第一章：数学基础回顾

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§1.1 矢量代数

一、矢量运算

矢量：既有大小又有方向的量，如：速度 $\vec{v}$，位置 $\vec{r}$ 等

标量：只有大小没有方向的量，如：质量 m，电荷 q 等

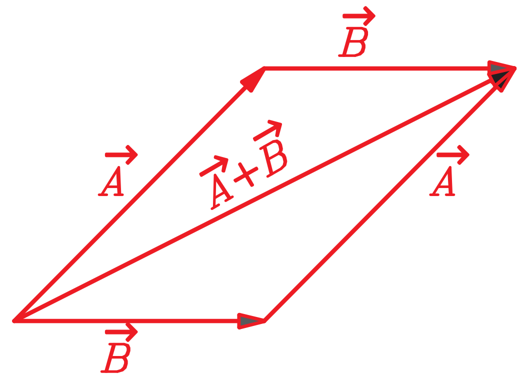
1. 矢量的加法

交换律（commutative）

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

结合律（associative）

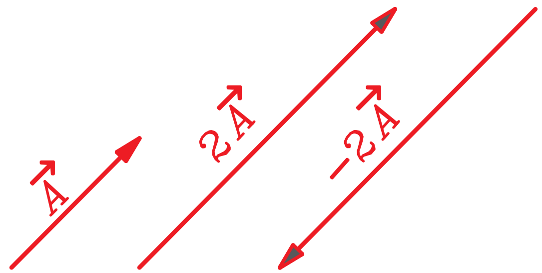
$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$$



2. 矢量与标量的乘积

分配律（distributive）

$$\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha\vec{B}$$

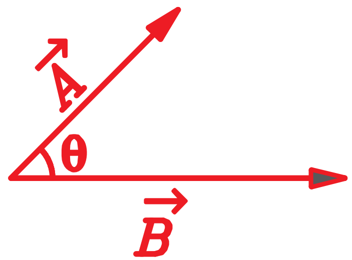


矢量的减法

$$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$$

3. 两个矢量的点积 (dot product, scalar product)

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$



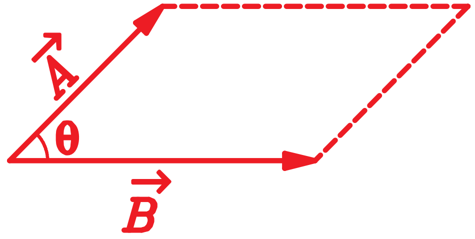
$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2 = A^2$$

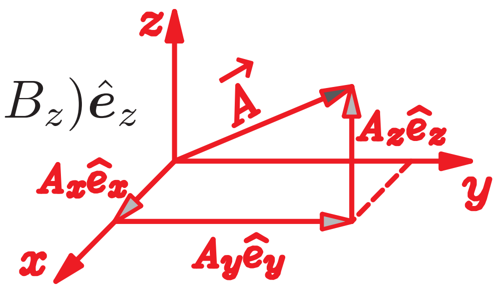
4. 两个矢量的叉积 (cross product, vector product)

$$\vec{A}\times\vec{B}=|\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$
$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$
$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$
$$\vec{A}\times\vec{A}=0$$



二、矢量代数：分量形式

$$\vec{A}=A_x\hat{e}_x+A_y\hat{e}_y+A_z\hat{e}_z$$
$$\vec{A}+\vec{B}=(A_x+B_x)\hat{e}_x+(A_y+B_y)\hat{e}_y+(A_z+B_z)\hat{e}_z$$
$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$$
$$A=\sqrt{\vec{A}\cdot\vec{A}}=\sqrt{A_x^2+A_y^2+A_z^2}$$



$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z = \sum_{i,j} \delta_{ij} A_i B_j$$

δ_{ij} 称为 Kronecker delta,

$$\delta_{ij} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j \hat{\boldsymbol{e}}_k$$

ε_{ijk} 称为 Levi-Civita symbol 或 Levi-Civita tensor

$$\varepsilon_{ijk} = \hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k) = \begin{cases} 1 & \text{if } ijk = 123,\ 231,\ 312 \\ -1 & \text{if } ijk = 132,\ 213,\ 321 \\ 0 & \text{otherwise} \end{cases}$$

Levi-Civita 张量的性质

1. 简单表示右手系中基矢量的矢积：$\hat{e}_i \times \hat{e}_j = \sum_k \varepsilon_{ijk}\hat{e}_k$
2. 任意两个下标互换，差一负号，如：$\varepsilon_{ijk} = -\varepsilon_{ikj}$
3. 单重求和（对重复下标求和，略去求和号）

$$\underbrace{\varepsilon_{ijk}\varepsilon_{mnk}}_{\text{隐含对 } k \text{ 求和}} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix}$$

4. 两重求和（上式中令 $n = j$）

$$\underbrace{\varepsilon_{ijk}\varepsilon_{mjk}}_{\text{隐含对 } j,\ k \text{ 求和}} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{mj} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$$

5. 三重求和（上式中令 $m = i$）

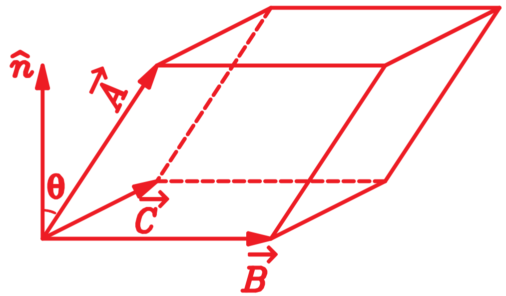
$$\underbrace{\varepsilon_{ijk}\varepsilon_{ijk}}_{\text{隐含对 } i,\ j,\ k \text{ 求和}} = 2\delta_{ii} = 6$$

三、三重积

1. 三重标积 (scalar triple product，混合积)

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



2. 三重矢积 (vector triple product)

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C}$$ not associative

$$\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=0$$

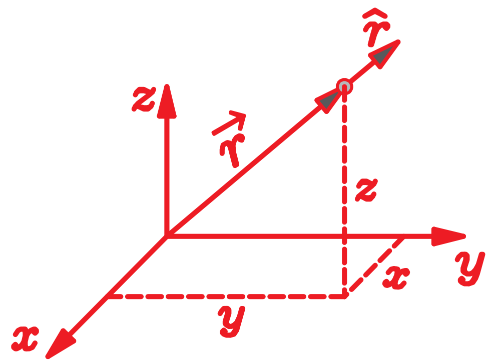
四、位置矢量、位移矢量、间距矢量

位置矢量

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$$

$$r = |\vec{\boldsymbol{r}}| = \sqrt{\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}} = \sqrt{x^2+y^2+z^2}$$

$$\hat{\boldsymbol{e}}_r = \vec{\boldsymbol{r}}/r$$



无限小位移矢量

$$d\vec{\boldsymbol{l}} \equiv d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{e}}_x + dy\,\hat{\boldsymbol{e}}_y + dz\,\hat{\boldsymbol{e}}_z$$

间距矢量

$$\vec{\boldsymbol{R}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,' = (x-x')\hat{\boldsymbol{e}}_x + (y-y')\hat{\boldsymbol{e}}_y + (z-z')\hat{\boldsymbol{e}}_z$$

$\vec{\boldsymbol{r}}$ 为场点（field point 观察点）的位置矢量，
$\vec{\boldsymbol{r}}\,'$ 为源点（source point）的位置矢量。
$R = |\vec{\boldsymbol{R}}| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

§ 1.2 微分

一、梯度

梯度是个矢量

$$\nabla T = \frac{\partial T}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial T}{\partial y}\hat{\boldsymbol{e}}_y + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

全微分

$$\begin{aligned} dT &= \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \\ &= (\nabla T)\cdot(d\vec{\boldsymbol{l}}) = |\nabla T|\,|d\vec{\boldsymbol{l}}|\cos\theta \end{aligned}$$

几何意义：
梯度 ∇T 的方向指向函数 T 的最大变化率（方向导数）方向，其大小即为函数 T 的最大变化率（方向导数）。

二、算符 ∇

梯度写成

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial T}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial T}{\partial z}\hat{\mathbf{e}}_z = \left(\hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right) T$$

del 矢量算符 ∇ （既是矢量又是算符）

$$\boxed{\nabla = \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}}$$

类比：

$$\vec{\boldsymbol{A}}\,a \quad\Longrightarrow\quad \nabla T \quad \text{梯度 (gradient)}$$

$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} \quad\Longrightarrow\quad \nabla\cdot\vec{\boldsymbol{v}} \quad \text{散度 (divergence)}$$

$$\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} \quad\Longrightarrow\quad \nabla\times\vec{\boldsymbol{v}} \quad \text{旋度 (curl)}$$

三、散度

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{v}} &= \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{\boldsymbol{e}}_x+v_y\hat{\boldsymbol{e}}_y+v_z\hat{\boldsymbol{e}}_z\right)\\ &= \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\end{aligned}$$

四、旋度

$$
\begin{aligned}
\nabla\times\vec{\boldsymbol{v}} &= \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\times(v_x\hat{\boldsymbol{e}}_x+v_y\hat{\boldsymbol{e}}_y+v_z\hat{\boldsymbol{e}}_z)\\
&= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z \end{vmatrix}\\
&= \hat{\boldsymbol{e}}_x\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{\boldsymbol{e}}_y\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{\boldsymbol{e}}_z\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)
\end{aligned}
$$

五、例题

$$
\nabla r = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}\hat{\boldsymbol{e}}_x+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y}\hat{\boldsymbol{e}}_y+\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z}\hat{\boldsymbol{e}}_z
$$

$$
\nabla r = \frac{x}{r}\hat{\boldsymbol{e}}_x+\frac{y}{r}\hat{\boldsymbol{e}}_y+\frac{z}{r}\hat{\boldsymbol{e}}_z = = \frac{\vec{\boldsymbol{r}}}{r} = \hat{\boldsymbol{e}}_r
$$

$$\nabla r = \frac{\vec{\boldsymbol{r}}}{r} = \hat{\boldsymbol{e}}_r, \qquad \nabla f(u) = \frac{d\,f}{du}\nabla u$$

$$\nabla\cdot\vec{\boldsymbol{A}}(u) = (\nabla u)\cdot\frac{d\,\vec{\boldsymbol{A}}(u)}{du}, \qquad \nabla\times\vec{\boldsymbol{A}}(u) = (\nabla u)\times\frac{d\,\vec{\boldsymbol{A}}(u)}{du}$$

$$\nabla r^2 = 2r\nabla r = 2\vec{\boldsymbol{r}}$$

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\nabla r = -\frac{1}{r^2}\hat{\boldsymbol{e}}_r = -\frac{\vec{\boldsymbol{r}}}{r^3}$$

$$\nabla\cdot\vec{\boldsymbol{r}} = \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\cdot(x\hat{\boldsymbol{e}}_x+y\hat{\boldsymbol{e}}_y+z\hat{\boldsymbol{e}}_z) = 3$$

$$\nabla\times\vec{\boldsymbol{r}} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

六、乘积的梯度、散度、旋度

∇ 既是矢量又是线性算符

分配率：

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla\cdot(\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}})=\nabla\cdot\vec{\boldsymbol{A}}+\nabla\cdot\vec{\boldsymbol{B}}$$

$$\nabla\times(\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}})=\nabla\times\vec{\boldsymbol{A}}+\nabla\times\vec{\boldsymbol{B}}$$

如果 k 是常数

$$\nabla(kf)=k\nabla f$$

$$\nabla\cdot(k\vec{\boldsymbol{A}})=k\nabla\cdot\vec{\boldsymbol{A}}$$

$$\nabla\times(k\vec{\boldsymbol{A}})=k\nabla\times\vec{\boldsymbol{A}}$$

乘积形式：$fg,\quad \vec{A}\cdot\vec{B},\quad f\vec{A},\quad \vec{A}\times\vec{B}$

$$\nabla(fg) = \nabla_f(fg)+\nabla_g(fg)$$

先利用▽的算符（求导）特性，乘积的求导分成两项；

1. ▽ 对 f 作用（求导），g 视为常量，记为 ∇_f
2. ▽ 对 g 作用（求导），f 视为常量，记为 ∇_g

$$= g\nabla_f f + f\nabla_g g$$

再利用▽矢量运算特性；$\vec{C}(fg) = g\vec{C}f = f\vec{C}g$

将受▽作用的函数移到▽的右边，
不受▽作用的函数移到▽的左边

$$\nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\vec{A}) = \nabla_f\cdot(f\vec{A})+\nabla_A\cdot(f\vec{A})$$

先利用▽的求导特性，乘积的求导分成两项；

1. ▽ 对 f 求导，$\vec{A}$ 视为常量，记为 ∇_f
2. ▽ 对 $\vec{A}$ 求导，f 视为常量，记为 ∇_A

$$= \vec{A}\cdot(\nabla_f f)+f(\nabla_A\cdot\vec{A})$$

再利用▽矢量运算特性；

$\vec{C}\cdot(f\vec{A}) = \vec{A}\cdot(\vec{C}f) = f(\vec{C}\cdot\vec{A})$

将受▽作用的函数移到▽的右边，
不受▽作用的函数移到▽的左边

$$\nabla\cdot(f\vec{A}) = \vec{A}\cdot\nabla f + f\nabla\cdot\vec{A}$$

$$\nabla \times (f\vec{\boldsymbol{A}}) = \nabla_f \times (f\vec{\boldsymbol{A}}) + \nabla_A \times (f\vec{\boldsymbol{A}})$$

先利用∇的求导特性，求导分成两项；
1. ∇ 对 f 求导，$\vec{\boldsymbol{A}}$ 视为常量，即 ∇_f
2. ∇ 对 $\vec{\boldsymbol{A}}$ 求导，f 视为常量，即 ∇_A

再利用∇矢量运算特性；

$$\vec{\boldsymbol{C}} \times (f\vec{\boldsymbol{A}}) = -\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{C}} f) = f(\vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}})$$

将受∇作用的函数移到∇的右边，
不受∇作用的函数移到∇的左边

$$= -\vec{\boldsymbol{A}} \times (\nabla_f f) + f(\nabla_A \times \vec{\boldsymbol{A}})$$

$$\nabla \times (f\vec{\boldsymbol{A}}) = -\vec{\boldsymbol{A}} \times \nabla f + f\nabla \times \vec{\boldsymbol{A}}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \nabla_A\cdot(\vec{A}\times\vec{B}) + \nabla_B\cdot(\vec{A}\times\vec{B})$$

先利用∇的求导特性，求导分成两项：

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= \vec{B}\cdot(\nabla_A\times\vec{A}) - \vec{A}\cdot(\nabla_B\times\vec{B})$$

再利用∇矢量运算特性：

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$$

将受∇作用的函数移到∇的右边，
不受∇作用的函数移到∇的左边

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla \times (\vec{A} \times \vec{B}) = \nabla_A \times (\vec{A} \times \vec{B}) + \nabla_B \times (\vec{A} \times \vec{B})$$

先利用∇的求导特性，求导分成两项；

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= (\vec{B} \cdot \nabla_A)\vec{A} - \vec{B}(\nabla_A \cdot \vec{A}) + \vec{A}(\nabla_B \cdot \vec{B}) - (\vec{A} \cdot \nabla_B)\vec{B}$$

再利用∇矢量运算特性；

$$\begin{aligned} \vec{C} \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \vec{C})\vec{A} - \vec{B}(\vec{C} \cdot \vec{A}) \\ &= \vec{A}(\vec{C} \cdot \vec{B}) - (\vec{A} \cdot \vec{C})\vec{B} \end{aligned}$$

将受∇作用的函数移到∇的右边，
不受∇作用的函数移到∇的左边

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B})$$

其中

$$
\begin{aligned}
(\vec{\boldsymbol{B}}\cdot\nabla) &= \left[(B_x\hat{\boldsymbol{e}}_x + B_y\hat{\boldsymbol{e}}_y + B_z\hat{\boldsymbol{e}}_z)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\right] \\
&= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z} \quad \Longrightarrow \quad \text{标量算符}
\end{aligned}
$$

从而

$$
\begin{aligned}
(\vec{\boldsymbol{B}}\cdot\nabla)\vec{\boldsymbol{A}} &= \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z) \\
&= \quad B_x\frac{\partial A_x}{\partial x}\hat{\boldsymbol{e}}_x + B_y\frac{\partial A_x}{\partial y}\hat{\boldsymbol{e}}_x + B_z\frac{\partial A_x}{\partial z}\hat{\boldsymbol{e}}_x \\
&\quad + B_x\frac{\partial A_y}{\partial x}\hat{\boldsymbol{e}}_y + B_y\frac{\partial A_y}{\partial y}\hat{\boldsymbol{e}}_y + B_z\frac{\partial A_y}{\partial z}\hat{\boldsymbol{e}}_y \\
&\quad + B_x\frac{\partial A_z}{\partial x}\hat{\boldsymbol{e}}_z + B_y\frac{\partial A_z}{\partial y}\hat{\boldsymbol{e}}_z + B_z\frac{\partial A_z}{\partial z}\hat{\boldsymbol{e}}_z
\end{aligned}
$$

$$\nabla(\vec{A}\cdot\vec{B}) = \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{A}\cdot\vec{B})$$

先利用∇的求导特性，求导分成两项；

1. ∇ 对 $\vec{A}$ 求导，$\vec{B}$ 视为常量，记为 ∇_A
2. ∇ 对 $\vec{B}$ 求导，$\vec{A}$ 视为常量，记为 ∇_B

$$= \vec{B}\times(\nabla_A\times\vec{A}) + (\vec{B}\cdot\nabla_A)\vec{A} + \vec{A}\times(\nabla_B\times\vec{B}) + (\vec{A}\cdot\nabla_B)\vec{B}$$

再利用∇矢量运算特性；

$$\vec{B}\times(\vec{C}\times\vec{A}) = \vec{C}(\vec{A}\cdot\vec{B}) - (\vec{B}\cdot\vec{C})\vec{A}$$

$$\Downarrow$$

$$\vec{C}(\vec{A}\cdot\vec{B}) = \vec{B}\times(\vec{C}\times\vec{A}) + (\vec{B}\cdot\vec{C})\vec{A}$$

$$\vec{C}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{C}\times\vec{B}) + (\vec{A}\cdot\vec{C})\vec{B}$$

将受∇作用的函数移到∇的右边，
不受∇作用的函数移到∇的左边

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B}$$

七、二重算符作用

可能的情况：

$$\left.\begin{array}{ll}\text{梯度的散度} & \nabla\cdot(\nabla f)\\ \text{梯度的旋度} & \nabla\times(\nabla f)\end{array}\right\}\quad\text{起始于标量函数}$$

$$\left.\begin{array}{ll}\text{散度的梯度} & \nabla(\nabla\cdot\vec{\boldsymbol{A}})\\ \text{旋度的散度} & \nabla\cdot(\nabla\times\vec{\boldsymbol{A}})\\ \text{旋度的旋度} & \nabla\times(\nabla\times\vec{\boldsymbol{A}})\end{array}\right\}\quad\text{起始于矢量函数}$$

$$\begin{aligned}\nabla\cdot(\nabla f) &= \left(\hat{\boldsymbol{e}}_x\frac{\partial}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{e}}_x\frac{\partial f}{\partial x}+\hat{\boldsymbol{e}}_y\frac{\partial f}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial f}{\partial z}\right)\\ &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\equiv\nabla^2 f\end{aligned}$$

标量函数 f 的 Laplacian，也是个标量

标量算符 *Laplacian*：∇^2

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

标量的Laplacian仍为标量

$$\begin{aligned}
\nabla^2\vec{\boldsymbol{A}} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_x\hat{\boldsymbol{e}}_x + A_y\hat{\boldsymbol{e}}_y + A_z\hat{\boldsymbol{e}}_z) \\
&= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{\boldsymbol{e}}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{\boldsymbol{e}}_y \\
&\quad + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{\boldsymbol{e}}_z
\end{aligned}$$

矢量的Laplacian仍为矢量

$$\nabla^2\vec{\boldsymbol{A}} = (\nabla^2 A_x)\hat{\boldsymbol{e}}_x + (\nabla^2 A_y)\hat{\boldsymbol{e}}_y + (\nabla^2 A_z)\hat{\boldsymbol{e}}_z$$

$\hat{e}_x$，$\hat{e}_y$，$\hat{e}_z$ 为常矢量。

梯度的旋度恒为零：

$$\nabla \times (\nabla f) = (\nabla \times \nabla) f = 0$$

注意：

$$(\vec{C} g) \times (\vec{C} f) = (\vec{C} \times \vec{C}) g f = 0$$

$$(\nabla g) \times (\nabla f) = (\nabla_g\, g) \times (\nabla_f\, f) = (\nabla_g \times \nabla_f)\, g f \neq 0$$

散度的梯度不常用：$\nabla(\nabla \cdot \vec{A}) \neq \nabla^2 \vec{A}$

旋度的散度恒为零：

$$\nabla \cdot (\nabla \times \vec{A}) = (\nabla \times \nabla) \cdot \vec{A} = 0$$

利用了：$\vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{B} \times \vec{C}) \cdot \vec{A}$

旋度的旋度常用于定义矢量的 Laplacian：

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-(\nabla\cdot\nabla)\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

利用了：$\vec{B}\times(\vec{C}\times\vec{A})=\vec{C}(\vec{B}\cdot\vec{A})-(\vec{B}\cdot\vec{C})\vec{A}$

一般矢量的 Laplacian：

$$\nabla^2\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla\times(\nabla\times\vec{A})$$

$\nabla^2\vec{A}=(\nabla^2A_x)\hat{e}_x+(\nabla^2A_y)\hat{e}_y+(\nabla^2A_z)\hat{e}_z$ 仅对直角坐标成立。

如果 $$\begin{cases}\vec{R}=\vec{r}-\vec{r}\,'\\ \vec{r}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z,\\ \vec{r}\,'=x'\hat{e}_x+y'\hat{e}_y+z'\hat{e}_z,\\ \nabla=\hat{e}_i\dfrac{\partial}{\partial x_i},\quad \nabla'=\hat{e}_i\dfrac{\partial}{\partial x_i'}\end{cases}\implies \nabla'\left[g(\vec{R})\right]=-\nabla\left[g(\vec{R})\right]$$

八、例题

以下 $\vec{A}$ 为任意矢量，$\vec{a}$ 为常矢量。

$$(\vec{A}\cdot\nabla)\vec{r} = (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z) = \vec{A}$$

$$\begin{aligned}(\vec{A}\cdot\nabla)r &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\sqrt{x^2+y^2+z^2} \\ &= (A_x x + A_y y + A_z z)/r = (\vec{A}\cdot\vec{r})/r\end{aligned}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \nabla(a_x x + a_y y + a_z z) = a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z = \vec{a}$$

$$\nabla\cdot(\vec{a}\times\vec{r}) = \nabla_r\cdot(\vec{a}\times\vec{r}) = -\vec{a}\cdot(\nabla\times\vec{r}) = 0$$

$$\nabla\times(\vec{a}\times\vec{r}) = \nabla_r\times(\vec{a}\times\vec{r}) = \vec{a}(\nabla\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r} = 2\vec{a}$$

利用 $\nabla\cdot\vec{r} = 3$ 和 $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$

$$
\begin{aligned}
\nabla\cdot[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\cdot\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\cdot\vec{\boldsymbol{r}} \\
&= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}+3\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}}=4\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}} \\
\nabla\times[(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}}] &= [\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})]\times\vec{\boldsymbol{r}}+(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{r}})\nabla\times\vec{\boldsymbol{r}} \\
&= \vec{\boldsymbol{a}}\times\vec{\boldsymbol{r}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] &= \nabla_r\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)]+\nabla_A\cdot[\vec{\boldsymbol{r}}\times\vec{\boldsymbol{A}}(r)] \\
&= (\nabla\times\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}}(r)-\vec{\boldsymbol{r}}\cdot[\nabla\times\vec{\boldsymbol{A}}(r)] \\
&= \vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\nabla r]=\vec{\boldsymbol{r}}\cdot[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}/r]=0
\end{aligned}
$$

利用 $[\vec{\boldsymbol{A}}'(r)\times\vec{\boldsymbol{r}}]\perp\vec{\boldsymbol{r}}$

$$\nabla\cdot[\phi(r)(\vec{\boldsymbol a}\times\vec{\boldsymbol r})] = [\nabla\phi(r)]\cdot(\vec{\boldsymbol a}\times\vec{\boldsymbol r})+\phi(r)\nabla\cdot(\vec{\boldsymbol a}\times\vec{\boldsymbol r})$$
$$= [\phi'(r)\nabla r]\cdot(\vec{\boldsymbol a}\times\vec{\boldsymbol r}) = \frac{\phi'(r)}{r}\,\vec{\boldsymbol r}\cdot(\vec{\boldsymbol a}\times\vec{\boldsymbol r}) = 0$$

$$\nabla\times[\vec{\boldsymbol E}_0\sin(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r}-\omega t)] = \underbrace{(\nabla\times\vec{\boldsymbol E}_0)}_{0}\sin(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r}-\omega t)-\vec{\boldsymbol E}_0\times[\nabla\sin(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r}-\omega t)]$$
$$= -\vec{\boldsymbol E}_0\times[\cos(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r}-\omega t)]\nabla(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r})$$
$$= \vec{\boldsymbol k}\times\vec{\boldsymbol E}_0\cos(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r}-\omega t)$$

利用 $\nabla(\vec{\boldsymbol k}\cdot\vec{\boldsymbol r})=\vec{\boldsymbol k}$

$$
\begin{aligned}
\nabla\left(\vec{a}\cdot\frac{\vec{r}}{r^3}\right) &= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)+\vec{a}\times\left(\nabla\times\frac{\vec{r}}{r^3}\right)\\
&= (\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left[(\vec{a}\cdot\nabla)\vec{r}\right]+\vec{r}\left[(\vec{a}\cdot\nabla)\frac{1}{r^3}\right]\\
&= \frac{\vec{a}}{r^3}+\vec{r}\left(-\frac{3}{r^4}\right)\left[(\vec{a}\cdot\nabla)r\right] = \frac{\vec{a}}{r^3}-\left(\frac{3\vec{r}}{r^4}\right)\frac{(\vec{a}\cdot\vec{r})}{r}\\
&= \frac{\vec{a}}{r^3}-\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\left(\vec{a}\times\frac{\vec{r}}{r^3}\right) &= \vec{a}\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)-(\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right) = -(\vec{a}\cdot\nabla)\left(\frac{\vec{r}}{r^3}\right)\\
&= -\frac{\vec{a}}{r^3}+\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}
\end{aligned}
$$

$$
\begin{aligned}
(\vec{A}\times\nabla)\times\vec{r} &= \underbrace{\Big[\overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]}_{\text{对重复下标求和}}\times(x_l\hat{e}_l) \\
&= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\times\hat{e}_l) && \text{利用}\begin{cases}\partial_j x_l=\delta_{jl}\\ \hat{e}_k\times\hat{e}_l=\varepsilon_{klm}\hat{e}_m\end{cases} \\
&= \varepsilon_{ijk}(A_i\delta_{jl})(\varepsilon_{klm}\hat{e}_m) && \text{有 }\delta_{jl}\text{ 项，}\begin{cases}\text{对 }l\text{ 先求和}\\ \text{只留 }l=j\text{ 项}\end{cases} \\
&= \varepsilon_{ijk}A_i\varepsilon_{kjm}\hat{e}_m && \text{连续利用 }\varepsilon_{ijk}=-\varepsilon_{ikj} \\
&= -[\varepsilon_{ijk}\varepsilon_{mjk}]A_i\hat{e}_m && [\ ]\text{ 内为Levi-Civita 符号的两重求和} \\
&= -2\delta_{im}A_i\hat{e}_m && \text{满足}\quad \varepsilon_{ijk}\varepsilon_{mjk}=2\delta_{im} \\
&= -2A_i\hat{e}_i = -2\vec{A}
\end{aligned}
$$

$$(\vec{A}\times\nabla)\times\vec{r} = -2\vec{A}$$

$\vec{A}$ 可以为任意矢量

$$(\vec{A}\times\nabla)\cdot\vec{r} = \underbrace{\Big[\ \overbrace{\varepsilon_{ijk}(A_i\partial_j)}^{\vec{A}\times\nabla\text{ 的 }k\text{ 分量}}\ \hat{e}_k\Big]\cdot(x_l\hat{e}_l)}_{\text{对重复下标求和}}$$

$$= \varepsilon_{ijk}(A_i\partial_j x_l)(\hat{e}_k\cdot\hat{e}_l) \qquad \text{利用}\begin{cases}\partial_j x_l = \delta_{jl}\\ \hat{e}_k\cdot\hat{e}_l = \delta_{kl}\end{cases}$$

$$= \varepsilon_{ijk}(A_i\delta_{jl})\delta_{kl} = \varepsilon_{ill}A_i = 0$$

$$(\vec{A}\times\nabla)\cdot\vec{r} = 0$$

$\vec{A}$ 可以为任意矢量

另证 $(\vec{A}\times\nabla)\cdot\vec{r} = \vec{A}\cdot(\nabla\times\vec{r}) = 0$

$$(\vec{A}\times\nabla)\times\vec{r} = (\vec{A}\times\nabla_r)\times\vec{r}$$

利用 $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{a}\,(\vec{b}\cdot\vec{c})$

$$= \nabla_r(\vec{A}\cdot\vec{r}) - \vec{A}(\nabla_r\cdot\vec{r})$$

$\nabla_r(\vec{A}\cdot\vec{r})$ 中 $\vec{A}$ 视为常矢量

利用 $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ 和 $\nabla\cdot\vec{r} = 3$

$$= -2\vec{A}$$

$$(\vec{A}\times\nabla)\times\vec{r} = -2\vec{A}$$

§1.3 积分

一、线积分、面积分和体积分

线积分：
$$\int_{a\mathcal{P}}^{b} \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{l}} = \int_{a\mathcal{P}}^{b} d\vec{\boldsymbol{l}} \cdot \vec{\boldsymbol{A}}$$

闭合路径线积分：
$$\oint_{a\mathcal{P}}^{a} d\vec{\boldsymbol{l}} \cdot \vec{\boldsymbol{A}}$$

物理应用：力 $\vec{\boldsymbol{F}}$ 所做的功 $W = \int_{a\mathcal{P}}^{b} d\vec{\boldsymbol{l}} \cdot \vec{\boldsymbol{F}}$

推广：

$$\int_{a\mathcal{P}}^{b} f\, d\vec{\boldsymbol{l}} = \int_{a\mathcal{P}}^{b} d\vec{\boldsymbol{l}}\, f, \qquad \int_{a\mathcal{P}}^{b} \vec{\boldsymbol{A}} \times d\vec{\boldsymbol{l}} = -\int_{a\mathcal{P}}^{b} d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{A}}$$

面积分： $$\int_{\mathcal{S}} \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{\sigma}} = \int_{\mathcal{S}} \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{A}}\, d\sigma$$

闭合曲面面积分： $$\oint_{\mathcal{S}} \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{A}}\, d\sigma$$

物理应用：通量 $Q = \int_{\mathcal{S}} \vec{\boldsymbol{n}} \cdot (\rho \vec{\boldsymbol{v}})\, d\sigma$

推广： $$\int_{\mathcal{S}} \vec{\boldsymbol{n}}\, f\, d\sigma, \qquad \int_{\mathcal{S}} \vec{\boldsymbol{n}} \times \vec{\boldsymbol{A}}\, d\sigma$$

体积分： $$\int_{\mathcal{V}} f\, d\tau = \int_{\mathcal{V}} d\tau\, f$$

物理应用：质量 $m = \int_{\mathcal{V}} \rho\, d\tau$

推广： $$\int_{\mathcal{V}} d\tau\, \vec{\boldsymbol{A}} = \hat{\boldsymbol{e}}_x \int_{\mathcal{V}} A_x\, d\tau + \hat{\boldsymbol{e}}_y \int_{\mathcal{V}} A_y\, d\tau + \hat{\boldsymbol{e}}_z \int_{\mathcal{V}} A_z\, d\tau$$

二、若干基本积分定理

梯度基本定理 (fundamental theorem for gradients)：

$$\int_{\boldsymbol{a}\mathcal{P}}^{\boldsymbol{b}} (\nabla f) \cdot d\vec{\boldsymbol{l}} = \int_{\boldsymbol{a}\mathcal{P}}^{\boldsymbol{b}} d\,f = f(\boldsymbol{b}) - f(\boldsymbol{a}) \qquad \text{与路径 } \mathcal{P} \text{ 无关}$$

散度定理 (divergence theorem)，也称高斯定理 (Gauss's theorem)：

$$\oint_{\mathcal{S}} \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{A}}\, d\sigma = \int_{\mathcal{V}} (\nabla \cdot \vec{\boldsymbol{A}})\, d\tau \qquad \text{散度定理}$$

推广：

$$\oint_{\mathcal{S}} \vec{\boldsymbol{n}} \times \vec{\boldsymbol{A}}\, d\sigma = \int_{\mathcal{V}} (\nabla \times \vec{\boldsymbol{A}})\, d\tau \qquad \text{旋度定理}$$

$$\oint_{\mathcal{S}} \vec{\boldsymbol{n}} f\, d\sigma = \int_{\mathcal{V}} (\nabla f)\, d\tau \qquad \text{梯度定理}$$

Stokes' 定理 (Stokes' theorem):

$$\oint_{\mathcal{L}} d\vec{l}\cdot\vec{A} = \int_S \left[\vec{n}\cdot(\nabla\times\vec{A})\right] d\sigma$$

利用 $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

改写:

$$\oint_{\mathcal{L}} d\vec{l}\cdot\vec{A} = \int_S \left[(\vec{n}\times\nabla)\cdot\vec{A}\right] d\sigma$$

推广:

$$\oint_{\mathcal{L}} d\vec{l}\times\vec{A} = \int_S \left[(\vec{n}\times\nabla)\times\vec{A}\right] d\sigma$$

$$\oint_{\mathcal{L}} d\vec{l}\, f = \int_S \left[(\vec{n}\times\nabla)f\right] d\sigma$$

Green 第二定理:

$$\int_{\mathcal{V}} (U\nabla^2 V - V\nabla^2 U)\, d\tau = \oint_{\mathcal{S}} \vec{n}\cdot(U\nabla V - V\nabla U)\, d\sigma$$

三、例题

试利用 $\oint_{\mathcal{P}} d\vec{\boldsymbol{l}}\cdot\vec{\boldsymbol{A}} = \int_{\mathcal{S}} [(\vec{\boldsymbol{n}}\times\nabla)\cdot\vec{\boldsymbol{A}}]\,d\sigma = \int_{\mathcal{S}} \vec{\boldsymbol{n}}\cdot(\nabla\times\vec{\boldsymbol{A}})\,d\sigma$ 证明

$$\vec{\boldsymbol{I}}_1 = \oint_{\mathcal{P}} d\vec{\boldsymbol{l}}\times\vec{\boldsymbol{A}} = \int_{\mathcal{S}} [(\vec{\boldsymbol{n}}\times\nabla)\times\vec{\boldsymbol{A}}]\,d\sigma = \vec{\boldsymbol{I}}_2$$

$\vec{\boldsymbol{a}}$ 为任意常矢量。

$$\begin{aligned}
\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{I}}_1 &= \vec{\boldsymbol{a}}\cdot\oint_{\mathcal{P}} d\vec{\boldsymbol{l}}\times\vec{\boldsymbol{A}} = \oint_{\mathcal{P}} \vec{\boldsymbol{a}}\cdot(d\vec{\boldsymbol{l}}\times\vec{\boldsymbol{A}}) \\
&= \oint_{\mathcal{P}} d\vec{\boldsymbol{l}}\cdot(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{a}}) = \int_{\mathcal{S}} \vec{\boldsymbol{n}}\cdot[\nabla\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{a}})]\,d\sigma \\
&= \int_{\mathcal{S}} \vec{\boldsymbol{n}}\cdot[(\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{A}} - \vec{\boldsymbol{a}}(\nabla\cdot\vec{\boldsymbol{A}})]\,d\sigma
\end{aligned}$$

$\vec{\boldsymbol{a}}$ 为任意常矢量

利用 $\nabla\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{a}}) = (\vec{\boldsymbol{a}}\cdot\nabla)\vec{\boldsymbol{A}} - \vec{\boldsymbol{a}}(\nabla\cdot\vec{\boldsymbol{A}}) + \vec{\boldsymbol{A}}(\nabla\cdot\vec{\boldsymbol{a}}) - (\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{a}}$

$$\vec{a}\cdot\vec{I}_1 = \boxed{\int_S \vec{n}\cdot\left[(\vec{a}\cdot\nabla)\vec{A} - \vec{a}(\nabla\cdot\vec{A})\right] d\sigma}$$

$$\begin{aligned}
\vec{a}\cdot\vec{I}_2 &= \vec{a}\cdot\int_S (\vec{n}\times\nabla)\times\vec{A}\, d\sigma = \int_S \vec{a}\cdot\left[(\vec{n}\times\nabla)\times\vec{A}\right] d\sigma \\
&= \int_S \left[\vec{a}\times(\vec{n}\times\nabla)\right]\cdot\vec{A}\, d\sigma \qquad \text{利用 } \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} \\
&= \int_S \underbrace{\left[\vec{n}\,\overbrace{(\vec{a}\cdot\nabla)}^{\text{标量算符}} - \overbrace{(\vec{n}\cdot\vec{a})}^{\text{标量}}\nabla\right]}_{\text{矢量算符}}\cdot\vec{A}\, d\sigma \qquad \text{利用 } \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - (\vec{b}\cdot\vec{a})\vec{c} \\
&= \int_S \left[\vec{n}\cdot(\vec{a}\cdot\nabla)\vec{A} - (\vec{n}\cdot\vec{a})(\nabla\cdot\vec{A})\right] d\sigma \\
&= \boxed{\int_S \vec{n}\cdot\left[(\vec{a}\cdot\nabla)\vec{A} - \vec{a}(\nabla\cdot\vec{A})\right] d\sigma} = \vec{a}\cdot\vec{I}_1
\end{aligned}$$

试利用 $\oint_{\mathcal{P}} d\vec{l} \times \vec{A} = \int_{\mathcal{S}_1} \left[(\vec{n} \times \nabla) \times \vec{A}\right] d\sigma$ 证明

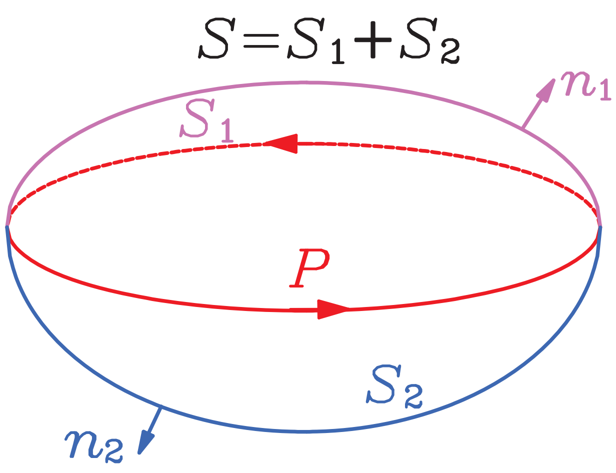
$\vec{S}_1 = \frac{1}{2}\oint \vec{r}\times d\vec{l}$ 若闭合路径 $\mathcal{P}$ 在一平面内，则 $|\vec{S}_1|$ 为 $\mathcal{P}$ 围成的面积

证：

$$\begin{aligned}\oint_{\mathcal{P}} \vec{r} \times d\vec{l} &= -\oint_{\mathcal{P}} d\vec{l} \times \vec{r} \\ &= -\int_{\mathcal{S}_1} \left[(\vec{n} \times \nabla) \times \vec{r}\right] d\sigma\end{aligned}$$

利用 $(\vec{A} \times \nabla) \times \vec{r} = -2\vec{A}$

$$= 2\int_{\mathcal{S}_1} \vec{n}\, d\sigma = 2\vec{S}_1$$



另：

$$\oint_{\mathcal{S}} d\vec{\sigma} = \oint_{\mathcal{S}} \vec{n}\, d\sigma = \oint_{\mathcal{S}} \vec{n}\, 1\, d\sigma$$

$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ 利用 $\oint_{\mathcal{S}} \vec{n}\, f\, d\sigma = \int_{\mathcal{V}} \nabla f\, d\tau$

$$\oint_{\mathcal{S}_1} d\vec{\sigma} + \oint_{\mathcal{S}_2} d\vec{\sigma} = \int_{\mathcal{V}} \nabla 1\, d\tau = 0$$

§1.4 并矢与张量

一、两个矢量的三种乘积运算

点积（dot product，也称标积）： (对重复下标求和)

$$\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}=a_ib_j\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j=a_ib_j\delta_{ij}=a_ib_i$$

结果为标量

物理用例：功 $W=\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{l}}$

叉积（cross product，也称矢积）： (对重复下标求和)

$$\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}=a_ib_j\hat{\boldsymbol{e}}_i\times\hat{\boldsymbol{e}}_j=a_ib_j\,\varepsilon_{ijk}\hat{\boldsymbol{e}}_k$$

结果为矢量

物理用例：力矩 $\vec{\boldsymbol{L}}=\vec{\boldsymbol{r}}\times\vec{\boldsymbol{F}}$

并积（dyadic product）： (对重复下标求和)

$$\vec{\boldsymbol{a}}\,\vec{\boldsymbol{b}}=a_ib_j\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$$

结果为并矢 (dyad)，用 $\overleftrightarrow{A}$ 表示

物理用例：动量流 $\vec{\boldsymbol{v}}\,\vec{\boldsymbol{p}}$、自旋流 $\vec{\boldsymbol{v}}\,\vec{\boldsymbol{s}}$

并积 (dyadic product) 和并矢 (dyad) 的性质

1. 不满足交换律

$$\vec{a}\,\vec{b} \neq \vec{b}\,\vec{a}$$

2. 矩阵表示，9 个分量，只有 6 个独立

$$\overrightarrow{\overrightarrow{A}} = \vec{a}\,\vec{b} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}, \quad \overrightarrow{\overrightarrow{A}} = A_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = a_ib_j\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$$

3. 并矢的转置 (transpose) 类似于矩阵之转置

$$(\overrightarrow{\overrightarrow{A}})^T = (\vec{a}\,\vec{b})^T = \vec{b}\,\vec{a} = \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{bmatrix}, \quad (\overrightarrow{\overrightarrow{A}})^T = a_jb_i\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$$

4. 并矢之和：$\vec{a}\,\vec{b} + \vec{c}\,\vec{d} = (a_ib_j + c_id_j)\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ 类似于矩阵之和

5. 并矢式 (dyadic)：两个或两个以上的并矢之和，9 个独立分量 A_{ij}

   也称张量 (tensor)，用 $\overleftrightarrow{A}$ 表示，$\overleftrightarrow{A} = A_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$

二、张量代数

1. 矢量与张量的点积：　注意：$\vec{a}\cdot\overleftrightarrow{A}\neq\overleftrightarrow{A}\cdot\vec{a}$

左点积：　相当于行矢量左乘矩阵

$$\begin{aligned}\vec{a}\cdot\overleftrightarrow{A} &= (a_i\hat{e}_i)\cdot(A_{jk}\hat{e}_j\hat{e}_k)\\ &= a_iA_{jk}\,\hat{e}_i\cdot\hat{e}_j\,\hat{e}_k\\ &= a_iA_{jk}\,\delta_{ij}\,\hat{e}_k = a_iA_{ik}\hat{e}_k\end{aligned}$$

右点积：　相当于列矢量右乘矩阵

$$\begin{aligned}\overleftrightarrow{A}\cdot\vec{a} &= (A_{jk}\hat{e}_j\hat{e}_k)\cdot(a_i\hat{e}_i)\\ &= a_iA_{jk}\hat{e}_j\,\hat{e}_k\cdot\hat{e}_i\\ &= a_iA_{jk}\hat{e}_j\,\delta_{ik} = a_iA_{ji}\hat{e}_j\end{aligned}$$

2. 矢量与并矢的点积：并矢 $\overrightarrow{\overrightarrow{A}}=\vec{b}\,\vec{c}=b_jc_k\hat{e}_j\hat{e}_k$，　注意：$\vec{a}\cdot(\vec{b}\,\vec{c})\neq(\vec{b}\,\vec{c})\cdot\vec{a}$

左点积：

$$\begin{aligned}\vec{a}\cdot\overrightarrow{\overrightarrow{A}} &= \vec{a}\cdot(\vec{b}\,\vec{c})\\ &= (a_i\hat{e}_i)\cdot(b_jc_k\hat{e}_j\hat{e}_k)\\ &= [a_i\,b_j\,c_k]\,\hat{e}_i\cdot\hat{e}_j\,\hat{e}_k\\ &= [a_i\,b_j\,\delta_{ij}]\,c_k\hat{e}_k = (\vec{a}\cdot\vec{b})\vec{c}\end{aligned}$$

右点积：

$$\begin{aligned}\overrightarrow{\overrightarrow{A}}\cdot\vec{a} &= (\vec{b}\,\vec{c})\cdot\vec{a}\\ &= (b_jc_k\hat{e}_j\hat{e}_k)\cdot(a_i\hat{e}_i)\\ &= [a_i\,b_j\,c_k]\,\hat{e}_j\,\hat{e}_k\cdot\hat{e}_i\\ &= [a_i\,c_k\delta_{ik}]\,b_j\hat{e}_j = \vec{b}(\vec{c}\cdot\vec{a})\end{aligned}$$

$$\vec{a}\cdot(\vec{b}\,\vec{c}) = (\vec{a}\cdot\vec{b})\vec{c} = (\vec{b}\cdot\vec{a})\vec{c} = \vec{b}\cdot(\vec{a}\,\vec{c})$$

$$(\vec{b}\,\vec{c})\cdot\vec{a} = \vec{b}\,(\vec{c}\cdot\vec{a}) = (\vec{b}\,\vec{a})\cdot\vec{c}$$

3. 矢量与张量的叉积： 注意： $\vec{a}\times\overleftrightarrow{A}\neq\overleftrightarrow{A}\times\vec{a}$

左叉积：

$$
\begin{aligned}
\vec{a}\times\overleftrightarrow{A} &= (a_i\hat{e}_i)\times(A_{jk}\hat{e}_j\hat{e}_k)\\
&= a_iA_{jk}\,\hat{e}_i\times\hat{e}_j\,\hat{e}_k\\
&= a_iA_{jk}\,\varepsilon_{ijl}\hat{e}_l\,\hat{e}_k\\
&= \varepsilon_{ijl}a_iA_{jk}\hat{e}_l\hat{e}_k
\end{aligned}
$$

右叉积：

$$
\begin{aligned}
\overleftrightarrow{A}\times\vec{a} &= (A_{jk}\hat{e}_j\hat{e}_k)\times(a_i\hat{e}_i)\\
&= A_{jk}a_i\hat{e}_j\,\hat{e}_k\times\hat{e}_i\\
&= a_iA_{jk}\hat{e}_j\,\varepsilon_{kil}\hat{e}_l\\
&= \varepsilon_{kil}A_{jk}a_i\hat{e}_j\hat{e}_l = \varepsilon_{ijl}A_{ki}a_j\hat{e}_k\hat{e}_l
\end{aligned}
$$

4. 矢量与并矢的叉积：并矢 $\vec{\vec{A}}=\vec{b}\,\vec{c}=b_jc_k\hat{e}_j\hat{e}_k$，即：$A_{jk}=b_jc_k$

左叉积：

$$
\begin{aligned}
\vec{a}\times\vec{\vec{A}} &= \varepsilon_{ijl}a_i\overbrace{(b_jc_k)}^{A_{jk}}\hat{e}_l\hat{e}_k\\
&= [\varepsilon_{ijl}\,a_i\,b_j\,\hat{e}_l]\,c_k\hat{e}_k\\
&= (\vec{a}\times\vec{b})\,\vec{c}
\end{aligned}
$$

$$\vec{a}\times(\vec{b}\,\vec{c})=(\vec{a}\times\vec{b})\,\vec{c}$$

右叉积：

$$
\begin{aligned}
\vec{\vec{A}}\times\vec{a} &= \varepsilon_{kil}\overbrace{(b_jc_k)}^{A_{jk}}a_i\hat{e}_j\hat{e}_l\\
&= b_j\hat{e}_j\,[\varepsilon_{kil}\,c_k\,a_i\,\hat{e}_l]\\
&= \vec{b}\,(\vec{c}\times\vec{a})
\end{aligned}
$$

$$(\vec{b}\,\vec{c})\times\vec{a}=\vec{b}\,(\vec{c}\times\vec{a})$$

5. 张量与张量的点积：相当于矩阵相乘

$$\overleftrightarrow{A}\cdot\overleftrightarrow{B} = (A_{ij}\hat{e}_i\hat{e}_j)\cdot(B_{kl}\hat{e}_k\hat{e}_l)$$
$$= A_{ij}B_{kl}\hat{e}_i\,\hat{e}_j\cdot\hat{e}_k\,\hat{e}_l$$
$$= A_{ij}B_{kl}\,\delta_{jk}\,\hat{e}_i\,\hat{e}_l = A_{ij}B_{jl}\,\hat{e}_i\,\hat{e}_l \qquad \text{仍是张量}$$

6. 张量与张量的双点积：相当于矩阵相乘再求迹 (trace)

张量的迹 (trace)：

$$\overleftrightarrow{A}_t = A_{ii} = A_{11}+A_{22}+A_{33}$$

$$\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) = (a_i\hat{e}_i)\cdot[(b_j\hat{e}_j)\cdot(C_{kl}\hat{e}_k\hat{e}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{e}_j\cdot\hat{e}_k)\,(\hat{e}_i\cdot\hat{e}_l)]$$
$$= a_ib_j\,C_{kl}\,[(\hat{e}_i\hat{e}_j):(\hat{e}_k\hat{e}_l)]$$

$$\overleftrightarrow{A}:\overleftrightarrow{B} = (A_{ij}\hat{e}_i\hat{e}_j):(B_{kl}\hat{e}_k\hat{e}_l)$$
$$= A_{ij}B_{kl}\,[(\hat{e}_i\hat{e}_j):(\hat{e}_k\hat{e}_l)]$$
$$= A_{ij}B_{kl}\,[(\hat{e}_j\cdot\hat{e}_k)\,(\hat{e}_i\cdot\hat{e}_l)]$$
$$= A_{ij}B_{kl}\,\delta_{jk}\delta_{il}$$
$$= A_{ij}B_{ji} \qquad \text{退化为标量}$$

$$\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) = (\vec{a}\vec{b}):\overleftrightarrow{C} = \vec{b}\cdot\overleftrightarrow{C}\cdot\vec{a}$$

$$\boxed{(\hat{e}_i\hat{e}_j):(\hat{e}_k\hat{e}_l) \equiv (\hat{e}_j\cdot\hat{e}_k)\,(\hat{e}_i\cdot\hat{e}_l)}$$

7. 单位张量：

当张量的分量 $A_{ij}=\delta_{ij}$ 时，这时的张量称为单位张量

$$\overleftrightarrow{\boldsymbol{I}}=\delta_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j=\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_i$$

$$\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}}=a_i\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_j=a_i\delta_{ij}\hat{\boldsymbol{e}}_j=a_j\hat{\boldsymbol{e}}_j=\vec{\boldsymbol{a}}$$

即：
$$\begin{cases}\vec{\boldsymbol{a}}\cdot\overleftrightarrow{\boldsymbol{I}}=\vec{\boldsymbol{a}}\\ \text{同理可证}\\ \overleftrightarrow{\boldsymbol{I}}\cdot\vec{\boldsymbol{a}}=\vec{\boldsymbol{a}}\end{cases}$$

既然张量的点积同矩阵的乘积，易证：

$$\overleftrightarrow{\boldsymbol{A}}\cdot\overleftrightarrow{\boldsymbol{I}}=\overleftrightarrow{\boldsymbol{I}}\cdot\overleftrightarrow{\boldsymbol{A}}=\overleftrightarrow{\boldsymbol{A}}$$

双点积：
$$\overleftrightarrow{\boldsymbol{I}}:\overleftrightarrow{\boldsymbol{A}}=(\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_i):(A_{jk}\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k)=A_{jk}(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k)$$

$$=A_{jk}(\delta_{ij})(\delta_{ik})=A_{ii}=\boxed{\overleftrightarrow{\boldsymbol{A}}_t=\overleftrightarrow{\boldsymbol{I}}:\overleftrightarrow{\boldsymbol{A}}=\overleftrightarrow{\boldsymbol{A}}:\overleftrightarrow{\boldsymbol{I}}}$$

如 $\overleftrightarrow{\boldsymbol{A}}=\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}$, 则 $A_{ii}=a_ib_i$，从而：
$$\boxed{\overleftrightarrow{\boldsymbol{I}}:(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}})=a_ib_i=\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}}$$

8. 若干常用恒等式

交换律：

$$\vec{u}\cdot\overleftrightarrow{A} = (\overleftrightarrow{A})^T\cdot\vec{u}$$

但是：
$$\vec{u}\times\overleftrightarrow{A} \neq (\overleftrightarrow{A})^T\times\vec{u}$$
$$\vec{u}\times\overleftrightarrow{A} \neq -(\overleftrightarrow{A})^T\times\vec{u}$$

$$\left[\overleftrightarrow{A}\times\vec{u}\right]^T = -\vec{u}\times(\overleftrightarrow{A})^T \qquad \left[\vec{u}\times\overleftrightarrow{A}\right]^T = -(\overleftrightarrow{A})^T\times\vec{u}$$

结合律：

$$(\vec{a}\cdot\overleftrightarrow{B})\cdot\vec{c} = \vec{a}\cdot(\overleftrightarrow{B}\cdot\vec{c}) = \vec{a}\cdot\overleftrightarrow{B}\cdot\vec{c} = \vec{c}\cdot(\overleftrightarrow{B})^T\cdot\vec{a}$$

$$\vec{u}\cdot(\overleftrightarrow{A}\times\vec{v}) = (\vec{u}\cdot\overleftrightarrow{A})\times\vec{v} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\vec{v} = \vec{u}\times(\overleftrightarrow{A}\cdot\vec{v})$$

$$\overleftrightarrow{B}\cdot(\overleftrightarrow{A}\times\vec{u}) = (\overleftrightarrow{B}\cdot\overleftrightarrow{A})\times\vec{u} \qquad (\vec{u}\times\overleftrightarrow{A})\cdot\overleftrightarrow{B} = \vec{u}\times(\overleftrightarrow{A}\cdot\overleftrightarrow{B})$$

混合积：

类似于 $(\vec{w}\times\vec{u})\cdot\vec{v} = \vec{w}\cdot(\vec{u}\times\vec{v}) \xRightarrow{\text{推广}} \begin{cases} (\overleftrightarrow{A}\times\vec{u})\cdot\vec{v} = \overleftrightarrow{A}\cdot(\vec{u}\times\vec{v}) \\ (\overleftrightarrow{A}\times\vec{u})\cdot\overleftrightarrow{B} = \overleftrightarrow{A}\cdot(\vec{u}\times\overleftrightarrow{B}) \end{cases}$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w} \xRightarrow{\text{推广}} \vec{u}\cdot(\vec{v}\times\overleftrightarrow{A}) = (\vec{u}\times\vec{v})\cdot\overleftrightarrow{A}$$

连续叉积：

$$\vec{u}\times(\vec{v}\times\vec{w}) = \vec{v}\,(\vec{u}\cdot\vec{w}) - (\vec{u}\cdot\vec{v})\,\vec{w} = \vec{v}\,(\vec{w}\cdot\vec{u}) - (\vec{u}\cdot\vec{v})\,\vec{w}$$

$$\Big\Downarrow \text{推广}$$

$$\vec{u}\times(\vec{v}\times\overleftrightarrow{A}) = \vec{v}\,(\vec{u}\cdot\overleftrightarrow{A}) - (\vec{u}\cdot\vec{v})\overleftrightarrow{A} \neq \vec{v}\,(\overleftrightarrow{A}\cdot\vec{u}) - (\vec{u}\cdot\vec{v})\overleftrightarrow{A}$$

$$\overleftrightarrow{A}\times(\vec{v}\times\vec{w}) = (\overleftrightarrow{A}\cdot\vec{w})\,\vec{v} - (\overleftrightarrow{A}\cdot\vec{v})\,\vec{w} \neq \vec{v}\,(\overleftrightarrow{A}\cdot\vec{w}) - (\overleftrightarrow{A}\cdot\vec{v})\,\vec{w}$$

$$(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\cdot\vec{w})\,\vec{v} - (\vec{w}\cdot\vec{v})\,\vec{u} = \vec{v}\,(\vec{u}\cdot\vec{w}) - \vec{u}\,(\vec{v}\cdot\vec{w})$$

$$\Big\Downarrow \text{推广}$$

$$(\vec{u}\times\vec{v})\times\overleftrightarrow{A} = \vec{v}\,(\vec{u}\cdot\overleftrightarrow{A}) - \vec{u}\,(\vec{v}\cdot\overleftrightarrow{A}) \neq (\overleftrightarrow{A}\cdot\vec{u})\,\vec{v} - \vec{u}\,(\vec{v}\cdot\overleftrightarrow{A})$$

$$(\overleftrightarrow{A}\times\vec{v})\times\vec{w} = (\overleftrightarrow{A}\cdot\vec{w})\,\vec{v} - \overleftrightarrow{A}(\vec{w}\cdot\vec{v}) \neq \vec{v}\,(\vec{w}\cdot\overleftrightarrow{A}) - (\vec{w}\cdot\vec{v})\,\overleftrightarrow{A}$$

没有保证 $\overleftrightarrow{A}$ 在最左边

三、张量分析（微分、积分）

以下公式在直角坐标下成立

矢量的梯度是张量：

$$\nabla\vec{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\partial_i)\,(A_j\hat{\boldsymbol{e}}_j) = (\partial_i A_j)\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial_i A_j \equiv \frac{\partial A_j}{\partial x_i}$$

张量的散度是矢量：

$$\nabla\cdot\overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\partial_i)\cdot(A_{jk}\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k) = (\partial_i A_{jk})\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k = (\partial_i A_{jk})\delta_{ij}\hat{\boldsymbol{e}}_k = \partial_i A_{ik}\hat{\boldsymbol{e}}_k$$

$$\nabla\cdot(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) = \partial_i(a_i b_k)\hat{\boldsymbol{e}}_k = (\partial_i a_i)b_k\hat{\boldsymbol{e}}_k + a_i\partial_i b_k\hat{\boldsymbol{e}}_k = (\nabla\cdot\vec{\boldsymbol{a}})\vec{\boldsymbol{b}} + \vec{\boldsymbol{a}}\cdot(\nabla\vec{\boldsymbol{b}})$$

张量的旋度仍为张量：

$$\nabla\times\overleftrightarrow{\boldsymbol{A}} = (\hat{\boldsymbol{e}}_i\partial_i)\times(A_{jk}\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k) = (\partial_i A_{jk})\hat{\boldsymbol{e}}_i\times\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k = (\partial_i A_{jk})\varepsilon_{ijl}\hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_k$$

$$\nabla\times(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) = [\partial_i(a_j b_k)]\varepsilon_{ijl}\hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_k = [\varepsilon_{ijl}\,(\partial_i a_j)\hat{\boldsymbol{e}}_l]\,(b_k\hat{\boldsymbol{e}}_k) - [\varepsilon_{jil}(a_j\partial_i)\hat{\boldsymbol{e}}_l]\,(b_k\hat{\boldsymbol{e}}_k)$$

$$= \nabla_a\times(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) + \nabla_b\times(\vec{\boldsymbol{a}}\vec{\boldsymbol{b}}) = (\nabla\times\vec{\boldsymbol{a}})\,\vec{\boldsymbol{b}} - (\vec{\boldsymbol{a}}\times\nabla)\,\vec{\boldsymbol{b}}$$

例题

$$\nabla\vec{\boldsymbol{r}} = \partial_i x_j \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = \delta_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = \overleftrightarrow{\boldsymbol{I}}, \qquad \text{单位张量}$$

$$(\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

$\vec{\boldsymbol{a}}$ 为常矢量

$$\nabla(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}}) = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{a}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\nabla\times\vec{\boldsymbol{B}}) + (\vec{\boldsymbol{A}}\cdot\nabla)\vec{\boldsymbol{B}} + \vec{\boldsymbol{B}}\times(\nabla\times\vec{\boldsymbol{A}}) + (\vec{\boldsymbol{B}}\cdot\nabla)\vec{\boldsymbol{A}}$$

$$\nabla(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \nabla_A(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) + \nabla_B(\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}}) = (\nabla\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{B}} + (\nabla\vec{\boldsymbol{B}})\cdot\vec{\boldsymbol{A}}$$

$$\vec{\boldsymbol{A}}\times(\nabla\times\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\nabla_B\times\vec{\boldsymbol{B}}) = \nabla_B(\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}}) - (\vec{\boldsymbol{A}}\cdot\nabla_B)\vec{\boldsymbol{B}} = (\nabla\vec{\boldsymbol{B}})\cdot\vec{\boldsymbol{A}} - \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}})$$

因此若 $\nabla\times\vec{\boldsymbol{B}} = 0$，则有

$$(\nabla\vec{\boldsymbol{B}})\cdot\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}\cdot(\nabla\vec{\boldsymbol{B}}) \qquad \text{if} \quad \nabla\times\vec{\boldsymbol{B}} = 0$$

$$
\begin{aligned}
(\vec{\boldsymbol{A}}\times\nabla)\times\vec{\boldsymbol{r}} &= (\vec{\boldsymbol{A}}\times\nabla_r)\times\vec{\boldsymbol{r}} \\
&= \nabla_r(\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}(\nabla\cdot\vec{\boldsymbol{r}}) \\
&= (\nabla_r\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{A}} - 3\vec{\boldsymbol{A}} \\
&= -2\vec{\boldsymbol{A}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\vec{\boldsymbol{r}}\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) &= \nabla(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2) \\
&= \nabla_1(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2) + \nabla_2(\vec{\boldsymbol{r}}_1\cdot\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2) \\
&= \nabla_1\Big[\vec{\boldsymbol{r}}_1\cdot\underbrace{(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}_2)}_{\text{常矢量}}\Big] + \nabla_2\Big\{\vec{\boldsymbol{r}}_2\cdot\underbrace{[(\overleftrightarrow{\boldsymbol{C}})^T\cdot\vec{\boldsymbol{r}}_1]}_{\text{常矢量}}\Big\} \\
&= (\nabla\vec{\boldsymbol{r}})\cdot(\overleftrightarrow{\boldsymbol{C}}\cdot\vec{\boldsymbol{r}}) + \nabla\vec{\boldsymbol{r}}\cdot[(\overleftrightarrow{\boldsymbol{C}})^T\cdot\vec{\boldsymbol{r}}] \qquad \text{利用 } \nabla\vec{\boldsymbol{r}} = \overleftrightarrow{\boldsymbol{I}} \\
&= [\overleftrightarrow{\boldsymbol{C}} + (\overleftrightarrow{\boldsymbol{C}})^T]\cdot\vec{\boldsymbol{r}}
\end{aligned}
$$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(\vec{r}\cdot\vec{c}) = (\nabla\vec{r})\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\nabla(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r})\cdot(\overleftrightarrow{C})^T = \overleftrightarrow{I}\cdot(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T$$

$$\nabla\cdot(\overleftrightarrow{C}\cdot\vec{r}) = \nabla\cdot\left[\vec{r}\cdot(\overleftrightarrow{C})^T\right] = (\nabla\vec{r}):(\overleftrightarrow{C})^T = \overleftrightarrow{I}:(\overleftrightarrow{C})^T = (\overleftrightarrow{C})^T_t = \overleftrightarrow{C}_t$$

矢量 $\nabla\frac{1}{r} = -\frac{\vec{r}}{r^3}$ 的梯度：

$$\begin{aligned}\nabla\nabla\frac{1}{r} = \nabla(-\frac{\vec{r}}{r^3}) \quad &= -\frac{(\nabla\vec{r})}{r^3} - \nabla\left(\frac{1}{r^3}\right)\vec{r} = -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\nabla r\right]\vec{r} \\ &= -\frac{\overleftrightarrow{I}}{r^3} + \left[\frac{3}{r^4}\frac{\vec{r}}{r}\right]\vec{r} = \frac{1}{r^3}\left(\frac{3\vec{r}\vec{r}}{r^2} - \overleftrightarrow{I}\right)\end{aligned}$$

$\boxed{\text{仅对 } \vec{r}\neq 0 \text{ 成立}}$

$$\begin{aligned}(\nabla\nabla\frac{1}{r}):\overleftrightarrow{I} \quad &= \overleftrightarrow{I}:\left(\nabla\nabla\frac{1}{r}\right) \\ &= \left(\overleftrightarrow{I}:\nabla\nabla\right)\frac{1}{r} \\ &= \nabla^2\frac{1}{r} = 0 \qquad \text{if } r\neq 0\end{aligned}$$

利用 $\overleftrightarrow{I}:\overleftrightarrow{A} = \overleftrightarrow{A}:\overleftrightarrow{I}$

利用 $\overleftrightarrow{A}:(\overleftrightarrow{B}\phi) = (\overleftrightarrow{A}:\overleftrightarrow{B})\phi$

利用 $\overleftrightarrow{I}:\nabla\nabla = \nabla^2$

例：如果 $\overleftrightarrow{\boldsymbol{T}}$ 为对称张量，即 $(\overleftrightarrow{\boldsymbol{T}})^T=\overleftrightarrow{\boldsymbol{T}}$，$T_{ij}=T_{ji}$，试证明：

$$\nabla\cdot[\overleftrightarrow{\boldsymbol{T}}\times\vec{\boldsymbol{r}}]=[\nabla\cdot\overleftrightarrow{\boldsymbol{T}}]\times\vec{\boldsymbol{r}}=-\vec{\boldsymbol{r}}\times[\nabla\cdot\overleftrightarrow{\boldsymbol{T}}]$$

利用分量形式

$$\begin{aligned}
\nabla\cdot[\overleftrightarrow{\boldsymbol{T}}\times\vec{\boldsymbol{r}}] &= (\hat{\boldsymbol{e}}_i\partial_i)\cdot[(T_{jk}\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k)\times(x_l\hat{\boldsymbol{e}}_l)] \\
&= (\hat{\boldsymbol{e}}_i\partial_i)\cdot[x_lT_{jk}\hat{\boldsymbol{e}}_j(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l)] \\
&= \partial_i[x_lT_{jk}]\,[(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)\,(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l)] && \text{利用 } \partial_i x_l=\delta_{il} \\
&= [\delta_{il}T_{jk}+x_l(\partial_iT_{jk})]\,[\delta_{ij}(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l)] \\
&= \underbrace{T_{ik}(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_i)}_{0}+(x_l\,\partial_iT_{ik})\,(\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_l) \\
&= (\partial_iT_{ik}\hat{\boldsymbol{e}}_k)\times(x_l\hat{\boldsymbol{e}}_l)=(\nabla\cdot\overleftrightarrow{\boldsymbol{T}})\times\vec{\boldsymbol{r}}
\end{aligned}$$

$$\begin{aligned}
& T_{ik}\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_i &&= T_{ki}\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_i \\
& \text{交换 } i\leftrightarrow k &&= T_{ik}\hat{\boldsymbol{e}}_i\times\hat{\boldsymbol{e}}_k \\
& \text{交换叉积次序} &&= -T_{ik}\hat{\boldsymbol{e}}_k\times\hat{\boldsymbol{e}}_i
\end{aligned}$$

对于张量的 Gauss 定理和 Stokes 定理，应特别注意点积、叉积的次序:

Gauss 定理：
$$\oint_{\mathcal{S}} d\sigma\, \vec{\boldsymbol{n}} \cdot \overleftrightarrow{\boldsymbol{T}} = \int_{\mathcal{V}} d\tau\, \nabla \cdot \overleftrightarrow{\boldsymbol{T}}$$

推广：
$$\oint_{\mathcal{S}} d\sigma\, \vec{\boldsymbol{n}} \times \overleftrightarrow{\boldsymbol{T}} = \int_{\mathcal{V}} d\tau\, \nabla \times \overleftrightarrow{\boldsymbol{T}}$$

$$\oint_{\mathcal{S}} d\sigma\, \vec{\boldsymbol{n}}\, \vec{\boldsymbol{A}} = \int_{\mathcal{V}} d\tau\, \nabla\, \vec{\boldsymbol{A}}$$

Stokes 定理：
$$\oint_{\mathcal{P}} d\vec{\boldsymbol{l}} \cdot \overleftrightarrow{\boldsymbol{T}} = \int_{\mathcal{S}} d\sigma \left[(\vec{\boldsymbol{n}} \times \nabla) \cdot \overleftrightarrow{\boldsymbol{T}}\right] = \int_{\mathcal{S}} d\sigma \left[\vec{\boldsymbol{n}} \cdot (\nabla \times \overleftrightarrow{\boldsymbol{T}})\right]$$

推广：
$$\oint_{\mathcal{P}} d\vec{\boldsymbol{l}} \times \overleftrightarrow{\boldsymbol{T}} = \int_{\mathcal{S}} d\sigma \left[(\vec{\boldsymbol{n}} \times \nabla) \times \overleftrightarrow{\boldsymbol{T}}\right]$$

$$\oint_{\mathcal{P}} d\vec{\boldsymbol{l}}\, \vec{\boldsymbol{A}} = \int_{\mathcal{S}} d\sigma \left[(\vec{\boldsymbol{n}} \times \nabla)\vec{\boldsymbol{A}}\right]$$

例：如果在区域 $\mathcal{V}$ 内矢量 $\vec{j}$ 满足 $\nabla\cdot\vec{j}=0$，在 $\mathcal{V}$ 边界 $\mathcal{S}$ 上 $j_n=\vec{n}\cdot\vec{j}=0$，试证：$\displaystyle\int_{\mathcal{V}}\vec{j}\,d\tau=0$。 — 教材第 30 页 1.4 题

由于在 $\mathcal{V}$ 内 $\nabla\cdot\vec{j}=0$，故

$$
\begin{aligned}
\vec{j} &= \vec{j}\cdot\overleftrightarrow{I} && \text{利用 } \nabla\vec{r}=\overleftrightarrow{I}\\
&= \vec{j}\cdot(\nabla\vec{r}) && \text{利用 } (\vec{a}\cdot\nabla)\vec{b}=\vec{a}\cdot(\nabla\vec{b})\\
&= (\vec{j}\cdot\nabla)\vec{r} && \text{利用 } \nabla\cdot(\vec{a}\,\vec{b})=(\nabla\cdot\vec{a})\,\vec{b}+(\vec{a}\cdot\nabla)\vec{b}\\
&= \nabla\cdot(\vec{j}\,\vec{r})-\underbrace{(\nabla\cdot\vec{j})}_{0}\vec{r} \quad = \nabla\cdot(\vec{j}\,\vec{r})
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathcal{V}}\vec{j}\,d\tau &= \int_{\mathcal{V}}\nabla\cdot(\vec{j}\,\vec{r})\,d\tau && \text{利用了上式 } \vec{j}=\nabla\cdot(\vec{j}\vec{r}) \text{ if } \nabla\cdot\vec{j}=0\\
&= \oint_{\mathcal{S}}\vec{n}\cdot(\vec{j}\,\vec{r})\,d\sigma && \text{利用了高斯定理 } \oint_{\mathcal{S}}\vec{n}\cdot\overleftrightarrow{T}\,d\sigma=\int_{\mathcal{V}}(\nabla\cdot\overleftrightarrow{T})\,d\tau\\
&= \int_{\mathcal{S}}(\vec{n}\cdot\vec{j})\,\vec{r}\,d\sigma && \text{利用了 } \vec{n}\cdot(\vec{a}\vec{b})=(\vec{n}\cdot\vec{a})\vec{b}\\
&= 0 && \text{因为在边界 } \mathcal{S} \text{ 上 } \vec{n}\cdot\vec{j}=0
\end{aligned}
$$

$$\vec{\boldsymbol{F}} = \oint_{\mathcal{P}} I d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} = \int_{\mathcal{S}} I d\sigma (\vec{\boldsymbol{n}} \times \nabla) \times B$$

$$= \int_{\mathcal{S}} I d\sigma [\nabla_B (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{n}}) - \vec{\boldsymbol{n}} (\nabla \cdot \vec{\boldsymbol{B}})]$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \qquad = \int_{\mathcal{S}} I d\sigma (\nabla \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{n}}$$

if $\nabla \times \vec{\boldsymbol{B}} = 0$
$(\nabla \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{n}} = \vec{\boldsymbol{n}} \cdot (\nabla \vec{\boldsymbol{B}})$

$$= \int_{\mathcal{S}} I d\sigma \vec{\boldsymbol{n}} \cdot (\nabla \vec{\boldsymbol{B}})$$

$$d\vec{\boldsymbol{\mu}} = I \vec{\boldsymbol{n}}\, d\sigma \qquad = \int_{\mathcal{S}} d\vec{\boldsymbol{\mu}} \cdot (\nabla \vec{\boldsymbol{B}})$$

§1.5 曲线坐标系

一、正交曲线坐标系

空间任意一点 P 的直角坐标：(x, y, z)，有时也记为 (x_1, x_2, x_3)

如果存在一组独立、连续、单值函数：

$$u_1 = f_1(x, y, z), \quad u_2 = f_2(x, y, z), \quad u_3 = f_3(x, y, z)$$

并且其反函数

$$x = x_1 = \varphi_1(u_1, u_2, u_3), \quad y = x_2 = \varphi_2(u_1, u_2, u_3), \quad z = x_3 = \varphi_3(u_1, u_2, u_3)$$

也独立、连续、单值，那么 P 点坐标：$(x, y, z) \quad \Longleftrightarrow \quad (u_1, u_2, u_3)$

(u_1, u_2, u_3) 称为空间点 P 的曲线坐标 (curvilinear coordinates)

用曲线坐标来描述空间点位置的坐标系称为一般曲线坐标系

曲线坐标系中，

位置矢量： $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z = \vec{\boldsymbol{r}}(x, y, z) = \vec{\boldsymbol{r}}(u_1, u_2, u_3)$

微分线元： $d\,\vec{\boldsymbol{l}} \equiv d\,\vec{\boldsymbol{r}} = \hat{\boldsymbol{e}}_x\,dx + \hat{\boldsymbol{e}}_y\,dy + \hat{\boldsymbol{e}}_z\,dz = \vec{\boldsymbol{a}}_1\,du_1 + \vec{\boldsymbol{a}}_2\,du_2 + \vec{\boldsymbol{a}}_3\,du_3$

如果对任意 $\vec{\boldsymbol{r}}$，$\vec{\boldsymbol{a}}_1$, $\vec{\boldsymbol{a}}_2$, $\vec{\boldsymbol{a}}_3$ 两两垂直：$\vec{\boldsymbol{a}}_1 \cdot \vec{\boldsymbol{a}}_2 = 0$, $\vec{\boldsymbol{a}}_2 \cdot \vec{\boldsymbol{a}}_3 = 0$, $\vec{\boldsymbol{a}}_3 \cdot \vec{\boldsymbol{a}}_1 = 0$,

则称为正交曲线坐标系(orthogonal curvilinear coordinates)。

或者曲线坐标系中，

$$\left.\begin{array}{ll} u_2 = f_2(x, y, z) = c_2 & \text{曲面} \\ u_3 = f_3(x, y, z) = c_3 & \text{曲面} \end{array}\right\}$$

两曲面相交的曲线称为坐标曲线 u_1

（因为这时 u_2 和 u_3 已取定值，只有 u_1 可变）

类似定义坐标曲线 u_2 和 u_3

以 $\hat{e}_1$, $\hat{e}_2$ 和 $\hat{e}_3$ 分别表示坐标曲线 u_1，u_2 和 u_3的切线方向（方向分别沿u_1, u_2 和 u_3 增加的方向）的单位矢量。$\hat{e}_1$, $\hat{e}_2$ 和 $\hat{e}_3$ 与空间位置有关。

如果在空间任一点，$\hat{e}_1$, $\hat{e}_2$ 和 $\hat{e}_3$ 都两两垂直，则称为正交曲线坐标系。

坐标曲线 u_1 的方程:

表达为两曲面相交:

$$\begin{cases} u_2 = f_2(x,y,z) = c_2 \\ u_3 = f_3(x,y,z) = c_3 \end{cases}$$

表达为参数方程（以 u_1 为参数）:

$$\begin{cases} x = \varphi_1(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ y = \varphi_2(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \\ z = \varphi_3(u_1,u_2,u_3)\Big|_{\substack{u_2=c_2\\u_3=c_3}} \end{cases}$$

从其参数方程 φ_i 易求坐标曲线 u_1 切线方向的单位矢量 $\hat{\boldsymbol{e}}_1$:

$$\begin{aligned} \hat{\boldsymbol{e}}_1 &= \frac{1}{h_1}\left(\frac{\partial\varphi_1}{\partial u_1}\hat{\boldsymbol{e}}_x + \frac{\partial\varphi_2}{\partial u_1}\hat{\boldsymbol{e}}_y + \frac{\partial\varphi_3}{\partial u_1}\hat{\boldsymbol{e}}_z\right) = \frac{1}{h_1}\left(\frac{\partial x}{\partial u_1}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_1}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_1}\hat{\boldsymbol{e}}_z\right) \\ h_1 &= \left[\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2\right]^{1/2} \end{aligned}$$

（h_1 用于归一化，使 $\hat{\boldsymbol{e}}_1$ 长度为1。）

同理:

$$\hat{\boldsymbol{e}}_i = \frac{1}{h_i}\left(\frac{\partial x}{\partial u_i}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_i}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_i}\hat{\boldsymbol{e}}_z\right), \quad h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}$$

保持 u_2, u_3 不变，而 u_1 有一微小变化：$u_1 \implies u_1 + du_1$

$$\text{位置矢量：}\quad \vec{\boldsymbol{r}}(u_1, u_2, u_3) \implies \vec{\boldsymbol{r}}(u_1 + du_1, u_2, u_3)$$

$$\begin{aligned}
\text{微分线元：}\quad d\vec{\boldsymbol{l}}_1 &= \vec{\boldsymbol{r}}(u_1 + du_1, u_2, u_3) - \vec{\boldsymbol{r}}(u_1, u_2, u_3) \\
&= \frac{\partial\, \vec{\boldsymbol{r}}(u_1, u_2, u_3)}{\partial u_1}\, du_1 \\
&= \frac{\partial\, (x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z)}{\partial u_1}\, du_1
\end{aligned}$$

$$\text{从而}\quad d\vec{\boldsymbol{l}}_1 = \underbrace{\left(\frac{\partial x}{\partial u_1}\hat{\boldsymbol{e}}_x + \frac{\partial y}{\partial u_1}\hat{\boldsymbol{e}}_y + \frac{\partial z}{\partial u_1}\hat{\boldsymbol{e}}_z\right)}_{\text{坐标曲线 } u_1 \text{ 的切线方向 } h_1\hat{\boldsymbol{e}}_1} du_1 = h_1\hat{\boldsymbol{e}}_1\, du_1$$

$$\text{因此}\quad d\vec{\boldsymbol{l}}_1 = \frac{\partial\, \vec{\boldsymbol{r}}(u_1, u_2, u_3)}{\partial u_1}\, du_1 = h_1\hat{\boldsymbol{e}}_1\, du_1,$$

$$h_1 = \left[\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2\right]^{1/2}$$

类似可得

$$d\vec{l}_i = \frac{\partial \vec{r}(u_1, u_2, u_3)}{\partial u_i} du_i = h_i \hat{e}_i\, du_i, \qquad i = 1, 2, 3$$

其中： $h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}, \qquad i = 1, 2, 3$ 称为度量因子

微分线元

$$\begin{aligned} d\vec{l} = d\vec{r} &= \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = \vec{a}_1\, du_1 + \vec{a}_2\, du_2 + \vec{a}_3\, du_3 \\ &= d\vec{l}_1 + d\vec{l}_2 + d\vec{l}_3 \\ &= h_1 \hat{e}_1\, du_1 + h_2 \hat{e}_2\, du_2 + h_3 \hat{e}_3\, du_3 \end{aligned}$$

$\hat{e}_i$: 坐标曲线 u_i 切线方向的单位矢量（方向沿 u_i 增加的方向）

微分线元： $\boxed{d\vec{l} = h_1 \hat{e}_1\, du_1 + h_2 \hat{e}_2\, du_2 + h_3 \hat{e}_3\, du_3}$

二、正交曲线坐标系中的梯度、散度、旋度

直角坐标系

全微分

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$dT = (\nabla T)\cdot(d\vec{l})$$

微分线元

$$d\vec{l} = \hat{e}_x\,dx + \hat{e}_y\,dy + \hat{e}_z\,dz$$

比较：

$$\nabla T = \frac{\partial T}{\partial x}\hat{e}_x + \frac{\partial T}{\partial y}\hat{e}_y + \frac{\partial T}{\partial z}\hat{e}_z$$

正交曲线坐标系

全微分

$$dT = \frac{\partial T}{\partial u_1}du_1 + \frac{\partial T}{\partial u_2}du_2 + \frac{\partial T}{\partial u_3}du_3$$

$$dT = (\nabla T)\cdot(d\vec{l})$$

微分线元

$$d\vec{l} = h_1\hat{e}_1\,du_1 + h_2\hat{e}_2\,du_2 + h_3\hat{e}_3\,du_3$$

比较：

$$\boxed{\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{e}_3}$$

$$\nabla\cdot(\vec{A}) = \nabla\cdot(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)$$

与直角坐标系不同，这时$\hat{e}_i$不是常矢量

$$= \nabla\cdot(A_1\hat{e}_1) + \nabla\cdot(A_2\hat{e}_2) + \nabla\cdot(A_3\hat{e}_3)$$

u_2 梯度：$\nabla u_2 = \frac{1}{h_1}\frac{\partial u_2}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial u_2}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial u_2}{\partial u_3}\hat{e}_3 = \frac{\hat{e}_2}{h_2}$　同理：$\nabla u_3 = \frac{\hat{e}_3}{h_3}$

得：$\hat{e}_1 = \hat{e}_2\times\hat{e}_3 = h_2h_3\nabla u_2\times\nabla u_3$　右手系

$$\nabla\cdot(A_1\hat{e}_1) = \nabla\cdot[A_1\,\underbrace{h_2h_3(\nabla u_2\times\nabla u_3)}_{\text{即为 }\hat{e}_1}] = \nabla\cdot[h_2h_3A_1\,(\nabla u_2\times\nabla u_3)]$$

$$= [\nabla(h_2h_3A_1)]\cdot(\nabla u_2\times\nabla u_3) + (h_2h_3A_1)\,[\nabla\cdot(\nabla u_2\times\nabla u_3)]$$

[...]为$h_2h_3A_1$的梯度

$$= \left[\frac{1}{h_1}\frac{\partial(h_2h_3A_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial(h_2h_3A_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial(h_2h_3A_1)}{\partial u_3}\hat{e}_3\right]\cdot\frac{\hat{e}_1}{h_2h_3}$$

$$+ (h_2h_3A_1)\,[\underbrace{(\nabla\times\nabla u_2)}_{0}\cdot\nabla u_3 - \underbrace{(\nabla\times\nabla u_3)}_{0}\cdot\nabla u_2]$$

$$\nabla\cdot(A_1\hat{e}_1) = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3A_1)}{\partial u_1}$$

利用了 $\hat{e}_2\cdot\hat{e}_1 = 0$ 和 $\hat{e}_3\cdot\hat{e}_1 = 0$

类似可求得

$$\nabla\cdot(A_2\hat{\boldsymbol{e}}_2) = \frac{1}{h_1h_2h_3}\frac{\partial(h_3h_1A_2)}{\partial u_2} \qquad \nabla\cdot(A_3\hat{\boldsymbol{e}}_3) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2A_3)}{\partial u_3}$$

$$\begin{aligned}\nabla\cdot\vec{\boldsymbol{A}} &= \nabla\cdot(A_1\hat{\boldsymbol{e}}_1)+\nabla\cdot(A_2\hat{\boldsymbol{e}}_2)+\nabla\cdot(A_3\hat{\boldsymbol{e}}_3)\\ &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1}+\frac{\partial(h_3h_1A_2)}{\partial u_2}+\frac{\partial(h_1h_2A_3)}{\partial u_3}\right]\end{aligned}$$

旋度

$$\begin{aligned}\nabla\times\vec{\boldsymbol{A}} &= \nabla\times(A_1\hat{\boldsymbol{e}}_1+A_2\hat{\boldsymbol{e}}_2+A_3\hat{\boldsymbol{e}}_3) && \text{不同于直角坐标系，这里}\hat{\boldsymbol{e}}_i\text{不是常矢量}\\ &= \nabla\times(A_1\hat{\boldsymbol{e}}_1)+\nabla\times(A_2\hat{\boldsymbol{e}}_2)+\nabla\times(A_3\hat{\boldsymbol{e}}_3) && \text{利用 } \nabla u_i=\frac{\hat{\boldsymbol{e}}_i}{h_i}\\ &= \nabla\times(A_1h_1\nabla u_1)+\nabla\times(A_2h_2\nabla u_2)+\nabla\times(A_3h_3\nabla u_3)\end{aligned}$$

利用 $\nabla\times(f\vec{a}) = (\nabla f)\times\vec{a} + f\nabla\times\vec{a}$

$$\nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\underbrace{\nabla u_1}_{\frac{\hat{e}_1}{h_1}} + (A_1h_1)\underbrace{[\nabla\times(\nabla u_1)]}_{0}$$

$$\nabla(A_1h_1) = \frac{1}{h_1}\frac{\partial(A_1h_1)}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial(A_1h_1)}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial(A_1h_1)}{\partial u_3}\hat{e}_3$$

$$\nabla\times(A_1\hat{e}_1) = \nabla\times(A_1h_1\nabla u_1) = [\nabla(A_1h_1)]\times\frac{\hat{e}_1}{h_1}$$

利用 $\hat{e}_1\times\hat{e}_1 = 0$, $\hat{e}_2\times\hat{e}_1 = -\hat{e}_3$ 和 $\hat{e}_3\times\hat{e}_1 = \hat{e}_2$

$$\nabla\times(A_1\hat{e}_1) = \frac{\hat{e}_2}{h_1h_3}\frac{\partial(A_1h_1)}{\partial u_3} - \frac{\hat{e}_3}{h_1h_2}\frac{\partial(A_1h_1)}{\partial u_2}$$

类似可得

$$\nabla\times(A_2\hat{e}_2) = \frac{\hat{e}_3}{h_2h_1}\frac{\partial(A_2h_2)}{\partial u_1} - \frac{\hat{e}_1}{h_2h_3}\frac{\partial(A_2h_2)}{\partial u_3}$$

$$\nabla\times(A_3\hat{e}_3) = \frac{\hat{e}_1}{h_3h_2}\frac{\partial(A_3h_3)}{\partial u_2} - \frac{\hat{e}_2}{h_3h_1}\frac{\partial(A_3h_3)}{\partial u_1}$$

$$\nabla \times \vec{\boldsymbol{A}} = \nabla \times (A_1 \hat{\boldsymbol{e}}_1) + \nabla \times (A_2 \hat{\boldsymbol{e}}_2) + \nabla \times (A_3 \hat{\boldsymbol{e}}_3)$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\boldsymbol{e}}_1 & h_2 \hat{\boldsymbol{e}}_2 & h_3 \hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \longrightarrow [\text{直角坐标系}]\text{比较} \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (h_2 h_3 A_1)}{\partial u_1} + \frac{\partial (h_3 h_1 A_2)}{\partial u_2} + \frac{\partial (h_1 h_2 A_3)}{\partial u_3} \right]$$

$$\nabla T = \frac{1}{h_1} \frac{\partial T}{\partial u_1} \hat{\boldsymbol{e}}_1 + \frac{1}{h_2} \frac{\partial T}{\partial u_2} \hat{\boldsymbol{e}}_2 + \frac{1}{h_3} \frac{\partial T}{\partial u_3} \hat{\boldsymbol{e}}_3$$

最后

$$\nabla^2 T = \nabla \cdot (\nabla T) = \nabla \cdot \left(\overbrace{\frac{1}{h_1} \frac{\partial T}{\partial u_1}}^{A_1} \hat{\boldsymbol{e}}_1 + \overbrace{\frac{1}{h_2} \frac{\partial T}{\partial u_2}}^{A_2} \hat{\boldsymbol{e}}_2 + \overbrace{\frac{1}{h_3} \frac{\partial T}{\partial u_3}}^{A_3} \hat{\boldsymbol{e}}_3 \right)$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial T}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial T}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial T}{\partial u_3} \right) \right]$$

三、球坐标系中的梯度、散度、旋度

要求梯度、散度、旋度，关键在于求

$$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}$$

为求 h_i，需要知道：

$$\begin{aligned} x &= \varphi_1(u_1, u_2, u_3) = \varphi_1(r, \theta, \phi) \\ y &= \varphi_2(u_1, u_2, u_3) = \varphi_2(r, \theta, \phi) \\ z &= \varphi_3(u_1, u_2, u_3) = \varphi_3(r, \theta, \phi) \end{aligned}$$

对球坐标系：

$u_1 = r$ 为 P 点到坐标原点的距离，

$u_2 = \theta$ 称为极角(polar angle)，

$u_3 = \phi$ 称为方位角(azimuthal angle)。

对球坐标系，熟知

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta \end{cases} \qquad \text{从而：} \qquad \begin{cases} h_1 = 1 \\ h_2 = r \\ h_3 = r\sin\theta \end{cases}$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$$

$$= \frac{\partial T}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi$$

$$\nabla\cdot\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3A_1)}{\partial u_1} + \frac{\partial(h_3h_1A_2)}{\partial u_2} + \frac{\partial(h_1h_2A_3)}{\partial u_3}\right] \quad \text{代入} \begin{cases} A_1 = A_r \\ A_2 = A_\theta \\ A_3 = A_\phi \end{cases}$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial(r^2A_r)}{\partial r} + r\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right]$$

$$\nabla\times\vec{\boldsymbol{A}} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & r\sin\theta\hat{\boldsymbol{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\hat{\boldsymbol{e}}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{e}}_\phi$$

$$\nabla^2 T = \frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial T}{\partial u_1}\Big)+\frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial T}{\partial u_2}\Big)+\frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial T}{\partial u_3}\Big)\Big]$$

代入 $\begin{cases} u_1 = r,\ u_2=\theta,\ u_3=\phi \\ h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{r^2\sin\theta}\Big[\sin\theta\frac{\partial}{\partial r}\Big(r^2\frac{\partial T}{\partial r}\Big)+\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial T}{\partial\theta}\Big)+\frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\phi^2}\Big]$$

四、柱坐标系中的梯度、散度、旋度

对柱坐标系，有

$\begin{cases} u_1 = s & \text{为 } P \text{ 点到 } z \text{ 轴的距离} \\ u_2 = \phi & \text{同球坐标的 } \phi \\ u_3 = z & \text{同直角坐标的 } z \end{cases}$ 且 $\begin{cases} x = s\cos\phi, \\ y = s\sin\phi, \\ z = z \end{cases}$ 从而 $\begin{cases} h_1 = 1 \\ h_2 = s \\ h_3 = 1 \end{cases}$

利用了 $h_i = \Big[\Big(\frac{\partial x}{\partial u_i}\Big)^2+\Big(\frac{\partial y}{\partial u_i}\Big)^2+\Big(\frac{\partial z}{\partial u_i}\Big)^2\Big]^{1/2}$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{\boldsymbol{e}}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{\boldsymbol{e}}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{\boldsymbol{e}}_3 \quad \text{代入} \begin{cases} u_1 = s, u_2 = \phi, u_3 = z \\ h_1 = 1, h_2 = s, h_3 = 1 \end{cases}$$

$$= \frac{\partial T}{\partial s}\hat{\boldsymbol{e}}_s + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial T}{\partial z}\hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial(h_1 h_2 A_3)}{\partial u_3}\right] \quad \text{代入} \begin{cases} A_1 = A_s \\ A_2 = A_\phi \\ A_3 = A_z \end{cases}$$

$$= \frac{1}{s}\left[\frac{\partial(sA_s)}{\partial s} + \frac{\partial(A_\phi)}{\partial \phi} + s\frac{\partial A_z}{\partial z}\right]$$

$$\nabla \times \vec{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{s}\begin{vmatrix} \hat{\boldsymbol{e}}_s & s\hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$$

$$= \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{e}}_s + \left[\frac{\partial A_s}{\partial z} - \frac{\partial(A_z)}{\partial s}\right]\hat{\boldsymbol{e}}_\phi + \frac{1}{s}\left[\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial \phi}\right]\hat{\boldsymbol{e}}_z$$

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial T}{\partial u_3}\right)\right]$$

代入 $\begin{cases} u_1 = r, u_2 = \theta, u_3 = \phi \\ h_1 = 1, h_2 = r, h_3 = r\sin\theta \end{cases}$

$$= \frac{1}{s}\frac{\partial}{\partial r}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

§1.6 Dirac delta 函数

一、$\vec{\boldsymbol{v}} = \dfrac{\vec{\boldsymbol{r}}}{r^3}$ 的散度

如 $\vec{\boldsymbol{r}} \neq 0$，

$$\begin{aligned} \nabla \cdot \vec{\boldsymbol{v}} &= \nabla \cdot \frac{\vec{\boldsymbol{r}}}{r^3} = \frac{1}{r^3} \nabla \cdot \vec{\boldsymbol{r}} + (\nabla \frac{1}{r^3}) \cdot \vec{\boldsymbol{r}} \\ &= \frac{3}{r^3} - \frac{3}{r^4} (\nabla r) \cdot \vec{\boldsymbol{r}} = 0 \end{aligned}$$

（利用了 $\nabla r = \dfrac{\vec{\boldsymbol{r}}}{r}$）

对 $\vec{\boldsymbol{r}} = 0$点，

$$\begin{aligned} \nabla \cdot \vec{\boldsymbol{v}} &= \lim_{\Delta V \to 0} \frac{\int_{\Delta S} \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{v}}\, d\sigma}{\Delta V} \\ &= \lim_{\Delta V \to 0} \frac{\int_{\Delta S} \hat{e}_r \cdot \frac{\vec{\boldsymbol{r}}}{r^3} r^2 d\Omega}{\Delta V} = \lim_{\Delta V \to 0} \frac{4\pi}{\frac{4\pi}{3} \eta^3} = \infty \end{aligned}$$

取 ΔV 为球心于原点半径为 η 的小球
故 ΔS 为球心于原点半径为 η 的球面

$$\int_{\mathcal{V}} \nabla\cdot\vec{\boldsymbol{v}}\,d\tau = \int_{\Delta V} \nabla\cdot\vec{\boldsymbol{v}}\,d\tau$$

$\mathcal{V}$ 为包围原点的任意体积
ΔV 为球心于原点半径为 η 的小球

$$= \int_{\Delta S} \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}\,d\sigma$$

ΔS 为球心于原点半径为 η 的球面

$$= \int \hat{\boldsymbol{e}}_r\cdot\frac{\vec{\boldsymbol{r}}}{r^3}r^2 d\Omega = 4\pi$$

因此 $w(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\nabla\cdot\vec{\boldsymbol{v}} = \frac{1}{4\pi}\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3}$ 满足下列性质：

$$w(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3} = \begin{cases} 0 & \vec{\boldsymbol{r}}\neq 0 \\ \infty & \vec{\boldsymbol{r}} = 0 \end{cases}$$

$$\int_{\mathcal{V}} w(\vec{\boldsymbol{r}})\,d\tau = 1$$

$\mathcal{V}$ 为包围原点的任意体积

$w(\vec{\boldsymbol{r}})$ 即为 Dirac delta 函数： $w(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3} = \delta(\vec{\boldsymbol{r}})$

数学上称为广义函数 (generalized function) 或分布 (distribution)

1935 Paul Dirac 从物理上引进　　1950 Laurent Schwartz 从数学上证明

二、一维 Dirac delta 函数

$$\delta(x-a) = \left\{ \begin{array}{ll} 0 & x \neq a \\ \infty & x = a \end{array} \right\}$$

且

$$\int_{-\infty}^{+\infty} \delta(x-a)\, dx = \int_{a-\eta}^{a+\eta} \delta(x-a)\, dx = 1$$

$\delta(x)$ 称为 Dirac delta 函数

数学上称为广义函数 (generalized function) 或分布 (distribution)

广义函数专著：Lighthill “Fourier Analysis and Generalized Functions” (Cambridge University 1964)

对任意平滑函数，由于 $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$，故有

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\, dx = \int_{a-\eta}^{a+\eta} f(a)\delta(x-a)\, dx = f(a) \int_{a-\eta}^{a+\eta} \delta(x-a)\, dx = f(a)$$

δ 函数的性质

1. $\delta(-x) = \delta(x)$

2. $\delta'(-x) = -\delta'(x)$

3. $x\delta'(x) = -\delta(x)$

4. $\delta(kx) = \dfrac{1}{|k|}\delta(x)$

5. $f(x)\delta(x-a) = f(a)\delta(x-a), \qquad x\delta(x) = 0$

6. $\delta\left[g(x)\right] = \sum\limits_i \dfrac{\delta(x-x_i)}{|g'(x_i)|},$ 函数 $g(x)$ 只有单重根，$\sum\limits_i$ 对 $g(x)$ 的所有单重根求和

7. $\delta(x) = \dfrac{d\theta(x)}{d\,x}, \qquad \theta(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$ 阶跃函数 (step function)

涉及δ 函数性质的证明，只须证明对任意平滑函数 $f(x)$，下式成立：

$$\int_{-\infty}^{+\infty} f(x)\,(\text{左边})\,dx = \int_{-\infty}^{+\infty} f(x)\,(\text{右边})\,dx \qquad \text{— “Physicist's proofs”}$$

例如，要证明 $x\delta'(x) = -\delta(x)$，只须

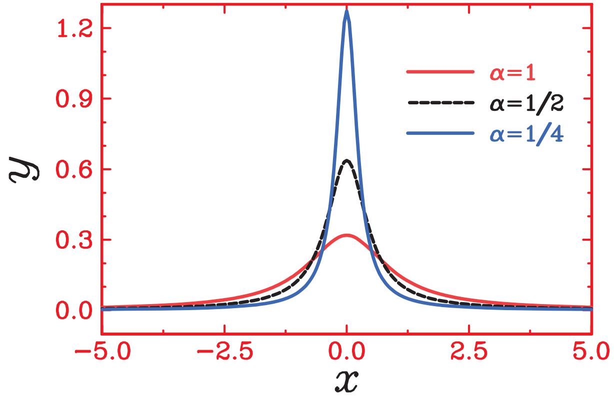
$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)\,(\text{左边})\,dx &= \int_{-\infty}^{+\infty} f(x)\,[x\delta'(x)]\,dx \\
&= \int_{-\infty}^{+\infty} f(x)\,x\,d\,\delta(x) \qquad\qquad \text{利用分部积分} \\
&= \underbrace{\Big[f(x)\,x\,\delta(x)\Big]_{-\infty}^{+\infty}}_{=0\ \text{因为}x\delta(x)=0} - \int_{-\infty}^{+\infty} \delta(x)\,d\,[f(x)x] \\
&= -\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx - \underbrace{\int_{-\infty}^{+\infty} x\delta(x)\,d\,f(x)}_{=0\ \text{因为}x\delta(x)=0} \\
&= \int_{-\infty}^{+\infty} [-\delta(x)] f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,(\text{右边})\,dx
\end{aligned}$$

讨论 δ 函数时，常用到 δ 函数的极限形式 (the limits of sequences of smooth functions)：

$$\delta(x) = \lim_{\alpha\to 0+} \delta_\alpha(x)$$

$\delta_\alpha(x)$ 有许多形式，例如可表为

$$y = \delta_\alpha(x) = \frac{\alpha}{\pi(x^2+\alpha^2)}$$



显然：

$$\delta(x) = \lim_{\alpha\to 0} \delta_\alpha(x) = \begin{cases} \lim\limits_{\alpha\to 0} \dfrac{\alpha}{\pi x^2} = 0 & \text{for } x \neq 0 \\ \lim\limits_{\alpha\to 0} \dfrac{1}{\pi\alpha} = \infty & \text{for } x = 0 \end{cases}$$

且

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(x)\, dx &= \lim_{\alpha\to 0} \int_{-\infty}^{+\infty} \frac{\alpha}{\pi(x^2+\alpha^2)}\, dx \\ &= \lim_{\alpha\to 0} \left[\frac{1}{\pi} \tan^{-1}\frac{x}{\alpha}\right]_{-\infty}^{+\infty} = 1 \end{aligned}$$

δ 函数有许多积分表示，最简单的是表为 Fourier 积分。

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-\alpha|k|}\,dk &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{(ix+\alpha)k}\,dk + \int_{0}^{+\infty} e^{(ix-\alpha)k}\,dk\right)\\ &= \frac{1}{2\pi}\left(\frac{1}{ix+\alpha}-\frac{1}{ix-\alpha}\right) = \frac{\alpha}{\pi(x^2+\alpha^2)} = \delta_\alpha(x)\end{aligned}$$

因此

$$\begin{aligned}\delta_\alpha(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-\alpha|k|}\,dk\\ \delta(x) &= \lim_{\alpha\to 0}\delta_\alpha(x)\\ &= \lim_{\alpha\to 0}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-\alpha|k|}\,dk\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk\end{aligned}$$

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk}$$

三、三维 Dirac delta 函数

$$
\begin{aligned}
\delta(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,') &= \delta(x-x')\delta(y-y')\delta(z-z') && \text{直角坐标}\\
&= \frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') && \text{球坐标}\\
&= \frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') && \text{球坐标}\\
&= \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') && \text{柱坐标}\\
&= \frac{1}{h_1h_2h_3}\delta(u_1-u_1')\delta(u_2-u_2')\delta(u_3-u_3') && \text{正交曲线坐标}
\end{aligned}
$$

积分：

$$
\int_{\mathcal{V}_\infty}\delta(\vec{\boldsymbol{r}}-\vec{\boldsymbol{a}})\,d\tau = 1
$$

$$
\int_{\mathcal{V}_\infty} f(\vec{\boldsymbol{r}})\delta(\vec{\boldsymbol{r}}-\vec{\boldsymbol{a}})\,d\tau = f(\vec{\boldsymbol{a}})
$$

积分表示：

$$\delta(\vec{\boldsymbol{r}}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}\, d^3k$$

常用公式：

$$\nabla\cdot\frac{\vec{\boldsymbol{r}}}{r^3} = -\nabla^2\frac{1}{r} = 4\pi\delta(\vec{\boldsymbol{r}})$$

$$\nabla\cdot\frac{\vec{\boldsymbol{R}}}{R^3} = -\nabla^2\frac{1}{R} = 4\pi\delta(\vec{\boldsymbol{R}}),$$ 其中：$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'$

三、例题

例 1. 试求张量 $\overleftrightarrow{\boldsymbol{W}} = \nabla\nabla\dfrac{1}{r}$ 的第 ij 元：$W_{ij} = \partial_i\partial_j\dfrac{1}{r}$，其中 $\partial_i = \dfrac{\partial}{\partial x_i}$

对 $\vec{\boldsymbol{r}} \neq 0$，已求得

$$\overleftrightarrow{\boldsymbol{W}} = \nabla\nabla\frac{1}{r} = \frac{1}{r^3}\left(\frac{3\vec{\boldsymbol{r}}\vec{\boldsymbol{r}}}{r^2} - \overleftrightarrow{\boldsymbol{I}}\right)$$ 即：$$W_{ij} = \frac{3x_ix_j - r^2\delta_{ij}}{r^5}$$

现求对于包括 $\vec{r}=0$ 情况的 W_{ij}。

由于 $\vec{r}\neq 0$ 时： $$W_{ij}=\frac{3x_ix_j-r^2\delta_{ij}}{r^5}$$

可设一般情况： $$W_{ij}=a\delta_{ij}+b\,x_ix_j$$ a, b 是 r 的函数以保证 $\overleftrightarrow{W}$ 为张量

对重复指标求和： $$W_{ii}=3a+b\,r^2$$ 其中 $\delta_{ii}=3$, $x_i\,x_i=r^2$

从 $\overleftrightarrow{W}$ 的定义知: $$W_{ii}=\partial_i\partial_i\frac{1}{r}=\nabla^2\frac{1}{r}=-4\pi\delta(\vec{r})$$

从而: $$3a+br^2=-4\pi\delta(\vec{r}) \qquad (1)$$

又，对 $\vec{r}\neq 0$ 且 $i\neq j$ $$W_{ij}=\frac{3x_ix_j}{r^5}$$

因此: $$W_{ij}=b\,x_ix_j=\frac{3x_ix_j}{r^5} \qquad (2)$$

联立 (1) 和 (2) 得： $$b=\frac{3}{r^5},\qquad a=-\frac{4\pi}{3}\delta(\vec{r})-\frac{1}{r^3}$$

求得： $$W_{ij}=-\frac{4\pi}{3}\delta(\vec{r})\delta_{ij}+\frac{3x_ix_j-r^2\delta_{ij}}{r^5}$$

例 2. 位于 $\vec{r}=\vec{r}_0$ 处的点电偶极子的电荷密度 $\rho(\vec{r})=-\vec{p}\cdot\nabla\,\delta(\vec{r}-\vec{r}_0)$

点电荷 q 的体电荷密度： $\rho(\vec{r})=q\,\delta(\vec{r}-\vec{a})$ if 点电荷位于$\vec{r}=\vec{a}$ 处

两个点电荷 $+q$ 和 $-q$, 分别位于 $\vec{r}_0+\vec{l}$ 和 $\vec{r}_0$

电偶极矩：
$$\vec{p}=\lim_{\substack{q\to\infty\\ \vec{l}\to 0}} q\,\vec{l} \quad\Longrightarrow\quad \lim_{\substack{q\to\infty\\ \vec{l}\to 0}} q\times(\vec{l}\text{的高阶项})=0 \tag{1}$$

电荷密度：
$$\rho(\vec{r})=q\,\delta[\vec{r}-(\vec{r}_0+\vec{l})]-q\,\delta(\vec{r}-\vec{r}_0)$$
$$=q\left[\delta(\vec{r}-\vec{r}_0-\vec{l})-\delta(\vec{r}-\vec{r}_0)\right] \tag{2}$$

Taylor 展开：
$$f(\vec{r}-\vec{r}_0-\vec{l})=f(\vec{r}-\vec{r}_0)+[\nabla f(\vec{r}-\vec{r}_0)]\cdot(-\vec{l})+\vec{l}\text{的高阶项}$$

故：
$$\delta(\vec{r}-\vec{r}_0-\vec{l})=\delta(\vec{r}-\vec{r}_0)-\vec{l}\cdot[\nabla\delta(\vec{r}-\vec{r}_0)]+\vec{l}\text{的高阶项}$$

代入 (2)
$$\rho(\vec{r})=-q\,\vec{l}\cdot[\nabla\delta(\vec{r}-\vec{r}_0)]+q\,(\vec{l}\text{的高阶项})$$

利用 (1)
$$=-\vec{p}\cdot[\nabla\delta(\vec{r}-\vec{r}_0)]=-[\nabla\delta(\vec{r}-\vec{r}_0)]\cdot\vec{p}$$

对 δ 函数，从数学上看，只有其积分性质才有明确的意义。因此，对 δ 函数的导数（包括梯度），也是仅在计算其积分性质时，才有意义。

例 3. 求：$\displaystyle\int_{-\infty}^{+\infty} f(x)\delta'(x-a)\,dx$ 和 $\displaystyle\int_{\mathcal{V}} \vec{\boldsymbol{g}}(\vec{\boldsymbol{r}})\cdot\nabla\delta(\vec{\boldsymbol{r}}-\vec{\boldsymbol{a}})\,d\tau$

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)\,\delta'(x-a)\,dx &= \int_{-\infty}^{+\infty} f(x)\,d\delta(x-a) \qquad \text{分部积分} \\
&= \Big[f(x)\delta(x-a)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(x-a)\,df(x) \\
&= -\int_{-\infty}^{+\infty} \delta(x-a)\,f'(x)\,dx = -f'(a)
\end{aligned}$$

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta'(x-a)\,dx = -f'(a)}$$

利用： $\nabla\cdot(\vec{g}f) = f(\nabla\cdot\vec{g}) + \vec{g}\cdot(\nabla f)$

$$\int_{\mathcal{V}} \vec{g}(\vec{r})\cdot\nabla\delta(\vec{r}-\vec{a})\,d\tau = \int_{\mathcal{V}}\Big\{\nabla\cdot\big[\,\vec{g}(\vec{r})\delta(\vec{r}-\vec{a})\,\big] - \delta(\vec{r}-\vec{a})\big[\,\nabla\cdot\vec{g}(\vec{r})\,\big]\Big\}\,d\tau$$

利用： $\int_{\mathcal{V}}\nabla\cdot(\vec{g}f)\,d\tau = \int_{\mathcal{S}}\vec{n}\cdot(\vec{g}f)\,d\sigma$

$$= \underbrace{\int_{\mathcal{S}}\vec{n}\cdot\big[\,\vec{g}(\vec{r})\delta(\vec{r}-\vec{a})\,\big]\,d\sigma}_{0} - \int_{\mathcal{V}}\delta(\vec{r}-\vec{a})\big[\,\nabla\cdot\vec{g}(\vec{r})\,\big]\,d\tau$$

$$= -\int_{\mathcal{V}}\delta(\vec{r}-\vec{a})\big[\,\nabla\cdot\vec{g}(\vec{r})\,\big]\,d\tau = -\Big[\nabla\cdot\vec{g}(\vec{r})\Big]_{\vec{r}=\vec{a}}$$

$$\boxed{\int_{\mathcal{V}} \vec{g}(\vec{r})\cdot\nabla\delta(\vec{r}-\vec{a})\,d\tau = -\Big[\nabla\cdot\vec{g}(\vec{r})\Big]_{\vec{r}=\vec{a}}}$$

练习：

$$\boxed{\int_{\mathcal{V}}[\nabla\delta(\vec{r}-\vec{a})]\cdot\overleftrightarrow{t}(\vec{r})\,d\tau = -\Big[\nabla\cdot\overleftrightarrow{t}(\vec{r})\Big]_{\vec{r}=\vec{a}}}$$

$$\text{偶极矩} = \int_{\mathcal{V}}\rho(\vec{r})\,\vec{r}\,d\tau = \int_{\mathcal{V}}\underbrace{\Big\{-\big[\nabla\delta(\vec{r}-\vec{a})\big]\cdot\vec{p}\Big\}}_{\{\ldots\}\text{ 内为 }\rho(\vec{r})}\!\!\underbrace{\vec{p}\,\}\,\vec{r}}_{\text{并矢}}\,d\tau = \Big[\nabla\cdot(\vec{p}\,\vec{r})\Big]_{\vec{r}=\vec{a}} = \vec{p}$$

§ 1.7 矢量场理论

自 Faraday 之后，电磁规律用矢量场 $\vec{\boldsymbol{E}}$，$\vec{\boldsymbol{B}}$ 来描述，而 $\vec{\boldsymbol{E}}$，$\vec{\boldsymbol{B}}$ 的规律则用微分方程描述。后者通常基于矢量场的散度和旋度。

问题一：给定（已知）某矢量场 $\vec{\boldsymbol{F}}$ 的散度和旋度，能否唯一确定 $\vec{\boldsymbol{F}}$

$$\begin{cases} \nabla\times\vec{\boldsymbol{F}}=\vec{\boldsymbol{C}} \\ \nabla\cdot\vec{\boldsymbol{F}}=D \end{cases} \overset{?}{\Longrightarrow} \vec{\boldsymbol{F}}$$

反例：

$$\vec{\boldsymbol{F}}_1=y\hat{\boldsymbol{e}}_x+x\hat{\boldsymbol{e}}_y \qquad \neq \qquad \vec{\boldsymbol{F}}_2=yz\hat{\boldsymbol{e}}_x+zx\hat{\boldsymbol{e}}_y+xy\hat{\boldsymbol{e}}_z$$

$$\nabla\cdot\vec{\boldsymbol{F}}_1=\nabla\cdot\vec{\boldsymbol{F}}_2=0, \qquad \nabla\times\vec{\boldsymbol{F}}_1=\nabla\times\vec{\boldsymbol{F}}_2=0$$

还需要边界（初始）条件

有些矢量场满足： $\nabla \cdot \vec{\boldsymbol{F}}_e = 0$ 称为无散(度)场 (divergence-less or solenoidal fields)

有些矢量场满足： $\nabla \times \vec{\boldsymbol{F}}_l = 0$ 称为无旋(度)场 (curl-less or irrotational fields)

一般矢量场： $\nabla \cdot \vec{\boldsymbol{F}} \neq 0 \qquad \nabla \times \vec{\boldsymbol{F}} \neq 0$

问题二：一般矢量场能否表为一个无散场和一个无旋场之和？

$$\vec{\boldsymbol{F}} \overset{?}{=} \vec{\boldsymbol{F}}_e + \vec{\boldsymbol{F}}_l$$

一、 Helmholtz 定理

在单连通有限区域 $\mathcal{V}$ 内，如果：

1. 矢量场 $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ 的散度 $D(\vec{\boldsymbol{r}})$ 和旋度 $\vec{\boldsymbol{C}}(\vec{\boldsymbol{r}})$ 给定
2. 有限区域 $\mathcal{V}$ 的边界 $\mathcal{S}$ 上，$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ 的法向和切向分量给定

则：$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ 由下式唯一确定

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = -\nabla U + \nabla \times \vec{\boldsymbol{W}}$$

$$U(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{D(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau' - \frac{1}{4\pi}\oint_{\mathcal{S}} \frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\sigma'$$

$$\vec{\boldsymbol{W}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\vec{\boldsymbol{C}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau' - \frac{1}{4\pi}\oint_{\mathcal{S}} \frac{\vec{\boldsymbol{n}}\times\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\sigma'$$

证明：

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \quad = \int_{\mathcal{V}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')\, d\tau'$$ 积分对 $\vec{\boldsymbol{r}}\,'$ 进行

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = \int_{\mathcal{V}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')\, d\tau'$$

利用 $\nabla^2 \dfrac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = -4\pi\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')$

$$= -\frac{1}{4\pi} \int_{\mathcal{V}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\nabla^2 \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'$$

由于 ∇^2 对 $\vec{\boldsymbol{r}}$ 进行，$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')$ 视为常矢量

利用：$\nabla^2(f\vec{\boldsymbol{c}}) = \vec{\boldsymbol{c}}\,\nabla^2 f$

思考：如 $\vec{\boldsymbol{a}}$ 是 $\vec{\boldsymbol{r}}$ 的函数，$\nabla^2(f\vec{\boldsymbol{a}}) =?$

$$= -\frac{1}{4\pi} \int_{\mathcal{V}} \nabla^2 \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'$$

积分对 $\vec{\boldsymbol{r}}\,'$ 进行，∇^2对 $\vec{\boldsymbol{r}}$ 进行，可交换次序

$$= -\frac{1}{4\pi} \nabla^2 \int_{\mathcal{V}} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'$$

利用：$\nabla \times (\nabla \times \vec{\boldsymbol{a}}) = \nabla(\nabla \cdot \vec{\boldsymbol{a}}) - \nabla^2 \vec{\boldsymbol{a}}$

$$= -\frac{1}{4\pi} \nabla \left[\nabla \cdot \int_{\mathcal{V}} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'\right] + \frac{1}{4\pi} \nabla \times \left[\nabla \times \int_{\mathcal{V}} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'\right]$$

$$= -\nabla U + \nabla \times \vec{\boldsymbol{W}}$$

$$U = \frac{1}{4\pi} \left[\nabla \cdot \int_{\mathcal{V}} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'\right], \qquad \vec{\boldsymbol{W}} = \frac{1}{4\pi} \left[\nabla \times \int_{\mathcal{V}} \frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\tau'\right]$$

$$U = \frac{1}{4\pi}\left[\nabla\cdot\int_{\mathcal{V}}\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'\right]$$

积分对 $\vec{\boldsymbol{r}}\,'$ 进行，$\nabla\cdot$ 对 $\vec{\boldsymbol{r}}$ 进行，可交换次序

$$= \frac{1}{4\pi}\int_{\mathcal{V}}\nabla\cdot\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'$$

$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')$ 与 $\vec{\boldsymbol{r}}$ 无关，可利用 $\nabla\cdot(\vec{\boldsymbol{c}}f) = \vec{\boldsymbol{c}}\cdot\nabla f$

$$= \frac{1}{4\pi}\int_{\mathcal{V}}\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\cdot\nabla\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'$$

利用 $\nabla'\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = -\nabla\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}$

$$= -\frac{1}{4\pi}\int_{\mathcal{V}}\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\cdot\nabla'\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'$$

利用 $\nabla'\cdot(\vec{\boldsymbol{a}}f) = f(\nabla'\cdot\vec{\boldsymbol{a}}) + \vec{\boldsymbol{a}}\cdot\nabla' f$

$$= -\frac{1}{4\pi}\int_{\mathcal{V}}\nabla'\cdot\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau' + \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\cdot\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'$$

第一项利用散度定理

$$\oint_{\mathcal{S}}\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}\,d\sigma = \int_{\mathcal{V}}\nabla\cdot\vec{\boldsymbol{v}}\,d\tau$$

$$= -\frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\sigma' + \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\cdot\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,d\tau'$$

$= U$

$$\vec{\boldsymbol{W}} = \frac{1}{4\pi}\left[\nabla\times\int_{\mathcal{V}}\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau'\right]$$

积分对 $\vec{\boldsymbol{r}}\,'$ 进行，$\nabla\times$ 对 $\vec{\boldsymbol{r}}$ 进行，可交换次序

$$= \frac{1}{4\pi}\int_{\mathcal{V}}\nabla\times\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau'$$

$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')$ 与 $\vec{\boldsymbol{r}}$ 无关，有 $\nabla\times(\vec{\boldsymbol{c}}f) = -\vec{\boldsymbol{c}}\times\nabla f$

$$= -\frac{1}{4\pi}\int_{\mathcal{V}}\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\times\nabla\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau'$$

利用 $\nabla'\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = -\nabla\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}$

$$= \frac{1}{4\pi}\int_{\mathcal{V}}\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')\times\nabla'\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau'$$

利用 $\nabla'\times(\vec{\boldsymbol{a}}f) = f(\nabla'\times\vec{\boldsymbol{a}}) - \vec{\boldsymbol{a}}\times\nabla' f$

$$= -\frac{1}{4\pi}\int_{\mathcal{V}}\nabla'\times\frac{\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau' + \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\times\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau'$$

第一项利用旋度定理

$$\oint_{\mathcal{S}}\vec{\boldsymbol{n}}\times\vec{\boldsymbol{v}}\,d\sigma = \int_{\mathcal{V}}\nabla\times\vec{\boldsymbol{v}}\,d\tau$$

$$= -\frac{1}{4\pi}\oint_{\mathcal{S}}\frac{\vec{\boldsymbol{n}}\times\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\sigma' + \frac{1}{4\pi}\int_{\mathcal{V}}\frac{\nabla'\times\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d\tau' \qquad = \vec{\boldsymbol{W}}$$

利用 U 和 $\vec{W}$ 得 Helmholtz 定理：

$$
\begin{aligned}
\vec{F}(\vec{r}) &= -\nabla U + \nabla \times \vec{W} \\
&= -\nabla \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\sigma'\right]}_{U} \\
&\quad + \nabla \times \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\sigma'\right]}_{\vec{W}}
\end{aligned}
$$

故，由 Helmholtz 定理知：

1. 矢量场的散度、旋度加上边界条件，才能唯一确定矢量场
2. 由 $\nabla \times (\nabla U) = 0$ 和 $\nabla \cdot (\nabla \times \vec{W}) = 0$ 知，任意的矢量场均能表为一个无旋场和一个无散场之和

二、Helmholtz 定理的一些结论

$$\begin{aligned}\vec{F}(\vec{r}) &= -\nabla \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,d\sigma'\right]}_{U} \\ &\quad + \nabla \times \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\,d\sigma'\right]}_{\vec{W}}\end{aligned}$$

1. 任意的矢量场 $\vec{F}$ 均可表为一个标量函数 U 的梯度（无旋场）和一个矢量函数 $\vec{W}$ 的旋度（无散场）之和。其中标量函数 U 由 $\vec{F}$ 的散度及 $\vec{F}$ 在边界上的法向分量完全确定，矢量函数 $\vec{W}$ 则由 $\vec{F}$ 的旋度及 $\vec{F}$ 在边界上的切向分量完全确定；

2. 矢量场 $\vec{F}$ 的散度和旋度可视为体源，其在边界上的值则可视为表面源，矢量场 $\vec{F}$ 由体源和表面源共同确定；

3. 如果在某有限区域 $\mathcal{V}$ 内不存在体源，则该区域的矢量场 $\vec{F}$ 完全由 $\vec{F}$ 在边界上的值决定，因此在求解矢量场时可以用所谓“等效源法”，在求解区之外或边界上人为地引进等效源，只要等效源产生的场与矢量场 $\vec{F}$ 在边界上的值相等，则等效源在区域 $\mathcal{V}$ 内产生的场必为 $\vec{F}$。

$$\vec{F}(\vec{r}) = -\nabla \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \cdot \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\sigma'\right]}_{U}$$
$$+ \nabla \times \underbrace{\left[\int_{\mathcal{V}} \frac{\nabla' \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\tau' - \oint_{\mathcal{S}} \frac{\vec{n} \times \vec{F}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} d\sigma'\right]}_{\vec{W}}$$

4. 对无界空间 $\mathcal{V}_\infty$，如果：

(a) 当 $r \to \infty$ 时，$D(\vec{r})$ 和 $\vec{C}(\vec{r})$ 比 $\dfrac{1}{r^2}$ 更快趋向 0

$(\lim_{r\to\infty} r^2 D(\vec{r}) = 0,\ \lim_{r\to\infty} r^2 \vec{C}(\vec{r}) = 0)$， 此条件已保证面积分项为0

(b) 当 $r \to \infty$ 时，$\vec{F}(\vec{r})$ 趋于 0 $(\lim_{r\to\infty} \vec{F}(\vec{r}) = 0)$, 此条件排除 $\vec{F}$ 差一常矢量

则：$\vec{F}(\vec{r})$ 由下式唯一确定

$$\vec{F}(\vec{r}) = -\nabla\left[\frac{1}{4\pi}\int_{\mathcal{V}_\infty} \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\tau'\right] + \nabla \times \left[\frac{1}{4\pi}\int_{\mathcal{V}_\infty} \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\tau'\right]$$

三、势函数存在定理

定理一：对无旋场 $\vec{\boldsymbol{F}}_l$，下面说法等价

1. 场的旋度处处为零，即：$\nabla \times \vec{\boldsymbol{F}}_l = 0$ everywhere
2. $\displaystyle\int_{\boldsymbol{a}\mathcal{P}}^{\boldsymbol{b}} \vec{\boldsymbol{F}}_l \cdot d\vec{\boldsymbol{l}}$ 的值与积分路径 $\mathcal{P}$ 无关
3. 对任意闭合路径都有 $\displaystyle\oint \vec{\boldsymbol{F}}_l \cdot d\vec{\boldsymbol{l}} = 0$
4. $\vec{\boldsymbol{F}}_l$ 可表为某标量函数的梯度，$\vec{\boldsymbol{F}}_l = -\nabla V$，其中 V 称为标（量）势。

标（量）势不是唯一的，可差一常数。

该定理也称为标量势存在定理。

定理二：对无散场 $\vec{\boldsymbol{F}}_e$，下面说法等价

1. 场的散度处处为零，即：$\nabla \cdot \vec{\boldsymbol{F}}_e = 0$ everywhere
2. 给定积分表面 $\mathcal{S}$ 的边界线，面积分 $\displaystyle\int_{\mathcal{S}} \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{F}}_e \, d\sigma$ 的值与积分表面 $\mathcal{S}$ 无关
3. 对任意闭合面都有 $\displaystyle\oint \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{F}}_e \, d\sigma = 0$
4. $\vec{\boldsymbol{F}}_e$ 可表为某矢量函数的旋度，$\vec{\boldsymbol{F}}_e = \nabla \times \vec{\boldsymbol{A}}$，其中 $\vec{\boldsymbol{A}}$ 称为矢（量）势。

矢（量）势不是唯一的，可差一任意标量函数的梯度。

该定理也称为矢量势存在定理。