CD - Further algebra and functions

How do the roots relate to coefficients for quadratics, cubics, and quartics?	For $ax^2+bx+c=0$ with roots α and β then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ For $ax^3+bx^2+cx+d=0$ with roots α, β and γ then $\sum\alpha=\alpha+\beta+\gamma=-\frac{b}{a}$, $\sum\alpha\beta=\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$ and $\alpha\beta\gamma=-\frac{d}{a}$ For $ax^4+bx^3+cx^2+dx+e=0$ with roots α, β, γ and δ then $\sum\alpha=-\frac{b}{a}$, $\sum\alpha\beta=\frac{c}{a}$, $\sum\alpha\beta\gamma=-\frac{d}{a}$ and $\alpha\beta\gamma\delta=\frac{e}{a}$ *The proof for the quadratic roots is shown below:* **Quadratic equations** Let the quadratic equation $ax^2+bx+c=0$ have roots $x=\alpha$ and $x=\beta$ Dividing through by a gives $x^2+\frac{b}{a}x+\frac{c}{a}=0$ Since $x=\alpha$ and $x=\beta$ are the roots of this quadratic, you can write the equation in the form $(x-\alpha)(x-\beta)=0$ Expanding the brackets gives $x^2-(\alpha+\beta)x+\alpha\beta=0$ Comparing the two versions of the quadratic equation gives $x^2+\frac{b}{a}x+\frac{c}{a}\equiv x^2-(\alpha+\beta)x+\alpha\beta=0$ So, comparing the coefficients for x and the constant gives $(\alpha+\beta)=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ *The proof for the cubic and quartics are similar.*
What results should you know for the roots topic?	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$
When can L'Hopital's rule be used?	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$, provided that $f(x)=g(x)=0$ or $f(x)=g(x)=\pm\infty$ and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists This may be used alongside Maclaurin expansion.

How can you transform linear and reciprocal roots?

Key point

If the roots are transformed in a linear way, so that $y = mx + c$,

then you transform the equation by substituting $x = \frac{y-c}{m}$

If the new roots are reciprocals, so that $y = \frac{1}{x}$, then you transform the equation by substituting $x = \frac{1}{y}$

__Example:__

E.g., $x^3 - 3x^2 + 4 = 0$ has roots α, β, γ. Find a related cubic with roots 2α, 2β, 2γ.

Let $y = 2\alpha \Rightarrow \alpha = \frac{1}{2}y$

$\boxed{\text{SOMETIMES FASTER TO USE THIS}}$

$(\frac{1}{2}y)^3 - 3(\frac{1}{2}y)^2 + 4 = 0$

essentially subbing in $\frac{1}{2}\alpha$

$\frac{1}{8}y^3 - \frac{3}{4}y^2 + 4 = 0$

$\times 8$

$y^3 - 6y^2 + 32 = 0$

Let $f(x) = x^3 - 3x^2 + 4$, $f(\frac{1}{2}x)$ will have double roots ∴ stretch in x-direction s.f. 2

How can you find maximums and minimums without calculus?

1. Say f(x) intersects y = k so f(x) = k.
2. Consider the determinant.

The function $f(x) = \frac{x^2 - 4x + 4}{x}$ intersects the straight line $y = k$

a Form a quadratic equation in x and k

b Hence find the values of k for which $f(x)$ has real roots.

c Use a graphical calculator or graph sketching software on your computer to confirm your answer.

a $k = \frac{x^2 - 4x + 4}{x}$

$kx = x^2 - 4x + 4$

$x^2 - (k+4)x + 4 = 0$

(1) Multiply both sides by x and simplify to form a quadratic equation in x

b The function has real roots if $b^2 - 4ac \geq 0$

$(k+4)^2 \geq 4 \times 1 \times 4$

$k^2 + 8k + 16 - 16 \geq 0$

$k^2 + 8k \geq 0$

$k(k+8) \geq 0$

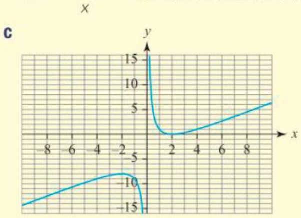
$k = \frac{x^2 - 4x + 4}{x}$ has real roots if $k \geq 0$ or $k \leq -8$

The discriminant must be ≥ 0

Substitute values and solve the inequality.

(2)(3) k and $(k+8)$ must either both be positive or both be negative.

c



(2) The graph of $y = \frac{x^2 - 4x + 4}{x}$ shows that x does not exist if $-8 \leq y \leq 0$

So x can only have real roots if $y \geq 0$ or $y \leq -8$

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<td>How should you deal with double inequalities?</td>
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Consider each case seperately.

Example 7

Find the values of x for which $0 \le \frac{4+x}{3-x} < 1$

Consider $0 \le \frac{4+x}{3-x}$

Critical values are $x = -4$ and $x = 3$

For $x < -4$, $\frac{4+x}{3-x}$ is negative.

For $-4 < x < 3$, $\frac{4+x}{3-x}$ is positive.

For $x > 3$, $\frac{4+x}{3-x}$ is negative.

So $\frac{4+x}{3-x} \ge 0$ when $-4 \le x < 3$

Now consider $\frac{4+x}{3-x} < 1$

When $x < 3$

$4 + x < 3 - x$

$x < -\frac{1}{2}$

When $x > 3$

$4 + x > 3 - x$

$x > -\frac{1}{2}$, but we have the more restrictive condition $x > 3$

So $\frac{4+x}{3-x} < 1$ when $x < -\frac{1}{2}$ or $x > 3$

Full solution: $0 \le \frac{4+x}{3-x} < 1$ when $-4 \le x < -\frac{1}{2}$

(2) Consider the signs of the numerator and denominator for each set of values.

Do not include 3 as $\frac{4+x}{3-x}$ is not defined at $x = 3$

$(3 - x)$ is positive.

(1) $(3 - x)$ is negative so reverse the inequality.

(3) Find the values which satisfy both inequalities.
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<td>How can conics be translated?</td>
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- Replacing f(x) by f(x - c) in an equation translates the curve c units in the x-direction. Similarly, replacing g(y) by g(y - c) translates the curve c units in they-direction.
- Replacing f(x) by f(x/k) in an equation will stretch the curve by scale factor k in the x-direction. Similarly, replacing g(y) by g(y/k) will stretch the curve by scale factor k in they-direction.

<u>**Examples of this includes:**</u>

Key point

A parabola with equation $(y-y_1)^2 = 4a(x-x_1)$ will have its vertex on the point (x_1, y_1)

Key point

An ellipse with equation $\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} = 1$ will be centred on the point (x_1, y_1) and have radius of a in the x-direction and b in the y-direction.

Key point

A rectangular hyperbola with equation $(x-x_1)(y-y_1) = c^2$ will be centred on the point (x_1, y_1) and have asymptotes $x = x_1$, $y = y_1$
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	Key point A hyperbola with equation $\frac{(x-x_1)^2}{a^2}-\frac{(y-y_1)^2}{b^2}=1$ will be centred on the point (x_1, y_1) and have asymptotes $y-y_1=\pm\frac{b}{a}(x-x_1)$
How can you rotate conics by multiples of 90°?	**Key point** To rotate a conic by $\frac{\pi}{2}$ radians, replace x by y and y by $-x$ To rotate a conic by π radians, replace x by $-x$ and y by $-y$ To rotate a conic by $\frac{3\pi}{2}$ radians, replace x by $-y$ and y by x *This relates nicely to matrices.*