Analysis 2

Incalculas

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1 Differentiation

1.1 Definition

Definition 1.1 (Differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ be a function from J to $\mathbb{R}$. The function is said to be differentiable at c if

$$\lim_{x\to c} \frac{f(x) - f(x)}{x - c}$$

exists.

Notation 1.2 (Differentiation) If f is differentiable at c, we define

$$\lim_{x\to c} \frac{f(x) - f(x)}{x - c} = \left.\frac{df}{dx}\right|_{x=c} = f'(c)$$

Let's now discuss about interpretation of differentiation

Let J be an interval in R with $c \in J$ and let $f : J \to \mathbb{R}$ be a function.

Aim is to find an approximation for f near 0, $f(x) \approx a + bx$ for some $a, b \in \mathbb{R}$ near 0.

$$\lim_{x\to c} [f(x) - (a + bx)] = 0$$

$$E(x) = f(x) - (a + bx)$$

Find $a, b \in \mathbb{R}$ such that,

$$\lim_{x\to c} E(x) = 0$$

Then, $a = f(c)$ and $b = f'(c)$, this tells us that we know first order approximation from differentiation.

Definition 1.3 (Alternate definition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$, we say f is differentiable if and only if

$$\lim_{h\to 0} \frac{f(c + h) - f(c)}{h}$$

exists.

Take $J = [a, b]$ then

- *$c = a$, we only check for one sided limit (right hand limit)*
- *$a < c < b$, we check both left hand and right hand limits*
- *$c = b$, we only check for one sided limit (left hand limit)*

Definition 1.5 (Differentiable on an interval) Let J be an interval on R and $f : J \to \mathbb{R}$ be real function on J, we say f is differentiable on J if f is differentiable for every point on J.

Example 1.6 $f(x) = x$,

$$f'(x) = \lim_{h\to 0} \frac{(x + h) - (x)}{h} = \lim_{h\to 0} 1 = 1 \implies f'(x) = 1$$

Theorem 1.7 (Equivalent condition for differentiation) Let J be an interval in $\mathbb{R}$ with $c \in J$ and let $f : J \to \mathbb{R}$ then f is differentiable at c if and only if $\exists f_1 : J \to \mathbb{R}$ such that

- *i)* $f(x) = f(c) + f_1(x) + (x - c) \forall x in J$.
- *ii)* f_1 is continous at c.

and $f_1(c) = f'(c)$

Proof | Let f be differentiable at c then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Now define,

$$f_1(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

This now proves i) and ii).

Assume that f_1 exists and satisfies i) and ii) then,

$$f_1(x) = \frac{f(x) - f(c)}{x - c}, x \neq c$$

and since $f_1(x)$ is continous at c

$$\lim_{x \to c} f_1(x)$$

exists and therefore,

$$f_1(x) = \lim_{x \to c} f_1(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This proves that f is differentiable at c and $f'(c) = f_1(c)$. ∎

Example 1.8 Define $f : \mathbb{R} \to \mathbb{R}, x \to x^n$ for some fixed $n \in \mathbb{N}$, for $c \in \mathbb{R}$, $f'(c) = nc^{n-1}$.

Proof |

$$f(x) - f(c) = x^n - c^n$$

$$f(x) - f(c) = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

Let $f_1(x) = x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}$, given it's a polynomial, we know $f_1(x)$ is continous in $\mathbb{R}$.

$$f_1(c) = nc^n - 1$$

Therefore f is differentiable in $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

∎

Example 1.9 Define $f : \mathbb{R} \to \mathbb{R}, x \to e^x$, for $c \in \mathbb{R}$, $f'(c) = e^c$.

Proof

$$f(x) - f(c) = e^x - e^c$$

$$f(x) - f(c) = e^c\,(e^x - 1)$$

Since,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Define,

$$f_1(x) = e^c \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{n!}$$

Assume that f_1 is continous at c,

$$f_1(c) = e^c$$

This proves that f is differentiable at c and $f'(c) = e^c$.

(Note: this proof uses power series of exponential function which uses differentiation, this proof is just a circular argument and we will get back to an actual proof at the end of the course)

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Example 1.10 Define $f : \mathbb{R} \to \mathbb{R}, x \to |x|$, for $c \in \mathbb{R}$, $f'(c) = 1$ if $c > 0$, $f'(c) = -1$ if $x < 0$ and $f'(c)$ does not exist for $c = 0$.

Proof Both $c < 0$ and $c > 0$ cases are trivial. Now assume $f(x)$ is differentiable at 0 then the following limit exists.

$$\lim_{h\to 0} \frac{|0+h| - |0|}{h} = |h|/h$$

Clearly, left hand limit is -1 and right hand limit is 1 and therefore limit does not exist.

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Theorem 1.11 (Algebraic properties) Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be 2 differentiable functions, then for $c \in J$.

- $i)\,(f+g)'(c) = f'(c) + g'(c)$
- $ii)$ for $\alpha \in \mathbb{R}$, $(\alpha f)'(c) = \alpha f'(c)$
- $iii)\,(fg)'(c)$ exists
- $iv)$