# CB - Complex Numbers

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| **Give 3 different forms of a complex number** | $z = x + yi$<br>$= r(\cos(\theta) + i\sin(\theta))$<br>$= re^{i\theta}$ |
| **Prove $\lvert z_1z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$** | $(1)\ \lvert z_1z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$<br>$\arg(z_1z_2) = \arg z_1 + \arg z_2$<br>***NOTE:*** Multiplication rotates $z_1$ by $\arg z_2$<br>***Proof:*** let $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$<br>$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$<br>$= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2)$<br>$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$<br>$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$<br>$\therefore \lvert z_1z_2\rvert = r_1r_2$ $\quad$ $\arg(z_1z_2) = \theta_1 + \theta_2$<br>$= \lvert z_1\rvert\lvert z_2\rvert$ $\quad$ $= \arg z_1 + \arg z_2$<br>*Note, it follows that…*<br>$\lvert z_1z_2z_3\dots z_n\rvert = \lvert z_1\rvert\lvert z_2\rvert\lvert z_3\rvert\dots\lvert z_n\rvert$<br>$\arg(z_1z_2z_3\dots z_n) = \arg z_1 + \arg z_2 + \arg z_3 + \dots + \arg z$<br>$\left\lvert\frac{z_1z_2}{z_3z_4}\right\rvert = \frac{\lvert z_1\rvert\lvert z_2\rvert}{\lvert z_3\rvert\lvert z_4\rvert}$<br>$\arg\left(\frac{z_1z_2}{z_3z_4}\right) = \arg z_1 + \arg z_2 - \arg z_3 - \arg z_4$<br>$(3)\ \lvert z^n\rvert = \lvert z\rvert^n$<br>$\arg(z^n) = n\arg z$ |
| **What does De Moivre's Theorem state?** | If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ for all $n \in R$ |

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| | de Moivre's Theorem states that if $z=r(\cos\theta+i\sin\theta)$ then $z^n=r^n(\cos n\theta+i\sin n\theta)$ for all n $\in\mathbb{R}$ |
| **How can you express sin(nθ) and cos(nθ) in powers of sinθ and cosθ?** | 1. Rewrite sin(nθ) and cos(nθ) as [sin(θ) and cos(θ)]$^n$ using De Moivre's Theorem.<br>2. Expand the brackets.<br>3. Equate real and imaginary parts.<br>4. Use any identities, if necessary.<br><br>Express $\cos 4\theta$ in terms of $\cos\theta$<br><br>de Moivre's Theorem<br>if $z=(\cos\theta+i\sin\theta)$<br>then $z^n=(\cos n\theta+i\sin n\theta)$<br><br>$\cos 4\theta + i\sin 4\theta \equiv (\cos\theta + i\sin\theta)^4$<br>$\equiv (c+is)^4$<br>$\equiv c^4 + 4c^3(is)^1 + 6c^2(is)^2 + 4c^1(is)^3 + (is)^4$<br>$\equiv c^4 - 6c^2s^2 + s^4 + i(4c^3s - 4cs^3)$<br>$\equiv c^4 - 6c^2(1-c^2) + (1-c^2)^2 + i(4c^3s - 4cs^3)$<br>$\equiv c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 + i(4c^3s - 4cs^3)$<br>$\equiv 8c^4 - 8c^2 + 1 + i(4c^3s - 4cs^3)$<br>Compare Re: $\cos 4\theta \equiv 8\cos^4\theta - 8\cos^2\theta + 1$<br>Note! Im: $\sin 4\theta \equiv 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ |
| **How can sin(nθ) and cos(nθ) be expressed in terms of z = cos(θ) + isin(θ) and e's?** | If $z = \cos\theta + i\sin\theta$<br><br>then<br><br>$$z_n + z^{-n} = z^n + \frac{1}{z^n} = 2\cos n\theta$$<br>$$\left[z_n - z^{-n} = z^n - \frac{1}{z^n} = 2i\sin n\theta\right]$$<br>and<br><br>$$z + z^{-1} = z + \frac{1}{z} = 2\cos\theta$$<br>$$z - z^{-1} = z - \frac{1}{z} = 2i\sin\theta$$<br>Thus… |

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$$\cos x = \mathrm{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2},$$
$$\sin x = \mathrm{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}.$$

*This can be shown using De Moivre's and the fact that cosine is odd and sine is even.*

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**How can you express $\sin^n\theta$ and $\cos^n\theta$ in terms of $\sin(n\theta)$ or $\cos(n\theta)$?**

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1. Use the fact that $2\cos\theta = z + z^{-1}$ or $2i\sin\theta = z - z^{-1}$
2. Raise both sides the required power
3. Use the fact that $2\cos(n\theta) = z^n + z^{-n}$ or $2i\sin(n\theta) = z^n - z^{-n}$

Express $\cos^3\theta$ in the form $a\cos 3\theta + b\cos\theta$

$\boxed{z^n + z^{-n} = z^n + \frac{1}{z^n} = 2\cos n\theta}$ $\boxed{z + z^{-1} = z + \frac{1}{z} = 2\cos\theta}$

$\boxed{(a+b)^n \equiv {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^0 b^n}$

$$(2\cos\theta)^3 \equiv (z + z^{-1})^3$$
$$\equiv {}^3C_0 (z)^3 (z^{-1})^0 + {}^3C_1 (z)^2 (z^{-1})^1 + {}^3C_2 (z)^1 (z^{-1})^2 + {}^3C_3 (z)^0 (z^{-1})^3$$
$$\equiv z^3 + 3z + 3z^{-1} + z^{-3}$$
$$\equiv (z^3 + z^{-3}) + 3(z + z^{-1})$$
$$\therefore 8\cos^3\theta \equiv 2\cos 3\theta + 3(2\cos\theta)$$
$$\therefore \cos^3\theta \equiv \frac{2}{8}\cos 3\theta + \frac{6}{8}\cos\theta$$
$$\therefore \cos^3\theta \equiv \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

Express $\sin^4\theta$ in terms of the form $\cos k\theta$

$\boxed{z^n + z^{-n} = z^n + \frac{1}{z^n} = 2\cos n\theta \quad z^n - z^{-n} = z^n - \frac{1}{z^n} = 2i\sin n\theta}$ $\boxed{z + z^{-1} = z + \frac{1}{z} = 2\cos\theta \quad z - z^{-1} = z - \frac{1}{z} = 2i\sin\theta}$

$\boxed{(a+b)^n \equiv {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_n a^0 b^n}$

$$(2i\sin\theta)^4 \equiv (z - z^{-1})^4$$
$$\equiv {}^4C_0 (z)^4(-z^{-1})^0 + {}^4C_1 (z)^3(-z^{-1})^1 + {}^4C_2 (z)^2(-z^{-1})^2 + {}^4C_3 (z)^1(-z^{-1})^3 + {}^4C_4 (z)^0(-z^{-1})^4$$
$$\equiv z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$
$$\equiv (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6$$
$$16\sin^4\theta \equiv 2\cos 4\theta - 8\cos 2\theta + 6$$
$$\therefore \sin^4\theta \equiv \frac{2}{16}\cos 4\theta - \frac{8}{16}\cos 2\theta + \frac{6}{16}$$
$$\therefore \sin^4\theta \equiv \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

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**What 2 results arise from the roots of unity?**

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1. The sum of all roots to zero (by symmetry of vectors)
2. If $z_2 = \omega$ then multiplying it by itself gives $\omega^2 = z_3$
    - Since it rotates the complex number (keeping magnitude the same) as $\arg(z^n) = n\arg(z)$

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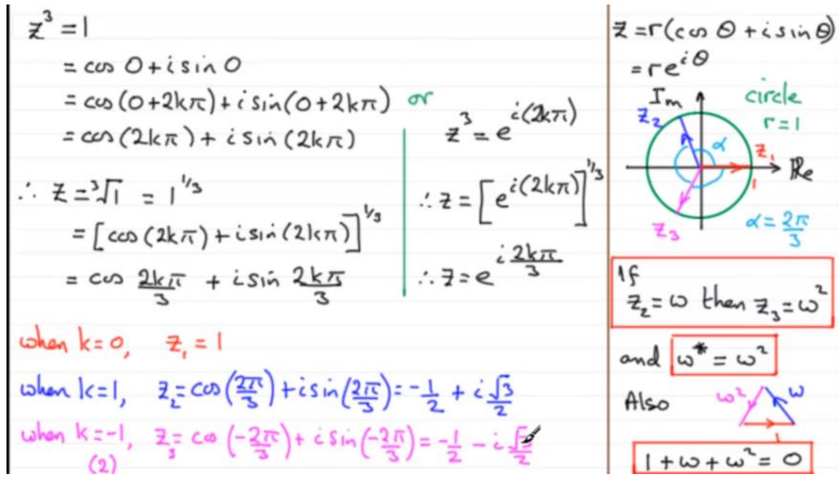

$z^3 = 1$

$= \cos 0 + i \sin 0$

$= \cos(0+2k\pi) + i\sin(0+2k\pi)$ or $z^3 = e^{i(2k\pi)}$

$= \cos(2k\pi) + i\sin(2k\pi)$

$\therefore z = \sqrt[3]{1} = 1^{1/3}$

$= [\cos(2k\pi) + i\sin(2k\pi)]^{1/3}$

$= \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$

$\therefore z = \left[e^{i(2k\pi)}\right]^{1/3}$

$\therefore z = e^{i\frac{2k\pi}{3}}$

when $k=0$, $z_1 = 1$

when $k=1$, $z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

when $k=-1$ (2), $z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

Im, Re, $z_1$, $z_2$, $z_3$, circle $r=1$, $\alpha$, $\alpha = \frac{2\pi}{3}$

$\boxed{\text{If } z_2 = \omega \text{ then } z_3 = \omega^2}$

and $\boxed{\omega^* = \omega^2}$

Also $\omega^2$, $\omega$

$\boxed{1 + \omega + \omega^2 = 0}$
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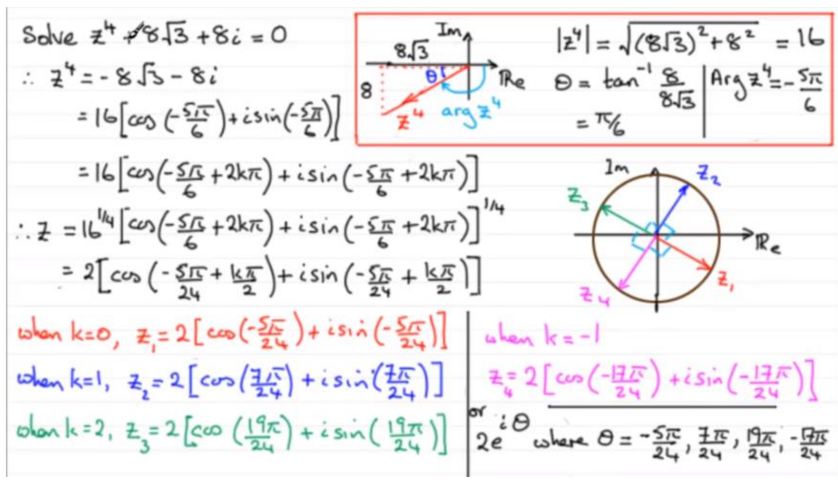
<tr><td>How do you find the nth root of a complex number?</td><td>

1. Set $z^n$ = (complex number)
2. Add 2kπ (or 2kπi if Euler's Formula is used) to the complex number
3. nth root both sides
4. Find the value for z for different values of k



Solve $z^4 + 8\sqrt{3} + 8i = 0$

$\therefore z^4 = -8\sqrt{3} - 8i$

$= 16\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$

$|z^4| = \sqrt{(8\sqrt{3})^2 + 8^2} = 16$

$\theta = \tan^{-1}\frac{8}{8\sqrt{3}} = \pi/6$

$\text{Arg}\, z^4 = -\frac{5\pi}{6}$

Im, Re, $8\sqrt{3}$, 8, $\theta$, $z^4$, $\arg z^4$

$= 16\left[\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right]$

$\therefore z = 16^{1/4}\left[\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right]^{1/4}$

$= 2\left[\cos\left(-\frac{5\pi}{24} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{5\pi}{24} + \frac{k\pi}{2}\right)\right]$

Im, Re, $z_1$, $z_2$, $z_3$, $z_4$

when $k=0$, $z_1 = 2\left[\cos\left(-\frac{5\pi}{24}\right) + i\sin\left(-\frac{5\pi}{24}\right)\right]$

when $k=1$, $z_2 = 2\left[\cos\left(\frac{7\pi}{24}\right) + i\sin\left(\frac{7\pi}{24}\right)\right]$

when $k=2$, $z_3 = 2\left[\cos\left(\frac{19\pi}{24}\right) + i\sin\left(\frac{19\pi}{24}\right)\right]$

when $k=-1$

$z_4 = 2\left[\cos\left(-\frac{17\pi}{24}\right) + i\sin\left(-\frac{17\pi}{24}\right)\right]$

or $2e^{i\theta}$ where $\theta = -\frac{5\pi}{24}, \frac{7\pi}{24}, \frac{19\pi}{24}, -\frac{17\pi}{24}$
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<tr><td>How a circle loci written using complex numbers?</td><td>

$|z - z_1| = r$ where $z_1$ is the centre and $r$ the radius
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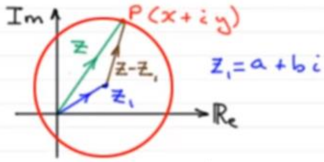
$$\boxed{\begin{array}{c}\text{Circle, centre } z_1, \text{ radius } r \\ |z - z_1| = r\end{array}}$$


$|z - 3 + 2i| = 4$<br>
$\therefore |z - (3 - 2i)| = 4$<br>
$\therefore$ centre $(3, -2)$, radius 4<br>
<br>
$|4 + 5i - z| = 3$<br>
$\therefore |-1(z - 4 - 5i)| = 3$<br>
$\therefore |-1||z - 4 - 5i| = 3$<br>
$\therefore |z - (4 + 5i)| = 3$<br>
centre $(4, 5)$, radius 3<br>
<br>
Find the cartesian equation<br>
$|x + iy - (a + bi)| = r$<br>
$\therefore |(x - a) + i(y - b)| = r$<br>
$\therefore \sqrt{(x-a)^2 + (y-b)^2} = r$<br>
$\therefore (x - a)^2 + (y - b)^2 = r^2$
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<td>How is a perpendicular bisector loci written using complex numbers?</td>
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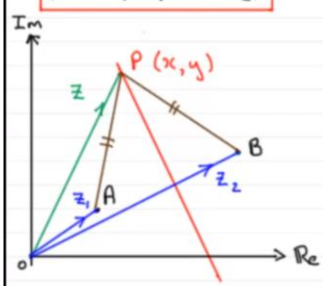
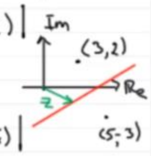
$|z - z_1| = |z - z_2|$ where $z_1$ and $z_2$ are the fixed points<br>
<br>
The locus of a point which is equidistant from two fixed points A and B in the complex plane is the perpendicular bisector of A and B.<br>
$$\boxed{|z - z_1| = |z - z_2|}$$


Sketch the locus of $z$ and find the cartesian equation for which $|z-3-2i| = |5-3i-z|$<br>
$\therefore |z - (3 + 2i)| = |-1(z - 5 + 3i)|$<br>
$\therefore |z - (3 + 2i)| = |z - (5 - 3i)|$<br>


let $z = x + iy$<br>
$|x + iy - (3 + 2i)| = |x + iy - (5 - 3i)|$<br>
$\therefore |(x-3) + i(y-2)| = |(x-5) + i(y+3)|$<br>
$\therefore \sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x-5)^2 + (y+3)^2}$<br>
$\therefore \cancel{x^2} - 6x + 9 + \cancel{y^2} - 4y + 4 = \cancel{x^2} - 10x + 25 + \cancel{y^2} + 6y + 9$<br>
$\therefore 4x - 10y - 21 = 0$
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<td>How is the half-line loci written using complex numbers?</td>
<td>$arg(z - z_1) = \theta$ which represents a half-line from the fixed point $z_1$ making an angle $\theta$ with a line from the fixed point $z_1$ parallel to the real axis.</td>
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<td>How does z and z* relate on an Argand Diagram and in mod-arg form?</td>
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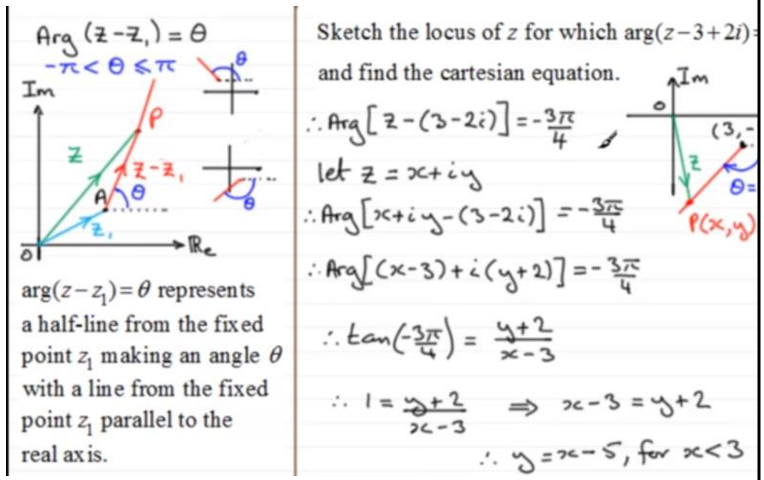
- z and z* are mirror images in the real axis.




$$z = x + iy,\ \ z^* = x - iy$$
$$\Rightarrow z = \cos(\theta) + i\sin(\theta)$$
$$\Rightarrow z^* = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \cos(\theta) - i\sin(\theta)$$

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