Summary - income project

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2023-06-27

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1 Factor Model (with idiosyncratic factor)

We consider a p-factor model for the 3 parameters of the Singh-Maddala distribution. $N = 10$ countries; $T = 21$ years. Macro regressors: gdp, cpi change, unemployment.

Let $y_{it} = (\ln a_{i,t}, \ln q_{i,t}, \ln \mu_{it})'$, $i = 1, \ldots, N$, $t = 1, \ldots, T$ and let $\hat{y}_{it}$ denote the 'realized' counterpart of y_{it} with asymptotic QML covariance matrix $\widehat{V}_{it}$.

We impose the following model structure

$$
\begin{aligned}
\hat{y}_{it} &= y_{it} + u_{it}, \qquad u_{it} \sim N(0, \underbrace{\widehat{V}_{it}^{1/2} A \widehat{V}_{it}^{1/2'}}_{=\tilde{V}_{it}}) && (1) \\
y_{it} &= c_i + B_i^{(c)} f_t^{(c)} + B_i^{(i)} f_t^{(i)} + B_i^{(m)} X_{i,t} && (2)
\end{aligned}
$$

$$
\begin{aligned}
f_t^{(c)} &= f_{t-1}^{(c)} + \eta_t^{(c)}, \qquad \eta_t^{(c)} \sim N\left(0, I_p\right) && (3) \\
f_t^{(i)} &= f_{t-1}^{(i)} + \eta_t^{(i)}, \qquad \eta_t^{(i)} \sim N\left(0, I_3\right) && (4)
\end{aligned}
$$

with diagonal idiosyncratic loading matrix $B_i^{(i)}$ and p.d. bias adjustment matrix A. Let $B^{(c)} = [B_1^{(c)'}, \ldots, B_N^{(c)'}]'$ with the upper triangular $p \times p$ submatrix of $B^{(c)}$ restricted to be lower triangular with positive diagonal elements. The diagonal elements of $B_i^{(i)}$ are also restricted to be nonnegative. We impose conjugate Gaussian and Wishart priors on all model parameters and simulate the common and idiosyncratic factors via FFBS with initial factor value for $t = 0$ set to zero.

Problems:

- The Gibbs draws of some of the common loadings show very strong serial correlation: converged?
- $\hookrightarrow$ possible reasons:
  - (Weak) identification problem due to sum of two random walk factors for each parameter.
  - Very low measurement error covariance induced by $\widehat{V}_{it}$.

– Sorting of assets?

1.1 State-Space Representation (with missing data W_t, Durbin and Koopman, 2012)

$$W_t \hat{y}_t = W_t c + W_t B f_t + W_t u_t, \qquad \text{(measurement equation)}$$

with $\underset{3N\times 1}{\hat{y}_t} = (\hat{y}'_{1t}, \dots, \hat{y}'_{Nt})'$, $\underset{3N\times 1}{c} = (c'_1, \dots, c'_N)'$, $\underset{3N\times p}{B} = [B'_1 \cdots B'_N]'$,\ $\underset{3N\times 1}{u_t} = (u'_{1t}, \dots, u'_{Nt})'$, and $W_t u_t \overset{iid}{\sim} N(0, W_t \underbrace{\operatorname{diag}\{V_i\}}_{:=V} W'_t)$,\ $\underset{3N_t^* \times 3N}{W_t}$ is a known matrix whose rows are a subset of the rows of the identity matrix (N_t^*: # non-missing countries at time t).

$$f_t = \Phi f_{t-1} + \eta_t, \qquad \text{(transition equation)}$$

Number of parameters to be estimated:

$$\underset{(c)}{3N} + \underset{(B)}{N \cdot 3p - \frac{p(p-1)}{2}} + \underset{(V)}{3N}$$

1.2 Kalman Filter Recursion

$$f_t|\mathcal{I}_{t-1} \sim N(f_{t|t-1}, P_{t|t-1})$$

Initialization: Choose $f_{1|0}$ and $P_{1|0}$.

%Case1 1:

For $t = 1, \dots, T$:

$$\begin{aligned}
W_t v_t &= W_t(\hat{y}_t - \hat{y}_{t|t-1}) = W_t(\hat{y}_t - c - Bf_{t|t-1}), \\
W_t F_t W_t' &= \mathrm{Var}(W_t v_t|\mathcal{I}_{t-1}) = W_t(BP_{t|t-1}B' + \underbrace{\mathrm{Var}(u_t)}_{=V})W_t', \\
f_{t|t} &= f_{t|t-1} + P_{t|t-1}(W_t B)'(W_t F_t W_t')^{-1} W_t v_t, \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1}(W_t B)'(W_t F_t W_t')^{-1} W_t B P_{t|t-1}, \\
(K_t W_t^{-1} &= \Phi P_{t|t-1}(W_t B)'(W_t F_t W_t')^{-1}), \\
(L_t &= \Phi - K_t W_t^{-1} W_t B), \\
f_{t+1|t} &= \Phi f_{t|t} = \Phi f_{t|t-1} + K_t W_t^{-1} W_t v_t, \\
P_{t+1|t} &= \Phi P_{t|t}\Phi' + I_p = \Phi P_{t|t-1} L_t' + I_p \\
&= \Phi P_{t|t-1}\left(\Phi - \Phi P_{t|t-1}(W_t B)'(W_t')^{-1}(BP_{t|t-1}B' + V)^{-1} W_t^{-1} W_t B\right)' + I_p
\end{aligned}$$

1.3 Gibbs-Sampler

Step 1: Simulate f conditional on $\hat{y}$, B and A. (FFBS).

1.1 Generate f_T from $f_T|\hat{y}, B, A \sim N(f_{T|T}, P_{T|T})$

1.2 For $t = T-1, T-2, \ldots, 1$. Generate f_t from

$$f_t|\tilde{y}_t, f_{t+1}, B, A \sim N\left(\mu_{f_{t+1}}, \Sigma_{f_{t+1}}\right),$$

where

$$\mu_{f_{t+1}} = f_{t|t} + P_{t|t}\Phi'(\Phi P_{t|t}\Phi' + I_p)^{-1}(f_{t+1} - \Phi f_{t|t}),$$

$$\Sigma_{f_{t+1}} = P_{t|t} - P_{t|t}\Phi'(\Phi P_{t|t}\Phi' + I_p)^{-1}\Phi P_{t|t},$$

$$\tilde{y}_t = (\hat{y}_1, \ldots, \hat{y}_t)'.$$

Step 2: Simulate B conditional on $f, \hat{y}$ and A.\ For $i = 1, \ldots, N$:

Cond. posterior:

$$\mathrm{vec}(B_i)|A, f, \hat{y} \sim N\left(\mathrm{vec}(B_1), \Omega_1\right),$$

$$\Omega_1^{-1} = \Omega_0^{-1} + \sum_{t=1}^{T}\left(f_t f_t' \otimes \tilde{V}_{it}^{-1}\right),$$

$$\mathrm{vec}(B_1) = \Omega_1\left\{\mathrm{vec}\left(\sum_{t=1}^{T}\tilde{V}_{it}^{-1} y_{it} f_t'\right) + \Omega_0^{-1}\,\mathrm{vec}(B_0)\right\},$$

$$\Rightarrow R\,\mathrm{vec}(B_i)|A, f, \hat{y} \sim N\left(R\,\mathrm{vec}(B_1), R\Omega_1 R'\right),$$

{

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$$\Omega_1^{-1} = \Omega_0^{-1} + R\sum_{t=1}^{T}\left(f_t f_t' \otimes \tilde{V}_{it}^{-1}\right)R',$$

$$\mathrm{vec}(B_1) = \Omega_1 R\left\{\mathrm{vec}\left(\sum_{t=1}^{T}\tilde{V}_{it}^{-1} y_{it} f_t'\right) + \Omega_0^{-1}\,\mathrm{vec}(B_0)\right\},$$

$$\Rightarrow R\,\mathrm{vec}(B_i)|A, f, \hat{y} \sim N\left(R\,\mathrm{vec}(B_1), \Omega_1\right),$$

} where R is a $3p \times 3p$ matrix whose rows are a subset of the rows of the identity matrix. $R\,\mathrm{vec}(B_i)$ retains only the non-zero elements of $\mathrm{vec}(B_i)$ to fulfill the zero-restrictions. For each i sampling is repeated until the sign restrictions for identification are satisfied.

Step 3: Simulate A conditional on $f, \hat{y}$ and B.

$$\widehat{V}_{it}^{-1/2'} \hat{y}_{it} = \widehat{V}_{it}^{-1/2'} y_{it} + \underbrace{\widehat{V}_{it}^{-1/2'} u_{it}}_{\tilde{u}_{it}}, \qquad \tilde{u}_{it} \overset{iid}{\sim} N(0, A). \tag{5}$$

Cond. posterior:

$$A|B, f, \hat{y} \sim W^{-1}\left(\Psi_0 + \sum_{t=1}^{T}\sum_{i=1}^{N} \hat{\tilde{u}}_{it}\hat{\tilde{u}}_{it}', TN + \nu_0\right).$$