Analysis 2

Incalculas

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1 Differentiation

1.1 Definition

Definition 1.1 (Differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ be a function from J to $\mathbb{R}$. The function is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists.

Notation 1.2 (Differentiation) If f is differentiable at c, we define

$$\lim_{x \to c} \frac{f(x) - f(x)}{x - c} = \left.\frac{df}{dx}\right|_{x=c} = f'(c)$$

Let's now discuss about interpretation of differentiation

Let J be an interval in R with $c \in J$ and let $f : J \to \mathbb{R}$ be a function.

Aim is to find an approximation for f near 0, $f(x) \approx a + bx$ for some $a, b \in \mathbb{R}$ near 0.

$$\lim_{x \to c} [f(x) - (a + bx)] = 0$$

$$E(x) = f(x) - (a + bx)$$

Find $a, b \in \mathbb{R}$ such that,

$$\lim_{x \to c} E(x) = 0$$

Then, $a = f(c)$ and $b = f'(c)$, this tells us that we know first order approximation from differentiation.

Definition 1.3 (Alternate definition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$, we say f is differentiable if and only if

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists.

Take $J = [a, b]$ then

- *$c = a$, we only check for one sided limit (right hand limit)*
- *$a < c < b$, we check both left hand and right hand limits*
- *$c = b$, we only check for one sided limit (left hand limit)*

Definition 1.5 (Differentiable on an interval) Let J be an interval on R and $f : J \to \mathbb{R}$ be real function on J, we say f is differentiable on J if f is differentiable for every point on J.

Example 1.6 $f(x) = x$,

$$f'(x) = \lim_{h \to 0} \frac{(x + h) - (x)}{h} = \lim_{h \to 0} 1 = 1 \implies f'(x) = 1$$

Theorem 1.7 (Equivalent condition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ then f is differentiable at c if and only if $\exists f_1 : J \to \mathbb{R}$ such that

- $i) f(x) = f(c) + f_1(x)(x-c) \forall x in J.$
- $ii) f_1$ is continous at c.

and $f_1(c) = f'(c)$

Proof Let f be differentiable at c then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Now define,

$$f_1(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

This now proves i) and ii).

Assume that f_1 exists and satisfies i) and ii) then,

$$f_1(x) = \frac{f(x) - f(c)}{x - c}, x \neq c$$

and since $f_1(x)$ is continous at c

$$\lim_{x \to c} f_1(x)$$

exists and therefore,

$$f_1(x) = \lim_{x \to c} f_1(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This proves that f is differentiable at c and $f'(c) = f_1(c)$. ∎

Corollary 1.8 Let $f : J \to \mathbb{R}$ be function from interval $J \in \mathbb{R}$ to $\mathbb{R}$ be differential at $c \in J$ then f is continous at c.

Proof From Theorem 1.7,

$$f(x) = f(c) + f_1(x)(x - c)$$

f_1 is contious at c and hence f is differentiable at c ∎

Example 1.9 Define $f : \mathbb{R} \to \mathbb{R}, x \to x^n$ for some fixed $n \in \mathbb{N}$, for $c \in \mathbb{R}$, $f'(c) = nc^{n-1}$.

Proof

$$f(x) - f(c) = x^n - c^n$$

$$\implies f(x) - f(c) = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

Let $f_1(x) = x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}$, given it's a polynomial, we know $f_1(x)$ is continous in $\mathbb{R}$.

$$f_1(c) = nc^{n-1}$$

Therefore f is differentiable in $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

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EXAMPLE 1.10 Define $f : \mathbb{R} \to \mathbb{R}, x \to e^x$, for $c \in \mathbb{R}$, $f'(c) = e^c$.

Proof

$$f(x) - f(c) = e^x - e^c$$

$$\implies f(x) - f(c) = e^c\,(e^x - 1)$$

Since,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Define,

$$f_1(x) = e^c \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{n!}$$

Assume that f_1 is continous at c,

$$f_1(c) = e^c$$

This proves that f is differentiable at c and $f'(c) = e^c$.

(Note: this proof uses power series of exponential function which uses differentiation, this proof is just a circular argument and we will get back to an actual proof at the end of the course)

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EXAMPLE 1.11 Define $f : \mathbb{R} \to \mathbb{R}, x \to |x|$, for $c \in \mathbb{R}$, $f'(c) = 1$ if $c > 0$, $f'(c) = -1$ if $x < 0$ and $f'(c)$ does not exist for $c = 0$.

Proof Both $c < 0$ and $c > 0$ cases are trivial. Now assume $f(x)$ is differentiable at 0 then the following limit exists.

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h} = |h|/h$$

Clearly, left hand limit is -1 and right hand limit is 1 and therefore limit does not exist.

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THEOREM 1.12 (Algebraic properties) Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be 2 functions differentaible at $c \in J$.

- $i)$ $(f+g)$ definied as

$$(f+g)(x) = f(x) + g(x)$$

Then,

$$(f+g)'(c) = f'(x) + g'(x)$$

- *ii)* for $\alpha \in \mathbb{R}$, (αf) definied as

$$(\alpha f)(x) = \alpha f(x)$$

Then,

$$(\alpha f)'(c) = \alpha f'(x)$$

- *iii)* (fg) definied as

$$(fg)(x) = f(x)g(x)$$

Then,

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

- *iv)* Assume $f(c) \neq 0$, $(\frac{1}{f})$ definied as

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$$

Then,

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

Proof | For *i*):
From theorem 1.7 we have that there exists continous functions $f_1 : J \to \mathbb{R}$ and $g_1 : J \to \mathbb{R}$ such that,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$g(x) = g(c) + g_1(x)(x-c)$$

From these equations we have,

$$f(x) + g(x) = f(c) + g(c) + f_1(x)(x-c) + g_1(x)(x-c)$$

$$\implies (f+g)(x) = (f+g)(c) + (f_1(x) + g_1(x))(x-c)$$
$$\implies (f+g)(x) = (f+g)(c) + ((f_1+g_1)(x))(x-c)$$

Now applying theorem 1.7, we have that

$$(f+g)'(c) = (f_1+g_1)(c) = f_1(c) + g_1(c) = f'(c) + g'(c)$$

$$\implies (f+g)'(c) = f'(c) + g'(c)$$

We shall apply the same structure for the proofs of other algebraic properties,
For *ii*):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$

From this equation we have,

$$\alpha f(x) = \alpha f(c) + \alpha f_1(x)(x-c)$$
$$\implies (\alpha f)(x) = (\alpha f)(c) + ((\alpha f_1)(x))(x-c)$$

Now applying theorem 1.7, we have that

$$(\alpha f)'(c) = (\alpha f_1)(c) = \alpha f_1(c) = \alpha f'(c)$$

$$\implies (\alpha f)'(c) = \alpha f'(c)$$

For *iii*):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$g(x) = g(c) + g_1(x)(x-c)$$

From these equations we have,

$$f(x)g(x) = f(c)g(c) + (x-c)\left[f(c)g_1(x) + f_1(x)g(c) + f_1(x)g_1(x)(x-c)\right]$$

Now applying theorem 1.7,

$$(fg)'(c) = f(c)g_1(c) + f_1(c)g(c)$$
$$\implies (fg)'(c) = f(c)g'(c) + f'(c)g(c)$$

For *iv*):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$\implies \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{1}{f(c) + f_1(x)(x-c)} - \frac{1}{f(c)}$$
$$\implies \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{f(c) - f(c) - f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\frac{1}{f(x)} = \frac{1}{f(c)} - \frac{-f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\implies \left(\frac{1}{f}\right)(x) = \left(\frac{1}{f}\right)(c) - \frac{-f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\implies \left(\frac{1}{f}\right)(x) = \left(\frac{1}{f}\right)(c) - (x-c)\frac{-f_1(x)}{(f(c))(f(c) + f_1(x)(x-c))}$$

Now applying theorem 1.7,

$$\left(\frac{1}{f}\right)'(c) = -\frac{f_1(c)}{(f(c))(f(c) + f_1(c)(c-c))}$$
$$\implies \left(\frac{1}{f}\right)'(c) = -\frac{f_1(c)}{(f(c))^2}$$
$$\implies \left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

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1.3 Differentiation as a linear map

Let $J \in \mathbb{R}$ be an interval and let $c \in J$,
We define vector space $\mathfrak{D}_c$ as,

$$\mathfrak{D}_c = \{f : J \to \mathbb{R} : f \text{ is differentiable at } c\}$$

$\mathfrak{D}_c$ is a vector space over $\mathbb{R}$,
Let's define,

$$\left.\frac{d}{dx}\right|_{x=c} : \mathfrak{D}_c \to \mathbb{R}, \left.\frac{df}{dx}\right|_{x=c} = f'(c)$$

It is trivial to check that, $\left.\frac{d}{dx}\right|_{x=c}$ is a linear map from the vector space $\mathfrak{D}_c$ to itself.

$$\mathfrak{D} = \{f : j \to \mathbb{R} : f \text{ is differential on } J\}$$

$$\mathfrak{F} = \{f : J \to \mathbb{R}\}$$

$$\frac{d}{dx} : \mathfrak{D} \to \mathfrak{F}, f \to \frac{df}{dx}$$

It is again trivial to check that, $\frac{d}{dx}$ is a linear map from the vector space $\mathfrak{D}$ to itself.

Notation 1.13 (Continous functions) Given interval J in $\mathbb{R}$

$$\mathcal{C}(J) = \{f : J \to \mathbb{R} : f \text{ is continous on } J\}$$

Notation 1.14 (Continously differentiable functions) Given interval J in $\mathbb{R}$

$$\mathcal{C}^1(J) = \{f : J \to \mathbb{R} : f \text{ is differentiable on } J \text{ and } f' \in \mathcal{C}(J)\ \}$$

1.4 Chain Rule

Theorem 1.15 (Chain Rule) Given intervals J and J_1 in $\mathbb{R}$ and functions $f : J \to \mathbb{R}$ and $g : J_1 \to \mathbb{R}$, such that $f(J) \subseteq J_1$. Let $c \in J$, if f is differentiable at c and g is differentiable at $f(c)$ then (gof) is differentiable at c and $(gof)'(c)$ is given by

$$(gof)'(c) = g'(f(c))f'(c)$$

Proof | From theorem 1.7 we have that there exists $f_1 : J \to \mathbb{R}$ such that,

$$f(x) = f(c) + f_1(x)(x - c)\forall x \in J$$

where f_1 is continous at c and $f'(c) = f_1(c)$
From theorem 1.7 we have that there exists $g_1 : J_1 \to \mathbb{R}$ such that,

$$g(y) = g(f(c)) + g_1(y)(y - f(c))\forall y \in J_1$$

where g_1 is continous at $f(c)$ and $g'(f(c)) = g_1(f(c))$
Now,

$$(gof)(x) = g(f(x))$$
$$\implies (gof)(x) = g(f(c)) + g_1(f(x))(f(x) - f(c))$$
$$\implies (gof)(x) = g(f(c)) + g_1(f(c) + f_1(x)(x - c))f_1(x)(x - c)$$

Since g_1 and f_1 are continous, $g_1(f(c) + f_1(x)(x - c))f_1(x)$ is contious, therefore we can apply theorem 1.7 now and we get,

$$(gof)_1(x) = g_1(f(c) + f_1(x)(x - c))f_1(x)$$

$$(gof)'(c) = (gof)_1(c) = g_1(f(c) + f_1(c)(c - c))f_1(c)$$

$$(gof)'(c) = g'(f(c))f'(c)$$

We shall now declare few conjectures to give ourselves oppertunity to look at better examples (just going along with how the course was thought, I the student do not like this),

CONJECTURE 1.16 $f(x) = e^x$ is differentiable in $\mathbb{R}$ and $f'(x) = e^x$

CONJECTURE 1.17 $f(x) = \log(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = 1/x$

CONJECTURE 1.18 $f(x) = \sin(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = \cos(x)$

CONJECTURE 1.19 $f(x) = \cos(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = -\sin(x)$

EXAMPLE 1.20 (This example uses one of the cojectures)

$$f(x) = x^\alpha, x > 0, \alpha \in \mathbb{R}$$

Then,

$$f(x) = e^{\alpha \log(x)}$$

let, $q(x) = \alpha \log(x)$ and $p(x) = e^x$, therefore,

$$f(x) = (poq)(x)$$

Now using chain rule we get,

$$f'(x) = p'(q(x))q'(x)$$
$$\implies f'(x) = e^{\alpha \log(x)} \frac{\alpha}{x}$$
$$\implies f'(x) = \alpha x^{\alpha - 1}$$

1.5 Local minima and maxima

DEFINITION 1.21 (Local Minima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a local minima of f if $\exists \delta \in \mathbb{R}$ such that,

- *i)*
$$(c - \delta, c + \delta) \subseteq J$$
- *ii)*
$$f(x) \geq f(c) \forall x \in (c - \delta, c + \delta)$$

DEFINITION 1.22 (Local Maxima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a local maxima of f if $\exists \delta \in \mathbb{R}$ such that,

- *i)*
$$(c - \delta, c + \delta) \subseteq J$$
- *ii)*
$$f(x) \leq f(c) \forall x \in (c - \delta, c + \delta)$$

Definition 1.23 (Global Minima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a global minima of f if,

$$f(x) \geq f(c) \forall x \in J$$

Definition 1.24 (Global Maxima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a global maxima of f if,

$$f(x) \leq f(c) \forall x \in J$$