DD - Linear Programming

<table>
<tr>
<td>How should you structure a linear programming problem?</td>
<td>
1. State the the variables and what they represent.
    ■ Eg, “let x = the number of frogs, y = the number of rocks”.
2. Find constraints (including x, y ≥ 0).
3. Find the objective function.
4. Write the conclusion.
    ■ “Maximise/minimise [objective function] subject to”
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<td>What is a basic variable under simplex?</td>
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A variable that has a 1 in their row and zeros in the rest of their column.

An example is shown below:
<table>
<tr><th>Basic variable</th><th>P</th><th>x</th><th>y</th><th>s</th><th>t</th><th>Value</th><th>Row</th></tr>
<tr><td></td><td>1</td><td>-2</td><td>0</td><td>0</td><td>4</td><td>2000</td><td>$R4 = R1 + 8 \times R6$</td></tr>
<tr><td>s</td><td>0</td><td>$2\frac{1}{2}$</td><td>0</td><td>1</td><td>$-1\frac{1}{2}$</td><td>750</td><td>$R5 = R2 - 3 \times R6$</td></tr>
<tr><td>y</td><td>0</td><td>$\frac{1}{2}$</td><td>1</td><td>0</td><td>$\frac{1}{2}$</td><td>250</td><td>$R6 = \frac{R3}{2}$</td></tr>
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<td>How is the Simplex Algorithm performed? (with example)</td>
<td>
1. Write the constraints and objective functions as equations in standard form using slack variables.
2. Transfer (1) to a simplex tableau where the slack variables form the basis.
3. Choose column with most negative coefficient in object row. This is the PIVOT COLUMN.
    ■ This generally tells you the corresponding variables have to be increased to the optimal solution.
4. Choose the row giving the smallest θ-value. This is the PIVOT.
    ■ This is found by dividing the each value by the positive numbers in the pivot column.
5. Divide the pivot row by the pivot
6. Combine suitable multiples of the new row with other rows to make all other values zeros in the pivot column.
    ■ This replaces the basic variable, eg, it may go from s to x.
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7. If there are **NO NEGATIVE** coefficients in the object row, the solution is optimal. Otherwise go to step 3.

<u>A DETAILED EXAMPLE IS SHOWN BELOW:</u>

$$\begin{aligned} &\text{Maximise} && P = 6x + 8y \\ &\text{subject to} && 4x + 3y \leq 1500 \\ & && x + 2y \leq 500 \\ & && x \geq 0, y \geq 0 \end{aligned}$$

Becomes…

$$\begin{aligned} &\text{Maximise} && P = 6x + 8y \\ &\text{subject to} && 4x + 3y + s = 1500 \\ & && x + 2y + t = 500 \\ & && x \geq 0, y \geq 0, s \geq 0, t \geq 0 \end{aligned}$$

Hence...

$P - 6x - 8y - 0s - 0t = 0$

$4x + 3y + s + 0t = 1500$

$x + 2y + 0s + t = 500$

and the simplex tableau is

Basic variable	P	x	y	s	t	Value	Row
	1	−6	−8	0	0	0	R1
s	0	4	3	1	0	1500	R2
t	0	1	2	0	1	500	R3

And thus the pivot is chosen…

Pivot column

Basic variable	P	x	y	s	t	Value	Row
	1	−6	−8	0	0	0	R1
s	0	4	3	1	0	1500	R2
Pivot row → t	0	1	$\boxed{2}$	0	1	500	R3

Pivot

Since the ϑ-value for s = 1500 / 3 = 500 and t = 500 / 2 so the final row is chosen.

Dividing by the pivot row by the pivot and getting rid of the remaining numbers in the pivot column yields…

Basic variable	P	x	y	s	t	Value	Row
	1	–2	0	0	4	2000	$R4 = R1 + 8 \times R6$
s	0	$2\frac{1}{2}$	0	1	$-1\frac{1}{2}$	750	$R5 = R2 - 3 \times R6$
y	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	250	$R6 = \frac{R3}{2}$

Repeating the change of basis for the x-column. We get…

Basic variable	P	x	y	s	t	Value	Row
	1	0	0	$\frac{4}{5}$	$2\frac{4}{5}$	2600	$R7 = R4 + 2 \times R8$
x	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	300	$R8 = \frac{R5}{2\frac{1}{2}}$
y	0	0	1	$-\frac{1}{5}$	$\frac{4}{5}$	100	$R9 = R6 - \frac{1}{2} \times R8$

You now have $x = 300$, $y = 100$, $s = 0$, $t = 0$, giving $P = 2600$

How are solutions for the Simplex Algorithm given?

With the basis variables equalling the value of the row and the rest equalling 0 as shown:

Step 6

B.V.	P	x	y	z	s	t	u	Value	Row
	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	44	$R9 = R5 + \frac{R11}{3}$
y	0	0	1	$\frac{4}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	16	$R10 = R6 - \frac{R11}{3}$
x	0	1	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0	12	$R11 = \frac{R7}{1\frac{2}{3}}$
u	0	0	0	$3\frac{2}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$	1	18	$R12 = R8 - R11$

Step 7 There are no negative numbers in the top row so the solution is optimal.

The optimal solution is $y = 16$, $x = 12$, $u = 18$, $z = 0$, $s = 0$, $t = 0$, giving $P = 44$

When can the standard Simplex Algorithm be used?

1. You need to maximise the object function.
2. Every non-trivial constraint is an inequality using ≤.
3. All variables are ≥ 0 (including the slack).
4. The origin is a vertex of the feasible region.

How can you apply the standard Simplex Algorithm when ‘the objective function is being minimised’ or ‘an inequality involves ≥ instead’?

Key point

Minimising the objective function C is equivalent to maximising the objective function $P = -C$

1.
2. You rewrite the inequality using ≤ by multiply through by -1.

Both of these are applied below:

Example 3

Minimise $C = 3x - 4y - 3z$

subject to $x + y - z \geq -2$

$x + 2y + z \leq 3$

$x \geq 0, y \geq 0, z \geq 0$

Maximise $P = -3x + 4y + 3z$

subject to $-x - y + z \leq 2$

$x + 2y + z \leq 3$

$x, y, z \geq 0$

First restate the problem.

$P + 3x - 4y - 3z = 0$

$-x - y + z + s = 2$

$x + 2y + z + t = 3$

$x, y, z, s, t \geq 0$

Write in standard form with slack variables.

P	x	y	z	s	t	Value	Row
1	3	−4	−3	0	0	0	R1
0	−1	−1	1	1	0	2	R2
0	1	2	1	0	1	3	R3

Complete the simplex tableau.