_m)$, if $k = 2^m$;

(iii) $(Y)_k = \{\langle x,y\rangle : \exists z(\{m\}^X(z) = y \,\&\, x \in (Y)_y\}$, if $k = 3 \cdot 5^m$.

$H(X,Y,e)$ says that Y is the jump hierarchy starting with X and iterated along the ordering coded by e. One can easily show that the principle (ATR) of arithmetic transfinite recursion is equivalent to the assertion that Y exists for all X and $e \in \mathcal{O}^X$; such a jump hierarchy is unique in the sense below (cf. [25,52]).

Lemma 5.3. ATR_0 proves

(a) $\mathcal{O}^X(e) \to \exists Y H(X,Y,e)$;

(b) $\mathcal{O}^X(e) \ \&\ H(X,Y,e) \ \&\ H(X,Z,e) \to (Y)_e = (Z)_e$.

Given an element e in $\mathcal{O}^X$ and $H(X,Y,e)$, we write H_e^X for $(Y)_e$. By the previous lemma the sets H_e^X uniquely exist for all e in $\mathcal{O}^X$. Following Kleene we call Y hyperarithmetic in X if for some e in $\mathcal{O}^X$, Y is recursive in H_e^X; Y is hyperarithmetic if it is hyperarithmetic in the empty set.

Definition. $HYP(X,Y) :\Longleftrightarrow$

$$\exists e \exists k[\mathcal{O}^X(e) \ \&\ Tot^X(k) \ \&\ \forall Z(H(X,Z,e) \to Y = \{k;(Z)_e\})] .$$

Using the Σ_1^1 axiom of choice, we may consider $HYP(X,Y)$ as a Π_1^1 formula in ATR_0. In order to increase readibility we often write $e \in \mathcal{O}^X$, $Y \in HYP(X)$, $e \in \mathcal{O}$ and $Y \in HYP$ instead of $\mathcal{O}^X(e)$, $HYP(X,Y)$, $\mathcal{O}(e)$ and $HYP(\emptyset,Y)$, respectively, although these formulas do not define sets in ATR_0. From the definition of $HYP(X,Y)$ and Lemma 5.1 we conclude:

Lemma 5.4. For every Π_1^1 formula $A[X,Y,\underline{u}]$ there exists a Π_1^1 formula $B[X,\underline{u}]$ such that

$$ATR_0 \vdash (\exists Y \in HYP(X))A[X,Y,\underline{u}] \leftrightarrow B[X,\underline{u}] .$$

The Suslin-Kleene theorem is one of the most important results in the theory of hyperarithmetic sets since it characterizes the sets hyperarithmetic in X as the sets Δ_1^1

in X. It is known from Friedman [22] and Simpson [52] that Suslin-Kleene is provable in ATR_0.

Theorem 5.1 (Suslin-Kleene theorem).

(a) There exist a Π^1_1 formula $A[X,u,v]$ and a Σ^1_1 formula $B[X,u,v]$ such that ATR_0 proves:

(i) $Y \in HYP(X) \rightarrow \exists x \forall y (y \in Y \leftrightarrow A[X,x,y])$;

(ii) $Y \in HYP(X) \rightarrow \exists x \forall y (y \in Y \leftrightarrow B[X,x,y])$.

(b) If $A[X,\underline{u},v]$ is a Π^1_1 formula and $B[X,\underline{u},v]$ a Σ^1_1 formula, then the following is a theorem of ATR_0:

$$\forall y (A[X,\underline{u},y] \leftrightarrow B[X,\underline{u},y]) \rightarrow (\exists Y \in HYP(X)) \forall y (y \in Y \leftrightarrow A[X,\underline{u},y]) .$$

Corollary 5.1. Suppose that $A[X,Y,\underline{u}]$ is a Σ^1_1 formula. Then

$$ATR_0 \vdash \exists ! Y A[X,Y,\underline{u}] \rightarrow (\exists Y \in HYP(X)) A[X,Y,\underline{u}] .$$

Proof. We work in ATR_0 and assume $\exists ! Y A[X,Y,\underline{u}]$. Then we have for all k that

$$\exists Y (A[X,Y,\underline{u}] \ \& \ k \in Y) \leftrightarrow \forall Y (A[X,Y,\underline{u}] \rightarrow k \in Y) .$$

Therefore the set $Z := \{y : \exists Y (A[X,Y,\underline{u}] \ \& \ y \in Y)\}$ is an element of $HYP(X)$ by Suslin-Kleene. ./.

Now we come back to the relations $\tilde{\in}$, $\approx$ and REP introduced in section 1.6. We use Corollary 5.1 to show that $\tilde{\in}$ and $\approx$ are Δ^1_1 on the class of representation trees.

Corollary 5.2. There exist Σ^1_1 formulas $A_0[X,Y]$, $A_1[X,Y]$ and Π^1_1 formulas $B_0[X,Y]$, $B_1[X,Y]$ such that ATR_0 proves:

(a) $REP(S)\ \&\ REP(T) \rightarrow (S \tilde{\in} T \leftrightarrow A_0[S,T])$;

(b) $REP(S)\ \&\ REP(T) \rightarrow (S \tilde{\in} T \leftrightarrow B_0[S,T])$;

(c) $REP(S)\ \&\ REP(T) \rightarrow (S \simeq T \leftrightarrow A_1[S,T])$;

(d) $REP(S)\ \&\ REP(T) \rightarrow (S \simeq T \leftrightarrow B_1[S,T])$.

Proof. We show (a) and (b), the proofs of (c) and (d) are similar. Define

(1) $A'[X,Y] :\Longleftrightarrow \exists x(\langle 2x\rangle \in Y\ \&\ X \simeq Y_{\langle 2x\rangle})$,

(2) $B'[X,Y] :\Longleftrightarrow (\exists f \in HYP(X \dot{\cup} Y))\exists x[\langle 2x\rangle \in Y\ \&\ f : X \simeq Y_{\langle 2x\rangle}]$

and choose $A_0[X,Y]$ and $B_0[X,Y]$ to be Σ^1_1 and Π^1_1 formulas provably (in ATR_0) equivalent to $A'[X,Y]$ and $B'[X,Y]$, respectively; $\dot{\cup}$ is the symbol for the disjoint union of two sets. From the definition of $\tilde{\in}$ and Lemma 1.6(d) we conclude:

(3) $ATR_0 \vdash REP(S)\ \&\ REP(T) \rightarrow (S \tilde{\in} T \leftrightarrow A_0[S,T])$,

(4) $ATR_0 \vdash REP(S)\ \&\ REP(T) \rightarrow (B_0[S,T] \rightarrow S \tilde{\in} T)$,

(5) $ATR_0 \vdash REP(S)\ \&\ REP(T)\ \&\ S \tilde{\in} T \rightarrow$
$$\exists! f \exists x(\langle 2x\rangle \in T\ \&\ f : S \simeq T_{\langle 2x\rangle}) .$$

Now Corollary 5.1 can be applied to give

(6) $ATR_0 \vdash REP(S)\ \&\ REP(T) \rightarrow (S \tilde{\in} T \rightarrow B_0[S,T])$.

Then (a) and (b) follow from (3), (4), (6). ./.

Lemma 5.5. The following is provable in ATR_0. Let T be a special tree in HYP(X) with the property

$$\forall x(\langle 2x\rangle \in T \rightarrow REP(T_{\langle 2x\rangle})) .$$

Then there exists a representation tree T' in HYP(X) such that for all representation trees S

$$S \tilde{\in} T' \leftrightarrow S \tilde{\in} T .$$

Proof. If T is an N-tree, then take T' = T; otherwise define:

(1) $B(u) :\Longleftrightarrow \langle 2u\rangle \in T \ \& \ \forall y(\langle 2y\rangle \in T \ \& \ y <_N u \rightarrow T_{\langle 2y\rangle} \not\equiv T_{\langle 2u\rangle})$,

(2) $T' := \{\langle\rangle\} \cup \{\langle 2x\rangle * t : B(x) \ \& \ t \in T_{\langle 2x\rangle}\}$.

T' is an element of HYP(X) by Corollary 5.2 and Theorem 5.1. It is easy to check that T' is a representation tree and $S \tilde{\in} T' \leftrightarrow S \tilde{\in} T$ for all representation trees S. ./.

Theorem 5.2. Let A(X,Y,u) be a Π^1_1 formula with just the free set variables X, Y and possibly additional number variables. Then ATR_0 proves

$$\forall x(\exists Y \in HYP(X))A(X,Y,x) \rightarrow (\exists Z \in HYP(X))\forall x A(X,(Z)_x,x) .$$

Proof. Reasoning in ATR_0, assume that

(1) $\forall x(\exists Y \in HYP(X))A(X,Y,x)$

and define B(u,v,X) to be a Π^1_1 formula provably equivalent to

$$\exists e\exists k[v = \langle e,k\rangle \ \& \ e \in \mathcal{O}^X \ \& \ Tot^X(k) \ \&$$
$$\forall Y\forall Z(H(X,Z,e) \ \& \ Y = (k;(Z)_e) \rightarrow A(X,Y,u))$$

From (1) we obtain $\forall x \exists y B(x,y,X)$. Then by Lemma 5.2 we can find a Π^1_1 formula $B'(u,v,X)$ such that

(2) $\forall x \forall y (B'(x,y,X) \rightarrow B(x,y,X))$,

(3) $\forall x \exists ! y B'(x,y,X)$.

The next step is to define two formulas $C_0(u,v,X)$ and $C_1(u,v,X)$ as follows

(4) $C_0(u,v,X) :\Longleftrightarrow \exists y \exists e \exists k [B'(u,y,X) \;\&\; y = \langle e,k \rangle \;\&$
$\forall Y \forall Z (H(X,Z,e) \;\&\; Y = (k;(Z)_e) \rightarrow v \in Y)]$;

(5) $C_1(u,v,X) :\Longleftrightarrow \forall y \forall e \forall k [B'(u,y,X) \;\&\; y = \langle e,k \rangle \rightarrow$
$\exists Y \exists Z (H(X,Z,e) \;\&\; Y = (k;(Z)_e) \;\&\; v \in Y)]$

From (2), (3) and Lemma 5.3 we conclude

(6) $\forall x \forall z [C_0(x,z,X) \leftrightarrow C_1(x,z,X)]$.

Since $C_0(u,v,X)$ is equivalent to a Π^1_1 formula and $C_1(u,v,X)$ to a Σ^1_1 formula, we can apply the Suslin-Kleene theorem to find a set $Z \in HYP(X)$ such that $Z = \{\langle z,x \rangle : C_0(x,z,X)\}$. It is clear that $\forall x A(X,(Z)_x,x)$. ./.

Our next goal is to show that the hyperarithmetic representation trees determine a model of KPu^r. The strategy is clear: Sets are represented by hyperarithmetic representation trees, the membership relation $\in$ is interpreted by $\tilde{\in}$ and the remaining symbols are translated according to the following definition.

Definition.

(a) $\tilde{N} := \{<>\} \cup \{t : \exists x(t = <2x> \vee t = <2x,2x+1>)\}$;

(b) $\mathcal{U}(X,Y) :\Longleftrightarrow \mathrm{REP}(Y)\ \&\ \mathrm{HYP}(X,Y)$;

(c) $\tilde{S}(X) :\Longleftrightarrow \forall x(X \neq [x])$;

(d) $\tilde{R}(X) :\Longleftrightarrow 0 =_N 1$;

(e) $\tilde{J}(X_1,\dots,X_n) :\Longleftrightarrow$

$$\exists x_1 \dots \exists x_n (X_1 = [x_1]\ \& \dots \&\ X_n = [x_n]\ \&\ J(x_1,\dots,x_n)),$$

if J is a relation symbol of the language L_1 different from R.

We often write $Y \in \mathcal{U}(X)$ instead of $\mathcal{U}(X,Y)$.

Definition of $\tilde{a}$ for every term a of L_*.

$$\tilde{a} := \begin{cases} [n] & \text{, if a is the number constant n;} \\ \tilde{N} & \text{, if a is the constant N;} \\ X & \text{, if a is the variable x.} \end{cases}$$

Inductive definition of $A^{\mathcal{U}(V)}$ for every L_* formula A.

1. $(a \in b)^{\mathcal{U}(V)} := (\tilde{a}\ \tilde{\in}\ \tilde{b})$; $S(a)^{\mathcal{U}(V)} := \tilde{S}(\tilde{a})$; $J(a_1,\dots,a_n)^{\mathcal{U}(V)}$ $:= \tilde{J}(\tilde{a}_1,\dots,\tilde{a}_n)$, if J is a relation symbol of L_1.

2. $(\neg A)^{\mathcal{U}(V)} := \neg A^{\mathcal{U}(V)}$ for every atomic formula A.

3. $(A\ \&\ B)^{\mathcal{U}(V)} := A^{\mathcal{U}(V)}\ \&\ B^{\mathcal{U}(V)}$; $(A \vee B)^{\mathcal{U}(V)} := A^{\mathcal{U}(V)} \vee B^{\mathcal{U}(V)}$.

4. $(\forall x \in a)A(x)^{\mathcal{U}(V)} := \forall x[<2x> \in \tilde{a} \rightarrow A^{\mathcal{U}(V)}(\tilde{a}_{<2x>})]$;

   $(\exists x \in a)A(x)^{\mathcal{U}(V)} := \exists x[<2x> \in \tilde{a}\ \&\ A^{\mathcal{U}(V)}(\tilde{a}_{<2x>})]$.

5. $\forall x A(x)^{\mathcal{U}(V)} := (\forall X \in \mathcal{U}(V))A^{\mathcal{U}(V)}(X)$;

   $\exists x A(x)^{\mathcal{U}(V)} := (\exists X \in \mathcal{U}(V))A^{\mathcal{U}(V)}(X)$.

Remark. It is clear from Corollary 5.2 that the translation $A^{\mathcal{U}(V)}$ of a Δ_0 formula A of L_* is Δ^1_1 on the class $\mathcal{U}(V)$. More precisely: With any Δ_0 formula $A[\underline{x}]$ of L_* we can associate a Π^1_1 formula $A_1[\underline{X}]$ and a Σ^1_1 formula $A_2[\underline{X}]$ such that ATR_0 proves for $i = 1,2$

$$\underline{X} \in \mathcal{U}(V) \to (A^{\mathcal{U}(V)}[\underline{X}] \leftrightarrow A_i[\underline{X}]) .$$

Theorem 5.3. Let $A[\underline{x}]$ be an axiom of KPu^r. Then ATR_0 proves

$$\underline{X} \in \mathcal{U}(V) \to A^{\mathcal{U}(V)}[\underline{X}] .$$

Proof. 1. If $A[\underline{x}]$ is an equality axiom, then the conclusion follows by Lemma 1.6 and Lemma 5.1; if $A[\underline{x}]$ is an ontological or number-theoretic axiom, then the conclusion is trivially satisfied.

2. Now we check the set-theoretic axioms of KPu^r.

2.1. (Pair) and (Transitive Hull). For $S_0, S_1 \in \mathcal{U}(V)$ we define

(1) $T_0 := \{<>\} \cup \{<0,t> : t \in S_0\} \cup \{<2,t> : t \in S_1\}$,

(2) $T_1 := \{<>\} \cup \{<2s>*t : s*t \in S_0\}$.

T_0 and T_1 are special trees in $\mathcal{U}(V)$ which satisfy

(3) $S_0 \tilde{\in} T_0$ & $S_1 \tilde{\in} T_0$,

(4) $(\forall X \tilde{\in} S_0)(X \tilde{\in} T_1)$ & $(\forall X \tilde{\in} T_1)(\forall Y \tilde{\in} X)(Y \tilde{\in} T_1)$.

Hence, by Lemma 5.5, there are trees $T'_0, T'_1 \in \mathcal{U}(V)$ which are witnesses for (Pair) and (Transitive Hull).

2.2. (Δ_0-Sep). Suppose that $\underline{S}, T$ are elements of $\mathcal{U}(V)$ and

$A[\underline{x},y]$ is a Δ_0 formula of L_*. By Theorem 5.1 and the remark above the set

(5) $X := \{\langle\rangle\} \cup \{\langle 2x\rangle * t : \langle 2x\rangle * t \in T \ \& \ A^{U(V)}[\underline{S}, T_{\langle 2x\rangle}]\}$

is a special tree in HYP(V). It follows that

(6) $\forall Y(Y \tilde{\in} X \leftrightarrow Y \tilde{\in} T \ \& \ A^{U(V)}[\underline{S},Y])$.

We apply Lemma 5.5 in order to find a copy of X in $U(V)$.

2.3. (Δ_0-Col). Suppose again that $\underline{S}$,T are elements of $U(V)$ and $A[\underline{x},y,z]$ is a Δ_0 formula of L_*. Assume also that

(7) $(\forall X \tilde{\in} T)(\exists Y \in U(V))A^{U(V)}[\underline{S},X,Y]$.

This is equivalent to

(8) $\forall x(\exists Y \in HYP(V))[\langle 2x\rangle \in T \rightarrow (REP(Y) \ \& \ A^{U(V)}[\underline{S}, T_{\langle 2x\rangle}, Y])]$.

The formula within the square brackets is provably equivalent to a Π^1_1 formula. Therefore we can apply Theorem 5.2 in order to find a set $Z \in HYP(V)$ such that

(9) $\forall x[\langle 2x\rangle \in T \rightarrow (REP((Z)_x) \ \& \ A^{U(V)}[\underline{S}, T_{\langle 2x\rangle}, (Z)_x])]$.

Hence HYP(V) contains the special tree

(10) $Z' := \{\langle\rangle\} \cup \{\langle 2x\rangle * t : \langle 2x\rangle \in T \ \& \ t \in (Z)_x\}$

and Lemma 5.5 gives a set $Z'' \in U(V)$ such that

(11) $(\forall X \tilde{\in} T)(\exists Y \tilde{\in} Z'')A^{U(V)}[\underline{S},X,Y]$.

This completes the interpretation of the set-theoretic axioms of KPu^r in ATR_0.

3. Induction axioms. The translation of $(I_\in)$ is provable in ATR_0 since representation trees are well-founded. The translation of (I_N) is trivially provable. ./.

<u>Corollary 5.3</u>. We have for every sentence A of L_*:

$$KPu^r \vdash A \implies ATR_0 \vdash A^{\mathcal{U}(V)} .$$

§6. Γ_0 REVISITED.

The ordinal Γ_0 has gained some importance in the foundations of mathematics as being the ordinal generally associated with theories which formalize the predicative part of mathematics.

The notion of predicativity goes back to Poincaré who objects to the Platonistic conception of set. Following his standpoint, only the natural numbers with the unlimited principle of complete induction can be regarded as given. Sets do not exist a priori but have to be introduced by carefully chosen definitions. In order to avoid a vicious circle in the definition $\forall x(x \in a \leftrightarrow A(x))$ of a set a, we have to require that the meaning of the formula A does not refer to a totality where a might belong to. Typical predicative sets are all sets $a = \{x \in N : \mathcal{N} \models A(x)\}$ with A Π^1_0; sets of the form $b = \{x \in N : Pow(N) \models B(x)\}$, where B is a Π^1_1 formula, are prima facie impredicative.

Russell's ramified theory of types was the first attempt to formalize predicative reasoning. Later Feferman, Kreisel and Schütte were able to characterize predicativity in the framework of second order arithmetic. It was shown by Feferman and Schütte that Γ_0 is the proof-theoretic ordinal of predicative analysis.

In this section we will survey some known approaches to predicativity and establish connections to theories of ordinal Γ_0 introduced so far. For a detailed discussion of the philosophical background and the classical mathematical realization of predicativity we suggest Feferman [8,14], Kreisel [38] and Schütte [47].

6.1. Autonomous iterations.

The original examples in this direction are autonomous progressions of ramified systems à la Feferman, Kreisel and Schütte [8,36,47] and autonomous progressions of theories according to Feferman and Turing [7,59] based on reflection principles. Here we present an approach in the same vein fitting precisely into our framework.

We consider a natural process of iteration of Π^1_0 comprehension. If α is a countable ordinal, we write $(\Pi^1_0\text{-CA})_{<\alpha}$ for elementary analysis EA together with the axioms

$$\forall X \exists Y H_B(X,Y,\mathcal{Q}) \quad \text{and} \quad TI(\mathcal{Q},A)$$

for all primitive recursive well-orderings $\mathcal{Q}$ of order-type $<\alpha$, all arithmetic formulas $B[X,Y,u]$ and all L_2 formulas A.

The theories $(\Pi^1_0\text{-CA})_{<\alpha}$ are predicative modulo well-orderings of order-type less than α, whereas the property of being a well-ordering is impredicative. The set Aut of Π^1_0-autonomous or predicative ordinals is generated by

(i) $0 \in \text{Aut}$;

(ii) if $\beta \in \text{Aut}$ and α is provable in $(\Pi^1_0\text{-CA})_{<\beta}$, then $\alpha \in \text{Aut}$.

From now on we consider $(\Pi^1_0\text{-CA})_{<\Gamma_0}$ as the theory which formalizes predicative reasoning in terms of second order arithmetic. This is justified by the following famous result of Feferman [8] and Schütte [46].

<u>Theorem 6.1</u>. $\Gamma_0 = \min\{\xi : \xi \notin \mathrm{Aut}\} = \min\{\xi : \xi = |(\Pi^1_0\text{-CA})_{<\xi}|\}$.

6.2. <u>The bar rule and systems of ordinal Γ_0</u>.

In order to improve the formalism, we introduce a new theory $\mathrm{AUT}(\Pi^1_0)$ which is equivalent to $(\Pi^1_0\text{-CA})_{<\Gamma_0}$ and avoids the direct reference to the set Aut and the ordinal Γ_0. The first example of a theory with these characteristics is the system IR introduced in Feferman [8]. It will be described below. By the bar rule (BR) we understand the following rule of inference

$$\text{(BR)} \qquad \frac{\mathrm{WF}(\mathcal{Q})}{\mathrm{TI}(\mathcal{Q},A)}$$

for all binary primitive recursive relations $\mathcal{Q}$ and all formulas A. In addition to elementary analysis EA and the bar rule (BR), $\mathrm{AUT}(\Pi^1_0)$ uses the rule of inference

$$\frac{\mathrm{WF}(\mathcal{Q})}{\forall X \exists Y H_B(X,Y,\mathcal{Q})}$$

for every binary primitive recursive relation $\mathcal{Q}$ and every Π^1_0 formula $B[X,Y,u]$. The system IR of [8] is just $\mathrm{AUT}(\Pi^1_0)$ + (Δ^1_1 comprehension rule). IR can be shown to prove the same

Π^1_2 sentences as $AUT(\Pi^1_0)$. The following Lemma 6.1 is a trivial consequence of Theorem 6.1.

Lemma 6.1.

(a) $AUT(\Pi^1_0) \equiv (\Pi^1_0\text{-CA})_{<\Gamma_0}$.

(b) $|AUT(\Pi^1_0)| = \Gamma_0$.

Feferman [11] describes a subsystem PS_1 of set theory which is a conservative extension of IR. The minimal standard model of $AUT(\Pi^1_0)$ and IR is the set $L_{\Gamma_0} \cap Pow(N)$, the minimal standard model of PS_1 is L_{Γ_0} ; therefore these theories have predicative interpretations.

It is a well known result due to Kreisel that the minimal standard model of $(\Sigma^1_1\text{-AC})$ and $(\Delta^1_1\text{-CA})$ is the collection of all hyperarithmetic sets $HYP = L_{\Omega_1} \cap Pow(N)$; the same holds directly for the extensions of these systems by BR, and hence the resulting theories have no standard predicative interpretations. Nevertheless it follows from Feferman [14] and Feferman-Jäger [19] that both theories are reducible to predicative mathematics

Lemma 6.2.

(a) $(\Sigma^1_1\text{-AC}) + (BR)$ and $(\Delta^1_1\text{-CA}) + (BR)$ are conservative extensions of $AUT(\Pi^1_0)$ for Π^1_2 sentences.

(b) $|(\Sigma^1_1\text{-AC}) + (BR)| = |(\Delta^1_1\text{-CA}) + (BR)| = \Gamma_0$.

The corresponding systems in the framework of admissible sets are the theories $KPu^0 + (BR)$ and $KPu^r + (BR)$. By Corollary

3.1, which implies $(\Delta^1_1\text{-CA}) + (\mathrm{BR}) \subset \mathrm{KPu}^0 + (\mathrm{BR})$, and Lemmas 6.9, 6.10 below we obtain that $(\Delta^1_1\text{-CA}) + (\mathrm{BR})$ and $\mathrm{KPu}^0 + (\mathrm{BR})$ prove the same Π^1_1 sentences. With some extra work this result can be extended to $\mathrm{KPu}^r + (\mathrm{BR})$ (proof not given here).

Lemma 6.3.

(a) $\mathrm{KPu}^r + (\mathrm{BR})$, $\mathrm{KPu}^0 + (\mathrm{BR})$ and $(\Delta^1_1\text{-CA}) + (\mathrm{BR})$ prove the same Π^1_1 sentences.

(b) $|\mathrm{KPu}^r + (\mathrm{BR})| = |\mathrm{KPu}^0 + (\mathrm{BR})| = |(\Delta^1_1\text{-CA}) + (\mathrm{BR})| = \Gamma_0$.

6.3. Martin-Löf's intuitionistic theory of finite types.

Martin-Löf [41] is concerned with constructivity rather than predicativity. His intuitionistic theory of finite types, here denoted FT, is 'intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop [2]' (quoted from the introduction of [41]).

Feferman [16] establishes the connections between FT and the iterated inductive fixed point theory $\bigcup_{n<\omega} \widehat{\mathrm{ID}}_n$. The paper [16] also contains a proof of the following result which may be compared with Theorem 4.4.

Lemma 6.4. $|\mathrm{FT}| = |\bigcup_{n<\omega} \widehat{\mathrm{ID}}_n| = \Gamma_0$.

6.4. The theory ATR_0.

Friedman's theory ATR_0 is one of the five fundamental systems of the Friedman-Simpson program of Reverse Mathematics, a program which is guided by the question: which set existence axioms are needed to prove the theorems of ordinary mathematics?

It can be shown that ATR_0 is just strong enough to prove a series of well-known results in classical descriptive set theory (cf. [25]). Over EA_0, the principle (ATR) is equivalent to:

(i) comparability of well-orderings;

(ii) every uncountable closed set of reals has a perfect subset;

(iii) every uncountable analytic set of reals has a perfect subset;

(iv) every clopen (open) game in ω^ω is determined.

Prima facie, ATR_0 has nothing to do with predicative mathematics and it seems questionable whether a direct predicative justification of the principle (ATR) is possible. Nevertheless it follows from [25] or Corllary 4.6 that the proof-theoretic ordinal of ATR_0 is Γ_0.

<u>Lemma 6.5.</u>

(a) ATR_0 proves the same Π^1_1 sentences as $AUT(\Pi^1_0)$.

(b) $|ATR_0| = \Gamma_0$.

6.5. The theory KPi^0.

The results concerning theories for admissible sets without foundation are spread over section 4. Among other things we have considered there the role of the axiom β and proved the following two lemmas:

Lemma 6.6.

(a) $ES^0 + (\text{Axiom } \beta) \subset KPu^0 + (\text{Axiom } \beta) \subset KPi^0$.

(b) $ES^0 + (\text{Axiom } \beta) \subset ES^r + (\text{Axiom } \beta)$.

Lemma 6.7. For every sentence A of L_2 we have:

$$ES^r + (\text{Axiom } \beta) \vdash A \implies KPi^0 \vdash A .$$

The next lemma is a consequence of statement (i) in the previous section and of the remark subsequent to Theorem 4.7.

Lemma 6.8. $ATR_0 \subset ES^0 + (\text{Axiom } \beta)$.

Remark. In Simpson [52] a set-theoretic version ATR_0^S of ATR_0 is presented. ATR_0^S is formulated in the language of ZF without urelements and is a conservative extension of ATR_0. It is easy to see that ATR_0^S corresponds to our theory ES^r + (Axiom β) where the set of urelements is interpreted by ω.

Now we turn to the relations between $KPu^0 + (BR)$ and KPl^0. Both are theories of ordinal Γ_0 but they formalize completely different universes. It is interesting to see how the admissibles in KPl^0 are used in order to reduce $KPu^0 + (BR)$ to KPl^0.

The following lemma is proved by induction on the length of derivations in $KPu^0 + (BR)$.

Lemma 6.9. If A is a formula of $\mathcal{L}_*$, then

$$KPu^0 + (BR) \vdash A \implies KPl^0 \vdash \forall x(Ad(x) \rightarrow A^x) .$$

Fortunately, section 4 also provides a predicative justification of KPi^0, at least in a liberal sense.

Lemma 6.10.
(a) KPi^0 is a conservative extension of $AUT(\Pi^1_0)$ for Π^1_1 sentences.
(b) $|KPi^0| = \Gamma_0$.
(c) We have for every Π_2 sentence A of $\mathcal{L}_*$:

$$KPi^0 \vdash \forall x(Ad(x) \rightarrow A^x) \implies L_{\Gamma_0} \models A .$$

The assertions (b) and (c) are proved in section 4; (a) follows from (b) and Lemma 6.1 by standard proof-theoretic techniques.

6.6. The reflective closure of a theory.

A few years ago Feferman developed the general concepts of reflective closure and schematic theory. Applied to PA, the reflective closure considered as a schematic theory gives a theory $PA[P]^{\mathfrak{d}}$ of proof-theoretic strength Γ_0. We claim that $PA[P]^{\mathfrak{d}}$ can be reduced to KPi^0 easily. A proof of this claim and a formal definition of $PA[P]^{\mathfrak{d}}$ are not included here. We refer to Feferman [17] for a detailed discussion of the general notion of reflective closure of a schematic theory S and the special case S = PA.

6.7. Summary.

The previous considerations support the thesis that KPi^0 is the 'strongest' theory of predicative strength. We strongly believe that any reasonable theory of proof-theoretic strength $\leq \Gamma_0$ can be reduced to KPi^0 in a very natural way.

To conclude, a listing is provided of the most important theories of proof-theoretic strength Γ_0. The theory $\bigcup_{n<\omega} ID_n^*$ mentioned below was introduced by Friedman and is studied in Cantini [6] in detail.

Theorem 6.2.

(a) The following subsystems of analysis and set theory prove the same Π_1^1 sentences and have proof-theoretic ordinal Γ_0:

A. Subsystems of analysis. $AUT(\Pi_0^1)$, IR, $(\Delta_1^1\text{-CA}) + (BR)$, $(\Sigma_1^1\text{-AC}) + (BR)$, ATR_0;

B. Subsystems of set theory. PS_1, $KPu^0 + (BR)$, $KPu^r + (BR)$, $ES^0 + (\text{Axiom } \beta)$, $ES^r + (\text{Axiom } \beta)$, $KPu^0 + (\text{Axiom } \beta)$, KPl^0, KPi^0.

(b) FT, $\bigcup_{n<\omega} \widehat{ID}_n$, $\bigcup_{n<\omega} ID_n^*$ and $PA[P]^{\circlearrowleft}$ are further theories of proof-theoretic strength Γ_0.

§7. Π^1_1 COMPREHENSION, (Lim) AND ($I_\in$).

The remainder of this Habilitationsschrift is devoted to impredicative subsystems of set theory and analysis. We begin with establishing connections between the principle of Π^1_1 comprehension and the set-theoretic axioms (Lim) and ($I_\in$).

The set theories KPl^r and KPl are formulated in the language L_{Ad}. KPl^r is ES^r + (Lim) or, put in other terms, KPl^0 + ($I_\in$). KPl is KPl^r + (IND_N) + ($IND_\in$). By $(\Pi^1_1\text{-CA})_0$ we denote the second order theory EA_0 plus the scheme of Π^1_1 comprehension

(Π^1_1-CA) $\exists Z \forall x(x \in Z \leftrightarrow A(x))$

for all Π^1_1 formulas A(u) of L_2. The second order theory (Π^1_1-CA) is $(\Pi^1_1\text{-CA})_0$ + (IND_N), i.e. EA + Π^1_1 comprehension. The classical version of bar induction is given by the scheme

(BI) $\forall Z(WF(Z) \rightarrow TI(Z,A))$

for arbitrary L_2 formulas A(u). EA + (BI) is an interesting subsystem of analysis which is proof-theoretically equivalent to ID_1. We will not say more about this system here; the mathematical and foundational significance of (BI) is discussed in Feferman [13] and Kreisel [37].

<u>Theorem 7.1 (Quantifier theorem)</u>.
(a) For every Π^1_1 formula A[x,Y] of L_2 there exists a Δ_0 formula $A_\Delta[d,x,y]$ of L_* such that KPl^r proves

$$Ad(d) \ \& \ x \in N \ \& \ y \subset N \ \& \ y \in d \ \rightarrow \ (A[x,y] \leftrightarrow A_\Delta[d,x,y]) \ .$$

(b) For every Σ^1_2 formula $B[x,Y]$ of L_2 there exists a Σ_1 formula $B_\Sigma[x,y]$ of L_{Ad} such that KPl^r proves

$$x \in N \ \&\ y \subset N \ \rightarrow\ (B[x,y] \leftrightarrow B_\Sigma[x,y]) .$$

Proof. (a) Let $A[x,Y]$ be a Π^1_1 formula of L_2. By standard recursion-theoretic arguments we find a binary relation $\mathcal{Q}_{x,Y}$ with parameters x and Y, defined by the arithmetic formula $\mathcal{Q}[u,v,x,Y]$, such that

(1) $KPl^r \vdash x \in N \ \&\ y \subset N \ \rightarrow\ (A[x,y] \leftrightarrow Wf(N,\mathcal{Q}_{x,y}))$.

Next, using Theorem 4.6 and $(I_\in)$, we obtain

(2) $KPl^r \vdash Ad(d) \ \&\ x \in N \ \&\ y \subset N \ \&\ y \in d \ \rightarrow$

$$(Wf(N,\mathcal{Q}_{x,y}) \leftrightarrow A_\Delta[d,x,y])$$

for $A_\Delta[d,x,y]$ being the Δ_0 formula

(3) $A_\Delta[d,x,y] \ :\Longleftrightarrow$

$$(\exists f \in d)(f \text{ collapsing function for } (N,\mathcal{Q}_{x,y})) .$$

Hence the conclusion follows from (1) and (2).

(b) If $B[x,Y]$ is the Σ^1_2 formula $\exists Z C[x,Y,Z]$, where C is Π^1_1, we can, by (a), find a Δ_0 formula $C_\Delta[d,x,y,z]$ corresponding to C. By (Lim) and (a) the claim follows for the Σ_1 formula $B_\Sigma[x,y]$,

(4) $B_\Sigma[x,y] \ :\Longleftrightarrow$

$$\exists d(Ad(d) \ \&\ y \in d \ \&\ (\exists z \in d)(z \subset N \ \&\ C_\Delta[d,x,y,z])) \ . \ ./.$$

<u>Remark</u>. This theorem is called quantifier theorem since it proves the equivalence of Σ^1_2 predicates of analysis with Σ_1

predicates of set theory. Hence the number of unrestricted quantifiers is lowered by one.

The essential step in the preceding argument is the reduction of the well-foundedness of a relation on N, a Π^1_1 statement, to the well-foundedness of the $\in$-relation, which is given axiomatically. We now apply the quantifier theorem in order to prove Π^1_1 comprehension in KPl^r.

<u>Corollary 7.1</u>. If $A[u,v,Y]$ is a Π^1_1 formula of L_2, then

$$KPl^r \vdash v \in N \;\&\; y \subset N \;\rightarrow\; \exists z(z = \{x \in N : A[x,v,y]\}) .$$

Proof. Choose a Δ_0 formula $A_\Delta[d,u,v,y]$ according to Theorem 7.1(a). Given $y \subset N$, we find an admissible d which contains y and define by Δ_0 separation in d

$$z = \{x \in N : A_\Delta[d,x,v,y]\} = \{x \in N : A[x,v,y]\} . \qquad ./.$$

<u>Lemma 7.1</u>. KPl proves bar induction (BI).

Proof. Let A(u) be an arbitrary formula. We assume WF(z) for a $z \subset N$ and show TI(z,A) by arguing in KPl. Using Δ_0 separation, we define the set

(1) $r := \{(x,y) \in N \times N : \langle x,y\rangle \in z\}$.

This implies Wf(N,r). Since the axiom β is provable in KPl, there exists a collapsing function f for (N,r):

(2) $dom(f) = N \;\&\; (\forall y \in N)(f(y) = \{f(u) : (u,y) \in r\})$.

Now take

(3) $B(x) :\Longleftrightarrow (\forall y \in N)(f(y) = x \rightarrow A(y))$.

It is shown by $(\mathrm{IND}_{\in})$, and here we work in KPl and not in KPl^r, that

(4) $\forall x((\forall y \in x)B(y) \rightarrow B(x)) \rightarrow \forall x B(x)$.

By the definition of $B(x)$ this implies $TI(z,A)$. ./.

<u>Theorem 7.2.</u>

(a) $(\Pi^1_1\text{-CA})_0$ is contained in KPl^r.

(b) $(\Pi^1_1\text{-CA})$ is contained in $\mathrm{KPl}^r + (\mathrm{IND}_N)$.

(c) $(\Pi^1_1\text{-CA}) + (\mathrm{BI})$ is contained in KPl.

This theorem follows from Corollary 7.1 and Lemma 7.1. It gives one half of the equivalences between the set theories and subsystems of analysis mentioned there. In order to reduce these set theories to theories in L_2, we will employ the model of representation trees. Two important lemmas about $(\Pi^1_1\text{-CA})_0$ will be used without proof.

<u>Lemma 7.2.</u>

(a) $(\Pi^1_1\text{-CA})_0$ proves the Σ^1_1 axiom of choice.

(b) ATR_0 is contained in $(\Pi^1_1\text{-CA})_0$.

<u>Lemma 7.3 (Π^1_1 uniformization)</u>. For every Π^1_1 formula $A[\underline{X},Y,\underline{u}]$ there exists a Π^1_1 formula $A'[\underline{X},Y,\underline{u}]$ such that $(\Pi^1_1\text{-CA})_0$ proves:

(a) $\forall Y(A'[\underline{X},Y,\underline{u}] \rightarrow A[\underline{X},Y,\underline{u}])$;

(b) $\exists Y A[\underline{X},Y,\underline{u}] \rightarrow \exists! Y A'[\underline{X},Y,\underline{u}]$.

Lemma 7.3 is a recent result of Simpson [53] where he proves that, over ATR_0, Π^1_1 uniformization is equivalent to Π^1_1 comprehension. Part (a) of Lemma 7.2 is standard; cf. e.g. Feferman [18], Simpson [53] and Tait [55]. By (a) it is easy to show that $(\Pi^1_1\text{-CA})_0$ contains ATR_0. Set parameters are allowed in the Π^1_1 comprehension scheme; therefore $(\Pi^1_1\text{-CA})_0$ proves comprehension for formulas arithmetic in Π^1_1.

<u>Inductive definition of the class of Π^1_0-Π^1_1 formulas</u>.

1. If A is a Σ^1_1 or Π^1_1 formula, then A is a Π^1_0-Π^1_1 formula.
2. If A and B are Π^1_0-Π^1_1 formulas, then $(A \& B)$ and $(A \vee B)$ are Π^1_0-Π^1_1 formulas.
3. If $A(u)$ is a Π^1_0-Π^1_1 formula, then $\forall x A(x)$ and $\exists x A(x)$ are Π^1_0-Π^1_1 formulas.

<u>Lemma 7.4 (Π^1_0-Π^1_1 comprehension)</u>. If $A(u)$ is a Π^1_0-Π^1_1 formula, then $(\Pi^1_1\text{-CA})_0$ proves

$$\exists Z \forall x (x \in Z \leftrightarrow A(x)) .$$

The translation of L_{Ad} into L_2 is an extension of the translation in section 5. In order to find an interpretation of the additional relation symbol Ad, we call a representation tree an admissible representation tree if it codes a model of KPu^0 in the sense explained below. First we define the L_2 formula $A^{(T)}$ for every L_* formula A and special tree T. This definition is basically the definition of $A^{\mathcal{U}(V)}$ in §5 with $\mathcal{U}(V)$ replaced by $\{T_t : t \in T\}$.

<u>Inductive definition of $A^{(T)}$ for every L_* formula A.</u>

1. $A^{(T)} := A^{\mathcal{U}(\emptyset)}$ for every atomic and negated atomic formula of L_*.

2. $(A \& B)^{(T)} := A^{(T)} \& B^{(T)}$; $(A \vee B)^{(T)} := A^{(T)} \vee B^{(T)}$.

3. $(\forall x \in a)A(x)^{(T)} := \forall x[<2x> \in \tilde{a} \rightarrow A^{(T)}(\tilde{a}_{<2x>})]$;
   $(\exists x \in a)A(x)^{(T)} := \exists x[<2x> \in \tilde{a} \& A^{(T)}(\tilde{a}_{<2x>})]$.

4. $\forall x A(x)^{(T)} := \forall t[t \in T \rightarrow A^{(T)}(T_t)]$;
   $\exists x A(x)^{(T)} := \exists t[t \in T \& A^{(T)}(T_t)]$.

Let For(k) (Δ_0-For(k)) be an arithmetic formula which expresses that k is the Gödel number of a formula (Δ_0 formula) of L_*. Let Ax(k) be an arithmetic formula which expresses that k is the Gödel number of the universal closure of an axiom of KPu^0. Given a representation tree T, truth of an L_* formula according to the definition above is Δ_1^1 in T. Following for example Feferman [18], there exist a Π_1^1 formula $Tr_\Pi(T,t,k)$ and a Σ_1^1 formula $Tr_\Sigma(T,t,k)$ which express for every representation tree T: (i) k is the Gödel number of an L_* formula with lh(t) many free variables; (ii) $(\forall i < lh(t))((t)_i \in T)$; and (iii) the formula coded by k is true in T if its free variables are interpreted by $T_{(t)_0}, \ldots, T_{(t)_{lh(t)-1}}$. We may assume that these formulas have been chosen so that the following is provable in $(\Pi_1^1\text{-CA})_0$:

(Tr1) $REP(T) \rightarrow (Tr_\Pi(T,t,k) \leftrightarrow Tr_\Sigma(T,t,k))$;

(Tr2) $REP(T) \& REP(S) \& f : T \simeq S \& lh(t) = lh(s) \&$
$(\forall i < lh(t))((s)_i = f((t)_i)) \rightarrow (Tr_\Pi(T,t,k) \leftrightarrow Tr_\Pi(S,s,k))$;

(Tr3) Δ_o-For(k) & REP(S) & REP(T) & $(\forall i < lh(t))(S_{(t)_i} \simeq T_{(t)_i})$

$\rightarrow (Tr_\Pi(S,t,k) \leftrightarrow Tr_\Pi(T,t,k))$;

(tr4) REP(T) & $s = \langle t_1,\ldots,t_n\rangle$ & $(\forall i < n)((s)_i \in T)$ $\rightarrow$

$(A^{(T)}[T_{t_1},\ldots,T_{t_n}] \leftrightarrow Tr_\Pi(T,s,k))$,

if k is the Gödel number of the L_* formula $A[u_1,\ldots,u_n]$.

Definition. Take $\widetilde{Ad}(T)$ to be the following Π^1_o-Π^1_1 formula

$$\widetilde{Ad}(T) :\Longleftrightarrow \begin{cases} REP(T) \mathbin{\&} \tilde{N} \mathrel{\tilde{\in}} T \mathbin{\&} \forall t(t \in T \rightarrow T_t \mathrel{\tilde{\in}} T) \\ \mathbin{\&} \forall x(Ax(x) \rightarrow Tr_\Pi(T,\langle\rangle,x)) . \end{cases}$$

Lemma 7.5. We can prove in $(\Pi^1_1\text{-CA})_o$:

(a) $\widetilde{Ad}(T)$ & REP(S) & $S \simeq T$ $\rightarrow$ $\widetilde{Ad}(S)$;

(b) REP(T) $\rightarrow \exists S(\widetilde{Ad}(S) \mathbin{\&} T \mathrel{\tilde{\in}} S)$.

Proof. (a) follows from (Tr2) of the truth definition Tr_Π. (b) The sets in $\mathcal{U}(T)$ are coded into a representation tree. Each tree in $\mathcal{U}(T)$ is represented by a pair $\langle e,k\rangle$ such that $e \in \mathcal{O}^T$ and $Tot^T(k)$. So define

(1) $A(T,e,k,t) :\Longleftrightarrow e \in \mathcal{O}^T \mathbin{\&} Tot^T(k) \mathbin{\&}$

$$\forall Y \forall Z[H(T,Z,e) \mathbin{\&} Y = (k;(Z)_e) \rightarrow REP(Y) \mathbin{\&} t \in Y] .$$

$A(T,e,k,t)$ is a Π^1_o-Π^1_1 formula, hence by Π^1_o-Π^1_1 comprehension we can define the tree

(2) $S := \{\langle\rangle\} \cup \{\langle 2\cdot\langle e,k\rangle\rangle * t : A(T,e,k,t)\}$

which has the properties

(3) $X \in \mathcal{U}(T) \rightarrow X \tilde{\in} S$,

(4) $\langle 2n\rangle \in S \rightarrow REP(S_{\langle 2n\rangle})\ \&\ \exists X(X \in \mathcal{U}(T)\ \&\ X \simeq S_{\langle 2n\rangle})$.

The set S' defined according to Lemma 5.5 is a representation tree that codes $\mathcal{U}(T)$; in particular we have $T \tilde{\in} S'$. With essentially the same methods as in the proof of Theorem 5.3 we obtain $\widetilde{Ad}(S')$. ./.

Finally we define the translation A^{REP} of a formula A of L_{Ad}. From now on we often write $X \in REP$ instead of $REP(X)$.

Inductive definition of A^{REP} for every L_{Ad} formula A.

1. $A^{REP} := A^{\mathcal{U}(\emptyset)}$ for every atomic and negated atomic formula A of L_*.
2. $Ad(a)^{REP} := \widetilde{Ad}(\tilde{a})$; $(\neg Ad(a))^{REP} := \neg\widetilde{Ad}(\tilde{a})$.
3. $(A\ \&\ B)^{REP} := A^{REP}\ \&\ B^{REP}$; $(A \vee B)^{REP} := A^{REP} \vee B^{REP}$.
4. $(\forall x \in a)A(x)^{REP} := \forall x[\langle 2x\rangle \in \tilde{a} \rightarrow A^{REP}(\tilde{a}_{\langle 2x\rangle})]$;
   $(\exists x \in a)A(x)^{REP} := \exists x[\langle 2x\rangle \in \tilde{a}\ \&\ A^{REP}(\tilde{a}_{\langle 2x\rangle})]$.
5. $\forall x A(x)^{REP} := \forall X[X \in REP \rightarrow A^{REP}(X)]$;
   $\exists x A(x)^{REP} := \exists X[X \in REP\ \&\ A^{REP}(X)]$.

Remark. If A is a Δ_0 formula of L_{Ad}, then A^{REP} is a Π^1_0-Π^1_1 formula of L_2.

Theorem 7.3. Let $A[\underline{x}]$ be an axiom of KPl^r. Then $(\Pi^1_1\text{-}CA)_0$ proves

$$\underline{X} \in REP \rightarrow A^{REP}[\underline{X}].$$

Proof. If $A[\underline{x}]$ is the axiom (Lim), then the conclusion follows from Lemma 7.5(b); if $A[\underline{x}]$ is an Ad-axiom, then the conclusion follows from the definition of the predicate $\widetilde{Ad}$. Now suppose that $\underline{S},T$ are representation trees and $A[\underline{x},y]$ is a Δ_o formula of L_{Ad}. By Π^1_o-Π^1_1 comprehension we define the set

(1) $X := \{<>\} \cup \{<2x>*t : <2x>*t \in T \ \& \ A^{REP}[\underline{S},T_{<2x>}]\}$

which is a representation tree such that

(2) $(\forall Y \in REP)(Y \tilde{\in} X \leftrightarrow Y \tilde{\in} T \ \& \ A^{REP}[\underline{S},Y])$.

This proves Δ_o separation. For the remaining axioms of KPl^r see the proof of Theorem 5.3. ./.

<u>Corollary 7.2</u>. We have for every sentence A of L_{Ad}:

(a) $KPl^r \vdash A \implies (\Pi^1_1\text{-CA})_o \vdash A^{REP}$;

(b) $KPl^r + (IND_N) \vdash A \implies (\Pi^1_1\text{-CA}) \vdash A^{REP}$;

(c) $KPl \vdash A \implies (\Pi^1_1\text{-CA}) + (BI) \vdash A^{REP}$.

Proof. (a) and (b) follow from Theorem 7.3. For (c) we have to check that the translation of every instance of $(IND_\in)$ is provable from (BI). Let $A(x)$ be an L_{Ad} formula and assume $\neg A^{REP}(S)$ for some representation tree S. The arithmetic relation $\mathcal{Q}$,

(1) $n \mathcal{Q} m :\iff n \in S \ \& \ m \in S \ \& \ m \subset n$,

is well-founded since S is well-founded, and therefore we have by (BI) that

(2) $\forall x[\forall y(y \mathcal{Q} x \rightarrow B(y)) \rightarrow B(x)] \rightarrow \forall x B(x)$

for all L_2 formulas $B(x)$. By choosing $x \in S \rightarrow A^{REP}(S_x)$ as $B(x)$, this implies the existence of a $t \in S$ such that

(3) $\neg A^{REP}(S_t)$ & $\forall s(s \mathcal{Q} t \rightarrow A^{REP}(S_s))$.

Then S_t is a representation tree with the property

(4) $\neg A^{REP}(S_t)$ & $\forall X(X \tilde{\in} S_t \rightarrow A^{REP}(X))$.

So $(IND_\in)$ is proved. ./.

Theorem 7.2 and Corollary 7.2 settle the relations between subsystems of analysis with Π^1_1 comprehension and subsystems of KPl. The ordinal analysis of the theories considered in this section will not be given here. A detailed discussion of this aspect and the connections to theories for iterated inductive definitions can be found in Buchholz-Feferman-Pohlers-Sieg [4] and in Buchholz [3], Feferman [10], Pohlers [43] and Sieg [50]. The ordinal analysis of KPl^r, $KPl^r + (IND_N)$ and KPl is carried through in Jäger [29]. For completeness we list the proof-theoretic ordinals.

Theorem 7.4.

(a) $|KPl^r| = |(\Pi^1_1\text{-}CA)_0| = \overline{\theta}\Omega_\omega 0$.

(b) $|KPl^r + (IND_N)| = |(\Pi^1_1\text{-}CA)| = \overline{\theta}(\Omega_\omega \cdot \varepsilon_0)0$.

(c) $|KPl| = |(\Pi^1_1\text{-}CA) + (BI)| = \overline{\theta}\varepsilon_{\Omega_\omega+1}0$.

In the rest of this section we study the class of sets provably coded by representation trees. Within KPl^r we define a predicate Rep(d,a,T) which expresses that the representation tree T in the admissible set d codes the set $a \in d$.

Lemma 7.6. There exists a Δ_0 formula $Rep_0(d,a,T)$ such that KPl^r proves for all admissible sets d, all sets $a \in d$ and all special trees $T \in d$:

$$Rep_0(d,a,T) \leftrightarrow (a \in N \;\&\; T = [a]) \vee (a \notin N \;\&\; (\forall x \in a)(\exists y \in N)[<2y> \in T \;\&\; Rep_0(d,x,T_{<2y>})] \;\&\; (\forall y \in N)[<2y> \in T \rightarrow (\exists x \in a)Rep_0(d,x,T_{<2y>})]) .$$

The predicate Rep_0 is defined by using the second recursion theorem in each admissible set d according to Lemma 3.6 and Lemma 4.1. The crucial definition, however, is the following:

Definition. Rep(d,a,T) :⟺

$$Ad(d) \;\&\; a \in d \;\&\; T \in d \;\&\; REP(T) \;\&\; Rep_0(d,a,T) .$$

We call a set a hereditarily countable in d if there exists a transitive $x \supset a$ which is countable in d. By (Lim) a is hereditarily countable iff it is hereditarily countable in some d.

Definition.

(a) $Count(f,a) :\Longleftrightarrow Fun_{1-1}(f) \;\&\; dom(f) = a \;\&\; rng(f) \subset N$;

(b) $HC(d,a) :\Longleftrightarrow (\exists x \in d)(\exists f \in d)[Tran(x) \;\&\; a \subset x \;\&\; Count(f,x)]$;

(c) $HC(a) :\Longleftrightarrow \exists x \exists f(a \subset x \;\&\; Tran(x) \;\&\; Count(f,x))$.

<u>Lemma 7.7</u>. KPl^r proves:

(a) $Ad(d)$ & $a \in d$ & $HC(d,a) \rightarrow \exists X Rep(d,a,X)$;

(b) $Rep(d,a,S)$ & $Rep(d,b,T) \rightarrow (a = b \leftrightarrow S \approx T)$;

(c) $Rep(d,a,S)$ & $Rep(d,b,T) \rightarrow (a \in b \leftrightarrow S \mathrel{\tilde{\in}} T)$;

(d) $Rep(d,a,T) \rightarrow HC(d,a)$;

(e) $Ad(d)$ & $T \in d$ & $REP(T) \rightarrow \exists x Rep(d,x,T)$.

Proof. (a) is proved similar to Lemma 1.10; (b) - (e) follow from the definition of Rep. ./.

<u>Theorem 7.5</u>. In KPl^r we can show that the elements of HC are exactly the sets coded by representation trees; i.e.

$$KPl^r \vdash HC(a) \leftrightarrow \exists d(\exists T \in d) Rep(d,a,T) .$$

This theorem follows from Lemma 7.7 and has an interesting consequence for the axiom of countability (C). Since HC satisfies (C), we conclude from Theorem 7.2, Corollary 7.2, Lemma 7.7 and Theorem 7.5 that the addition of (C) does not increase the proof-theoretic strength of KPl^r, $KPl^r + (IND_N)$ and KPl.

<u>Corollary 7.3</u>. We have for every sentence A of L_{Ad}:

(a) $KPl^r + (C) \vdash A \implies KPl^r \vdash A^{HC}$;

(b) $KPl^r + (IND_N) + (C) \vdash A \implies KPl^r + (IND_N) \vdash A^{HC}$;

(c) $KPl + (C) \vdash A \implies KPl \vdash A^{HC}$.

§8. THEORIES FOR RECURSIVELY INACCESSIBLE UNIVERSES.

In this section we study the relationships between subsystems of analysis with Δ^1_2 comprehension and set theories which formalize recursively inaccessible universes. The following formal systems will be considered:

1. KPi^r, $KPi^r + (IND_N)$, KPi. The theory KPi^r is $KPu^r + (Lim)$, KPi is $KPu + (Lim)$. Hence KPi^r and KPi are $KPu^r + KPl^r$ and $KPu + KPl$, respectively.
2. $KP\beta^r$, $KP\beta^r + (IND_N)$, $KP\beta$. The theory $KP\beta^r$ is $KPu^r + (\text{Axiom } \beta)$, $KP\beta$ is $KPu + (\text{Axiom } \beta)$.
3. $(\Delta^1_2\text{-CA})_0$, $(\Delta^1_2\text{-CA})$, $(\Delta^1_2\text{-CA}) + (BI)$. The theory $(\Delta^1_2\text{-CA})_0$ consists of EA_0 plus Δ^1_2 comprehension. $(\Delta^1_2\text{-CA})$ is EA plus Δ^1_2 comprehension.

Before we compare these theories proof-theoretically, we consider their intended models. Recall that α is called recursively inaccessible if α is an admissible limit of admissibles. We have:

$$(*) \quad \begin{array}{l} \alpha \text{ recursively inaccessible} \implies L_\alpha \models KPi \implies \\ L_\alpha \models KP\beta \implies L_\alpha \cap Pow(N) \models (\Delta^1_2\text{-CA}) + (BI) . \end{array}$$

The statement (*) follows from Theorem 8.1 and Theorem 8.2 below. The converse implications are not correct in general.

Example 1. Let κ be $\aleph^L_1$. It is clear that L_{κ^+} is not a model of KPi. On the other hand, L_{κ^+} is a model of the axiom β.

Example 2. Let α be the smallest admissible ordinal such that

$$L_\alpha \models \exists\xi(\xi \text{ uncountable cardinal}) .$$

Then α is of the form κ^+, where $\kappa = \aleph_1^{L_\alpha}$. L_α is a model of full analysis A_2 and therefore of $(\Delta^1_2\text{-CA}) + (\mathrm{BI})$. L_α is not a model of the axiom β.

The first example goes back to Platek [42], the second one is due to Simpson [private communication]. However, if L_α is a locally countable admissible set, then α is recursively inaccessible iff $L_\alpha \cap \mathrm{Pow}(N)$ is a model of $(\Delta^1_2\text{-CA}) + (\mathrm{BI})$. We omit the proof of this assertion.

Theorem 8.1.

(a) $\mathrm{KP}\beta^r$ is contained in KPi^r.

(b) $\mathrm{KP}\beta^r + (\mathrm{IND}_N)$ is contained in $\mathrm{KPi}^r + (\mathrm{IND}_N)$.

(c) $\mathrm{KP}\beta$ is contained in KPi.

These inclusions are obvious from Theorem 4.6 where we proved the axiom β in KPl^0. The axiom β is also crucial for embedding Δ^1_2 comprehension into $\mathrm{KP}\beta^r$.

Lemma 8.1. For every Σ^1_2 formula $A[x,Y]$ of L_2 there exists a Σ_1 formula $B[x,y]$ of L_* such that $\mathrm{KP}\beta^r$ proves

$$x \in N \ \&\ y \subset N \ \rightarrow\ (A[x,y] \leftrightarrow B[x,y]) .$$

Proof. This lemma boils down to showing that each Π^1_1 formula of L_2 is provably equivalent to a Σ_1 formula of L_*. So assume that $C[x,Y]$ is Π^1_1. By the Π^1_1 normal form theorem there exists a binary relation $\mathcal{Q}_{x,Y}$ with parameters x and Y, defined by the arithmetic formula $\mathcal{Q}[u,v,x,Y]$, such that

(1) $KP\beta^r \vdash x \in N \ \& \ y < N \ \rightarrow \ (C[x,y] \leftrightarrow Wf(N,\mathcal{Q}_{x,y}))$.

By the axiom β and the well-foundedness of the $\in$-relation this implies

(2) $KP\beta^r \vdash x \in N \ \& \ y < N \ \rightarrow \ (C[x,y] \leftrightarrow D[x,y])$,

where $D[x,y]$ is the Σ_1 formula stating the existence of the collapsing function for $(N,\mathcal{Q}_{x,y})$. ./.

Theorem 8.2.
(a) $(\Delta^1_2\text{-CA})_0$ is contained in $KP\beta^r$.
(b) $(\Delta^1_2\text{-CA})$ is contained in $KP\beta^r + (IND_N)$.
(c) $(\Delta^1_2\text{-CA}) + (BI)$ is contained in $KP\beta$.

Proof. By the previous lemma we reduce each Σ^1_2 (Π^1_2) formula of L_2 to a Σ (Π) formula of L_* and prove Δ^1_2 comprehension from Δ separation. The remaining cases of this embedding theorem are treated as in §7 (cf. Lemma 7.1, Theorem 7.2). ./.

For the reduction of KPi to $(\Delta^1_2\text{-CA}) + (BI)$ we come back to representation trees and the translation A^{REP} of every L_{Ad} formula A. In order to handle Δ_0 collection, we need the following lemma.

Lemma 8.2.
(a) $(\Delta^1_2\text{-CA})_0$ proves the Σ^1_2 axiom of choice.
(b) For every Π^1_0-Π^1_1 formula $A[x,Y]$ there exists a Σ^1_2 formula $C[x,Y]$ such that

$$(\Delta^1_2\text{-CA})_0 \vdash A[x,Y] \leftrightarrow C[x,Y].$$

Proof. (a) is a standard consequence of the provability of Π^1_1 uniformization in $(\Pi^1_1\text{-CA})_0$ and hence in $(\Delta^1_2\text{-CA})_0$; cf. Lemma 7.3 and Feferman [18]. (b) follows from (a) by induction on the length of the formula $A[x,Y]$. ./.

<u>Theorem 8.3</u>. We have for every sentence A of L_{Ad}:

(a) $KPi^r \vdash A \implies (\Delta^1_2\text{-CA})_0 \vdash A^{REP}$;

(b) $KPi^r + (IND_N) \vdash A \implies (\Delta^1_2\text{-CA}) \vdash A^{REP}$;

(c) $KPi \vdash A \implies (\Delta^1_2\text{-CA}) + (BI) \vdash A^{REP}$.

Proof. In view of Corollary 7.2 we have only to take care of Δ_0 collection. So suppose that $A[\underline{a},x,y]$ is a Δ_0 formula of L_{Ad} and $\underline{S},T$ are representation trees. We have to show:

$$(1)\quad (\forall X \,\tilde{\in}\, T)(\exists Y \in REP)A^{REP}[\underline{S},X,Y] \rightarrow$$
$$(\exists Z \in REP)(\forall X \,\tilde{\in}\, T)(\exists Y \,\tilde{\in}\, Z)A^{REP}[\underline{S},X,Y].$$

The premise of this implication translates into

$$(2)\quad \forall x \exists Y(\langle 2x\rangle \in T \rightarrow REP(Y) \;\&\; A^{REP}[\underline{S},T_{\langle 2x\rangle},Y]),$$

where the formula within the round brackets is $\Pi^1_0\text{-}\Pi^1_1$ and therefore provably equivalent to a Σ^1_2 formula by the lemma above. With the Σ^1_2 axiom of choice we find a set W such that

$$(3)\quad \forall x(\langle 2x\rangle \in T \rightarrow REP((W)_x) \;\&\; A^{REP}[\underline{S},T_{\langle 2x\rangle},(W)_x]).$$

By $\Pi^1_0\text{-}\Pi^1_1$ comprehension we define the tree Z,

$$(4)\quad Z := \{\langle\rangle\} \cup \{\langle 2x\rangle * t : \langle 2x\rangle \in T \;\&\; t \in (W)_x\},$$

which satisfies

(5) $(\forall X \tilde{\in} T)(\exists Y \tilde{\in} Z)A^{REP}[\underline{S},X,Y]$.

By Lemma 5.5 there exists a copy of Z in REP. ./.

In the ordinal analysis of the systems considered in this section, we have to distinguish between theories with full induction principles and theories with restricted induction. The latter correspond to iterations of the hyperjump less than ω or ε_0 times, in the first case we have a virtual iteration of the hyperjump up to the first recursively inaccessible ordinal. Pohlers [44] and Jäger [31] discuss the iteration of hyperjump-like operations in proof theory and emphasize the importance of iterations along virtual well-orderings in the proof-theoretic analysis of theories like $(\Delta^1_2\text{-CA})+(\mathrm{BI})$ and KPi. The exact proof-theoretic bounds for $(\Delta^1_2\text{-CA})_0$ and $(\Delta^1_2\text{-CA})$ follow from the work of Buchholz, Feferman, Pohlers and Sieg, put together in [4]. Cut elimination and ordinal analysis for KPi^r and $\mathrm{KPi}^r+(\mathrm{IND}_N)$ are carried through in Jäger [29].

Theorem 8.4.

(a) $|\mathrm{KPi}^r| = |\mathrm{KP}\beta^r| = |(\Delta^1_2\text{-CA})_0| = \overline{\theta}\Omega_\omega 0$.

(b) $|\mathrm{KPi}^r+(\mathrm{IND}_N)| = |\mathrm{KP}\beta^r+(\mathrm{IND}_N)| = |(\Delta^1_2\text{-CA})| = \overline{\theta}\Omega_{\varepsilon_0} 0$.

The strongest result obtained so far in the field of proof-theoretic ordinals is the calculation of the ordinal of KPi and equivalent theories; see Jäger-Pohlers [35]. An interesting application of the ordinal analysis for KPi and $(\Delta^1_2\text{-CA})+(\mathrm{BI})$ is the reduction of these classical systems to

Feferman's constructive theory T_0 of functions and classes. T_0 is introduced in [12,15], the reduction of KPi to T_0 is presented in Jäger [32].

Theorem 8.5.
$|KPi| = |KP\beta| = |(\Delta^1_2\text{-}CA) + (BI)| = |T_0| = \overline{\theta}0(\overline{\theta}1\varepsilon_{i_0+1}0)0$.

The last remark refers to the axiom of countability (C) and to the class HC. The axiom of countability appears natural in our context, especially since

$$L_\alpha \models KPi \iff L_\alpha \models KP\beta \iff L_\alpha \cap Pow(N) \models (\Delta^1_2\text{-}CA)$$

for all admissibles L_α that satisfy (C). From a proof-theoretic standpoint it is therefore satisfactory that adding (C) as a further axiom does not increase the strength of these theories. As in section 7 we obtain:

Theorem 8.6. We have for every sentence A of $\mathcal{L}_{Ad}$:

(a) $KPi^r + (C) \vdash A \implies KPi^r \vdash A^{HC}$;

(b) $KPi^r + (IND_N) + (C) \vdash A \implies KPi^r + (IND_N) \vdash A^{HC}$;

(c) $KPi + (C) \vdash A \implies KPi \vdash A^{HC}$.

APPENDIX. RAMIFIED SET THEORY.

Now we go back to section 2 and add the proofs of Theorem 2.2 and Theorem 2.4 concerning the admissible extension Th^e of a theory Th. Theorem 2.1 and Theorem 2.3 are proved analogously.

The strategy for the proof of Theorem 2.4 is the following: First we construct a system RS of ramified set theory over the ground language L_*. Then we show that each sentence of L_* which is provable in Th^e with length α can be proved in RS by employing 3^α many levels of the ramified system. Cut elimination removes the ramified part but increases the length of the derivation. For the proof of Theorem 2.2 we use finite subsystems RS^k, $k < \omega$, in order to ensure finite cut ranks and finite derivation trees.

A.1. The system RS.

Throughout this section we assume that Th is a theory in L_* which contains ES^o. The more general case that Th is formulated in the extension $L_*(e_1,\dots,e_n)$ of L_* can be treated literally in the same way, but we dispense with the additional set constants $e_1,\dots,e_n$ for notational simplicity.

The vocabulary of RS consists of the vocabulary of L_*, a constant M_α for every ordinal α and some auxiliary symbols. The terms of RS are defined inductively from the terms of L_*.

Inductive definition of the terms of RS.

1. Every term of L_* is a term of level 0.
2. Every M_α is a term of level $\alpha+1$.
3. If $\underline{a}$ are terms of level $\leq \alpha$ and $A[u,\underline{v}]$ is an L_* formula, then $\{x \in M_\alpha : A^\alpha[x,\underline{a}]\}$ is a term of level $\alpha+1$.

Here A^α results from A by replacing each unrestricted quantifier $\forall x(\ldots)$ and $\exists x(\ldots)$ in A by $(\forall x \in M_\alpha)(\ldots)$ and $(\exists x \in M_\alpha)(\ldots)$, respectively. If we write $\{x \in M_\alpha : B(x)\}$, then we always assume that $B(x)$ has the form $A^\alpha[x,\underline{a}]$ for some L_* formula A and terms $\underline{a}$ of level $\leq \alpha$. The level of a term a is denoted by $\mathrm{lev}(a)$.

Definition. The formulas of RS are all expressions $A[\underline{a}]$ where A is a Δ_0 formula of L_* and $\underline{a}$ are terms of RS.

Notational conventions.

1. u, v, w range over the free variables of L_*;
   $a^\alpha, b^\alpha, c^\alpha, d^\alpha$ range over the RS terms of level α;
   a, b, c, d range over arbitrary RS terms.

2. $$a \dot{\in} b \;:\Longleftrightarrow\; \begin{cases} a \in b, & \text{if } \mathrm{lev}(b) = 0; \\ 0 =_N 0, & \text{if } b \text{ is } M_\beta; \\ B(a), & \text{if } b \text{ is } \{x \in M_\beta : B(x)\}. \end{cases}$$

3. If $\underline{a} = a_1,\ldots,a_n$ and $\underline{b} = b_1,\ldots,b_n$, then
   $\underline{a} = \underline{b} \;:\Longleftrightarrow\; (a_1 = b_1 \,\&\ldots\&\, a_n = b_n)$;
   $\underline{a} \neq \underline{b} \;:\Longleftrightarrow\; \neg\, \underline{a} = \underline{b}$.

4. $\alpha \# \beta$:= natural sum of α and β.

5. $\alpha \mathrel{\dot{<}} \beta \; :\Longleftrightarrow \begin{cases} \alpha = 0 , & \text{if } \beta = 0; \\ \alpha < \beta , & \text{if } \beta \neq 0. \end{cases}$

6. $\alpha \dot{-} 1 \; := \begin{cases} \beta , & \text{if } \alpha = \beta+1; \\ \alpha , & \text{otherwise;} \end{cases}$

$\alpha \dot{-} (n+1) \; := \; (\alpha \dot{-} n) \dot{-} 1 .$

Inductive definition of the rank rn(A) of an RS formula A.

1. If a and b are terms of levels α and β, then

$rn(a \in b) := rn(a \notin b) := \max(\omega\cdot(3\alpha \dot{-} 1) , \omega\cdot(3\beta \dot{-} 2) + 2) .$

2. If $\underline{a} = a_1,\dots,a_n$ are terms of levels $\alpha_1,\dots,\alpha_n$ and J is an n-ary relation symbol different from $\in$, then

$rn(J(\underline{a})) := rn(\neg J(\underline{a})) := \omega\cdot\max(3\alpha_1 \dot{-} 2 , \dots , 3\alpha_n \dot{-} 2) .$

3. $rn(A \,\&\, B) := rn(A \vee B) := \max(rn(A) , rn(B)) + 2 .$

4. $rn((\forall x \in M_\alpha)A(x)) := rn((\exists x \in M_\alpha)A(x)) :=$

$\max(\omega\cdot 3\alpha , rn(A(u)) + 6) .$

5. If a is not of the form M_β and $lev(a) = \alpha$, then

$rn((\forall x \in a)A(x)) := rn((\exists x \in a)A(x)) :=$

$\max(\omega\cdot(3\alpha \dot{-} 2) , rn(A(u)) + 6) .$

Lemma A.1.

(a) $rn(\neg A) = rn(A)$.

(b) If $\underline{a}$ are terms of level $\leq \alpha$ and $A[\underline{u}]$ is an L_* formula, then $rn(A^\alpha[\underline{a}]) < \omega\cdot 3\alpha + \omega$.

(c) $rn((\forall x \in a^\alpha)(x \in b^\beta)) \leq \max(\omega\cdot(3\alpha \dot{-} 2) , \omega\cdot(3\beta \dot{-} 2) + 8) .$

(d) $rn(a^\alpha = b^\beta) < \max(\omega\cdot(3\alpha \dot{-} 2) + \omega , \omega\cdot(3\beta \dot{-} 2) + \omega) .$

(e) $rn(A_i) + 1 < rn(A_0 \,\&\, A_1)$ for $i = 0$ or $i = 1$.

(f) $rn((\exists x \in b)(a = x)) + 1 < rn(a \in b)$, if $lev(a) + lev(b) \geq 1$.

(g) $\mathrm{rn}(A(a)) + 1 < \max(\omega\cdot 3\beta, \mathrm{rn}(A(u)) + 2)$, if $\mathrm{lev}(a) \leq \beta$.

(h) $\mathrm{rn}(a \dot{\in} b \ \& \ A(a)) + 1 < \mathrm{rn}((\exists x \in b)A(x))$, if $\mathrm{lev}(a) \dot{<} \mathrm{lev}(b)$.

The proof of this lemma is straightforward and left to the reader. Now we turn to provability in RS which is defined for finite sets of RS formulas.

Basic rules of RS.

(A1) $\Gamma, \neg A, A$, if A is an atomic formula of L_*;

(A2) Γ, A , if A is in TRUE;

(A3) $\Gamma, n \in N$, if n is a number constant;

(A4) $\Gamma, a^0 \notin N$, if a^0 is not a number constant;

(A5) $\Gamma, S(a)$, if $\mathrm{lev}(a) > 0$;

(A6) $\Gamma, \neg J(a_1,\ldots,a_n)$, if J is a relation symbol of L_1 and $\mathrm{lev}(a_i) > 0$ for some i.

Normal rules of RS.

(&), (v) as usual;

$$(\forall)\ \frac{\Gamma, a \dot{\in} b \rightarrow A(a) \quad \text{for all } a \text{ such that } \mathrm{lev}(a) \dot{<} \mathrm{lev}(b)}{\Gamma, (\forall x \in b)A(x)}$$

$$(\exists)\ \frac{\Gamma, a \dot{\in} b \ \& \ A(a) \quad \text{for some } a \text{ such that } \mathrm{lev}(a) \dot{<} \mathrm{lev}(b)}{\Gamma, (\exists x \in b)A(x)}$$

$$(\in)\ \frac{\Gamma\,,\ (\exists x \in b)(a = x)}{\Gamma\,,\ a \in b}\ ,\ \text{if } lev(a) + lev(b) > 0$$

$$(\notin)\ \frac{\Gamma\,,\ (\forall x \in b)(a \neq x)}{\Gamma\,,\ a \notin b}\ ,\ \text{if } lev(a) + lev(b) > 0$$

Cut rule of RS.

$$(cut)\ \frac{\Gamma\,,\ A \qquad \Gamma\,,\ \neg A}{\Gamma}$$

The rank of this cut is the rank $rn(A) = rn(\neg A)$ of its cut formulas.

Inductive definition of RS $\vdash^{\alpha}_{\beta} \Gamma$.

1. If Γ is a basic rule of RS, then we have RS $\vdash^{\alpha}_{\beta} \Gamma$ for all ordinals α and β.
2. If RS $\vdash^{\alpha_i}_{\beta} \Gamma_i$ and $\alpha_i < \alpha$ for every premise Γ_i of a normal rule or a cut of rank $< \beta$, then we have RS $\vdash^{\alpha}_{\beta} \Gamma$ for the conclusion Γ of that inference.

RS $\vdash^{\alpha} \Gamma$ abbreviates RS $\vdash^{\alpha}_{0} \Gamma$; RS $\vdash^{<\alpha} \Gamma$ means RS $\vdash^{\beta} \Gamma$ for some $\beta < \alpha$. By Lemma A.1 and standard techniques of predicative proof theory (cf. e.g. [47,54]) we obtain cut elimination for RS.

Theorem A.1 (Cut elimination).

(a) RS $\vdash^{\alpha}_{\beta+1} \Gamma \implies$ RS $\vdash^{2^{\alpha}}_{\beta} \Gamma$;

(b) RS $\vdash^{\alpha}_{\beta} \Gamma \implies$ RS $\vdash^{\phi\beta\alpha} \Gamma$.

Definition. Assumptions: $A[\underline{u}]$ is a formula of the language $L_*(M)$ of Th^e; $\Gamma[\underline{u}]$ is the set $\{B_1[\underline{u}],\dots,B_n[\underline{u}]\}$ of $L_*(M)$ formulas; $\alpha \leq \beta$; $\underline{a} = a_1,\dots,a_m$ are RS terms such that $lev(a_i) \leq \alpha$ for $i = 1,\dots,m$.

(a) $A^{(\alpha,\beta)}[\underline{a}]$ is the RS formula obtained from $A[\underline{u}]$ by

(i) replacing the set constant M by M_0 and the free variables $\underline{u}$ by $\underline{a}$;

(ii) replacing each unrestricted quantifier $\forall x(\dots)$ by $(\forall x \in M_\alpha)(\dots)$;

(iii) replacing each unrestricted quantifier $\exists x(\dots)$ by $(\exists x \in M_\beta)(\dots)$.

(b) $A^\alpha[\underline{a}]$ is $A^{(\alpha,\alpha)}[\underline{a}]$.

(c) $\Gamma^{(\alpha,\beta)}[\underline{a}]$ is the set $\{B_1^{(\alpha,\beta)}[\underline{a}],\dots,B_n^{(\alpha,\beta)}[\underline{a}]\}$.

(d) $\Gamma^\alpha[\underline{a}]$ is the set $\Gamma^{(\alpha,\alpha)}[\underline{a}]$.

Remark. The previous definition must not be confused with the definition of $A^{(m,n)}$ and $\Gamma^{(m,n)}$ in section 4.3.

Theorem A.2. If Γ is a finite set of L_* sentences, then

$$RS \vdash^\alpha \Gamma^0 \implies Z_*^\infty \vdash^\alpha \Gamma .$$

Theorem A.2 is easily proved by induction on α. In combination with the cut elimination theorem it says that the ramified part of RS is not essential as far as provability of L_* formulas is considered. On the other hand, we need the ramified part of RS for the (partial) embedding of Th^e into RS.

Lemma A.2 (Logical completeness). If $\alpha = rn(A)$, then

$$RS \vdash^{\alpha} \neg A , A \ .$$

The proof of this lemma is by straightforward induction on α. It is not possible to prove in RS all equality axioms $a = b \to (A(a) \to A(b))$, since RS does not comprise equality at level 0. The rules of RS, however, are adapted for lifting the equality axioms from level 0 to arbitrary levels.

Definition. Let OE be the (finite) set of all formulas

$\neg (\forall x \in M_0)(x = x)$,

$\neg (\forall x \in M_0)(S(x) \leftrightarrow x \notin N)$,

$\neg (\forall x \in M_0)(\forall z \in M_0)(x \in z \to S(z))$,

$\neg (\forall \underline{x} \in M_0)(I(\underline{x}) \to \underline{x} \in N)$,

$\neg (\forall \underline{x} \in M_0)(\forall \underline{z} \in M_0)(\underline{x} = \underline{z} \ \& \ J(\underline{x}) \to J(\underline{z}))$

for any relation symbol I of L_1 and J of L_*.

Lemma A.3 (Equality). Suppose that $\sigma = rn(A(\underline{a})) \# rn(A(\underline{b}))$ and $\alpha = \omega \cdot (\sigma+1)$. Then we have:

$$RS \vdash^{\alpha} OE , \underline{a} \neq \underline{b} , \neg A(\underline{a}) , A(\underline{b}) \ .$$

Proof by induction on $\max(rn(A(\underline{a})), rn(A(\underline{b})))$. If $\underline{a}$, $\underline{b}$ are terms of level 0 and $A(\underline{u})$ is an atomic formula, then the set OE takes effect. Otherwise the assertion follows from the I.H., with some additional considerations if $A(u)$ is of the form $(\forall x \in u_i)B(x,\underline{u})$ or $(\exists x \in u_i)B(x,\underline{u})$ and u_i is from the list $\underline{u}$. By symmetry we can confine ourselves to the second case. Depending on the form of a and b, we have to distinguish 9 cases. We carry through the argument for the most complicated one. Let c and d be arbitrary terms such that $lev(c) \dot{<} lev(a_i)$ and $lev(d) \dot{<} lev(b_i)$. Define

(1) $\sigma_c := rn(c \dot{\in} a_i \ \& \ B(c,\underline{a})) \# rn(A(\underline{b}))$, $\alpha_c := \omega \cdot (\sigma_c + 1)$

and assume that $a_i = \{x \in M_\gamma : C(x)\}$ and $b_i = \{x \in M_\delta : D(x)\}$. By the I.H., Lemma A.1 and Lemma A.2 we have:

(2) $RS \vdash^{<\alpha_c} OE , \underline{a} \neq \underline{b} , c \neq d , \neg B(c,\underline{a}) , B(d,\underline{b})$;

(3) $RS \vdash^{<\alpha_c} \neg D(d) , D(d)$;

(4) $RS \vdash^{<\alpha} C(c) , \neg C(c)$.

From (2) and (3) we obtain by (&), (∃) and (v)

(5) $RS \vdash^{<\alpha_c} OE , \underline{a} \neq \underline{b} , \neg D(d) \vee c \neq d , \neg B(c,\underline{a}) , A(\underline{b})$

which yields by (∀) and (∉)

(6) $RS \vdash^{<\alpha} OE , \underline{a} \neq \underline{b} , c \notin b_i , \neg B(c,\underline{a}) , A(\underline{b})$.

From (4) and (6) we conclude by (&), (∃) and (v) in the following order:

(7) $RS \vdash^{<\alpha} OE , \underline{a} \neq \underline{b} , C(c) \ \& \ c \notin b_i , \neg C(c) , \neg B(c,\underline{a}) , A(\underline{b})$;

(8) $RS \vdash^{<\alpha} OE , \underline{a} \neq \underline{b} , (\exists x \in a_i)(x \notin b_i) , \neg C(c) , \neg B(c,\underline{a}) , A(\underline{b})$;

(9) $RS \vdash^{<\alpha} OE , \underline{a} \neq \underline{b} , c \dot{\in} a_i \rightarrow \neg B(c,\underline{a}) , A(\underline{b})$.

An application of (∀) implies the conclusion. ./.

Lemma A.4. If a is a term of level α, $\beta = \omega\cdot 3\alpha + \omega$, $\gamma \geqq \alpha$ and $\delta = \omega\cdot 3\gamma + \omega + \omega$, then we have:

(a) $RS \vdash^{<\beta} OE\ ,\ a = a$;

(b) $RS \vdash^{<\beta} OE\ ,\ a \in M_\gamma$;

(c) $RS \vdash^{<\delta} OE\ ,\ (\forall x \in M_\gamma)(\forall y \in x)(y \in M_\gamma)$;

(d) $RS \vdash^{<\omega} OE\ ,\ a \in b \rightarrow S(b)$.

Lemma A.5. If $\sigma = rn((\exists x \in M_\beta)(x \in a\ \&\ A(x))) \# rn((\exists x \in a)A(x))$, then

$$RS \vdash^{\omega\cdot(\sigma+1)} OE\ ,\ (\forall x \in M_\beta)(x \notin a \vee \neg A(x))\ ,\ (\exists x \in a)A(x)\ .$$

Lemma A.6. If $A[u,\underline{v}]$ is a Δ_0 formula of L_* and $a,\underline{b}$ are terms of level $\leqq \alpha$, then we have for all $\beta > \alpha$ and $\gamma = \omega\cdot 3\alpha + \omega + \omega$:

$$RS \vdash^{<\gamma} (\exists z \in M_\beta)(z = \{x \in a : A[x,\underline{b}]\})\ .$$

The proof of Lemma A.4 is obvious; Lemma A.5 follows from Lemma A.2 and Lemma A.3; for the proof of Lemma A.6 we choose $c^{\alpha+1} := \{x \in M_\alpha : x \in a\ \&\ A[x,\underline{b}]\}$ as witness for z.

Definition. A finite set Λ of $L_*(M)$ formulas is called suitable if each element of Λ has the form

$$\exists x_1 \ldots \exists x_n \neg A[x_1,\ldots,x_n,\underline{a}]\ ,$$

where $A[u_1,\ldots,u_n,\underline{v}]$ is a Kripke-Platek axiom of Th^e.

Lemma A.7. Assume that $\Lambda[\underline{u}]$ is a suitable set of $L_*(M)$ formulas. If

$$Z_*^\infty \vdash^\alpha \Lambda[\underline{u}]\ ,\ \Gamma[\underline{u}]\ ,$$

then we have, for all β, $\sigma \geqq \beta+3^\alpha$, $\rho = \omega\cdot 3\sigma + \omega$, $\gamma = \varepsilon_{\sigma\#\alpha}$ and all terms $\underline{a}$ of level $\leqq \beta$,

$$RS \vdash_\rho^\gamma OE\ ,\ \Gamma^{(\beta,\sigma)}[\underline{a}]\ .$$

The proof of this lemma proceeds by induction on α and uses Lemma A.4 - Lemma A.6. It is analogous to the proofs of Lemma 3.4 and Lemma 4.3 , and therefore we can omit details.

Lemma A.8. Suppose that $Th_\infty^e \vdash^\alpha \Gamma[\underline{u}]$. Then there exist universal closures $A_1,\ldots,A_n$ of axioms of the theory Th such that

$$RS \vdash_\rho^{\gamma+\omega} \{\neg A_1^0,\ldots,\neg A_n^0\}\ ,\ \Gamma^{(\beta,\sigma)}[\underline{a}]$$

for all β, $\sigma \geqq \beta+3^\alpha$, $\rho = \omega\cdot 3\sigma + \omega$, $\gamma = \varepsilon_{\sigma\#\alpha}$ and all terms $\underline{a}$ of level $\leqq \beta$.

Proof. Since $Th_\infty^e \vdash^\alpha \Gamma[\underline{u}]$, there exist finite sets Λ_0, Λ_1, Λ_2, Λ_3 of $L_*(M)$ formulas such that

(1) $Z_*^\infty \vdash^\alpha \Lambda_0\ ,\ \Lambda_1\ ,\ \Lambda_2\ ,\ \Lambda_3\ ,\ \Gamma[\underline{u}]$

and

(i) each element of Λ_0 is the negation of the universal closure of an ontological or equality axiom of ES;

(ii) each element of Λ_1 is the negation of the universal closure of an M-axiom of Th^e;

(iii) each element of Λ_2 is the negation of a Th-axiom of Th^e;

(iv) each element of Λ_3 is the negation of the universal closure of a Kripke-Platek axiom of Th^e.

By the previous lemma this implies

(2) $RS \vdash^{\gamma}_{\rho} OE , \Lambda_0^{(\beta,\sigma)} , \Lambda_1^{(\beta,\sigma)} , \Lambda_2^{(\beta,\sigma)} , \Gamma^{(\beta,\sigma)}[\underline{a}]$.

Observe that $\Lambda_2^{(\beta,\sigma)} = \Lambda_2^0$ and $RS \vdash^{<\gamma} OE , A$ for all formulas A in $\Lambda_0^{(\beta,\sigma)} \cup \Lambda_1^{(\beta,\sigma)}$. Therefore several cuts yield

(3) $RS \vdash^{\gamma+\omega}_{\rho} OE , \Lambda_2^0 , \Gamma^{(\beta,\sigma)}[\underline{a}]$.

This completes the proof. ./.

<u>Theorem A.3</u>. Let A be a sentence of L_* and α an ε-number. Then

$$Th^e_\infty \vdash^{<\alpha} A^M \implies Th_\infty \vdash^{<\phi\alpha 0} A .$$

Proof. Assume $Th^e_\infty \vdash^{<\alpha} A^M$. By Lemma A.8 there exist universal closures $B_1,\dots,B_n$ of axioms of Th and a $\beta < \alpha$ such that

(1) $RS \vdash^{\gamma+\omega}_{\rho} \{\neg B_1^0,\dots,\neg B_n^0\} , A^0$

for $\sigma = 3^\beta$, $\rho = \omega\cdot 3\sigma + \omega$ and $\gamma = \varepsilon_{\sigma \# \alpha}$. Since $\phi\rho(\gamma+\omega) < \phi\alpha 0$, cut elimination in RS yields

(2) $RS \vdash^{<\phi\alpha 0} \{\neg B_1^0,\dots,\neg B_n^0\} , A^0$.

The claim follows by Theorem A.2. ./.

A.2. The finite subsystems RS^k, $k < \omega$.

The way through the ramified system RS does not work for proving Theorem 2.2, even if we consider finite levels only. In this case we would have to deal with infinite cut ranks and infinite derivations and could not go back to the formal theory Th directly. In the following we therefore consider finite subsystems RS^k of RS for all $k < \omega$ where the term formation is severely restricted.

The vocabulary of RS^k is that of RS. We start with a fixed but arbitrary enumeration $(F_i : i < \omega)$ of all formulas of L_*. Depending on this enumeration, we define the k-terms and k-formulas for every $k < \omega$.

Inductive definition of the k-terms.

1. Every term of L_* is a k-term of level 0.
2. Every M_n with $n < \omega$ is a k-term of level n+1.
3. If $A[u,\underline{v}]$ is a formula of the list $(F_0,\dots,F_k)$ and $\underline{a}$ are k-terms of level $\leq n$, then $\{x \in M_n : A^n[x,\underline{a}]\}$ is a k-term of level n+1.

Ter_n^k denotes the class of all k-terms a such that $lev(a) \dot{<} n$.

Remark. Every k-term is a term of RS.

Definition. A formula A of RS is called a k-formula if all terms in A are k-terms.

<u>Inductive definition of the k-rank $rn^k(A)$ of a k-formula A.</u>

1. If a and b are k-terms of level 0, then
$rn^k(a \in b) := rn^k(a \notin b) := 0$.

2. If a and b are k-terms such that $lev(a) + lev(b) > 0$, then
$rn^k(a \in b) := rn^k(a \notin b) := rn^k((\exists x \in b)(a = x)) + 2$.

3. If $\underline{a}$ are k-terms and J is a relation symbol different from $\in$, then
$rn^k(J(\underline{a})) := rn^k(\neg J(\underline{a})) := 0$.

4. $rn^k(A \& B) := rn^k(A \vee B) := \max(rn^k(A), rn^k(B)) + 2$.

5. $rn^k((\forall x \in a^n)A(x)) := rn^k((\exists x \in a^n)A(x)) :=$
$\sup(rn^k(d \dot{\in} a^n \; \& \; A(d)) + 2 : d \in Ter^k_n)$.

In order to show that $rn^k(A) < \omega$ for every k-formula A, we define an equivalence relation ~ on the set of all k-terms.

<u>Inductive definition of $a \sim b$ for all k-terms a , b.</u>

1. $a \sim b$ if $lev(a) = lev(b) = 0$.

2. $M_n \sim M_n$.

3. If $\underline{a} = a_1, \ldots, a_m$ and $\underline{b} = b_1, \ldots, b_m$ are k-terms of level $\leq n$ and $a_1 \sim b_1, \ldots, a_m \sim b_m$, then $\{x \in M_n : A^n[x, \underline{a}]\} \sim \{y \in M_n : A^n[y, \underline{b}]\}$.

<u>Remark</u>. The equivalence classes of the terms in Ter^k_n obviously form a finite partition of Ter^k_n. $rn^k(A) < \omega$ for all k-formulas A therefore follows from the lemma below, which is clear by the previous definition.

<u>Lemma A.9</u>. If $A(\underline{u})$ is a k-formula, $a_1, \ldots, a_m \in Ter^k_n$ and $a_i \sim b_i$ for $i = 1, \ldots, m$, then $rn^k(A(a_1, \ldots, a_m)) = rn^k(A(b_1, \ldots, b_m))$.

Corollary. $rn^k(A) < \omega$ for every k-formula A.

RS^k is ramified set theory for k-formulas. It is similar to RS with all axioms and rules restricted to k-formulas.

Basic rules of RS^k.

(A1) $\Gamma , \neg A , A$, if $rn^k(A) = 0$;

(A2) Γ , A , if A is in TRUE;

(A3) $\Gamma , S(a)$, if $lev(a) > 0$;

(A4) $\Gamma , \neg J(a_1, \ldots, a_n)$, if J is a relation symbol of L_1 and $lev(a_i) > 0$ for some i.

Normal rules of RS^k.

(&), (v) as usual;

$$(\forall)\ \frac{\Gamma , a \dot{\in} b \to A(a) \quad \text{for all } a \in Ter^k_{lev(b)}}{\Gamma , (\forall x \in b)A(x)}$$

$$(\exists)\ \frac{\Gamma , a \dot{\in} b \ \&\ A(a) \quad \text{for some } a \in Ter^k_{lev(b)}}{\Gamma , (\exists x \in b)A(x)}$$

$$(\in)\ \frac{\Gamma , (\exists x \in b)(a = x)}{\Gamma , a \in b} \quad , \text{ if } lev(a) + lev(b) > 0$$

$$(\notin)\ \frac{\Gamma , (\forall x \in b)(a \neq x)}{\Gamma , a \notin b} \quad , \text{ if } lev(a) + lev(b) > 0$$

Cut rule of RS^k.

$$(\mathrm{cut})\ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

The rank of this cut is the k-rank $rn^k(A) = rn^k(\neg A)$ of its cut formulas.

$RS^k \vdash^m_n \Gamma$ is defined according to $RS \vdash^\alpha_\beta \Gamma$. $RS^k \vdash \Gamma$ means $RS^k \vdash^m_n \Gamma$ for some $m, n < \omega$; $RS^k \vdash_0 \Gamma$ is written, if there exists an $m < \omega$ such that $RS^k \vdash^m_0 \Gamma$. Cut elimination for RS^k is obvious by the definition of the k-rank and the corollary above.

Theorem A.4 (Cut elimination).

(a) $RS^k \vdash^m_{n+1} \Gamma \implies RS^k \vdash^{2^m}_n \Gamma$;

(b) $RS^k \vdash \Gamma \implies RS^k \vdash_0 \Gamma$.

Theorem A.5. If Γ is a finite set of L_* formulas, then

$$RS^k \vdash^m_0 \Gamma^0 \implies Z_* \vdash \Gamma .$$

The proof of this lemma is by straightforward induction on m. The following Lemma A.10 - Lemma A.13 and Theorem A.6 are the finite versions of Lemma A.2 - Lemma A.8 and Theorem A.3; the proofs are literally the same.

Lemma A.10. For all k-formulas A, $B(u)$, all k-terms a, b, all natural numbers n and all $m \geq lev(a)$:

(a) $RS^k \vdash \neg A, A$;

(b) $RS^k \vdash 0E, a \nmid b, \neg B(a), B(b)$;

(c) $RS^k \vdash OE\ ,\ a = a\ \&\ (b \in a \to S(a))$;

(d) $RS^k \vdash OE\ ,\ a \in M_m\ \&\ Tran(M_m)$;

(e) $RS^k \vdash OE\ ,\ (\forall x \in M_n)(x \notin a \vee \neg B(x))\ ,\ (\exists x \in a)B(x)$.

<u>Lemma A.11</u>. Let $A[u,\underline{v}]$ be a Δ_0 formula of L_* such that the formula $u \in w\ \&\ A[u,\underline{v}]$ belongs to the list $(F_0,\ldots,F_k)$. Then we have for all k-terms $a,\underline{b}$ of level $\leq m$:

$$RS^k \vdash (\exists z \in M_{m+1})(z = \{x \in a : A[x,\underline{b}]\}) .$$

<u>Definition</u>.

(a) Let $A[u,\underline{v}]$ be a Δ_0 formula of L_*. The formula

$$u \in w\ \&\ A[u,\underline{v}]$$

is called the critical formula of every instance of $(\Delta_0\text{-Sep})$

$$\exists z(z = \{x \in a : A[x,\underline{b}]\})$$

for arbitrary $L_*(M)$ terms $a,\underline{b}$.

(b) For every suitable set Λ of $L_*(M)$ formulas we define $h(\Lambda)$ to be the least $k < \omega$ with the following property:
If $\exists\underline{x}\neg A[\underline{x},\underline{a}]$ is in Λ and $A[\underline{u},\underline{a}]$ is an instance of $(\Delta_0\text{-Sep})$, then its critical formula belongs to the list $(F_0,\ldots,F_k)$.

<u>Lemma A.12</u>. Assume that $\Lambda[\underline{u}]$ is a suitable set of $L_*(M)$ formulas and $\Gamma[\underline{u}]$ an arbitrary finite set of $L_*(M)$ formulas. If

$$Z_* \vdash_0^n \Lambda[\underline{u}]\ ,\ \Gamma[\underline{u}]\ ,$$

then we have, for all m, $s \geq m+3^n$, $k \geq h(\Lambda[\underline{u}])$ and all k-terms

$\underline{a}$ of level $\leq m$,

$$RS^k \vdash OE\ ,\ \Gamma^{(m,s)}[\underline{a}]\ .$$

Lemma A.13. If $Th^e \vdash_0^n \Gamma[\underline{u}]$, then there exist universal closures $A_1,\ldots,A_t$ of axioms of the theory Th and a number $k_0 < \omega$ such that

$$RS^k \vdash \{\neg A_1^0,\ldots,\neg A_t^0\}\ ,\ \Gamma^{(m,s)}[\underline{a}]$$

for all numbers m, $s \geq m+3^n$, $k > k_0$ and all k-terms $\underline{a}$ of level $\leq m$.

Theorem A.6. If A is a sentence of L_*, then

$$Th^e \vdash A^M \implies Th \vdash A\ .$$

Proof. By Lemma A.13 and Theorem A.5; c.f. proof of Theorem A.3. ./.

The next remarks refer to $Th^e + (I_N)$ and $Th^e + (I_N) + (I_\in)$. Theorem 2.2(b) is proved as Theorem A.6, with Lemma A.14 below used to reduce (I_N) in Th^e to (IND_N) in Th. This lemma also reduces $(I_\in)$ in Th^e to $(IND_\in)$ in Th since all terms in RS^k are of finite level.

Lemma A.14. For every k-formula A(u) there exists an L_* formula $A_*(u)$ such that

$$RS^k \vdash (\forall x \in M_0)(A(x) \leftrightarrow A_*^0(x))\ .$$

Proof. We define $A_*(u)$ by induction on $rn^k(A(u))$.

1. If J is an arbitrary relation symbol, $\underline{a}$ a sequence of k-terms of level 0 and A(u) the formula $J(\underline{a})$ or $\neg J(\underline{a})$, then $A_*(u) := A(u)$.

2. If J is a relation symbol different from $\in$ and $\underline{a}$ a sequence of k-terms not all of level 0, then

$$A_*(u) := \begin{cases} 0 =_N 1 \text{ , if } A(u) \text{ is } J(\underline{a}); \\ 0 =_N 0 \text{ , if } A(u) \text{ is } \neg J(\underline{a}). \end{cases}$$

3. If a and b are k-terms such that $lev(a) + lev(b) > 0$, then

$$A_*(u) := \begin{cases} (\exists x \in b)(a = x)_* \text{ , if } A(u) \text{ is } a \in b; \\ (\forall x \in b)(a \neq x)_* \text{ , if } A(u) \text{ is } a \notin b. \end{cases}$$

4. $$A_*(u) := \begin{cases} B_*(u) \;\&\; C_*(u) \text{ , if } A(u) \text{ is } B(u) \;\&\; C(u); \\ B_*(u) \vee C_*(u) \text{ , if } A(u) \text{ is } B(u) \vee C(u). \end{cases}$$

5. Suppose that A(u) is the formula $(\forall x \in b)B(x,u)$ and b a k-term of level m. We remember that Ter_m^k splits into finitely many equivalence classes. For every equivalence class K there exists an expression $t_K(\underline{v})$ such that every element of K can be written as $t_K(\underline{c})$ for some terms $\underline{c}$ of level 0. Then we define:

$$A_*(u) := \bigwedge_{\substack{K \text{ equiv.} \\ \text{class}}} (\forall \underline{x} \in M_0)(t_K(\underline{x}) \mathrel{\dot{\in}} b \rightarrow A(t_K(\underline{x}),u))_*$$

6. If A(u) is the formula $(\exists x \in b)B(x,u)$, we define $A_*(u)$ accordingly.

It is easy to see that the claim is satisfied in each case. ./.

Theorem A.7. If A is a sentence of L_*, then

(a) $Th^e + (I_N) \vdash A^M \implies Th + (IND_N) \vdash A$;

(b) $Th^e + (I_N) + (I_\in) \vdash A^M \implies Th + (IND_N) + (IND_\in) \vdash A$.

Proof. By Theorem A.6 and Lemma A.14. ./.

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