Vectors

Scalar/dot Product

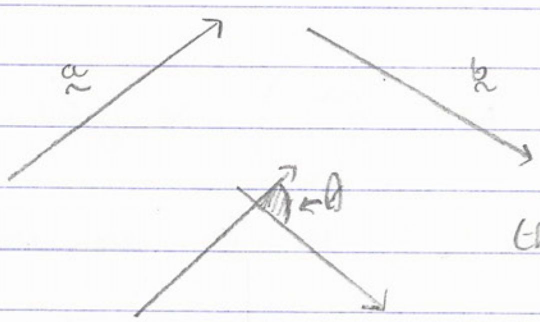
$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\cdot\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

(Note: position vectors refers to the bracket form as above & are relative to the origin)

Length of a vector

$$|\mathbf{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Angle between 2 vectors



Finding it only involves using the direction vectors.

the angle coming out of the direction

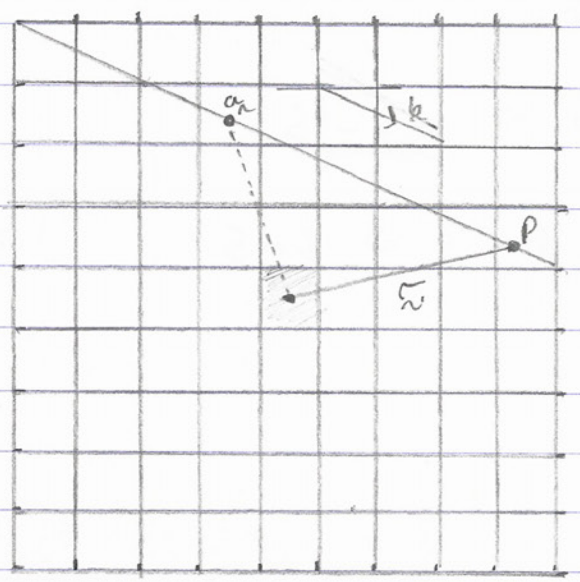
$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

dot product of the vectors

multiply length of each

Perpendicular when $\cos\theta = 0$ ∴ $\theta = 90°$ ⇒ when $\mathbf{a}\cdot\mathbf{b} = 0$ ∴ $|\mathbf{a}||\mathbf{b}| \neq 0$ (never ever)

Vector eqn of a line



You can get to any point on a line by using $\mathbf{a}$ (a fixed point on the line) & travelling a scalar multiple of $\mathbf{b}$ (a vector parallel to the line)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

direction vector

(Remember to sketch even 3D vectors)

Vector Line Joining 2 points

$A(-1,5,2)$ & $B(2,1,-3)$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\begin{pmatrix}-1\\5\\2\end{pmatrix} + \begin{pmatrix}2\\1\\-3\end{pmatrix}$

direction vector

$= \begin{pmatrix}3\\-4\\-5\end{pmatrix}$

$\Rightarrow \underset{\sim}{r}_1 = \begin{pmatrix}-1\\5\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix}$

Ratios

Let $\underset{\sim}{b}_1 = \begin{pmatrix}a-1\\-a-1\\b-1\end{pmatrix}$ & $\underset{\sim}{b}_2 = \begin{pmatrix}2a\\3-5a\\15\end{pmatrix}$ so find the values of a & b for which $\underset{\sim}{b}_1$ & $\underset{\sim}{b}_2$ are parallel.

ratios must equal

$\frac{a-1}{-a-1} = \frac{2a}{3-5a} \Rightarrow (a-1)(3-5a) = 2a(-a-1)$

$\Rightarrow 3a^2 - 10a + 3 = 0$

$\Rightarrow a = 1/3, 3$

$\Rightarrow$ Let β = some scalar multiple

$\beta(a-1) = 2a \Rightarrow$ when $a = 1/3$ $\beta(-2/3) = 2/3$

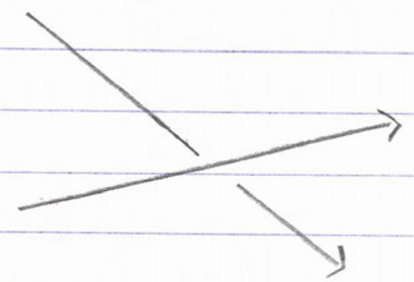
& $\Rightarrow \beta = -1$ so...

$\beta(b) = 15 \Rightarrow \underline{b = -15}$

& when $a = 3$, $\beta = 3 \Rightarrow b = 5$

Skew lines

- When lines aren't parallel or perpendicular. E.g.



One travelling under the other.

Example:

$$L_1 = \begin{pmatrix}1\\5\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix} \quad \& \quad L_2 = \begin{pmatrix}0\\3\\5\end{pmatrix} + \mu\begin{pmatrix}1\\0\\1\end{pmatrix}$$

When $L_1 = L_2 \Rightarrow$ ① $1+2\lambda = \mu$ $\Rightarrow \mu = 5$

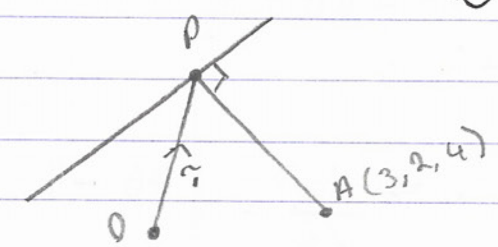
② $5-\lambda = 3$ $\Rightarrow \lambda = 2$ sub into ①

③ $4+\lambda = 5+\mu$

Check with ③ $\Rightarrow 6 \neq 10$

$\Rightarrow$ ∵ it doesn't satisfy the 3rd, it's skew

Distance to a point



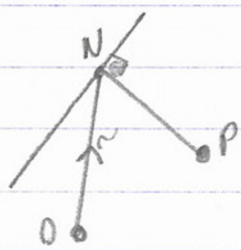
A line is closest to the point in question when the vector using the point A to some point P on the line is perpendicular to the line's direction

$$\Rightarrow \overline{AP} = \overline{AO} + r_1$$
$$= -\overline{OA} + r_1$$

Now $\overline{AP}$. direction vector $= 0 \Rightarrow$ use to find the value of λ and use this to find the point P & finally distance to it.

Example of distance from point to line (closest)

$r = i + j + t(j - k)$ & $P(2, 1, -2)$



$\overline{PN} = r - \overline{OP}$

$= i + \cancel{j} + t(j-k) - 2i - \cancel{j} + 2k$

$= -i + tj - tk + 2k$

$= -i + tj - (t-2)k$

$= \begin{pmatrix} -1 \\ t \\ -t+2 \end{pmatrix}$

$\Rightarrow \overline{PN} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ t \\ -t+2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = +t + t - 2 = 0$

$\Rightarrow 2t - 2 = 0$

$\underline{\underline{t = 1}}$

$\Rightarrow \overline{PN} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow |\overline{PN}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \underline{\underline{\sqrt{3}}}$

<u>Q2</u> let a = length

$a^2 = (-1)^2 + t^2 + \cancel{(-1)^2}(t-2)^2$

$= 1 + t^2 + t^2 - 4t + 4$

$= 2t^2 - 4t + 5$

minimising a^2 will minimise $a \Rightarrow \frac{da^2}{dt} = 4t - 4$

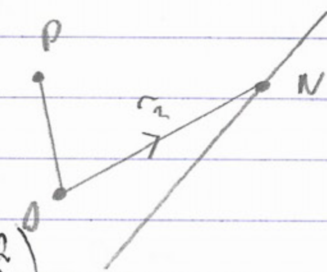
$4t - 4 = 0 \Rightarrow \underline{\underline{t = 1}}$

sub into $\sqrt{2t^2 - 4t + 5} = \sqrt{2 - 4 + 5} = \sqrt{7-4} = \underline{\underline{\sqrt{3}}}$

Perpendicular Distance Between 2 Lines (parallel)

$$r_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \quad \& \quad r_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

Pick some point on r_1, e.g., $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$



$$\Rightarrow \overline{PN} = \overline{PO} + \overline{ON}\ (r_2)$$
$$= r_2 - \overline{OP}$$
$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 - 2\lambda \\ 2 + 3\lambda \\ -1 - \lambda \end{pmatrix}$$

$$\Rightarrow x = (-1)(2\lambda + 1), \quad y = (3\lambda + 2), \quad z = (-1)(\lambda + 1)$$

$$a^2 = (2\lambda+1)^2 + (3\lambda+2)^2 + (\lambda+1)^2$$
$$= 4\lambda^2 + 4\lambda + 1 + 3^2\lambda + \dots$$
$$= 14\lambda^2 + 18\lambda + 6$$

CALCULUS ONLY WORKS IF PARALLE

$$\frac{da^2}{d\lambda} = 28\lambda + 18 \Rightarrow \lambda = \frac{-18}{28} = -\frac{9}{14}$$

Sub into $a^2 = \frac{3}{14} \Rightarrow a = \frac{\sqrt{42}}{14}$

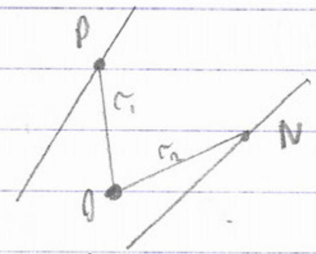
OR $\begin{pmatrix} -2\lambda - 1 \\ 3\lambda + 2 \\ -\lambda - 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = 4\lambda + 2 + 9\lambda + 6 + \lambda + 1 = 0$

$$\Rightarrow 14\lambda + 9 = 0$$
$$\lambda = -9/14$$

$$\Rightarrow \overline{PN} = \begin{pmatrix} 2/7 \\ 1/14 \\ -5/14 \end{pmatrix} \Rightarrow |\overline{PN}| = \frac{\sqrt{42}}{14}$$

Shortest Distance Between 2 Skew Lines

$$r_1 = \begin{pmatrix}5\\0\\-4\end{pmatrix} + m\begin{pmatrix}-3\\3\\0\end{pmatrix} \qquad r_2 = \begin{pmatrix}0\\-3\\2\end{pmatrix} + n\begin{pmatrix}5\\-4\\-1\end{pmatrix}$$



$$\overline{PN} = \overline{PO} + \overline{ON}$$
$$= -\underset{\sim}{P} + \underset{\sim}{N}$$
$$= \begin{pmatrix}5n + 3m - 5\\-3m - 4n - 3\\-n + 6\end{pmatrix}$$

$$\therefore \underset{\sim}{P} = \begin{pmatrix}5-3m\\3m\\-4\end{pmatrix} \text{ \& } \underset{\sim}{N} = \begin{pmatrix}5n\\-3-4n\\2-n\end{pmatrix}$$

Perpendicular to both r_1 & r_2

$$\begin{pmatrix}5n+3m-5\\-3m-4n-3\\-n+6\end{pmatrix} \cdot \begin{pmatrix}-3\\3\\0\end{pmatrix} = -18m - 27n + 6 = 0$$

$$'' \cdot \begin{pmatrix}5\\-4\\-1\end{pmatrix} = \quad 42n + 27m = 19$$

$$m = \frac{-29}{3} \text{ \& } n = 20/3$$

$$\Rightarrow \text{Vector} = \begin{pmatrix}-2/3\\-2/3\\-2/3\end{pmatrix} \qquad |\overline{PN}| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \sqrt{\frac{4}{3}} = \boxed{\frac{2\sqrt{3}}{3}}$$