Probability and Elements of Real Analysis: Hw1

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Problem 1

According to the countable additivity of definition 1.1.2, let

$$\begin{aligned} A_1 &= \Omega \\ A_n &= \phi \text{ for } n \in Z^+/\{1\} \end{aligned} \tag{1}$$

then

$$P(\Omega) = P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) = P(\Omega) + \sum_{n=2}^{\infty} P(A_n) \tag{2}$$

Therefore we have $\sum_{n=2}^{\infty} P(A_n) = 0$ which implies $P(\phi) = 0$ since probability is always non-negative. Using this proposition and countable additivity again, let

$$A_n = \phi \text{ for } n \in Z^+/\{1, 2, 3, ..., k\} \tag{3}$$

We derive the finite additivity as

$$P(\bigcup_{n=1}^{k} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{k} P(A_n) \tag{4}$$

(i)

We apply finite additivity to prove this problem,

$$P(B) = P((B - A) \cup A) = P(B - A) + P(A) \geq P(A). \tag{5}$$

(ii)

Using the conclusion of part (i), since $A \subset A_n$, then

$$P(A) \leq P(A_n) \ \forall n \in Z^+ \tag{6}$$

Therefore

$$P(A) \leq \lim_{n\to\infty} P(A_n) = 0 \Rightarrow P(A) = 0 \tag{7}$$

Problem 2

First we construct a uniformly distributed random variable taking values in [0,1] and defined on infinite coin-space Ω_∞, as the way Example 1.2.5 did. According to the assumption of this problem, the probability for head on each toss is $p = \frac{1}{2}$. For $n \in Z^+$,

$$Y_n(\omega) := \begin{cases} 1 & \text{if } \omega_n = H \\ 0 & \text{if } \omega_n = T \end{cases} \tag{8}$$

We set

$$X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n} \tag{9}$$

Problem 2 continued on next page. . .

In terms of the distribution measure μ_X of X, we write it as

$$\mu_X[\frac{k}{2^n}, \frac{k+1}{2^n}] = \frac{1}{2^n} \tag{10}$$

for all k and n are integers such that $0 \le k \le 2^n - 1$. It can be shown that

$$\mu_X[a, b] = b - a \tag{11}$$

for $0 \le a \le b \le 1$. That is, the distribution of X is uniform on [0,1].

(i)

We consider cdf of normal distribution,

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi \tag{12}$$

We construct random variable Z as $Z = N^{-1}(X)$, then

$$P(Z \le a) = P(X \le N(a)) = N(a) \tag{13}$$

for any real number a. That is, Z is a standard normal random variable on the coin-toss space $(\Omega_\infty, \mathcal{F}_\infty, \mathcal{P})$.

(ii)

We define Z as

$$Z := N^{-1}(X) \tag{14}$$

Then

$$\lim_{n\to\infty} Z_n(\omega) = \lim_{n\to\infty} N^{-1}(X_n(\omega)) = N^{-1}(\lim_{n\to\infty} X_n(\omega)) = N^{-1}(X(\omega)) = Z(\omega) \tag{15}$$

for every $\omega \in \Omega_\infty$. Therefore $\{Z_n\}_{n=1}^{\infty}$ is the expected sequence.

Problem 3

To derive the smallest algebra contains the union of the two given algebra $\mathcal{F}_\infty = \phi, \mathcal{U}, \mathcal{U}^\rfloor, \otimes$ and $\mathcal{F}_\in = \phi, \mathcal{V}, \mathcal{V}^\rfloor, \otimes$, we need to generate all the atoms. An atom of $\mathcal{F}$ is a set $A \in \mathcal{F}$ such that the only subsets of A which are also in $\mathcal{F}$ are the empty set ϕ and A itself.
Therefore in our problem, we have four atoms at most, which are

$$A_1 = U \cap V^c, A_2 = V \cap U^c, A_3 = U \cap V, A_4 = U^c \cap V^c \tag{16}$$

This can be easily observed from a Wayne Figure. In general the smallest algebra is just the power set of these four sets, that is

$$\begin{aligned}
\mathcal{F} = \{\phi, & \\
& A_1, A_2, A_3, A_4, \\
& A_1 \cup A_2, A_1 \cup A_3, A_1 \cup A_4, A_2 \cup A_3, A_2 \cup A_4, A_3 \cup A_4, \\
& A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_4, A_1 \cup A_3 \cup A_4, A_2 \cup A_3 \cup A_4, \\
& \Omega\}
\end{aligned} \tag{17}$$

But above is the most general case, which implies that $U \cap V \neq \phi$ and $U \cup V \neq \Omega$. There are 3 more cases,

- $U \cap V \neq \phi$ and $U \cup V = \Omega$
  3 atoms: $A_1 = U \cap V^c, A_2 = V \cap U^c, A_3 = U \cap V$
  $2^3 = 8$ elements: $\mathcal{F} = \{\phi, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3, \Omega\}$
- $U \cap V = \phi$ and $U \cup V \neq \Omega$ 3 atoms: $A_1 = U, A_2 = V, A_3 = U^c \cap V^c$
  $2^3 = 8$ elements: $\mathcal{F} = \{\phi, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3, \Omega\}$
- $U \cap V = \phi$ and $U \cup V = \Omega$
  2 atoms: $A_1 = U, A_2 = V = U^c$
  $2^2 = 4$ elements: $\mathcal{F} = \{\phi, A_1, A_2, \Omega\}$