Linear Algebra

Incalculas

Updated on: October 26, 2022

Contents

1 Vector Space

1.1 What is a Vector Space?

Vector space is a set with a binary operation

$$+ : V \times V \to V$$

Key properties of a vector space:

- *i)Commutativity*

$$A + B = B + A$$

- *ii)Associativity*

$$A + (B + C) = (A + B) + C$$

- *iii)Identity*

$$\exists 0 \in V$$

$$\text{s.t } v + 0 = v, \forall v \in V$$

- *iv)Inverse*

$$\text{given } v \in V, \exists v' \in V$$

$$\text{s.t } v + v' = 0$$

A set of scalars associated with a vector space is a Field

1.2 Definition of Fields

Definition 1.1 (Fields)

$$(F, +, \cdot)$$

- $(F, +)$ Satifies i) to iv) [an abelian group]
- $(F - \{0\}, \cdot)$ Satifies i) to iv) [an abelian group]
- $\cdot$ is distributive over $+$

Example 1.2 (Binary field)

$$\{0, 1\}$$

Addition:

$$0 + 1 = 1 = 1 + 0$$

$$1 + 1 = 0$$

Multiplication:

$$1 \cdot 0 = 0 = 0 \cdot 1$$

$$1 \cdot 1 = 1$$

$$0 \cdot 0 = 0$$

What a field does to a vector in a vector space:
$\gamma, \mu \in F, v, w \in V$

$$\gamma \cdot (v + w) = (\gamma v + \gamma w)$$

$$(\gamma + \mu) v = \gamma v + \mu v$$

$$\mu(\gamma \cdot v) = (\mu\gamma) v$$

Properties of a vector space V over field F:
$u, v, w \in V$ and $a, b \in F$

1) $u + v = w = v + u$
2) $u + (v + w) = (u + v) + w$
3) $\exists 0 \in V$, s.t $0 + v = v$
4) $\exists v' \in V$, s.t $v + v' = 0$
5) $\exists 1 \in F$, s.t $1 \cdot v = v$
6) $(ab) \cdot v = a \cdot (bv)$
7) $a \cdot (v + w) = a \cdot v + a \cdot w$
8) $(a + b) \cdot v = a \cdot v + b \cdot v$

EXAMPLE 1.3 set $V = \{0\}$ over the field $\mathbb{R}$

EXAMPLE 1.4 set $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \ldots \mathbb{R}^n$ over the field $\mathbb{R}$

If F is a field, then F is also a Vector space over itself

1.3 Definition of Vector Spaces

DEFINITION 1.6 (Vector Space)

$$(V, F, +, \cdot)$$

- $(V, +)$ is an Abelian group
- $\cdot : F \times V \to V$ is associative and follows distributivity

EXAMPLE 1.7

$$\mathbb{Q}, \mathbb{R}, \mathbb{C} \text{ are all fields}$$

PROPOSITION 1.8 Any field $F \subset \mathbb{R}$ contains $\mathbb{Q}$

Proof | We assume normal addition and multiplication

$$\forall a \in \mathbb{Q}, a = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$
$0, 1 \in F$, as $(F, +)$ and $(F - \{0\}, \times)$ is an abelian groups. As $+$ is closed $1 + 1 = 2 \in F$ and $2 + 1 = 3 \in F$ and so on. By inverse property of $+$, we know $-1, -2, -3 \cdots \in F$, therefore $\mathbb{Z} \in F$. By inverse property of $\times$, we know $\{1/a : a \in \mathbb{Z}\} \in F$. Since $\times$ is closed, $S = \{a/b : a, b \in \mathbb{Z}\} \in F$, set S is just $\mathbb{Q}$. Therefore $Z \in F$ if $F \subset \mathbb{R}$ is a field. ∎

Properties of fields:

1) *Cancellation law,* If $a + b = a + c$ then $b = c$

2) Additive identity is unique
3) Additive inverse is unique
4) Multiplicative identiy is unique
5) Multiplicative inverse is unique
6) if $a \cdot b = a \cdot c$ then $b = c$ if $a \neq 0$

Properties of Vector spaces:

1) *Cancellation law*, If $a + b = a + c$ then $b = c$
2) Additive identity is unique
3) Additive inverse is unique
4) $0 \cdot v = 0 \rightarrow$ is the null vector

Matrices as Vector spaces:

$$M_{n \times m} \left(\mathbb{R} \right)$$

- Addition is entry wise
- $0_{n \times m}$ the additive identity
- Scaling is entry wise

$M_{n \times m}$ looks like $\mathbb{R}^{n.m}$

1.4 Vector Subspaces

Definition 1.9 (Vector Subspace) W is said to be a vector subspace of V if

- $W \subseteq V$
- W is a vector subspace

Example 1.10 $P \left(\mathbb{R} \right)$ is the set of all polynomials.
$P_n \left(\mathbb{R} \right)$ is the set of all polynominals of degree at most n.

- $P_0 \left(\mathbb{R} \right) \subset P_1 \left(\mathbb{R} \right) \subset P_1 \left(\mathbb{R} \right) \subset P_3 \left(\mathbb{R} \right) \subset \ldots P_n \left(\mathbb{R} \right)$
- $\{a_0 + a_2 x^2 + a_4 x^4 \cdots : a_i \in \mathbb{R}\}$

Example 1.11 $M_n \left(\mathbb{R} \right)$ is the set of all $n \times n$ matrices

- Set of all symmetric matrices, $S = \{A : A \in M_n \left(\mathbb{R} \right), A = A^t\}$
- $T = \{A : A \in M_n \left(\mathbb{R} \right), Tr \left(A \right) = 0\}$

2 Span and Linear Independence

2.1 Definiton of Span

Definition 2.1 (Span)

$$\text{span} \left(S \right) = \{\gamma_1 v_1 + \gamma_2 v_2 \ldots \gamma_k v_k : v_i \in S, \gamma_i \in F\}$$

Example 2.2

$$S = \{1, x, x^2\}$$

$$\text{span}(S) = P_2(\mathbb{R})$$

Example 2.3

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\text{span}(S) = \mathbb{R}^2$$

Example 2.4

$$S = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\}$$

$$\text{span}(S) = M_2(\mathbb{R})_{sym}$$

Properties of Span:

1) span(S) is always a subspace containing S as,
   $+$ is closed and
   Scaling operation is also closed.
2) For a subspace $S \subseteq W$, $S \subseteq \text{span}(S) \subseteq W$
3) span (S) is the smallest subspace containg S.
   That is, span (S) is the intersection of all subspaces containing S.
4) span $(\emptyset) = \{0\}$
   This is more of a convention, or you can arrive at this by following the above definition of Span.
5) $S_1 \subseteq S_2$ implies span $(S_1) \subseteq$ span (S_2)
6) span (span (S)) = span (S)
7) span $(W) = W$ if and only if W is a subspace.
8) span $(S_1 \cup S_2) =$ span (S_1) + span (S_2)
   where $A + B = \{a + b : a \in A, b \in B\}$

if V_1 and V_2 are subspaces then, $V_1 + V_2$ is also a subspace.

2.2 Linear (In)dependence

Given a non empty set S, consider span (S). Is there an element $v \in S$ such that, span (S) = span $(S - \{v\})$?

Then S is linearly dependent.

Example 2.6

$$S = \{(0,0,0), (0,1,0), (0,0,e)\}$$

$$\text{span}(S) = \{(0,a,b) : a, b \in \mathbb{R}\} = \text{span}(S - \{(0,0,0)\})$$

For span (S) = span $(S - \{v\})$, there exists $v_1, v_2, \ldots, v_k \in S - \{v\}$ and scalars $\gamma_1, \gamma_2, \ldots, \gamma_k \in F$ such that,

$$V = \sum_{i=1}^{k} \gamma_i v_i$$

as $0 \notin S$, not all γ_i can be 0.

Definition 2.7 (Linear Dependence) A subset $S \subseteq V$ is said to be linearly dependent if there exists scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and v_i are all distinct.

Definition 2.8 (Linear Independence) A subset $S \subseteq V$ is said to be linearly independent if there exists no scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and v_i are all distinct.

Example 2.9 $V = P(\mathbb{R}), S = \{1+x, 1-x, x^2, 2x^2-1\}$

$$\frac{1}{2}(1+x) + \frac{1}{2}(1-x) + (2x^2-1) + (-2)x^2 = 0$$

$S' = \{1+x, 1-x, x^2\}$

if S is linearly dependent, then $\exists v \in S$ such that $span(S) = span(S - \{v\})$

Proposition 2.11 if S is linearly independent and $\exists v \notin \text{span}(S)$ then $S \cup \{v\}$ is linearly independent.

Proof let $\gamma, \gamma_1, \ldots, \gamma_k \in F$ and $v_1, \ldots, v_k \in S$

$$\gamma_1 v1 + \cdots + \gamma_k v_k + \gamma v = 0$$

case 1: $\gamma = 0$,

$$\gamma_1 v1 + \cdots + \gamma_k v_k = 0$$

as S is linearly independent $\gamma_i = 0$ therefore,

$$\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma = 0$$

therefore $S \cup \{v\}$ is linearly independent.
case 2: $\gamma \neq 0$,

$$v = \frac{-\gamma_1}{\gamma} v_1 + \cdots + \frac{-\gamma_k}{\gamma} v_k$$

but $v \notin \text{span}(S)$, therefore this case is not possible. ■

3 Basis and Dimension of a vector space

3.1 Basis and Dimension

Definition 3.1 (Basis) set $\beta \subseteq V$ is called the basis of V if

1) β is linearly independent.
2) β spans V

Definition 3.2 (Dimension) For β being a basis of V, $\dim_F V = |\beta|$

Theorem 3.3 (Replacement Theorem) Let $S \subseteq V$ be a finite set which spans V and let $L \subseteq V$ be a finite linearly independent subset, then,

1) $|L| \leq |S|$
2) $\exists T \subseteq S$ (T is of size $|S| - |L|$) such that $T \cup L$ spans V

Proof

$$L = \{u_1, \ldots, u_m\}$$
$$S = \{v_1, \ldots, v_n\}$$
$$u_m \in V = \text{span}\,(S)$$

therefore,

$$u_m = a_1 v_1 + \cdots + a_n v_n$$

as u_m is in L a linearly independent set, u_m cannot be 0, therefore, all of a_i cannot be zero as well. Let $a_n \neq 0$.

$$v_n = \frac{1}{a_n} u_m + \frac{-a_1}{a_n} v_1 + \cdots + \frac{-a_{n-1}}{a_n} v_{n-1}$$
$$v_n \in \text{span}\{v_1, \ldots, v_{n-1}, u_m\} = \text{span}\,(S) = V$$

similarly,

$$u_{m-1} \in V = \text{span}\{v_1, \ldots, v_{n-1}, u_m\}$$

therefore,

$$u_{m-1} = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b_m u_m$$

as u_{m-1} and u_m are linearly independent, therefore all of a_i cannot be 0

$$v_{n-1} = \frac{1}{a_{n-1}} u_{m-1} + \frac{-b_m}{a_{n-1}} u_m + \frac{-a_1}{a_{n-1}} v_1 \cdots + \frac{-a_{n-2}}{a_{n-1}} v_{n-2}$$

$$v_{n-1} \in \text{span}\{v_1, \ldots, v_{n-1}, u_{m-1}, u_m\} = \text{span}\,(S) = V$$

Now iterate this process to get,

$$S' = \{v_1, \ldots, v_{n-m}, u_1, \ldots, u_m\}$$

Where S' spans V,

$$\text{span}\,(S') = V$$

If $n < m$ then, there will be a point when iterating the steps where a proper subset of L will span V, which would not be possible since this breaks the linear independence of L.
The set T from the theorem is,

$$T = S' - L$$

■

If $S \subseteq V$ spans V and $L \subseteq V$ is an linearly independent subset, then $|L| \leq |S|$.

Corollary 3.5 Given 2 basis of the same vector space V, β and β'.

$$|\beta| = |\beta'|$$

Proof | Set $S = \beta$ and $L = \beta'$. β spans V by definition of basis and β' is linearly independent by definition of basis.
Now according to 1) of replacement theorem, we get.

$$|\beta| \leq |\beta'|$$

Similarly, we can swap S and L to get,

$$|\beta| \geq |\beta'|$$
$$\therefore |\beta| = |\beta'|$$

■

Going back to the definition of dimension, we can see that basis need not be unique but the dimension of a vector space is unique regardless of which basis we use to find the dimension

Corollary 3.7 Let V be a vector space of finite dimension n, then

1) Any finite spanning set S, $|S| \geq n$, if $|S| = n$ then S is a basis.
2) Any linearly independent set L, $|L| \leq n$, if $|L| = n$, then L spans V.
3) Any linearly independent set L can be extended to form a basis of V.

Corollary 3.8 Let V be a finite dimensional vector space over the field F andlet $W \subseteq V$ be a subspace, then,

$$\dim_F(W) \leq \dim_F(V)$$

If equality holds, then, $V = W$.

Proof | Let β be a basis of V of size n. Now look at $\text{span}\{\emptyset\} = \{0\}$, if it is W then we are done, if not, we choose $u_1 \in W$ which is non zero, $L_1 = \{u_1\} \subseteq W$ is linearly independent. We check again if $\text{span}(L_1) = W$, if it is then we are donem, if it isn't then we choose $u_2 \in W \setminus \text{span}(L_1)$. We iterate this process until we get $\beta_\circ = \{u_1, \ldots, u_k\}$ which spans W. This set is linearly independent by construction (refer to proposition 2.11). Using 1) from replacement theorem, using β as S and $\beta_c irc$ as L, we get,

$$|\beta_\circ| \leq |\beta|$$

which is the same as,

$$\dim_F(W) \leq \dim_F(V)$$

And using corollary 3.8 we can say that the equality implies $V = W$. ■

Given subspace W of vector space V, with $\beta_\circ$ as the basis of W, we can extend the basis $\beta_\circ$ to β as a basis for V

3.3 Direct Sum

Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Then $W_1 + W_2$ is also a subspace.

Case A: $W_1 \cap W_2 = \{0\}$

Definition 3.10 (Direct Sum) Given subspaces W_1 and W_2 of a finite dimensional vector space V, if $W_1 \cap W_2 = \{0\}$ then the vector subspace $W_1 + W_2$ is known as the direct sum.

Case B: $W_1 \cap W_2 \neq \{0\}$

$$\dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2)$$

Proof | Start with basis of $W_1 \cap W_2$, $\beta_\circ = \{u_1, \ldots, u_k\}$. Now extend this basis to a basis of W_1, $\beta_1 = \{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ and similarly for W_2, $\beta_2 = \{u_1, \ldots, u_k, w_1', \ldots, w_m\}$

$$\beta = \beta_1 \cup \beta_2$$

$$\beta = \{u_1, \ldots, u_k, w_1, \ldots, w_l, w_1', \ldots, w_m'\}$$

As any element in $W_1 + W_2$ can be represented by a linear combination of elements of β, β spans $W_1 + W_2$. To show that β is linearly independent,

$$c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l + \gamma_1w_1' + \cdots + \gamma_mw_m' = 0$$

$$=> c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l = -\gamma_1w_1' - \cdots - \gamma_mw_m' = v$$

Clearly $v \in W_1$ and $v \in W_2$, therefore $v \in W_1 \cap W_2$, therefore v can be written as,

$$v = c_1'u_1 + \cdots + c_k'u_k$$

$$=> c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l = -\gamma_1w_1' - \cdots - \gamma_mw_m' = v = c_1'u_1 + \cdots + c_k'u_k$$

From these 2 equations, we are able to determine that β is linearly independent,
We have,

$$-\gamma_1w_1' - \cdots - \gamma_mw_m' = c_1'u_1 + \cdots + c_k'u_k$$

$\{u_1, \ldots, u_k, w_1', \ldots, w_m'\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1' = \cdots = c_k' = 0$$

Similarly, we have,

$$c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l = c_1'u_1 + \cdots + c_k'u_k$$

We know $c_1' = \cdots = c_k' = 0$, therefore,

$$c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l = 0$$

$\{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1 = \cdots = c_l = 0$$

Therefore the only solution for

$$c_1u_1 + \cdots + c_ku_k + d_1w_1 + \cdots + d_lw_l + \gamma_1w_1' + \cdots + \gamma_mw_m' = 0$$

is all the scalars being equal to zero, which implies linear independence of β. Therefore β is both linearly independent and spans $W_1 + W_2$, which makes it a basis for $W_1 + W_2$.

$$\dim_F(W_1) = |\beta_1| = k + l$$

$$\dim_F(W_2) = |\beta_1| = k + m$$

$$\dim_F(W_1 + W_2) = |\beta| = k + l + m$$

$$\dim_F(W_1 \cap W_2) = |\beta_\circ| = k$$

$$\therefore \dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2)$$

■

Proposition 3.11 If V is the direct sum, $W_1 + W_2$ then $v \in V$ can be uniquely expressed as $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

Proof Let, $v = w_1 + w_2 = w_1' + w_2'$ then, $w_1 - w_1' = w_2 - w_2' = x$. therefore $x \in W_1 \cap W_2$ and by definition of direct sum, $x \in \{0\}$, therefore $x = 0$. Which gives us, $w_1 = w_1'$ and $w_2 = w_2'$. ■

4 Quotient Spaces

4.1 Quotient Spaces

Goal is to create a new vector space out of $W \subseteq V$, define an equvalence relation $\sim_W$ on V. $v_1 \sim_W v_2$ if and only if $v_1 - v_2 \in W$

- *Reflexive* Yes, if $0 \in W$.
- *Symmetric* Yes, if whenever, $v_1 - v_2 \in W$ implies $v_2 - v_1 \in W$, ie, $-(v_1 - v_2) \in W$.
- *transitivity* Yes, if whenever $v_1 - v_2 \in W$ and $v_2 - v_3 \in W$ implies $v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \in W$.

Starting with any arbitary subset of V we end with a subspace of V due to the conditions that needs to be satisfied for $\sim_W$ to be an equivalence relation.

For any eqivalence relation $\sim$ on an arbitary set S, an equivalence class is given by.

$$[s] = \{s' : s' \in S, s' \sim s\}$$

Equivalence relations are disjoined and partition the set. that is either $[s] \cap [s'] = \emptyset$ or $[s] = [s']$.

For $\sim_W$,

$$[v] = \{v_1 : v_1 - v \in W\}$$
$$[v] = \{v_1 = v + w : w \in W\}$$
$$[v] = v + W$$

$[v_1] = [v_2]$ we translate via W,

$$v_1 + W = v_2 + W$$
$$v_1 = v_2 + W$$
$$v_1 - v_2 = w \in W$$

Therefore $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.

We can create a new vector space, where each element is an equivalence class of the equivalence relation $\sim_W$. Consider the set of all equivalence class of the relation $\sim_W$, V/W.

Addition under the vector space V/W,

$$+ : V/W \times V/W \to V/W$$

$$[v_1] + [v_2] = [v_1 + v_2]$$

Scaling,

$$\times : F \times V/W \to V/W$$

$$(\gamma, [v]) \to [\gamma v]$$

5 Linear Maps

5.1 Linear Maps

Given vector spaces V, W (over the same field F) and linear map T.

$$T : V \to W$$

T should be compatible with the structures of the vector spaces $\forall v_1, v_2, v \in V$ and $\forall \gamma \in F$.

- $T(v_1 + v_2) = Tv_1 + Tv_2$
- $T(\gamma v) = \gamma T v$

Example 5.1 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$T(x+y) = \gamma_\circ (x+y) = \gamma_\circ x + \gamma_\circ y = Tx + Ty$$
$$T(\gamma y) = \gamma_\circ (\gamma y) = \gamma(\gamma_\circ y) = \gamma(Tx)$$

$T(0_v) = 0_w$

Example 5.3 (Dilation)

$$T : V \to V, v \to 2v \, (= v + v)$$

$$\gamma v \to 2(\gamma v) = \gamma(2v) = \gamma(Tv)$$

If in the field, $2 = 0$ then the map is the zero map, if not then it's a bijective map.

Example 5.4 (Identity map)

$$I : V \to V, v \to v$$

Example 5.5 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$

Example 5.6 (Matrices)

$$A \in M_{n \times n}(\mathbb{R})$$

$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$L_A \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + L_A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$L_A \left(\gamma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \to \gamma L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example 5.7 (Reflection)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

This is for reflection across x-axis, for any other reflection the line as to pass through the origin because 0 has to be mapped to 0 and in general $T.T$ gives us the identity map.

Example 5.8 (Rotation)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (-y, x)$$

This is rotaion of $\pi/2$ clockwise, for a more general rotaion,

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$||T(x, y)|| = ||(x, y)||$$

This is a property of every rotation linear map.

Example 5.9 (Projection)

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$

$$P^2 = P.P = P$$

$$Q^2 = Q.Q = Q$$

This is the defining property of a projection map.

Example 5.10 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$

$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$

5.2 Null Space

Definition 5.11 (Null Space) Given $T : V \to W$, null space $N(T)$ is given by,

$$N(T) = \{v \in V : T(v) = 0_w\}$$

Example 5.12 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$N(T) = \{0\}$$

Example 5.13 (Dilation)

$$T : V \to V, v \to 2v\,(= v + v)$$

$$N(T) = \begin{matrix} V & 2 = 0 \text{ in } F \\ \{0\} & 2 \neq 0 \text{ in } F \end{matrix}$$

Example 5.14 (Identity map)

$$I : V \to V, v \to v$$

$$N(T) = \{0\}$$

Example 5.15 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$

$$N(T) = V$$

Example 5.16 (Reflection)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

$$N(T) = \{0\}$$

Example 5.17 (Rotation)

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$N(T) = \{0\}$$

Example 5.18 (Projection)

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$

$$N(P) = \{(0, y)\}$$

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$

$$N(Q) = \{(x, y) : x = y\}$$

Example 5.19 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$
$$N(T) = \{0\}$$
$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$
$$N(T) = \{0\}$$

5.3 Range

Definition 5.20 (Range) Given $T : V \to W$, null space $R(T)$ is given by,

$$R(T) = \text{Image of } V \text{ under } T$$

Example 5.21 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$
$$R(T) = \mathbb{R}$$

Example 5.22 (Dilation)

$$T : V \to V, v \to 2v\,(= v + v)$$
$$R(T) = \begin{matrix} \{0\} & 2 = 0 \text{ in } F \\ V & 2 \neq 0 \text{ in } F \end{matrix}$$

Example 5.23 (Identity map)

$$I : V \to V, v \to v$$
$$R(T) = V$$

Example 5.24 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$
$$R(T) = \{0\}$$

Example 5.25 (Reflection)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$
$$R(T) = \mathbb{R}^2$$

Example 5.26 (Rotation)

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$R(T) = \mathbb{R}^2$$

Example 5.27 (Projection)

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$
$$R(P) = \{(x, 0)\}$$
$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$
$$R(Q) = \{(x, 0)\}$$

Example 5.28 (Inclusion)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$
$$R(T) = \{(x, y, 0)\}$$
$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$
$$R(T) = \{(x, y, ax + by)\}$$

5.4 Null space and Range as vector subspaces

$$v_1, v_2 \in N(T), \gamma \in F$$

$$T(v_1 + v_2) = Tv_1 + Tv_2$$

$$T(v_1 + v_2) = 0 + 0 = 0$$

$$\therefore v_1 + v_2 \in N(T)$$

$$T(\gamma v_1) = \gamma T v_1 = \gamma 0 = 0$$

$$\therefore \gamma v_1 \in N(T)$$

Therefore $N(T)$ is a subspace.

$$w_1, w_2 \in R(T), \gamma \in F$$

$$w_1 = Tv_1$$

$$w_2 = Tv_2$$

$$w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$$

$$\therefore w_1 + w_2 \in R(T)$$

$$w_1 = Tv_1$$

$$\gamma T v_1 = \gamma w_1 = T(\gamma v_1)$$

$$\therefore \gamma w_1 \in R(T)$$

Therefore $R(T)$ is a subspace.

EXAMPLE 5.29 (Differentiation)

$$D : P(\mathbb{R}) \to P(\mathbb{R}), P \to P'$$

$$a_0 + a_1 x + \cdots + a_k x^k \to a_1 + 2a_2 x + \cdots + k a_k x^{k-1}$$

$$D.(P_1 + P_2) = D.P_1 + D.P_2$$

$$D.(\gamma P_1) = \gamma D.P_1$$

$$D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$$

$$N(D) = \{a_0 \in \mathbb{R}\}$$

$$R(D) = P_{n-1}(\mathbb{R})$$

EXAMPLE 5.30 (Integration)

$$I : P(\mathbb{R}) \to P(\mathbb{R}), P \to \int P dx$$

$$a_0 + \cdots + a_k x^k \to a_0 x + \cdots + \frac{a_k x^{k+1}}{k+1}$$

$$N(I) = \{0\}$$

$$R(I) = \{J(x) \in P_{n+1}(\mathbb{R}) : J(0) = 0\}$$

DEFINITION 5.31 (Nullity) Given a linear map, nullity of the linear map is the dimension of the null space.

DEFINITION 5.32 (Rank) Given a linear map, rank of the linear map is the dimension of the range.

5.5 Rank Nullity Theorem

THEOREM 5.33 (Rank Nullity) Let V be a finite dimensional vector space and W be any vector space, if $T : V \to W$ is a linear map then,

$$\dim_F(R(T)) + \dim_F(N(T)) = \dim_F V$$

Proof | Let $\beta_\circ = \{v_1, \ldots, v_k\}$ be a basis of $N(T)$, now extend this to a basis of V, β

$$\beta = \{v_1, \ldots, v_k, v_1', \ldots, v_l'\}$$

We know,

$$T(\beta) = \{0_w, T(v_1'), \ldots, T(v_l')\}$$

$$L = \{T(v_1'), \ldots, T(v_l')\}$$

Let's take,

$$\gamma_1 T(v_1') + \cdots + \gamma_l T(v_l') = 0$$

$$T\left(\gamma_1 v_1' + \cdots + \gamma_l v_l'\right) = 0$$
$$\gamma_1 v_1' + \cdots + \gamma_l v_l' \in N\left(T\right)$$
$$\gamma_1 v_1' + \cdots + \gamma_l v_l' = \gamma_1' v_1 + \cdots + \gamma_l' v_k$$
$$\gamma_1 v_1' + \cdots + \gamma_l v_l' - \gamma_1' v_1 - \cdots - \gamma_l' v_k = 0$$

By the linear independence of β we can conclude that,

$$\gamma_1 = \cdots = v_l' = \gamma_1' = \gamma_l' = 0$$

Which gives us that L is linearly independent and this is a set which spans $R\left(T\right)$ which makes it a basis of $R\left(T\right)$

$$\dim_F\left(R\left(T\right)\right) + \dim_F\left(N\left(T\right)\right) = |L| + |\beta_\circ| = l + k$$
$$\dim_F V = |\beta| = l + k$$
$$\therefore \dim_F\left(R\left(T\right)\right) + \dim_F\left(N\left(T\right)\right) = \dim_F V$$

■

Given a linear map $T : V \to W$, if $S \subseteq V$ spans V, then $T\left(S\right)$ spans $R\left(T\right)$.

Proof | Given $w = T\left(v\right) \in R\left(T\right)$, express $v = \sum \gamma_i v_i$, where $v_i \in S$ and $\gamma_i \in F$. $T\left(v\right) = \sum \gamma_i T\left(v_i\right) \implies T\left(v\right) \in \text{span}\left(T\left(S\right)\right)$ ■

Corollary 5.35 Given a linear map, $T : V \to V$, where V is a finite dimensional vector space, then the following are equivalent.

1) T is injective
2) T is surjective

Proof | T is injective implies, $N\left(T\right) = \{0\} \implies \text{Nullity} = 0 \implies \text{Rank} = \dim_F V - 0 = \dim_F V \implies R\left(T\right) = V \implies T$ is surjective.
T is surjective implies, $R\left(T\right) = V \implies \text{rank} = \dim_F V \implies \text{Nullity} = 0 \implies N\left(0\right) = \{0\}$.
If $Tv_1 = Tv_2$ and $N\left(0\right) = \{0\}$, then, $T\left(v_1 - v_2\right) = 0 \implies v_1 - v_2 \in N\left(T\right) \implies v_1 - v_2 \in \{0\} \implies v_1 = v_2$.
Therefore, T is surjective implies injectivity as well. ■

As part of the above proof, we also proved that Null space being zero implies injectivity.

6 Isomorphism, Matrices and Linear maps as vector spaces

6.1 Linear Isomorphism

Definition 6.1 (Linear Isomorphism) A linear map $T : V \to W$ between vector spaces V and W both over a common field F is called an isomorphism if T is bijective.

Example 6.2

$$T : P_2\left(\mathbb{R}\right) \to \mathbb{R}^3, a + bx + cx^2 \to \left(a, b, c\right)$$

Zero in $\mathbb{R}^3$ is $(0,0,0)$, clearly only polynominal which is mapped to it is 0. Therefore $N(T) = \{0\}$ and Nullity $= 0$. If $Tp_1 = Tp_2$, then, $T(p_1 - p_2) = (0,0,0) \implies T\left(a_1 - a_2 + (b_1 - b_2)x + (c_1 - c_2)x^2\right) = (0,0,0) \implies (a_1 - a_2, b_1 - b_2, c_1 - c_2) = (0,0,0) \implies a_1 = a_2, b_1 = b_2, c_1 = c_2 \implies p_1 = p_2$, therefore T is injective. Rank $= \dim_F P_2(\mathbb{R}) - 0 = 3 \implies R(T) = \mathbb{R}^3$, therefore T is surjective. Therefore T is bijective.

When we say a vector space looks like another vector space, we mean that there exists a linear isomorphism from one vector space to another, ie V is isomorphic to W as a vector space.

Example 6.4

$$T : M_2(\mathbb{R}) \to \mathbb{R}^4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Properties of isomorphic linear maps (we will take $T : V \to W$ as our isomorphic linear map to discuss the properties):

1) $T^{-1} : W \to V$ is a linear map

Proof | For $w_1, w_2 \in W$ there exists unique $v_1, v_2 \in V$ such that, $Tv_1 = w_1$ and $Tv_2 = w_2$ because of bijectivity of T, there also exists unique $v \in V$ such that, $Tv = w_1 + w_2$.

$$T^{-1}w_1 = v1, T^{-1}w_2 = v_2$$

$$\implies v_1 + v_2 = T^{-1}w_1 + T^{-1}w_2$$

$$\implies T(v_1 + v_2) = T\left(T^{-1}w_1 + T^{-1}w_2\right)$$

$$\implies T(v_1 + v_2) = T\left(T^{-1}w_1\right) + T\left(T^{-1}w_2\right) = w_1 + w_2$$

$$\implies T(v_1 + v_2) = w_1 + w_2$$

Therefore unique $v = v_1 + v_2$. Therefore $T^{-1}(w_1 + w_2) = T^{-1}w_1 + T^{-1}w_2$.
For $w \in W$ there exists unique $v \in V$ such that, $Tv = w$ because of bijectivity of T, there also exists unique $v' \in V$ such that, $Tv = \gamma w$.

$$T^{-1}w = v$$

$$\gamma T^{-1}w = \gamma v$$

$$\implies T(\gamma v) = T\left(\gamma T^{-1}w\right)$$

$$\implies T(\gamma v) = \gamma T\left(T^{-1}w\right) = \gamma w$$

$$\implies T(\gamma v) = \gamma w$$

Therefore unique $v' = \gamma v$. Therefore $T^{-1}(\gamma w) = \gamma T^{-1}w$
Showing that addition and scaling is closed is suffice to show that T^{-1} is a linear map. ∎

2) $\gamma T : V \to W, V \to \gamma Tv$ is an isomorphic linear map.

Proof | Take $v_1, v_2 \in V$, such that,

$$\gamma Tv_1 = \gamma Tv_2$$

$$\implies \gamma T(v_1 - v_2) = 0$$
$$\implies T(v_1 - v_2) = 0$$
$$\implies v_1 = v_2$$

This implies injectivity of γT.
Suppose γT is not surjective, then $\exists w \in W$ such that, $\forall v \in V$,

$$\gamma T v \neq w$$

$$T v \neq w/\gamma$$

But this is a contradiction given we know T to be a bijection, therefore γT has to be surjective, therefore is bijective and an isomorphism. ■

3) • *a*)If $T : V \to W$ is an injective linear map and $\{v_1, \dots, v_n\}$ is a linearly independent set, then $\{Tv_1, \dots, Tv_n\}$ is also a linearly independent set.

Proof

$$\gamma_1 T v_1 + \cdots + \gamma_n T v_2 = 0$$
$$\implies T(\gamma_1 v_2 + \cdots + \gamma_n v_n) = 0$$

Injectivity of the linear map implies,

$$\gamma_1 v_2 + \cdots + \gamma_n v_n = 0$$

Linear independence of $\{v_1, \dots, v_n\}$

$$\gamma_1 = \cdots = \gamma_n = 0$$

■

• *b*)If $T : V \to W$ is an surjective linear map and let $\{w_1, \dots, w_n\}$ be a subset of W and choose $v_i \in V$ such that $Tv_i = w_i$ and if $\{v_1, \dots, v_n\}$ spans V, then $\{w_1, \dots, w_n\}$ spans W.

Proof | We know for a given $w \in W$ there exists at one $v \in V$ such that, $Tv = w$.

$$v = \gamma_1 v_1 + \cdots + \gamma_n v_n$$

$$Tv = T(\gamma_1 v_1 + \cdots + \gamma_n v_n)$$
$$w = T(\gamma_1 v_1) + \cdots + T(\gamma_n v_n)$$
$$w = \gamma_1 T v_1 + \cdots + \gamma_n T v_n$$
$$w = \gamma_1 w_1 + \cdots + \gamma_n w_n$$

Since we choose an arbitart w, $\{w_1, \dots, w_n\}$ spans the set W. ■

6.2 Linear maps as vectors

CONVENTION 6.5 $\mathcal{L}(V, W)$ represents set of all linear maps from $V \to W$ and $\mathcal{L}(V)$ represents set of all linear maps from $V \to V$.

$$+ : \mathcal{L}(V, W) \times \mathcal{L}(V, W) \to \mathcal{L}(V, W)$$

$$(T_1 + T_2) : V \to W, v \to T_1 v + T_2 v$$

$$(T_1 + T_2) \in \mathcal{L}(V, W)$$

$$\times : F \times \mathcal{L}(V, W) \to \mathcal{L}(V, W)$$

$$(\gamma T) : V \to W, v \to \gamma T v$$

$$(\gamma T) \in \mathcal{L}(V, W)$$

Example 6.6 (Dual of V) $W = \mathbb{R}$, where V is a vector space over R. Represented by $V^* = \mathcal{L}(V, \mathbb{R})$ and $\dim_F(V^*) = \dim_F(V)$.

Example 6.7 $V = \mathbb{R}$, where W is a vector space over R. $\mathcal{L}(\mathbb{R}, W)$ is a vector space, any element in $\mathcal{L}(\mathbb{R}, W)$ has unique property where the map is determined by just where 1 is mapped to.

$$T : \mathbb{R} \to W$$

$$T(\gamma) = \gamma T(1)$$

$$\Phi : \mathcal{L}(\mathbb{R}, W) \to W$$

Φ is an isomorphism.

Proof For every $w \in W$ there is an unique $T_w \in \mathcal{L}(\mathbb{R}, W)$ such that, $T_w : \mathbb{R} \to W, \gamma \to \gamma w$ or $T_w(\gamma) = \gamma w$. set of all T_w is the same as $\mathcal{L}(\mathbb{R}, W)$ since if there exists an element T‘ which is not in the set of all T_w then the property that the map is determined by where 1 is mapped to will yield a contradiction. Therefore, we have a bijective between W and $\mathcal{L}(\mathbb{R}, W)$. Therefore, Φ is an isomorphism. ∎

Conjecture 6.8 $\dim_F \mathcal{L}(V, W) = \dim_F V \times \dim_F W$

6.3 Ordered Basis

Definition 6.9 (Ordered Basis) An ordered basis of a vector space V is $\beta = \{v_1, \ldots, v_n\}$ where β is an ordered set.

Example 6.10 $V = \mathbb{R}^2$, $\beta = \{(1, 0), (0, 1)\}$

Example 6.11 $V = P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

6.4 Matrix Representation

Goal is to create a matrix to represent $T : V \to W$, let $\beta = \{v_1, \ldots, v_m\}$ be an ordered basis of V, let $\alpha = \{w_1, \ldots, w_n\}$ be an ordered basis of W.

It will be assumed you know basics of matrices, although everything will be revisted later in the notes with formal definitions.

$$Tv_j = \sum_{i=1}^{n} \gamma_{ij} w_i$$

$$\begin{pmatrix} Tv_1 \\ Tv_2 \\ \vdots \\ Tv_m \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} \end{pmatrix}$$

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} \end{pmatrix} \in M_{m \times n}(F)$$

Aim is to associate a matrix $[T]_\beta^\alpha$ for a linear map $T : V \to W$, where $\beta = \{v_1, \ldots, v_m\}$ is an ordered basis of V and $\alpha = \{w_1, \ldots, w_n\}$ is an ordered basis of W.

$$S_\beta : V \to F^m$$

Where S_β is given by, for any $v \in V$,

$$v = \sum_{i=1}^{m} \gamma_i v_i \to \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} \in F^m$$

$$S_\alpha : W \to F^n$$

Where S_α is given by, for any $w \in W$,

$$w = \sum_{i=1}^{n} \gamma_i w_i \to \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \in F^n$$

$$T : V \to W$$
$$S_\beta : V \to F^m$$
$$S_\alpha : W \to F^n$$
$$S_\alpha \cdot T \cdot S_\beta^{-1} : F^m \to F^n$$

We now define $[T]_\beta^\alpha$ to be the matrix associated with the map $S_\alpha \cdot T \cdot S_\beta^{-1} : F^m \to F^n$

Conjecture 6.12 $[Tv]^\alpha = [T]_\beta^\alpha [v]^\beta$

Example 6.13

$$I : V \to V$$

For $\beta = \alpha$,

$$[I]_\beta^\alpha = I_n$$

For $\beta = \{e_1, e_2\}$ and $\alpha = \{e_2, e_1\}$,

$$[I]^{\alpha}_{\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 6.14 (Differentiation)

$$D : P_2(\mathbb{R}) \to P_2(\mathbb{R}), P(x) \to P'(x)$$

$$\beta = \{1, x, x^2\}$$

$$[D]^{\beta}_{\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = \{2x, 1, x^2\}$$

$$[D]^{\alpha}_{\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To associate a matrix to a linear map,

$$T : V \to W$$

$$w\,(\in W) = \sum_{i=1}^{n} = \gamma_i w_i$$

$[T]^{\alpha}_{\beta}$ is the matrix obtaining by evaulting T on V with basis β with respect to α.

$$\left([T]^{\alpha}_{\beta}\right)_{ij} = \gamma_{ij}$$

Where,

$$Tv_J = \gamma_{1j} w_1 + \cdots + \gamma_{ij} w_i + \cdots + \gamma_{nj} w_n$$

$$[Tv]^{\alpha} = [Tv]^{\alpha}_{\beta}\,[v]^{\beta}$$

Convention 6.15 $[a]^b$ represents a vector a represented as a linear combination of elements of the set β, where β is a basis for a vector space with a in it.

$\beta = \{v_1, \ldots, v_m\}$, represents ordered basis of V and $\alpha = \{w_1, \ldots, w_n\}$, represents ordered basis of W.

The j$^{\text{th}}$ column of $[T]^{\alpha}_{\beta}$ is given by the coefficients obtained by expanding $T(v_j)$ in terms of α

Convention 6.16 Just $[a]^b$ will be used instead of $[a]^b_b$

Example 6.17

$$I : V \to V, v \to v$$

Given $\dim_F V = n$,

$$[I]^{\beta}_{\beta} = I_n$$

Example 6.18

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

Rotation anti clock-wise by θ,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\beta = \{(1,0),(0,1)\}$$
$$\gamma = \{(1,1),(1,-1)\}$$
$$[R_\theta]_\beta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$
$$[R_\theta]_\beta^\gamma = [I_2]_\beta^\gamma [R_\theta]_\beta^\beta$$
$$[R_\theta]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$
$$[R_\theta]_\beta^\gamma = \begin{pmatrix} cos\theta + sin\theta & -sin\theta + cos\theta \\ cos\theta - sin\theta & -sin\theta - cos\theta \end{pmatrix}$$
$$[R_\theta]_\gamma = \begin{pmatrix} cos\theta & sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Sum of diagonal elements of $[R_\theta]_\gamma$ and $[R_\theta]_\beta$ are the same, and so is the determinant of both matrices.

Example 6.20

$$D : P_n(\mathbb{R}) \to P_n(\mathbb{R}), P(x) \to P'(x)$$
$$\beta = \{1, x, \ldots, x^n\}$$
$$[D]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We could look at $P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ but let's stick to square matrices for now, and $P_{n-1}(\mathbb{R})$ is clearly a subspace of $P_n(\mathbb{R})$.

We can see that the sum of diagonal elements is clearly 0 and the determinant is 0 as well.

Proposition 6.21 Given $\dim_F V = m$ and $\dim_F W = n$ and given $\beta = \{v_1, \ldots, v_m\}$ a basis of V and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W.

$$\Phi : \mathcal{L}(V, W) \to M_{n\times m}(F), T \to [T]_\beta^\gamma$$

Φ is an isomorphism.

This statement says that the matrix representation of linear maps between 2 finite dimensional vector spaces is "equivalent" (i.e: isomorphic) to the linear map itself.

Proof | Is $[S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$
J$^{\text{th}}$column of $[S+T]_\beta^\gamma =$ coefficient of $(S+T)\,v_j$
J$^{\text{th}}$column of $[S]_\beta^\gamma =$ coefficient of Sv_j
J$^{\text{th}}$column of $[T]_\beta^\gamma =$ coefficient of Tv_j

$$\therefore [S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$$

Is $[\lambda T]_\beta^\gamma = \lambda\,[T]_\beta^\gamma$.
J$^{\text{th}}$column of $[\lambda T]_\beta^\gamma =$ coefficient of $\lambda T v_j = \lambda$ coefficient of $Tv_j =$ j$^{\text{th}}$ coefficient of $\lambda\,[T]_\beta^\gamma$.
Therefore Φ is a linear map.
Zero in $M_{n\times m}(F)$ is the zero matrix.

$$\forall v \in V, [v]^\gamma \times 0_{n\times m} = [0]^\beta$$

Therefore the only preimage of $0_{n\times m}$ is the zero map between V and W, therefore the null space of Φ is $\{0\}$, now we know that Φ is injective.
Given a matrix $A \in M_{n\times m}$, define $T_A : V \to W, [v]_\beta \to [v]_\beta A$, where$[v]_\beta A$ (a $1\times n$ matrix) represents a vector w as a linear sum of the vectors in basis γ. T_A is a well defined linear map from $V \to W$ (proof of this is trivial but skipped here), therefore $T_A \in \mathcal{L}(V,W)$ is a preimage for $A \in M_{n\times m}$. Therefore Φ is bijective and hence an isomorphism. ∎

This also proves conjecture 6.8

7 Dual spaces and Dual maps

7.1 Dual Spaces

Definition 7.1 (Dual Space) Given a finite dimensional vector space V over field F, dual space is given by $V^* = \mathcal{L}(V,F)$, called as dual of V.

$$\dim_F V^* = \dim_F M_{1\times n}(F) = n = \dim_F V$$

There is no natural linear isomorphism between V and V^*.

"There cannot be a cannonical or God-given isomorphism"

We can take dual of a dual, known as double dual.

"In a sense, double duals are more well behaved"

7.2 Dual basis

let $\beta = \{v_1, \dots, v_n\}$ be a basis of V.

$$v_i^* : V \to F, a_1v_1 + \cdots + a_nv_n \to a_i$$

$$v_i^* v_j = \delta_{ij}$$

$\{v_1^*, \dots, v_n^*\}$ is a basis of V^*.

Proof | Given $T \in V^*, T : V \to F$ and $v \in V$, where $\beta = \{v_1, \ldots, v_n\}$ is a basis of V,

$$Tv = T\,(a_1 v_1, \ldots, a_n v_n)$$

$$Tv = a_1 T v_1 + \cdots + a_n T v_n$$
$$Tv = T v_1 a_1 + \cdots + T v_n a_n$$
$$Tv = T v_1 v_1^* v + \cdots + T v_n v_n^* v$$
$$Tv = (T v_1 v_1^* + \cdots + T v_n v_n^*)\, v$$
$$T = T v_1 v_1^* + \cdots + T v_n v_n^*$$

$T v_i$'s are fixed scalars, therefore the set $\beta^* \{v_1^*, \ldots, v_n^*\}$ spans V^*, linear independence follows from the fact that $\dim_F V^* = n$

■

examples for dual spaces to be added later

7.3 Dual Maps