Solutions to Stochastic Calculus for Finance II (Steven Shreve)

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Introduction

This solution manual will be updated anytime, and is NOT intended for any business use. The author suggests this manual as a reference book to the metioned book by Steven Shreve. Also anyone involved in any mathematical finance courses shall not copy the solutions in this book directly. This is ideally for self-check purpose.

If anyone find any mistakes or misprints in this book, please inform me. Thanks.

1 Chapter 1

1.1 Exercise 1.4

(i) On the infinite coin-toss space Ω_∞, for $n = 1, 2, \dots$ we define

$$Y_n(\omega) = \begin{cases} 1, & \text{if } \omega_n = H \\ 0, & \text{if } \omega_n = T \end{cases}$$

We set $X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n}$ Then by Example 1.2.5, X is uniformly distributed on $[0, 1]$. Furthermore, let

$$N(z) = \int_{-\infty}^{z} \phi(\zeta) d\zeta \qquad \text{where} \qquad \phi(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}$$

Then for $Z = N^{-1}(X)$, by Example 1.2.6,

$$\mu_Z[a, b] = P(\omega : a \leq Z(\omega) \leq b) = P(\omega : a \leq N^{-1}(X(\omega)) \leq b) = P(\omega : N(a) \leq X \leq N(b)) = N(b) - N(a)$$

for any $-\infty < a \leq b < \infty$. So we conculde that $Z = N^{-1}(X)$ is a standard normal distributionon on $(\Omega_\infty, \mathcal{F}_\infty, P)$.

(ii) Let

$$Z_n(\omega) = N^{-1}(X_n(\omega)) \qquad \text{where} \qquad X_n(\omega) = \sum_{k=1}^{n} \frac{Y_k(\omega)}{2^k}$$

It is clear that

$$\lim_{n\to\infty} Z_n(\omega) = \lim_{n\to\infty} N^{-1}(X_n(\omega)) = N^{-1}(\lim_{n\to\infty} X_n(\omega)) = N^{-1}(X(\omega)) = Z(\omega) \qquad \text{for every } \omega \in \Omega_\infty$$

Since $Y_k(\omega)$ only depends on the $k-$th coin toss, X_n will only depend on the first n coin tosses, which means Z_n will also only depend on the first n coin tosses.

1.2 Exercise 1.5

First, since

$$I_{(0,X(\omega))}(x) = \begin{cases} 0, & \text{if } x \notin (0, X(\omega)) \\ 1, & \text{if } x \in (0, X(\omega)) \end{cases} \tag{1.1}$$

we can compute the double integral as:

$$\int_\Omega \int_0^\infty I_{(0,X(\omega))}(x) dx dP(\omega) = \int_\Omega [X(\omega) - 0] dP(\omega) = E[X] \tag{1.2}$$

In the other hand, we may also compute the same double integral by changing the order f the integration as:

$$\int_\Omega \int_0^\infty I_{(0,X(\omega))}(x) dx dP(\omega) = \int_0^\infty \int_\Omega I_{(0,X(\omega))}(x) dP(\omega) dx$$

Now let us consider the innner integral with respect to $P(\omega)$, in which we shall consider the function $I_{(0,X(\omega))}(x)$ as a function of ω instead of x. Now note the condition $x \in (0, X(\omega))$ in (1.1) is equivalent to $X(\omega) \geq x$. So

$$\int_\Omega I_{(0,X(\omega))}(x) dP(\omega) = \int_\Omega I_{\omega:X(\omega)\geq x}(\omega) dP(\omega) = P(\omega : X(\omega) \geq x) = P(X \geq x) = 1 - P(X \leq x) = 1 - F(x)$$

Then

$$\int_\Omega \int_0^\infty I_{(0,X(\omega))}(x) dx dP(\omega) = \int_0^\infty \int_\Omega I_{(0,X(\omega))}(x) dP(\omega) dx = \int_0^\infty [1 - F(x)] dx \tag{1.3}$$

By equating (1.1) and (1.3), we conculde that

$$E[X] = \int_0^\infty [1 - F(x)] dx$$

1.3 Exercise 1.6

1.

$$
\begin{aligned}
E[e^{uX}] &= \int e^{ux} f(x) dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left(ux - \frac{(x-\mu)^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left(ux - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left[-\frac{1}{2\sigma^2} \left(x^2 - (2\mu + 2u\sigma^2)x \right) - \frac{\mu^2}{2\sigma^2} \right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left[-\frac{1}{2\sigma^2} \left(x - (\mu + u\sigma^2) \right)^2 + \frac{1}{2\sigma^2}(\mu + u\sigma^2)^2 - \frac{\mu^2}{2\sigma^2} \right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left[-\frac{1}{2\sigma^2} \left(x - (\mu + u\sigma^2) \right)^2 \right] dx \cdot \exp\left[\frac{1}{2\sigma^2}(\mu + u\sigma^2)^2 - \frac{\mu^2}{2\sigma^2} \right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int \exp\left[-\frac{1}{2\sigma^2} y^2 \right] dy \cdot \exp\left[\frac{2u\mu\sigma^2}{2\sigma^2} + \frac{\mu^2\sigma^4}{2\sigma^2} \right] \\
&= 1 \cdot \exp\left(u\mu + \frac{1}{2} u^2 \sigma^2 \right)
\end{aligned}
$$

2.

$$E[\phi(X)] = E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2} \geq e^{u\mu} = \phi(\mu) = \phi(EX)$$

1.4 Exercise 1.7

1. $f(x) = 0$ for any x;
2. $\lim_{n\to\infty} \int f_n(x)dx = 1 \neq 0 = \int f(x)dx$;
3. The sequence f_i is not monotone. Refer to the following diagram for the graphs of the functions:

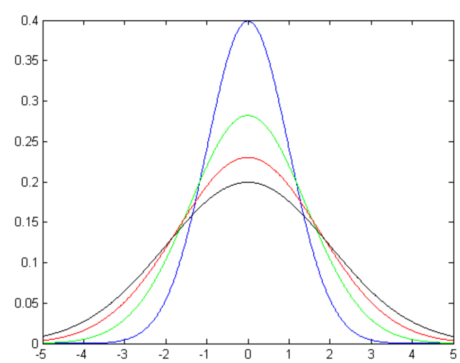

Figure 1: Graphs of Exercise 1.7 with n=1,2,3,4 from the top respectively

1.5 Exercise 1.8

(i) We consider the sequence

$$Y_n(\omega) = \frac{e^{tX(\omega)} - e^{s_n X(\omega)}}{t - s_n}$$

First note that

$$\lim_{s_n \to t} Y_n(\omega) = \frac{d}{dt}[e^{tX(\omega)}] = X(\omega)e^{tX(\omega)}$$

Also by the Mean value Theorem, there exists $\theta_n \in [s_n, t]$ (or $[t, s_n]$) such that

$$Y_n(\omega) = \frac{e^{tX(\omega)} - e^{s_n X(\omega)}}{t - s_n} = X(\omega)e^{\theta_n(\omega)X(\omega)}$$

Note that for n sufficiently large, $\theta_n \leq \max\{t, s_n\} \leq 2|t|$ since $s_n \to t$, so

$$|Y_n(\omega)| = X(\omega)e^{\theta_n(\omega)X(\omega)} \leq X(\omega)e^{2|t|X(\omega)}$$

Also note that for any n and $E[Xe^{tX}] < \infty$ for any t, so by Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty} EY_n = \lim_{n\to\infty} \int_{\omega\in\Omega} Y_n(\omega)dP(\omega) = \int_{\omega\in\Omega} \lim_{n\to\infty} Y_n(\omega)dP(\omega) = \int_{\omega\in\Omega} X(\omega)e^{tX(\omega)}dP(\omega) \\ = E[Xe^{tX}]\end{aligned}$$

i.e.,

$$\lim_{n\to\infty} EY_n = E[\lim_{n\to\infty} Y_n] = E[Xe^{tX}]$$

(ii) If X can take both positive and negative values, then write $X = X^+ - X^-$, where $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$. Then by the Mean value Theorem, there exists $\theta_n \in [s_n, t]$ (or $[t, s_n]$) such that

$$Y_n(\omega) = X(\omega)e^{\theta_n(\omega)X(\omega)}$$

Then

$$|Y_n(\omega)| = \left|X(\omega)e^{\theta_n(\omega)X(\omega)}\right| \leq |X|\, e^{\theta_n(\omega)X(\omega)} \leq |X|\, e^{2|t||X|} \tag{1.4}$$

Now we need to show that $E[|X|\, e^{t|X|}] < \infty$ for any t to apply Dominated Convergence Theorem. For any t,

$$\begin{aligned}E[|X|\, e^{t|X|}] &= \int_{\omega\in\Omega} |X|\, e^{t|X|}dP(\omega) \\ &= \int_{\omega\in\Omega} 1_{X(\omega)\geq 0} \cdot |X|\, e^{t|X|}dP(\omega) + \int_{\omega\in\Omega} 1_{X(\omega)\leq 0} \cdot |X|\, e^{t|X|}dP(\omega) \\ &= \int_{\omega\in\Omega} 1_{X(\omega)\geq 0} \cdot X^+ e^{tX^+}dP(\omega) + \int_{\omega\in\Omega} 1_{X(\omega)\leq 0} \cdot X^- e^{tX^-}dP(\omega) \\ &= E[1_{X(\omega)\geq 0} \cdot X^+ e^{tX^+}] + E[1_{X(\omega)\leq 0} \cdot X^- e^{tX^-}]\end{aligned}$$

Since $E[|X|\, e^{tX}] < \infty$ for any t, so

$$\begin{aligned}E[|X|\, e^{tX}] &= \int_{\omega\in\Omega} |X|\, e^{tX}dP(\omega) \\ &= \int_{\omega\in\Omega} 1_{X(\omega)\geq 0} \cdot |X|\, e^{tX}dP(\omega) + \int_{\omega\in\Omega} 1_{X(\omega)\leq 0} \cdot |X|\, e^{tX}dP(\omega)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\omega\in\Omega} 1_{X(\omega)\ge 0}\cdot X^+ e^{tX^+} dP(\omega) + \int_{\omega\in\Omega} 1_{X(\omega)\le 0}\cdot X^- e^{-tX^-} dP(\omega)\\
&= E[1_{X(\omega)\ge 0}\cdot X^+ e^{tX^+}] + E[1_{X(\omega)\le 0}\cdot X^- e^{-tX^-}] < \infty
\end{aligned}
$$

which implies for any t

$$
E[1_{X(\omega)\ge 0}\cdot X^+ e^{tX^+}] < \infty \qquad \text{and} \qquad E[1_{X(\omega)\le 0}\cdot X^- e^{-tX^-}] < \infty
$$

Or equivalently for any t, we have

$$
E[1_{X(\omega)\ge 0}\cdot X^+ e^{tX^+}] < \infty \qquad \text{and} \qquad E[1_{X(\omega)\le 0}\cdot X^- e^{tX^-}] < \infty
$$

Hence

$$
E[|X|\, e^{t|X|}] = E[1_{X(\omega)\ge 0}\cdot X^+ e^{tX^+}] + E[1_{X(\omega)\le 0}\cdot X^- e^{tX^-}] < \infty
$$

for any t. Then $E[|X|\, e^{|t||X|}] < \infty$ for any t. Then by (1.4) and Dominated Convergence Theorem, we have

$$
\lim_{n\to\infty} EY_n = E[\lim_{n\to\infty} Y_n] = E[Xe^{tX}]
$$

which is $\phi'(t) = E[Xe^{tX}]$.

1.6 Exercise 1.10

1. Since Z is a nonnegative random variable and

$$
E[Z] = \int_\Omega Z(\omega)d\omega = 0\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 1
$$

by Theorem 1.6.1, $\tilde{P}$ is a probability measure.

2.

$$
\begin{aligned}
\tilde{P}(A) = \int_A Z(\omega)dP(\omega) &= \int_{A\cap\{\omega:0\le\omega\le 1/2\}} Z(\omega)dP(\omega) + \int_{A\cap\{\omega:1/2\le\omega\le 1\}} Z(\omega)dP(\omega)\\
&= 0\cdot P(A\cap\{\omega: 0\le\omega\le 1/2\}) + 2\cdot P(A\cap\{\omega: 1/2\le\omega\le 1\})\\
&\le 0 + 2\cdot P(A) = 0
\end{aligned}
$$

since $\tilde{P}(A)\ge 0$, $\tilde{P}(A) = 0$.

3. Let $A = \{\omega: 0\le\omega\le 1/2\}$, then $P(A) = 1/2 > 0$, $\tilde{P}(A) = \int_0^{1/2} Z(\omega)d\omega = \int_0^{1/2} 0 d\omega = 0$.

1.7 Exercise 1.11

By (1.6.4) of Text, we have for any $u\in R$

$$
\begin{aligned}
\tilde{E}\left[e^{uY}\right] &= E\left[e^{uY}Z\right] = E\left[e^{u(X+\theta)}e^{-\theta X-\frac{1}{2}\theta^2}\right]\\
&= E\left[e^{(u-\theta)X}\right]\cdot e^{u\theta-\frac{1}{2}\theta^2} = e^{\frac{1}{2}(u-\theta)^2}\cdot e^{u\theta-\frac{1}{2}\theta^2}\\
&= e^{\frac{1}{2}u^2-u\theta+\frac{1}{2}\theta^2+u\theta-\frac{1}{2}\theta^2}\\
&= e^{u^2/2}
\end{aligned}
$$

therefore Y is standard normal under $\tilde{P}$.

1.8 Exercise 1.14

Since for $a \geq 0$

$$P(X \leq a) = 1 - e^{-\lambda a} = \int_{-\infty}^{a} f(x)dx$$

where

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$f(x)$ is the density function for such expoential r.v.

1.

$$\begin{aligned}
\tilde{P}(\Omega) &= \int_{\Omega} Z(\omega)dP(\omega) \\
&= \int_{-\infty}^{\infty} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} f(x)dx \\
&= \int_{0}^{\infty} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \cdot (\lambda e^{-\lambda x})dx \\
&= \int_{0}^{\infty} \tilde{\lambda} e^{-\tilde{\lambda} x} dx \\
&= 1
\end{aligned}$$

2.

$$\begin{aligned}
\tilde{P}(X \leq a) &= \int_{\Omega} \chi_{\omega : X(\omega) \leq a} Z(\omega)dP(\omega) \\
&= \int_{0}^{a} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \cdot (\lambda e^{-\lambda x})dx \\
&= \int_{0}^{a} \tilde{\lambda} e^{-\tilde{\lambda} x} dx \\
&= 1 - e^{-\tilde{\lambda} a}
\end{aligned}$$

2 Chapter 2

2.1 Exercise 2.2

1. By the definition, $\sigma(X) = \{\{X \in B\}; B$ ranges over the subsets of $\mathbf{R}\}$.

$$\begin{aligned}
&\text{when } B = \{1\}, && \{X \in B\} = \{X = 1\} = \{\omega \in \Omega_2 : S_2(\omega) = 4\} = \{HT, TH\} \\
&\text{when } B = \{0\}, && \{X \in B\} = \{X = 0\} = \{\omega \in \Omega_2 : S_2(\omega) \neq 4\} = \{HH, TT\}
\end{aligned}$$

   So all the sets in $\sigma(X)$ are $\phi, \Omega_2, \{HT, TH\}$ and $\{HH, TT\}$.

2. By the definition, $\sigma(S_1) = \{\{S_1 \in B\}; B$ ranges over the subsets of $\mathbf{R}\}$.

$$\begin{aligned}
&\text{when } B = \{8\}, && \{S_1 \in B\} = \{S_1 = 8\} = \{\omega \in \Omega_2 : S_1(\omega) = 4\} = \{HT, HH\} \\
&\text{when } B = \{2\}, && \{S_1 \in B\} = \{S_1 = 2\} = \{\omega \in \Omega_2 : S_1(\omega) \neq 4\} = \{TH, TT\}
\end{aligned}$$

   So all the sets in $\sigma(X)$ are $\phi, \Omega_2, \{HT, HH\}$ and $\{TH, TT\}$.

3. Here we only investigate the non-trivial cases:

$A \in \sigma(X)$	$B \in \sigma(S_1)$	$A \cap B$	$\tilde{P}(A)\tilde{P}(B)$	$\tilde{P}(A \cap B)$
$\{HT, TH\}$	$\{HT, HH\}$	{HT}	$(\frac{1}{4}+\frac{1}{4}) \cdot (\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HT, TH\}$	$\{TH, TT\}$	{TH}	$(\frac{1}{4}+\frac{1}{4}) \cdot (\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HH, TT\}$	$\{HT, HH\}$	{HH}	$(\frac{1}{4}+\frac{1}{4}) \cdot (\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$
$\{HH, TT\}$	$\{TH, TT\}$	{TT}	$(\frac{1}{4}+\frac{1}{4}) \cdot (\frac{1}{4}+\frac{1}{4}) = \frac{1}{4}$	$\frac{1}{4}$

therefore for any $A \in \sigma(X)$ and $B \in \sigma(S_1)$, we have $\tilde{P}(A \cap B) = \tilde{P}(A)\tilde{P}(B)$. So $\sigma(X)$ and $\sigma(S_1)$ are independent under $\tilde{P}$.

4. Again we only investigate the non-trivial cases:

$A \in \sigma(X)$	$B \in \sigma(S_1)$	$A \cap B$	$P(A)P(B)$	$P(A \cap B)$
$\{HT, TH\}$	$\{HT, HH\}$	{HT}	$(\frac{2}{9}+\frac{2}{9}) \cdot (\frac{2}{9}+\frac{4}{9}) = \frac{8}{27}$	$\frac{2}{9}$
$\{HT, TH\}$	$\{TH, TT\}$	{TH}	$(\frac{2}{9}+\frac{2}{9}) \cdot (\frac{2}{9}+\frac{1}{9}) = \frac{4}{27}$	$\frac{2}{9}$
$\{HH, TT\}$	$\{HT, HH\}$	{HH}	$(\frac{4}{9}+\frac{1}{9}) \cdot (\frac{2}{9}+\frac{4}{9}) = \frac{10}{27}$	$\frac{4}{9}$
$\{HH, TT\}$	$\{TH, TT\}$	{TT}	$(\frac{4}{9}+\frac{1}{9}) \cdot (\frac{2}{9}+\frac{1}{9}) = \frac{5}{27}$	$\frac{1}{9}$

therefore for some $A \in \sigma(X)$ and $B \in \sigma(S_1)$ (indeed for any non-trivial elements as shown), we have $P(A \cap B) \neq P(A)P(B)$. So $\sigma(X)$ and $\sigma(S_1)$ are not independent under P.

5. If we are told that $X = 1$, then we have $S_2 = 4$, i.e., the event must be HT or TH. But since $P(HT) = P(TH) = 2/9$, we canot use the un-conditioned probability for S_1 any longer. In other words, based on the current information, we shall estimate $P(S_1 = 8|X = 1) = 1/2 = P(S_1 = 2|X = 1)$.

2.2 Exercise 2.3

Since V is normal random variable and

$$E[V] = E[X\cos\theta + Y\sin\theta] = \cos\theta E[X] + \sin\theta E[Y] = \cos\theta \cdot 0 + \sin\theta \cdot 0 = 0$$
$$Var[V] = Var[X\cos\theta + Y\sin\theta] = \cos^2\theta \cdot Var[X] + \sin^2\theta \cdot Var[Y] = \cos^2\theta \cdot 1 + \sin^2\theta \cdot 1 = 1$$

so V is standard normal. Similarly W is also standard normal.

To show the independence, we quote the fact that if Z is a standard normal random variable, then for any u, $E[e^{uZ}] = \exp(u^2/2)$ (Exercise 1.6). Then for any real numbers a and b,

$$\begin{aligned}
E[e^{aV+bW}] &= E[e^{a(X\cos\theta+Y\sin\theta)+b(-X\sin\theta+Y\cos\theta)}] \\
&= E[e^{X(a\cos\theta-b\sin\theta)+Y(a\cos\theta+b\sin\theta)}] \\
&= E[e^{X(a\cos\theta-b\sin\theta)}] \cdot E[e^{Y(a\cos\theta+b\sin\theta)}] \qquad \text{by the independence of } X \text{ and } Y \\
&= \exp\left[\frac{(a\cos\theta-b\sin\theta)^2}{2} + \frac{(a\cos\theta+b\sin\theta)^2}{2}\right] \\
&= \exp\left[\frac{a^2\cos^2\theta+b^2\sin^2\theta-2ab\cos\theta\sin\theta}{2} + \frac{a^2\cos^2\theta+b^2\sin^2\theta+2ab\cos\theta\sin\theta}{2}\right] \\
&= \exp\left[\frac{a^2}{2}+\frac{b^2}{2}\right] \\
&= E[e^{aV}]E[e^{bW}]
\end{aligned}$$

So V and W are independent.

2.3 Exercise 2.4

1. By the definition,

$$\begin{aligned}
E\left[e^{uX+vY}\right] =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ux+vxz} d\mu_Z(z) d\mu_X(x) \\
=& \int_{-\infty}^{\infty}\left[\frac{1}{2}e^{ux+vx\cdot 1}+\frac{1}{2}e^{ux+vx\cdot(-1)}\right] d\mu_X(x) \\
=& \frac{1}{2}\int_{-\infty}^{\infty} e^{(u+v)x} d\mu_X(x)+\frac{1}{2}\int_{-\infty}^{\infty} e^{(u-v)x} d\mu_X(x) \\
=& E[e^{(u+v)X}]+E[e^{(u-v)X}] \\
=& \frac{1}{2}e^{(u+v)^2/2}+\frac{1}{2}e^{(u-v)^2/2} \\
=& e^{(u^2+v^2)/2}\cdot\frac{e^{uv}+e^{-uv}}{2}
\end{aligned}$$

   for any u and v.

2. Let $u=0$ in the last result, then

$$E[e^{vY}]=e^{(0^2+v^2)/2}\cdot\frac{e^{0\cdot v}+e^{-0\cdot v}}{2}=e^{v^2/2}$$

   So Y must be a standard normal random variable.

3. If X and Y are independent, then by Theorem 2.2.7 of text, the joint moment generating function must factor as:

$$Ee^{uX+vY}=Ee^{uX}Ee^{vY}$$

   But apparently

$$Ee^{uX+vY}=e^{(u^2+v^2)/2}\cdot\frac{e^{uv}+e^{-uv}}{2}\neq e^{u^2/2}\cdot e^{v^2/2}=Ee^{uX}Ee^{vY}$$

   So they are not independent.

2.4 Exercise 2.5

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\
&= \int_{-|x|}^{\infty}\frac{2|x|+y}{\sqrt{2\pi}}\exp\left[-\frac{(2|x|+y)^2}{2}\right]dy \\
&= \int_{|x|}^{\infty}\frac{z}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz \qquad (z=y+2|x|) \\
&= -\frac{1}{\sqrt{2\pi}}\exp[-\frac{z^2}{2}]\Big|_{z=|x|}^{z=\infty} \\
&= \frac{1}{\sqrt{2\pi}}\exp[-\frac{x^2}{2}] \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\
&= \int_{|x|\geq -y}\frac{2|x|+y}{\sqrt{2\pi}}\exp\left[-\frac{(2|x|+y)^2}{2}\right]dx
\end{aligned}$$

if $y \geq 0$, then

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left[-\frac{(2|x|+y)^2}{2}\right] dx \\
&= 2\int_{0}^{\infty} \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dx \\
&= 2\int_{y}^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] \frac{dz}{2} \qquad (z = 2x+y) \\
&= \frac{1}{\sqrt{2\pi}} \exp[-\frac{y^2}{2}]
\end{aligned}$$

if $y \leq 0$, then

$$\begin{aligned}
f_Y(y) &= \int_{-y}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left[-\frac{(2|x|+y)^2}{2}\right] dx + \int_{-\infty}^{y} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left[-\frac{(2|x|+y)^2}{2}\right] dx \\
&= \int_{-y}^{\infty} \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dx + \int_{-\infty}^{y} \frac{-2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(-2x+y)^2}{2}\right] dx \\
&= 2\int_{-y}^{\infty} \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dx \\
&= 2\int_{-y}^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] \frac{dz}{2} \qquad (z = 2x+y) \\
&= \frac{1}{\sqrt{2\pi}} \exp[-\frac{y^2}{2}]
\end{aligned}$$

Therefore both X and Y are standard random variable.

It is clear that $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$, so they are not independent.

And it is also clear that $E[X] = E[Y] = 0$, and

$$\begin{aligned}
& E[XY] \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dydx \\
=& \int_{-\infty}^{\infty}\int_{-|x|}^{\infty} xy \cdot \frac{2|x|+y}{\sqrt{2\pi}} \exp\left[-\frac{(2|x|+y)^2}{2}\right] dydx \\
=& \int_{0}^{\infty}\int_{-x}^{\infty} xy \cdot \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dydx + \int_{-\infty}^{0}\int_{x}^{\infty} xy \cdot \frac{-2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(-2x+y)^2}{2}\right] dydx \\
=& \int_{0}^{\infty}\int_{-x}^{\infty} xy \cdot \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dydx + \int_{\infty}^{0}\int_{-t}^{\infty} (-t)y \cdot \frac{2t+y}{\sqrt{2\pi}} \exp\left[-\frac{(2t+y)^2}{2}\right] dy(-dt) \\
& \qquad (t = -x) \\
=& \int_{0}^{\infty}\int_{-x}^{\infty} xy \cdot \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dydx - \int_{0}^{\infty}\int_{-x}^{\infty} xy \cdot \frac{2x+y}{\sqrt{2\pi}} \exp\left[-\frac{(2x+y)^2}{2}\right] dydx \\
=& 0
\end{aligned}$$

so $E[XY] = E[X]E[Y]$, which implies X and Y are uncoorelated.

2.5 Exercise 2.7

Let $\mu = E[Y - X]$, then we consider

$$\begin{aligned}
Var(Y-X) =& E\left[(Y-X-\mu)^2\right] \\
=& E\left[\left((Y-E[Y|\mathcal{G}]) + (E[Y|\mathcal{G}] - X - \mu)\right)^2\right] \\
=& E\left[(Y-E[Y|\mathcal{G}])^2 + (E[Y|\mathcal{G}] - X - \mu)^2 + 2\left(Y-E[Y|\mathcal{G}]\right)\left(E[Y|\mathcal{G}] - X - \mu\right)\right] \\
\geq& E\left[(Y-E[Y|\mathcal{G}])^2\right] + 0 + 2E\left[\left(Y-E[Y|\mathcal{G}]\right)\left(E[Y|\mathcal{G}] - X - \mu\right)\right] \\
=& Var(Err) + 2E\left[\left(Y-E[Y|\mathcal{G}]\right)\left(E[Y|\mathcal{G}] - X - \mu\right)\right]
\end{aligned} \tag{2.1}$$

The last step is since $E(Err) = 0$, so $Var(Err) = E\left[(Y - E[Y|\mathcal{G}])^2\right]$.

Now note that both X and $E[Y|\mathcal{G}]$ are $\mathcal{G}-$measurable random variables,

$$\begin{aligned}
& E\left[\left(Y-E[Y|\mathcal{G}]\right)\left(E[Y|\mathcal{G}] - X - \mu\right)\right] \\
=& E\left[YE[Y|\mathcal{G}] - XY - \mu Y - E[Y|\mathcal{G}]E[Y|\mathcal{G}] + XE[Y|\mathcal{G}] + \mu E[Y|\mathcal{G}]\right] \\
=& E\left[YE[Y|\mathcal{G}] - XY - \mu Y - E[YE[Y|\mathcal{G}]|\mathcal{G}] + E[XY|\mathcal{G}] + \mu E[Y|\mathcal{G}]\right] \qquad \text{by } \mathcal{G}-\text{measurable} \\
=& E[YE[Y|\mathcal{G}]] - E[XY] - \mu E[y] - E[E[YE[Y|\mathcal{G}]|\mathcal{G}]] + E[E[XY|\mathcal{G}]] + \mu E[E[Y|\mathcal{G}]] \\
=& E[YE[Y|\mathcal{G}]] - E[XY] - \mu E[y] - E[YE[Y|\mathcal{G}]] + E[XY] + \mu E[Y] \qquad \text{by iterated conditioning } E[E[Z|\mathcal{G}]] = E[Z] \\
=& 0
\end{aligned}$$

By substituting this result into (2.1), we will have

$$Var(Y-X) \geq Var(Err)$$

2.6 Exercise 2.10

By the notations, for any $A \in \sigma(X)$,

$$\begin{aligned}
& \int_A g(X)dP \\
=& \int_{\{\omega\in\Omega : X(\omega)\in B\}} g(X(\omega))dP(\omega) \\
=& \int_\Omega g(X(\omega)) \cdot \mathbb{1}(X \in B)dP(\omega) \\
=& \int_{-\infty}^{\infty} g(x)f_X(x) \cdot \mathbb{1}(x \in B)dx \\
=& \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{yf_{X,Y}(x,y)}{f_X(x)}dy\right] \cdot f_X(x) \cdot \mathbb{1}(x \in B)dx \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot \mathbb{1}(x \in B)f_{X,Y}(x,y)dxdy \\
=& E[Y \cdot \mathbb{1}(X \in B)] \qquad \text{by the notation of (2.6.3)} \\
=& E[Y \cdot \mathbb{1}(A)] \qquad \text{by the notation of (2.6.2) and } A = \{X \in B\} \\
=& \int_A YdP
\end{aligned}$$

whic verifies the partial averaging property, hence proves $E[Y|X] = g(X)$.

3 Chapter 3

3.1 Exercise 3.2

For $0 \leq s \leq t$,

$$\begin{aligned}
&E[W^2(t) - t|\mathcal{F}(s)] \\
=&E[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t|\mathcal{F}(s)] \\
=&E[(W(t) - W(s))^2 - t|\mathcal{F}(s)] + E[(2W(t)W(s)|\mathcal{F}(s)] + E[-W^2(s)|\mathcal{F}(s)] \\
=&[(t-s) - t] + 2W(s)E[W(t)|\mathcal{F}(s)] - W^2(s) \\
=&-s + 2W^2(s) - W^2(s) \\
=&W^2(s) - s
\end{aligned}$$

so martingale.

3.2 Exercise 3.3

$$\begin{aligned}
\phi(u) &= E\left[e^{u(X-\mu)}\right] = e^{\frac{1}{2}u^2\sigma^2} \\
\phi'(u) &= E\left[(X-\mu)e^{u(X-\mu)}\right] = u\sigma^2 e^{\frac{1}{2}u^2\sigma^2} \\
\phi''(u) &= E\left[(X-\mu)^2 e^{u(X-\mu)}\right] = (\sigma^2 + u^2\sigma^4)e^{\frac{1}{2}u^2\sigma^2} \\
\phi^{(3)}(u) &= E\left[(X-\mu)^3 e^{u(X-\mu)}\right] = \left[2u\sigma^4 + (\sigma^2 + u^2\sigma^4)\cdot u\sigma^2\right] e^{\frac{1}{2}u^2\sigma^2} \\
&= \left(3u\sigma^4 + u^3\sigma^6\right) e^{\frac{1}{2}u^2\sigma^2} \\
\phi^{(4)}(u) &= E\left[(X-\mu)^4 e^{u(X-\mu)}\right] = \left[(3\sigma^4 + 3u2\sigma^6) + (3u\sigma^4 + u^3\sigma^6)\cdot u\sigma^2\right] e^{\frac{1}{2}u^2\sigma^2} \\
&= \left(3\sigma^4 + 6u^2\sigma^6 + u^4\sigma^8\right) e^{\frac{1}{2}u^2\sigma^2}
\end{aligned}$$

In particular, $\phi^{(3)}(0) = 0$ and $\phi^{(4)}(0) = 3\sigma^4$. The latter implies the kurtosis of a normal random variable is $\phi^{(4)}(0)/(\sigma^2)^2 = 3$.

3.3 Exercise 3.4

1. Since

$$\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \leq \max_{0\leq k\leq n-1} |W(t_{k+1} - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

By re-writing it, we have

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \geq \frac{\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max_{0\leq k\leq n-1} |W(t_{k+1}) - W(t_k)|} \longrightarrow \frac{T}{0} = \infty$$

for almost every path of the Brownina motion W as $n \to \infty$ and $|\Pi| \to 0$.

2.

$$
\begin{aligned}
0 \leq & \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \\
= & \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 \\
\to & 0 \cdot T = 0
\end{aligned}
$$

for almost every path of the Brownina motion W as $n \to \infty$ and $|\Pi| \to 0$.

3.4 Exercise 3.5

Note that $W(T) \sim N(0, T)$, with probability density function $f(x) = \frac{1}{\sqrt{2\pi T}} \exp(-x^2/2T)$. Then by the definition of the expectation

$$
\begin{aligned}
& E\left[e^{-rT}(S(T) - K)^+\right] \\
= & \int_{-\infty}^{\infty} e^{-rT} \left(S(0) e^{(r - \frac{1}{2}\sigma^2)T + \sigma x} - K\right)^+ f(x) dx \\
= & \int_{A}^{\infty} e^{-rT} \left(S(0) e^{(r - \frac{1}{2}\sigma^2)T + \sigma x} - K\right) f(x) dx
\end{aligned}
$$

where A is the constant such that $S(T) \geq K$ for any $x \geq A$ and $S(T) \leq K$ for any $x \leq A$. More precisely,

$$
\begin{aligned}
A = & \min\{x : (r - \frac{1}{2}\sigma^2)T + \sigma x \geq K\} \\
= & \min\{x : x \geq \frac{\log\left(\frac{K}{S(0)}\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma}\} \\
= & \frac{\log\left(\frac{K}{S(0)}\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma} \\
= & -\sqrt{T} d_-(T, S(0))
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\left[e^{-rT}(S(T) - K)^+\right] \\
= & \int_{-\sqrt{T}d_-}^{\infty} e^{-rT} S(0) e^{(r - \frac{1}{2}\sigma^2)T + \sigma x} f(x) dx - \int_{-\sqrt{T}d_-}^{\infty} e^{-rT} K f(x) dx \\
= & e^{-rT} S(0) e^{(r - \frac{1}{2}\sigma^2)T} \underbrace{\int_{-\sqrt{T}d_-}^{\infty} e^{\sigma x} f(x) dx}_{I} - e^{-rT} K \underbrace{\int_{-\sqrt{T}d_-}^{\infty} f(x) dx}_{II}
\end{aligned}
$$

Now we compute the two integrals separately:

$$
\begin{aligned}
I =& \frac{1}{\sqrt{2\pi T}} \int_{-\sqrt{T}d_-}^{\infty} e^{\sigma x} \cdot e^{-\frac{x^2}{2T}} dx \\
=& \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} \exp\left(\sigma\sqrt{T}t - \frac{t^2}{2}\right) dt && (t = \frac{x}{\sqrt{T}}) \\
=& \frac{1}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-d_-}^{\infty} \exp\left[-\frac{1}{2}(t-\sigma\sqrt{T})^2\right] dt && \text{(completing the square)} \\
=& \frac{1}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-d_- - \sigma\sqrt{T}}^{\infty} e^{-\frac{x^2}{2}} dx && (x = t - \sigma\sqrt{T}) \\
=& \frac{1}{\sqrt{2\pi}} e^{\sigma^2 T/2} \int_{-d_+}^{\infty} e^{-\frac{x^2}{2}} dx && (d_+ = d_- + \sigma\sqrt{T}) \\
=& e^{\sigma^2 T/2} N(d_+)
\end{aligned}
$$

and

$$
\begin{aligned}
II =& \frac{1}{\sqrt{2\pi T}} \int_{-\sqrt{T}d_-}^{\infty} e^{-\frac{x^2}{2T}} dx \\
=& \frac{1}{\sqrt{2\pi}} \int_{-d_-}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt && (t = \frac{x}{\sqrt{T}}) \\
=& N(d_-)
\end{aligned}
$$

By substituting I and II,we have the expected payoff of the option

$$
\begin{aligned}
& E\left[e^{-rT}(S(T)-K)^+\right] \\
=& e^{-rT} S(0) e^{(r-\frac{1}{2}\sigma^2)T} e^{\sigma^2 T/2} N(d_+) - e^{-rT} K N(d_-) \\
=& S(0) N(d_+) - e^{-rT} K N(d_-)
\end{aligned}
$$

3.5 Exercise 3.6

1. Write

$$E[f(X(t))|\mathcal{F}(s)] = E[f(X(t) - X(s) + X(s))|\mathcal{F}(s)]$$

We define $g(x) = E[f(X(t)-X(s)+x)]$, then we have $E[f(X(t)-X(s)+X(s))|\mathcal{F}(s)] = g(X(s))$. Now since

$$X(t) - X(s) = \mu(t-s) + W(t) - W(s)$$

is normally distibuted with mean $\mu(t-s)$ and variance $t-s$,

$$
\begin{aligned}
g(x) &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(\omega + x) \exp\left[-\frac{(\omega - \mu(t-s))^2}{2(t-s)}\right] d\omega \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(y - x - \mu(t-s))^2}{2(t-s)}\right] dy
\end{aligned}
$$

by changing variable $y = \omega + x$.

Therefore $E[f(X(t))|\mathcal{F}(s)] = g(X(s))$, i.e. X has Markov property.

2. Write

$$E[f(S(t))|\mathcal{F}(s)] = E\left[f\left(\frac{S(t)}{S(s)} \cdot S(s)\right)|\mathcal{F}(s)\right]$$

We define $g(x) = E\left[f\left(\frac{S(t)}{S(s)} \cdot x\right) | \mathcal{F}(s)\right]$, then since

$$\log \frac{S(t)}{S(s)} = \nu(t-s) + \sigma(W(t) - W(s))$$

is normally distibuted with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$,

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi}\sigma\sqrt{t-s}} \int_{-\infty}^{\infty} f(e^{\omega}x) \exp\left[-\frac{(\omega - \nu(t-s))^2}{2\sigma^2(t-s)}\right] d\omega \\ &= \frac{1}{\sqrt{2\pi}\sigma\sqrt{t-s}} \int_{0}^{\infty} \frac{1}{y} f(y) \exp\left[-\frac{(\log \frac{y}{x} - \nu(t-s))^2}{2\sigma^2(t-s)}\right] dy \end{aligned}$$

by changing variable $y = e^{\omega}x$ ($d\omega = dy/y$ and $-\infty \leq x \leq \infty$ implies $0 \leq y \leq \infty$).
Therefore

$$g(x) = \int_0^{\infty} f(y)p(\tau, x, y)dy \qquad \text{with} \qquad p(\tau, x, y) = \frac{1}{\sqrt{2\pi}\sigma y\sqrt{\tau}} \exp\left[-\frac{(\log \frac{y}{x} - \nu\tau)^2}{2\sigma^2\tau}\right]$$

satisfies $E[f(S(t))|\mathcal{F}(s)] = g(S(s))$ for any f. Hence S has Markov property.

3.6 Exercise 3.7

1.

$$\begin{aligned} & E\left[Z(t)\middle|\mathcal{F}(s)\right] \\ =& E\left[\frac{Z(t)}{Z(s)}Z(s)\middle|\mathcal{F}(s)\right] \\ =& Z(s)E\left[\frac{Z(t)}{Z(s)}\middle|\mathcal{F}(s)\right] \\ =& Z(s)E\left[\exp\left(\sigma\mu(t-s) + \sigma(W(t) - W(s)) - (\sigma\mu + \frac{1}{2}\sigma^2)(t-s)\right)\middle|\mathcal{F}(s)\right] \\ =& Z(s)e^{\sigma\mu(t-s)}E\left[e^{\sigma(W(t)-W(s))}\right] \cdot e^{-(\sigma\mu+\frac{1}{2}\sigma^2)(t-s)} \qquad \text{(by independence)} \\ =& Z(s)e^{\sigma\mu(t-s)} \cdot e^{\frac{1}{2}\sigma^2(t-s)} \cdot e^{-(\sigma\mu+\frac{1}{2}\sigma^2)(t-s)} \qquad (\text{since } W(t) - W(s) \sim N(0, s-t)) \\ =& Z(s) \end{aligned}$$

2. since τ_m is a stopping time, and a martingale which is stoped at a stopping time is still a martingale,

$$E[Z(t \wedge \tau_m)] = E[Z(0 \wedge \tau_m)] = E[Z(0)] = Z(0) = 1$$

which is exactly

$$E[\exp\{\sigma X(t \wedge \tau_m) - \left(-\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\}] = 1 \tag{3.1}$$

3. For $\tau_m < \infty$,

$$\lim_{t\to\infty} Z(t \wedge \tau_m) = Z(\tau_m) = \exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right] \tag{3.2}$$

For $\tau_m = \infty$,

$$\lim_{t\to\infty} Z(t \wedge \tau_m) = \lim_{t\to\infty} \exp\left[\sigma X(t) - (\sigma\mu + \frac{1}{2}\sigma^2)t\right] \tag{3.3}$$

since

$$0 \le e^{\sigma X(t)} \le e^{\sigma m} \qquad \lim_{t\to\infty} \exp\left[-(\sigma\mu + \frac{1}{2}\sigma^2)t\right] = 0 \tag{3.4}$$

$$\lim_{t\to\infty} Z(t\wedge\tau_m) = \lim_{t\to\infty} \exp\left[\sigma X(t)\right] \cdot \lim_{t\to\infty} \exp\left[-(\sigma\mu + \frac{1}{2}\sigma^2)t\right] = \lim_{t\to\infty} e^{\sigma X(t)} \cdot 0 = 0 \tag{3.5}$$

therefore

$$\lim_{t\to\infty} \exp\left[\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right] = I_{\tau_m<\infty} \exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right] \tag{3.6}$$

By taking the limits in (3.1), we have

$$\begin{aligned} & E\left[I_{\tau_m<\infty} \exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right] \\ =& E\left[\lim_{t\to\infty} \exp\left[\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right]\right] \\ =& \lim_{t\to\infty} E\left[\exp\left[\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right]\right] \\ =& 1 \end{aligned} \tag{3.7}$$

Furthermore

$$1 = \lim_{\sigma\to 0+} E\left[I_{\tau_m<\infty} \exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right] = E[I_{\tau_m<\infty}]$$

which is equivalent to $P[\tau_M < \infty] = 1$.

Therefore,

$$1 = E\left[I_{\tau_m<\infty} \exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right] = E\left[\exp\left[\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right]$$

or equivalentely, for any $\sigma > 0$

$$E\left[\exp\left[-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right] = e^{-\sigma m}$$

Let $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$, then $\alpha > 0$ and $\sigma = -\mu \pm \sqrt{u^2 + 2\alpha}$. But since $\sigma > 0$, $\sigma = -\mu + \sqrt{u^2 + 2\alpha}$, so for any $\alpha > 0$

$$\begin{aligned} & E[e^{-\alpha\tau_m}] \\ =& E\left[\exp\left[-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\right]\right] \\ =& e^{-\sigma m} \\ =& e^{-(-\mu+\sqrt{u^2+2\alpha})m} \\ =& e^{\mu-\sqrt{u^2+2\alpha}m} \end{aligned}$$

4. By differentiating

$$\begin{aligned}
\frac{d}{d\alpha}E\left[e^{-\alpha\tau_m}\right] &= \frac{d}{d\alpha}\left[e^{m\mu-m\sqrt{2\alpha+\mu^2}}\right] \\
E\left[-\tau_m\cdot e^{-\alpha\tau_m}\right] &= \frac{-m}{2\sqrt{2\alpha+\mu^2}}\cdot 2\cdot e^{m\mu-m\sqrt{2\alpha+\mu^2}} \\
\lim_{\alpha\to 0+}E\left[-\tau_m\cdot e^{-\alpha\tau_m}\right] &= \lim_{\alpha\to 0+}\frac{-m}{\sqrt{2\alpha+\mu^2}}\cdot e^{m\mu-m\sqrt{2\alpha+\mu^2}} \\
E\left[-\tau_m\right] &= \frac{-m}{\sqrt{0+\mu^2}}\cdot e^{m\mu-m\sqrt{0+\mu^2}} \\
E\left[\tau_m\right] &= \frac{m}{\mu} \qquad\qquad \text{since } \mu>0
\end{aligned}$$

And $E\left[\tau_m\right]=\frac{m}{\mu}<\infty$ follows immediately.

5. For $\mu<0$ and $\sigma>-2\mu>0$, we still have (3.2) and (3.3) respectively. The slight difference to the argument in (3.4) will be,

$$0\le\lim_{t\to\infty}\exp\left[-(\sigma\mu+\frac{1}{2}\sigma^2)t\right]\le\lim_{t\to\infty}\exp\left[-\frac{\sigma(\sigma+2\mu)}{2}t\right]=0$$

as both $\sigma>0$ and $\sigma+2\mu>0$.

Then (3.5)-(3.7) follow directly as part (iii), which is

$$E\left[I_{\tau_m<\infty}\exp\left[\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m\right]\right]=1 \tag{3.8}$$

Insteading of taking limits $\sigma\to 0+$ in (ref130724g), we can only take the limit $\sigma\to -2\mu$ as:

Furthermore

$$1=\lim_{\sigma\to -2\mu}E\left[I_{\tau_m<\infty}\exp\left[\sigma m-(\sigma\mu+\frac{1}{2}\sigma^2)\tau_m\right]\right]=E[e^{-2\mu m}\cdot I_{\tau_m<\infty}]$$

which is equivalent to $P[\tau_M<\infty]=E[I_{\tau_m<\infty}]=e^{2m\mu}=e^{-2m|\mu|}$.

The Laplace transform argument is exactly the same as part (iii).

3.7 Exercise 3.8

(i)

$$\begin{aligned}
\phi_n(u) =& E\left[\exp\left(u\frac{1}{\sqrt{n}}M_{nt,n}\right)\right] \\
=& E\left[\exp\left(\frac{u}{\sqrt{n}}\sum_{k=1}^{nt}X_{k,n}\right)\right] \\
=& \prod_{k=1}^{nt}E\left[\exp\left(\frac{u}{\sqrt{n}}X_{k,n}\right)\right] \qquad \text{by independence} \\
=& \prod_{k=1}^{nt}\left[\tilde{p_n}e^{u/\sqrt{n}}+\tilde{q_n}e^{-u/\sqrt{n}}\right] \\
=& \left[\tilde{p_n}e^{u/\sqrt{n}}+\tilde{q_n}e^{-u/\sqrt{n}}\right]^n \qquad \text{by i.i.d}
\end{aligned}$$

$$=\left[e^{u/\sqrt{n}}\left(\frac{\frac{r}{n}+1-e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}}-e^{-\sigma/\sqrt{n}}}\right)-e^{-u/\sqrt{n}}\left(\frac{\frac{r}{n}+1-e^{\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}}-e^{-\sigma/\sqrt{n}}}\right)\right]^n$$

(ii) First by the substitution $\sqrt{n}=1/x$, we have

$$\begin{aligned}\log\phi_{1/x^2}(u)=&\frac{t}{x^2}\log\left[e^{ux}\frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x}-e^{-\sigma x}}-e^{-ux}\frac{rx^2+1-e^{\sigma x}}{e^{\sigma x}-e^{-\sigma x}}\right]\\=&\frac{t}{x^2}\log\left[(rx^2+1)\frac{e^{ux}-e^{-ux}}{e^{\sigma x}-e^{-\sigma x}}+\frac{e^{(\sigma-u)x}-e^{(u-\sigma)x}}{e^{\sigma x}-e^{-\sigma x}}\right]\\=&\frac{t}{x^2}\log\left[\frac{(rx^2+1)\sinh ux+\sinh(\sigma-u)x}{\sinh\sigma x}\right]\end{aligned}$$

Then by $\sinh(A-B)=\sinh A\cosh B-\cosh A\sinh B$, we have

$$\begin{aligned}\log\phi_{1/x^2}(u)=&\frac{t}{x^2}\log\left[\frac{(rx^2+1)\sinh ux+\sinh(\sigma-u)x}{\sinh\sigma x}\right]\\=&\frac{t}{x^2}\log\left[\frac{(rx^2+1)\sinh ux+\sinh\sigma x\cosh ux-\cosh\sigma x\sinh ux}{\sinh\sigma x}\right]\\=&\frac{t}{x^2}\log\left[\cosh ux+\frac{(rx^2+1-\cosh\sigma x)\sinh ux}{\sinh\sigma x}\right]\end{aligned}$$

(iii) Since $\sinh z=z+\mathcal{O}(z^3)$,

$$\frac{\sinh ux}{\sinh\sigma x}=\frac{ux+\mathcal{O}(x^3)}{\sigma x+\mathcal{O}(x^3)}=\frac{u}{\sigma}+\mathcal{O}(x^3)$$

This is because of the identity

$$(\sigma x+\mathcal{O}(x^3))\cdot(\frac{u}{\sigma}+\mathcal{O}(x^3))=ux+\mathcal{O}(x^3)+\mathcal{O}(x^6)=ux+\mathcal{O}(x^3)$$

And since $\cosh z=1+\frac{1}{2}z^2+\mathcal{O}(z^4)$,

$$1-\cosh z=-\frac{1}{2}z^2+\mathcal{O}(z^4)$$

Therefore

$$\begin{aligned}&\cosh ux+\frac{(rx^2+1-\cosh\sigma x)\sinh ux}{\sinh\sigma x}\\=&1+\frac{u^2x^2}{2}+\mathcal{O}(x^4)+(rx^2-\frac{\sigma^2x^2}{2}+\mathcal{O}(x^4))\cdot(\frac{u}{\sigma}+\mathcal{O}(x^3))\\=&1+\frac{u^2x^2}{2}+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+\mathcal{O}(x^4)+\mathcal{O}(x^5)+\mathcal{O}(x^7)\\=&1+\frac{u^2x^2}{2}+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+\mathcal{O}(x^4)\end{aligned}$$

(iv) Since

$$\log\phi_{1/x^2}(u)=\frac{t}{x^2}\log\left[1+\frac{u^2x^2}{2}+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+\mathcal{O}(x^4)\right]$$

$$
\begin{aligned}
&= \frac{t}{x^2} \log \left[1 + \left(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2} \right) x^2 + \mathcal{O}(x^4) \right] \\
&= \frac{t}{x^2} \left[\left(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2} \right) x^2 + \mathcal{O}(x^4) + \mathcal{O}(x^6) + \mathcal{O}(x^8) \right] \\
&= \frac{t}{x^2} \left[\left(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2} \right) x^2 + \mathcal{O}(x^4) \right] \\
&= \left(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2} \right) t + \mathcal{O}(x^2)
\end{aligned}
$$

we have the moment generating function

$$
\begin{aligned}
\lim_{x \downarrow 0} \log \phi_{1/x^2}(u) &= \left(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2} \right) t \\
&= u \left(\frac{rt}{\sigma} - \frac{\sigma t}{2} \right) + \frac{u^2}{2} t
\end{aligned}
$$

This implies that

$$
\frac{1}{\sqrt{n}} M_{nt,n} \quad \sim \quad N\left(\left(\frac{rt}{\sigma} - \frac{\sigma t}{2} \right), t \right) \quad \text{as } n \to \infty
$$

Now by the following property of the normal random variables

$$
Z \sim N(a, b) \quad \Longrightarrow \quad cZ \sim N(ca, c^2 b)
$$

we have

$$
\frac{\sigma}{\sqrt{n}} M_{nt,n} \quad \sim \quad N\left(\sigma \left(\frac{rt}{\sigma} - \frac{\sigma t}{2} \right), \sigma^2 t \right) = N\left(\left(r - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right)
$$

as $n \to \infty$, i.e., the limiting distribution of $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ is normal with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$.

4 Chapter 4

Theorem 1. *If $dx = a(x,t)dt + b(x,t)dW$, where dW is a Wiener process/Brownian motion then for $G(x,t)$, we have*

$$
dG = \left(a \frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 G}{\partial x^2} \right) dt + b \frac{\partial G}{\partial x} dW \tag{4.1}
$$

4.1 Exercise 4.1

For any $0 \le s \le t \le T$, without loss of generality, we assume $s = t_l$ and $t = t_k$ [1]. Now

$$
\begin{aligned}
&E[I(t)|\mathcal{F}(s)] \\
&= E\left[\sum_{j=0}^{l-1} \Delta t_j [M(t_{j+1} - M(t_j)) + \sum_{j=k-1}^{l} \Delta t_j [M(t_{j+1} - M(t_j))] \Big| \mathcal{F}(t_l) \right] \\
&= \sum_{j=0}^{l-1} E\left[\Delta t_j [M(t_{j+1} - M(t_j)) \Big| \mathcal{F}(t_l) \right] + \sum_{j=l}^{k-1} E\left[\Delta t_j [M(t_{j+1} - M(t_j)) \Big| \mathcal{F}(t_l) \right]
\end{aligned}
$$

[1] since we can always re-arrange the indices to have a new partition over 0 to T

$$
\begin{aligned}
&= \sum_{j=0}^{l-1} [\Delta t_j [M(t_{j+1} - M(t_j))] + \sum_{j=l}^{k-1} E\left[E\left[\Delta t_j [M(t_{j+1} - M(t_j)) \middle| \mathcal{F}(t_j) \right] \middle| \mathcal{F}(t_l) \right] \\
&= I(s) + \sum_{j=l}^{k-1} E\left[E\left[\Delta t_j [M(t_{j+1} - M(t_j)) \middle| \mathcal{F}(t_j) \right] \middle| \mathcal{F}(t_l) \right] \\
&= I(s) + \sum_{j=l}^{k-1} E\left[[\Delta t_j [M(t_j - M(t_j))] \middle| \mathcal{F}(t_l) \right] \\
&= I(s) + \sum_{j=l}^{k-1} E\left[0 \middle| \mathcal{F}(t_l) \right] \\
&= I(s) + 0 \\
&= I(s)
\end{aligned}
$$

so $I(t)$ is a martingale for $0 \leq t \leq T$.

4.2 Exercise 4.2

(i) Without loss of generality, we assume $t = t_k$ and $s = t_l$ where $t_l < t_k$. In other words, to show that $I(t) - I(s)$ in independent of $\mathcal{F}(s)$ if $0 \leq s < t \leq T$, it is sufficient to show that $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$ for $0 \leq t_l < t_k \leq T$. Hence

$$
\begin{aligned}
I(t) - I(s) &= \sum_{j=0}^{k} \Delta(t_j)[W(t_{j+1}) - W(t_j)] - \sum_{j=0}^{l} \Delta(t_j)[W(t_{j+1}) - W(t_j)] \\
&= \sum_{j=l+1}^{k} \Delta(t_j)[W(t_{j+1}) - W(t_j)]
\end{aligned}
$$

Now since $\Delta(t)$ is a nonrandom simple process, each $\Delta(t_j)$ will be a constant; also since $W(t)$ is a brownian motion, each $W(t_{j+1}) - W(t_j)$ will be independent of $\mathcal{F}(t_j)$ for $j > l$. Futhermore, since $\mathcal{F}(t_l) \subset \mathcal{F}(t_j)$ for all $j < l$, each $W(t_{j+1}) - W(t_j)$ will be independent of $\mathcal{F}(t_l)$, which implies each $\Delta(t_j)[W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_l)$. BY taking the summation, we have $I(t) - I(s) = \sum_{j=l+1}^{k} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is independent of $\mathcal{F}(t_l)$.

(ii) Again, we consider

$$
I(t) - I(s) = \sum_{j=l+1}^{k} \Delta(t_j)[W(t_{j+1}) - W(t_j)]
$$

Now since each $W(t_{j+1}) - W(t_j)$ is a normal distribution random variable with mean 0 and variance $t_{j+1} - t_j$, $\Delta(t_j)[W(t_{j+1}) - W(t_j)]$ will have a normal distribution with mean 0 and variance $\Delta^2(t_j)(t_{j+1} - t_j)$. Since the summation of any normal random variables is still a normal random variable, then

$$
I(t) - I(s) \quad \sim \quad N\left(0, \sum_{j=l+1}^{k} \Delta^2(t_j)(t_{j+1} - t_j)\right)
$$

As $\Delta(t)$ is a nonrandom simple process, $\sim N\left(0, \sum_{j=l+1}^{k} \Delta^2(t_j)(t_{j+1} - t_j)\right)$ is exactly the definition of the Riemann sum $\int_{t_l}^{t_k} \Delta^2(u)du$. Hence

$$
I(t) - I(s) \quad \sim \quad N\left(0, \int_s^t \Delta^2(u)du\right)
$$

(iii) Since

$$
\begin{aligned}
E[I(t)|\mathcal{F}(s)] =& E[I(t)-I(s)+I(s)|\mathcal{F}(s)]\\
=& E[I(t)-I(s)|\mathcal{F}(s)]+E[I(s)|\mathcal{F}(s)]\\
=& E[I(t)-I(s)]+I(s) \qquad \text{since } I(t)-I(s) \text{ is independent of } \mathcal{F}(s)\\
=& 0+I(s) \qquad \text{since } I(t)-I(s) \text{ is normally distributed with mean 0}\\
=& I(s)
\end{aligned}
$$

I(t) is a martingale.

(iv) First we write

$$
\begin{aligned}
& I^2(t)-\int_0^t \Delta^2(u)du\\
=& (I(t)-I(s))^2+2I(t)I(s)-I^2(s)-\int_0^t \Delta^2(u)du\\
=& (I(t)-I(s))^2+2(I(t)-I(s))I(s)+2I^2(s)-I^2(s)-\int_0^t \Delta^2(u)du
\end{aligned}
$$

then for $0 \le s \le t \le T$,

$$
\begin{aligned}
& E\left[I^2(t)-\int_0^t \Delta^2(u)du|\mathcal{F}(s)\right]\\
=& E\left[(I(t)-I(s))^2+2(I(t)-I(s))I(s)+I^2(s)-\int_0^t \Delta^2(u)du|\mathcal{F}(s)\right]\\
=& E\left[(I(t)-I(s))^2+2(I(t)-I(s))I(s)|\mathcal{F}(s)\right]+I^2(s)-\int_0^t \Delta^2(u)du\\
=& E\left[(I(t)-I(s))^2\right]+2E\left[2(I(t)-I(s))|\mathcal{F}(s)\right]I(s)+I^2(s)-\int_0^t \Delta^2(u)du\\
=& \int_s^t \Delta^2(u)du+2\cdot 0\cdot I(s)+I^2(s)-\int_0^t \Delta^2(u)du\\
=& I^2(s)-\int_0^s \Delta^2(u)du
\end{aligned}
$$

which implies that $I^2(t)-\int_0^t \Delta^2(u)du$ is a martingale.

4.3 Exercise 4.5

1. In Theorem 1, $G(x,t)=\log x$, then

$$
\begin{aligned}
d\log S(t) &= \left[\alpha(t)S(t)\frac{1}{S(t)}+0+\frac{1}{2}\sigma^2 S^2(t)\cdot\left(-\frac{1}{S^2(t)}\right)\right]dt+\sigma S(t)\frac{1}{S(t)}dW(t)\\
&= \left(\alpha(t)-\frac{1}{2}\sigma^2\right)dt+\sigma(t)dW(t)
\end{aligned}
$$

2.

$$
\log S(t)-\log S(0)=\int_0^t\left(\alpha(s)-\frac{1}{2}\sigma^2\right)ds+\int_0^t\sigma(s)dW(s)
$$

so

$$
S(t)=S(0)\exp\left[\int_0^t\left(\alpha(s)-\frac{1}{2}\sigma^2\right)ds+\int_0^t\sigma(s)dW(s)\right]
$$

4.4 Exercise 4.6

Define Ito's process

$$X(t) = \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma(s)dW(s)$$

Then

$$\begin{aligned} dX(t) &= (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW(t) \\ dX(t)dX(t) &= \sigma^2 dt \end{aligned}$$

Now we use two different notations to compute $d(S^p(t))$:

1. Under the notations in Theorem 1, $G(x,t) = S^p(0)e^{px}$, $a = \alpha - \frac{1}{2}\sigma^2$ and $b = \sigma$. Then

$$\begin{aligned} d(S^p(t)) &= \left[(\alpha - \frac{1}{2}\sigma^2)\cdot pS^p(0)e^{pX(t)} + 0 + \frac{1}{2}\sigma^2 p^2 \cdot S^p(0)e^{pX(t)}\right]dt + \sigma \cdot S^p(0)pe^{pX(t)}dW(t) \\ &= \left(p\alpha - \frac{1}{2}p\sigma^2 + \frac{1}{2}p^2\sigma^2\right)S^p(t)dt + \sigma p S^p(t)dW(t) \end{aligned}$$

2. Under the notations in Theorem 4.4.6 of Text, $S^p(t) = f(X(t))$, then $f(x) = S^p(0)e^{px}$, $f'(x) = pS^p(0)e^{px}$ and $f''(x) = p^2 S^p(0)e^{px}$. Now

$$\begin{aligned} d(S^p(t)) &= df(X(t)) \\ &= pS^p(0)e^{pX(t)}dX(t) + \frac{1}{2}p^2 S^p(0)e^{pX(t)}dX(t)dX(t) \\ &= pS^p(t)dX(t) + \frac{1}{2}p^2 S^p(t)dX(t)dX(t) \\ &= pS^p(t)\cdot\left[(\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW(t)\right] + \frac{1}{2}p^2 S^p(t)\cdot\sigma^2 dt \\ &= \left(p\alpha - \frac{1}{2}p\sigma^2 + \frac{1}{2}p^2\sigma^2\right)S^p(t)dt + \sigma p S^p(t)dW(t) \end{aligned}$$

4.5 Exercise 4.7

Here we use the following version (4.4.1)of Ito's lemma:

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt \tag{4.2}$$

1. Set $f(x) = x^4$ in (4.2), then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Then (4.2) becomes

$$dW^4(t) = 4W^3(t)dW(t) + \frac{1}{2}\cdot 12W^2(t)dt = 4W^3(t)dW(t) + 6W^2(t)dt$$

We may also write it in the integral form as:

$$W^4(T) = W^4(0) + 4\int_0^T W^3(t)dW(t) + 6\int_0^T W^2(t)dt = 4\int_0^T W^3(t)dW(t) + 6\int_0^T W^2(t)dt \tag{4.3}$$

2. First we note that $W(t) \sim N(0,t)$, then by the symmetry of the integrand

$$E[W^3(t)] = \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 0$$

Now we take the expectation in (4.3),

$$\begin{aligned}E\left[W^4(T)\right] =&E\left[4\int_0^T W^3(t)dW(t)+6\int_0^T W^2(t)dt\right]\\=&\int_0^T 4E\left[W^3(t)\right]dW(t)+6\int_0^T E\left[W^2(t)\right]dt\\=&\int_0^T 40dW(t)+6\int_0^T t^2dt\\=&3T^2\end{aligned}$$

3. Set $f(x)=x^6$ in (4.2), then $f'(x)=6x^5$ and $f''(x)=30x^4$. Then (4.2) becomes

$$dW^6(t)=6W^5(t)dW(t)+\frac{1}{2}\cdot 30W^4(t)dt=6W^5(t)dW(t)+15W^4(t)dt$$

We may also write it in the integral form as:

$$W^6(T)=W^6(0)+6\int_0^T W^5(t)dW(t)+15\int_0^T W^4(t)dt=6\int_0^T W^5(t)dW(t)+15\int_0^T W^4(t)dt$$

It is easy to see that $E[W^5(t)]=0$. Then by taking expectation in the above equation, we have

$$\begin{aligned}E\left[W^6(T)\right] =&E\left[6\int_0^T W^5(t)dW(t)+15\int_0^T W^4(t)dt\right]\\=&\int_0^T 6E\left[W^5(t)\right]dW(t)+15\int_0^T E\left[W^4(t)\right]dt\\=&6\int_0^T 0dW(t)+15\int_0^T 3t^2dt\\=&15T^3\end{aligned}$$

4.6 Exercise 4.8

1. In (4.4.23) of text, let $f(t,x)=e^{\beta t}x$, then $f_t=e^{\beta t}\beta x$, $f_x=e^{\beta t}$ and $f_{xx}=0$. Then (4.4.23) becomes

$$\begin{aligned}d\left(e^{\beta t}R(t)\right) =&e^{\beta t}\beta R(t)dt+e^{\beta t}dX(t)\\=&e^{\beta t}\beta R(t)dt+e^{\beta t}\left[(\alpha-\beta R(t))dt+\sigma dW(t)\right]\\=&\alpha e^{\beta t}dt+\sigma e^{\beta t}dW(t)\end{aligned}$$

2. Integrating the both sides of the above equation, we have

$$\begin{aligned}e^{\beta\cdot t}R(t) =&e^{\beta\cdot 0}R(0)+\int_0^t \alpha e^{\beta s}ds+\int_0^t \sigma e^{\beta s}dW(s)\\=&R(0)+\frac{\alpha}{\beta}\left(e^{\beta t}-1\right)+\sigma\int_0^t e^{\beta s}dW(s)\end{aligned}$$

which implies

$$R(t)=e^{-\beta t}R(0)+\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)+\sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s)$$

4.7 Exercise 4.9

1. It is clear that

$$N'(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

and

$$d_\pm = d_\pm(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + (r \pm \frac{\sigma^2}{2})(T-t)\right]$$

Then

$$\begin{aligned}
\frac{N'(d_+)}{N'(d_-)} &= \exp\left[-\frac{d_+^2}{2} + \frac{d_-^2}{2}\right] \\
&= \exp\frac{1}{2\sigma^2(T-t)}\left[\left(\log\frac{x}{K} + (r - \frac{\sigma^2}{2})(T-t)\right)^2 - \left(\log\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)\right)^2\right] \\
&= \exp\frac{1}{2\sigma^2(T-t)}\left[-4\cdot\left(\log\frac{x}{K} + r(T-t)\right)\cdot\frac{\sigma^2}{2}(T-t)\right] \\
&= \exp\left[-\log\frac{x}{K} + r(T-t)\right] \\
&= \frac{K}{x}e^{-r(T-t)}
\end{aligned}$$

2. First

$$\frac{\partial d_\pm}{\partial x} = \frac{1}{\sigma\sqrt{T-t}x}$$

By the definition

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

$$\begin{aligned}
c_x &= \frac{\partial c}{\partial x} \\
&= N(d_+(T-t,x)) + xN'(d_+)\frac{\partial d_+}{\partial x} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial x} \\
&= N(d_+) + \left(xN'(d_+) - Ke^{-r(T-t)}N'(d_-)\right)\frac{1}{\sigma\sqrt{T-t}x} \\
&= N(d_+)
\end{aligned}$$

3.

$$\begin{aligned}
c_t &= \frac{\partial c}{\partial t} \\
&= xN'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}rN(d_-) - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial t} \\
&= -Ke^{-r(T-t)}rN(d_-) + N'(d_+)\left[x\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}\frac{\partial d_+}{\partial t}\partial xKe^{-r(T-t)}\right] \\
&= -Ke^{-r(T-t)}rN(d_-) + xN'(d_+)\left(\frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t}\right)
\end{aligned}$$

Now note that

$$d_+ - d_- = \sigma\sqrt{T-t}$$

$$\frac{\partial(d_+ - d_-)}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}$$

so

$$c_t = -Ke^{-r(T-t)}rN(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+)$$

4.

$$\begin{aligned}
&c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx}\\
=&-rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2 x^2 \frac{d}{dx}[N(d_+(x))]\\
=&-rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2 x^2 N'(d_+)\frac{\partial d_+}{\partial x}\\
=&-rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2 x^2 N'(d_+)\cdot\frac{1}{\sigma\sqrt{T-t}x}\\
=&rxN(d_+) - rKe^{-r(T-t)}N(d_-)\\
=&rc
\end{aligned}$$

5. As $t \to T-$, $\sqrt{T-t} \to 0$. And $\ln \frac{x}{K} > 0$ if $x > K$, and $\ln \frac{x}{K} < 0$ if $0 < x < K$. Therefore

$$\begin{aligned}
\lim_{t\to T-} d_\pm &= \lim_{t\to T-} \frac{1}{\sqrt{T-t}}\left[\log\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2(T-t))\right]\\
&= \lim_{t\to T-} \frac{1}{\sqrt{T-t}}\left[\log\frac{x}{K} + r\right] + \lim_{t\to T-}(r \pm \frac{1}{2}\sigma^2\sqrt{T-t})\\
&= \begin{cases} \infty, & \text{if } x > K\\ -\infty, & \text{if } 0 < x < k\end{cases}
\end{aligned}$$

Then

$$\lim_{t\to T-} N(d_\pm) \begin{cases} N(\infty) = 1, & \text{if } x > K\\ N(-\infty) = 0, & \text{if } 0 < x < k\end{cases}$$

So

$$\begin{aligned}
\lim_{t\to T-} c(t,x) =& \begin{cases} x\cdot 1 - Ke^{-r(T-T)}\cdot 1 = x - K, & \text{if } x > K\\ x\cdot 0 - Ke^{-r(T-T)}\cdot 0 = 0, & \text{if } 0 < x < K\end{cases}\\
=&(x-K)^+
\end{aligned}$$

6. For $0 \le t < T$, $\ln \frac{x}{K} \to -\infty$ as $x \to 0+$.

$$\begin{aligned}
\lim_{x\to 0+} d_\pm =& \lim_{x\to 0+} \frac{1}{\sqrt{T-t}}\left[\log\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2(T-t))\right]\\
=&-\infty
\end{aligned}$$

which implies $N(d_\pm) \to N(-\infty) = 0$ as $x \to 0+$. Therefore $c(t,x) \to x\cdot 0 - Ke^{-r(T-t)}\cdot 0 = 0$ as $x \to 0+$.

7. For $0 \le t < T$, $\ln \frac{x}{K} \to \infty$ as $x \to \infty$.

$$\begin{aligned}
\lim_{x\to \infty} d_\pm =& \lim_{x\to \infty} \frac{1}{\sqrt{T-t}}\left[\log\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2(T-t))\right]\\
=&\infty
\end{aligned}$$

which implies $N(d_\pm) \to N(\infty) = 1$ as $x \to \infty$.

$$\begin{aligned}
&\lim_{x\to\infty} \left[c(t,x) - \left(x - e^{-r(T-t)}K\right)\right] \\
=&\lim_{x\to\infty} \left[xN(d_+) - Ke^{-r(T-t)} - \left(x - e^{-r(T-t)}K\right)\right] \\
=&\lim_{x\to\infty} x(N(d_+) - 1) \\
=&\lim_{x\to\infty} \frac{N(d_+) - 1}{x^{-1}} \\
=&\lim_{x\to\infty} \frac{\frac{d}{dx}[N(d_+) - 1]}{-x^{-2}} \\
=&\lim_{x\to\infty} \frac{N'(d_+) \cdot \frac{x}{\sigma\sqrt{T-t}}}{-x^{-2}} \\
=&\frac{-1}{\sigma\sqrt{T-t}} \lim_{x\to\infty} x^3 N'(d_+) \\
=&\frac{-1}{\sigma\sqrt{T-t}} \lim_{x\to\infty} x^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-d_+^2/2} \\
=&\frac{-1}{\sigma\sqrt{2\pi}\sqrt{T-t}} \lim_{x\to\infty} K^3 \exp\left[3\sigma\sqrt{T-t}d_+ - 3(T-t)(r + \frac{\sigma^2}{2})\right] \cdot e^{-d_+^2/2} \\
=&\frac{-1}{\sigma\sqrt{2\pi}\sqrt{T-t}} \exp\left[-3(T-t)(r + \frac{\sigma^2}{2})\right] \cdot \lim_{x\to\infty} \exp\left[-\frac{d_+^2}{2} + 3\sigma\sqrt{T-t}d_+\right] \\
=&\frac{-1}{\sigma\sqrt{2\pi}\sqrt{T-t}} \exp\left[-3(T-t)(r + \frac{\sigma^2}{2})\right] \cdot 0 \\
=&0
\end{aligned}$$

4.8 Exercise 4.11

Without confusion, we shorten the notations when it is proper through this problem:

$$S = S(t), \qquad c = c(t, S(t)), \qquad c_x = c_x(t, S(t)), \qquad c_t = c_t(t, S(t)), \qquad c_{xx} = c_{xx}(t, S(t))$$

By

$$dS(t) = \alpha S dt + \sigma_2 S dW(t)$$

we have

$$dS(t)dS(t) = \alpha^2 S^2 dtdt + 2\alpha S\sigma_2 S dtdW(t) + \sigma_2^2 S dW(t)dW(t) = \sigma_2^2 S dt$$

and

$$dc(t, S(t)) = c_t dt + c_x dS + \frac{1}{2} c_{xx} dSdS$$

Now we compute

$$\begin{aligned}
d(e^{-rt}X(t)) &= -re^{-rt}X(t)dt + e^{-rt}dX(t) \\
&= -re^{-rt}X(t)dt + e^{-rt} \cdot \left[dc - c_x dS + r(X - c + Sc_x)dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx} dt\right] \\
&= e^{-rt} \left[\left(c_t dt + c_x dS + \frac{1}{2} c_{xx} dSdS\right) - c_x dS + r(X - c + Sc_x)dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx} dt\right] \\
&= e^{-rt} \left(c_t + \frac{1}{2}\sigma_1^2 S^2 c_{xx} - rc + rSc_x - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}\right) dt + e^{-rt} \left(c_x - c_x\right) dS \\
&= e^{-rt}(c_t - rc + rSc_x + \frac{1}{2}\sigma_1^2 S^2 c_{xx})dt \\
&= 0
\end{aligned}$$

which implies $e^{-rt}X(t)$ is a non-random constant and $e^{-rt}X(t) = e^{-r\cdot 0}X(0) = X(0) = 0$ for any t. Since $r > 0$, so $X(t) = 0$ for any t.

4.9 Exercise 4.13

Since $B_1(t)$ and $B_2(t)$ are Brownian motions, $dB_1(t)dB_1(t) = dt$ and $dB_2(t)dB_2(t) = dt$. By substituting the definition

$$\begin{aligned} dB_1(t) =& dW_1(t) \\ dB_2(t) =& \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t) \end{aligned}$$

Note that these equations imply that $W_1(t)$ is Brownian motion and $W_2(t)$ is a continuous martingale with $W_2(0) = 0$ (since we can write $dW_2(t) = dB_2(t)/\sqrt{1-\rho^2(t)} - \rho(t)dW_1(t)/\sqrt{1-\rho^2(t)}$ with $-1 < \rho < 1$). Furthermore we have

$$\begin{cases} dW_1(t)dW_1(t) = dt \\ (\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)) \cdot (\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)) = dt \end{cases} \tag{4.4}$$

Also by $dB_1(t)dB_2(t) = \rho(t)dt$, we have

$$\begin{aligned} dW_1(t)(\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)) = \rho(t)dt \\ \rho(t)dt + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t) = \rho(t)dt \end{aligned}$$

which implies

$$dW_1(t)dW_2(t) = 0 \tag{4.5}$$

Then substitute (4.5) into the second equation of (4.4),

$$\begin{aligned} \rho^2dt + (1-\rho^2)dW_2(t)dW_2(t) + \rho\sqrt{1-\rho^2}dW_1(t)dW_2(t) = dt \\ \rho^2dt + (1-\rho^2)dW_2(t)dW_2(t) + 0 = dt \end{aligned}$$

$$dW_2(t)dW_2(t) = dt \tag{4.6}$$

Based on the fact that $W_1(t)$ and $W_2(t)$ are continuous martingales with $W_1(0) = 0 = W_2(0)$, and (4.5)(4.4)(4.6), by applying Theorem 4.6.5, we conclude that $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

4.10 Exercise 4.15

1. Note that

$$dW_i(t)dW_j(t) = \begin{cases} dt & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{4.7}$$

and

$$dB_i(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t)$$

which implies $B_i(t)$ is a continuous martingale with $B_i(0) = 0$ for any $1 \leq i \leq d$.

Then we compute the quadratic variation, or equivalently,

$$\begin{aligned}
dB_i(t)dB_i(t) = & \left[\sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dW_j(t)\right]\cdot\sum_{k=1}^{d}\frac{\sigma_{ik}(t)}{\sigma_i(t)}dW_k(t) \\
= & \frac{1}{\sigma_i^2(t)}\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\left[\sum_{k=1}^{d}\sigma_{ik}(t)dW_k(t)\right] \\
= & \frac{1}{\sigma_i^2(t)}\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\sigma_{ij}(t)dW_j(t) \qquad \text{by (4.7)} \\
= & \frac{1}{\sigma_i^2(t)}\sum_{j=1}^{d}\sigma_{ij}^2(t)dt \\
= & 1
\end{aligned}$$

2.

$$\begin{aligned}
dB_i(t)dB_k(t) = & \left[\sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dW_j(t)\right]\cdot\sum_{l=1}^{d}\frac{\sigma_{kl}(t)}{\sigma_k(t)}dW_l(t) \\
= & \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\left[\sum_{l=1}^{d}\sigma_{kl}(t)dW_l(t)\right] \\
= & \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\sigma_{kj}(t)dW_j(t) \qquad \text{by (4.7)} \\
= & \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)\sigma_{kj}(t)dt \\
= & \rho_{ik}(t)
\end{aligned}$$

4.11 Exercise 4.18

1. In (4.4.13), let

$$f(t,x) = e^{-\theta x-(r+\theta^2/2)t}$$

then

$$\begin{aligned}
f_t(t,x) &= -(r+\theta^2/2)e^{-\theta x-(r+\theta^2/2)t} \\
f_x(t,x) &= -\theta e^{-\theta x-(r+\theta^2/2)t} \\
f_{xx}(t,x) &= \theta^2 e^{-\theta x-(r+\theta^2/2)t}
\end{aligned}$$

Then Ito'e Formula of (4.4.13) becomes

$$\begin{aligned}
d\zeta(t) = & -(r+\theta^2/2)e^{-\theta W(t)-(r+\theta^2/2)t}dt - \theta e^{-\theta W(t)-(r+\theta^2/2)t}dW(t) \\
& + \frac{1}{2}\theta^2 e^{-\theta W(t)-(r+\theta^2/2)t}dW(t)dW(t) \\
= & -(r+\theta^2/2)\zeta(t)dt - \theta\zeta(t)dW(t) + \frac{1}{2}\theta^2\zeta(t)dt \\
= & -\theta\zeta(t)dW(t) - r\zeta(t)dt
\end{aligned}$$

2. By Ito's Product rule,

$$\begin{aligned}
d(\zeta(t)Z(t)) =&\zeta(t)d(Z(t)) + Z(t)d(\zeta(t)) + d(\zeta(t))d(Z(t)) \\
=&\zeta \cdot [rXdt + \Delta(\alpha - r)Sdt + \Delta\sigma SdW(t)] + X \cdot [-\theta\zeta dW(t) - r\zeta dt] \\
&+ [rXdt + \Delta(\alpha - r)Sdt + \Delta\sigma SdW(t)] \cdot [-\theta\zeta dW(t) - r\zeta dt] \\
=&\left[\zeta rX + \zeta\Delta(\alpha - r)S - r\zeta X - \Delta\sigma S\theta\zeta\right] dt \\
&+ \left[\zeta\Delta\sigma S - X\theta\zeta\right] dW(t) \\
=&\Delta\zeta S\left[(\alpha - r) - \sigma\theta\right] dt + \zeta[\Delta\sigma S - X\theta]dW(t) \\
=&\zeta[\Delta\sigma S - X\theta]dW(t) \qquad \text{(since } \theta = (\theta - r)/\sigma)
\end{aligned}$$

i.e.,$d(\zeta(t)Z(t))$ has no dt term. Therefore $\zeta(t)Z(t)$ is a martingale.

3. If an investor begin with initial capital $X(0)$ amd has portfolio value $V(T)$ at time T, which is $X(T) = V(T)$, then

$$\zeta(T)X(T) = \zeta(T)V(T)$$

Since $X(T)$ and $V(T)$ are random, we take the expectation in the above formula

$$E[\zeta(T)X(T)] = E[\zeta(T)V(T)]$$

Now since $\zeta(t)X(t)$ is a martingale,

$$E[\zeta(T)X(T)] = \zeta(0)X(0) = X(0)$$

By equating the above two equations, we have $X(0) = E[\zeta(T)V(T)]$.

4.12 Exercise 4.19

1. By the definition, $dB(t) = sign(t)dW(t)$, $B(t)$ is a martingale with continuous path. Furthermore, since $dB(t)dB(t) = sign^2(t)dW(t)dW(t) = dt$, $B(t)$ is a Brownian motion.

2. By Ito's product formula,

$$\begin{aligned}
d[B(t)W(t)] =&d[B(t)]W(t) + d[W(t)]B(t) + d[B(t)]d[W(t)] \\
=&sign(W(t))W(t)dW(t) + B(t)dW(t) + sign(W(t))dW(t)dW(t) \\
=&[sign(W(t))W(t) + B(t)]dW(t) + sign(W(t))dt
\end{aligned}$$

Integrate both sides, we will have

$$\begin{aligned}
B(t)W(t) =&B(0)W(0) + \int_0^t sign(W(s))ds + \int_0^t [sign(W(s))W(s) + B(s)]dW(s) \\
=&\int_0^t sign(W(s))ds + \int_0^t [sign(W(s))W(s) + B(s)]dW(s)
\end{aligned}$$

Note that the expectation of an Ito's integral is 0 and $W(s) \sim N(0, s)$, which implies

$$E[sign(W(s))] = \int_{-\infty}^{\infty} sign(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2s}dx = \int_0^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2s}dx - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}e^{-x^2/2s}dx = 0$$

Then by taking expectation on both sides,

$$E[B(t)W(t)] = \int_0^t E[sign(W(s))]ds = 0$$

Note that $E[W(t)] = 0$, the above result also implies that $B(t)$ and $W(t)$ are uncorrelated.

3. By Ito's product formula,

$$\begin{aligned}d[W(t)W(t)] =& d[W(t)]W(t) + d[W(t)]W(t) + d[W(t)]d[W(t)]\\ =& W(t)dW(t) + W(t)dW(t) + dW(t)dW(t)\\ =& 2W(t)dW(t) + dt\end{aligned}$$

4. By Ito's product formula,

$$\begin{aligned}d[B(t)W^2(t)] =& d[B(t)]W^2(t) + d[W^2(t)]B(t) + d[B(t)]d[W^2(t)]\\ =& sign(W(t))W^2(t)dW(t) + [2W(t)dW(t) + dt]\,B(t)\\ &+ sign(W(t))dW(t)\,[2W(t)dW(t) + dt]\\ =& [sign(W(t))W^2(t) + 2B(t)W(t)]dW(t) + [B(t) + 2sign(W(t))W(t)]dt\end{aligned}$$

By taking the integral on both sides,

$$B(t)W^2(t) = \int_0^t [sign(W(s))W^2(s) + 2B(s)W(s)]dW(s) + \int_0^t [B(s) + 2sign(W(s))W(s)]ds$$

Note that again the expectation of an Ito's integral is 0 and $W(s) \sim N(0, s)$. Then by taking the expectation, we have

$$\begin{aligned}E[B(t)W^2(t)] =& \int_0^t E\left[[B(s) + 2sign(W(s))W(s)\right] ds\\ =& 2\int_0^t E\left[sign(W(s))W(s)\right] ds\end{aligned}$$

Now

$$\begin{aligned}E[sign(W(s))] =& \int_{-\infty}^{\infty} sign(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2s}dx\\ =& \int_0^{\infty} \frac{x}{\sqrt{2\pi}}e^{-x^2/2s}dx - \int_{-\infty}^{0} \frac{x}{\sqrt{2\pi}}e^{-x^2/2s}dx = 0\\ =& 2\int_0^{\infty} \frac{x}{\sqrt{2\pi}}e^{-x^2/2s}dx\\ >& 0\end{aligned}$$

Thererfore $E[B(t)W^2(t)] > 0 \neq 0 = E[B(t)]E[W^2(t)]$.

If $B(t)$ and $W(t)$ are independent, then $B(t)$ and $W^2(t)$ will also be independent, which implies $E[B(t)W^2(t)] = E[B(t)]E[W^2(t)]$. Contradiction. Therefore $B(t)$ and $W(t)$ are uncorrelated but not independent.

5 Chapter 5

5.1 Exercise 5.1

Recall that $dtdt = 0$, $dtdW = 0$ and $dW(t)dW(t) = dt$.

1. First note that

$$dX(t) = \sigma(t)dW(t) + (\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t))dt$$

For $f(x) = S(0)e^x$, we have $f'(x) = f''(x) = f(x)$ and $f(X(t)) = S(0)e^{X(t)} = D(t)S(t)$, therefore by Ito's Lemma,

$$
\begin{aligned}
d(D(t)S(t)) =& df(X(t)) \\
=& f'(X(t))dX(t) + \frac{1}{2}f'(X(t))dX(t)dX(t) \\
=& S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}dX(t)dX(t) \\
=& D(t)S(t)\left[\sigma(t)dW(t) + (\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t))dt + \frac{1}{2}\left(\sigma^2 dt + 0 + 0\right)\right] \\
=& D(t)S(t)\left[\sigma(t)dW(t) + (\alpha(t) - R(t))dt\right] \\
=& D(t)S(t)\sigma(t)\left[dW(t) + \Theta(t)dt\right]
\end{aligned}
$$

with $\Theta(t) = (\alpha(t) - R(t))/\sigma(t)$.

2. The other way is by Ito's Product Rule,

$$
\begin{aligned}
d(D(t)S(t)) =& S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
=& S(t)\left[-R(t)D(t)dt\right] + D(t)\left[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right] \\
& + \left[-R(t)D(t)dt\right] \cdot \left[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right] \\
=& \left[\alpha(t)S(t)D(t) - S(t)R(t)D(t)\right]dt + D(t)\sigma(t)S(t)dW(t) \\
=& D(t)S(t)\left[\sigma(t)dW(t) + (\alpha(t) - R(t))dt\right] \\
=& D(t)S(t)\sigma(t)\left[dW(t) + \Theta(t)dt\right]
\end{aligned}
$$

5.2 Exercise 5.4

1. Let us consider $d(\ln S(t))$, by Ito's Lemma

$$
\begin{aligned}
d(\ln S(t)) =& \frac{1}{S(t)}S(t) + \frac{1}{2}\cdot\left(-\frac{1}{S^2(t)}\right)dS(t)dS(t) \\
=& r(t)dt + \sigma(t)d\tilde{W}(t) - \frac{\sigma^2}{2}d\tilde{W}(t)d\tilde{W}(t) \\
=& \left(r(t) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)d\tilde{W}(t)
\end{aligned}
$$

By writing it in the integral form, we have

$$
\begin{aligned}
\ln S(T) =& \ln S(0) + \int_0^T \left(r(t) - \frac{\sigma^2(t)}{2}\right)dt + \int_0^T \sigma(t)d\tilde{W}(t) \\
S(T) =& S(0)\exp\left[\int_0^T \left(r(t) - \frac{\sigma^2(t)}{2}\right)dt + \int_0^T \sigma(t)d\tilde{W}(t)\right]
\end{aligned}
$$

Under the form $S(T) = S(0)e^X$,

$$
X = \int_0^T \left(r(t) - \frac{\sigma^2(t)}{2}\right)dt + \int_0^T \sigma(t)d\tilde{W}(t)
$$

By Theorem 4.4.9 of Text, $\int_0^T \sigma(t)d\tilde{W}(t)$ is a normal random variable with mean 0 and variance $\int_0^T \sigma^2(t)dt$, which implies X is normal distributed with mean $\int_0^T \left(r(t) - \frac{\sigma^2(t)}{2}\right)dt$ and variance $\int_0^T \sigma^2(t)dt$.

2. To simplify the notations, denote

$$\begin{aligned}
R =& \frac{1}{T}\int_0^T r(t)dt \\
\Sigma =& \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}
\end{aligned}$$

Then X is normal with mean $RT - T\Sigma^2/2$ and variance $T\Sigma^2$.

$$\begin{aligned}
& c(0, S(0)) \\
=& \tilde{E}\left[e^{-RT}(S(T)-K)^+\right] \\
=& e^{-RT}\int_{x:S(T)\geq K} \frac{1}{\sqrt{2\pi}\Sigma\sqrt{T}}\left(S(0)e^x - K\right)\exp\left[-\left(x - RT + \frac{T\Sigma^2}{2}\right)^2/2T\Sigma^2\right]dx \\
=& S(0)\int_{x:S(T)\geq K} \frac{e^{-RT}}{\sqrt{2\pi}\Sigma\sqrt{T}}e^x\exp\left[-\left(x - RT + \frac{T\Sigma^2}{2}\right)^2/2T\Sigma^2\right]dx \\
& - Ke^{-RT}\int_{x:S(T)\geq K} \frac{1}{\sqrt{2\pi}\Sigma\sqrt{T}}\exp\left[-\left(x - RT + \frac{T\Sigma^2}{2}\right)^2/2T\Sigma^2\right]dx
\end{aligned} \tag{5.1}$$

Now

$$\begin{aligned}
& \int_{x:S(T)\geq K} \frac{1}{\sqrt{2\pi}\Sigma\sqrt{T}}\exp\left[-\left(x - RT + \frac{T\Sigma^2}{2}\right)^2/2T\Sigma^2\right]dx \\
=& P(S(T)\geq K) \\
=& P\left(X \geq \ln\frac{S(0)}{K}\right) \\
=& P\left(\frac{X - (RT - \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}} \geq \frac{\ln\frac{K}{S(0)} - (RT - \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}}\right) \\
=& P\left(\frac{X - (RT - \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}} \leq \frac{\ln\frac{S(0)}{K} + (RT - \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}}\right) \\
=& N\left(\frac{1}{\Sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + (RT - \frac{1}{2}T\Sigma^2)\right]\right)
\end{aligned} \tag{5.2}$$

and

$$
\begin{aligned}
&\int_{x:S(T)\geq K} \frac{e^{-RT}}{\sqrt{2\pi}\Sigma\sqrt{T}} e^x \exp\left[-\left(x - RT + \frac{T\Sigma^2}{2}\right)^2 / 2T\Sigma^2\right] dx \\
=&\int_{x:S(T)\geq K} \frac{1}{\sqrt{2\pi}} \exp\left[-\left(x - RT - \frac{T\Sigma^2}{2}\right)^2 / 2T\Sigma^2\right] dx \\
=&\frac{1}{\sqrt{2\pi}} \int_{\ln\frac{K}{S(0)}}^{\infty} \exp\left[-\left(x - RT - \frac{T\Sigma^2}{2}\right)^2 / 2T\Sigma^2\right] dx \\
=&P\left(\frac{X - (RT + \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}} \geq \frac{\ln\frac{K}{S(0)} - (RT + \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}}\right) \\
=&P\left(\frac{X - (RT + \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}} \leq \frac{\ln\frac{S(0)}{K} + (RT + \frac{1}{2}T\Sigma^2)}{\Sigma\sqrt{T}}\right) \\
=&N\left(\frac{1}{\Sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + (RT + \frac{1}{2}T\Sigma^2)\right]\right)
\end{aligned}
\tag{5.3}
$$

By substituting (5.2) and (5.3) into (5.1), we have

$$
c(0, S(0)) = BSM\left(T, S(0); K, R, \Sigma\right)
$$

5.3 Exercise 5.5

1. By the definition

$$
\frac{1}{Z(t)} = \exp\left[\int_0^t \Theta(u)dW(u) + \frac{1}{2}\int_0^t \Theta^2(u)du\right]
$$

Let

$$
X(t) = \int_0^t \Theta(u)dW(u) + \frac{1}{2}\int_0^t \Theta^2(u)du
$$

then

$$
\begin{aligned}
d\left[\frac{1}{Z(t)}\right] =& \frac{d}{dt}\left[e^{X(t)}\right] \\
=& e^{X(t)}dX(t) + \frac{1}{2}e^{X(t)}dX(t)dX(t) \\
=& \frac{1}{Z(t)}\left[\Theta(t)dW(t) + \frac{1}{2}\Theta^2(t)dt\right] + \frac{1}{2Z(t)}\Theta^2(t)dt \\
=& \frac{1}{Z(t)}\left[\Theta(t)dW(t) + \Theta^2(t)dt\right]
\end{aligned}
$$

2. By Lemma 5.2.2 of text, since $\tilde{M}(t)$ is a martingale, for $0 \leq s \leq t$,

$$
E\left[Z(t)\tilde{M}(t)\middle|\mathcal{F}(s)\right] = Z(s)E\left[\frac{1}{Z(t)}Z(t)\tilde{M}(t)\middle|\mathcal{F}(s)\right] = Z(s)E\left[\tilde{M}(t)\middle|\mathcal{F}(s)\right] = Z(s)\tilde{M}(s)
$$

which means $\tilde{M}(t)Z(t)$ is a martingale.

3. Note that $dM(t) = \Gamma(t)dW(t)$, then

$$\begin{aligned}
&d\left[\tilde{M}(t)\right] \\
=&d\left[M(t)\cdot\frac{1}{Z(t)}\right] \\
=&M(t)d\left[\frac{1}{Z(t)}\right]+\frac{1}{Z(t)}d\left[M(t)\right]+d\left[M(t)\right]d\left[\frac{1}{Z(t)}\right] \\
=&M(t)\cdot\left[\frac{\Theta(t)}{Z(t)}dW(t)+\frac{\Theta^2(t)}{Z(t)}dt\right]+\frac{1}{Z(t)}\cdot\Gamma(t)dW(t)+\left[\frac{\Theta(t)}{Z(t)}dW(t)+\frac{\Theta^2(t)}{Z(t)}dt\right]\cdot\Gamma(t)dW(t) \\
=&\frac{1}{Z(t)}\left[\Gamma(t)+M(t)\Theta(t)\right]dW(t)+\frac{1}{Z(t)}\left[\Gamma(t)\Theta(t)+M(t)\Theta^2(t)M(t)\right]dt
\end{aligned}$$

4. Now since $d\tilde{W}(t) = dW(t) + \theta(t)dt$, if we define

$$\tilde{\Gamma}(t)=\frac{\Gamma(t)+M(t)\Theta(t)}{Z(t)}$$

then

$$\begin{aligned}
&d\left[\tilde{M}(t)\right] \\
=&\frac{1}{Z(t)}\left[\Gamma(t)+M(t)\Theta(t)\right]dW(t)+\frac{\Theta(t)}{Z(t)}\left[\Gamma(t)+M(t)\Theta(t)M(t)\right]dt \\
=&\tilde{\Gamma}(t)dW(t)+\tilde{\Gamma}(t)\Theta(t)dt \\
=&\tilde{\Gamma}(t)d\tilde{W}(t)
\end{aligned}$$

Integrate both sides, we will have

$$\tilde{M}(t)=\tilde{M}(0)+\int_0^t\tilde{\Gamma}(u)d\tilde{W}(u)$$

5.4 Exercise 5.12

1. First each $\tilde{B}_i(t)$ is a continuous martingale. Secondly, since $dB_i(t) = \sum_j \sigma_{ij}(t)/\sigma_i(t)dW_t(t)$, $dB_i(t)dt = 0$ and furthermore we have

$$d\tilde{B}_i(t)d\tilde{B}_i(t)=\left[dB_i(t)+\gamma_i(t)\right]\cdot\left[dB_i(t)+\gamma_i(t)\right]=dB_i(t)dB_i(t)=dt$$

$\tilde{B}_i(t)$ must be a brownian motion for every i.

2. By

$$d\tilde{B}_i(t)=dB_i(t)+\gamma_i(t)$$

we have

$$\begin{aligned}
dS_i(t)=&\alpha_i(t)S_i(t)dt+\sigma_i(t)S_i(t)dB_i(t) \\
=&\alpha_i(t)S_i(t)dt+\sigma_i(t)S_i(t)[d\tilde{B}_i(t)-\gamma_i(t)] \\
=&\left[\alpha_i(t)S_i(t)-\sigma_i(t)\gamma_i(t)S_i(t)\right]dt+\sigma_i(t)S_i(t)d\tilde{B}_i(t) \\
=&R(t)S_i(t)dt+\sigma_i(t)S_i(t)d\tilde{B}_i(t)
\end{aligned}$$

3. Since $dB_i(t)dB_k(t) = \rho_{ik}(t)dt$,

$$\begin{aligned}
&d\tilde{B}_i(t)d\tilde{B}_k(t)\\
=&[dB_i(t) + \gamma_i(t)dt] \cdot [dB_k(t) + \gamma_k(t)dt]\\
=&dB_i(t)dB_k(t) + \gamma_i(t)dtdB_k(t) + \gamma_k(t)dtdB_i(t) + \gamma_i(t)\gamma_k(t)dtdt\\
=&dB_i(t)dB_k(t)\\
=&\rho_{ik}(t)dt
\end{aligned}$$

4. By Ito's product rule,

$$d[B_i(t)B_k(t)] = B_i(t)dB_k(t) + B_k(t)dB_i(t) + dB_i(t)dB_k(t)$$

then since $dB_i(t)dB_k(t) = \rho_{ik}(t)dt$, we have

$$B_i(t)B_k(t) = \int_0^t B_i(u)dB_k(u) + \int_0^t B_k(u)dB_i(u) + \int_0^t \rho_{ik}(u)du$$

Now note the fact that the expectation of an Ito's integral is 0, if we take the expectation on the above formula, then

$$E[B_i(t)B_k(t)] = E\left[\int_0^t \rho_{ik}(u)du\right] = \int_0^t \rho_{ik}(u)du$$

Similarly we have $E[\tilde{B}_i(t)\tilde{B}_k(t)] = \int_0^t \rho_{ik}(u)du$.

5. For $E[B_1(t)B_2(t)]$, since $dB_1(t) = dW_2(t)$ and $dB_2(t) = sign(W_1(t))dW_2(t)$ by Ito's product rule,

$$\begin{aligned}
d[B_1(t)B_2(t)] =&B_1(t)dB_2(t) + B_2(t)dB_1(t) + dB_1(t)dB_2(t)\\
=&W_2(t)dB_2(t) + B_2(t)dW_2(t) + sign(W_1(t))dW_2(t)dW_2(t)
\end{aligned}$$

By $dW_2(t)dW_2(t) = dt$, we may write it in the integral version as:

$$B_1(t)B_2(t) = B_1(0)B_2(0) + \int_0^t W_2(u)dB_2(u) + \int_0^t B_2(u)dW_2(u) + \int_0^t sign(W_1(u))du$$

Note that the expectation of an Ito's integral is 0 and $W_1(u) \sim N(0, u)$, which implies

$$E[sign(W_1(u))] = \int_{-\infty}^{\infty} sign(x)\frac{1}{\sqrt{2\pi u}}e^{-x^2/2u}dx = \int_0^{\infty} \frac{1}{\sqrt{2\pi u}}e^{-x^2/2u}dx - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}e^{-x^2/2u}dx = 0$$

Then we have

$$E[B_1(t)B_2(t)] = 0 \tag{5.4}$$

For $E[\tilde{B}_1(t)\tilde{B}_2(t)]$, since

$$d\tilde{B}_1(t) = dB_1(t) = dW_2(t) = d\tilde{W}_2(t) \tag{5.5}$$

and

$$d\tilde{B}_2(t) = dB_2(t) = sign(W_1(t))dW_2(t) = sign(\tilde{W}_1(t) - t)d\tilde{W}_2(t) \tag{5.6}$$

by Ito's product rule,

$$\begin{aligned}
d[\tilde{B}_1(t)\tilde{B}_2(t)] =&\tilde{B}_1(t)d\tilde{B}_2(t) + \tilde{B}_2(t)d\tilde{B}_1(t) + d\tilde{B}_1(t)d\tilde{B}_2(t)\\
=&W_2(t)sign(\tilde{W}_1(t) - t)d\tilde{W}_2(t) + B_2(t)d\tilde{W}_2(t) + sign(W_1(t))dt\\
=&W_2(t)sign(\tilde{W}_1(t) - t)d\tilde{W}_2(t) + B_2(t)d\tilde{W}_2(t) + sign(\tilde{W}_1(t) - t)dt
\end{aligned}$$

we may write it in the integral version as:

$$\tilde{B}_1(t)\tilde{B}_2(t) = \tilde{B}_1(0)\tilde{B}_2(0) + \int_0^t [W_2(u)sign(\tilde{W}_1(u) - u) + B_2(u)]d\tilde{W}_2(u) + \int_0^t sign(\tilde{W}_1(u) - u)du$$

Note that the expectation of an Ito's integral is 0 and $\tilde{W}_1(u) \sim N(0, u)$, which implies $\tilde{W}_1(u) - u \sim N(-u, u)$ and

$$\begin{aligned}
E[sign(W_1(u) - u)] &= \int_{-\infty}^{\infty} sign(x)\frac{1}{\sqrt{2\pi u}}e^{-(x-u)^2/2u}dx \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi u}}e^{-(x-u)^2/2u}dx - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi u}}e^{-(x-u)^2/2u}dx \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi u}}e^{-(x-u)^2/2u}dx - \int_0^{\infty} \frac{1}{\sqrt{2\pi u}}e^{-(x+u)^2/2u}dx \\
&= \frac{1}{\sqrt{2\pi u}} \int_0^{\infty} \left[e^{-(x-u)^2/2u} - e^{-(x+u)^2/2u}\right] du \\
&> 0 \qquad\qquad \text{since } u > 0
\end{aligned}$$

Then we have

$$E[\tilde{B}_1(t)\tilde{B}_2(t)] = \int_0^t E[sign(\tilde{W}_1(u) - u)du] > 0 \tag{5.7}$$

(5.4) and (5.7) are the exmaples that $E[\tilde{B}_1(t)\tilde{B}_2(t)] > 0 \neq 0 = E[B_1(t)B_2(t)]$.

5.5 Exercise 5.13

1.

$$\begin{aligned}
\tilde{E}[W_1(t)] =& \tilde{E}[\tilde{W_1}(t)] = 0 \\
\tilde{E}[W_2(t)] =& \tilde{E}[\tilde{W}_2(t) - \int_0^t W_1(u)du] = \tilde{E}[\tilde{W}_2(t)] - \int_0^t \tilde{E}[W_1(u)]du = 0
\end{aligned}$$

2. Note that the covariance

$$\begin{aligned}
\tilde{Cov}[W_1(T), W_2(T)] =& \tilde{E}[W_1(T)W_2(T)] - \tilde{E}[W_1(T)]\tilde{E}[W_2(T)] \\
=& \tilde{E}[W_1(T)W_2(T)]
\end{aligned}$$

because of $\tilde{E}[W_1(T)] = 0 = \tilde{E}[W_2(T)]$.
Now by Ito's product rule, we have

$$W_1(T)W_2(T) = W_1(0)W_2(0) + \int_0^T W_1(t)dW_2(t) + \int_0^T W_2(t)dW_1(t)$$

Here since

$$\begin{aligned}
d\tilde{W}_1(t) =& dW_1(t) \\
d\tilde{W_2}(t) =& dW_2(t) + W_1(t)dt
\end{aligned}$$

therefore,

$$\begin{aligned}
dW_1(t) =& d\tilde{W}_1(t) \\
dW_2(t) =& d\tilde{W}_2 - \tilde{W}_1(t)dt
\end{aligned}$$

then

$$W_1(T)W_2(T) = \tilde{W_1}(0)\tilde{W_2}(0) + \int_0^T \tilde{W_1}(t)d\tilde{W_2}(t) - \int_0^T \tilde{W_1}(t)\tilde{W_1}(t)dt + \int_0^T \tilde{W_2}(t)d\tilde{W_1}(t)$$

Now since the expectation of any Ito's integration is 0, and $W_1(0) = 0 = W_2(0)$,

$$\tilde{E}[W_1(T)W_2(T)] = -\int_0^T \tilde{E}[\tilde{W_1}^2(t)]dt = -\int_0^T tdt = -\frac{T^2}{2}$$

here we used the fact that $\tilde{E}[\tilde{W_1}^2(t)] = Cov[\tilde{W_1}] = t$.

5.6 Exercise 5.14

1. Since

$$\begin{aligned}
&d\left(e^{-rt}X(t)\right)\\
=&-re^{-rt}X(t)dt + e^{-rt}dX(t)\\
=&-re^{-rt}X(t)dt + e^{-rt}\Delta(t)dS(t) - e^{-rt}a\Delta(t)dt + re^{-rt}\left(X(t) - \Delta(t)S(t)\right)dt\\
=&e^{-rt}\left[\Delta(t)\left(rS(t)dt + \sigma S(t)d\tilde{W}(t) + adt\right) - a\Delta(t)dt - r\Delta(t)S(t)dt\right]\\
=&\sigma S(t)e^{-rt}\Delta(t)d\tilde{W}(t)
\end{aligned} \tag{5.8}$$

contains no dt term, $e^{-rt}X(t)$ is a martingale.

2. (a) Let $Z(t) = \sigma\tilde{W}(t) + (r - \frac{1}{2}\sigma^2)t$, then

$$\begin{aligned}
dY(t) =&d\left(e^{Z(t)}\right)\\
=&e^{Z(t)}dZ(t) + \frac{1}{2}e^{Z(t)}dZ(t)dZ(t)\\
=&Y(t)\left[\sigma d\tilde{W}(t) + (r - \frac{1}{2}\sigma^2)dt\right] + \frac{1}{2}Y(t)\sigma^2(t)\\
=&rY(t)dt + \sigma d\tilde{W}(t)
\end{aligned}$$

   (b) Since

$$\begin{aligned}
&d\left(e^{-rt}Y(t)\right)\\
=&-re^{-rt}Y(t)dt + e^{-rt}dY(t)\\
=&-re^{-rt}Y(t)dt + e^{-rt}Y(t)dt + e^{-rt}\sigma Y(t)d\tilde{W}(t)\\
=&\sigma Y(t)e^{-rt}d\tilde{W}(t)
\end{aligned}$$

   contains no dt term, $e^{-rt}Y(t)$ is a martingale.

   (c) By Ito's product rule,

$$\begin{aligned}
dS(t) =&S(0)dY(t) + \int_0^t \frac{1}{Y(s)}dsdY(t) + Y(t)\cdot\frac{a}{Y(t)}dt + dY(t)\cdot\frac{a}{Y(t)}dt\\
=&\left[S(0) + \int_0^t \frac{1}{Y(s)}ds\right]dY(t) + adt \qquad \text{since } dY(t)dt = 0\\
=&\frac{S(t)}{Y(t)}dY(t) + adt\\
=&\frac{S(t)}{Y(t)}\cdot\left[rY(t)dt + \sigma Y(t)d\tilde{W}(t)\right] + adt\\
=&rS(t)dt + \sigma S(t)d\tilde{W}(t) + adt
\end{aligned}$$

which is (5.9.7) of Text.

3. First

$$\begin{aligned}
&\tilde{E}[S(T)|\mathcal{F}(t)] \\
=&\tilde{E}[S(0)Y(T)+Y(T)\int_0^T \frac{a}{Y(s)}ds|\mathcal{F}(t)] \\
=&S(0)\tilde{E}[Y(T)|\mathcal{F}(t)]+\tilde{E}\left[Y(T)\int_0^t \frac{a}{Y(s)}ds|\mathcal{F}(t)\right]+\tilde{E}\left[Y(T)\int_t^T \frac{a}{Y(s)}ds|\mathcal{F}(t)\right] \\
=&S(0)\tilde{E}[Y(T)|\mathcal{F}(t)]+\tilde{E}\left[Y(T)|\mathcal{F}(t)\right]\int_0^t \frac{a}{Y(s)}ds + a\int_t^T \tilde{E}\left[\frac{Y(T)}{Y(s)}|\mathcal{F}(t)\right]ds
\end{aligned} \tag{5.9}$$

by taking out the what is known.

Secondly, since $e^{-rt}Y(t)$ is a martingale under $\tilde{P}$, we have

$$\tilde{E}[Y(T)|\mathcal{F}(t)]=\tilde{E}[Y(T)e^{-rT}|\mathcal{F}(t)]e^{rT}=Y(t)e^{-rt}e^{rT}=Y(t)e^{r(T-t)}$$

And by the definition of $Y(t)$, we have

$$\begin{aligned}
\frac{Y(T)}{Y(s)}=&\exp\left[\sigma\left(\tilde{W}(T)-\tilde{W}(s)\right)+(r-\frac{1}{2}\sigma^2)(T-s)\right] \\
=&\exp\left[\sigma\left(\tilde{W}(T)-\tilde{W}(s)\right)\right]e^{(r-\frac{1}{2}\sigma^2)(T-s)}
\end{aligned} \tag{5.10}$$

Since $\tilde{W}(T)-\tilde{W}(s)$ is normally distributed with mean 0 and variance $T-s$, by taking the expectation on both sides of the above equation, we have

$$\begin{aligned}
E\left[\frac{Y(T)}{Y(s)}\right]=&E[e^{\sigma\left(\tilde{W}(T)-\tilde{W}(s)\right)}]e^{(r-\frac{1}{2}\sigma^2)(T-s)} \\
=&e^{\frac{1}{2}\sigma^2(T-s)}\cdot e^{(r-\frac{1}{2}\sigma^2)(T-s)} \\
=&e^{r(T-s)}
\end{aligned} \tag{5.11}$$

Now by substituting (5.10) and (5.11) into (5.9), we have

$$\begin{aligned}
&\tilde{E}\left[S(T)|\mathcal{F}(t)\right] \\
=&S(0)e^{r(T-t)}Y(t)+e^{r(T-t)}Y(t)\int_0^t \frac{a}{Y(s)}ds + a\int_t^T e^{r(T-s)}ds \\
=&e^{r(T-t)}S(t)+a\int_t^T e^{r(T-s)}ds \\
=&e^{r(T-t)}S(t)+\frac{a}{r}\left[e^{r(T-t)}-1\right]
\end{aligned} \tag{5.12}$$

4. Now we differentiate (5.12)

$$\begin{aligned}
&d(\tilde{E}\left[S(T)|\mathcal{F}(t)\right]) \\
=&d\left(e^{r(T-t)}S(t)\right)+\frac{a}{r}d\left(e^{r(T-t)}-1\right) \\
=&e^{rT}d\left(e^{-rt}S(t)\right)-ae^{r(T-t)}dt
\end{aligned} \tag{5.13}$$

Since

$$\begin{aligned} & d\left(e^{-rt}S(t)\right) \\ =& -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ =& -re^{-rt}S(t)dt + e^{-rt}rS(t)dt + e^{-rt}\sigma S(t)d\tilde{W}(t) + ae^{-rt}dt \\ =& e^{-rt}\sigma S(t)d\tilde{W}(t) + ae^{-rt}dt \end{aligned} \tag{5.14}$$

by substituting (5.14) into (5.13), we have

$$\begin{aligned} & d(\tilde{E}\left[S(T)|\mathcal{F}(t)\right]) \\ =& e^{rT}d\left(e^{-rt)}S(t)\right) + -ae^{r(T-t)}dt \\ =& e^{r(T-t)}\sigma S(t)d\tilde{W}(t) + ae^{r(T-t)}dt - ae^{r(T-t)}dt \\ =& e^{r(T-t)}\sigma S(t)d\tilde{W}(t) \end{aligned}$$

There is no dt term in $d(\tilde{E}\left[S(T)|\mathcal{F}(t)\right])$. Therefore $\tilde{E}\left[S(T)|\mathcal{F}(t)\right]$ is a martingale under $\tilde{P}$.

5. Since

$$\begin{aligned} & \tilde{E}\left[e^{-r(T-t)}(S(T)-K)|\mathcal{F}(t)\right] \\ =& \tilde{E}\left[e^{-r(T-t)}S(T)|\mathcal{F}(t)\right] - e^{-r(T-t)}K \qquad \text{by linearility} \\ =& e^{-r(T-t)}\tilde{E}\left[S(T)|\mathcal{F}(t)\right] - e^{-r(T-t)}K \qquad \text{by taking out what is known} \end{aligned}$$

$\tilde{E}\left[e^{-r(T-t)}(S(T)-K)|\mathcal{F}(t)\right] = 0$ implies that $K = \tilde{E}\left[S(T)|\mathcal{F}(t)\right]$, i.e.,

$$For_S(t,T) = K = \tilde{E}\left[S(T)|\mathcal{F}(t)\right] = Fut_S(t,T)$$

6. According to the hedgeing strategy, the value of the portfolio at time t is governed by (5.9.7) of Text with $\Delta(t) = 1$, i.e.,

$$dX(t) = dS(t) - adt + r(X(t) - S(t))dt$$

Then (5.8) implies that

$$d\left(e^{-rt}X(t)\right) = \sigma S(t)e^{-rt}d\tilde{W}(t)$$

By integrating both sides from 0 to T, we have

$$\begin{aligned} e^{-rT}X(T) - X(0) =& \int_0^T \sigma S(t)e^{-rt}d\tilde{W}(t)dt \\ =& \int_0^T e^{-rt}\left(dS(t) - rS(t)dt - adt\right) \qquad \text{by (5.9.7) of Text} \\ =& \int_0^T d\left(e^{-rt}S(t)\right) - \int_0^T ae^{-rt}dt \\ =& e^{-rT}S(T) - S(0) - \frac{a}{r}\left(1 - e^{-rT}\right) \end{aligned}$$

Associated with the fact that $X(0) = 0$, we can rewrite it as

$$X(T) = S(T) - S(0)e^{rT} - \frac{a}{r}\left(e^{rT} - 1\right)$$

By (5.12),

$$S(0)e^{rT} + \frac{a}{r}\left(e^{rT}-1\right) = Fut_S(0,T) = For_S(0,T)$$

So

$$X(T) = S(T) - For_S(0,T)$$

Therefore eventually, at time T, she delivers the asset at the price $For_S(0,T)$, which is exactly the amount that she needs to cover the debt in the money market $X(T) - S(T) = S(0)e^{rT} + \frac{a}{r}\left(e^{rT}-1\right)$.

6 Chapter 6

6.1 Exercise 6.1

1. It is easy to see $Z(t) = e^0 = 1$ since $\int_t^t \sigma(v)dW(v) = 0 = \int_t^t (b(v) - \sigma^2(v)/2)dv$. Let

$$S(u) = \int_t^u \sigma(v)dW(v) + \int_t^u (b(v) - \sigma^2(v)/2)dv$$

. Then $Z(u) = e^{S(u)}$ and

$$dS(u) = \sigma(u)dW(u) + \left(b(u) - \sigma^2(u)/2\right)dv$$

Now set $f(t,x) = e^x$ in Ito's Formula (4.4.23) of text, then $f_x = f_{xx} = e^x$ and $f_t = 0$. Then we have

$$\begin{aligned}
dZ(u) =& df(t, S(u)) \\
=& e^{S(u)}dS(u) + \frac{1}{2}e^{S(u)}dS(u)dS(u) \\
=& Z(u)\cdot\left[\sigma(u)dW(u) + \left(b(u) - \sigma^2(u)/2\right)dv\right] \\
&+ Z(u)\frac{1}{2}\cdot\left[\sigma(u)dW(u) + \left(b(u) - \sigma^2(u)/2\right)dv\right]\cdot\left[\sigma(u)dW(u) + \left(b(u) - \sigma^2(u)/2\right)dv\right] \\
=& Z(u)\cdot\left[\left(b(u) - \sigma^2(u)/2\right) + \frac{1}{2}\sigma^2(u)\right]du \qquad \text{(since } dW(u)dW(u) = du\text{)} \\
&+ Z(u)\cdot\left[\sigma(u) + 0\right]du \qquad \text{(since } dW(u)du = 0 = dudu\text{)} \\
=& b(u)Z(u)du + \sigma(u)Z(u)dW(u)
\end{aligned}$$

2. It is clear that if $X(u) = Y(u)Z(u)$, then $X(t) = Y(t)Z(t) = x\cdot 1 = x$. Furthermore, by Ito's product rule, $dW(u)dW(u) = du$ and $dW(u)du = 0 = dudu$, we have

$$\begin{aligned}
dX(u) =& d[Y(u)Z(u)] \\
=& Y(u)dZ(u) + Z(u)dY(u) + dZ(u)dY(u) \\
=& \left[bzdu + \sigma z dW(u)\right]\cdot Y(u) + \left[\frac{a-\sigma\gamma}{Z} + \frac{\gamma}{Z}dW(u)\right]\cdot Z(u) \\
&+ \left[bzdu + \sigma z dW(u)\right]\cdot\left[\frac{a-\sigma\gamma}{Z} + \frac{\gamma}{Z}dW(u)\right] \\
=& \left[(a-\sigma\gamma)du + \gamma dW(u)\right] + \left[bZYdu + \sigma ZYdW(u)\right] + \left[\frac{\gamma\sigma Z}{Z}du + 0\right] \\
=& \left[a - \sigma\gamma + bZY + \gamma\sigma\right]du + \left[\gamma + \sigma ZY\right]dW(u) \\
=& \left(a(u) + b(u)X(u)\right)du + \left(\gamma(u) + \sigma(u)X(u)\right)dW(u) \qquad \text{(since } X = ZY\text{)}
\end{aligned}$$

i.e. $X(u) = Z(u)Y(u)$ solve the stochastic differential equation (6.2.4).

6.2 Exercise 6.3

1.
$$\frac{d}{ds}\left[e^{-\int_0^s b(v)dv}C(s,T)\right] = -b(s)e^{-\int_0^s b(v)dv}C(s,T) + e^{-\int_0^s b(v)dv}C'(s,T) = -e^{-\int_0^s b(v)dv}$$

2.
$$\int_t^T \frac{d}{ds}\left[e^{-\int_0^s b(v)dv}C(s,T)\right]ds = \int_t^T -e^{-\int_0^s b(v)dv}ds$$
$$e^{-\int_0^T b(v)dv}C(T,T) - e^{-\int_0^t b(v)dv}C(t,T) = -\int_t^T e^{-\int_0^s b(v)dv}ds$$

By $C(T,T)=0$,
$$C(t,T) = \frac{\int_t^T e^{-\int_0^s b(v)dv}ds}{e^{-\int_0^t b(v)dv}} = \int_t^T \frac{e^{-\int_0^s b(v)dv}}{e^{-\int_0^t b(v)dv}}ds = \int_t^T e^{-\int_t^s b(v)dv}ds$$

3.
$$A(T,T) - A(t,T) = \int_t^T \left[-a(s)C(s,T) + \frac{\sigma^2(s)}{2}C^2(s,T)\right]ds$$

The result follows from $A(T,T)=0$.

6.3 Exercise 6.4

1.
$$\phi'(t) = \phi(t)\cdot\frac{d}{dt}\left[\frac{\sigma^2}{2}\int_t^T C(u,T)dt\right] = -\frac{\sigma^2}{2}\phi(t)C(t,T)$$
$$\phi''(t) = -\frac{\sigma^2}{2}\phi'(t)C(t,T) - \frac{\sigma^2}{2}\phi(t)C'(t,T)$$

therefore,
$$C'(t,T) = \frac{-\phi''(t) - \frac{\sigma^2}{2}\phi'(t)C(t,T)}{\frac{\sigma^2}{2}\phi(t)} = -\frac{2\phi''(t)}{\sigma^2\phi(t)} - \frac{\phi'(t)}{\phi(t)}C(t,T) = -\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{\sigma^2}{2}C^2(t,T)$$

2. Rearrange (6.5.14) into
$$C' - \frac{\sigma^2}{2}C^2 = b\cdot C - 1$$
we have
$$-\frac{2\phi''(t)}{\sigma^2\phi(t)} = -b\cdot\frac{2\phi'(t)}{\sigma^2\phi(t)} - 1$$
which implies (6.9.10)

3. By the characteristic equation $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$, we have
$$\lambda = \frac{b\pm\sqrt{b^2+2\sigma^2}}{2} = \frac{b}{2}\pm\gamma$$
Then
$$\phi(t) = a_1e^{(\frac{b}{2}+\gamma)t} + a_2e^{(\frac{b}{2}-\gamma)t}$$

Indeed, by the facts $\phi(T)=1$ and $\phi'(T)=-\frac{\sigma^2}{2}\phi(T)C(T,T)=0$, we have a set of equations:

$$\begin{cases} a_1 e^{(\frac{b}{2}+\gamma)T} + a_2 e^{(\frac{b}{2}-\gamma)T} & = 1 \\ a_1(\frac{b}{2}+\gamma)e^{(\frac{b}{2}+\gamma)T} + a_2(\frac{b}{2}-\gamma)e^{(\frac{b}{2}-\gamma)T} & = 0 \end{cases}$$

which imples

$$\begin{cases} a_1 = \frac{\gamma-\frac{b}{2}}{2\gamma}e^{-(\frac{b}{2}+\gamma)T} & = \frac{\sigma^2}{4\gamma(r+\frac{b}{2})}e^{-(\frac{b}{2}+\gamma)T} \\ a_2 = \frac{\gamma+\frac{b}{2}}{2\gamma}e^{-(\frac{b}{2}-\gamma)T} & = \frac{\sigma^2}{4\gamma(r-\frac{b}{2})}e^{-(\frac{b}{2}-\gamma)T} \end{cases}$$

so $c_1=c_2=\sigma^2/4\gamma$.

4. (6.9.12) is straightforward from (6.9.11) by differentiatin w.r.t t. Again, since $\phi'(T)=-\frac{\sigma^2}{2}\phi(T)C(T,T)=0$, we have

$$c_1\cdot e^0 - c_2\cdot e^0 = 0$$

i.e. $c_1=c_2$.

5. By $\gamma=\sqrt{b^2+2\sigma^2}/2$, we have

$$\gamma^2-\frac{b^2}{4}=(\gamma+\frac{b}{2})(\gamma-\frac{b}{2})=\frac{\sigma^2}{2}$$

then

$$\begin{aligned} \phi(t) &= c_1 e^{-\frac{b}{2}(T-t)}\left[\frac{1}{\frac{b}{2}+\gamma}e^{-\gamma(T-t)} - \frac{1}{\frac{b}{2}-\gamma}e^{\gamma(T-t)}\right] \\ &= c_1 e^{-\frac{b}{2}(T-t)}\left[\frac{\frac{b}{2}-\gamma}{-\gamma^2+\frac{b^2}{4}}e^{-\gamma(T-t)} - \frac{\frac{b}{2}+\gamma}{-\gamma^2+\frac{b^2}{4}}e^{\gamma(T-t)}\right] \\ &= c_1 e^{-\frac{b}{2}(T-t)}\frac{2}{\sigma^2}\left[-(\frac{b}{2}-\gamma)e^{-\gamma(T-t)} + (\frac{b}{2}+\gamma)e^{\gamma(T-t)}\right] \\ &= c_1 e^{-\frac{b}{2}(T-t)}\frac{2}{\sigma^2}\left[b\cdot\frac{e^{\gamma(T-t)}-e^{-\gamma(T-t)}}{2} + 2\gamma\cdot\frac{e^{\gamma(T-t)}+e^{-\gamma(T-t)}}{2}\right] \\ &= c_1 e^{-\frac{b}{2}(T-t)}\frac{2}{\sigma^2}\left[b\sinh(\gamma(T-t)) + 2\gamma\cosh(\gamma(T-t))\right] \end{aligned}$$

and

$$\begin{aligned} \phi'(t) &= c_1 e^{-\frac{b}{2}(T-t)}\left[e^{-\gamma(T-t)} - e^{\gamma(T-t)}\right] \\ &= -2c_1 e^{-\frac{b}{2}(T-t)}\sinh(\gamma(T-t)) \end{aligned}$$

Therefore

$$\begin{aligned} C(t,T) &= -\frac{2\phi'(t)}{\sigma^2\phi(t)} \\ &= -\frac{-4c_1 e^{-\frac{b}{2}(T-t)}\sinh(\gamma(T-t))}{\sigma^2 c_1 e^{-\frac{b}{2}(T-t)}\frac{2}{\sigma^2}\left[b\sinh(\gamma(T-t))+2\gamma\cosh(\gamma(T-t))\right]} \\ &= \frac{\sinh(\gamma(T-t))}{\frac{b}{2}\sinh(\gamma(T-t))+\gamma\cosh(\gamma(T-t))} \end{aligned}$$

6. By the fact $\phi(T) = 1$, we have

$$c_1 e^{-\frac{b}{2}(T-T)} \frac{2}{\sigma^2} \left[b \sinh(\gamma(T-T)) + 2\gamma \cosh(\gamma(T-T)) \right] = 1$$

$$c_1 \frac{2}{\sigma^2} \cdot 2\gamma = 1$$

$$c_1 = \frac{\sigma^2}{4\gamma}$$

Eventually,

$$\begin{aligned}
A(t,T) &= A(T,T) - \int_t^T A'(s,T) ds \\
&= 0 - \int_t^T \frac{2a\phi'(s)}{\sigma^2 \phi(s)} ds \\
&= \int_t^T a \cdot C(s,T) ds \\
&= a \cdot \frac{\log(\phi(t))}{\sigma^2/2} \\
&= \frac{2a}{\sigma^2} \log \left[\frac{2c_1}{\sigma^2} \cdot \frac{b \sinh(\gamma(T-t)) + 2\gamma \cosh(\gamma(T-t))}{e^{\frac{b}{2}(T-t)}} \right] \\
&= -\frac{2a}{\sigma^2} \log \left[\frac{\gamma e^{\frac{b}{2}(T-t)}}{b \sinh(\gamma(T-t)) + 2\gamma \cosh(\gamma(T-t))} \right]
\end{aligned}$$

6.4 Exercise 6.5

1. Since

$$g(t, x_1, x_2) = E^{t,x_1,x_2} h(x_1(T), x_2(T))$$

we have for any $0 \le s \le t \le T$ similarly as Theorem 6.3.1 of Text,

$$E\left[h(X_1(T), X_2(T)) \middle| \mathcal{F}(t) \right] = g(t, X_1(t), X_2(t))$$
$$E\left[h(X_1(T), X_2(T)) \middle| \mathcal{F}(s) \right] = g(s, X_1(s), X_2(s))$$

Then

$$\begin{aligned}
E\left[g(t, X_1(t), X_2(t)) \middle| \mathcal{F}(s) \right] &= E\left[E\left[h(X_1(T), X_2(T)) | \mathcal{F}(t) \right] \middle| \mathcal{F}(s) \right] \\
&= E\left[h(X_1(T), X_2(T)) \middle| \mathcal{F}(t) \right] \\
&= g(s, X_1(s), X_2(s))
\end{aligned}$$

Similarly, Since

$$e^{-rt} f(t, x_1, x_2) = E^{t,x_1,x_2} \left[e^{-rT} h(x_1(T), x_2(T)) \right] = e^{-rT} E^{t,x_1,x_2} \left[h(x_1(T), x_2(T)) \right]$$

$e^{-rt} f(t, X_1(t), X_2(t))$ is also a martingale.

2. If W_1 and W_2 are independent, $dW_1(t)dW_1(t) = dt = dW_1(t)dW_1(t)$ and $dW_1(t)dW_2(t) = 0$. By Ito's formula,

$$\begin{aligned}
&dg(t, X_1(t), X_2(t)) \\
=&g_t dt + g_{x_1} dX_1(t) + g_{x_2} dX_2(t) + \frac{1}{2} g_{x_1x_1} dX_1(t)dX_1(t) + g_{x_1x_2} dX_1(t)dX_2(t) + \frac{1}{2} g_{x_2x_2} dX_2(t)dX_2(t) \\
=&g_t dt + g_{x_1} \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \\
&+ g_{x_2} \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
&+ \frac{1}{2} g_{x_1x_1} [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \\
&+ g_{x_1x_2} \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
&+ \frac{1}{2} g_{x_2x_2} \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
=&\left[g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2} g_{x_1x_1} \left(\gamma_{11}^2 + \gamma_{12}^2\right) + g_{x_1x_2} \left(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}\right) + \frac{1}{2} g_{x_2x_2} \left(\gamma_{21}^2 + \gamma_{22}^2\right)\right] dt \\
&+ [\gamma_{11} g_{x_1} + \gamma_{21} g_{x_2}] dW_1(t) + [\gamma_{12} g_{x_1} + \gamma_{22} g_{x_2}] dW_2(t)
\end{aligned}$$

Since $g(t, X_1(t), X_2(t))$ is a martingale, there is no dt term in $dg(t, X_1(t), X_2(t))$. Therefore by setting the dt term to zero in the above differential form, we will have (6.6.3) of Text immediately.

Smilarly by Ito's formula,

$$\begin{aligned}
&d\left[e^{-rt} f(t, X_1(t), X_2(t))\right] \\
=&\left(-re^{-rt} f + e^{-rt} f_t\right) dt + e^{-rt} f_{x_1} dX_1(t) + e^{-rt} f_{x_2} dX_2(t) \\
&+ e^{-rt} \frac{1}{2} f_{x_1x_1} dX_1(t)dX_1(t) + e^{-rt} f_{x_1x_2} dX_1(t)dX_2(t) + \frac{1}{2} e^{-rt} f_{x_2x_2} dX_2(t)dX_2(t) \\
=&e^{-rt} \left[-rf + f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2} f_{x_1x_1} \left(\gamma_{11}^2 + \gamma_{12}^2\right) + f_{x_1x_2} \left(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}\right) + \frac{1}{2} f_{x_2x_2} \left(\gamma_{21}^2 + \gamma_{22}^2\right)\right] dt \\
&+ e^{-rt} [\gamma_{11} f_{x_1} + \gamma_{21} f_{x_2}] dW_1(t) + e^{-rt} [\gamma_{12} f_{x_1} + \gamma_{22} f_{x_2}] dW_2(t)
\end{aligned}$$

Since $e^{-rt} f(t, X_1(t), X_2(t))$ is a martingale, there is no dt term in $d[e^{-rt} f(t, X_1(t), X_2(t))]$. (6.6.4) of Text follows immediately by setting the dt term to zero in the above differential form.

3. If $dW_1(t)dW_2(t) = \rho dt$, $dW_1(t)dW_1(t) = dt = dW_1(t)dW_1(t)$, then by Ito's formula,

$$\begin{aligned}
&dg(t, X_1(t), X_2(t)) \\
=&g_t dt + g_{x_1} dX_1(t) + g_{x_2} dX_2(t) + \frac{1}{2} g_{x_1x_1} dX_1(t)dX_1(t) + g_{x_1x_2} dX_1(t)dX_2(t) + \frac{1}{2} g_{x_2x_2} dX_2(t)dX_2(t) \\
=&g_t dt + g_{x_1} \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \\
&+ g_{x_2} \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
&+ \frac{1}{2} g_{x_1x_1} [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \\
&+ g_{x_1x_2} \cdot [\beta_1 dt + \gamma_{11} dW_1(t) + \gamma_{12} dW_2(t)] \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
&+ \frac{1}{2} g_{x_2x_2} \cdot [\beta_2 dt + \gamma_{21} dW_1(t) + \gamma_{22} dW_2(t)] \\
=&\left[g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2} g_{x_1x_1} \left(\gamma_{11}^2 + \gamma_{12}^2\right) + g_{x_1x_2} \left(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}\right) + \frac{1}{2} g_{x_2x_2} \left(\gamma_{21}^2 + \gamma_{22}^2\right)\right] dt \\
&+ [g_{x_1x_1} \rho\gamma_{11}\gamma_{12} + g_{x_1x_2} (\rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21}) + g_{x_2x_2} \rho\gamma_{21}\gamma_{22}] dt \\
&+ [\gamma_{11} g_{x_1} + \gamma_{21} g_{x_2}] dW_1(t) + [\gamma_{12} g_{x_1} + \gamma_{22} g_{x_2}] dW_2(t)
\end{aligned}$$

Since $g(t, X_1(t), X_2(t))$ is a martingale, there is no dt term in $dg(t, X_1(t), X_2(t))$. Therefore by setting the dt term to zero in the above differential form, we will have (6.9.13) of Text immediately.

Smilarly by Ito's formula,

$$
\begin{aligned}
& d\left[e^{-rt} f(t, X_1(t), X_2(t))\right] \\
=& \left(-re^{-rt} f + e^{-rt} f_t\right) dt + e^{-rt} f_{x_1} dX_1(t) + e^{-rt} f_{x_2} dX_2(t) \\
& + e^{-rt} \frac{1}{2} f_{x_1x_1} dX_1(t) dX_1(t) + e^{-rt} f_{x_1x_2} dX_1(t) dX_2(t) + \frac{1}{2} e^{-rt} f_{x_2x_2} dX_2(t) dX_2(t) \\
=& e^{-rt}\left[-rf + f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2} f_{x_1x_1}\left(\gamma_{11}^2 + \gamma_{12}^2\right) + f_{x_1x_2}\left(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}\right) + \frac{1}{2} f_{x_2x_2}\left(\gamma_{21}^2 + \gamma_{22}^2\right)\right] dt \\
& + e^{-rt}\left[g_{x_1x_1}\rho\gamma_{11}\gamma_{12} + g_{x_1x_2}\left(\rho\gamma_{11}\gamma_{22} + \rho\gamma_{12}\gamma_{21}\right) + g_{x_2x_2}\rho\gamma_{21}\gamma_{22}\right] dt \\
& + e^{-rt}\left[\gamma_{11} f_{x_1} + \gamma_{21} f_{x_2}\right] dW_1(t) + e^{-rt}\left[\gamma_{12} f_{x_1} + \gamma_{22} f_{x_2}\right] dW_2(t)
\end{aligned}
$$

Since $e^{-rt} f(t, X_1(t), X_2(t))$ is a martingale, there is no dt term in $d\left[e^{-rt} f(t, X_1(t), X_2(t))\right]$. (6.9.14) of Text follows immediately by setting the dt term to zero in the above differential form.

6.5 Exercise 6.7

1. By the iterated conditioning, for $0 \le s \le t \le T$, we have

$$
\begin{aligned}
\tilde{E}\left[e^{-rt} c(t, S(t), V(t)) \middle| \mathcal{F}(s)\right] =& \tilde{E}\left[e^{-rt} \tilde{E}\left[e^{-r(T-t)} (S(T) - K)^+ \middle| \mathcal{F}(t)\right] \middle| \mathcal{F}(s)\right] \\
=& \tilde{E}\left[e^{-rt} e^{-r(T-t)} (S(T) - K)^+ \middle| \mathcal{F}(s)\right] \\
=& \tilde{E}\left[e^{-rT} (S(T) - K)^+ \middle| \mathcal{F}(s)\right] \\
=& e^{-rs} \tilde{E}\left[e^{-r(T-s)} (S(T) - K)^+ \middle| \mathcal{F}(s)\right] \\
=& e^{-rs} c(s, S(s), V(s))
\end{aligned}
$$

Then since it is a martingale, the dt term in the differential $d[e^{-rt} c(t, S(t), V(t))]$ should be 0,

$$
\begin{aligned}
& d[e^{-rt} c(t, S(t), V(t))] \\
=& e^{-rt}\left[-rc dt + c_t dt + c_s dS(t) + c_v dV(t) + \frac{1}{2} c_{ss} dS(t) dS(t) + c_{sv} dS(t) dV(t) + \frac{1}{2} c_{vv} dV(t) dV(t)\right] \\
=& e^{-rt}\left[-rc + c_t + c_s rS + c_v(a - bV) + \frac{1}{2} c_{ss} V S^2 + c_{sv} V S \sigma\rho + \frac{1}{2} c_{vv} \sigma^2 V\right] dt \\
& + e^{-rt}\left[c_s \sqrt{V} S\right] d\tilde{W}_1(t) + e^{-rt}\left[c_v \sigma \sqrt{V}\right] d\tilde{W}_2(t)
\end{aligned}
$$

which is

$$
-rc + c_t + c_s rs + c_v(a - bv) + \frac{1}{2} c_{ss} v s^2 + c_{sv} v s \sigma\rho + \frac{1}{2} c_{vv} \sigma^2 v = 0
$$

This is the same as (6.9.26) of text.

2. To simplify the notations, we denote $f = f(t, \log s, v)$, $g = g(t, \log s, v)$ and their paritial deriva-

tives as well. Then

$$\begin{aligned}
c(t,s,v) =& sf(t,\log s,v) - e^{-r(T-t)}Kg(t,\log s,v)\\
c_t =& sf_t - e^{-r(T-t)}Kg_t - re^{-r(T-t)}Kg\\
c_s =& f + sf_x\cdot\frac{1}{s} - e^{-r(T-t)}Kg_x\cdot\frac{1}{s} = f + f_x - e^{-r(T-t)}Kg_x\cdot\frac{1}{s}\\
c_v =& sf_v - e^{-r(T-t)}Kg_v\\
c_{ss} =& \frac{1}{s}f_x + \frac{1}{s}f_{xx} - e^{-r(T-t)}Kg_{xx}\frac{1}{s^2}\\
c_{sv} =& f_v + f_{xv} - e^{-r(T-t)}Kg_{xv}\frac{1}{s}\\
c_{vv} =& sf_{vv} - e^{-r(T-t)}Kg_{vv}
\end{aligned}$$

Then the left handside of (6.9.26) of Text becomes

$$\begin{aligned}
& c_t + rsc_s + (a-bv)c_v + \frac{1}{2}s^2vc_{ss} + \rho\sigma sv + \frac{1}{2}\sigma^2vc_{vv}\\
=& s\left[f_t + rf + rf_x + (a-bv)f_v + \frac{v}{2}f_x + \frac{v}{2}f_{xx} + \rho\sigma vf_v + \rho\sigma vf_{xv} + \frac{\sigma^2v}{2}f_{vv}\right]\\
& + e^{-r(T-t)}K\left[g_t + rg - rg_x + (a-bv)g_v + \frac{v}{2}g_{xx} + \rho\sigma vg_{xv} + \frac{\sigma^2v}{2}g_{vv}\right]\\
=& rsf - re^{-r(T-t)}Kg\\
=& rc
\end{aligned}$$

which verifies (6.9.26) of Text.

3. One method is to observe first that $f(t,X(t),V(t))$ is a martingale, and take differentiation and then set $dt=0$ to get the equation.

   Here we apply the Two Dimensional Feynman-Kac Theorem (Exercise 6.5 of Text)directly[2] Now by the definition of $dX(t)$ and $dV(t)$,

$$\begin{aligned}
\beta_1 = r + \frac{1}{2}v \qquad &, \gamma_{11} = \sqrt{v}, \qquad \gamma_{12} = 0\\
\beta_2 = a - bv + \rho\sigma v \qquad &, \gamma_{21} = 0, \qquad \gamma_{12} = \sigma\sqrt{v}
\end{aligned}$$

   Then (6.9.13) yields

$$f_t + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \frac{v}{2}f_{xx} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2vf_{vv} = 0$$

   Also for the boundary condition, in (6.6.1)of Text $h(x,v) = I_{x\geq\log K}$, then

$$f(T,x,v) = I_{x\geq\log K}$$

4. Here we still apply the Two Dimensional Feynman-Kac Theorem directly, by the definition of $dX(t)$ and $dV(t)$,

$$\begin{aligned}
\beta_1 = r - \frac{1}{2}v \qquad &, \gamma_{11} = \sqrt{v}, \qquad \gamma_{12} = 0\\
\beta_2 = a - bv \qquad &, \gamma_{21} = 0, \qquad \gamma_{12} = \sigma\sqrt{v}
\end{aligned}$$

[2]Here we cannot apply (6.6.3) or (6.6.4) of Text since they are for independent Brownian motions.

Then (6.9.13) yields

$$g_t + (r - \frac{1}{2}v)g_x + (a - bv)g_v + \frac{v}{2}g_{xx} + \rho\sigma v g_{xv} + \frac{1}{2}\sigma^2 v g_{vv} = 0$$

Also for the boundary condition, in (6.6.1)of Text $h(x, v) = I_{x \geq \log K}$, then

$$gT, x, v) = I_{x \geq \log K}$$

5. Apparently the function $c(t, x, v)$ defined by (6.9.34) satisfies the equation (6.9.26). Now by the precise definition

$$\begin{aligned}
&c(T, s, v) \\
=&sf(T, \log s, v) - e^{-r(T-T)}Kg(T, \log s, v) \\
=&\begin{cases} s - K, & \text{if } \log s \geq \log K \\ s \cdot 0 - K \cdot 0, & \text{otherwise} \end{cases} \\
=&(s - K)^+
\end{aligned}$$

which implies that such c satisfies the boundary condition.

7 Chapter 7

8 Chapter 8

9 Chapter 9

10 Chapter 10

11 Chapter 11

11.1 Exercise 11.1

Since

$$\begin{aligned}
&M^2(t) - M^2(s) \\
=&\left[N^2(t) - N^2(s)\right] + \lambda^2(t^2 - s^2) - 2\lambda t N(t) + 2\lambda s N(s) \\
=&\left[(N(t) - N(s))^2 + 2N(t)N(s) - 2N^2(s)\right] + \lambda^2(t^2 - s^2) - 2\lambda t[N(t) - N(s)] + 2\lambda N(s)(s - t) \\
=&[N(t) - N(s)]^2 + 2N(s)[N(t) - N(s)] + \lambda^2(t^2 - s^2) - 2\lambda t[N(t) - N(s)] + 2\lambda N(s)(s - t)
\end{aligned}$$

we have

$$\begin{aligned}
&E[M^2(t) - M^2(s)\Big|\mathcal{F}(s)] \\
=&E[[N(t) - N(s)]^2 + 2N(s)[N(t) - N(s)] + \lambda^2(t^2 - s^2) \\
&- 2\lambda t[N(t) - N(s)] + 2\lambda N(s)(s - t)\Big|\mathcal{F}(s)] \\
=&\lambda^2(t - s)^2 + \lambda(t - s) + 2N(s) \cdot \lambda(t - s) + \lambda^2(t^2 - s^2) \\
&- 2\lambda t \cdot \lambda(t - s) + 2\lambda N(s)(s - t) \\
=&\lambda^2(t^2 + s^2 - 2ts + t^2 - s^2 - 2t^2 + 2ts) + \lambda(t - s) \\
=&\lambda(t - s)
\end{aligned}$$

Therefore, for any $0 \leq s \leq t$,

1.

$$
\begin{aligned}
&E[M^2(t)\Big|\mathcal{F}(s)] \\
=&E[M^2(t) - M^2(s) + M^2(s)\Big|\mathcal{F}(s)] \\
=&E[M^2(t) - M^2(s)\Big|\mathcal{F}(s)] + M^2(s) \\
=&\lambda(t-s) + M^2(s) \\
\geq &M^2(s)
\end{aligned}
$$

i.e., $M^2(t)$ is submartingale.

2.

$$
\begin{aligned}
&E[M^2(t) - \lambda t\Big|\mathcal{F}(s)] \\
=&E[M^2(t) - M^2(s) - \lambda(t-s) + M^2(s) - \lambda s\Big|\mathcal{F}(s)] \\
=&E[M^2(t) - M^2(s)\Big|\mathcal{F}(s)] - \lambda(t-s) + M^2(s) - \lambda s \\
=&\lambda(t-s) - \lambda(t-s) + M^2(s) - \lambda s \\
=&M^2(s) - \lambda s
\end{aligned}
$$

i.e., $M^2(t) - t$ is martingale.