Two Picard-Like Theorems

Alex Rusciano

Complex Analysis
Department of Mathematics
Carleton College

Contact Information:
email: rusciana@carleton.edu

1 Introduction

Picard's Theorem is an intriguing and important theorem in the study of entire functions. In the first part of this paper, we prove a version of Picard's Theorem for entire functions with "gaps" in their power series. In the second, we establish a generalization of the Casorati-Weierstrass theorem that applies to entire functions essential singularity at ∞. The scope of the results we establish is not the most general, but the proof techniques are often useful.

2 Picard's Theorem for Functions with Gaps

Theorem 2.1 (Picard's Theorem for Functions with Gaps)**.** *Let f be a non-polynomial entire function defined by a power series about* 0

$$\sum_{1}^{\infty} a_k z^{n_k}$$

with the additional condition that $n_k - n_{k-1} \geq k^2$. Then f obtains every value infinitely many times.

The idea here is that for the term growth and entire conditions to be compatible, there have to be large stretches of 0 valued coefficients in the power series of the function.

It is easy to construct such functions through defining them by power series, though these will not resemble functions with well known names.

To prove this theorem, we analyze the function in terms of the most significant term of the power series.

Definition 2.2 (maximum term)**.** We define the maximum term of the entire functions f at $r > 0$ to be $sup_k |a_k| r^{n_k}$. We will denote this by $m(r)$. Note that because f is entire, the power series must go to 0 absolutely so $m(r)$ is a well-defined finite positive number. This also shows that $m(r)$ is realized by at least one index k. This brings us to another definition

Definition 2.3 (central index)**.** For integer k, define k to be a central index of r if $m(r) = |a_k| r^{n_k}$. We create the notation $I_k = r : |a_k| r^{n_k} = m(r)$, i.e. the set of r for which k is a central index.

Note that there may be more than one central index for a given r.

These are common notions, and a few remarks will be made here without proof. The proofs are short technical arguments and not particularly interesting. Each I_k is empty, a point, or a closed segment $[c, d]$. The I_k overlap only at endpoints. Furthermore, for $j < k$ we have I_j is not to the right of I_k (on the real line). Finally, all positive real r appear in the union of the I_k. We need one lemma specific to the functions satisfying the conditions of the above theorem. It says loosely that in big I_k, you can find values of r such that the central index k is big relative to the rest of the function. More precisely

Lemma 2.4. *Let f satisfy the conditions of the above theorem. Let k be an integer larger than 1 with corresponding $I_k = [c_k, d_k]$ be such that*

$$\log \frac{d_k}{c_k} > \frac{2\log(3k)}{k^2}$$

Then there is a point $r \in I_k$ such that $|f(z) - a_k z^{n_k}| < |a_k z^{n_k}|$ on the circle $|z| = r$.

Proof. By the triangle inequality we know that

$$|f(z) - a_k z^{n_k}| \leq \sum_{j=1}^{k-1} |a_j| r^{n_j} + \sum_{j=k+1}^{\infty} a_j z^{n_j}$$

We prove the claim by showing that for some choice of z

$$\sum_{j=1}^{k-1} \frac{|a_j|}{|a_k|} r^{n_j - n_k} + \sum_{j=k+1}^{\infty} \frac{|a_j|}{|a_k|} z^{n_j - n_k} < 1$$

We proceed first by establishing an upper bound on the two sums above which applies for all $r \in [c_k, d_k]$. Let's work on the second sum. The goal is to show that it is less than a geometric series sum that gives a good enough bound. Observe that $n_j - n_k > k^2(j-k)$ because there are $j - k$ terms between $n_j - n_k$ and the ith is separated by an amount $(k+i)^2 > k^2$. Because $j > k$, we know from I_k's definition that $|a_j| d_k^{n_j} \leq |a_k| d_k^{n_k}$. Hence we get a term by term bound $\frac{|a_j|}{|a_k|} z^{n_j - n_k} \leq \frac{r}{d_k} r^{n_j - n_k}$.

Because $r < d_k$ we have smaller exponents producing bigger numbers so it follows that $\frac{r}{d_k} r^{n_j - n_k} <$

$((\frac{r}{d_k})^{k^2})^{j-k}$ and

$$\sum_{j=k+1}^{\infty} \frac{|a_j|}{|a_k|} z^{n_j - n_k} < \sum_{j=k+1}^{\infty} \frac{r}{d_k} r^{n_j - n_k} = \frac{(r/d)^{k^2}}{1-(r/d)^{k^2}}$$

The last equality follows from the middle term being a geometric series. Now we produce a bound for the first sum.

Certainly for $j < k$ we have $n_k - n_j \geq k^2$ by the gap condition. And also because $r > c$ we get $\frac{|a_j|}{|a_k|} \leq c^{n_k - n_j}$. As a result we get that the first sum is less than or equal to

$$\sum_{j=1}^{k-1} (\frac{c}{r})^{k^2} < k(\frac{c}{r})^{k^2}$$

We now have both sums bounded. If we plug in $|z| = \sqrt{cd}$ it becomes

$$\frac{(\sqrt{\frac{c}{d}})^{k^2}}{1-(\sqrt{\frac{c}{d}})^{k^2}} + k(\sqrt{\frac{c}{d}})^{k^2}$$

We can show this is less than 1 as follows. Indeed, removing the logarithms from the conditions imposed upon k in the statement of the lemma, then it translates precisely into $k(\sqrt{(\frac{c}{d})})^{k^2} < \frac{1}{3}$. And then plugging this into the first term gives $\frac{\frac{1}{3}}{1-\frac{1}{3}}$ which is $\frac{1}{2}$. $1/3 + 1/2$ is less than 1, completing the proof. □

Rouche's Theorem is also needed to carry out the proof. The tools to prove it are only basic residue theory, but for brevity its proof will not be given here. It says

Theorem 2.5 (Rouche's Theorem (very weak version stated here))**.** *If f and g are analytic in the same region and there is a circle λ in the region on which $|h(z) - g(z)| < h(z)$ then f and g have the same number of zeroes on the disc encircled by λ.*

Now the proof of the theorem can be carried out.

Proof. WLOG we only show that 0 is obtained infinitely many times. Because $f - z_0$ certainly satisfies the conditions of the theorem as well.

Let k be such that I_k satisfies the conditions of the lemma. Then the conclusion of the theorem says precisely that Rouche's Theorem applies with f being g, $a_k z^{n_k}$ being h, and the lemmas promised r giving the circular path. This means $f(z)$ takes on the value zero n_k times in the disc

$D_r(0)$. In particular it takes on the value of 0. It needs only be shown that there are infinitely many k that satisfy the conditions of the lemma.

Suppose there are finitely many. Then we can choose a K so that none of the sets I_k for $k > K$ satisfy the conditions. Then

$$\sum_{k>K}^{\infty} \frac{d_k}{c_k} \leq \sum_{k>K}^{\infty} \frac{2\log(3k)}{k^2} < \infty$$

On the other hand, we know the LHS ought to be infinity, because it is precisely $\displaystyle\int_{c_K}^{\infty} \frac{dt}{t}$ which is ∞. This completes the proof of the main theorem of the section. $\square$

3 Weak Lines of Julia

Before we state the main theorem of the section, we must make clear what a weak line of Julia is.

Definition 3.1 (weak line of Julia)**.** For a given angle ϕ and a $\epsilon > 0$ we can define the rays and sectors by $R(\phi)$ and $S(\phi, \epsilon)$ in the obvious way; that is, the ray points in direction ϕ and the sector is centered about ϕ with arc measure ϵ radians. The ray $R(\phi)$ is called a weak line of Julia of a function f if for each $\epsilon, r > 0$ f maps $S(\phi, \epsilon) \cap \{z : |z| > r\}$ densely into the plane.

Now we can state the theorem

Theorem 3.2 (existance of weak lines of Julia)**.** *Entire transcendental functions have at least one weak line of Julia*

As before, the proof of this proposition hinges on a lemma. And as before, the proof of the lemma will be presented first.

Lemma 3.3. *Let f be an analytic function in $D_1(0)$ such that on $D_1(0)$ the lower bound $|f(u)-w| \geq \delta$ holds for some complex w and some positive δ. Then it must be the case that on the slightly smaller disc $D_{\frac{1}{5}}(0)$ the upper bound $|f(u)| < \frac{5M^2}{\delta}$ holds for $M = Max(|w|, |F(0)|, \delta)$.*

Proof. Suppose we show that analytic g on $D_1(0)$ satisfying $|g(u)| \geq 1$ must also satisfy the condition $|g(u)| < |g(0)|^2$ on $D_{\frac{1}{5}}(0)$. The conditions of the lemma show that $\frac{f(u)-w}{\delta} \geq 1$ so the above statement's condition is met for $g = \frac{f(u)-w}{\delta}$ and we assume we know its result is true. This would produce the inequality chain in $D_{\frac{1}{5}}(0)$

$$|f(u) - w| < \frac{|f(0) - w|^2}{\delta} \leq \frac{4M^2}{\delta}$$

Then using the triangle inequality on the leftmost term and using that $M \geq \delta$ gives

$$|f(u)| < \frac{4M^2}{\delta} + M \leq \frac{5M^2}{\delta}$$

This shows we must only show the statement about such g is true.

We proceed naively and smoothly till we hit a block. Then after we reach that block we tweak the argument so that the result follows. First set $h = \frac{1}{g}$. h is analytic on the unit disc and between 0 and 1 in absolute value, exclusive of 0. We also restrict ourselves to $u \in D_{\frac{1}{5}}(0)$. Then

$$|h(u)| \geq |h(0)| - |h(u) - h(0)| \geq |h(0)| - \int_0^u |h'(v)||dv|$$

Applying Cauchy's inequality with radius $r = \frac{4}{5}$ to bound $h'(v)$ uses 1 for the maximum of h, for the circle of radius r about u fits inside the unit circle. This shows that

$$|h(u)| \geq h(0) - \frac{5|u|}{4} > |h(0)| - \frac{1}{4}$$

This is progress but not good enough. Sometimes this RHS is small enough. In particular, it is small enough when $f(0)$ is closer to the 1 side than the 0 side. So we don't apply the inequality to h, but to a relative of h for which we can be sure the inequality is sharp enough. It turns out we the relative we want is the analytic branch of $h(u)^s$ where $s = -\frac{\log(4)}{\log(|h(0)|)}$.

There is such a branch. To see this, note that the argument I gave for the other journal article I summarized still applies, showing that $h(z)$ can be written as $e^{l(z)}$ for a $l(z)$ analytic on the unit disc. Then $h(u)^s$ defined by $\exp(sl(z))$ is a holomorphic branch. Call this branch H. Also, H remains between 0 and 1 so the inequality

$$|H(u)| > H(0) - \frac{1}{4}$$

still holds. Now direct calculation shows for the value of s chosen

$$(|H(0)| - \frac{1}{4})^{\frac{1}{s}} = |h(0)|^2$$

The fact $x^{\frac{1}{\log(x)}} = e$ was used to replace $|h(0)|^s$ with $\exp(-\log(2))$ and the rest is straightforward. Therefore $|h(u)| > |h(0)|^2$ for u in the disc of radius $\frac{1}{5}$ about 0. Taking the reciprocal yields the upper bound on g in the disc we were after. And recall earlier that this proved to be sufficient to establish the lemma we were originally after. $\square$

It would be somewhat silly of a lemma if it only applied to the disc $D_1(0)$. Instead, it really applies to any disc. For $f(w_0+ur)$ dilates and translates an arbitrary disc onto $D_1(0)$ and establishes the upper bound of F upon a subdisk of the chosen arbitrary disc. I mention this only for motivation so the following manipulations are easier. Now let's jump into the proof of the main result.

Proof. The proof proceeds in three main steps. As a first step, we construct a collection of discs $\{D_n\}_{n=1}^{\infty}$ where $D_n = D(r_n e^{i\lambda_n}; \frac{r_n}{n})$ with r_n going to ∞ and $0 \leq \lambda_n < 2\pi$. As a second, we show the sequence we construct has the property that the image under f of each infinite union of members of the sequence is dense in the plane. Then we show that the existance of any sequence of discs that satisfies the above two properties implies that f has a weak line of Julia.

We start with the first part. Take any countable, dense set of the complex plane (e.g. complex numbers with rational components). Enumerate this sequence $\{w_k\}_{k=1}^{\infty}$. The version of the Casorati-Weierstrass theorem relevent here is the one that says that entire functions have what is basically an essential singularity at infinity. This translates into f mapping $\{z : |z| > r\}$ densely into the plane. This guarentees the existance of a z_0 of absolute value r where r is arbitrarily large such that $|f(z_0)| < 1$. Now we want to show that, by taking r sufficiently big, there is a disc of radius $\frac{r}{n}$ centered about a point z_0 where $|z_0| = r$ such that the image of the disc under f intersects all of the discs $D(w_i; \frac{1}{n})$ for $i = 1$ to $i = n$. We proceed by contradiction.

Suppose this is not the case. Then for all r large enough, the discs centered along $|z| = r$ of radius $\frac{r}{n}$ all manage to all have images that miss the $D(w_i; \frac{1}{n})$. To start, this assumption is true for z_0. Then the lemma applied using $f(z_0 + ru/n)$ with $\delta = \frac{1}{n}$, $w = w_j$ (w_j being the ommitted disc's center) gives $|f(z)| < 5nA^2$ on $D(z_0; \frac{r}{5n})$. Here A can be taken to be $Max(|w_1|, .., |w_n|, 1)$. Now, inductively, given z_{i-1} previously defined, form z_i by rotating z_{i-1} by $\frac{\pi}{30n}$ radians. We see that $z_i \in D(z_{i-1}; \frac{r}{5n})$ as follows.

$$|z_i - z_{i-1}| = 2r\sin(\frac{\pi}{30n}) < (2r)\frac{\pi}{30n} < \frac{r}{5n}$$

where the the first inequality comes from $\sin(x) < x$ for positive x. This is important for two reasons. First, the discs from $i = 0$ to $i = 30n$ cover the circle of radius r. Second, we already knew that the lemma applies to $D(z_i; \frac{r}{5n})$ but the upper bound given to it by the lemma depends on $|f(z_i)|$. Now we can see by a simple induction, $|f(z_i)| < (5n)^{2^i-1}A^{2^i}$ and therefore z_i's disc is bound by $(5n)^{2^{i+1}-1}A^{2^{i+1}}$. Indeed, one can see from this how the induction proceeds.

On this circle of radius r, we have shown $f(z) < (5n)^{2^{30n}-1}A^{2^{30n}}$ This is big, but it does not depend on r. Recall that we assumed r could be arbitrarily large so that this bound still holds. And recall that the maximum modulus principle tells us that the maximum of f occurs on boundary. From these two facts it follows that f is bound. Therefore f is a bound function which is entire, and it must be a constant, which certainly makes it not transcendental. This is a contradiction.

It appears strange to use n in the above inequality when we could take a simple value of n like 1 and get the same effect. However, now it guides our selection of D_n which is the goal of this first (and hardest) section of the proof. By the above analysis, there's a $D(z_n, \frac{r_n}{n})$ such that $r_n = |z_n|$ is arbitrarily big (in particular, taking r_n going to infinity with n) and f maps this disc so that it intersects $D(w_i, \frac{1}{n})$ for $i = 1$ to $i = n$. This completes part one of the proof and provides a bridge into the second part.

Indeed, this is simple now. Given some w in the plane, w must be a approached by a subsequence $\{v_k\}$ of the $\{w_k\}$. Let U be an infinite union of our constructed discs. By construction, as the index of a disc gets bigger, there are points in that disc that intersect arbitrarily small neighborhoods of the v_k. Hence a sequence of points in the image of the union of the discs approaches w because v_k approaches w.

Now we must only show the third part; that is, such a disc sequence requires f to have a weak line of Julia. Because a circle is a compact set, the rays defined by going through centers of circles have a ray they accumulate towards. Call this $R(\lambda)$. Note that the ratio of distance from origin to radius of circle is $\frac{1}{n}$ which actually decreases with n towards 0. One helpful way to view this is that the line tangent to the circle approaches 0 radians relative to its direction ray. It is then apparent that for each sector $S(\lambda, \epsilon)$ and each subsector of this $S(\lambda, \delta)$, n can be taken large enough so that all D_n whose direction rays are within $S(\lambda, \delta)$ must themselves lie completely inside $S(\lambda, \epsilon)$. This completes part three of the proof and establishes the main theorem.

□

4 Bibliography

Two Relatives of Picard's Theorem on Entire Functions, by Robert M. Gethner, appearing in The American Mathematical Monthly, Vol. 99. No. 1, Jan., 1992, p. 13-19