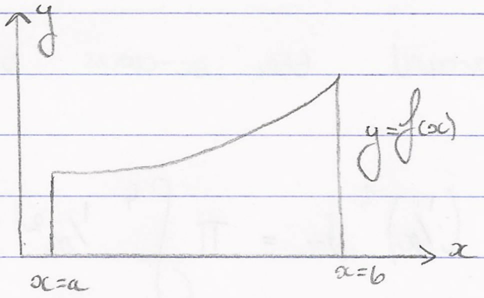
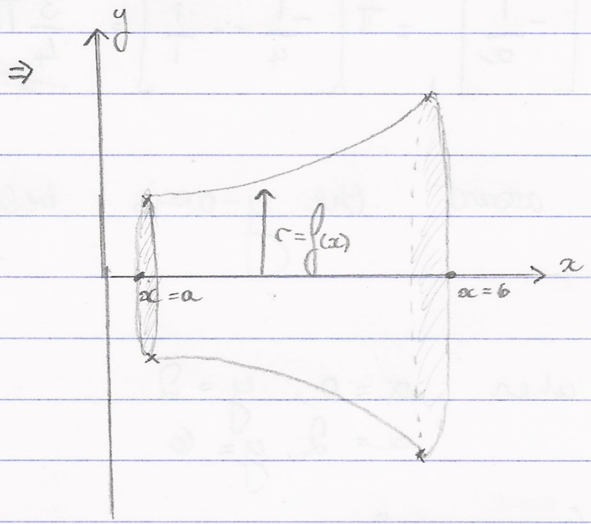
Volumes of Revolution & Mean Values

Derivation of Volumes of Revolution



Imagine rotating $y = f(x)$ around the x-axis.



To calculate the volume of the shape formed, you could split the curve into an infinite number of cylinders.

Using $\pi r^2 h$, $r = f(x)$ at that point & h is the h of a cylinder. Let $h = \delta x$ (some small value of x.

$\Rightarrow$ volume of one cylinder $= \pi (f(x))^2 \delta x$ however, you want to sum all the cylinders between $x=a$ & $x=b$

$$\Rightarrow \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi (f(x))^2 \delta x = \int_a^b \pi (f(x))^2 dx$$

$$= \int_a^b \pi y^2 dx$$

$$= \pi \int_a^b y^2 dx$$

Applying the same logic rotating around the y-axis, you get:

$$\pi \int_a^b x^2 dy$$

Example around x-axis

Rotating $y = 1/x$ $360°$ around the x-axis between $x=1$ & $x=4$

$$y = 1/x \Rightarrow V = \pi\int_1^4 (1/x)^2\,dx = \pi\int_1^4 1/x^2\,dx = \pi\int_1^4 x^{-2}\,dx$$

$$= \pi\left[-x^{-1}\right]_1^4 = \pi\left[-\frac{1}{x}\right]_1^4 = \pi\left[\frac{-1}{4} - -\frac{1}{1}\right] = \underline{\underline{\frac{3}{4}\pi \text{ units}^3}}$$

Example around y-axis

Rotating $y = x^2+2$ $360°$ around the y-axis between $x=0$ and $x=2$

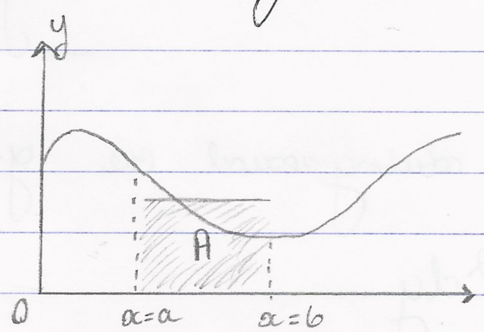
$\Rightarrow x = \sqrt{y-2}$ & when $x=0,\ y=2$
$x=2,\ y=6$

$$\Rightarrow \pi\int_a^b x^2\,dy \Rightarrow \pi\int_2^6 (\sqrt{y-2})^2\,dy$$

$$= \pi\int_2^6 y-2\,dy = \underline{8\pi \text{ units}^3}$$

Mean Values with (Example)

Essentially about finding a rectangle with the same area as the limits on the curves. The rectangle has the base of the limits.



E.g., $f(x) = t^2(3t+1)(t-2)$ between $1 \le t \le 3$

$$\int_1^3 f(x)\,dt = 418/5 \text{ units}^2$$

base $= 2 \Rightarrow$ height $= \frac{418}{15} \div 2 = \frac{209}{15}$