A Strategy to the Game “2048”

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Games are a good way to entertain someone. There are many factors behind how enjoyable a game can be, but the exhilaration when one demonstrates mastery over a game is certainly satisfying. Over the 20th century, mathematicians and the like have extensively studied popular games like chess in attempts to master them by finding a solution to game. While some games, like chess, do not have solutions yet, some games have been solved, showing the possibility of solution. Notably, the game of “2048” can be solved.

The Game

2048 is a game developed by Italian web developer Gabrielle Cirulli (Wikipedia, 2020). It is a one person (singleplayer) game, where the player is given a discrete plane of 4×4 squares which will be called as “the board”. Upon initialization of the game, two squares are randomly occupied by a tile each, either numbered “2” or “4”, randomly as well. The player can then choose one of four directions: up, down, left, or right, and any existing tiles on the plane move in the chosen direction much like the rook from conventional chess until it collides with other tiles or an edge of the plane. The conditions are as follows:

1. If a tile collides with an edge of the board, then nothing additional happens.
2. If a tile collides with another tile that is numbered differently then nothing additional happens. Say a “2” tile moves and hits a “4” tile, then the “2” tile stops beside the “4” tile as if the “4” is an edge of the board.
3. If a tile collides with another tile of the same numbering, then the tiles combine to a tile numbered as the sum, occupying the space of the collided tile. Say a “2” tile moves and hits another “2” tile, then the tiles combine into a “4” tile occupying the space of the stationary“2” tile.

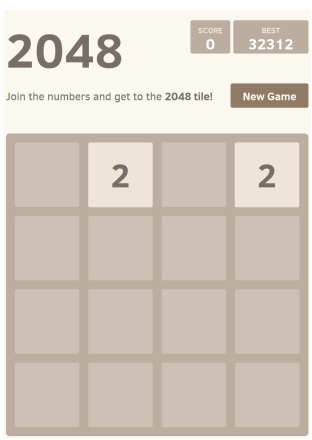


Figure 1: One of the many possible starting positions of the game.

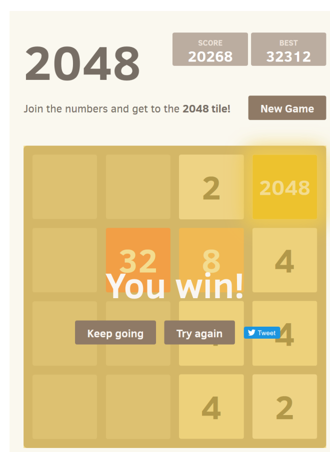


Figure 2: The goal of the game.

There are situations where the moving conditions become confusing, such as the situation where (without loss of generality) there are a row of identically numbered tiles, and the user chooses to move (again, without loss of generality) left. This case is resolved by combining the tiles pairwise into two identical tiles of a sum like condition 3. A valid move is any move where at least one tile is displaced, and upon each valid move the game will spawn either a “2” tile or a “4” tile in a random, unoccupied location of the board. The goal of the game is to obtain a “2048” tile, but the player can choose to continue playing until the board is completely filled up. With the “survive until the board is filled up” aspect, there is a scoring system to measure how far a player has gone until there are no more valid moves.

(a) A position with a row of "4" tiles.

(b) The new position after moving right.

Figure 3: An encounter with a confusing condition. The "4" tiles become merged pairwise.

The game is free-to-play online through a website or by downloading the application for free on a mobile phone, and the code to the game is open-source, being free to download, altered, and republished. As such, different permutations to the game are also online, such has changing the numbered tiles into images relating to internet culture, or increasing the size of the board. Gabrielle has said that the game of "2048" was a clone of another game of "1024". in which "1" tiles and "2" tiles spawn instead of "2" tiles and "4" tiles instead. Solutions to permutations of the game where the tiles are altered would be reducible to the original game, where permutations of the game where the dimensions of the board are altered may require a different solution.

When "2048" was released, it was met with a very positive response, receiving "over 4 million visitors in less than a week". It became the talk of the town; everyone was playing the game on their phones whenever they had some time to spare. People had come up with various ways to keep the game going for as long as possible. A popular idea was to limit the amount of directions one should choose on every move. For example, if a player chose to move the tiles downwards, the player should only move downwards until they could not any more, and move either left or right. It is then suggested that the player should move in the chosen two directions for a longer lasting game, and by extension a higher score. The reader is encouraged to play the game for themselves to understand the nuances of more esoteric collision conditions and the idea of creating a hierarchy of movement.

The Notion of "Solving a Game"

What does it mean to "solve" a game? In puzzles like Sudoku, there exists an answer to a particular puzzle. There is a "solution" to this game. However, it would be strange to claim that one can solve games like chess. Perhaps the winner can be considered as a "solution", but the loser sees the same game as the winner. Is being the loser in a chess game also a "solution"? Following through with this hypothesis of a vaguely defined idea of "solution", it does not aid in determining what a "solution" is to "2048". Fortunately, mathematicians have already defined such notions, and they have lead to formulating the mathematical field of Game Theory.

A game is considered "solved" if there is a mathematically defined way to reach the goal of a game at any given point of the game. The way to reach the goal of the game is called a "strategy". A way to visualize this is to examine the starting position of the game as a snapshot, or how the game looks like when a player initializes it. Upon a valid move, the position alters and the game now appears visually different. This position is another snapshot. It can be said that a valid move behaves similarily like what a function would do, but these “functions” are not well-defined for the case of “2048”, due to the randomness of the game spawning new tiles at unoccupied locations.

The Hunt for a Solution

Despite the randomness of the tiles spawning, it can be somewhat resolved by having a move lead to a set of outcomes instead. For example, take some starting position of two “2” tiles in two separate columns, and suppose the player moves up. The two tiles collide with the upper edge of the board, and a new tile spawns. There will be $16 - 2 = 14$ places where the new tile spawns, and the newly spawned tile could be a “2” or a “4”. In total, feeding the initial position into the “move up” “function” yields a set of $2(16 - 2) = 28$ possible positions.

From this thought process, if the positions were completely determined, then a strategy could be found by backward induction. Backward induction is to consider the ending position of the game, namely a position with a “2048” tile, and work backwards. This way the solution can always be found from any moment of the game. A “game tree” can be constructed with each node being a “snapshot” of how the game looks between each move, and the branches being a valid “move”.

This “game tree” idea seems to make sense, despite it mainly used for multiplayer games that take turns. One might even argue that the game tree for “2048” might be simpler than usual game trees since the player is in control of the board at all times, as opposed to certain nodes being positions in which only one of the players can access, and certain nodes being accessible only by the other player, known as a “partisan game” by Game Theorists.

However, a problem remains: there are a lot of positions to comprehend, and a human’s mind might not suffice to visualize all positions, let alone a portion of it. Would a computer be able to store all positions of the game? Beginning from the initializing positions, any two tiles would be randomly occupied with some combination of “2”’s and “4”’s. There are $\binom{16}{2} = 120$ possible ways to occupy two out of 16 squares on the board, and there are four cases as to how the tiles could be numbered: both “2”’s, both “4”’s, a “2” for the lefter and upper tile and a “4” for the righter and lower tile, and vice versa. To see why there are **four** cases instead of three, consider a “double corner” position, where a tile spawns on the top left corner, and another spawns on the top right corner of the board. Having the left tile be “2” and the right tile be “4” is different from the left being “4” and the right being “2”. Therefore, there are $120 \cdot 4 = 480$ starting positions to consider.

Then, for each position, possible moves must be considered. Sometimes there are 4 possible moves, but sometimes there are 3, in the cases of tiles being spawned either on the same edge row, or the same edge column. It is not possible that two tiles are spawned both on the same row and column since that would imply both tiles are occupying the same space, which violate the starting position requirement of two tiles being on the board. To find how many possible moves there are from the starting position a sum of the starting positions that only have 3 possible moves and those with 4 possible moves can be calculated.

For the case of 3 possible moves, consider the positions aforementioned where the two tiles are occupying the same row or columns such that only 3 of the 4 moves are valid. Firstly, consider the amount of positions there are with this case: Fix a tile in a corner, and permute the other tile by shifting along the column and row. Since there are 6 placements for the other tile for each of the 4 corners, accounting for the overcounted

4 duplicate "double corner" positions there would be $4 \cdot 6 - 4 = 20$ positions. After accounting for the permutations of tiles, $20 \cdot 4 = 80$ of the 480 starting positions have only 3 possible valid moves. Now consider the amount of possible positions generated from one of these positions. Consider the tiles were moved in a direction such that they did not combine. Since it is a valid move, one new tile would spawn. There are 14 spots where the tile could spawn, and there are 2 kinds of tiles to spawn. So moving in that direction would yield a set of 28 possible positions. If the tiles did combine, by the same idea moving in such a direction would yield a set of 30 possible positions instead. Note that by the conditions of the 4 possible tile combinations, and 3 possible valid moves, 2 of the 4 tile combinations would allow 2 of the 3 valid moves to combine tiles. Therefore, for the case of 3 possible moves:

- 40 of the beginning positions have same numbered tiles,
- $40 \cdot 2 = 80$ of the moves make 30 possible positions each, and 40 of the moves make 28 each, which means
- $40 \cdot 3 + 40 = 160$ of the moves make 28 possible positions each, leading to:

$$80 \cdot 30 + 160 \cdot 28 = 6880 \text{ new positions.}$$

Looking at the lower bound of one of these game trees (one starting position that yields the smallest amount of possible positions), all 3 directions yields 28 possible positions each, giving a total of 84 possible positions, or branches to the next level of the game tree. Hence there is an exponential growth of $\mathcal{O}(84^n)$ for these game trees. To put into perspective: Suppose each position of the game takes up 1 byte of memory on a computer. By considering all 480 starting positions, storing just 5 potential moves will take up around 2 terabytes of memory, let alone the unknown upper bound on how many moves one can make in this game.

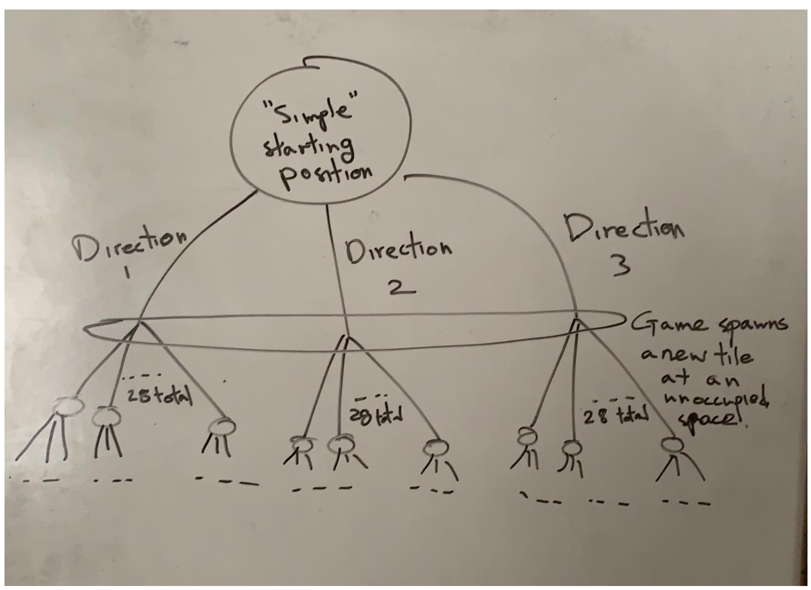
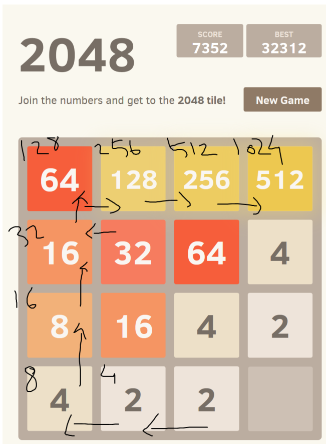
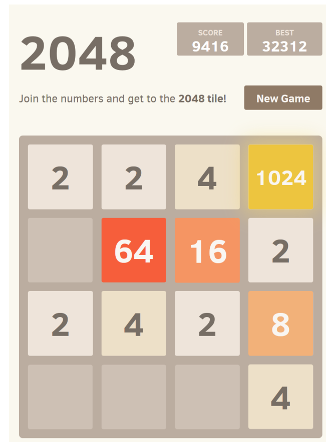


Figure 4: Visualization of a game tree with the smallest amount of possible positions. Many branches have been truncated. By the above paragraph, each node contains 1 byte of information.

From the sheer amount of positions to consider it might be good to conclude that comprehending all possible positions is a fool's errand. Perhaps there are other ways to tackle the problem. First note that all tiles will be a power of 2. It can be shown by induction: Spawned tiles are either "2^1" or "2^2". By induction hypothesis suppose this was true for some 2^k. In the event of two identically numbered tiles colliding with each other, by collision condition 3 above a new tile with the sum of the two tiles will take the place of the stationary tile. The sum is $2^k + 2^k = 2(2^k) = 2^{k+1}$. Now consider the necessary conditions to create a tile of 2^{k+1}: Two tiles of 2^k would be required. To create tile of 2^k, two tiles of 2^{k-1} would be required, and so on.



moving in the directions leads to...



(a) The combined number is written on the top left corner. Note the flexibility of the path at the "32" tile. The tile is not along the path, but rather merges to the path.

(b) A "1024" tile after traversing through the path.

Figure 5: A found path from "2" up to the largest "512" tile to create a "1024" tile.

With this intuition, a strategy can be formed: In any position of the game, take the tile with the largest number 2^k on it. Then, move in a way such that a finite sequence of 2^n is formed until two tiles of identical numbers are beside each other. Finally, move along the path to create a tile of 2^{k+1}. If $k = 10$, then the strategy creates a tile of 2048, thus beating "2048". To visualize this, consider some position of the game where the largest tile was "64". The player should then move until there is a "32" right beside the "64", a "16" beside the "32", an "8" beside the "16", a "4" beside the "8", a "2" beside the "4", and another "2" beside the "2". The player can stop this moving phase early if say a "16" tile beside the "16" tile instead. Then the player should move in the direction of this path, so that the two "2"'s become a "4", then the "4"'s can combine into "8", and so on until a tile of "64" merges with the largest tile of "64" to create a "128" tile.

Heuristics

There is a clear change in the direction in the hunt for the solution at some point, moving from the game tree to noticing a pattern one can follow despite not knowing where the player could be in the game tree. This new strategy is satisfying; it is algorithmic, and can be somewhat carried over any position in the game. The new strategy also feels microscopic, since it does not require full information of how the current position came to be, but it does not explicitly tell the player which position to go to. A direction, than a path.

This technique of developing a vague yet existant path between the starting position and the goal of obtaining the "2048" tile is called a heuristic. A heuristic can be thought of more like an "idea to solve" than the "answer" to the puzzle. It can solve a problem "more quickly when classic methods are too slow"

(Wikipedia, 2020). While the use of heuristics might be frowned upon by more conservative mathematicians as it does not concretely give a solution, but rather a “handwavey” solution, it could also be argued that heuristics is able to provide a solution when rigorous methods could not. Moreover, with the resolved controversy behind the use of computers to prove the Four Color Theorem, conservatives should give the heuristic technique a second thought.

The heuristic technique definitely applies here. The attempt at a solution through examining a game tree has proven to be too daunting with the amount of information needed to process, and as such, the heuristics of: finding the maximum tile, form a path of decreasing exponents until a repeat exponent is next to each other, and finally following through the path.

Recall the people’s tactic on limiting movement. While it is not techinically a heuristic in solving the game, as blindly moving in a couple of directions does not gurantee a “2048” tile, the tactic does enhance the heuristics for the game. With the tactic, the player is able to “lock” the maximum tile into a corner, easing the path making phase of the strategy. Since the maximum tile is in a corner, the limited movement tactic would not move the tile, and the player is free to construct a path reliably along the edges of the board.

While video games can be a great pastime, it can also be an invaluable resource in the visualization and application of various mathematical, problem solving concepts. It also teaches perserverance: when an approach that was thought to work turned out to fail, do not give up, but instead try another approach to a solution. Perhaps it also teaches to be content with potentially inelegant solutions, but be proud of the fact that a solution has been found.

References

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