Probability: Hw2

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Instructor: Elena Kosygina 6:00 pm

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Problem 1

Step 1. $E(X) = 0 \Rightarrow X = 0$

Given $E(X) = 0$, assume by contradiction that $X \neq 0$, then $X > 0$ according to the condition $X \geq 0$,

$$\mathrm{E}[X] = \int_\Omega X \,\mathrm{d}P = \int_\Omega X(\omega) P(\mathrm{d}\omega) > 0 \tag{1}$$

which contradicts to given condition $E(X) = 0$ since $X > 0$ and $dP > 0$. Therefore, $X = 0$.

Step 2. $X = 0 \Rightarrow E(X) = 0$

This is directly derived from the definition, given $X = 0$,

$$\mathrm{E}[X] = \int_\Omega X \,\mathrm{d}P = 0 \tag{2}$$

Problem 2

Step 1. Discrete distribution taking only non-negative integer values

When a random variable takes only values in 0, 1, 2, 3, ... we can use the following formula for computing its expectation (even when the expectation is infinite):

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} P(X \geq i) \tag{3}$$

Proof.

$$\sum_{i=1}^{\infty} \mathrm{P}(X \geq i) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} P(X = j) \tag{4}$$

Interchanging the order of summation, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \sum_{j=i}^{\infty} P(X = j) &= \sum_{j=1}^{\infty} \sum_{i=1}^{j} P(X = j) && (5)\\ &= \sum_{j=1}^{\infty} j\, P(X = j) && (6)\\ &= \mathrm{E}[X]. && (7)\end{aligned}$$

Step 2. Prove the left part of the inequality

Write X as

$$X = \lfloor X \rfloor + (X - \lfloor X \rfloor) \tag{8}$$

Problem 2 [Step 2. Prove the left part of the inequality] continued on next page...

where $\lfloor X \rfloor$ denotes the largest integer not greater than X, therefore $X - \lfloor X \rfloor \geq 0$. Then using the conclusion in step 1 for discrete distribution,

$$\begin{aligned} EX &= E(\lfloor X \rfloor) + E(X - \lfloor X \rfloor) \\ &= \sum_{n=1}^{\infty} P(\lfloor X \rfloor \geq n) + E(X - \lfloor X \rfloor) \\ &= \sum_{n=1}^{\infty} P(X \geq n) + E(X - \lfloor X \rfloor) \\ &\geq \sum_{n=1}^{\infty} P(X \geq n) \end{aligned} \tag{9}$$

Step 3. Prove the right part of the inequality

Write X as

$$X = \lceil X \rceil - (\lceil X \rceil - X) \tag{10}$$

where $\lceil X \rceil$ denotes the smallest integer not less than X, therefore $\lceil X \rceil - X \geq 0$. Then similarly using the conclusion in step 1 for discrete distribution,

$$\begin{aligned} EX &= E(\lceil X \rceil) - E(\lceil X \rceil - X) \\ &= \sum_{n=1}^{\infty} P(\lceil X \rceil \geq n) - E(\lceil X \rceil - X) \\ &= \sum_{n=1}^{\infty} P(X + 1 \geq n) - E(\lceil X \rceil - X) \\ &= \sum_{n=0}^{\infty} P(X \geq n) - E(\lceil X \rceil - X) \end{aligned} \tag{11}$$

According to the condition $X \geq 0$, we can go further

$$\begin{aligned} EX &= P(X \geq 0) + \sum_{n=1}^{\infty} P(X \geq n) - E(\lceil X \rceil - X) \\ &= 1 + \sum_{n=1}^{\infty} P(X \geq n) - E(\lceil X \rceil - X) \\ &\leq 1 + \sum_{n=1}^{\infty} P(X \geq n) \end{aligned} \tag{12}$$

Problem 3

(a)

We can prove this using a form of the Cauchy-Schwarz inequality for expectation, but that would be cheating, because C-S is equivalent to this property about ρ. What I will in fact do is to use the same proof technique for establishing C-S to also establish this property about ρ.
To this end, suppose that t is some real number that we will choose later, and consider the obvious inequality

$$E((V + tW)^2) \geq 0 \tag{13}$$

where $V = X - \mu_X$ and $W = Y - \mu_Y$. Expanding out the left-hand-side, and using the linearity of expectation, we find that

$$E(V^2) + 2tE(VW) + t^2E(W^2) \geq 0 \tag{14}$$

Note that the left-hand-side is just a quadratic polynomial in t. Now, clearly we have that

$$E(V^2) = \sigma_X^2, E(W^2) = \sigma_Y^2, E(VW) = Cov(X,Y) \tag{15}$$

and so, our polynomial inequality becomes

$$\sigma_Y^2 t^2 + 2Cov(X,Y)t + \sigma_X^2 \geq 0 \tag{16}$$

From this inequality we find that the only way the left-hand-side could be 0 is if the polynomial has a double-root (i.e. it touches the x-axis in a single point), which could only occur if the discriminant is 0. So, the discriminant must always be negative or 0, which means that

$$4Cov(X,Y)^2 - 4\sigma_X^2\sigma_Y^2 \leq 0 \tag{17}$$

In other words,

$$\rho^2 = \frac{Cov(X,Y)}{\sigma_X^2\sigma_Y^2} \leq 1 \tag{18}$$

provided, of course, that the denominator does not vanish.

(b)

The Cauchy-Schwarz inequality states that for all vectors x and y of an inner product space it is true that

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle \cdot \langle y,y\rangle \tag{19}$$

where $\langle \cdot,\cdot\rangle$ is the inner product also known as dot product, therefore,

$$[\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})]^2 \leq \sum_{i=1}^n (x_i - \overline{x})^2 \sum_{i=1}^n (y_i - \overline{y})^2 \tag{20}$$

Substituting it into the expression of $r(x,y)$, we have

$$\begin{aligned} r^2(x,y) &= \frac{[\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})]^2}{(n-1)^2 s^2(x) s^2(y)} \\ &\leq \frac{\sum_{i=1}^n (x_i - \overline{x})^2 \sum_{i=1}^n (y_i - \overline{y})^2}{(n-1)^2 \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2 \frac{1}{n-1}\sum_{i=1}^n (y_i - \overline{y})^2} \\ &= 1 \end{aligned} \tag{21}$$

Therefore $r(x,y) \in [-1,1]$.

Problem 4

(a)

Since $|X| \leq M$ for some constant M,

$$\lim_{n\to\infty} E(X1_{A_n}) \leq \lim_{n\to\infty} E(M1_{A_n}) = M \lim_{n\to\infty} E(1_{A_n}) = M \times 0 = 0 \tag{22}$$

On the other hand,

$$\lim_{n\to\infty} E(X1_{A_n}) \geq \lim_{n\to\infty} E(-M1_{A_n}) = -M \lim_{n\to\infty} E(1_{A_n}) = -M \times 0 = 0 \tag{23}$$

Therefore,

$$\lim_{n\to\infty} E(X1_{A_n}) = 0 \tag{24}$$

(b)

Write X as $X1_{|X|\leq M} + X1_{|X|>M}$, we will show that the second term goes to 0 a.s. as $M \to \infty$.
Assume by contradiction that for any constant $M > 0$, $P(|X| > M) > 0$, thus when $M \to \infty$, $|X| > M \to \infty$,

$$\int_\Omega |X|1_{|X|>M}dp \to \infty \Rightarrow \int_\Omega |X|dp \to \infty \tag{25}$$

which contradicts to the fact that X is an integrable random variable, that is

$$\int_\Omega |X|dp < \infty \tag{26}$$

Therefore we can say there exist some constant N, such that when $M > N$, $P(|X| > M) = 0$, which implies that

$$\lim_{M\to\infty} E(X1_{|X|>M}) = 0 \tag{27}$$

According to the dominated convergence theorem, with the condition X is integrable and $|X|1_{|X|>M}$,

$$E(\lim_{M\to\infty} |X|1_{|X|>M}) = \lim_{M\to\infty} E(X1_{|X|>M}) = 0 \tag{28}$$

In part(a),

$$\lim_{M\to\infty} |X|1_{|X|\leq M} = |X| \Rightarrow \lim_{M\to\infty} |X|1_{|X|\leq M}1_{A_n} = |X|1_{A_n} \tag{29}$$

Now we can apply monotone convergence theorem, with the condition $|X|1_{A_n}1_{|X|\leq M}$ is increasing w.r.t M,

$$E(|X|I_{A_n}) = E(\lim_{M\to\infty} |X|1_{A_n}1_{|X|\leq M}) = \lim_{M\to\infty} \int_{|X|\leq M} |X|1_{A_n}dp \tag{30}$$

With the conclusion of part(a), we can go further with

$$\begin{aligned}
\lim_{n\to\infty} E(XI_{A_n}) &\leq \lim_{n\to\infty} E(|X|I_{A_n}) \\
&= \lim_{n\to\infty} \lim_{M\to\infty} \int_{|X|\leq M} |X|1_{A_n}dp \\
&= \lim_{M\to\infty} (\lim_{n\to\infty} \int_{|X|\leq M} |X|1_{A_n}dp) \\
&= 0
\end{aligned} \tag{31}$$

Similary we can prove $\lim_{n\to\infty} E(XI_{A_n}) \geq 0$, therefore

$$\lim_{n\to\infty} E(XI_{A_n}) = 0 \tag{32}$$

Problem 5

Would it be reasonable to estimate p as the average of $1/N_1, 1/N_2, ..., 1/N_k$?

No. Recall the Jensen's inequality, it is generally stated in the following form: if X is a random variable and φ is a convex function, then

$$\varphi(E[X]) \leq E[\varphi(X)] \tag{33}$$

where in this problem, $X = N$ and $\varphi(X) = 1/X$. Therefore, it is not reasonable to estimate p as the average.

How would you estimate p from this information?

According to the strong law of large numbers, the sample average converges almost surely to the expected value

$$\overline{X}_n \xrightarrow{a.s.} \mu \qquad \text{when } n \to \infty \tag{34}$$

That is

$$\Pr\Big(\lim_{n\to\infty} \overline{X}_n = \mu\Big) = 1 \tag{35}$$

Therefore we are able to generate a more reasonable N as

$$\overline{N} = \frac{1}{k}\sum_{n=1}^{k} N_n \tag{36}$$

And then follow the same way in the question to estimate p, as

$$\hat{p} = \frac{1}{\overline{N}} = \frac{k}{\sum_{n=1}^{k} N_n} \tag{37}$$