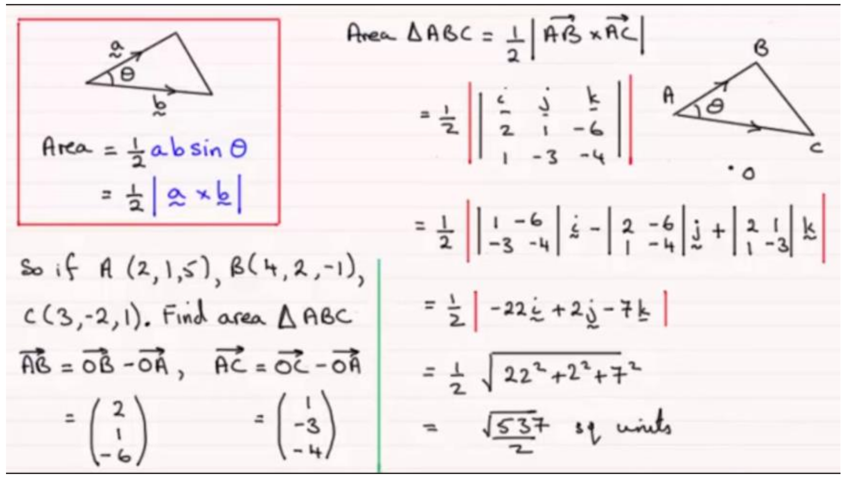
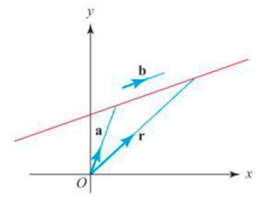
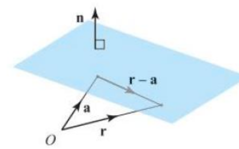
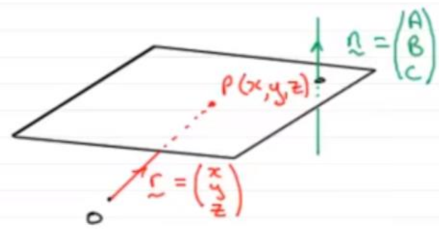
CF - Vectors

What are the 2 equations for the scalar product?	$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = a_1a_2 + b_1b_2 + c_1c_2$$ $$a.b = \lvert a \rvert \, \lvert b \rvert \cos \theta$$ *Where ϑ is the acute angle between the 2 vectors.*
When are two vectors perpendicular?	When **a**.**b** = 0
What is the vector/cross product?	• **a** x **b** = **n** where **n** is a vector normal to both **a** and **b**. $$\mathbf{a}\times\mathbf{b} = \left\lvert \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \right\rvert$$ • From the textbook: Key point The **vector product** $\mathbf{a}\times\mathbf{b}$ of vectors **a** and **b** is defined as $\mathbf{a}\times\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\hat{\boldsymbol{n}}$ where θ is the angle between the vectors **a** and **b** and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both **a** and **b** *In 2D, the cross product gives the area of a parallelogram made of the 2 vectors. This is since you can express the 2 vectors as a matrix and the determinant (change in area from a unit square) is its area. Whereas in 3D, the cross product of 2 vectors is another vector whose length is the area of the parallelogram.* *You can ensure you get the right direction by this:* https://www.youtube.com/watch?v=zGyfiOqiR4s.
When are 2 vectors parallel?	When **a** x **b** = **0**

How can you work out the area of a triangle using cross product?	 Since… $\mathbf{a} \times \boldsymbol{b} = \mathbf{ab} \sin \theta \hat{\mathbf{n}}$ and $\mathbf{a} \times \boldsymbol{b} = \mathbf{n}$ so… $\dfrac{\mathbf{a} \times \boldsymbol{b}}{\hat{\mathbf{n}}} = \lvert \mathbf{a} \times \boldsymbol{b} \rvert$
What is the vector equation of a plane?	$\mathbf{r} = \mathbf{a} + \alpha \mathbf{s} + \beta \mathbf{t}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \alpha \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \beta \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ *Where **a** is some point on the plane and where **s** and **t** are 2 non-parallel vectors used to define the plane.*
What is the vector product form of a line?	$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ Since…

	Two forms for the equation of a line in 3D are the Cartesian form $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}$ and the vector form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ A third possibility is the **vector product form**.  **Key point** A straight line passing through the point with position vector **a** and parallel to the vector **b** has equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ This follows from the fact that any vector on the line will be parallel to the vector **b** and the vector product of two parallel vectors is zero. You can expand the brackets of $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ to give $\mathbf{r}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}=\mathbf{0}$ which rearranges to $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ You should calculate the result of $\mathbf{a}\times\mathbf{b}$ using the vectors given.
What is the dot/scalar product equation of a plane?	$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}=p$$ Since… Consider a plane containing the point with position vector **a** and with a perpendicular vector **n**. Any vector on this plane will be perpendicular to the vector **n**. Using the definition of the scalar product, any point on the plane with position vector **r**, satisfies $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$. Expanding the bracket gives $\mathbf{r}\cdot\mathbf{n}-\mathbf{a}\cdot\mathbf{n}=0$, which you can rearrange to give $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$  **Key point** The scalar product equation of the plane perpendicular to the vector **n** and passing through the point with position vector **a** is $\mathbf{r}\cdot\mathbf{n}=p$, where $p=\mathbf{a}\cdot\mathbf{n}$
What is the Cartesian equation of a plane and how does it relate to another form?	 $\underset{\sim}{r}\cdot\underset{\sim}{n}=D$ $\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}A\\B\\C\end{pmatrix}=D$ $\therefore \boxed{Ax+By+Cz=D}$ Find the cartesian equation of a plane that passes through the point $(2,-4,-5)$ and is perpendicular to the vector $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

<table>
<tr>
<td>How do you show 2 planes are the same?</td>
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1. Show they have the normals which are parallel (so planes are parallel).
2. Check if the point of one lies on the other.

Parallel planes will have parallel normals but will not have a point in common.

Example 3

The planes Π_1 and Π_2 have equations $\mathbf{r}\cdot(\mathbf{i}-6\mathbf{j}+2\mathbf{k})=1$ and $\mathbf{r}=\mathbf{i}+\mathbf{j}+3\mathbf{k}+s(\mathbf{j}+3\mathbf{k})+t(2\mathbf{i}-\mathbf{k})$ respectively.

Show that Π_1 and Π_2 are, in fact, the same plane.

The normal to Π_2 is given by

$(\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-\mathbf{k})=-\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ — Calculate the normal vector to Π_2

$-\mathbf{i}+6\mathbf{j}-2\mathbf{k}=-(\mathbf{i}-6\mathbf{j}+2\mathbf{k})$ — 1 So their normal vectors are parallel.

The point $(1, 1, 3)$ lies on Π_2

$(\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}-6\mathbf{j}+2\mathbf{k})=1-6+6$ — Substitute into the equation of Π_1

$=1$ — 2 So the point (1, 1, 3) lies on Π_1

Therefore Π_1 and Π_2 represent the same plane. — 3 Since their normals are parallel and they have a point in common.

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<td>How can you work out the shortest distance from a point to a plane?</td>
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1. Form a line from that point (P) perpendicular to the plane.
2. Find where it intersects the plane (Q).
3. Calculate the distance PQ.

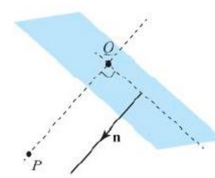
From the textbook:

You can find the shortest distance from a point to a plane.

Suppose you have the plane Π with equation $\mathbf{r}\cdot\mathbf{n}=p$ and the point P The shortest distance from P to the plane will be perpendicular to the plane. Therefore you need the length of $\overrightarrow{PQ}$

Q is the point of intersection of the plane and the line through P and Q. You can easily write down the equation of this line using the fact that it passes through P and has direction vector parallel to the vector $\mathbf{n}$ which is normal to the plane.

Once you've found the point of intersection Q you can find the length of the line segment PQ and hence the shortest distance from the point P to the plane Π



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<tr>
<td>How can you work out the angle between a line and plane?</td>
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1. Find the angle between the direction vector of the line and the normal of the plane using the dot product.
2. Do 90 - (angle above) (aka finding the complement).

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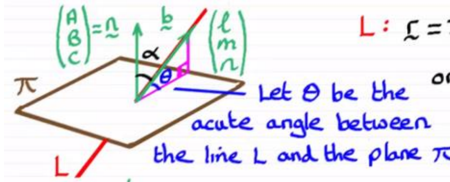
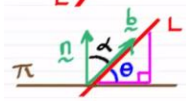
	 $L: \underset{\sim}{r} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k} + \lambda(l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k})$ or $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ $\pi: \underset{\sim}{r}\cdot(A\underset{\sim}{i} + B\underset{\sim}{j} + C\underset{\sim}{k}) = D$ or $Ax + By + Cz = D$  $\cos\alpha = \left\lvert \frac{\underset{\sim}{b}\cdot\underset{\sim}{n}}{\lvert\underset{\sim}{b}\rvert\lvert\underset{\sim}{n}\rvert} \right\rvert \Rightarrow \theta = 90^\circ - \alpha$
How can you find the angle between 2 planes?	Find the angle between their normals.
How can you find the distance between 2 skew lines using 2 different methods?	<u>**First Method:**</u> 1. Consider a general point on both lines. 2. Find a vector joining these 2 general point. 3. Dot this vector with the direction vector of both lines. 4. Use the newly found vector and calculate its magnitude. Find the minimum distance between the skew lines with equations $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(-\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j})$ A general point, P, on the first line has position vector, $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 1-\lambda \\ 2\lambda \end{pmatrix}$ A general point, Q, on the second line has position vector $\overrightarrow{OQ} = \begin{pmatrix} \mu \\ 1+2\mu \\ -2 \end{pmatrix}$. Therefore, the vector joining these two points is $\overrightarrow{PQ} = \begin{pmatrix} \mu \\ 1+2\mu \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1-\lambda \\ 2\lambda \end{pmatrix} = \begin{pmatrix} \mu-2 \\ 2\mu+\lambda \\ -2-2\lambda \end{pmatrix}$ Use $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

The distance PQ will be minimal when the vector $\overrightarrow{PQ}$ is perpendicular to both lines.

$$\begin{pmatrix} \mu-2 \\ 2\mu+\lambda \\ -2-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = 0$$

Since $\overrightarrow{PQ}$ is perpendicular to the first line.

So $-(2\mu+\lambda)+2(-2-2\lambda)=0$

$\Rightarrow -2\mu-5\lambda=4$ (1)

$$\begin{pmatrix} \mu-2 \\ 2\mu+\lambda \\ -2-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$$

Since $\overrightarrow{PQ}$ is perpendicular to the first line.

So $(\mu-2)+2(2\mu+\lambda)=0$

$\Rightarrow 5\mu+2\lambda=2$ (2)

Solve the equations simultaneously (you could use your calculator).

Solving equations (1) and (2) simultaneously gives, $\mu=\frac{6}{7}, \lambda=-\frac{8}{7}$

$$\overrightarrow{PQ} = \begin{pmatrix} \left(\frac{6}{7}\right)-2 \\ 2\left(\frac{6}{7}\right)+\left(-\frac{8}{7}\right) \\ -2-2\left(-\frac{8}{7}\right) \end{pmatrix} = \begin{pmatrix} -\frac{8}{7} \\ \frac{4}{7} \\ \frac{2}{7} \end{pmatrix}$$

Substitute the values of μ and λ into the vector $\overrightarrow{PQ}$

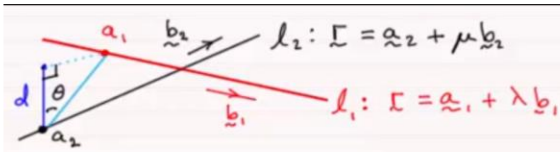
$$\overrightarrow{PQ} = \sqrt{\left(-\frac{8}{7}\right)^2+\left(\frac{4}{7}\right)^2+\left(\frac{2}{7}\right)^2} = \frac{2\sqrt{21}}{7} \text{ (1.31 to 3 sig. fig.)}$$

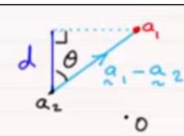
Calculate the length of $\overrightarrow{PQ}$

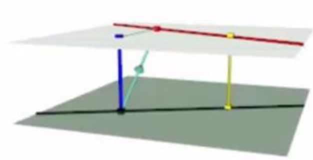
So the perpendicular distance between the two lines is 1.31 units

<u>Second Method:</u>

1. Consider the component of the general vector perpendicular to the lines.
2. Multiply by the magnitude of that unit vector.
3. Use the dot product in reverse.
4. Use the scalar product.







$$\begin{aligned} \text{distance } d &= |a_1 - a_2| \cos\theta \\ &= |a_1 - a_2|\,|\hat{d}| \cos\theta \\ &= \left|(a_1 - a_2) \cdot \hat{d}\right| \\ &= \left|(a_1 - a_2) \cdot \frac{b_1 \times b_2}{|b_1 \times b_2|}\right| \end{aligned}$$

How do you work out the distance between a point

<u>First Method:</u>

1. Calculate the vector between a general point on the line and the point in question (P).

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<td>and line (or 2 parallel lines) using 2 different methods?</td>
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2. Dot this vector with the direction vector of the line.
3. Calculate its magnitude.
<u>Second Method:</u>
Find the minimum distance between the line with equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + t(\mathbf{j} - \mathbf{k})$ and the point $(2, 1, -2)$
You are interested in the vector $\mathbf{i} + \mathbf{j} + t(\mathbf{j} - \mathbf{k}) - (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$
$= -\mathbf{i} + t\mathbf{j} + (2 - t)\mathbf{k}$
Find the vector joining the point to a general point on the line. (1)
The length of this vector is
$x = \sqrt{1^2 + t^2 + (2 - t)^2} = \sqrt{2t^2 - 4t + 5}$
Find the magnitude of the vector. (2)
Therefore $x^2 = 2t^2 - 4t + 5$
It is simpler to consider the expression for x^2
$\frac{d(x^2)}{dt} = 4t - 4$
Minimising x^2 will also minimise x
$\frac{dx}{dt} = 0 \Rightarrow \frac{d(x^2)}{dt} = 0$
$\Rightarrow t = 1$
Use the derivative to calculate t (3)
$x = \sqrt{1^2 + 1^2 + (2 - 1)^2} = \sqrt{3}$
Substitute t into the expression. (4)
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<td>How can you find the line of intersection of 2 planes?</td>
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1. Eliminate one of the variables.
2. Let another variable equal λ.
3. Find the remaining variables that define the line.

<u>Example:</u>
(A), x + y + z = -1
(B), x + 2y + 3z = -4

(B) - (A) gives (C), y + 2z = -3 which is true for all points on the line.

Let z = λ so y = -2λ - 3 from (C) and x = 2 + λ from (A). These all define the line of intersection.
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