A1 - Proofs

Consequence and Equivalence

- $a \Rightarrow b$ <u>means</u> if a is true then b is true. For example, if a = "p is a prime no. > 2" & b = "p is an odd no." then $a \Rightarrow b$.
  - <u>**HOWEVER**</u> check the converse to make sure you can write $\Leftrightarrow$ instead (meaning that a implies b and b implies a)
- Consequence is $\Rightarrow$ and equivalence is $\Leftrightarrow$.

Examples of Consequence and Equivalence

① "the object is a cube" $\Rightarrow$ "the object has six faces"

② "$x^3 = x$" $\Leftarrow$ "$x = -1$" <u>YET NOT</u> $\Leftrightarrow$ $\because$

$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x(x-1)(x+1) = 0$$
$$x = 0, 1, -1$$

③ "n is a prime no." $\Leftrightarrow$ "n has exactly 2 factors"

Proof by Exhaustion (with Examples)

- This will only work for certain examples, not for proving there are infinitely many primes.

① Conjecture: "47 is a prime no."

$47 \div 2 = 48.5 \Rightarrow$ not a factor
" " 3 " $32.\dot{3} \Rightarrow$ not a factor
" " 5 " $19.4 \Rightarrow$ not a factor
" " 7 " $13.85 \Rightarrow$ not a factor
" " 11 " $8.81 \Rightarrow$ not a factor

(don't need to try 4, 6 $\because$ already tried their factors)

We don't need to go any further $\because \sqrt{97} < 10$ but > 9 & if there is a factor above 10 then there must be a factor below because they come in pairs. For example: 144 has 2×72, 3×48, 4×36, 6×24 ... 12×12.

② Conjecture: "no square no. ends in 8"

1	2	3	4	5	6	7	8	9	10
1	4	9	16	25	36	49	64	81	100

⇒ If a no. ends in 1, its square will end in 1; if it ends in 3, its square will end in 9

(this is because of give multiplication where units will multiply by the other units to give the new units)

⇒ the conjecture is true ∴ no no. ends in 7

③ "Every integer that is a ~~multiple of 9~~ perfect cube is either a multiple of 9, 1 more than a multiple of 9 or 1 less than a multiple of 9" is the conjecture.

let $n_1 = 3k$, $n_2 = 3k-1$, $n_3 = 3k-2$ ∴ of 3 times tables

$n_1^3 = (3k)^3 = 27k^3 = 9(3k^3)$ ∴ is a multiple of 9.

$n_2^3 = (3k-1)^3 = (3k-1)(3k-1)(3k-1)$

$= (9k^2-6k+1)(3k-1)$

$= 27k^3 - 27k^2 + 9k - 1$

$= 9(3k^3 - 3k^2 + k) - 1$ ✓

$n_3^3 = (3k-2)^3 = 27k^3 - 54k^2 + 36k - 8$

$= 27k^3 - 54k^2 + 36k - 9 + 1$

$= 9(3k^3 - 6k^2 + 4k - 1) + 1$ ✓

Proof by Deduction (with Examples)

- It's going through a logical sequence of arguments starting with something you know to be true.

① "The sum of any 2 odd consecutive no.'s is always a multiple of 4" is the conjecture.

$2n-1$ & $2n+1$ will always be odd & consecutive

$\Rightarrow (2n-1)+(2n+1) = 4n = 4(n)$ ∴ multiple of 4 ∵ multiplied by 4.

② "$k^3 - k$ is divisible by 6 for integers $k > 1$"

$= k(k^2-1) = k(k+1)(k-1) = (k-1)k(k+1)$

this will be either odd × even × odd OR even × odd × even & ∵ at least one is even then $k^3 - k$ is divisible by two & ∵ at least one is odd (a multiple of 3), $k^3 - k$ must be divisible by 6 ∵ it is divisible by both 2 & 3

Disproof by Counter Example (with Examples)

① "n^2+n+11" is a prime for all integers $n > 0$"

substitute 11 $\Rightarrow 11^2 + 11 + 11 \Rightarrow 11(11+1+1) \Rightarrow 11(13)$ which is a prime ∵ divisible by 11 & 13

② "If $x^2 > x$ then $x > 1$"

take x as -2, $x^2 = 4$

x^2	x
4	2
1	1
0.01	0.1
4	-2

not > 1

Proof by Contradiction • Proving the opposite of the conjecture is false by deduction to prove the ~~ney~~ conjecture is true

Proving $\sqrt{2}$ is irrational • Assume $\sqrt{2}$ is rational $\Rightarrow \sqrt{2} = \frac{a}{b}$ where a & b are whole no.'s & the fraction is in its simplest form

$\Rightarrow$ a & b cannot both be even $\because$ you can cancel the fraction down further

$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \therefore a^2$ must be even $\because$ it is multiplied by 2 $\Rightarrow a$ is even

now let $a = 2k$ then $2b^2 = (2k)^2 \Rightarrow 2b^2 = 4k^2$

$\Rightarrow b^2 = 2k^2$

now b^2 must be even $\therefore$ b is even $\therefore$ a multiple of 2

now $\because$ a & b are both even, the original equation a/b cannot be right $\because$ its not in its simplest form $\Rightarrow \sqrt{2}$ is irrational as it isn't rational

Proving $\sqrt{3}$ is irrational • Assuming $\sqrt{3}$ is rational $\Rightarrow \sqrt{3} = \frac{a}{b}$ where a & b are whole no.'s ~~their~~ and a/b is in its simplest form

$\Rightarrow$ a & b cannot both be even

• $\sqrt{3} = \frac{a}{b} \Rightarrow 3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2$

If b is even $\Rightarrow$ b^2 is even $\Rightarrow$ $3b^2$ is even $\Rightarrow$ a^2 is even $\Rightarrow$ a is even $\Rightarrow$ $\sqrt{3}$ is irrational

<u>HOWEVER</u>

if b is odd $\Rightarrow$ b^2 is odd $\Rightarrow$ $3b^2$ is odd $\Rightarrow$ a^2 is odd $\Rightarrow$ a is odd

now let $b = 2m+1$ & $a = 2n+1$ (both odd)

$3(2m+1)^2 = (2n+1)^2$

$\Rightarrow 3(4m^2+4m+1) = 4n^2+4n+1$

$\Rightarrow 12m^2+12m+3 = 4n^2+4n+1$

$\Rightarrow 12m^2+12m+2 = 4n^2+4n$

$\Rightarrow 6m^2+6m+1 = 2n^2+2n$

$\Rightarrow \underbrace{2(3m^2+3m)}_{\text{even}}+1 = \underbrace{2(n^2+n)}_{\text{even}}$

<u>HOWEVER</u> even + 1 ≠ even $\Rightarrow$ contradiction meaning the original assumption is false

Proving Infinitely Many Primes

• Assume finitely many primes (i.e. $p_1, p_2, p_3 \dots p_n$)

let $P = p_1 \times p_2 \times p_3 \times p_4 \times \dots p_n + 1$

$\therefore$ P is > all primes but it doesn't mean P is a prime perhaps google to reword

If P is a prime & P is not in the list then we've found another prime $\Rightarrow$ infinitely many primes

If P is not prime, it divides by another prime in the list $\because$ of prime factor decomposition <u>YET</u> the p (eg., p_i) being a factor of P also means its a factor of 1 ($\because$ of $+1$) which is impossible as a prime cannot be a factor of 1 $\Rightarrow$ contradiction $\Rightarrow$ infinitely many primes