Analysis 2

Incalculas

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Contents

1 Differentiation

1.1 Definition

Definition 1.1 (Differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ be a function from J to $\mathbb{R}$. The function is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists.

Notation 1.2 (Differentiation) If f is differentiable at c, we define

$$\lim_{x \to c} \frac{f(x) - f(x)}{x - c} = \left.\frac{df}{dx}\right|_{x=c} = f'(c)$$

Let's now discuss about interpretation of differentiation

Let J be an interval in R with $c \in J$ and let $f : J \to \mathbb{R}$ be a function.

Aim is to find an approximation for f near 0, $f(x) \approx a + bx$ for some $a, b \in \mathbb{R}$ near 0.

$$\lim_{x \to c} [f(x) - (a + bx)] = 0$$

$$E(x) = f(x) - (a + bx)$$

Find $a, b \in \mathbb{R}$ such that,

$$\lim_{x \to c} E(x) = 0$$

Then, $a = f(c)$ and $b = f'(c)$, this tells us that we know first order approximation from differentiation.

Definition 1.3 (Alternate definition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$, we say f is differentiable if and only if

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists.

Take $J = [a, b]$ then

1) *$c = a$, we only check for one sided limit (right hand limit)*
2) *$a < c < b$, we check both left hand and right hand limits*
3) *$c = b$, we only check for one sided limit (left hand limit)*

Definition 1.5 (Differentiable on an interval) Let J be an interval on R and $f : J \to \mathbb{R}$ be real function on J, we say f is differentiable on J if f is differentiable for every point on J.

Example 1.6 $f(x) = x$,

$$f'(x) = \lim_{h \to 0} \frac{(x + h) - (x)}{h} = \lim_{h \to 0} 1 = 1 \implies f'(x) = 1$$

Theorem 1.7 (Equivalent condition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ then f is differentiable at c if and only if $\exists f_1 : J \to \mathbb{R}$ such that

1) $f(x) = f(c) + f_1(x)(x-c) \forall x \in J$.
2) f_1 is continous at c.

and $f_1(c) = f'(c)$

Proof | Let f be differentiable at c then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Now define,

$$f_1(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

This now proves i) and ii).

Assume that f_1 exists and satisfies i) and ii) then,

$$f_1(x) = \frac{f(x) - f(c)}{x - c}, x \neq c$$

and since $f_1(x)$ is continous at c

$$\lim_{x \to c} f_1(x)$$

exists and therefore,

$$f_1(x) = \lim_{x \to c} f_1(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This proves that f is differentiable at c and $f'(c) = f_1(c)$. ■

Corollary 1.8 Let $f : J \to \mathbb{R}$ be function from interval $J \in \mathbb{R}$ to $\mathbb{R}$ be differential at $c \in J$ then f is continous at c.

Proof | From Theorem 1.7,

$$f(x) = f(c) + f_1(x)(x - c)$$

f_1 is contious at c and hence f is differentiable at c ■

Example 1.9 Define $f : \mathbb{R} \to \mathbb{R}, x \to x^n$ for some fixed $n \in \mathbb{N}$, for $c \in \mathbb{R}$, $f'(c) = nc^{n-1}$.

Proof |

$$f(x) - f(c) = x^n - c^n$$

$$\implies f(x) - f(c) = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

Let $f_1(x) = x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}$, given it's a polynomial, we know $f_1(x)$ is continous in $\mathbb{R}$.

$$f_1(c) = nc^{n-1}$$

Therefore f is differentiable in $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

∎

Example 1.10 Define $f : \mathbb{R} \to \mathbb{R}, x \to e^x$, for $c \in \mathbb{R}$, $f'(c) = e^c$.

Proof

$$f(x) - f(c) = e^x - e^c$$

$$\implies f(x) - f(c) = e^c\left(e^x - 1\right)$$

Since,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Define,

$$f_1(x) = e^c \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{n!}$$

Assume that f_1 is continous at c,

$$f_1(c) = e^c$$

This proves that f is differentiable at c and $f'(c) = e^c$.

(Note: this proof uses power series of exponential function which uses differentiation, this proof is just a circular argument and we will get back to an actual proof at the end of the course)

∎

Example 1.11 Define $f : \mathbb{R} \to \mathbb{R}, x \to |x|$, for $c \in \mathbb{R}$, $f'(c) = 1$ if $c > 0$, $f'(c) = -1$ if $x < 0$ and $f'(c)$ does not exist for $c = 0$.

Proof Both $c < 0$ and $c > 0$ cases are trivial. Now assume $f(x)$ is differentiable at 0 then the following limit exists.

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h} = |h|/h$$

Clearly, left hand limit is -1 and right hand limit is 1 and therefore limit does not exist.

∎

Theorem 1.12 (Algebraic properties) Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be 2 functions differentaible at $c \in J$.

1) $(f+g)$ definied as

$$(f+g)(x) = f(x) + g(x)$$

Then,

$$(f+g)'(c) = f'(x) + g'(x)$$

2) for $\alpha \in \mathbb{R}$, (αf) definied as

$$(\alpha f)(x) = \alpha f(x)$$

Then,

$$(\alpha f)'(c) = \alpha f'(x)$$

3) (fg) definied as

$$(fg)(x) = f(x)g(x)$$

Then,

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

4) Assume $f(c) \neq 0$, $(\frac{1}{f})$ definied as

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$$

Then,

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

Proof | For i):
From theorem 1.7 we have that there exists continous functions $f_1 : J \to \mathbb{R}$ and $g_1 : J \to \mathbb{R}$ such that,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$g(x) = g(c) + g_1(x)(x-c)$$

From these equations we have,

$$f(x) + g(x) = f(c) + g(c) + f_1(x)(x-c) + g_1(x)(x-c)$$

$$\implies (f+g)(x) = (f+g)(c) + (f_1(x) + g_1(x))(x-c)$$
$$\implies (f+g)(x) = (f+g)(c) + ((f_1+g_1)(x))(x-c)$$

Now applying theorem 1.7, we have that

$$(f+g)'(c) = (f_1+g_1)(c) = f_1(c) + g_1(c) = f'(c) + g'(c)$$

$$\implies (f+g)'(c) = f'(c) + g'(c)$$

We shall apply the same structure for the proofs of other algebraic properties,
For ii):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$

From this equation we have,

$$\alpha f(x) = \alpha f(c) + \alpha f_1(x)(x-c)$$
$$\implies (\alpha f)(x) = (\alpha f)(c) + ((\alpha f_1)(x))(x-c)$$

Now applying theorem 1.7, we have that

$$(\alpha f)'(c) = (\alpha f_1)(c) = \alpha f_1(c) = \alpha f'(c)$$

$$\implies (\alpha f)'(c) = \alpha f'(c)$$

For *iii*):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$g(x) = g(c) + g_1(x)(x-c)$$

From these equations we have,

$$f(x)g(x) = f(c)g(c) + (x-c)\left[f(c)g_1(x) + f_1(x)g(c) + f_1(x)g_1(x)(x-c)\right]$$

Now applying theorem 1.7,

$$(fg)'(c) = f(c)g_1(c) + f_1(c)g(c)$$
$$\implies (fg)'(c) = f(c)g'(c) + f'(c)g(c)$$

For *iv*):
From theorem 1.7 we have,

$$f(x) = f(c) + f_1(x)(x-c)$$
$$\implies \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{1}{f(c) + f_1(x)(x-c)} - \frac{1}{f(c)}$$
$$\implies \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{f(c) - f(c) - f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\frac{1}{f(x)} = \frac{1}{f(c)} - \frac{-f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\implies \left(\frac{1}{f}\right)(x) = \left(\frac{1}{f}\right)(c) - \frac{-f_1(x)(x-c)}{(f(c))(f(c) + f_1(x)(x-c))}$$
$$\implies \left(\frac{1}{f}\right)(x) = \left(\frac{1}{f}\right)(c) - (x-c)\frac{-f_1(x)}{(f(c))(f(c) + f_1(x)(x-c))}$$

Now applying theorem 1.7,

$$\left(\frac{1}{f}\right)'(c) = -\frac{f_1(c)}{(f(c))(f(c) + f_1(c)(c-c))}$$
$$\implies \left(\frac{1}{f}\right)'(c) = -\frac{f_1(c)}{(f(c))^2}$$
$$\implies \left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

■

1.3 Differentiation as a linear map

Let $J \in \mathbb{R}$ be an interval and let $c \in J$,
We define vector space $\mathfrak{D}_c$ as,

$$\mathfrak{D}_c = \{f : J \to \mathbb{R} : f \text{ is differentiable at } c\}$$

$\mathfrak{D}_c$ is a vector space over $\mathbb{R}$,
Let's define,

$$\left.\frac{d}{dx}\right|_{x=c} : \mathfrak{D}_c \to \mathbb{R}, \left.\frac{df}{dx}\right|_{x=c} = f'(c)$$

It is trivial to check that, $\left.\frac{d}{dx}\right|_{x=c}$ is a linear map from the vector space $\mathfrak{D}_c$ to itself.

$$\mathfrak{D} = \{f : j \to \mathbb{R} : f \text{ is differential on } J\}$$

$$\mathfrak{F} = \{f : J \to \mathbb{R}\}$$

$$\frac{d}{dx} : \mathfrak{D} \to \mathfrak{F}, f \to \frac{df}{dx}$$

It is again trivial to check that, $\frac{d}{dx}$ is a linear map from the vector space $\mathfrak{D}$ to itself.

NOTATION 1.13 (Continous functions) Given interval J in $\mathbb{R}$

$$\mathcal{C}(J) = \{f : J \to \mathbb{R} : f \text{ is continous on } J\}$$

NOTATION 1.14 (Continously differentiable functions) Given interval J in $\mathbb{R}$

$$\mathcal{C}^1(J) = \{f : J \to \mathbb{R} : f \text{ is differentiable on } J \text{ and } f' \in \mathcal{C}(J)\ \}$$

1.4 Chain Rule

THEOREM 1.15 (Chain Rule) Given intervals J and J_1 in $\mathbb{R}$ and functions $f : J \to \mathbb{R}$ and $g : J_1 \to \mathbb{R}$, such that $f(J) \subseteq J_1$. Let $c \in J$, if f is differentiable at c and g is differentiable at $f(c)$ then (gof) is differentiable at c and $(gof)'(c)$ is given by

$$(gof)'(c) = g'(f(c))f'(c)$$

Proof | From theorem 1.7 we have that there exists $f_1 : J \to \mathbb{R}$ such that,

$$f(x) = f(c) + f_1(x)(x - c)\forall x \in J$$

where f_1 is continous at c and $f'(c) = f_1(c)$
From theorem 1.7 we have that there exists $g_1 : J_1 \to \mathbb{R}$ such that,

$$g(y) = g(f(c)) + g_1(y)(y - f(c))\forall y \in J_1$$

where g_1 is continous at $f(c)$ and $g'(f(c)) = g_1(f(c))$
Now,

$$(gof)(x) = g(f(x))$$

$$\implies (gof)(x) = g(f(c)) + g_1(f(x))(f(x) - f(c))$$

$$\implies (gof)(x) = g(f(c)) + g_1(f(c) + f_1(x)(x - c))f_1(x)(x - c)$$

Since g_1 and f_1 are continous, $g_1(f(c) + f_1(x)(x - c))f_1(x)$ is contious, therefore we can apply theorem 1.7 now and we get,

$$(gof)_1(x) = g_1(f(c) + f_1(x)(x - c))f_1(x)$$

$$(gof)'(c) = (gof)_1(c) = g_1(f(c) + f_1(c)(c - c))f_1(c)$$

$$(gof)'(c) = g'(f(c))f'(c)$$

We shall now declare few conjectures to give ourselves oppertunity to look at better examples (just going along with how the course was thought, I the student do not like this),

Conjecture 1.16 $f(x) = e^x$ is differentiable in $\mathbb{R}$ and $f'(x) = e^x$

Conjecture 1.17 $f(x) = \log(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = 1/x$

Conjecture 1.18 $f(x) = \sin(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = \cos(x)$

Conjecture 1.19 $f(x) = \cos(x)$ is differentiable in $\mathbb{R}$ and $f'(x) = -\sin(x)$

Example 1.20 (This example uses one of the cojectures)

$$f(x) = x^\alpha, x > 0, \alpha \in \mathbb{R}$$

Then,

$$f(x) = e^{\alpha \log(x)}$$

let, $q(x) = \alpha \log(x)$ and $p(x) = e^x$, therefore,

$$f(x) = (poq)(x)$$

Now using chain rule we get,

$$f'(x) = p'(q(x))q'(x)$$

$$\implies f'(x) = e^{\alpha \log(x)} \frac{\alpha}{x}$$

$$\implies f'(x) = \alpha x^{\alpha-1}$$

1.5 Local minima and maxima

Definition 1.21 (Local Minima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a local minima of f if $\exists \delta \in \mathbb{R}$ such that,

1)

$$(c-\delta, c+\delta) \subseteq J$$

2)

$$f(x) \geq f(c) \forall x \in (c-\delta, c+\delta)$$

Definition 1.22 (Local Maxima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a local maxima of f if $\exists \delta \in \mathbb{R}$ such that,

1)

$$(c-\delta, c+\delta) \subseteq J$$

2)

$$f(x) \leq f(c) \forall x \in (c-\delta, c+\delta)$$

Definition 1.23 (Global Minima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a global minima of f if,

$$f(x) \geq f(c) \forall x \in J$$

Definition 1.24 (Global Maxima) Let $f : J \to \mathbb{R}$ be a given real valued function from an interval in $\mathbb{R}$, we say that $c \in J$ is a global maxima of f if,

$$f(x) \leq f(c) \forall x \in J$$

Example 1.25 $f : [-1,1] \to \mathbb{R}, f(x) = x^2$, 0 is local and global minima.

Example 1.26 $f : [0,1] \to \mathbb{R}, f(x) = x$, 0 is global but not local minuma.

Theorem 1.27 For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$ and let c be a local minima, if f is differentiable at c then $f'(c) = 0$.

Proof | Since c is a local minima of f, there exists $\delta > 0$ such that,

$$(c-\delta, c+\delta) \subset J$$

and

$$f(c+h) \geq f(c), 0 < h < \delta$$

since f is differentiable at c we can write,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

For the above limit, the left hand limit is ≤ 0 and right hand limit is ≥ 0 but since f is differentiable, both must be equal and hence must be equal to zero. ∎

Theorem 1.28 For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$ and let c be a local maxima, if f is differentiable at c then $f'(c) = 0$.

Proof | The theorem trivially follows fromt the previous theorem. ∎

We shall see a more generalized form of these theorems later on which includes point of inflection after we prove Taylor's theorem.

1.6 Rolle's theorem

Theorem 1.29 (Rolle's theorem) For $a, b \in \mathbb{R}$, let $f : [a,b] \to \mathbb{R}$ be such that,

1) f is continous in $[a,b]$.
2) f is differentiable in (a,b).
3) $f(a) = f(b)$.

Then $\exists c \in (a,b)$ such that, $f'(c) = 0$.

Proof | Since f is continous and $[a,b]$ is a compact set in $\mathbb{R}$, f will map $[a,b]$ to a compact set in $\mathbb{R}$ and hence $\exists x_1, x_2 \in [a,b]$ such that,

$$f(x_1) \leq f(x) \leq f(x_2) \forall x \in [a,b]$$

1) *Case 1:* $f(x_1) = f(x_2)$
   f is a constant function and hence $f'(x) = 0 \forall x \in [a,b]$.
2) *Case 2:* $f(x_1) \leq f(x_2)$
   We can claim that, $\{x_1, x_2\} \neq \{a, b\}$ since $f(x_1) \neq f(x_2)$. This now proves that there exists a local minima or local maxima which now completes the proof.

■

1.7 Mean value theorem

THEOREM 1.30 (Mean value theorem) For $a, b \in \mathbb{R}$, let $f : [a,b] \to \mathbb{R}$ be such that,

1) f is continous in $[a,b]$.
2) f is differentiable in (a,b).

Then $\exists c \in (a,b)$ such that,

$$f'(c) = \frac{f(a) - f(b)}{b - a}$$

Proof | Define $g : [a,b] \to \mathbb{R}$,

$$g(x) = f(x) - f(a) - \left(\frac{x-a}{b-a}\right)(f(b) - f(a))$$

By algebraic properties of continous real functions, we know g is continous in $[a,b]$ and by algebraic properties of differentiation, we also know that g is differentiable in (a,b). Now by applying Rolle's theorem to g, $\exists c \in (a,b)$ such that

$$\begin{aligned} g'(c) &= 0 \\ \implies f'(c) - \frac{f(b) - f(a)}{b - a} &= 0 \\ \implies f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

EXAMPLE 1.31 (costant function) Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ be differentiable on J and assume $f'(x) = 0 \forall x \in J$, then f is a constant function.

Proof | for $x, y \in J$ by mean value theorem, $f(x) - f(y) = f'(c)(x - y)$, for some $c \in J$ and since $f' = 0$, $f(x) = f(y) \forall x, y \in J$. ■

EXAMPLE 1.32 (non decreasing function) Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ be differentiable on J and assume $f'(x) > 0 \forall x \in J$, then f is a non decreasing function.

Proof | for $x, y \in J$ by mean value theorem, $f(x) - f(y) = f'(c)(x - y)$, for some $c \in J$ and since $f' > 0$, $f(x) \geq f(y) \forall x, y \in J$ with $x \geq y$. ■

Definition 1.33 (Lipschitz functions) For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$. f is said to be lipshcitz if $\exists M > 0$ such that, $|f(x) - f(y)| \leq M|x - y|$.

By definition, it is clear that lipshitz functions are uniformly continous.

Theorem 1.34 For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$, if

1) $f : J \to \mathbb{R}$ is differentiable.
2) $f' : J \to \mathbb{R}$ is bounded.

Then f is lipschitz.

Proof | Let $x, y \in J$, assume $x < y$ now applying mean value theorem on the closed interval $[x, y]$ we get that $\exists c \in (x, y)$ such that,

$$f(x) - f(y) = f'(c)(x - y)$$

and now since f' is bounded, there exists $M > 0$ such that,

$$|f'(c)| \leq M \forall c \in J$$

From this we have,

$$f(x) - f(y) \leq M|x - y| \forall x, y \in J$$

■

Example 1.35 $f(x) = \sqrt{x}, x \in [0, 1]$, $f'(x) = x^{-1/2}/2$, f' is bounded for the given domain and hence is lipschitz.

Definition 1.36 (Continosly differentiable) For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$ be differentiable function, f is said to be continosly differentiable if $f' : J \to \mathbb{R}$ is a continous function.

1.8 Inverse function theorem

Theorem 1.37 (Inverse function theorem) For J interval in $\mathbb{R}$, let $f : J \to \mathbb{R}$, if f is continosly differentiable on J and assuming $f'(x) \neq 0 \forall x \in J$ then the following hold

1) f is strictly monotone.
2) $f(J)$ is an interval.
3) f has an inverse g.
4) g is differentiable.

Proof | f is differentiable in J and hence is continous on J. Continous images of connected sets are connected and, in $\mathbb{R}$ all connected sets are intervals and all intervals are connected sets hence concludes the proof of II).

If f is not strictly monotone then there exists $x, y, z \in J$ such that $x < y < z$ and,

$$f(y) \leq \min\{f(x), f(z)\} \text{ or } f(y) \geq \max\{f(x), f(z)\}$$

With out loss of generality assume,

$$f(y) \leq \min\{f(x), f(z)\}$$

Then either,

$$[f(y), f(x)] \subseteq [f(y), f(z)]$$

or

$$[f(y), f(z)] \subseteq [f(y), f(x)]$$

Without loss of generality assume,

$$[f(y), f(x)] \subseteq [f(y), f(z)]$$

Since $f(x) \in [f(y), f(z)]$ by intermediate value theorem we are able to tell that,

$$\exists z_1 \in [y, z] \text{ such that } f(z_1) = f(x)$$

and now by mean value theorem we are able to tell that,

$$\exists c \in [x, z_1] \text{ such that,}$$

$$f'(c) = \frac{f(x) - f(z_1)}{x - z_1} = 0$$

But this leads to a contradiction that f is such that, $f'(i) \neq 0 \forall i \in J$ and hence our assumption that f is not strictly monotone is wrong.

If $a, b \in J$ such that $a \neq b$ then without loss of generality assume $a < b$ then $f(a) < f(b)$ or $f(b) < f(a)$ and hence $f(a) \neq f(b)$ if $a \neq b$ which makes f an injective mapping and hence,

$$g : f(J) \to J, f(x) \to x$$

is a valid function and hence inverse of f exists.

Let $c \in f(j)$, we need to show that

$$\lim_{y \to c} \frac{g(y) - g(c)}{y - c}$$

exists to show that g is differentiable.

Let $y_n \to c$ with $y_n \in f(J)$ therefore there exists sequence $\{x_n\}$ in J such that,

$$f(x_n) = y_n$$

and there exists $d \in J$ such that,

$$f(d) = c$$

Now,

$$\frac{g(y_n) - g(c)}{y_n - c} = \frac{g(f(x_n)) - g(f(d))}{f(x_n) - f(d)} = \frac{x_n - d}{f(x_n) - f(d)} = \frac{1}{\frac{f(x_n) - f(d)}{x_n - d}}$$

$$\implies \lim_{n\to\infty} \frac{g(y_n)-g(c)}{y_n-c} = \lim_{n\to\infty} \frac{1}{\frac{f(x_n)-f(d)}{x_n-d}}$$

$$\implies g'(c) = 1/f'(d) = 1/f'(g(c))$$

This proves that g the inverse of f is differentiable.

■

1.9 Cauchy's mean value theorem

Theorem 1.38 (Cauchy's mean value theorem) For $a, b \in \mathbb{R}$ let $f, g : [a, b] \to \mathbb{R}$ be functions both differentiable on (a, b) and if $g(x) \neq 0 \forall x \in [a, b]$ then,

1) $g(b) \neq g(a)$
2) $\exists c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof The proof of I) follows from previous theorem now let,

$$G(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$$

Now we have,

$$G(a) = (f(b)-f(a))g(a) - (g(b)-g(a))f(a)$$

$$\implies G(a) = f(b)g(a) - g(b)f(a)$$

$$G(b) = (f(b)-f(a))g(b) - (g(b)-g(a))f(b)$$

$$\implies G(b) = f(b)g(a) - g(b)f(a)$$

Since $G(a) = G(b)$ we are able to use Rolle's theorem for f hence $\exists c \in (a, b)$ such that,

$$G'(c) = 0$$

$$\implies G'(c) = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c)$$

$$\implies (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = 0$$

$$\implies (f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$$

$$\implies \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

■

Theorem 1.39 (Darboux theorem) Given $a, b \in \mathbb{R}$ let $f : [a, b] \to \mathbb{R}$ be a differential function and assume that $f'(a) \leq k \leq f'(b)$ then $\exists c \in [a, b]$ such that $f'(c) = k$.

Proof Let,

$$F : [a, b] \to \mathbb{R}, x \to f(x) - kx$$

F is differentiable and hence a continous function and since continous mapping of a compact connected sets are compact and connected, the image of F has a minimum and a maximum.

Since F is differentiable at a we have that given $\epsilon > 0 \,\exists\, \delta > 0$ such that,

$$\left| \frac{F(x) - F(a)}{x - a} - F'(a) \right| < \epsilon, \forall\, x \in (a, a + \delta)$$

$$\implies F'(a) - \epsilon < \frac{F(x) - F(a)}{x - a} < F'(a) + \epsilon, \forall\, x \in (a, a + \delta)$$

Since $F'(a) < 0$ we can let $\epsilon = -F'(a)/2$, then there exists δ_1 such that

$$\frac{F(x) - F(a)}{x - a} < \frac{F'(a)}{2}, \forall\, x \in (a, a + \delta_1)$$

$$\implies F(x) - F(a) > x - a > 0, \forall\, x \in (a, a + \delta_1)$$

Therefore a is not the point of global maximum for F.

Since F is differentiable at b we have that given $\epsilon > 0 \,\exists\, \delta > 0$ such that,

$$\left| \frac{F(x) - F(b)}{x - b} - F'(b) \right| < \epsilon, \forall\, x \in (b - \delta, b)$$

$$\implies F'(b) - \epsilon < \frac{F(x) - F(b)}{x - b} < F'(b) + \epsilon, \forall\, x \in (b - \delta, b)$$

Since $F'(b) > 0$ we can let $\epsilon = F'(b)/2$, then there exists δ_2 such that

$$\frac{F'(b)}{2} < \frac{F(x) - F(b)}{x - b}, \forall\, x \in (b - \delta, b)$$

$$\implies F(x) - F(a) > x - a > 0, \forall\, x \in (a, a + \delta_1)$$

Therefore a is not the point of global maximum for F.

Then there must exist a maximum of F in the interior of $F([a, b])$ making it also a local maximum, therefore there exists a point $c \in [a, b]$ such that,

$$F'(c) = 0$$

$$\implies f'(c) = k$$

■

Theorem 1.40 (L'Hospital rule) Suppose f and g are real and differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$ for $a, b \in \mathbb{R}$. Suppose,

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$

If,

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a$$

or if,

$$g(x) \to \infty \text{ as } x \to a$$

then,

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a$$

Proof | *Case 1:* $A \neq \pm\infty$
Choose a real number $q > A$ and choose another real number r such that $A < q < r$, because $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$, $\exists\, c \in (a, b)$ such that $a < x < c$ and

$$\frac{f'(x)}{g'(x} < r.$$

If $a < x < y < c$ then by cauchy's mean value theorem

(to be continued)

■