Linear Algebra

Incalculas

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Contents

1 Vector Space

1.1 What is a Vector Space?

Vector space is a set with a binary operation

$$+ : V \times V \to V$$

Key properties of a vector space:

- *i) Commutativity*

$$A + B = B + A$$

- *ii) Associativity*

$$A + (B + C) = (A + B) + C$$

- *iii) Identity*

$$\exists 0 \in V$$

$$\text{s.t } v + 0 = v, \forall v \in V$$

- *iv) Inverse*

$$\forall v \in V, \exists v' \in V$$

$$\text{s.t } v + v' = 0$$

A set of scalars associated with a vector space is a Field

1.2 Definition of Fields

Definition 1.1 (Fields)

$$(F, +, \cdot)$$

- $(F, +)$ Satifies i) to iv) [an abelian group]
- $(F - \{0\}, \cdot)$ Satifies i) to iv) [an abelian group]
- $\cdot$ is distributive over $+$

Example 1.2 (Binary field)

$$\{0, 1\}$$

Addition:

$$0 + 1 = 1 = 1 + 0$$

$$1 + 1 = 0$$

Multiplication:

$$1 \cdot 0 = 0 = 0 \cdot 1$$

$$1 \cdot 1 = 1$$

$$0 \cdot 0 = 0$$

What a field does to a vector in a vector space:
$\gamma, \mu \in F, v, w \in V$

$$\gamma \cdot (v + w) = (\gamma v + \gamma w)$$

$$(\gamma + \mu) v = \gamma v + \mu v$$

$$\mu(\gamma \cdot v) = (\mu\gamma)v$$

Properties of a vector space V over field F:
$u, v, w \in V$ and $a, b \in F$

1) $u + v = w = v + u$
2) $u + (v + w) = (u + v) + w$
3) $\exists 0 \in V$, s.t $0 + v = v$
4) $\exists v' \in V$, s.t $v + v' = 0$
5) $\exists 1 \in F$, s.t $1 \cdot v = v$
6) $(ab) \cdot v = a \cdot (bv)$
7) $a \cdot (v + w) = a \cdot v + a \cdot w$
8) $(a + b) \cdot v = a \cdot v + b \cdot v$

EXAMPLE 1.3 set $V = \{0\}$ over the field $\mathbb{R}$

EXAMPLE 1.4 set $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \ldots \mathbb{R}^n$ over the field $\mathbb{R}$

If F is a field, then F is also a Vector space over itself

1.3 Definition of Vector Spaces

DEFINITION 1.6 (Vector Space)

$$(V, F, +, \cdot)$$

- $(V, +)$ is an Abelian group
- $\cdot : F \times V \to V$ is associative and follows distributivity

EXAMPLE 1.7

$$\mathbb{Q}, \mathbb{R}, \mathbb{C} \text{ are all fields}$$

PROPOSITION 1.8 Any field $F \subset \mathbb{R}$ contains $\mathbb{Q}$

Proof We assume normal addition and multiplication

$$\forall a \in \mathbb{Q}, a = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$
$0, 1 \in F$, as $(F, +)$ and $(F - \{0\}, \times)$ is an abelian groups. As $+$ is closed $1 + 1 = 2 \in F$ and $2 + 1 = 3 \in F$ and so on. By inverse property of $+$, we know $-1, -2, -3 \cdots \in F$, therefore $\mathbb{Z} \in F$. By inverse property of $\times$, we know $\{1/a : a \in \mathbb{Z}\} \in F$. Since $\times$ is closed, $S = \{a/b : a, b \in \mathbb{Z}\} \in F$, set S is just $\mathbb{Z}$. Therefore $Z \in F$ if $F \subset \mathbb{R}$ is a field.

■

Properties of fields:

1) *Cancellation law,* If $a + b = a + c$ then $b = c$

2) Additive identity is unique
3) Additive inverse is unique
4) Multiplicative identiy is unique
5) Multiplicative inverse is unique
6) if $a \cdot b = a \cdot c$ then $b = c$ if $a \neq 0$

Properties of Vector spaces:

1) *Cancellation law*, If $a + b = a + c$ then $b = c$
2) Additive identity is unique
3) Additive inverse is unique
4) $0 \cdot v = 0 \rightarrow$ is the null vector

Matrices as Vector spaces:

$$M_{n \times m} (\mathbb{R})$$

- Addition is entry wise
- $0_{n \times m}$ the additive identity
- Scaling is entry wise

$M_{n \times m}$ looks like $\mathbb{R}^{n.m}$

1.4 Vector Subspaces:

Definition 1.9 (Vector Subspace) W is said to be a vector subspace of V if

- $W \subseteq V$
- W is a vector subspace

Example 1.10 $P(\mathbb{R})$ is the set of all polynomials.
$P_n(\mathbb{R})$ is the set of all polynominals of degree at most n.

- $P_0(\mathbb{R}) \subset P_1(\mathbb{R}) \subset P_1(\mathbb{R}) \subset P_3(\mathbb{R}) \subset \ldots P_n(\mathbb{R})$
- $\{a_0 + a_2 x^2 + a_4 x^4 \cdots : a_i \in \mathbb{R}\}$

Example 1.11 $M_n(\mathbb{R})$ is the set of all $n \times n$ matrices

- Set of all symmetric matrices, $S = \{A : A \in M_n(\mathbb{R}), A = A^t\}$
- $T = \{A : A \in M_n(\mathbb{R}), Tr(A) = 0\}$

2 Span and Linear Independence

2.1 Definiton of Span

Definition 2.1 (Span)

$$\text{span}(S) = \{\gamma_1 v_1 + \gamma_2 v_2 \ldots \gamma_k v_k : v_i \in S, \gamma_i \in F\}$$

Example 2.2

$$S = \{1, x, x^2\}$$
$$\text{span}(S) = P_2(\mathbb{R})$$

Example 2.3

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$
$$\text{span}(S) = \mathbb{R}^2$$

Example 2.4

$$S = \{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\}$$
$$\text{span}(S) = M_2(\mathbb{R})_{sym}$$

Properties of Span:

1) span(S) is always a subspace containing S as,
   $+$ is closed and
   Scaling operation is also closed.
2) For a subspace $S \subseteq W$, $S \subseteq \text{span}(S) \subseteq W$
3) span (S) is the smallest subspace containg S.
   That is, span (S) is the intersection of all subspaces containing S.
4) span $(\emptyset) = \{0\}$
   This is more of a convention, or you can arrive at this by following the above definition of Span.
5) $S_1 \subseteq S_2$ implies span $(S_1) \subseteq$ span (S_2)
6) span (span (S)) = span (S)
7) span $(W) = W$ if and only if W is a subspace.
8) span $(S_1 \cup S_2)$ = span (S_1) + span (S_2)
   where $A + B = \{a + b : a \in A, b \in B\}$

if V_1 and V_2 are subspaces then, $V_1 + V_2$ is also a subspace.

2.2 Linear (In)dependence

Given a non empty set S, consider span (S). Is there an element $v \in S$ such that, span (S) = span $(S - \{v\})$?

Then S is linearly dependent.

Example 2.6

$$S = \{(0,0,0), (0,1,0), (0,0,e)\}$$
$$\text{span}(S) = \{(0,a,b) : a, b \in \mathbb{R}\} = \text{span}(S - \{(0,0,0)\})$$

For span (S) = span $(S - \{v\})$, there exists $v_1, v_2, \ldots, v_k \in S - \{v\}$ and scalars $\gamma_1, \gamma_2, \ldots, \gamma_k \in F$ such that,

$$V = \sum_{i=1}^{k} \gamma_i v_i$$

as $0 \notin S$, not all γ_i can be 0.

Definition 2.7 (Linear Dependence) A subset $S \subseteq V$ is said to be linearly dependent if there exists scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and v_i are all distinct.

Definition 2.8 (Linear Independence) A subset $S \subseteq V$ is said to be linearly independent if there exists no scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and v_i are all distinct.

Example 2.9 $V = P(\mathbb{R}), S = \{1+x, 1-x, x^2, 2x^2-1\}$

$$\frac{1}{2}(1+x) + \frac{1}{2}(1-x) + (2x^2-1) + (-2)x^2 = 0$$

$S' = \{1+x, 1-x, x^2\}$

if S is linearly dependent, then $\exists v \in S$ such that $span(S) = span(S - \{v\})$

Proposition 2.11 if S is linearly independent and $\exists v \notin \operatorname{span}(S)$ then $S \cup \{v\}$ is linearly independent.

Proof | let $\gamma, \gamma_1, \ldots, \gamma_k \in F$ and $v_1, \ldots, v_k \in S$

$$\gamma_1 v1 + \cdots + \gamma_k v_k + \gamma v = 0$$

case 1: $\gamma = 0$,

$$\gamma_1 v1 + \cdots + \gamma_k v_k = 0$$

as S is linearly independent $\gamma_i = 0$ therefore,

$$\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma = 0$$

therefore $S \cup \{v\}$ is linearly independent.
case 2: $\gamma \neq 0$,

$$v = \frac{-\gamma_1}{\gamma} v_1 + \cdots + \frac{-\gamma_k}{\gamma} v_k$$

but $v \notin \operatorname{span}(S)$, therefore this case is not possible. ■

3 Basis and Dimension of a vector space

3.1 Basis and Dimension

Definition 3.1 (Basis) set $\beta \subseteq V$ is called the basis of V if

1) β is linearly independent.
2) β spans V

Definition 3.2 (Dimension) For β being a basis of V, $\dim_F V = |\beta|$

Theorem 3.3 (Replacement Theorem) Let $S \subseteq V$ be a finite set which spans V and let $L \subseteq V$ be a finite linearly independent subset, then,

1) $|L| \leq |S|$
2) $\exists T \subseteq S$ (T is of size $|S| - |L|$) such that $T \cup L$ spans V

Proof

$$L = \{u_1, \dots, u_m\}$$
$$S = \{v_1, \dots, v_n\}$$
$$u_m \in V = \text{span}\,(S)$$

therefore,

$$u_m = a_1 v_1 + \cdots + a_n v_n$$

as u_m is in L a linearly independent set, u_m cannot be 0, therefore, all of a_i cannot be zero as well. Let $a_n \neq 0$.

$$v_n = \frac{1}{a_n} u_m + \frac{-a_1}{a_n} v_1 + \cdots + \frac{-a_{n-1}}{a_n} v_{n-1}$$
$$v_n \in \text{span}\{v_1, \dots, v_{n-1}, u_m\} = \text{span}\,(S) = V$$

similarly,

$$u_{m-1} \in V = \text{span}\{v_1, \dots, v_{n-1}, u_m\}$$

therefore,

$$u_{m-1} = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b_m u_m$$

as u_{m-1} and u_m are linearly independent, therefore all of a_i cannot be 0

$$v_{n-1} = \frac{1}{a_{n-1}} u_{m-1} + \frac{-b_m}{a_{n-1}} u_m + \frac{-a_1}{a_{n-1}} v_1 \cdots + \frac{-a_{n-2}}{a_{n-1}} v_{n-2}$$

$$v_{n-1} \in \text{span}\{v_1, \dots, v_{n-1}, u_{m-1}, u_m\} = \text{span}\,(S) = V$$

Now iterate this process to get,

$$S' = \{v_1, \dots, v_{n-m}, u_1, \dots, u_m\}$$

Where S' spans V,

$$\text{span}\,(S') = V$$

If $n < m$ then, there will be a point when iterating the steps where a proper subset of L will span V, which would not be possible since this breaks the linear independence of L.
The set T from the theorem is,

$$T = S' - L$$

■

If $S \subseteq V$ spans V and $L \subseteq V$ is an linearly independent subset, then $|L| \leq |S|$.

Corollary 3.5 Given 2 basis of the same vector space V, β and β'.

$$|\beta| = |\beta'|$$

Proof | Set $S = \beta$ and $L = \beta'$. β spans V by definition of basis and β' is linearly independent by definition of basis.
Now according to 1) of replacement theorem, we get.

$$|\beta| \leq |\beta'|$$

Similarly, we can swap S and L to get,

$$|\beta| \geq |\beta'|$$

$$\therefore |\beta| = |\beta'|$$

■

Going back to the definition of dimension, we can see that basis need not be unique but the dimension of a vector space is unique regardless of which basis we use to find the dimension

Corollary 3.7 Let V be a vector space of finite dimension n, then

1) Any finite spanning set S, $|S| \geq n$, if $|S| = n$ then S is a basis.
2) Any linearly independent set L, $|L| \leq n$, if $|L| = n$, then L spans V.
3) Any linearly independent set L can be extended to form a basis of V.

Corollary 3.8 Let V be a finite dimensional vector space over the field F andlet $W \subseteq V$ be a subspace, then,

$$\dim_F(W) \leq \dim_F(V)$$

If equality holds, then, $V = W$.

Proof | Let β be a basis of V of size n. Now look at $\text{span}\{\emptyset\} = \{0\}$, if it is W then we are done, if not, we choose $u_1 \in W$ which is non zero, $L_1 = \{u_1\} \subseteq W$ is linearly independent. We check again if $\text{span}(L_1) = W$, if it is then we are donem, if it isn't then we choose $u_2 \in W \setminus \text{span}(L_1)$. We iterate this process until we get $\beta_\circ = \{u_1, \ldots, u_k\}$ which spans W. This set is linearly independent by construction (refer to proposition 2.11). Using 1) from replacement theorem, using β as S and β_circ as L, we get,

$$|\beta_\circ| \leq |\beta|$$

which is the same as,

$$\dim_F(W) \leq \dim_F(V)$$

And using corollary 3.8 we can say that the equality implies $V = W$. ■

Given subspace W of vector space V, with $\beta_\circ$ as the basis of W, we can extend the basis $\beta_\circ$ to β as a basis for V

3.3 Direct Sum

Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Then $W_1 + W_2$ is also a subspace.
Case A: $W_1 \cap W_2 = \{0\}$

Definition 3.10 (Direct Sum) Given subspaces W_1 and W_2 of a finite dimensional vector space V, if $W_1 \cap W_2 = \{0\}$ then the vector subspace $W_1 + W_2$ is known as the direct sum.

Case B: $W_1 \cap W_2 \neq \{0\}$

$$\dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2)$$

Proof | Start with basis of $W_1 \cap W_2$, $\beta_\circ = \{u_1, \ldots, u_k\}$. Now extend this basis to a basis of W_1, $\beta_1 = \{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ and similarly for W_2, $\beta_2 = \{u_1, \ldots, u_k, w_1', \ldots, w_m\}$

$$\beta = \beta_1 \cup \beta_2$$

$$\beta = \{u_1, \ldots, u_k, w_1, \ldots, w_l, w_1', \ldots, w_m'\}$$

As any element in $W_1 + W_2$ can be represented by a linear combination of elements of β, β spans $W_1 + W_2$. To show that β is linearly independent,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l + \gamma_1 w_1' + \cdots + \gamma_m w_m' = 0$$

$$=> c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = -\gamma_1 w_1' - \cdots - \gamma_m w_m' = v$$

Clearly $v \in W_1$ and $v \in W_2$, therefore $v \in W_1 \cap W_2$, therefore v can be written as,

$$v = c_1' u_1 + \cdots + c_k' u_k$$

$$=> c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = -\gamma_1 w_1' - \cdots - \gamma_m w_m' = v = c_1' u_1 + \cdots + c_k' u_k$$

From these 2 equations, we are able to determine that β is linearly independent,
We have,

$$-\gamma_1 w_1' - \cdots - \gamma_m w_m' = c_1' u_1 + \cdots + c_k' u_k$$

$\{u_1, \ldots, u_k, w_1', \ldots, w_m'\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1' = \cdots = c_k' = 0$$

Similarly, we have,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = c_1' u_1 + \cdots + c_k' u_k$$

We know $c_1' = \cdots = c_k' = 0$, therefore,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = 0$$

$\{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1 = \cdots = c_l = 0$$

Therefore the only solution for

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l + \gamma_1 w_1' + \cdots + \gamma_m w_m' = 0$$

is all the scalars being equal to zero, which implies linear independence of β. Therefore β is both linearly independent and spans $W_1 + W_2$, which makes it a basis for $W_1 + W_2$.

$$\dim_F(W_1) = |\beta_1| = k + l$$

$$\dim_F(W_2) = |\beta_1| = k + m$$

$$\dim_F(W_1 + W_2) = |\beta| = k + l + m$$

$$\dim_F(W_1 \cap W_2) = |\beta_\circ| = k$$

$$\therefore \dim_F (W_1 + W_2) = \dim_F (W_1) + \dim_F (W_2) - \dim_F (W_1 \cap W_2)$$

■

Proposition 3.11 If V is the direct sum, $W_1 + W_2$ then $v \in V$ can be uniquely expressed as $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

Proof | Let, $v = w_1 + w_2 = w_1' + w_2'$ then, $w_1 - w_1' = w_2 - w_2' = x$. therefore $x \in W_1 \cap W_2$ and by definition of direct sum, $x \in \{0\}$, therefore $x = 0$. Which gives us, $w_1 = w_1'$ and $w_2 = w_2'$. ■

4 Quotient Spaces

4.1 Quotient Spaces

Goal is to create a new vector space out of $W \subseteq V$, define an equvalence relation $\sim_W$ on V. $v_1 \sim_W v_2$ if and only if $v_1 - v_2 \in W$

- *Reflexive* Yes, if $0 \in W$.
- *Symmetric* Yes, if whenever, $v_1 - v_2 \in W$ implies $v_2 - v_1 \in W$, ie, $-(v_1 - v_2) \in W$.
- *transitivity* Yes, if whenever $v_1 - v_2 \in W$ and $v_2 - v_3 \in W$ implies $v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \in W$.

Starting with any arbitary subset of V we end with a subspace of V due to the conditions that needs to be satisfied for $\sim_W$ to be an equivalence relation.

For any eqivalence relation $\sim$ on an arbitary set S, an equivalence class is given by.

$$[s] = \{s' : s' \in S, s' \sim s\}$$

Equivalence relations are disjoined and partition the set. that is either $[s] \cap [s'] = \emptyset$ or $[s] = [s']$.

For $\sim_W$,

$$[v] = \{v_1 : v_1 - v \in W\}$$
$$[v] = \{v_1 = v + w : w \in W\}$$
$$[v] = v + W$$

$[v_1] = [v_2]$ we translate via W,

$$v_1 + W = v_2 + W$$
$$v_1 = v_2 + W$$
$$v_1 - v_2 = w \in W$$

Therefore $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.

We can create a new vector space, where each element is an equivalence class of the equivalence relation $\sim_W$. Consider the set of all equivalence class of the relation $\sim_W$, V/W.

Addition under the vector space V/W,

$$+ : V/W \times V/W \to V/W$$

$$[v_1] + [v_2] = [v_1 + v_2]$$

Scaling,

$$\times : F \times V/W \to V/W$$

$$(\gamma, [v]) \to [\gamma v]$$

5 Linear Maps

5.1 Linear Maps

Given vector spaces V, W (over the same field F) and linear map T.

$$T : V \to W$$

T should be compatible with the structures of the vector spaces $\forall v_1, v_2, v \in V$ and $\forall \gamma \in F$.

- $T(v_1 + v_2) = Tv_1 + Tv_2$
- $T(\gamma v) = \gamma Tv$

Example 5.1 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$T(x+y) = \gamma_\circ (x+y) = \gamma_\circ x + \gamma_\circ y = Tx + Ty$$
$$T(\gamma y) = \gamma_\circ (\gamma y) = \gamma(\gamma_\circ y) = \gamma(Tx)$$

$T(0_v) = 0_w$

Example 5.3 (Dilation)

$$T : V \to V, v \to 2v \, (= v + v)$$
$$\gamma v \to 2(\gamma v) = \gamma(2v) = \gamma(Tv)$$

If in the field, $2 = 0$ then the map is the zero map, if not then it's a bijective map.

Example 5.4 (Identity map)

$$I : V \to V, v \to v$$

Example 5.5 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$

Example 5.6 (Matrices)

$$A \in M_{n \times n}(\mathbb{R})$$
$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$L_A \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + L_A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$L_A \left(\gamma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \to \gamma L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

EXAMPLE 5.7 (Reflection)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

This is for reflection across x-axis, for any other reflection the line as to pass through the origin because 0 has to be mapped to 0 and in general $T.T$ gives us the identity map.

EXAMPLE 5.8 (Rotation)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (-y, x)$$

This is rotaion of $\pi/2$ clockwise, for a more general rotaion,

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$||T\,(x, y)\,|| = ||\,(x, y)\,||$$

This is a property of every rotation linear map.

EXAMPLE 5.9 (Projection)

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$

$$P^2 = P.P = P$$

$$Q^2 = Q.Q = Q$$

This is the defining property of a projection map.

EXAMPLE 5.10 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$

$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$

Definition 5.11 (Null Space) Given $T: V \to W$, null space $N(T)$ is given by,

$$N(T) = \{v \in V : T(v) = 0_w\}$$

Example 5.12 (Scaling Linear Map)

$$T: \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$N(T) = \{0\}$$

Example 5.13 (Dilation)

$$T: V \to V, v \to 2v\,(= v + v)$$

$$N(T) = \begin{matrix} V & 2 = 0 \text{ in } F \\ \{0\} & 2 \neq 0 \text{ in } F \end{matrix}$$

Example 5.14 (Identity map)

$$I: V \to V, v \to v$$

$$N(T) = \{0\}$$

Example 5.15 (Trivial map)

$$T_0: V \to W, v \to 0_w$$

$$N(T) = V$$

Example 5.16 (Reflection)

$$T: \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

$$N(T) = \{0\}$$

Example 5.17 (Rotation)

$$T_\theta: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$N(T) = \{0\}$$

Example 5.18 (Projection)

$$P: \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$

$$N(P) = \{(0, y)\}$$

$$Q: \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$

$$N(Q) = \{(x, y) : x = y\}$$

Example 5.19 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x,y) \to (x,y,0)$$
$$N(T) = \{0\}$$
$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x,y) \to (x,y,ax+by)$$
$$N(T) = \{0\}$$

5.3 Range

Definition 5.20 (Range) Given $T : V \to W$, null space $R(T)$ is given by,

$$R(T) = \text{Image of } V \text{ under } T$$

Example 5.21 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$
$$R(T) = \mathbb{R}$$

Example 5.22 (Dilation)

$$T : V \to V, v \to 2v \,(= v + v)$$
$$R(T) = \begin{matrix} \{0\} & 2 = 0 \text{ in } F \\ V & 2 \neq 0 \text{ in } F \end{matrix}$$

Example 5.23 (Identity map)

$$I : V \to V, v \to v$$
$$R(T) = V$$

Example 5.24 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$
$$R(T) = \{0\}$$

Example 5.25 (Reflection)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \to (x,-y)$$
$$R(T) = \mathbb{R}^2$$

Example 5.26 (Rotation)

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$R(T) = \mathbb{R}^2$$

Example 5.27 (Projection)

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$
$$R(P) = \{(x, 0)\}$$
$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$
$$R(Q) = \{(x, 0)\}$$

Example 5.28 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$
$$N(T) = \{(x, y, 0)\}$$
$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$
$$N(T) = \{(x, y, ax + by)\}$$

5.4 Null space and Range as vector subspaces

$$v_1, v_2 \in N(T), \gamma \in F$$

$$T(v_1 + v_2) = Tv_1 + Tv_2$$

$$T(v_1 + v_2) = 0 + 0 = 0$$

$$\therefore v_1 + v_2 \in N(T)$$

$$T(\gamma v_1) = \gamma T v_1 = \gamma 0 = 0$$

$$\therefore \gamma v_1 \in N(T)$$

Therefore $N(T)$ is a subspace.