CI - Differential Equations

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<td>What is separation of variables?</td>
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$$\frac{dy}{dx} = 3x^2y$$

$$\frac{1}{y}\frac{dy}{dx} = 3x^2$$

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int 3x^2dx$$

$$\int \frac{1}{y}dy = \int 3x^2dx$$

$$\ln|y| = x^3 + c$$

$$y = Ae^{x^3}$$

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<td>How is an exact first order differential equation solved?</td>
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1. Use the product rule in reverse.
2. Integrate both sides.

$\frac{d}{dx}[f(x)y] = f(x)\frac{dy}{dx} + f'(x)y$

So if

$f(x)\frac{dy}{dx} + f'(x)y = g(x)$

$\therefore \frac{d}{dx}[f(x)y] = g(x)$

$\therefore f(x)y = \int g(x)\,dx$

Example 1

$x^4\frac{dy}{dx} + 4x^3y = e^{2x}$

$\therefore \frac{d}{dx}[x^4y] = e^{2x}$

$\therefore x^4y = \int e^{2x}dx$

$\therefore x^4y = \frac{1}{2}e^{2x} + c$

Example 2

$\cos x\frac{dy}{dx} - y\sin x = x^2$

given $y=1$, when $x=0$

$\frac{d}{dx}(y\cos x) = x^2$

$\therefore y\cos x = \int x^2dx$

$\therefore y\cos x = \frac{1}{3}x^3 + c$

when $y=1, x=0$

$\therefore 1 = c \Rightarrow$

May require some manipulation to get it into the form required.

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<td>How is the integrating factor used to solve 1st order linear differential equations?</td>
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1. Get the differential equation into the following form:

$$\frac{dy}{dx} + Py = Q$$

2. Find the integrating factor:

$$I = e^{\int P dx}$$

3. Multiply the equation in (1) by the integrating factor.
4. Use the product rule in reverse.
5. Integrate both sides.

Example:

Example 3

a Find the general solution to the equation $x\frac{dy}{dx} - y = 2x^3$

b Given that $y = 4$ when $x = 1$, find the particular solution.

a $\frac{dy}{dx} - \frac{1}{x}y = 2x^2$ — Rearrange the equation into the form $\frac{dy}{dx} + P(x)y = f(x)$

$e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$ — Find the integrating factor.

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2x$ — Multiply throughout by $\frac{1}{x}$

$\frac{d}{dx}\left[\frac{1}{x}y\right] = 2x$ — Use the fact it is now an exact equation.

$\frac{y}{x} = x^2 + c$ — Integrate both sides.

Nonlinear may be $dy/dx = x^2 + y^2$.

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<td>What is the general solution for a 2nd order differential equation made up of? Define both parts</td>
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- General Solution = Complementary Function + Particular Integral.
- GS = CF + PI

- Complementary function - solution when 2^{nd} order differential equations = 0.
- Particular integral - particular solution for what it equals.

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What are the 3 possible complementary functions?

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ 2nd order linear differential equation

The equation has a solution based on the form $y = Ae^{mx}$

where $am^2 + bm + c = 0$ (known as the Auxillary Equation)

Roots are real and different ($b^2-4ac>0$) $m = m_1, m_2$

General Solutions $y = Ae^{m_1 x} + Be^{m_2 x}$

Roots are equal ($b^2-4ac=0$) $m = m_1, m_1$

General Solutions $y = e^{mx}(Ax+B)$

Roots are imaginary ($b^2-4ac<0$) $m = p \pm iq$

General Solutions $y = e^{px}(A\cos qx + B\sin qx)$

Also shown below:

Table 1

Roots of auxiliary equation $ax^2+bx+c=0$	General solution
Real distinct roots, m_1 and m_2	$y = Ae^{m_1 x} + Be^{m_2 x}$
Repeated root, m	$y = (Ax+B)e^{mx}$
Complex roots $m \pm in$	$y = e^{mx}(A\cos nx + B\sin nx)$

What is a homogeneous differential equation?

When it equals **ZERO**.

$$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + P_0 y = Q(t)$$

Non - homogeneous equation

$$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + P_0 y = 0$$

Homogeneous equation

What possible particular integrals can you have? How do these depend on the complementary function?

$f(x) = ce^{\lambda x}$	y_c - contains $ae^{\lambda x}$ but **not** $axe^{\lambda x}$	$y = axe^{\lambda x}$
	y_c - contains $axe^{\lambda x}$	$y = ax^2e^{\lambda x}$
	y_c - does not contain $ae^{\lambda x}$ or $axe^{\lambda x}$	$y = ae^{\lambda x}$
$f(x) = c\cos\lambda x$ or	y_c - $A\cos\lambda x + B\sin\lambda x$	$y = ax\sin\lambda x$ if $f(x) = c\cos\lambda x$ $y = ax\cos\lambda x$ if $f(x) = c\sin\lambda x$
$f(x) = c\sin\lambda x$	y_c - does **not** contain $A\cos\lambda x + B\sin\lambda x$	$y = a\cos\lambda x + b\sin\lambda x$
f(x) polynomial f degree n		$y = ax^n + bx^{n-1} + \cdots$

You need to ensure you don't have the same constants on either side of the complementary function. Eg,

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$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, general solution $y = C.F. + P.I$

Find the general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 2$ ①

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$
Auxillary Equation
$m^2 - 4m = 0$
$\therefore m(m-4) = 0$
$\therefore m = 0$ or $m = 4$
$\therefore$ C.F. $y = A + Be^{4x}$
$\therefore$ P.I. $y = \lambda x$

$\therefore \frac{dy}{dx} = \lambda$, $\frac{d^2y}{dx^2} = 0$ Sub into ①
$\therefore -4\lambda = 2$
$\therefore \lambda = -\frac{1}{2}$
$\therefore$ P.I. is $-\frac{1}{2}x$
$\therefore$ General Solution:
$y = A + Be^{4x} - \frac{1}{2}x$

Or…

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, general solution $y = C.F. + P.I$

Find the general solution to $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$ ①

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$
Auxillary Equation
$m^2 - 3m + 2 = 0$
$\therefore (m-2)(m-1) = 0$
$\therefore m = 2$, $m = 1$
$\therefore$ C.F. $y = Ae^{2x} + Be^{x}$
$\therefore$ P.I. let $y = \lambda xe^{2x}$

$\therefore \frac{dy}{dx} = \lambda e^{2x} + 2\lambda xe^{2x}$
$\therefore \frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{2x} + 4\lambda xe^{2x}$
$= 4\lambda e^{2x} + 4\lambda xe^{2x}$
Sub. into ①
$4\lambda e^{2x} + 4\lambda xe^{2x} - 3\lambda e^{2x} - 6\lambda xe^{2x} + 2\lambda xe^{2x} = e^{2x}$
$\therefore \lambda e^{2x} = e^{2x} \Rightarrow \lambda = 1 \Rightarrow$ P.I. is xe^{2x}
General solution
$y = Ae^{2x} + Be^{x} + xe^{2x}$
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<td>When does a body undergo SHM?</td>
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Key point
Motion that satisfies a differential equation of the form $\frac{d^2x}{dt^2} = -\omega^2 x$ is called simple harmonic motion (SHM).

This is a requirement and gives the following solution:
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	$$\frac{\mathrm{d}x^2}{(dt)^2} = -\omega^2 x$$ $$\frac{\mathrm{d}x^2}{(dt)^2} + \omega^2 x = 0$$ $$m^2 + \omega^2 = 0$$ $$m = \pm\omega i$$ $$x = Pcos(\omega t) + Qsin(\omega t)$$ $$= a sin(\omega t + \varepsilon)$$
How can you derive the standard results for SHM?	There are certain standard results that apply to SHM. These are derived below. Using $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}$ gives $v\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 x$ $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \times v$ So $\int v\mathrm{d}v = -\int \omega^2 x \mathrm{d}x$ Separate the variables. $\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$ (1) Let the amplitude of the motion be a, then $\frac{0^2}{2} = -\frac{\omega^2 a^2}{2} + c \Rightarrow c = \frac{\omega^2 a^2}{2}$ Use the fact that when $x = a$ the velocity is zero. So $\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$ Substitute for c into (1) i.e. $v^2 = \omega^2(a^2 - x^2)$ or $v = \omega\sqrt{a^2 - x^2}$ **Standard results** $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x \Rightarrow v = \omega\sqrt{a^2 - x^2}$ $x = a\cos\omega t$ or $x = a\sin\omega t$ $T = \frac{2\pi}{\omega}$ It is the starting conditions that determine which form of x to use. where a is the amplitude and T is the period. *You can have 2 different values for x, it depends on when the clock starts ticking.*

	The value of ε depends on when the clock starts ticking, i.e. where we measure $t=0$ from. • If $t=0$ when $x=0$, then $\varepsilon=0$, and $x=a\sin\omega t$ • If $t=0$ when $x=a$, then $\varepsilon=\frac{\pi}{2}$, and $x=a\sin\left(\omega t+\frac{\pi}{2}\right)=a\cos\omega t$ The period of a simple $\sin t$ or $\cos t$ function is 2π. Hence the period of $x=a\sin(\omega t+\varepsilon)$ is $\frac{2\pi}{\omega}$
What is the form of a differential equation under damping? How does it link to the different types of damping?	$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\omega^2x=0$$ *Something in this form with k in front of dx/dt. The larger k is, the stronger resistance to motion.* If the discriminant of the auxiliary equals is… ● > 0, heavy damping ● = 0, critical damping ○ Here, the amplitude reduces as fast as possible. ● < 0, light damping.