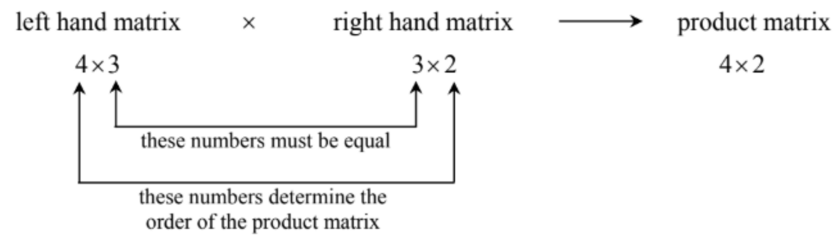
CC - Matrices

When is matrix multiplication allowed?	The column number of the left matrix = row number right matrix.  You say they're **conformable to multiplication**.
What is the determinant of a 2x2 matrix?	$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
What is det(AB) equal to?	det(**A**) x det(**B**)
How do you find the inverse of a 2x2 matrix?	If… $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ Then… $$A^{-1} = \frac{1}{\lvert A \rvert} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
Describe the order of composite transformations	$$C \cdot B \cdot A$$ Described as A followed by B followed by C.
What does the determinant represent?	The change in area/volume of a shape under the transformation.
When is orientation preserved under a transformation? What does it mean when it isn't?	When… $$det(T) > 0$$

<table>
<tr><td></td><td>Less than 0 means some reflection is involved in the transformation.</td></tr>
<tr><td>What is the determinant of a 3x3 matrix?</td><td>

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

You can also used any other row / column yet have to follow these +'s and -'s.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

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<tr><td>When is a matrix singular?
When is this useful?</td><td>

When…

$$det(A) = 0$$

Can be used to see whether a system of equations has a solution.

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<tr><td>How do you find the inverse of a 3x3 matrix?</td><td>

1. Calculate the determinant.
2. Find the minor of each value with alternating +'s and -'s.
3. Transpose this matrix of cofactors.
4. Divide by the determinant.

eliminate the element & find the minors

$$\begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 6 \\ -5 & -2 & -4 \\ 1 & -5 & -1 \end{bmatrix}$$

THEN TRANSPOSE

$$\Rightarrow \begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{9}\begin{bmatrix} 3 & -5 & 1 \\ 3 & -2 & -5 \\ 6 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -1/3 & 5/9 & -1/9 \\ -1/3 & 2/9 & 5/9 \\ -2/3 & 4/9 & 1/9 \end{bmatrix}$$

Cofactors are denoted by capital letters:

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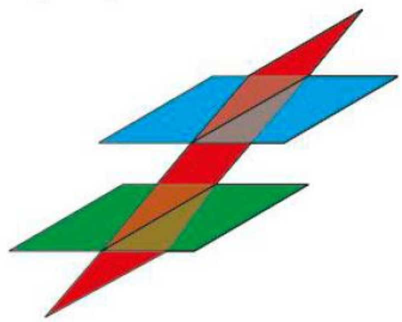
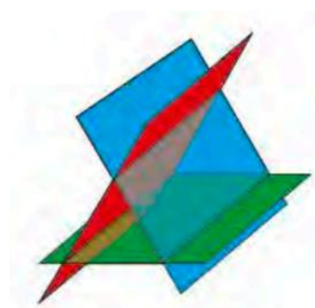
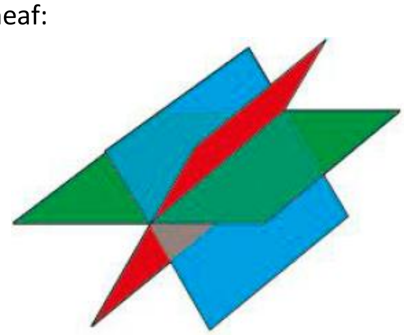
<tr><td></td><td>$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$'s cofactor matrix is $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$
Hence why | matrix | = $a_1A_1 + b_1B_1 + c_1C_1$ yet B_1 is negative.</td></tr>
<tr><td>**When are equations inconsistent in ‘systems of equations’?**</td><td>When they don’t have a unique point of intersection.</td></tr>
<tr><td>**Give the 3 cases where systems of equations have no unique point and the conditions of each**</td><td>1. When two of the planes are parallel:

Easy to check if any 2 are parallel.

2. Form a triangular prism:

Eliminating 1 variable will show they are inconsistent.

3. They form a sheaf:

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What are invariant points?	Points which remain unchanged under a transformation.
What is a line of invariant points?	A whole line of points, each which remains unchanged under a transformation.
Describe what happens to invariant lines under a transformation	When every point on the line is mapped to **ANOTHER** point on the same line.
How can you find invariant points / lines of invariant points OR invariant lines?	Invariant points / lines of points: $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$ Invariant lines: $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} x' \\ mx' + c \end{bmatrix}$$
What is $(AB)^T$ equal to?	$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
What is $\det(A^{-1})$ equal to? How can this be proved?	**Key point** For any square, non-singular matrix $\mathbf{A}$: $\det(\mathbf{A}^{-1}) \equiv \frac{1}{\det(\mathbf{A})}$ Can be proved using the fact $\det(\mathbf{AA}^{-1}) = \det(\mathbf{I})$
What is $(AB)^{-1}$ equal to? Why?	$\mathbf{B}^{-1}\mathbf{A}^{-1}$ since... **Example 6** Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ $(AB)^{-1}(AB) = I$ $(AB)^{-1}AB = I$ — Since matrix multiplication is associative. $(AB)^{-1}ABB^{-1} = IB^{-1}$ — Post-multiply both sides of the equation by B^{-1} $(AB)^{-1}A = B^{-1}$ — Since $BB^{-1} = I$ and $IB^{-1} = B^{-1}$ $(AB)^{-1}AA^{-1} = B^{-1}A^{-1}$ — Post-multiply both sides of the equation by A^{-1} $(AB)^{-1} = B^{-1}A^{-1}$ *as required* — Since $AA^{-1} = I$
What is $\det(M^T)$ equal to?	$\det(\mathbf{M})$

What row operations can be carried out on a matrix determinant? What effect can these have?	• No effect on determinant value:
	  ◦ Adding or subtracting any multiple of a row to another row or column to another column.
	• Changes sign of determinant value:
	  ◦ Swapping two rows or two columns.

	- Changes determinant value by scalar: - Multiplying/dividing a row or column by a scalar will multiply/divide (respectively) the determinant by that same scalar. *Thus to ensure the value isn't changed, you will have to add a minus if swapping or multiply when dividing or dividing when multiplying.*
How can you that something is a factor of a determinant?	To show (x - y) is a factor of the determinant, we substitute x = y into it and show that the determinant is now equal to zero. $$\begin{vmatrix} x & y & z \\ yz & zx & xy \\ y+z & x+z & x+y \end{vmatrix}$$ Becomes… $$\begin{vmatrix} x & y & z \\ yz & yz & y^2 \\ y+z & y+z & 2y \end{vmatrix}$$ Which becomes (when col 1 - col 2): $$\begin{vmatrix} 0 & y & z \\ 0 & yz & y^2 \\ 0 & y+z & 2y \end{vmatrix} = 0$$ *Hence, we've also shown when 2 columns or rows are equal, the determinant is always 0.*
What is an eigenvector and an eigenvalue?	- Eigenvector - a vector whose direction is maintained under a transformation. - Eigenvalue - the value by which the eigenvector is scaled under that transformation. This satisfies the equation **Ax** = λ**x**.

What is the characteristic equation for an eigenvector? And how is it derived?	You can rearrange the equation $\mathbf{Ax} = \lambda\mathbf{x}$ to give $\mathbf{Ax} - \lambda\mathbf{Ix} = 0$ since $\mathbf{x} = \mathbf{Ix}$ then factorise to give $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ Since $\mathbf{x}$ is a non-zero vector, it must be the case that the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular. Therefore $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ **Key point** The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is the **characteristic equation** of **A** and is used to find the eigenvalues.
How is a matrix diagonalised? What is important here?	**Key point** A matrix, **M**, can be **diagonalised** by finding **P** and **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. It can be shown that: • **D** is a diagonal matrix with the eigenvalues of M along the leading diagonal • **P** is a matrix where the columns are the eigenvectors of M The eigenvectors in the columns of **P** must occur in the same order as their corresponding eigenvalues in **D** *The equation above comes from* $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$. *This works because we first apply transformation P which turns our basis vectors into the eigenvectors. Now these eigen basis vectors are scaled using the transformation **M** (which is responsible for the eigenvectors in **P**). Their direction doesn't change. Now they are transformed back into our basis vectors of [(1, 0) (0,1)]. This is identical to purely having the eigenvalues (the values by which they're scaled) as the basis vectors.*
How is diagonalisation of a matrix useful? (with derivation)	If $\mathbf{A} = \mathbf{PDP}^{-1}$ then $\mathbf{A}^n = (\mathbf{PDP}^{-1})^n$ $= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\ldots(\mathbf{PDP}^{-1})$ $= (\mathbf{PD})(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\ldots(\mathbf{P}^{-1}\mathbf{P})(\mathbf{DP}^{-1})$ $= (\mathbf{PD})\mathbf{DD}\ldots\mathbf{D}(\mathbf{DP}^{-1})$ $= \mathbf{PD}^n\mathbf{P}^{-1}$ Since matrix multiplication is associative. Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$ **Key point** If $\mathbf{A} = \mathbf{PDP}^{-1}$ then $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ You can use this to solve problems involving $\mathbf{A}^n$, since $\mathbf{D}^n = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$ You can use proof by induction to prove this result.
How do eigenvectors relate to lines of invariant points?	From **Tx** = λ**x** … If a transformation given by a matrix **T** has an eigenvalue of 1 then the corresponding eigenvectors determine the direction of a line of invariant points through the origin.

How is a shear parallel to one axis represented?	$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$ Left is a shear parallel to the x-axis. Right is a shear parallel to the y-axis.