Bounds on the Number of the Perfect Rainbow Matchings in Balanced Bipartite Graphs

Dr. Zur Luria, Inbar Shmaya

January 2022

Abstract

A subgraph H of a given edge-colored graph is *rainbow* if the edge colors are all distinct. How many rainbow perfect matchings are there in a given balanced bipartite properly colored graph? This is a #P-hard problem and so it is unlikely that there exists an efficient solution. Instead, we seek tight bounds on this number.

Our main result is the following upper bound: Let $G = \langle V, E\rangle$ be a balanced bipartite graph with n vertices and f edges, and let $c : E \to \{1, ..., k\}$ be a proper edge-coloring of G in k colors. Then the number of rainbow perfect matchings in G is at most

$$\binom{k}{n} \cdot \left(\frac{1+o(1)}{e^2}\right)^n \cdot \left(\frac{f}{k}\right)^n$$

In order to prove this upper-bound we use the entropy method, which was already used to give a tight upper bound on the number of transversals in a Latin square.

1 Introduction

A Latin square is an $n \times n$ matrix containing the numbers $1, ..., n$, where every number appears exactly once in each row and column. A Latin matrix is an $n \times n$ matrix where every row and column contains n distinct numbers from a set containing m numbers with no repetitions ($n \leqslant m$). A transversal in an $n \times n$ matrix is a set of n cells, one from each row and column, containing distinct numbers.

In 1967, Ryser[1] made the following conjecture:

Conjecture 1.1 *There is a transversal in any Latin square of odd-order.*

Ryser's conjecture is a main open question in Combinatorics. In order to get closer to a solution, it is natural to seek generalizations. One way to do this

[1]H. J. Ryser, Neuere Probleme der Kombinatorik, Vortrage uber Kombinatorik Ober-Wolfach, Mathematisches Forschungsinstitut Oberwolfach, pp. 24-29 (1967)

is to find sufficient conditions for a Latin matrix to have a transversal. To this end, in 1991, Erdős and Spencer[2] proved the following theorem:

Theorem 1.2 *Let M be an $n \times n$ Latin matrix in which each distinct value appears at most $\frac{n-1}{16}$ times. Then M contains a transversal.*

An edge-coloring of a given graph is proper if every pair of edges that share a vertex have different colors. A subgraph of a given edge-colored graph is *rainbow* if its colors are distinct. Latin matrices are equivalent to proper edge-colorings of the complete bipartite graph $K_{n,n}$. In this representation, a transversal corresponds to a rainbow perfect matching. Ryser's conjecture is equivalent to the claim that every properly colored $K_{n,n}$ with n colors (when n is odd) has a rainbow perfect matching.

An edge-coloring of a graph is X-bounded if the number of edges with a certain color is at most X. Erdős and Spencer's result is equivalent to the statement that every $\frac{n-1}{16}$-bounded coloring of $K_{n,n}$ has a rainbow perfect matching. In 2019, Coulson and Perarnau[3] generalized this statement to the class of Dirac graphs. They showed that there exists a constant $\mu > 0$ such that every μn-bounded coloring of a Dirac graph has a rainbow perfect matching.

In this paper, we are interested in the number of such rainbow matchings. Our main result is the following upper bound on the number of rainbow matchings in a properly colored balanced bipartite graph.

Theorem 1.3 *Let $G = \langle V, E \rangle$ be a balanced bipartite graph with n vertices and f edges, and let $c : E \to \{1, ..., k\}$ be a proper edge-coloring of G in k colors. Then the number of rainbow perfect matchings in G is at most*

$$\binom{k}{n} \cdot \left(\frac{1+o(1)}{e^2}\right)^n \cdot \left(\frac{f}{k}\right)^n$$

Theorem 1.4 *With high probability, a uniformly random coloring of $K_{n,n}$ has at least*

$$\binom{k}{n} \cdot \left(\frac{1-o(1)}{e^2}\right)^n \cdot \left(\frac{n^2}{k}\right)^n$$

rainbow perfect matchings.

In section 2, we show that this theorem is a consequence of two lemmas. In sections 3 and 4, we prove these lemmas.

2 Proof of Theorem 1.3

Let T be the set of all of G's rainbow matchings. For a subset $C \subset \{1, ..., k\}$ of size n, let T_C be the set of rainbow matchings in G that use precisely the

[2] P. Erdős, J. Spencer, Lopsided Lovász Local Lemma and Latin transversals, Discrete Applied Mathematics 30, 151-154 (1991)

[3] M. Coulson, G. Perarnau, Rainbow matchings in Dirac bipartite graphs, ResearchGate, Random Structures and Algorithms (2019)

colors in C. Clearly, T is the disjoint union of all of the T_C's, and therefore $|T| = \sum_{C\in\binom{[k]}{n}} |T_C|$.

For $z \in \{1, ..., k\}$, let A_z be the set of edges in G whose color is z, and let $a_z = |A_z|$. Our first lemma upper bounds $|T_C|$ in terms of the numbers a_z.

Lemma 2.1

$$|T_C| \leq \left(\frac{1+o(1)}{e^2}\right)^n \prod_{z\in C} a_z.$$

As a consequence of this lemma, we get the bound

$$|T| \leq \sum_{C\in\binom{[k]}{n}} \left(\frac{1+o(1)}{e^2}\right)^n \prod_{z\in C} a_z.$$

This formula is easy to write down. Unfortunately, actually computing the above sum would take exponential time, and so it has limited practical use from a computational point of view.

We define the function $F : (\mathbb{R}^+)^k \to \mathbb{R}$ by

$$F(x_1, ..., x_k) = \sum_{C\in\binom{[k]}{n}} \prod_{z\in C} x_z.$$

Note that our bound can be stated in terms of F. We have $|T| \leq \left(\frac{1+o(1)}{e^2}\right)^n \cdot F(a_1, ..., a_k)$. Furthermore, the number a_z satisfy the equation $\sum_{z=1}^k a_z = f$. Our second lemma establishes a global maximum for F subject to this constraint.

Lemma 2.2 *The maximal value of F subject to the constraint $\sum_{z=1}^k x_z = f$ is attained when $x_1 = ... = x_k = \frac{f}{k}$.*

The lemma immediately implies Theorem 1.3:

$$\begin{aligned}
|T| &\leq \left(\frac{1+o(1)}{e^2}\right)^n \cdot F(a_1, ..., a_k) \\
&\leq \left(\frac{1+o(1)}{e^2}\right)^n \cdot \sum_{C\in\binom{[k]}{n}} \prod_{z\in C} \left(\frac{f}{k}\right) \\
&= \binom{k}{n} \cdot \left(\frac{1+o(1)}{e^2}\right)^n \cdot \left(\frac{f}{k}\right)^n.
\end{aligned}$$

3 Proof of Lemma 2.1

The entropy of a random variable X is the measure of the eff-ective size of its probability space. Let X be a random variable with the distribution $p_1, p_2, ..., p_n$, so its entropy is:

$$H(X) = \sum_i p_i \cdot \log\left(\frac{1}{p_i}\right).$$

In this proof, we assume that the base of the log function above is e.

Given random variables X and Y, $H(Y|X)$ is the amount of information X has given the value of Y is already known. So according to the chain rule:

$$H(X,Y) = H(X) + H(Y|X)$$

Let X be a random variable that is uniformly distributed on T_C, so its entropy is:

$$H(X) = log\big(|T_C|\big).$$

For each $z \in C$, let X_z be the edge in X of color z. We have:

$$H(X) = H(X_z : z \in C).$$

We associate each color $z \in C$ with a random variable $\alpha_z \sim U\big((0,1]\big)$, all independent. The X_zs are revealed in descending order of α_z. Let N_z be the number of legal options for a given X_z after every $X_{z'}$ has been revealed for $z' \in C$ and $\alpha_{z'} > \alpha_z$.

$$\begin{aligned}
H(X) &= \sum_z H\left(X_z | X_{z'} : \alpha_{z'} > \alpha_z\right) \\
&= \sum_z \mathop{\mathbb{E}}_{X_{z'}:\alpha_{z'}>\alpha_z} \left[H\left(X_z\right)|X_{z'} = e\right] && \text{(For a random variable } Z\text{: } H(Z) \le \log\left(|Range(Z)|\right)) \\
&\le \sum_z \mathop{\mathbb{E}}_{X_{z'}:\alpha_{z'}>\alpha_z} \left[\log\left(N_z\right)\right] && (N_z \text{ depends only on } z'\text{s that come before } z) \\
&= \sum_z \mathop{\mathbb{E}}_X \left[\log\left(N_z\right)\right] \\
&= \mathop{\mathbb{E}}_X \left[\sum_z \log\left(N_z\right)\right] \\
&\le \mathop{\mathbb{E}}_X \left[\sum_{z\in C} \mathop{\mathbb{E}}_{\alpha_z} \left[log(N_z)\right]\right] \\
&\le \mathop{\mathbb{E}}_X \left[\sum_{z\in C} \mathop{\mathbb{E}}_{\alpha_z} \left[log\left(\mathop{\mathbb{E}}_{\alpha_{z'}:z'\ne z} [N_z]\right)\right]\right] \\
&= \mathop{\mathbb{E}}_X \left[\sum_{z\in C} \mathop{\mathbb{E}}_{\alpha_z} \left[log\left(\sum_{\{i,j\}:\{i,j\}\in A_z} Pr\left(\{i,j\} \text{ is legal for } X_z\right)\right)\right]\right]
\end{aligned}$$

The first inequality follows from Jensen's inequality and the fact that $\log(x)$ is concave. There are two cases:

1. $X_z = \{i,j\} \Rightarrow Pr\left(\{i,j\} \text{ is legal for } X_z\right) = 1.$

2. $X_z \neq \{i,j\} \Rightarrow$ There are two edges in X with colors in C sharing the vertices i, j individually: $\{i, j'\}, \{i', j\} \in E(G)$. Let the colors of the edges $\{i, j'\}, \{i', j\}$ be: z_1, z_2 respectively. Note that $z_1 \neq z_2$ since X is rainbow.

   The probability of the color z to be chosen before the color z_1 and the probability of the color z to be chosen before the color z_2 are both equal to α_z, and the associated events are independent. Therefore:

$$\begin{aligned}
&Pr\left(\{i,j\} \text{ is legal for } X_z\right) \\
&= Pr\left(\alpha_z > \alpha_{z_1} \wedge \alpha_z > \alpha_{z_2}\right) \\
&= Pr\left(\alpha_z > \alpha_{z_1}\right) \cdot Pr\left(\alpha_z > \alpha_{z_2}\right) \\
&= \alpha_z^2
\end{aligned}$$

Hence:

$$\mathop{\mathbb{E}}_{\alpha_{z'}:z' \neq z} [N_z] = 1 + (a_z - 1) \cdot {\alpha_z}^2$$

Assigning this result in the equation above yields:

$$H(X) \leq \sum_{z \in C} \mathop{\mathbb{E}}_{\alpha_z} \left[\log\left(1 + (a_z - 1) \cdot {\alpha_z}^2\right)\right]$$

The variable α_z is a continuous random variable between 0 and 1. For convenience, in what follows we denote α_z by x:

$$\mathop{\mathbb{E}}_{\alpha_z} \left[\log\left(1 + (a_z - 1) \cdot {\alpha_z}^2\right)\right] = \int_0^1 log\left(1 + (a_z - 1) \cdot x^2\right)\, dx$$

In order to compute the integral above, we assume that the base of the logarithm function is e, hence: $\log_e \equiv \ln$.

In addition, we divide the solving of it into two cases:

1. $a_z = 1$
2. $a_z > 1$

Assigning $a_z = 1$ in the integral above leads to:

$$\begin{aligned}
&\int_0^1 \ln\left(1 + (1 - 1) \cdot x^2\right)\, dx \\
&= \int_0^1 \ln(1)\, dx = \int_0^1 0\, dx = 0\Big|_0^1 = 0
\end{aligned}$$

For the second case:

$$
\begin{aligned}
&\int_0^1 \ln\left(1+(1-a_z)\cdot x^2\right)\, dx \\
&= \begin{bmatrix} u = \ln\left(1+(a_z-1)\cdot x^2\right) & dv = 1 \\ du = \dfrac{2\cdot(a_z-1)\cdot x}{(a_z-1)\cdot x^2+1} & v = x \end{bmatrix} \\
&= x\cdot\ln\left(1+(a_z-1)\cdot x^2\right)\Big|_0^1 - \int_0^1 \frac{2\,(a_z-1)\cdot x^2}{(a_z-1)\cdot x^2+1}\,dx \\
&= \left[1\cdot\ln\left(1+(a_z-1)\cdot 1^2\right)\right] - \left[0\cdot\ln\left(1+(a_z-1)\cdot 0^2\right)\right] - 2\,(a_z-1)\cdot\int_0^1 \frac{x^2}{(a_z-1)\cdot x^2+1}\,dx \\
&= \ln(a_z) - 2\,(a_z-1)\cdot\int_0^1 \frac{x^2}{(a_z-1)\cdot x^2+1}\,dx \qquad (1)
\end{aligned}
$$

Now focus on the integral (1):

$$
\int_0^1 \frac{x^2}{(a_z-1)\cdot x^2+1}\,dx
$$

We can also write x^2 as:

$$
x^2 = \frac{a_z-1}{a_z-1}\cdot x^2 + \frac{1}{a_z-1} - \frac{1}{a_z-1} = \frac{(a_z-1)\cdot x^2+1}{a_z-1} - \frac{1}{a_z-1}
$$

Now assign:

$$
\begin{aligned}
&\int_0^1 \frac{x^2}{(a_z-1)\cdot x^2+1}\,dx \\
&= \int_0^1 \left(\frac{\frac{1}{a_z-1}\left((a_z-1)\cdot x^2+1\right)}{(a_z-1)\cdot x^2+1} - \frac{\frac{1}{a_z-1}}{(a_z-1)\cdot x^2+1} \right) dx \\
&= \int_0^1 \frac{\frac{1}{a_z-1}\left((a_z-1)\cdot x^2+1\right)}{(a_z-1)\cdot x^2+1}\,dx - \int_0^1 \frac{\frac{1}{a_z-1}}{(a_z-1)\cdot x^2+1}\,dx \\
&= \int_0^1 \frac{1}{a_z-1}\,dx - \int_0^1 \frac{\frac{1}{a_z-1}}{(a_z-1)\cdot x^2+1}\,dx \\
&= \frac{x}{a_z-1}\Big|_0^1 - \frac{1}{a_z-1}\cdot\int_0^1 \frac{1}{(a_z-1)\cdot x^2+1}\,dx \\
&= \frac{1}{a_z-1} - \frac{1}{a_z-1}\cdot\int_0^1 \frac{1}{(a_z-1)\cdot x^2+1}\,dx \qquad (2)
\end{aligned}
$$

Now focus on the integral (2):

$$\begin{aligned}
&\int_0^1 \frac{1}{(a_z-1)\cdot x^2+1}\,dx \\
&= \left[u=\sqrt{a_z-1}\cdot x \qquad \frac{du}{dx}=\sqrt{a_z-1}\right] \\
&= \int_0^{\sqrt{a_z-1}} \frac{1}{(u^2+1)}\cdot\frac{1}{\sqrt{a_z-1}}\,du \\
&= \frac{1}{\sqrt{a_z-1}}\cdot\int_0^{\sqrt{a_z-1}} \frac{1}{(u^2+1)}\,du \\
&= \frac{1}{\sqrt{a_z-1}}\cdot \arctan(u)\Big|_0^{\sqrt{a_z-1}} \\
&= \left[\frac{1}{\sqrt{a_z-1}}\cdot\arctan(\sqrt{a_z-1})\right]-\left[\frac{1}{\sqrt{a_z-1}}\cdot\arctan(0)\right] \\
&= \frac{1}{\sqrt{a_z-1}}\cdot\arctan(\sqrt{a_z-1})
\end{aligned}$$

Assign the result in (2):

$$\frac{1}{a_z-1}-\frac{1}{a_z-1}\cdot\frac{\arctan(\sqrt{a_z-1})}{\sqrt{a_z-1}}$$

Assign the result in (1):

$$\begin{aligned}
&\ln(a_z)-2\,(a_z-1)\cdot\left(\frac{1}{a_z-1}-\frac{1}{a_z-1}\cdot\frac{\arctan(\sqrt{a_z-1})}{\sqrt{a_z-1}}\right) \\
&= \ln(a_z)-2\cdot\left(1-\frac{\arctan(\sqrt{a_z-1})}{\sqrt{a_z-1}}\right) \\
&= \ln(a_z)-2+\frac{2\cdot\arctan(\sqrt{a_z-1})}{\sqrt{a_z-1}}
\end{aligned}$$

This result implies:

$$\begin{aligned}
H(X) &\le \sum_{z\in C}\left(\ln(a_z)-2+\frac{2\cdot\arctan(\sqrt{a_z-1})}{\sqrt{a_z-1}}\right) \\
&= \log_e\left(\prod_{z\in C}\frac{a_z}{e^2}\right)
\end{aligned}$$

And therefore:

$$|T_C| \le \prod_{z\in C}\frac{a_z}{e^2}=\left(\frac{1+o(1)}{e^2}\right)^n\prod_{z\in C}a_z.$$

4 Proof of Lemma 2.2

Our task is to solve the following optimization problem:

$$\begin{cases} \text{Maximize:} \quad F(x_1, ..., x_k) = \sum\limits_{C \in \binom{[k]}{n}} \prod_{z \in C} x_z \\ \text{subject to:} \quad g\left(x_1, ..., x_k\right) = 0, \quad \text{where } g(x_1, ..., x_k) = \left(\sum\limits_{z=1}^{k} x_z\right) - f \end{cases}$$

The Lagrange multiplier theorem states that the equation $\nabla F(x) = \lambda \nabla g(x)$ is a necessary condition for x to be a local extremum.

We first compute the local derivatives of F with respect to x_i:

$$\frac{\partial F}{\partial x_i} = \sum_{C \in \binom{[k] \setminus i}{n-1}} \prod_{z \in C} x_z.$$

Clearly, ∇g is the all ones vector, so $\nabla F = \lambda \nabla g$ at a point x if and only if $\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_j}$ at x for all $i, j \in [k]$.

Consider the difference $\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}$:

$$\begin{aligned} \frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j} &= \sum_{C \in \binom{[k] \setminus i}{n-1}} \prod_{z \in C} x_z - \sum_{C \in \binom{[k] \setminus j}{n-1}} \prod_{z \in C} x_z \\ &= (x_j - x_i) \sum_{C \in \binom{[k] \setminus \{i,j\}}{n-2}} \prod_{z \in C} x_z. \end{aligned}$$

Since by definition all the xs are positive, the sum on the right is clearly positive [4]. Hence the above expression is zero if and only if $x_i = x_j$. Since this must hold for every pair $i, j \in [k]$, there is a unique suspicious point: $x_1 = ... = x_k = \frac{f}{k}$. In order to classify this suspicious point, we use the following theorem.

Theorem 4.1 *Let x be a suspicious point of F subject to the constraint $g(x) = 0$, and let H be the Hessian of F. If for every $\vec{v} \in \mathbb{R}^k$ such that $\nabla g \cdot \vec{v} = 0$ there holds $\vec{v}^T H \cdot \vec{v} < 0$ then x is a local maximum of F.*

At the suspicious point where all x_z's are equal to $\frac{f}{k}$, there are two possibilities for the second order derivative of F with respect to its variables:

1. $\frac{\partial^2 F}{\partial^2 x_i} = 0.$

2. $\frac{\partial^2 F}{\partial x_i \partial x_j} = \sum_{C \in \binom{[k] \setminus \{i,j\}}{n-2}} \prod_{z \in C} x_z = \binom{k-2}{n-2} \left(\frac{f}{k}\right)^{n-2}$, for $i \neq j$.

[4] Let's redefine the domain of F such that the x's are all positive. Make sure this doesn't screw up the rest of the proof.

Let $\beta = \binom{k-2}{n-2}\left(\frac{f}{k}\right)^{n-2}$. Hence, H is β times the $k \times k$ matrix whose main diagonal is zero and the rest of its cells are equal to 1.

$$H = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{bmatrix}$$

The only vectors that satisfy $\nabla g \cdot \vec{v} = 0$ are precisely those whose coordinates sum to 0.

$$\vec{v} = (v_1, ..., v_k) \qquad\qquad \sum_{i=1}^{k} v_i = 0$$

According to theorem 4.1, it should holds $\vec{v}^T H \cdot \vec{v} < 0$ for x to be a local maximum of F. For the H matrix above and $\vec{v}$:

$$(H\vec{v})_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \vec{v} = \left(\sum_{j=1}^{k} v_j\right) - v_i = -v_i$$

Therefore:

$$H\vec{v} = (-v_1, ..., -v_k) = -\vec{v}$$

So for $\vec{v} \neq \vec{0}$:

$$\vec{v}^T H\vec{v} = \vec{v}^T(-\vec{v}) = -\left(\vec{v}^T\vec{v}\right) = -|\vec{v}|^2 < 0$$

We see that there is one and only one local maximum, which establishes a global maximum for F. The maximal value of F subject to the constraint $\sum\limits_{z=1}^{k} x_z = f$ is attained when: $x_1 = ... = x_k = \frac{f}{k}$.

5 References