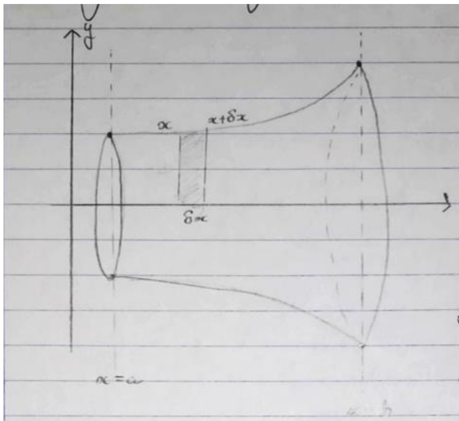
CE - Further calculus

Derive the volumes of revolution formula	 The volume of a ~~cylinder~~ cylinder is denoted by $V = \pi r^2 h$ The volume of the solid formed by ~~rotated~~ rotating the shaded area 2π radians around the x-axis is denoted by δV. This lies between: $\pi y^2 \delta x \le \delta V \le \pi (y+\delta y)^2 \delta x$ $\Rightarrow \pi y^2 \le \frac{\delta V}{\delta x} \le \pi (y+\delta y)^2$ as $\delta x \to 0$, $\delta y \to 0$ too & $\frac{\delta V}{\delta x} \to \frac{dV}{dx}$ $\therefore \frac{dV}{dx} = \pi y^2$ $\int \frac{dV}{dx}\, dx = \int 1\, dV$ $\int \pi y^2\, dx = V$
When is an improper integral?	**Key point** An **improper integral** is a definite integral where either: • one or both of the limits is $\pm\infty$ • the integrand (expression to be integrated) is undefined at one of the limits of the integral • the integrand is undefined at some point between the limits of the integral.
How can you evaluate an improper integral? What outcomes can you have?	1. By replacing the limit with a variable and taking the limit of that variable.

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2. You can have convergent (if the limit exists) or divergent (if the limit does not exist) integral.

Key point

To evaluate an improper integral with a limit of $\pm\infty$ use

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \quad \text{or} \quad \int_{\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$$

If the limit exists, then the improper integral is called **convergent**.

If the limit does not exist, then the improper integral is called **divergent**.

<u>Examples:</u>

Example 2

When splitting the integral, you need to use different variables for ∞ and $-\infty$

This is because both parts of the integral must be convergent for the original integral to exist.

a Find the value of $\int_{-\infty}^{\infty}\frac{x}{e^{x^2}}dx$

b Show that the improper integral $\int_{-\infty}^{\infty}e^{x}$ is divergent.

a $\int_{-\infty}^{\infty}\frac{x}{e^{x^2}}dx = \lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$

Choose to split the integral at 0

$= \lim_{a\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_a^0 + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b$

Since $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$ using the chain rule.

$= \lim_{a\to-\infty}\left(-\frac{1}{2}e^0 - -\frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} - -\frac{1}{2}e^0\right)$

$= \lim_{a\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right)$

Since $e^0 = 1$

$= \left(-\frac{1}{2}\right)+\left(\frac{1}{2}\right)$

Both limits exist, therefore the improper integral is convergent.

Since as $a\to-\infty$, $e^{-a^2}\to 0$ and as $b\to\infty$, $e^{-b^2}\to 0$

So the improper integral $\int_{-\infty}^{\infty}\frac{x}{e^{x^2}}dx = -\frac{1}{2}+\frac{1}{2} = 0$

Example 3

Find the value of $\int_0^4 \frac{2}{\sqrt{x}}dx$

$\int_0^4 \frac{2}{\sqrt{x}}dx = \lim_{t\to 0}\int_t^4 x^{-\frac{1}{2}}dx$

Replace 0 with t since $\frac{1}{\sqrt{x}}$ is undefined at $x = 0$

$= \lim_{t\to 0}\left[2x^{\frac{1}{2}}\right]_t^4$

$= \lim_{t\to 0}(4 - 2\sqrt{t})$

$= 4$

Since $2\sqrt{t}\to 0$ as $t\to 0$, the value of the integral is 4

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What 2 limit results do you need to know for the integration topic?

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Key point

For any real number k, $x^k e^{-x}\to 0$ when $x\to\infty$

Key point

For any real number k, $x^k \ln x\to 0$ when $x\to 0+$ (this means that x approaches zero from above as x must be positive for $\ln x$ to be defined).

Where k > 0.

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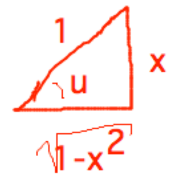
What is the formulae for 'Volumes of Revolution'?	$$\pi \int_a^b y^2 dx$$ For rotation around the x-axis. $$\pi \int_a^b x^2 dy$$ For rotation around the y-axis. *This is because, when you have a small change in x then the radius of the cylinders are y and following πr^2 you get the top formula with limits a b (both x-values with b - a representing the total height). Whereas, rotating around the y-axis means a small change in y with a radius x. This gives you the bottom formula with limits a b (both y-values).*
How can you calculate the mean value of a function?	$$y_M = \frac{1}{b-a} \int_a^b f(x)\ \mathrm{dx}$$
What substitutions do you need to know for what what integrals? $\frac{1}{\sqrt{a^2-x^2}}$ $\sin^{-1}\left(\frac{x}{a}\right)+c \quad (\lvert x\rvert < a)$ $\frac{1}{a^2+x^2}$ $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ $\frac{1}{\sqrt{x^2-a^2}}$ $\cosh^{-1}\left(\frac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2-a^2}\}+c \quad (x > a)$ $\frac{1}{\sqrt{a^2+x^2}}$ $\sinh^{-1}\left(\frac{x}{a}\right)$ or $\ln\{x+\sqrt{x^2+a^2}\}+c$	For integrals involving … try the substitution … • $\sqrt{a^2-x^2},\ \text{try } x = a\sin u$ • $a^2+x^2,\ \text{try } x = a\tan u$ • $\sqrt{x^2+a^2},\ \text{try } x = a\sinh u$ • $\sqrt{x^2-a^2},\ \text{try } x = a\cosh u$ **<u>Examples:</u>** $$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx$$ Would involve using x = sin(u).

	And so would the one below: $\int \sqrt{1-x^2}\,dx$ $x = \sin u$, $dx = \cos u\,du$ or $x = \cos u$, $dx = -\sin u\,du$ $= \int \sqrt{1-\sin^2 u}\,\cos u\,du$  $= \int \cos^2 u\,du$ $= \int \frac{1+\cos 2u}{2}\,du$ $= \frac{1}{2}\int 1 + \cos 2u\,du$ $= \frac{1}{2}\int du + \frac{1}{2}\int \cos 2u\,du$ $v = 2u$, $dv = 2du$ $= \frac{u}{2} + \frac{1}{4}\sin 2u + C = \frac{\sin^{-1}x}{2} + \frac{\sin u \cos u}{2} + C$ $= \frac{\sin^{-1}x}{2} + x\sqrt{1-x^2}/2 + C$ Yet, this particular integral can also be done by parts: $\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$ $\Longrightarrow \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx$ $\Longrightarrow 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \arcsin x + C$ $\Longrightarrow \int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C.$
What are the derivatives of sinh and cosh?	$\frac{d(\sinh x)}{dx} = \cosh x$ $\frac{d(\cosh x)}{dx} = \sinh x$
How should you differentiate inverse trigonometric and hyperbolic functions?	1. Let y = (inverse function). 2. Take the function of both sides. 3. Use the chain rule. 4. Use an identity. 5. Substitute either x or y back in.

	Example 3 Use the identity $\cosh^2 x - \sinh^2 x \equiv 1$ Differentiate arcosh$x$ with respect to x Let $y = \text{arcosh}x$ Then $x = \cosh y$ $\frac{dx}{dy} = \sinh y$ So $\frac{dy}{dx} = \frac{1}{\sinh y}$ — Use the chain rule: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ $= \frac{1}{\sqrt{\cosh^2 y - 1}}$ — Use the identity $\cosh^2 y - \sinh^2 y \equiv 1$ $= \frac{1}{\sqrt{x^2 - 1}}$
How can you split nonlinear partial fractions?	$\frac{f(x)}{(\alpha x^2 + \beta)(\gamma x + \delta)}$ can be split into partial fractions of the form $\frac{Ax + B}{\alpha x^2 + \beta} + \frac{C}{\gamma x + \delta}$ *An example is below…* **Example 2** the denominator yourself. Work out $\int \frac{2x+12}{(x+1)(x^2+9)} dx$ $\frac{2x+12}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$ $2x + 12 = A(x^2+9) + (Bx+C)(x+1)$ $= Ax^2 + 9A + Bx^2 + Bx + Cx + C$ — Multiply both sides by $(x+1)(x^2+9)$ Equating coefficients $x^2: 0 = A + B$ so $A = -B$ — Or use an alternative method to find the values of A, B and C $x: 2 = B + C$ (equation 1) $1: 12 = 9A + C$ (equation 2) Subtract equation 1 from equation 2 to give: $9A - B = 10$ $A = -B \Rightarrow -9B - B = 10 \Rightarrow B = -1$ — Solve the three equations simultaneously. So $A = 1$ and $C = 3$
What is a reduction formula and how can you find it?	A reduction formula for I_n is an equation that relates I_n to I_{n-1} and/or I_{n-2} which is used to reduce an integral.

Start by setting:

$$I_n = \int \cos^n x \, \mathrm{d}x.$$

Now re-write as:

$$I_n = \int \cos^{n-1} x \cos x \, \mathrm{d}x,$$

Integrating by this substitution:

$$\cos x \, \mathrm{d}x = \mathrm{d}(\sin x),$$

$$I_n = \int \cos^{n-1} x \, \mathrm{d}(\sin x).$$

Now integrating by parts:

$$\begin{aligned}
\int \cos^n x \, \mathrm{d}x &= \cos^{n-1} x \sin x - \int \sin x \, \mathrm{d}(\cos^{n-1} x) \\
&= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, \mathrm{d}x - (n-1) \int \cos^n x \, \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n,
\end{aligned}$$

Reduction Formula

$\int x^n e^x \, dx$ for any positive integer n

$$\int \underbrace{x^n}_{u} \underbrace{e^x \, dx}_{dv} = x^n e^x - n \int x^{n-1} e^x \, dx$$

EX 2.6 Using a Reduction Formula

Evaluate the integral $\int x^4 e^x \, dx$

$$n = 4 \qquad \int x^4 e^x \, dx = x^4 e^x - 4 \int x^{4-1} e^x \, dx = x^4 e^x - 4 \int x^3 e^x \, dx$$

$$n = 3 \qquad \int x^4 e^x \, dx = x^4 e^x - 4\left(x^3 e^x - 3 \int x^2 e^x \, dx\right)$$

$$\vdots$$

$$\int x^4 e^x \, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x + c$$