Matrices

Basics/Introduction

- The order of a matrix is denoted as $n \times m$ where n = row no. & m = col. no.
- When adding or subtracting, the matrices have to be of the same order.

- A zero matrix is a matrix where all elements = 0.
- For matrix multiplication, of $n_1 \times m_1$ & $n_2 \times m_2$. $m_1 = n_2$ & resulting matrix order is $n_1 \times m_2$.
- There is no such thing as matrix division:

$$4 \times 5 = 20 \Rightarrow 20 \div 4 = 5 \quad \underline{\text{or}} \quad 20 \times \tfrac{1}{4} = 5$$

- You have to use inverses, $4^{-1} = 1/4$.
- Dividing is essentially multiplying by an inverse.

Multiplication

- By convention, matrix multiplication is done as...

$$\begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 2\times -1 + -2 \times 7 & \ldots \\ \ldots & \ldots \end{bmatrix}$$

first row × first column = first row, first column.
first row × second column = first row, second column
...

- In general, $AB \neq BA$ (where A & B are matrices) unless it involves the identities or in other cases.
- Matrix multiplication is associative ∴ $A(BC) = (AB)C$ as transformations are applied in reverse order.
- It is also distributive where $A(B+C) = AB + AC$.

Transposing

- Transposing works as...
$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 7 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 5 \end{bmatrix}$$

Determinant & Inverses

- Written as $|A|$, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$.

- Multiplying a matrix by its inverse gives the identity matrix:

$$\begin{bmatrix} 5 & 2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be found by...

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

make these negative to this

flip these to this

- If a matrix A has $|A| = 0$, there is no inverse. In simultaneous equations... no solutions!

Basic Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$ stretch in x-direction by s.f. k.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$ stretch in y-direction by s.f. k.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$ enlargement centre $(0, 0)$ s.f. k.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ rotation form.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$ reflection form.

Examples of Rotations

① $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\cos\theta = 0 \Rightarrow \theta = 90°$ & $\sin\theta = 1 \Rightarrow \theta = 90°$

$\Rightarrow$ anticlockwise rotation of $90°$ centre $(0, 0)$

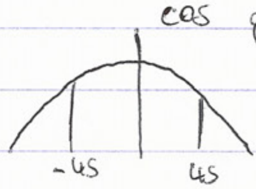
Reflections (Examples)

① $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$\cos 2\theta = 0 \Rightarrow 2\theta = 90 \Rightarrow \theta = 45$

$\sin 2\theta = -1 \Rightarrow 2\theta = -90 \Rightarrow \theta = -45$

cos graph:



$\Rightarrow y = \tan(-45)x = -x$

② $y = 3x \Rightarrow \tan\theta = 3 \Rightarrow \theta = 71.565... \Rightarrow 2\theta = 143.13...$

$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} -4/5 & 6/10 \\ -6/10 & -4/5 \end{bmatrix}$

Composite Transformations

- If A represents a 2×2 transformation & B is another. $B \cdot A$ represents 'A followed by B' OR $C \cdot B \cdot A$ is 'A followed by B followed by C'

$\because$ $f(g(x))$ is also read backwards $\Rightarrow$ matrix compositions are too.

Matrix Composition Transformation Example

① A: $y = \frac{1}{\sqrt{3}} x \Rightarrow \theta = 30$

$$A = \begin{bmatrix} 0.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & -0.5 \end{bmatrix}$$

B: $y = \sqrt{3} x \Rightarrow \theta = 60$

$$B = \begin{bmatrix} -0.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0.5 \end{bmatrix}$$

$$B \circ A = \begin{bmatrix} 0.5 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0.5 \end{bmatrix}$$ of A followed by B

Which is also a rotation of $60°$ anticlockwise centre $(0,0)$

② Enlargement & rotation $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

$k\cos\theta = -1$ then $-k\sin\theta = -1$

$\Rightarrow k\sin\theta = 1$

$\tan\theta = \frac{k\sin\theta}{k\cos\theta} = \frac{-1}{1} \Rightarrow \theta = -45°$

$k\sin -45 = 1 \Rightarrow k = 1/\sin -45 = -\sqrt{2}$

enlargement of s.f. $-\sqrt{2}$ & rotation $45°$ clockwise

Faster Method of Transforming

$$\begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix}\begin{array}{c} \begin{array}{cccc} O & F & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & 3 & 0 & 3 \end{bmatrix}\end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 0 & 0 & 9 & 9 \\ 0 & 15 & 0 & 15 \end{bmatrix} \\ \begin{array}{cccc} O & F' & D' & E' \end{array}\end{array}$$

Determinants

- area of image = area of original $\times$ $|\det(T)|$
- If $\det(T) > 0$, the orientation is preserved.
- A 3×3 determinant can be found as follows:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

watch out!

Delete all the rows & columns associated with a then multiply by the minors

- An example is below:

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{pmatrix} = 0\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - 0 + 1(2+1)$$

$$= 3$$

- If $\det(A) = 0$, A is a singular matrix.

Inverses & Equations

- Inverse of A is A^{-1} $\Rightarrow$ $A^{-1}A = AA^{-1} = I$
- Useful for simultaneous equations

Inverses of 3x3

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\det A = 2\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} + -3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 2(-1 - -4) - 1(-3) + -3(6)$$

$$= 2(3) + 3 - 18 = 9 - 18 = -9$$

eliminate the element & find the minors

$$\begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 6 \\ -5 & -2 & -4 \\ 1 & -5 & -1 \end{bmatrix}$$

THEN TRANSPOSE

$$\Rightarrow \begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{9}\begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -1/3 & 5/9 & -1/9 \\ -1/3 & 2/9 & 5/9 \\ -2/3 & 4/9 & 1/9 \end{bmatrix}$$

Systems of Equations

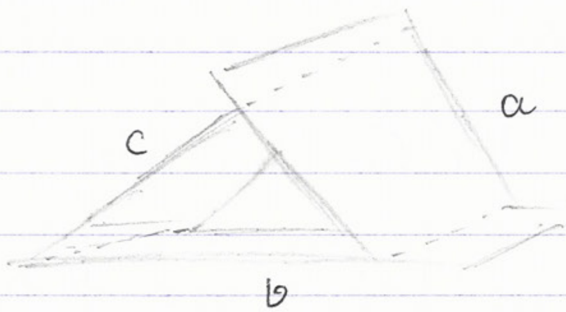
$$\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{pmatrix} 11 \\ 6 \end{pmatrix} = \frac{-1}{14} \begin{pmatrix} -1 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 11 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ -16/7 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

If 3 planes don't have a single point of intersection, they are <u>inconsistent</u>.

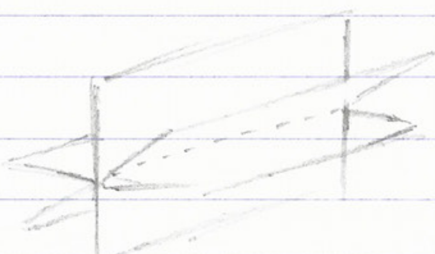
Two parallel planes

Triangular prism (where 2 planes add like vectors to the other)

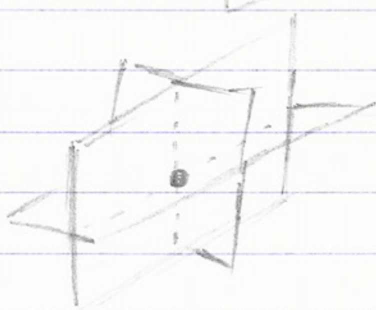


$a + b = c$

<u>OTHERWISE</u>



a sheaf along which are infinitely many solutions



a single point of intersection where there is <u>exactly</u> one solution.

(Ignore what the equation equals when checking if two are parallel)

If the det = 0, there are no solutions or infinitely many solutions (a sheaf).

3D Transformations

Reflections in $x = 0$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

" " $y = 0$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

" " $z = 0$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(These are not in the formula book)

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + y\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + z\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Invariant Lines & Points

- If every point on a line is mapped to another point on the same line, it is an invariant line.

YOU USE $Y = mx + c$ FOR LINES

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$ OR USE $Y = mx' + c$

$x + mx + c = x'$

$3(mx + c) = y'$

THEN WRITE IN $y' = mx' + c$ FORM

$3(mx + c) = m(x + mx + c) + c$

$3mx + 3c = mx + m^2 x + mc + c$

$2mx - m^2 x - mc + 2c = 0$

$(2m - m^2)x - mc + 2c = 0$

EQUAT BOTH SIDES

$2m - m^2 = 0 \Rightarrow m = 0$ or 2

$2c - mc = 0 \Rightarrow m = 0,\ c = 0.$ **or** $m = 2,\ c =$ any value

i.e. substituting 2 into m gives $2c - 2c = 0$

$\Rightarrow$ line $y = 0$ & $y = 2x + c$

- If T (transformation) & x (vector), $Tx = x$ then x is an invariant point.
- Example:

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$2x + 3y = x \Rightarrow x - 3y$

$-x - 2y = y \Rightarrow x = -3y$

WHICH gives a line of invariant points of $x = -3y$

OR

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$2x + 2y = 0$

$y - x = 0$

$2x + y = 0$

$-2x + 2y = 0 \quad +$

$3y = 0 \Rightarrow y = 0 \Rightarrow x = 0$

3Blue1Brown's Videos

Basics

- Transformations are functions with vector inputs & outputs.
- A whole grid moves under a transformation, imagine every point on the plane moving to a new point.

- Linear : • all lines remain lines & the origin is fixed under the transformations
  • grid lines remain parallel & evenly spaced too

- Transformations are based of $\hat{i}$ & $\hat{j}$, if you know where they go after a transformation, you know where every point goes.

If $\hat{i} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ & $\hat{j} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ then for any vector:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

The transformation can be described as...

$$\begin{array}{cc} \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \\ \hat{i} \quad \hat{j} \end{array}$$

why matrix multiplication is this way

- In the general case : $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} a \\ c \end{bmatrix}$ is where the first base vector lands & $\begin{bmatrix} b \\ d \end{bmatrix}$ is where the last base vector lands.

Rotations - A 90° anticlockwise rotation

$\hat{i}$ lands on $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & $\hat{j}$ lands on $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Shear - A shear moves $\hat{j}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\Rightarrow$ shearing everything.

- $\hat{i}$ remains at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ the transformation is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Compositions - They are one single action <u>rather</u> than 2 successive ones.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{composition}}$$

$\longleftarrow$

read right to left $\Rightarrow$ rotation followed by shear or $f(g(x))$ is read right to left (inner most)

- You break it down in multiplication:

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ for column 1}$$

& $$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ for column 2}$$

- This isn't commutative ∵ $90°$ ac → shear ≠ shear → $90°$ ac
- Yet, is associative ∵
- $(AB)C = A(BC)$, you're still doing $C \to B \to A$

Determinant - Under the transformation, $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, a 1×1 block's area has increased $6\times$.
- The determinant is by how much the area of a shape is changed under a transformations.
- If $\det(A) = 0$, its been squished onto a line.
- If $\det(A) < 0$, $\hat{i}$ & $\hat{j}$ flip.

Inverses - $A\underset{\sim}{x} = \underset{\sim}{v}$ where A = transformation causing $\underset{\sim}{x}$ to land on $\underset{\sim}{v}$.

$$A^{-1}A = I \implies \underset{\times A^{-1}}{I\underset{\sim}{x} = A^{-1}\underset{\sim}{v}}$$

which transformation $\underset{\sim}{v}$ back onto $\underset{\sim}{x}$