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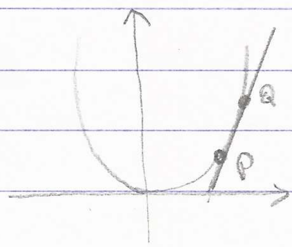
Differentiation and Integration

25-10-17 (Introduction)



The chord join P & Q has a gradient that gets closer and closer to the tangent at P as Q grows closer to P.

Let $P = (1, 1)$ & $Q = (1 + dx, (1+dx)^3)$ for $y = x^2$

$$\Rightarrow m = \frac{\Delta y}{\Delta x} = \frac{(1+dx)^2 - 1}{1 + dx - 1} = \frac{\cancel{1} + 2dx + (dx)^2 \cancel{-1}}{\cancel{1} + \cancel{dx} \cancel{-1}} = 2 + dx$$

so as $2 + dx$ is the m & $\lim\limits_{dx \to 0} 2 + dx = 2$

(this can be done for a general case by using x instead of 1)

$y = x^3 \Rightarrow P(x, \overset{x^3}{y})$ to $Q(x+h, (x+h)^3)$

$$m = \frac{\Delta y}{\Delta x} = \frac{(x+h)^3 - x}{\cancel{x} + h \cancel{-x}} = \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}2} \cancel{-x^3}}{\cancel{h}}$$
$$= 3x^2 + 3xh + h^2$$

$\text{Limit}_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$ as h approaches zero, the gradient at the point approaches $3x^2$

Format: $\lim\limits_{dx \to 0} \frac{dy}{dx} = \frac{dy}{dx}$ OR $f'(x) = \frac{dy}{dx}$

General Case: $y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$ for each term in a polynomial

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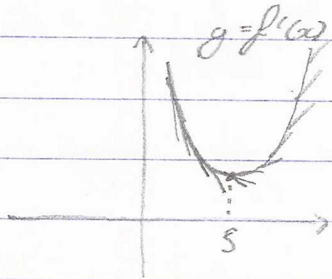
Stationary Points

- The idea is to find $f'(x)$ for a polynomial <u>AND</u> as the gradient of any stationary point $= 0$, solve $f'(x) = 0$

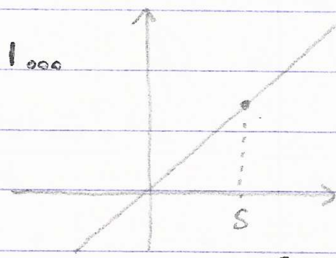
Second Derivative Test

- Use to find if a point is a maximum or minimum.
- If $\frac{d^2y}{dx^2} < 0$, maximum point ~~of~~ if $\frac{d^2y}{dx^2} > 0$, minimum point.
- This works as shown below:



As you approach the minimum point, the gradient increases from negative to positive. Thus $y = f''(x)$ can be plotted as:

$-3 \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \ldots$



And as any point here has the same positive gradient of a point,

The converse argument is used for maximum points (gradient goes positive to negative $\Rightarrow \frac{d^2y}{dx^2}$ will have a negative slope with same gradient at all points)

It doesn't work if it $= 0$, it could be a max or point of inflection. You don't know what's happening!

Gradient Either Side Test

- Always test both sides to be sure!
- You can devise a method by logic.

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Integration Introduction

also called antiderivatives

Indefinite integrals can be found using $\frac{dy}{dx} = ax^n \Rightarrow y = \frac{a}{n+1}x^{n+1} + C$

Increase power by 1 & divide coefficient by power.

- E.g., $\int x^3 \, dx = \frac{1}{4}x^4 + c$

Read as "the integral of x^3 with respect to x"

important as you don't know the final value removed under differentiation

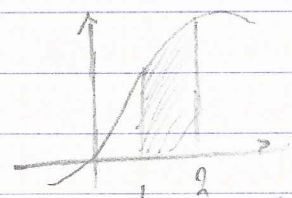
Definite integrals $\left(\int_a^b f(x)\,dx\right)$ find the area under the curve between a & b.

Fundamental Theorem of Calculus

$$\int_1^2 8x^3 + 6x + 2 \, dx = \left[2x^4 + 3x^2 + 2x + c\right]_1^2$$

the $+c$ doesn't matter here as it will be cancelled anyway.

$= 48 - 7 = 41$ units^2



OR more generalised $\int_a^b f'(x)\,dx = f(b) - f(a)$

& $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

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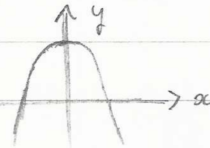
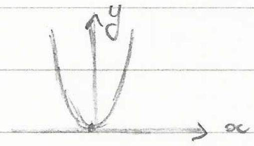
A2 Differentiation & Curve Sketching

- If $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave



- If $\frac{d^2y}{dx^2} > 0 \Rightarrow$ convex

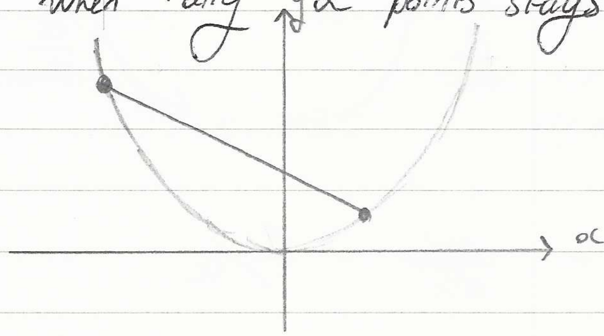


- If $\frac{d^2y}{dx^2} = 0 \Rightarrow$ point of inflection

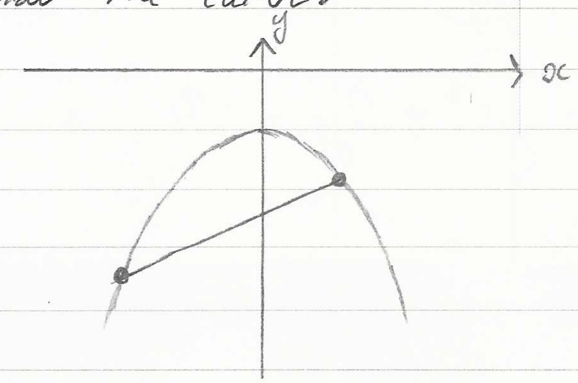
- A point of inflection is when a graph changes from concave to convex or vice versa (i.e., a change of concavity).

Note:
- a point of inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ YET not vice versa
- If $f''(x) = 0$ and $f'(x) \neq 0$ at some value x then it is definitely a point of inflection
- If $f''(x) = 0$ and $f'(x) = 0$ ~~then~~ at some value x then check the second derivative either side for a change in concavity

- Convex is when a line connecting any 2 points stays above above the curve:



- Concave is when a line connecting any 2 points stays below the curve:



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Product Rule

$$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= uv' + vu'$$

Quotient Rule

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{vu' - uv'}{v^2}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Differentiating common functions

This only works in radians as the derivation is based on small angle approximation.

$$\begin{tikzcd}
& \sin x \arrow[dr, "\frac{d}{dx}"] & \\
-\cos x \arrow[ur, "\frac{d}{dx}"] & & \cos x \arrow[dl, "\frac{d}{dx}"] \\
& -\sin x \arrow[ul, "\frac{d}{dx}"] &
\end{tikzcd}$$

$$y = \ln x, \quad \frac{dy}{dx} = 1/x$$

Inverse Differentiation

$$\left(\frac{dy}{dx}\right)^{-1} = \frac{dx}{dy}$$

E.g. $x = y^3 + 2y + 4 \quad \therefore \frac{dx}{dy} = 3y^2 + 2$

$$\therefore \frac{dy}{dx} = \frac{1}{3y^2+2}$$

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Differentiating arcsin x

$y = \arcsin x$

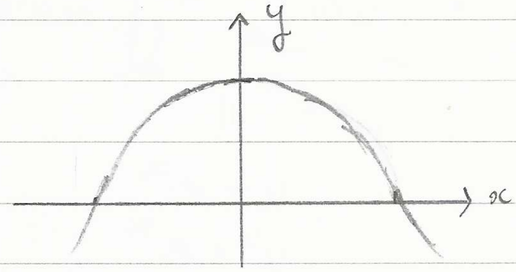
$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$\Rightarrow \frac{dy}{dx} \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

We take the positive root $\because$ $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$



$-\frac{\pi}{2} \leq y \leq \pi/2$

$0 \leq \cos y \leq 1$ $\therefore$ always positive

Differentiating $y = a^x$

$y = a^x$

$\ln y = x \ln a$

$y = e^{x \ln a}$, let $u = \ln a \Rightarrow y = e^{ux}$

$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = ue^{ux}$

$= \ln a \, e^{\ln a \, x}$

A2

Differentiating $\ln x$

- $\frac{d}{dx}(\ln x) = \frac{1}{x} \Leftrightarrow \int \frac{1}{x}\,dx = \ln|x| + C$

Derivation for integration by parts

$$y = uv \Rightarrow y' = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx$$

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\therefore \int uv'\,dx = uv - \int vu'\,dx$$

<u>OR</u> $$\int_a^b uv'\,dx = \left[uv\right]_a^b - \int_a^b vu'\,dx$$

Special case?

- $\int \overset{u}{e^x}\,\overset{v'}{\cos(x)}\,dx$ $\quad u = e^x \quad v = \sin(x)$ $\quad u' = e^x \quad v' = \cos(x)$

$$= e^x\sin(x) - \int \overset{u}{e^x}\,\overset{v'}{\sin(x)}\,dx \qquad u = e^x \quad v = -\cos x \quad u' = e^x \quad v' = \sin x$$

$$= e^x\sin(x) - \left[e^x \cdot -\cos(x) - \int -\cos(x)e^x\,dx\right.$$

$$= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$

$$\therefore 2\int e^x\cos(x)\,dx = e^x\left[\sin(x) + \cos(x)\right]$$

$$\int e^x\cos(x)\,dx = \frac{1}{2}e^x\left[\sin(x) + \cos(x)\right]$$

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Implicit differentiation example

$$x^2 + y^2 + 2x - 7 = 0$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 + \frac{d}{dx}2x + \frac{d}{dx}(-7) = \frac{d}{dx}0$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} + 2 = 0$$

$$2x + 2y\frac{dy}{dx} + 2 = 0$$

$$x + y\frac{dy}{dx} + 1 = 0$$

$$\frac{dy}{dx} = \frac{-x-1}{y}$$ (Function of both x & y)

For a tangent with gradient 1 $\therefore$ $\frac{-x-1}{y} = 1$ $\therefore$ $y = -x-1$
(a relationship between the 2 variables)