# Linear Algebra

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## Contents

# 1 VECTOR SPACE

## 1.1 What is a Vector Space?

Vector space is a set with a binary operation

$$+ : V \times V \to V$$

Key properties of a vector space:

- i) *Commutativity*

$$A + B = B + A$$

- ii) *Associativity*

$$A + (B + C) = (A + B) + C$$

- iii) *Identity*

$$\exists 0 \in V$$

$$\text{s.t } v + 0 = v, \forall v \in V$$

- iv) *Inverse*

$$\text{given } v \in V, \exists v' \in V$$

$$\text{s.t } v + v' = 0$$

A set of scalars associated with a vector space is a Field

## 1.2 Definition of Fields

> **DEFINITION 1.1 (Fields)**
> $$(F, +, \cdot)$$
> - $(F, +)$ Satifies i) to iv) [an abelian group]
> - $(F - \{0\}, \cdot)$ Satifies i) to iv) [an abelian group]
> - $\cdot$ is distributive over $+$

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**EXAMPLE 1.2 (Binary field)**

$$\{0, 1\}$$

Addition:

$$0 + 1 = 1 = 1 + 0$$

$$1 + 1 = 0$$

Multiplication:

$$1 \cdot 0 = 0 = 0 \cdot 1$$

$$1 \cdot 1 = 1$$

$$0 \cdot 0 = 0$$

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What a field does to a vector in a vector space:
$\gamma, \mu \in F, v, w \in V$

$$\gamma \cdot (v + w) = (\gamma v + \gamma w)$$

$$(\gamma + \mu) v = \gamma v + \mu v$$

$$\mu\left(\gamma \cdot v\right) = \left(\mu\gamma\right) v$$

Properties of a vector space $V$ over field $F$:
$u, v, w \in V$ and $a, b \in F$

1) $u + v = w = v + u$
2) $u + (v + w) = (u + v) + w$
3) $\exists 0 \in V, \text{s.t } 0 + v = v$
4) $\exists v' \in V, \text{s.t } v + v' = 0$
5) $\exists 1 \in F, \text{s.t } 1 \cdot v = v$
6) $(ab) \cdot v = a \cdot (bv)$
7) $a \cdot (v + w) = a \cdot v + a \cdot w$
8) $(a + b) \cdot v = a \cdot v + b \cdot v$

EXAMPLE 1.3  set $V = \{0\}$ over the field $\mathbb{R}$

EXAMPLE 1.4  set $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \ldots \mathbb{R}^n$ over the field $\mathbb{R}$

*If F is a field, then F is also a Vector space over itself*

## 1.3  Definition of Vector Spaces

DEFINITION 1.6 (Vector Space)
$$(V, F, +, \cdot)$$

- $(V, +)$ is an Abelian group
- $\cdot : F \times V \to V$ is associative and follows distributivity

EXAMPLE 1.7
$$\mathbb{Q}, \mathbb{R}, \mathbb{C} \text{ are all fields}$$

PROPOSITION 1.8  Any field $F \subset \mathbb{R}$ contains $\mathbb{Q}$

*Proof* | We assume normal addition and multiplication

$$\forall a \in \mathbb{Q}, a = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$
$0, 1 \in F$, as $(F, +)$ and $(F - \{0\}, \times)$ is an abelian groups. As $+$ is closed $1 + 1 = 2 \in F$ and $2 + 1 = 3 \in F$ and so on. By inverse property of $+$, we know $-1, -2, -3 \cdots \in F$, therefore $\mathbb{Z} \in F$. By inverse property of $\times$, we know $\{1/a : a \in \mathbb{Z}\} \in F$. Since $\times$ is closed, $S = \{a/b : a, b \in \mathbb{Z}\} \in F$, set $S$ is just $\mathbb{Q}$. Therefore $Z \in F$ if $F \subset \mathbb{R}$ is a field.

Properties of fields:

1) *Cancellation law,* If $a + b = a + c$ then $b = c$

2) Additive identity is unique
3) Additive inverse is unique
4) Multiplicative identiy is unique
5) Multiplicative inverse is unique
6) if $a \cdot b = a \cdot c$ then $b = c$ if $a \neq 0$

Properties of Vector spaces:

1) *Cancellation law,* If $a + b = a + c$ then $b = c$
2) Additive identity is unique
3) Additive inverse is unique
4) $0 \cdot v = 0 \rightarrow$ is the null vector

Matrices as Vector spaces:

$$M_{n \times m}(\mathbb{R})$$

- Addition is entry wise
- $0_{n \times m}$ the additive identity
- Scaling is entry wise

$M_{n \times m}$ looks like $\mathbb{R}^{n.m}$

## 1.4 Vector Subspaces

DEFINITION 1.9 (Vector Subspace) $W$ is said to be a vector subspace of $V$ if

- $W \subseteq V$
- $W$ is a vector subspace

EXAMPLE 1.10 $P(\mathbb{R})$ is the set of all polynomials.
$P_n(\mathbb{R})$ is the set of all polynominals of degree at most n.

- $P_0(\mathbb{R}) \subset P_1(\mathbb{R}) \subset P_1(\mathbb{R}) \subset P_3(\mathbb{R}) \subset \ldots P_n(\mathbb{R})$
- $\{a_0 + a_2 x^2 + a_4 x^4 \cdots : a_i \in \mathbb{R}\}$

EXAMPLE 1.11 $M_n(\mathbb{R})$ is the set of all $n \times n$ matrices

- Set of all symmetric matrices, $S = \{A : A \in M_n(\mathbb{R}), A = A^t\}$
- $T = \{A : A \in M_n(\mathbb{R}), Tr(A) = 0\}$

## 2 SPAN AND LINEAR INDEPENDENCE

## 2.1 Definiton of Span

DEFINITION 2.1 (Span)

$$\text{span}(S) = \{\gamma_1 v_1 + \gamma_2 v_2 \ldots \gamma_k v_k : v_i \in S, \gamma_i \in F\}$$

**EXAMPLE 2.2**

$$S = \{1, x, x^2\}$$

$$\text{span}\,(S) = P_2\,(\mathbb{R})$$

**EXAMPLE 2.3**

$$S = \{(1,0,0)\,,(0,1,0)\,,(0,0,1)\}$$

$$\text{span}\,(S) = \mathbb{R}^2$$

**EXAMPLE 2.4**

$$S = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\}$$

$$\text{span}\,(S) = M_2\,(\mathbb{R})_{sym}$$

Properties of Span:

1) span($S$) is always a subspace containing $S$ as,
   $+$ is closed and
   Scaling operation is also closed.
2) For a subspace $S \subseteq W$, $S \subseteq \text{span}\,(S) \subseteq W$
3) span$\,(S)$ is the smallest subspace containg S.
   That is, span$\,(S)$ is the intersection of all subspaces containing $S$.
4) span$\,(\varnothing) = \{0\}$
   This is more of a convention, or you can arrive at this by following the above definition of Span.
5) $S_1 \subseteq S_2$ implies span$\,(S_1) \subseteq \text{span}\,(S_2)$
6) span$\,(\text{span}\,(S)) = \text{span}\,(S)$
7) span$\,(W) = W$ if and only if $W$ is a subspace.
8) span$\,(S_1 \cup S_2) = \text{span}\,(S_1) + \text{span}\,(S_2)$
   where $A + B = \{a + b : a \in A, b \in B\}$

💡 *if $V_1$ and $V_2$ are subspaces then, $V_1 + V_2$ is also a subspace.*

## 2.2 Linear (In)dependence

Given a non empty set $S$, consider span$\,(S)$. Is there an element $v \in S$ such that, span$\,(S) = \text{span}\,(S - \{v\})$?

Then $S$ is linearly dependent.

**EXAMPLE 2.6**

$$S = \{(0,0,0)\,,(0,1,0)\,,(0,0,e)\}$$

$$\text{span}\,(S) = \{(0,a,b) : a,b \in \mathbb{R}\} = \text{span}\,(S - \{(0,0,0)\})$$

For span$\,(S) = \text{span}\,(S - \{v\})$, there exists $v_1, v_2, \ldots, v_k \in S - \{v\}$ and scalars $\gamma_1, \gamma_2, \ldots, \gamma_k \in F$ such that,

$$V = \sum_{i=1}^{k} \gamma_i v_i$$

as $0 \notin S$, not all $\gamma_i$ can be 0.

**DEFINITION 2.7** (Linear Dependence)  A subset $S \subseteq V$ is said to be linearly dependent if there exists scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and $v_i$ are all distinct.

**DEFINITION 2.8** (Linear Independence)  A subset $S \subseteq V$ is said to be linearly independent if there exists no scalars, $\gamma_1, \gamma_2, \ldots \gamma_k \in F$ such that for $v_1, v_2, \ldots, v_k \in S$

$$\gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_k v_k = 0$$

Where not all $\gamma_i = 0$ and $v_i$ are all distinct.

**EXAMPLE 2.9**  $V = P(\mathbb{R})$, $S = \{1 + x, 1 - x, x^2, 2x^2 - 1\}$

$$\frac{1}{2}(1 + x) + \frac{1}{2}(1 - x) + (2x^2 - 1) + (-2)x^2 = 0$$

$S' = \{1 + x, 1 - x, x^2\}$

💡 *if S is linearly dependent, then $\exists v \in S$ such that span $(S) = span\,(S - \{v\})$*

**PROPOSITION 2.11**  if $S$ is linearly independent and $\exists v \notin span\,(S)$ then $S \cup \{v\}$ is linearly independent.

*Proof* | let $\gamma, \gamma_1, \ldots, \gamma_k \in F$ and $v_1, \ldots, v_k \in S$

$$\gamma_1 v1 + \cdots + \gamma_k v_k + \gamma v = 0$$

case 1: $\gamma = 0$,
$$\gamma_1 v1 + \cdots + \gamma_k v_k = 0$$

as $S$ is linearly independent $\gamma_i = 0$ therefore,

$$\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma = 0$$

therefore $S \cup \{v\}$ is linearly independent.
case 2: $\gamma \neq 0$,
$$v = \frac{-\gamma_1}{\gamma} v_1 + \cdots + \frac{-\gamma_k}{\gamma} v_k$$

but $v \notin span\,(S)$, therefore this case is not possible.

## 3  BASIS AND DIMENSION OF A VECTOR SPACE

### 3.1  Basis and Dimension

**DEFINITION 3.1** (Basis)  set $\beta \subseteq V$ is called the basis of $V$ if

1) $\beta$ is linearly independent.
2) $\beta$ spans $V$

**DEFINITION 3.2** (Dimension)  For $\beta$ being a basis of $V$, $\dim_F V = |\beta|$

## 3.2 Replacement Theorem

> **THEOREM 3.3** (Replacement Theorem) Let $S \subseteq V$ be a finite set which spans $V$ and let $L \subseteq V$ be a finite linearly independent subset, then,
>
> 1) $|L| \leq |S|$
> 2) $\exists T \subseteq S$ (T is of size $|S| - |L|$) such that $T \cup L$ spans V

*Proof*

$$L = \{u_1, \ldots, u_m\}$$

$$S = \{v_1, \ldots, v_n\}$$

$$u_m \in V = \operatorname{span}(S)$$

therefore,

$$u_m = a_1 v_1 + \cdots + a_n v_n$$

as $u_m$ is in $L$ a linearly independent set, $u_m$ cannot be 0, therefore, all of $a_i$ cannot be zero as well. Let $a_n \neq 0$.

$$v_n = \frac{1}{a_n} u_m + \frac{-a_1}{a_n} v_1 + \cdots + \frac{-a_{n-1}}{a_n} v_{n-1}$$

$$v_n \in \operatorname{span}\{v_1, \ldots, v_{n-1}, u_m\} = \operatorname{span}(S) = V$$

similarly,

$$u_{m-1} \in V = \operatorname{span}\{v_1, \ldots, v_{n-1}, u_m\}$$

therefore,

$$u_{m-1} = a_1 v_1 + \cdots + a_{n-1} v_{n-1} + b_m u_m$$

as $u_{m-1}$ and $u_m$ are linearly independent, therefore all of $a_i$ cannot be 0

$$v_{n-1} = \frac{1}{a_{n-1}} u_{m-1} + \frac{-b_m}{a_{n-1}} u_m + \frac{-a_1}{a_{n-1}} v_1 \cdots + \frac{-a_{n-2}}{a_{n-1}} v_{n-2}$$

$$v_{n-1} \in \operatorname{span}\{v_1, \ldots, v_{n-1}, u_{m-1}, u_m\} = \operatorname{span}(S) = V$$

Now iterate this process to get,

$$S' = \{v_1, \ldots, v_{n-m}, u_1, \ldots, u_m\}$$

Where $S'$ spans $V$,

$$\operatorname{span}(S') = V$$

If $n < m$ then, there will be a point when iterating the steps where a proper subset of L will span V, which would not be possible since this breaks the linear independence of L.
The set $T$ from the theorem is,

$$T = S' - L$$

💡 *If $S \subseteq V$ spans $V$ and $L \subseteq V$ is an linearly independent subset, then $|L| \leq |S|$.*

> **COROLLARY 3.5** Given 2 basis of the same vector space $V$, $\beta$ and $\beta'$.
>
> $$|\beta| = |\beta'|$$

*Proof* | Set $S = \beta$ and $L = \beta'$. $\beta$ spans $V$ by definition of basis and $\beta'$ is linearly independent by definition of basis.

Now according to 1) of replacement theorem, we get.

$$|\beta| \leq |\beta'|$$

Similarly, we can swap $S$ and $L$ to get,

$$|\beta| \geq |\beta'|$$
$$\therefore |\beta| = |\beta'|$$

◼

💡 *Going back to the definition of dimension, we can see that basis need not be unique but the dimension of a vector space is unique regardless of which basis we use to find the dimension*

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COROLLARY 3.7 Let $V$ be a vector space of finite dimension $n$, then

1) Any finite spanning set $S$, $|S| \geq n$, if $|S| = n$ then $S$ is a basis.
2) Any linearly independent set $L$, $|L| \leq n$, if $|L| = n$, then $L$ spans $V$.
3) Any linearly independent set $L$ can be extended to form a basis of $V$.

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COROLLARY 3.8 Let $V$ be a finite dimensional vector space over the field $F$ andlet $W \subseteq V$ be a subspace, then,

$$\dim_F (W) \leq \dim_F (V)$$

If equality holds, then, $V = W$.

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*Proof* | Let $\beta$ be a basis of $V$ of size $n$. Now look at $\text{span}\{\emptyset\} = \{0\}$, if it is $W$ then we are done, if not, we choose $u_1 \in W$ which is non zero, $L_1 = \{u_1\} \subseteq W$ is linearly independent. We check again if $\text{span}(L_1) = W$, if it is then we are donem, if it isn't then we choose $u_2 \in W \setminus \text{span}(L_1)$. We iterate this process until we get $\beta_\circ = \{u_1, \ldots, u_k\}$ which spans $W$. This set is linearly independent by construction (refer to proposition 2.11). Using 1) from replacement theorem, using $\beta$ as S and $\beta_c irc$ as L, we get,

$$|\beta_\circ| \leq |\beta|$$

which is the same as,

$$\dim_F (W) \leq \dim_F (V)$$

And using corollary 3.8 we can say that the equality implies $V = W$.  ◼

💡 *Given subspace W of vector space V, with $\beta_\circ$ as the basis of W, we can extend the basis $\beta_\circ$ to $\beta$ as a basis for V*

## 3.3 Direct Sum

Let $W_1$ and $W_2$ be subspaces of a finite dimensional vector space $V$. Then $W_1 + W_2$ is also a subspace.

Case A: $W_1 \cap W_2 = \{0\}$

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DEFINITION 3.10 (Direct Sum) Given subspaces $W_1$ and $W_2$ of a finite dimensional vector space $V$, if $W_1 \cap W_2 = \{0\}$ then the vector subspace $W_1 + W_2$ is known as the direct sum.

Case B: $W_1 \cap W_2 \neq \{0\}$

$$\dim_F (W_1 + W_2) = \dim_F (W_1) + \dim_F (W_2) - \dim_F (W_1 \cap W_2)$$

*Proof* | Start with basis of $W_1 \cap W_2$, $\beta_\circ = \{u_1, \ldots, u_k\}$. Now extend this basis to a basis of $W_1$, $\beta_1 = \{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ and similarly for $W_2$, $\beta_2 = \{u_1, \ldots, u_k, w_1', \ldots, w_m\}$

$$\beta = \beta_1 \cup \beta_2$$

$$\beta = \{u_1, \ldots, u_k, w_1, \ldots, w_l, w_1', \ldots, w_m'\}$$

As any element in $W_1 + W_2$ can be represented by a linear combination of elements of $\beta$, $\beta$ spans $W_1 + W_2$. To show that $\beta$ is linearly independent,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l + \gamma_1 w_1' + \cdots + \gamma_m w_m' = 0$$

$$=> c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = -\gamma_1 w_1' - \cdots - \gamma_m w_m' = v$$

Clearly $v \in W_1$ and $v \in W_2$, therefore $v \in W_1 \cap W_2$, therefore $v$ can be written as,

$$v = c_1' u_1 + \cdots + c_k' u_k$$

$$=> c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = -\gamma_1 w_1' - \cdots - \gamma_m w_m' = v = c_1' u_1 + \cdots + c_k' u_k$$

From these 2 equations, we are able to determine that $\beta$ is linearly independent,
We have,

$$-\gamma_1 w_1' - \cdots - \gamma_m w_m' = c_1' u_1 + \cdots + c_k' u_k$$

$\{u_1, \ldots, u_k, w_1', \ldots, w_m'\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1' = \cdots = c_k' = 0$$

Similarly, we have,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = c_1' u_1 + \cdots + c_k' u_k$$

We know $c_1' = \cdots = c_k' = 0$, therefore,

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l = 0$$

$\{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ is a linear independent set, therefore,

$$\gamma_1 = \cdots = w_m' = c_1 = \cdots = c_l = 0$$

Therefore the only solution for

$$c_1 u_1 + \cdots + c_k u_k + d_1 w_1 + \cdots + d_l w_l + \gamma_1 w_1' + \cdots + \gamma_m w_m' = 0$$

is all the scalars being equal to zero, which implies linear independence of $\beta$. Therefore $\beta$ is both linearly independent and spans $W_1 + W_2$, which makes it a basis for $W_1 + W_2$.

$$\dim_F (W_1) = |\beta_1| = k + l$$

$$\dim_F (W_2) = |\beta_1| = k + m$$

$$\dim_F (W_1 + W_2) = |\beta| = k + l + m$$

$$\dim_F (W_1 \cap W_2) = |\beta_\circ| = k$$

$$\therefore \dim_F (W_1 + W_2) = \dim_F (W_1) + \dim_F (W_2) - \dim_F (W_1 \cap W_2)$$

> **PROPOSITION 3.11** If $V$ is the direct sum, $W_1 + W_2$ then $v \in V$ can be uniquely expressed as $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

*Proof* | Let, $v = w_1 + w_2 = w_1' + w_2'$ then, $w_1 - w_1' = w_2 - w_2' = x$. therefore $x \in W_1 \cap W_2$ and by definition of direct sum, $x \in \{0\}$, therefore $x = 0$. Which gives us, $w_1 = w_1'$ and $w_2 = w_2'$. ∎

## 4 QUOTIENT SPACES

### 4.1 Quotient Spaces

Goal is to create a new vector space out of $W \subseteq V$, define an equvalence relation $\sim_W$ on $V$. $v_1 \sim_W v_2$ if and only if $v_1 - v_2 \in W$

- *Reflexive* Yes, if $0 \in W$.
- *Symmetric* Yes, if whenever, $v_1 - v_2 \in W$ implies $v_2 - v_1 \in W$, ie, $-(v_1 - v_2) \in W$.
- *transitivity* Yes, if whenever $v_1 - v_2 \in W$ and $v_2 - v_3 \in W$ implies $v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \in W$.

Starting with any arbitary subset of $V$ we end with a subspace of $V$ due to the conditions that needs to be satisfied for $\sim_W$ to be an equivalence relation.

For any eqivalence relation $\sim$ on an arbitary set $S$, an equivalence class is given by.

$$[s] = \{s' : s' \in S, s' \sim s\}$$

Equivalence relations are disjoined and partition the set. that is either $[s] \cap [s'] = \varnothing$ or $[s] = [s']$.

For $\sim_W$,

$$[v] = \{v_1 : v_1 - v \in W\}$$
$$[v] = \{v_1 = v + w : w \in W\}$$
$$[v] = v + W$$

$[v_1] = [v_2]$ we translate via $W$,

$$v_1 + W = v_2 + W$$
$$v_1 = v_2 + W$$
$$v_1 - v_2 = w \in W$$

Therefore $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.

We can create a new vector space, where each element is an equivalence class of the equivalence relation $\sim_W$. Consider the set of all equivalence class of the relation $\sim_W$, $V/W$.

Addition under the vector space $V/W$,

$$+ : V/W \times V/W \rightarrow V/W$$

$$[v_1] + [v_2] = [v_1 + v_2]$$

Scaling,

$$\times : F \times V/W \to V/W$$

$$(\gamma, [v]) \to [\gamma v]$$

# 5 LINEAR MAPS

## 5.1 Linear Maps

Given vector spaces $V$, $W$ (over the same field $F$) and linear map $T$.

$$T : V \to W$$

$T$ should be compatible with the structures of the vector spaces $\forall v_1, v_2, v \in V$ and $\forall \gamma \in F$.

- $T(v_1 + v_2) = Tv_1 + Tv_2$
- $T(\gamma v) = \gamma Tv$

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EXAMPLE 5.1 (Scaling Linear Map)

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$T(x + y) = \gamma_\circ (x + y) = \gamma_\circ x + \gamma_\circ y = Tx + Ty$$

$$T(\gamma y) = \gamma_\circ (\gamma y) = \gamma (\gamma_\circ y) = \gamma (Tx)$$

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$T(0_v) = 0_w$

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EXAMPLE 5.3 (Dilation)

$$T : V \to V, v \to 2v \,(= v + v)$$

$$\gamma v \to 2(\gamma v) = \gamma (2v) = \gamma (Tv)$$

If in the field, $2 = 0$ then the map is the zero map, if not then it's a bijective map.

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EXAMPLE 5.4 (Identity map)

$$I : V \to V, v \to v$$

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EXAMPLE 5.5 (Trivial map)

$$T_0 : V \to W, v \to 0_w$$

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EXAMPLE 5.6 (Matrices)

$$A \in M_{n \times n}(\mathbb{R})$$

$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \rightarrow A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$L_A \left( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + L_A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$L_A \left( \gamma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \rightarrow \gamma L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

EXAMPLE 5.7 (Reflection)

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \rightarrow (x, -y)$$

This is for reflection across x-axis, for any other reflection the line as to pass through the origin because $0$ has to be mapped to $0$ and in general $T.T$ gives us the identity map.

EXAMPLE 5.8 (Rotation)

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \rightarrow (-y, x)$$

This is rotaion of $\pi/2$ clockwise, for a more general rotaion,

$$T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$||T(x, y)|| = ||(x, y)||$$

This is a property of every rotation linear map.

EXAMPLE 5.9 (Projection)

$$P : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \rightarrow (x, 0)$$

$$Q : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \rightarrow (x - y, 0)$$

$$P^2 = P.P = P$$

$$Q^2 = Q.Q = Q$$

This is the defining property of a projection map.

EXAMPLE 5.10 (Inclusion)

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \rightarrow (x, y, 0)$$

$$T' : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \rightarrow (x, y, ax + by)$$

## 5.2 Null Space

> **DEFINITION 5.11 (Null Space)** Given $T : V \to W$, null space $N(T)$ is given by,
> $$N(T) = \{v \in V : T(v) = 0_w\}$$

**EXAMPLE 5.12 (Scaling Linear Map)**

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$N(T) = \{0\}$$

**EXAMPLE 5.13 (Dilation)**

$$T : V \to V, v \to 2v\,(= v + v)$$

$$N(T) = \begin{array}{ll} V & 2 = 0 \text{ in } F \\ \{0\} & 2 \neq 0 \text{ in } F \end{array}$$

**EXAMPLE 5.14 (Identity map)**

$$I : V \to V, v \to v$$

$$N(T) = \{0\}$$

**EXAMPLE 5.15 (Trivial map)**

$$T_0 : V \to W, v \to 0_w$$

$$N(T) = V$$

**EXAMPLE 5.16 (Reflection)**

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

$$N(T) = \{0\}$$

**EXAMPLE 5.17 (Rotation)**

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$N(T) = \{0\}$$

**EXAMPLE 5.18 (Projection)**

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, 0)$$

$$N(P) = \{(0, y)\}$$

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x - y, 0)$$

$$N(Q) = \{(x, y) : x = y\}$$

**EXAMPLE 5.19 (Inclusion)**

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, 0)$$

$$N(T) = \{0\}$$

$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \to (x, y, ax + by)$$

$$N(T) = \{0\}$$

## 5.3 Range

**DEFINITION 5.20 (Range)** Given $T : V \to W$, null space $R(T)$ is given by,

$$R(T) = \text{Image of } V \text{ under } T$$

**EXAMPLE 5.21 (Scaling Linear Map)**

$$T : \mathbb{R} \to \mathbb{R}, x \to \gamma_\circ x$$

$$R(T) = \mathbb{R}$$

**EXAMPLE 5.22 (Dilation)**

$$T : V \to V, v \to 2v \, (= v + v)$$

$$R(T) = \begin{matrix} \{0\} & 2 = 0 \text{ in } F \\ V & 2 \neq 0 \text{ in } F \end{matrix}$$

**EXAMPLE 5.23 (Identity map)**

$$I : V \to V, v \to v$$

$$R(T) = V$$

**EXAMPLE 5.24 (Trivial map)**

$$T_0 : V \to W, v \to 0_w$$

$$R(T) = \{0\}$$

**EXAMPLE 5.25 (Reflection)**

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x, -y)$$

$$R(T) = \mathbb{R}^2$$

**EXAMPLE 5.26 (Rotation)**

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$R(T) = \mathbb{R}^2$$

$$P : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \to (x,0)$$

$$R(P) = \{(x,0)\}$$

$$Q : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \to (x-y,0)$$

$$R(Q) = \{(x,0)\}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x,y) \to (x,y,0)$$

$$R(T) = \{(x,y,0)\}$$

$$T' : \mathbb{R}^2 \to \mathbb{R}^3, (x,y) \to (x,y,ax+by)$$

$$R(T) = \{(x,y,ax+by)\}$$

## 5.4  Null space and Range as vector subspaces

$$v_1, v_2 \in N(T), \gamma \in F$$

$$T(v_1 + v_2) = Tv_1 + Tv_2$$

$$T(v_1 + v_2) = 0 + 0 = 0$$

$$\therefore v_1 + v_2 \in N(T)$$

$$T(\gamma v_1) = \gamma Tv_1 = \gamma 0 = 0$$

$$\therefore \gamma v_1 \in N(T)$$

Therefore $N(T)$ is a subspace.

$$w_1, w_2 \in R(T), \gamma \in F$$

$$w_1 = Tv_1$$

$$w_2 = Tv_2$$

$$w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$$

$$\therefore w_1 + w_2 \in R(T)$$

$$w_1 = Tv_1$$

$$\gamma Tv_1 = \gamma w_1 = T(\gamma v_1)$$

$$\therefore \gamma w_1 \in R\left(T\right)$$

Therefore $R\left(T\right)$ is a subspace.

---

EXAMPLE 5.29 (Differentiation)

$$D : P\left(\mathbb{R}\right) \to P\left(\mathbb{R}\right), P \to P'$$

$$a_0 + a_1 x + \cdots + a_k x^k \to a_1 + 2a_2 x + \cdots + k a_k x^{k-1}$$

$$D.\left(P_1 + P_2\right) = D.P_1 + D.P_2$$

$$D.\left(\gamma P_1\right) = \gamma D.P_1$$

$$D : P_n\left(\mathbb{R}\right) \to P_n\left(\mathbb{R}\right)$$

$$N\left(D\right) = \{a_0 \in \mathbb{R}\}$$

$$R\left(D\right) = P_{n-1}\left(\mathbb{R}\right)$$

---

EXAMPLE 5.30 (Integration)

$$I : P\left(\mathbb{R}\right) \to P\left(\mathbb{R}\right), P \to \int P dx$$

$$a_0 + \cdots + a_k x^k \to a_0 x + \cdots + \frac{a_k x^{k+1}}{k+1}$$

$$N\left(I\right) = \{0\}$$

$$R\left(I\right) = \{J\left(x\right) \in P_{n+1}\left(\mathbb{R}\right) : J\left(0\right) = 0\}$$

---

DEFINITION 5.31 (Nullity)  Given a linear map, nullity of the linear map is the dimension of the null space.

DEFINITION 5.32 (Rank)  Given a linear map, rank of the linear map is the dimension of the range.

## 5.5  Rank Nullity Theorem

THEOREM 5.33 (Rank Nullity)  Let $V$ be a finite dimensional vector space and $W$ be any vector space, if $T : V \to W$ is a linear map then,

$$\dim_F\left(R\left(T\right)\right) + \dim_F\left(N\left(T\right)\right) = \dim_F V$$

*Proof* | Let $\beta_\circ = \{v_1, \ldots, v_k\}$ be a basis of $N\left(T\right)$, now extend this to a basis of $V$, $\beta$

$$\beta = \{v_1, \ldots, v_k, v_1', \ldots, v_l'\}$$

We know,

$$T\left(\beta\right) = \{0_w, T\left(v_1'\right), \ldots, T\left(v_l'\right)\}$$

$$L = \{T\left(v_1'\right), \ldots, T\left(v_l'\right)\}$$

Let's take,

$$\gamma_1 T\left(v_1'\right) + \cdots + \gamma_l T\left(v_l'\right) = 0$$

$$T\left(\gamma_1 v_1' + \cdots + \gamma_l v_l'\right) = 0$$

$$\gamma_1 v_1' + \cdots + \gamma_l v_l' \in N\left(T\right)$$

$$\gamma_1 v_1' + \cdots + \gamma_l v_l' = \gamma_1' v_1 + \cdots + \gamma_l' v_k$$

$$\gamma_1 v_1' + \cdots + \gamma_l v_l' - \gamma_1' v_1 - \cdots - \gamma_l' v_k = 0$$

By the linear independence of $\beta$ we can conclude that,

$$\gamma_1 = \cdots = v_l' = \gamma_1' = \gamma_l' = 0$$

Which gives us that $L$ is linearly independent and this is a set which spans $R\left(T\right)$ which makes it a basis of $R\left(T\right)$

$$\dim_F\left(R\left(T\right)\right) + \dim_F\left(N\left(T\right)\right) = |L| + |\beta_\circ| = l + k$$

$$\dim_F V = |\beta| = l + k$$

$$\therefore \dim_F\left(R\left(T\right)\right) + \dim_F\left(N\left(T\right)\right) = \dim_F V$$

💡 *Given a linear map $T : V \to W$, if $S \subseteq V$ spans $V$, then $T\left(S\right)$ spans $R\left(T\right)$.*

*Proof* | Given $w = T\left(v\right) \in R\left(T\right)$, express $v = \sum \gamma_i v_i$, where $v_i \in S$ and $\gamma_i \in F$. $T\left(v\right) = \sum \gamma_i T\left(v_i\right) \implies T\left(v\right) \in \operatorname{span}\left(T\left(S\right)\right)$

> **COROLLARY 5.35** Given a linear map, $T : V \to V$, where $V$ is a finite dimensional vector space, then the following are equivalent.
>
> 1) $T$ is injective
> 2) $T$ is surjective

*Proof* | $T$ is injective implies, $N\left(T\right) = \{0\} \implies$ Nullity $= 0 \implies$ Rank $= \dim_F V - 0 = \dim_F V \implies R\left(T\right) = V \implies T$ is surjective.
$T$ is surjective implies, $R\left(T\right) = V \implies$ rank $= \dim_F V \implies$ Nullity $= 0 \implies N\left(0\right) = \{0\}$.
If $Tv_1 = Tv_2$ and $N\left(0\right) = \{0\}$, then, $T\left(v_1 - v_2\right) = 0 \implies v_1 - v_2 \in N\left(T\right) \implies v_1 - v_2 \in \{0\} \implies v_1 = v_2$.
Therefore, $T$ is surjective implies injectivity as well.

💡 *As part of the above proof, we also proved that Null space being zero implies injectivity.*

## 6 ISOMORPHISM, MATRICES AND LINEAR MAPS AS VECTOR SPACES

### 6.1 Linear Isomorphism

> **DEFINITION 6.1** (Linear Isomorphism) A linear map $T : V \to W$ between vector spaces $V$ and $W$ both over a common field $F$ is called an isomorphism if $T$ is bijective.

> **EXAMPLE 6.2**
> $$T : P_2\left(\mathbb{R}\right) \to \mathbb{R}^3, a + bx + cx^2 \to \left(a, b, c\right)$$

Zero in $\mathbb{R}^3$ is $(0,0,0)$, clearly only polynominal which is mapped to it is 0. Therefore $N(T) = \{0\}$ and Nullity $= 0$. If $Tp_1 = Tp_2$, then, $T(p_1 - p_2) = (0,0,0) \implies T(a_1 - a_2 + (b_1 - b_2)x + (c_1 - c_2)x^2) = (0,0,0) \implies (a_1 - a_2, b_1 - b_2, c_1 - c_2) = (0,0,0) \implies a_1 = a_2, b_1 = b_2, c_1 = c_2 \implies p_1 = p_2$, therefore $T$ is injective. Rank $= \dim_F P_2(\mathbb{R}) - 0 = 3 \implies R(T) = \mathbb{R}^3$, therefore $T$ is surjective. Therefore $T$ is bijective.

💡 *When we say a vector space looks like another vector space, we mean that there exists a linear isomorphism from one vector space to another, ie V is isomorphic to W as a vector space.*

EXAMPLE 6.4

$$T : M_2(\mathbb{R}) \to \mathbb{R}^4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Properties of isomorphic linear maps (we will take $T : V \to W$ as our isomorphic linear map to discuss the properties):

1) $T^{-1} : W \to V$ is a linear map

*Proof* | For $w_1, w_2 \in W$ there exists unique $v_1, v_2 \in V$ such that, $Tv_1 = w_1$ and $Tv_2 = w_2$ because of bijectivity of $T$, there also exists unique $v \in V$ such that, $Tv = w_1 + w_2$.

$$T^{-1}w_1 = v1, T^{-1}w_2 = v_2$$

$$\implies v_1 + v_2 = T^{-1}w_1 + T^{-1}w_2$$

$$\implies T(v_1 + v_2) = T\left(T^{-1}w_1 + T^{-1}w_2\right)$$

$$\implies T(v_1 + v_2) = T\left(T^{-1}w_1\right) + T\left(T^{-1}w_2\right) = w_1 + w_2$$

$$\implies T(v_1 + v_2) = w_1 + w_2$$

Therefore unique $v = v_1 + v_2$. Therefore $T^{-1}(w_1 + w_2) = T^{-1}w_1 + T^{-1}w_2$.
For $w \in W$ there exists unique $v \in V$ such that, $Tv = w$ because of bijectivity of $T$, there also exists unique $v' \in V$ such that, $Tv = \gamma w$.

$$T^{-1}w = v$$

$$\gamma T^{-1}w = \gamma v$$

$$\implies T(\gamma v) = T\left(\gamma T^{-1}w\right)$$

$$\implies T(\gamma v) = \gamma T\left(T^{-1}w\right) = \gamma w$$

$$\implies T(\gamma v) = \gamma w$$

Therefore unique $v' = \gamma v$. Therefore $T^{-1}(\gamma w) = \gamma T^{-1}w$
Showing that addition and scaling is closed is suffice to show that $T^{-1}$ is a linear map. ■

2) $\gamma T : V \to W, V \to \gamma Tv$ is an isomorphic linear map.

*Proof* | Take $v_1, v_2 \in V$, such that,

$$\gamma Tv_1 = \gamma Tv_2$$

$$\implies \gamma T(v_1 - v_2) = 0$$

$$\implies T(v_1 - v_2) = 0$$

$$\implies v_1 = v_2$$

This implies injectivity of $\gamma T$.

Suppose $\gamma T$ is not surjective, then $\exists w \in W$ such that, $\forall v \in V$,

$$\gamma Tv \neq w$$

$$Tv \neq w/\gamma$$

But this is a contradiction given we know $T$ to be a bijection, therefore $\gamma T$ has to be surjective, therefore is bijective and an isomorphism. ∎

3) • *a)* If $T : V \to W$ is an injective linear map and $\{v_1, \ldots, v_n\}$ is a linearly independent set, then $\{Tv_1, \ldots, Tv_n\}$ is also a linearly independent set.

*Proof*

$$\gamma_1 Tv_1 + \cdots + \gamma_n Tv_2 = 0$$

$$\implies T(\gamma_1 v_2 + \cdots + \gamma_n v_n) = 0$$

Injectivity of the linear map implies,

$$\gamma_1 v_2 + \cdots + \gamma_n v_n = 0$$

Linear independence of $\{v_1, \ldots, v_n\}$

$$\gamma_1 = \cdots = \gamma_n = 0$$

■

• *b)* If $T : V \to W$ is an surjective linear map and let $\{w_1, \ldots, w_n\}$ be a subset of $W$ and choose $v_i \in V$ such that $Tv_i = w_i$ and if $\{v_1, \ldots, v_n\}$ spans $V$, then $\{w_1, \ldots, w_n\}$ spans $W$.

*Proof* We know for a given $w \in W$ there exists at one $v \in V$ such that, $Tv = w$.

$$v = \gamma_1 v_1 + \cdots + \gamma_n v_n$$

$$Tv = T(\gamma_1 v_1 + \cdots + \gamma_n v_n)$$

$$w = T(\gamma_1 v_1) + \cdots + T(\gamma_n v_n)$$

$$w = \gamma_1 Tv_1 + \cdots + \gamma_n Tv_n$$

$$w = \gamma_1 w_1 + \cdots + \gamma_n w_n$$

Since we choose an arbitart $w$, $\{w_1, \ldots, w_n\}$ spans the set $W$. ■

## 6.2 Linear maps as vectors

CONVENTION 6.5 $\mathcal{L}(V, W)$ represents set of all linear maps from $V \to W$ and $\mathcal{L}(V)$ represents set of all linear maps from $V \to V$.

$$+ : \mathcal{L}(V, W) \times \mathcal{L}(V, W) \to \mathcal{L}(V, W)$$

$$(T_1 + T_2) : V \to W, v \to T_1 v + T_2 v$$

$$(T_1 + T_2) \in \mathcal{L}(V, W)$$

$$\times : F \times \mathcal{L}(V, W) \to \mathcal{L}(V, W)$$

$$(\gamma T) : V \to W, v \to \gamma T v$$

$$(\gamma T) \in \mathcal{L}(V, W)$$

EXAMPLE 6.6 (Dual of $V$) $W = \mathbb{R}$, where $V$ is a vector space over $R$. Represented by $V^* = \mathcal{L}(V, \mathbb{R})$ and $\dim_F(V^*) = \dim_F(V)$.

EXAMPLE 6.7 $V = \mathbb{R}$, where $W$ is a vector space over $R$. $\mathcal{L}(\mathbb{R}, W)$ is a vector space, any element in $\mathcal{L}(\mathbb{R}, W)$ has unique property where the map is determined by just where 1 is mapped to.

$$T : \mathbb{R} \to W$$
$$T(\gamma) = \gamma T(1)$$
$$\Phi : \mathcal{L}(\mathbb{R}, W) \to W$$

$\Phi$ is an isomorphism.

*Proof* | For every $w \in W$ there is an unique $T_w \in \mathcal{L}(\mathbb{R}, W)$ such that, $T_w : \mathbb{R} \to W, \gamma \to \gamma w$ or $T_w(\gamma) = \gamma w$. set of all $T_w$ is the same as $\mathcal{L}(\mathbb{R}, W)$ since if there exists an element $T'$ which is not in the set of all $T_w$ then the property that the map is determined by where 1 is mapped to will yield a contradiction. Therefore, we have a bijective between $W$ and $\mathcal{L}(\mathbb{R}, W)$. Therefore, $\Phi$ is an isomorphism. ∎

**?** CONJECTURE 6.8 $\dim_F \mathcal{L}(V, W) = \dim_F V \times \dim_F W$

## 6.3 Ordered Basis

DEFINITION 6.9 (Ordered Basis) An ordered basis of a vector space $V$ is $\beta = \{v_1, \ldots, v_n\}$ where $\beta$ is an ordered set.

EXAMPLE 6.10 $V = \mathbb{R}^2, \beta = \{(1, 0), (0, 1)\}$

EXAMPLE 6.11 $V = P_2(\mathbb{R}), \beta = \{1, x, x^2\}$

## 6.4 Matrix Representation

Goal is to create a matrix to represent $T : V \to W$, let $\beta = \{v_1, \ldots, v_m\}$ be an ordered basis of $V$, let $\alpha = \{w_1, \ldots, w_n\}$ be an ordered basis of $W$.

It will be assumed you know basics of matrices, although everything will be revisted later in the notes with formal definitions.

$$Tv_j = \sum_{i=1}^{n} \gamma_{ij} w_i$$

$$\begin{pmatrix} Tv_1 \\ Tv_2 \\ \vdots \\ Tv_m \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} \end{pmatrix}$$

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} \end{pmatrix} \in M_{m \times n}(F)$$

Aim is to associate a matrix $[T]_\beta^\alpha$ for a linear map $T : V \to W$, where $\beta = \{v_1, \ldots, v_m\}$ is an ordered basis of $V$ and $\alpha = \{w_1, \ldots, w_n\}$ is an ordered basis of $W$.

$$S_\beta : V \to F^m$$

Where $S_\beta$ is given by, for any $v \in V$,

$$v = \sum_{i=1}^{m} \gamma_i v_i \to \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} \in F^m$$

$$S_\alpha : W \to F^n$$

Where $S_\alpha$ is given by, for any $w \in W$,

$$w = \sum_{i=1}^{n} \gamma_i w_i \to \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \in F^n$$

$$T : V \to W$$
$$S_\beta : V \to F^m$$
$$S_\alpha : W \to F^n$$
$$S_\alpha \cdot T \cdot S_\beta^{-1} : F^m \to F^n$$

We now define $[T]_\beta^\alpha$ to be the matrix associated with the map $S_\alpha \cdot T \cdot S_\beta^{-1} : F^m \to F^n$

> **CONJECTURE 6.12** $[Tv]^\alpha = [T]_\beta^\alpha [v]^\beta$

**EXAMPLE 6.13**

$$I : V \to V$$

For $\beta = \alpha$,

$$[I]_\beta^\alpha = I_n$$

For $\beta = \{e_1, e_2\}$ and $\alpha = \{e_2, e_1\}$,

$$[I]_\beta^\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

EXAMPLE 6.14 (Differentiation)

$$D : P_2\left(\mathbb{R}\right) \to P_2\left(\mathbb{R}\right), P\left(x\right) \to P'\left(x\right)$$

$$\beta = \{1, x, x^2\}$$

$$[D]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = \{2x, 1, x^2\}$$

$$[D]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To associate a matrix to a linear map,

$$T : V \to W$$

$$w\left(\in W\right) = \sum_{i=1}^{n} = \gamma_i w_i$$

$[T]_\beta^\alpha$ is the matrix obtaining by evaulting $T$ on $V$ with basis $\beta$ with respect to $\alpha$.

$$\left([T]_\beta^\alpha\right)_{ij} = \gamma_{ij}$$

Where,

$$Tv_J = \gamma_{1j} w_1 + \cdots + \gamma_{ij} w_i + \cdots + \gamma_{nj} w_n$$

$$[Tv]^\alpha = [Tv]_\beta^\alpha [v]^\beta$$

CONVENTION 6.15 $[a]^b$ represents a vector $a$ represented as a linear combination of elements of the set $\beta$, where $\beta$ is a basis for a vector space with $a$ in it.

$\beta = \{v_1, \ldots, v_m\}$, represents ordered basis of $V$ and $\alpha = \{w_1, \ldots, w_n\}$, represents ordered basis of $W$.

The $j^{\text{th}}$ column of $[T]_\beta^\alpha$ is given by the coefficients obtained by expanding $T\left(v_j\right)$ in terms of $\alpha$

CONVENTION 6.16 Just $[a]^b$ will be used instead of $[a]_b^b$

EXAMPLE 6.17

$$I : V \to V, v \to v$$

Given $\dim_F V = n$,

$$[I]_\beta^\beta = I_n$$

EXAMPLE 6.18

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

Rotation anti clock-wise by $\theta$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\beta = \{(1,0),(0,1)\}$$

$$\gamma = \{(1,1),(1,-1)\}$$

$$[R_\theta]_\beta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$[R_\theta]_\beta^\gamma = [I_2]_\beta^\gamma [R_\theta]_\beta^\beta$$

$$[R_\theta]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$[R_\theta]_\beta^\gamma = \begin{pmatrix} cos\theta + sin\theta & -sin\theta + cos\theta \\ cos\theta - sin\theta & -sin\theta - cos\theta \end{pmatrix}$$

$$[R_\theta]_\gamma = \begin{pmatrix} cos\theta & sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

💡 *Sum of diagonal elements of $[R_\theta]_\gamma$ and $[R_\theta]_\beta$ are the same, and so is the determinant of both matrices.*

EXAMPLE 6.20

$$D : P_n(\mathbb{R}) \to P_n(\mathbb{R}), P(x) \to P'(x)$$

$$\beta = \{1, x, \dots, x^n\}$$

$$[D]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We could look at $P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ but let's stick to square matrices for now, and $P_{n-1}(\mathbb{R})$ is clearly a subspace of $P_n(\mathbb{R})$.

We can see that the sum of diagonal elements is clearly 0 and the determinant is 0 as well.

PROPOSITION 6.21 Given $\dim_F V = m$ and $\dim_F W = n$ and given $\beta = \{v_1, \dots, v_m\}$ a basis of $V$ and $\gamma = \{w_1, \dots, w_m\}$ a basis of $W$.

$$\Phi : \mathcal{L}(V, W) \to M_{n \times m}(F), T \to [T]_\beta^\gamma$$

$\Phi$ is an isomorphism.

💡 *This statement says that the matrix representation of linear maps between 2 finite dimensional vector spaces is "equivalent" (i.e: isomorphic) to the linear map itself.*

*Proof* | Is $[S + T]^\gamma_\beta = [S]^\gamma_\beta + [T]^\gamma_\beta$

J$^{\text{th}}$column of $[S + T]^\gamma_\beta$ = coefficient of $(S + T) v_j$

J$^{\text{th}}$column of $[S]^\gamma_\beta$ = coefficient of $Sv_j$

J$^{\text{th}}$column of $[T]^\gamma_\beta$ = coefficient of $Tv_j$

$$\therefore [S + T]^\gamma_\beta = [S]^\gamma_\beta + [T]^\gamma_\beta$$

Is $[\lambda T]^\gamma_\beta = \lambda [T]^\gamma_\beta$.

J$^{\text{th}}$column of $[\lambda T]^\gamma_\beta$ = coefficient of $\lambda Tv_j$ = $\lambda$ coefficient of $Tv_j$ = j$^{\text{th}}$ coefficient of $\lambda [T]^\gamma_\beta$.
Therefore $\Phi$ is a linear map.

Zero in $M_{n \times m}(F)$ is the zero matrix.

$$\forall v \in V, [v]^\gamma \times 0_{n \times m} = [0]^\beta$$

Therefore the only preimage of $0_{n \times m}$ is the zero map between $V$ and $W$, therefore the null space of $\Phi$ is $\{0\}$, now we know that $\Phi$ is injective.

Given a matrix $A \in M_{n \times m}$, define $T_A : V \to W, [v]_\beta \to [v]_\beta A$, where $[v]_\beta A$ (a $1 \times n$ matrix) represents a vector $w$ as a linear sum of the vectors in basis $\gamma$. $T_A$ is a well defined linear map from $V \to W$ (proof of this is trivial but skipped here), therefore $T_A \in \mathcal{L}(V, W)$ is a preimage for $A \in M_{n \times m}$. Therefore $\Phi$ is bijective and hence an isomorphism. ■

💡 *This also proves conjecture 6.8*

## 7 Dual spaces and Dual maps

### 7.1 Dual Spaces

> **DEFINITION 7.1** (Dual Space) Given a finite dimensional vector space $V$ over field $F$, dual space is given by $V^* = \mathcal{L}(V, F)$, called as dual of V.

$$\dim_F V^* = \dim_F M_{1 \times n}(F) = n = \dim_F V$$

There is no natural linear isomorphism between $V$ and $V^*$.

"There cannot be a cannonical or God-given isomorphism"

We can take dual of a dual, known as double dual.

"In a sense, double duals are more well behaved"

### 7.2 Dual basis

let $\beta = \{v_1, \ldots, v_n\}$ be a basis of $V$.

$$v_i^* : V \to F, a_1 v_1 + \cdots + a_n v_n \to a_i$$

$$v_i^* v_j = \delta_{ij}$$

$\{v_1^*, \ldots, v_n^*\}$ is a basis of $V^*$.

*Proof* | Given $T \in V^*, T : V \to F$ and $v \in V$, where $\beta = \{v_1, \ldots, v_n\}$ is a basis of $V$,

$$Tv = T(a_1 v_1, \ldots, a_n v_n)$$

$$Tv = a_1 Tv_1 + \cdots + a_n Tv_n$$

$$Tv = Tv_1 a_1 + \cdots + Tv_n a_n$$

$$Tv = Tv_1 v_1^* v + \cdots + Tv_n v_n^* v$$

$$Tv = (Tv_1 v_1^* + \cdots + Tv_n v_n^*) v$$

$$T = Tv_1 v_1^* + \cdots + Tv_n v_n^*$$

$Tv_i$'s are fixed scalars, therefore the set $\beta^* \{v_1^*, \ldots, v_n^*\}$ spans $V^*$, linear independence follows from the fact that $\dim_F V^* = n$

---

EXAMPLE 7.2 For fixed $v \in \mathbb{R}^n$,

$$L_v : \mathbb{R}^n \to \mathbb{R}, w \to v \cdot w = \sum_{i=1}^{n} v_i w_i$$

From properties of dot product, it follows that $L_v$ is a linear map, hence $L_v \in (\mathbb{R}^n)^*$

---

EXAMPLE 7.3

$$T_{\text{tr}} : M_n(\mathbb{R}), A \to \text{tr}(A)$$

Since trace of a matrix is a linear map, $T_{\text{tr}} \in (M_n(\mathbb{R}))^*$.

## 7.3 Dual Maps

DEFINITION 7.4 (Dual Map) Given a linear map $T : V \to W$, Dual map $T^*$ of $T$ is given by,

$$T^* : W^* \to V^*, L \to L \cdot T$$

---

EXAMPLE 7.5

$$T_\gamma : \mathbb{R} \to \mathbb{R}, x \to \gamma x$$

$$T_\gamma^* : \mathbb{R}^* \to \mathbb{R}^*, (L : \mathbb{R} \to \mathbb{R}, x \to Lx) \to (\gamma L : \mathbb{R} \to \mathbb{R}, x \to \gamma Lx)$$

We know dimension of $\mathbb{R}^*$ to be 1, and the identity map is not equal to the zero map, therefore it is a basis for $\mathbb{R}^*$

---

EXAMPLE 7.6

$$T_{\text{tr}} : M_n(\mathbb{R}), A \to \text{tr}(A)$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\gamma = \{1\}$$

$\beta$ and $\gamma$ are basis for $M_n(\mathbb{R})$ and $\mathbb{R}$ respectively. $\gamma^* = \{1^*\}$ where $1^*(c) = 1^*(c \cdot 1) = c \cdot 1^*(c) = c$, therefore $\gamma^*$ is the dual basis of $\mathbb{R}$ ($1^*(1) = 1$ is by definition/construction of dual basis.)

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{\text{tr}}^* : \mathbb{R}^* \to M_n \left( \mathbb{R} \right)^*, L_\lambda \to L_\lambda \cdot T_{\text{tr}}$$

Where $L_\lambda \in \mathbb{R}^*$, $L_\lambda$ is given by $L_\lambda : \mathbb{R} \to \mathbb{R}, a \to \lambda a$, therefore,

$$T_{\text{tr}}^* : \mathbb{R}^* \to M_n \left( \mathbb{R} \right)^*, L_\lambda \to \lambda T_{\text{tr}}$$

$$[T_{\text{tr}}^*]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We can see that $[T_{\text{tr}}^*]_{\gamma^*}^{\beta^*}$ is the transpose of $[T_{\text{tr}}]_\beta^\gamma$.

Now let's see proof for matrix representation of dual of a map to be the transpose of the matrix representaion of the map.

*Proof* | Let $T : V \to W$ be a linear map from a finite dimensional vector space to another finite dimensional vector space. Let $\beta = \{v_1, \ldots, v_m\}$ be a basis of $V$ and let $\gamma = \{w_1, \ldots, w_n\}$ be a basis of vector space $W$.

$$\left( [T]_\beta^\gamma \right)_{ij} = \text{i}^{\text{th}} \text{ coefficient in the expansion of } Tv_j \text{ in terms of } \gamma$$

$\left( [T]_\beta^\gamma \right)_{ij} = C_{ij}$ if $Tv_j = \sum_{k=i}^n C_{kj} w_k$

$$\left( [T^*]_{\gamma^*}^{\beta^*} \right)_{ji} = \text{j}^{\text{th}} \text{ coefficient in the expansion of } T^* w_i^* \text{ in terms of } \beta^*$$

$\left( [T^*]_{\gamma^*}^{\beta^*} \right)_{ji} = \widetilde{C}_{ji}$ if $Tv_j = \sum_{k=j}^m \widetilde{C}_{ki} v_k^*$

$$(T^* w_i^*) v_j^* = \left( \widetilde{C}_{1i} v_1^* + \cdots + \widetilde{C}_{mi} v_m^* \right) v_j$$

$$(T^* w_i^*) v_j^* = \widetilde{C}_{ji} v_j^* v_j$$

$$(T^* w_i^*) v_j^* = \widetilde{C}_{ji}$$

$$(T^* w_i^*) v_j^* = w_i^* (Tv_j) = w_i^* (C_{1j} w_1 + \cdots + C_{nj} w_n)$$

$$(T^* w_i^*) v_j^* = C_{ij}$$

$$\therefore \widetilde{C}_{ji} = C_{ij}$$

$$\therefore \left( [T^*]_{\gamma^*}^{\beta^*} \right)^t = [T]_\beta^\gamma$$

# 8 MATRICES

## 8.1 Expected properties of matrix multiplication

Let $R : U \to V$, $S : V \to W$ and $T : W \to X$ be linear maps between finite dimensional vector spaces and let an ordered basis of $U$ be $\alpha$, of $V$ be $\beta$, of $W$ be $\gamma$ and of $X$ be $\delta$.

- i) *Associativity*

$$[T]_\gamma^\delta \left([S]_\beta^\gamma [R]_\alpha^\beta\right) = \left([T]_\gamma^\delta [S]_\beta^\gamma\right) [R]_\alpha^\beta$$

- ii)

$$[T \cdot S]_\beta^\delta = [T]_\gamma^\delta [S]_\beta^\gamma$$

ii) implies i)

*Proof*

$$[T]_\gamma^\delta \left([S]_\beta^\gamma [R]_\alpha^\beta\right) = [T]_\gamma^\delta [S \cdot R]_\alpha^\gamma = [T \cdot (S \cdot R)]_\alpha^\delta$$

$$\left([T]_\gamma^\delta [S]_\beta^\gamma\right) [R]_\alpha^\beta = [T \cdot S]_\beta^\delta [R]_\alpha^\beta = [(T \cdot S) \cdot R]_\alpha^\delta$$

i)

$$S : V \to W$$

$$T : W \to X$$

Let $\beta = \{v_1, \ldots, v_l\}$ be an ordered basis of $V$, $\gamma = \{w_1, \ldots, w_m\}$ be an ordered basis of $W$ and $\delta = \{x_1, \ldots, x_n\}$ be an ordered basis of $X$.

$$[T]_\gamma^\delta \implies t_{ij}$$

$$[S]_\beta^\gamma \implies s_{ij}$$

$$(T \cdot S) \, v_j = T \left(S v_j\right) = T \left(\sum_{i=1}^m s_{ij} w_i\right)$$

$$T \left(\sum_{i=1}^m s_{ij} w_i\right) = \sum_{i=1}^m s_{ij} T w_i = \sum_{i=1}^m s_{ij} \left(\sum_{k=1}^n t_{ki}\right) x_k = \sum_{k=1}^n \left(\sum_{i=1}^m t_{ki} s_{ij}\right) x_k$$

Therefore $(k,j)^{th}$ entry of $[T \cdot S]_\beta^\delta$ is equal to the $(k,j)^{th}$ entry of $[T]_\gamma^\delta [S]_\beta^\gamma$

## 8.2 Change of coordinates

> **COROLLARY 8.1** Let $T : V \to V$ be a linear map from a vector space of dimension n to itself and let $\beta$ and $\gamma$ be two ordered basis of $V$ then.
>
> - i) $[I_V]_\beta^\gamma [I_V]_\gamma^\beta = I_n$
> - ii) $[T]_\beta^\beta [I_V]_\gamma^\beta = [T]_\gamma^\beta$ and $[I_V]_\gamma^\beta [T]_\beta^\beta = [T]_\gamma^\beta$

*Proof*

- i) $[I_V]_\beta^\gamma [I_V]_\gamma^\beta = [I_V \cdot I_V]_\gamma^\gamma = I_n$
- ii) $[T]_\beta^\beta [I_V]_\gamma^\beta = [T \cdot I_V]_\gamma^\beta = [T]_\gamma^\beta$ and $[I_V]_\gamma^\beta [T]_\beta^\beta = [I_V \cdot T]_\gamma^\beta = [T]_\gamma^\beta$

**DEFINITION 8.2 (Change of basis)** The matrix $[I_V]_\beta^\gamma$ associated to the indentity map $I_V : V \to V, v \to v$ and the two ordered basis $\beta$ and $\gamma$ of $V$ is called the change of coordinate (or change of basis) matrix.

Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{v_1', \ldots, v_n'\}$ be an ordered basis of $V$ (a vector space over the field $F$) and define linear maps from $F^n$ to $V$,

$$\phi_\beta : F^n \to V, (c_1, \ldots, c_n) \to c_1 v_1 + \cdots + c_n v_n$$

$$\phi_\gamma : F^n \to V, (c_1, \ldots, c_n) \to c_1 v_1' + \cdots + c_n v_n'$$

The above 2 map are linear isomorphisms. (Hint: same dimsion and nullity is zero)
Consider the map,

$$T = \phi_\gamma^{-1} \cdot \phi_\beta : F^n \to F$$

Let's find the matrix representation of $T$ with respect to the standard basis of $F^n$. The j$^\text{th}$ column is given by $\phi_\gamma^{-1} \cdot \phi_\beta (e_j) = \phi_\gamma^{-1} v_j$.

$$\phi_\gamma^{-1} v_j = \phi_\gamma^{-1} \left( a_{1j} v_1' + \cdots + a_{nj} v_n' \right)$$

$$\phi_\gamma^{-1} v_j = a_{1j} \phi_\gamma^{-1} v_1' + \cdots + a_{nj} \phi_\gamma^{-1} v_n'$$

$$\phi_\gamma^{-1} v_j = a_{1j} e_1 + \cdots + a_{nj} e_n$$

Entries of the above matrix is precisely the entries of $[I_V]_\beta^\gamma$.

## 8.3 Invertibility

**DEFINITION 8.3 (Invertible maps)** A linear map $T : V \to W$ is called invertible if there exists $S : W \to V$ such that $T \cdot S = I_W$ and $S \cdot T = I_V$.

💡 *Due to rank-nullity, if $T : V \to W$ is invertible then both $V$ and $W$ are of same dimension.*

**DEFINITION 8.5 (Invertible matrices)** A matrix $A \in M_{m \times n} (\mathbb{R})$ is called invertible if there exists $B \in M_{n \times m}$ such that $AB = I_m$ and $BA = I_n$.

💡 *If $A \in M_{m \times n} (\mathbb{R})$ is invertible then $m = n$.*

**THEOREM 8.7** Linear map $T : V \to W$ with $\beta$ and $\gamma$ being ordered basis of $V$ and $W$ is invertible if and only if $[T]_\beta^\gamma$ is invertible.

*Proof* | If $T$ is invertible then dim $V =$ dim $W = n$ and $[T]_\beta^\gamma \in M_n (\mathbb{R})$. As $T \cdot T^{-1} = I_W$ and $T^{-1} \cdot T = I_V$, we get.

$$I_n = [I_V]_\beta = \left[ T^{-1} \cdot T \right]_\beta = \left[ T^{-1} \right]_\gamma^\beta [T]_\beta^\gamma$$

$$I_n = [I_W]_\gamma = \left[ T \cdot T^{-1} \right]_\gamma = [T]_\beta^\gamma \left[ T^{-1} \right]_\gamma^\beta$$

This proves, $[T]_\beta^\gamma$ is invertible if $T$ is invertible.

For converse, assume $A = [T]_\beta^\gamma$ is invertible with $B$ as its invertible, we know $m = n$. It suffices to show that $T$ is injective. If $v \in N(T)$, then $Tv = 0$. As,

$$A[v]^\beta = [T]_\beta^\gamma [v]^\beta = [Tv]^\gamma = 0$$

Multiplying by $B$ on the left, we conclude that $[v]^\beta = 0$. This implies that $v = 0$ and that $T$ is injective. ∎

## 8.4 Similar matrices

> **DEFINITION 8.8 (Similar matrices)** We say $A, B \in M_n(\mathbb{R})$ are similar if there exists an invertible matrix $Q \in M_n(\mathbb{R})$ such that $QAQ^{-1} = B$.

$Q$ is not unique, as $\lambda Q$ also will satisfy the condition if $Q$ does.

💡 *$A \sim B$ if $A$ is similar to $B$. $\sim$ is an equivalence relation on $M_n(\mathbb{R})$.*

- i)[*Reflexive*] $I_n A I_n^{-1} = A$.
- ii)[*Symmetric*] If $QAQ^{-1} = B$ then $Q^{-1}BQ = A$.
- iii)[*Transitive*] If $QAQ^{-1} = B$ and $PBP^{-1} = C$ then, $(PQ) A (PQ)^{-1} = PQAQ^{-1}P^{-1} = PBP^{-1} = C$.

---

**EXAMPLE 8.10 (Cross product)** $a \in \mathbb{R}^3$, $a = (x_1, x_2, x_3)$ and $||a|| = x_1^2 + x_2^2 + x_3^2 = 1$.

$$X_a : \mathbb{R}^3 \to \mathbb{R}^3, v \to a \times v$$

$$a \times (v + w) = a \times v + a \times w$$

$$a \times (\lambda v) = \lambda a \times v$$

Therefore $X_a$ is a linear map.

$$a^\perp = \{w \in \mathbb{R}^3 | a \cdot w = 0\}$$

Range of $X_a = a^\perp$, Null space of $X_a =$ span$(a)$.

$$\beta = \{e_1, e_2, e_3\}$$

$$\gamma = \{a, b, a \times b$$

Where $b \in a^\perp$

$$[X_a]_\beta = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

$$[X_a]_\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[X_a]_\gamma^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$[X_a]_g$ and $[X_a]_b$ are similar matrices where $Q = [I_{\mathbb{R}^3}]^\beta_\gamma$.

$$Q = [I_{\mathbb{R}^3}]^\beta_\gamma = \begin{pmatrix} | & | & | \\ a & b & a \times b \\ | & | & | \end{pmatrix}$$

## 8.5 Trace

**DEFINITION 8.11 (Trace)** For $A \in M_n(F)$, $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. Where,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

**DEFINITION 8.12 (Trace)** Given a linear map $T : V \to V$ from a finite dimensional vector space to itself, trace of the linear map is given by $\mathrm{tr}(T) = \mathrm{tr}\left([T]_\beta\right)$ where $\beta$ is a basis of $V$.

*Observation:* $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

*Proof*

$$\mathrm{tr}(AB) = (AB)_{11} + \cdots + (AB)_{nn} = \sum_{i=1}^n a_{1i}b_{i1} + \cdots + \sum_{i=1}^n a_{ni}b_{in} = \sum_{i,j=1}^n a_{ij}b_{ij}$$

$$\sum_{i,j=1}^n a_{ij}b_{ij} = \sum_{i,j=1}^n b_{ij}a_{ij} = \mathrm{tr}(BA)$$

Therefore,

$$\mathrm{tr}\left([T]^\gamma_\gamma\right) = \mathrm{tr}\left([I_V]^\gamma_\beta [T]^\beta_\beta [I_V]^\beta_\gamma\right) = \mathrm{tr}\left(\left([I_V]^\gamma_\beta [T]^\beta_\beta\right)[I_V]^\beta_\gamma\right) = \mathrm{tr}\left([I_V]^\beta_\gamma \left([I_V]^\gamma_\beta [T]^\beta_\beta\right)\right)$$

$$\mathrm{tr}\left([I_V]^\beta_\gamma \left([I_V]^\gamma_\beta [T]^\beta_\beta\right)\right) = \mathrm{tr}\left([I_V]^\beta_\gamma [I_V]^\gamma_\beta [T]^\beta_\beta\right) = \mathrm{tr}\left([T]^\beta_b\right)$$

This proves that the definition of trace of a linear map is well defined.

**EXAMPLE 8.14**

$$I_V : V \to V, v \to v$$

$[I_V] = I_n$ where n is the dimension of $V$, trace = n.