Probability: Homework 3

Due on July 28, 2014

Instructor: Elena Kosygina 6:00 pm

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Problem 1

Find the distribution function of X

Denote Y as Jereny's medical expenses in one year, which have exponential distribution with mean 1, which implies that

$$\begin{aligned} f(y;\lambda=1) &= \begin{cases} \lambda e^{-\lambda y}, & \text{if } y \geq 0 \\ \lambda 0, & \text{if } y < 0 \end{cases} = \begin{cases} e^{-y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases} \\ F(y;\lambda=1) &= \begin{cases} 1-e^{-\lambda y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases} = \begin{cases} 1-e^{-y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases} \end{aligned} \tag{1}$$

where $\lambda = 1^{-1} = 1$.
Now we consider X, X cannot have density since

$$P(X > 1) = 0 \tag{2}$$

because the maximum of the amount of unreimbursed medical expenses is $1 + 20\% \times (5-1) = 1.8$ (in thousand of dollars). Moreover, X is not simple as it can take any value in $[0, 1.8]$.
Let us find $F_X(x) = P(X \leq x)$. If $x < 0$,

$$F_X(x) = 0 \tag{3}$$

If $0 \leq x < 1$, that is equivalent to $0 \leq y < 1$ since $X = 1$ in this interval

$$F_X(x) = P(X \leq x) = P(Y \leq 1) = F_Y(1) = 1 - e^{-1} \tag{4}$$

If $1 \leq x < 1.8$, that is equivalent to $1 \leq y < 5$ since $X = 1 + 0.2(Y-1) = 0.8 + 0.2Y$ in this interval

$$F_X(x) = P(X \leq x) = P(0.8 + 0.2Y \leq x) = F_Y(5x-4) = 1 - e^{-(5x-4)} \tag{5}$$

If $x \geq 1$,

$$F_X(x) = P(X \leq x) = 1 \tag{6}$$

Find the distribution measure of X

Note that there is a jump in $F_X(x)$ at $x = 1$, since

$$\lim_{x\to 1^-} F_X(x) = 1 - e^{-5}, \lim_{x \to 1^+} F_X(x) = 1 \tag{7}$$

where the gap is e^{-5}.
Distribution measure of X is

$$\mu_X = \delta_0 + \mu_0 + e^{-5}\delta_1 \tag{8}$$

where

$$\begin{aligned} \mu_0([a,b]) &= \int_a^b 1_{[1,1.8]}(x) f(x) dx \\ \mu_X([a,b]) &= \delta_0([a,b]) \int_a^b 1_{[1,1.8]}(x) f(x) dx + e^{-5}\delta_1([a,b]) \end{aligned} \tag{9}$$

where $f(x) = 5e^{-(5x-4)}$.

Find $E(X)$

$$EX = \delta_0(R) + \int_1^{1.8} xf(x)dx + e^{-5}\delta_1(R) = 1 + \int_1^{1.8} 5xe^{-(5x-4)}dx + e^{-5} = 0.6e^{-5} - 1.6e^{-1} + 1.8 = 1.2154 \quad (10)$$

thousand dollars.

Problem 2

The distribution of X under $\tilde{P}$ is

$$\tilde{F}_X(x) = \int_\Omega I_{X(\omega)\leq x} Z(\omega)d\omega = \int_{-\infty}^x f(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \quad (11)$$

where $f(x)$ is given as

$$f(x) = e^{-\frac{(x+\theta)^2}{2}+\frac{x^2}{2}} \quad (12)$$

Then

$$\tilde{F}_X(x) = \int_\infty^x \frac{1}{\sqrt{2\pi}}e^{\frac{(x+\theta)^2}{2}}dx \quad (13)$$

Therefore, under $\tilde{P}$, $X \sim N(-\theta, 1)$.

Problem 3

For any $u \in R$,

$$\begin{aligned}\tilde{E}(e^{uY}) &= E(e^{uY}Z) = E(e^{u(X+\theta)}e^{-\theta X - \frac{1}{2}\theta^2}) \\ &= E(e^{(u-\theta)X})e^{u\theta-\frac{1}{2}\theta^2} = e^{\frac{1}{2}(u-\theta)^2}e^{u\theta-\frac{1}{2}\theta^2} \\ &= e^{u^2/2}\end{aligned} \quad (14)$$

Therefore Y is standard normal under $\tilde{P}$.

Problem 4

(i)

$$\frac{1}{\epsilon}P(X \in B(x,\epsilon)) = \frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{x-\epsilon}^{x+\epsilon} e^{-u^2/2}du \approx \frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}e^{x^2/2}\epsilon = \frac{1}{\sqrt{2\pi}}e^{x^2/2} \quad (15)$$

(ii)

$$\frac{1}{\epsilon}P(Y \in B(y,\epsilon)) = \frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{y-\epsilon}^{y+\epsilon} e^{-u^2/2}du \approx \frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}e^{y^2/2}\epsilon = \frac{1}{\sqrt{2\pi}}e^{y^2/2} \quad (16)$$

Problem 4 continued on next page...

(iii)

As given

$$\begin{aligned}\{X \in B(x,\epsilon)\} &= \{w : x - \epsilon/2 \leq X(w) \leq x + \epsilon/2\}\\ \{Y \in B(y,\epsilon)\} &= \{w : y - \epsilon/2 \leq Y(w) \leq y + \epsilon/2\}\end{aligned} \tag{17}$$

By substituting y with $x+\theta$ and $Y(w) = X(w) + \theta$, we find the second formula becomes

$$\{w : x + \theta - \epsilon/2 \leq X(w) + \theta \leq x + \theta + \epsilon/2\} = \{w : x - \epsilon/2 \leq X(w) \leq x + \epsilon/2\} \tag{18}$$

Therefore they are the same set.

(iv)

$$\begin{aligned}\frac{\tilde{P}(A)}{P(A)} &\approx \exp[-Y^2(\overline{w})/2 + X^2(\overline{w})/2]\\ &= \exp[-(X(\overline{w}) + \theta)^2/2 + X^2(\overline{w})/2]\\ &= \exp[-\theta X(\overline{w}) - \theta^2/2]\end{aligned} \tag{19}$$

Problem 5

Poisson distribution

A discrete random variable X is said to have a Poisson distribution with parameter $\lambda > 0$, if, for $k = 0, 1, 2, \ldots$ the probability mass function of X is given by:

$$f_P(k;\lambda) = \Pr_P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{20}$$

where

- e is Euler's number (e = 2.71828...)
- k! is the factorial of k.

The positive real number λ is equal to the expected value of X and also to its variance.

$$\lambda = \mathrm{E}(X) = \mathrm{Var}(X). \tag{21}$$

Binomial distribution

In general, if the random variable X follows the binomial distribution with parameters n and p, we write $X \sim B(n,p)$. The probability of getting exactly k successes in n trials is given by the probability mass function:

$$f_B(k;n,p) = \Pr_B(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{22}$$

for $k = 0, 1, 2, \ldots, n$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{23}$$

is the binomial coefficient.

Distribution of S and F

The probability of getting exactly k successes in X trials is given by the probability mass function:

$$\Pr(S = k|X) = f_B(k; X, p) \tag{24}$$

Therefore

$$\begin{aligned}
\Pr(S = k) &= \sum_{i=k}^{\infty} f_P(i; \lambda) f_B(k; i, p) \\
&= \sum_{i=k}^{\infty} \Pr_P(X = i) \Pr_B(S = k) \\
&= \sum_{i=k}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \binom{i}{k} p^k (1-p)^{i-k} \\
&= \sum_{i=k}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \frac{i!}{k!(i-k)!} p^k (1-p)^{i-k} \\
&= \frac{(p\lambda)^k e^{-\lambda}}{k!} \sum_{i=k}^{\infty} \frac{\lambda^{i-k}(1-p)^{i-k}}{(i-k)!}
\end{aligned} \tag{25}$$

where I didn't consider $i < k$ because it is impossible to generate k successes with $i < k$ trials. We note that the last term is exactly the taylor series of exponential function, that is

$$\sum_{i=k}^{\infty} \frac{\lambda^{i-k}(1-p)^{i-k}}{(i-k)!} = e^{(1-p)\lambda} \tag{26}$$

Therefore,

$$\Pr(S = k) = \frac{(p\lambda)^k e^{-\lambda}}{k!} e^{(1-p)\lambda} = \frac{(p\lambda)^k e^{-p\lambda}}{k!} \tag{27}$$

which implies that $S \sim \text{Poisson}(p\lambda)$.
Similarly we can substitute $(1-p)$ with p to find the distribution of F, follow the same process and conclude that $F \sim \text{Poisson}((1-p)\lambda)$.

Independence of S and F

For any positive integers k_1 and k_2,

$$\begin{aligned}
\Pr(S = k_1) &= f_P(k_1; p\lambda) = \frac{(p\lambda)^{k_1} e^{-p\lambda}}{(k_1)!}, \\
\Pr(F = k_2) &= f_P(k_2; (1-p)\lambda) = \frac{[(1-p)\lambda]^{k_2} e^{-(1-p)\lambda}}{(k_2)!}
\end{aligned} \tag{28}$$

For the event $S = k_1, F = k_2$, the number of Bernoulli trials is exactly $k_1 + k_2$, then

$$\Pr(S = k_1, F = k_2) = f_P(k_1 + k_2; \lambda) f_B(k_1; k_1 + k_2, p) = \frac{\lambda^{k_1+k_2} e^{-\lambda}}{(k_1+k_2)!} p^{k_1} (1-p)^{k_2} \binom{k_1+k_2}{k_1} \tag{29}$$

Since

$$\frac{e^{-p\lambda}}{(k_1)!} \frac{e^{-(1-p)\lambda}}{(k_2)!} = \frac{e^{-\lambda}}{k_1! k_2!} = \frac{e^{-\lambda}(k_1+k_2)!}{k_1! k_2! (k_1+k_2)!} = e^{-\lambda} \frac{1}{(k_1+k_2)!} \binom{k_1+k_2}{k_1} \tag{30}$$

We have exactly

$$\Pr(S = k_1, F = k_2) = \Pr(S = k_1) \Pr(F = k_2) \tag{31}$$

So we can conclude S and F are independent.

Problem 6

Let $\mathcal{G}$ be generated by the partition $E_1 = [0, 1/3), E_2 = \{1/3\}, E_3 = (1/3, 1/2), E_4 = [1/2, 1]$, then $\mathcal{G}$ is the power set of them. Now let us compute $E(X|\mathcal{G})$.

(i) $X(w) = 1_{(0,1/4)}(w)$

$$\begin{aligned}
c_1 &= \frac{\int_{E_1} XdP}{P(E_1)} = \frac{1/4}{1/3} = \frac{3}{4} \\
c_2 &= \frac{\int_{E_2} XdP}{P(E_2)} = 0 \\
c_3 &= \frac{\int_{E_3} XdP}{P(E_3)} = 0 \\
c_4 &= \frac{\int_{E_4} XdP}{P(E_4)} = 0
\end{aligned} \tag{32}$$

For any other elements E_i of $\mathcal{G}$,

$$c_i = \sum_{n=1}^{4} 1_{E_n \subset E_i} c_n \tag{33}$$

(ii) $X(w) = w$

$$\begin{aligned}
c_1 &= \frac{\int_{E_1} XdP}{P(E_1)} = \frac{1/18}{1/3} = \frac{1}{6} \\
c_2 &= \frac{\int_{E_2} XdP}{P(E_2)} = 0 \\
c_3 &= \frac{\int_{E_3} XdP}{P(E_3)} = \frac{5/72}{1/6} = \frac{5}{12} \\
c_4 &= \frac{\int_{E_4} XdP}{P(E_4)} = \frac{3}{4}
\end{aligned} \tag{34}$$

For any other elements E_i of $\mathcal{G}$,

$$c_i = \sum_{n=1}^{4} 1_{E_n \subset E_i} c_n \tag{35}$$

(ii) $X(w) = (w - 1/2)^2$

$$\begin{aligned}
c_1 &= \frac{\int_{E_1} XdP}{P(E_1)} = \frac{13}{108} \\
c_2 &= \frac{\int_{E_2} XdP}{P(E_2)} = 0 \\
c_3 &= \frac{\int_{E_3} XdP}{P(E_3)} = \frac{1}{108} \\
c_4 &= \frac{\int_{E_4} XdP}{P(E_4)} = \frac{1}{12}
\end{aligned} \tag{36}$$

For any other elements E_i of $\mathcal{G}$,

$$c_i = \sum_{n=1}^{4} 1_{E_n \subset E_i} c_n \tag{37}$$

Problem 7

Recall that $W_n := X_1 + X_2 + \cdots + X_n$ and $R_n := \frac{X_2+\cdots+X_n}{X1}$ are independent. This is well-known and can be proved by the usual method of derived distributions for two random variables. the mapping $(X_1, X_2 + X_3 + \cdots + X_n) \to (R_n, W_n)$ is $1-1$ and easily inverted.
Then, noting that $\frac{X_1}{W_n} = \frac{1}{1+R_n}$,

$$E(X_1|W_n) = E(\frac{W_n}{1+R_n}|W_n) = W_n E(\frac{1}{1+R_n}|W_n) = W_n E(\frac{1}{1+R_n}) \tag{38}$$

The last equality follows from independence. Also, on repeating the argument, or just taking unconditional expectations,

$$EX_1 = E(\frac{W_n}{1+R_n}) = E(\frac{1}{1+R_n})EW_n = E\frac{1}{1+R_n}nE(X_1) \tag{39}$$

So

$$E\frac{1}{1+R_n} = \frac{1}{n} \tag{40}$$

which can be plugged to get the result,

$$E(X_1|W_n) = \frac{W_n}{n} \tag{41}$$

Problem 8

Since X and Y are standard normal random variables, then $\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$, and the parameters of W is

$$\begin{aligned} \mu_3 &= EW = \mu_2 - \frac{\rho\sigma_2}{\sigma_1}\mu_1 = 0 \\ \sigma_3 &= \sqrt{Var(W)} = \sqrt{1-\rho^2} \end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned} g(x) &= Ef(x, \rho x + W) \\ &= \frac{1}{\sigma_3\sqrt{2\pi\sigma_3}} \int_{-\infty}^{\infty} f(x, \rho x + w) \exp(-\frac{w^2}{2\sigma_3^2}) dw \end{aligned} \tag{43}$$

where

$$f(x, \rho x + w) = \exp(x + \rho x + w) = \exp((1+\rho)x + w) \tag{44}$$

So,

$$\begin{aligned} g(x) &= \frac{1}{\sigma_3\sqrt{2\pi\sigma_3}} \int_{-\infty}^{\infty} \exp[(1+\rho)x + w - \frac{w^2}{2\sigma_3^2}]dw \\ &= \frac{1}{\sigma_3\sqrt{2\pi\sigma_3}} \int_{-\infty}^{\infty} \exp[\frac{-(w-\sigma_3^2)^2 + 2(1+\rho)\sigma_3^2 x + \sigma_3^4}{2\sigma_3^2}]dw \\ &= \frac{\exp[(1+\rho)x + \sigma_3^2/2]}{\sqrt{\sigma_3}} \int_{-\infty}^{\infty} \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{(w-\sigma_3^2)^2}{2\sigma_3^2}} dw \\ &= \frac{\exp[(1+\rho)x + \sigma_3^2/2]}{\sqrt{\sigma_3}} \end{aligned} \tag{45}$$

Problem 8 continued on next page. . .

Substitute σ_3, we have

$$g(x) = \frac{\exp[(1+\rho)x + (1-\rho^2)/2]}{(1-\rho^2)^{1/4}} \tag{46}$$

Then

$$E(f(X,Y)|X) = g(X) = \frac{\exp[(1+\rho)X + (1-\rho^2)/2]}{(1-\rho^2)^{1/4}} \tag{47}$$