自由相对论性粒子的运动方程

> 从自由粒子的作用量 (2.21) 出发，导出其三维和四维形式的运动方程，即 (2.24) 和 (2.25) 式，并验证它们都给出同样的运动模式，即匀速直线运动.

自由粒子的作用量 (2.21) 式为

$$S=-mc\int \mathrm{d}s=-mc\int \mathrm{d}\tau\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\right)^{1/2}, \tag{1}$$

其中 $\mathrm{d}s=(\sum_\mu \mathrm{d}x_\mu \mathrm{d}x^\mu)^{1/2}$, 按照书中将其简写为 $\mathrm{d}s=(\mathrm{d}x_\mu \mathrm{d}x^\mu)^{1/2}$

导出其三维形式的运动方程:

首先写出三维形式的拉格朗日量, 注意到 $\mathrm{d}s=\sqrt{c^2\mathrm{d}t^2-\mathrm{d}x^2}=c\mathrm{d}t\sqrt{1-\mathbf{v}^2/c^2}$,

即

$$S=-mc^2\int \mathrm{d}t\sqrt{1-\frac{\mathbf{v}^2}{c^2}}, \tag{2}$$

因此我们得到一个相对论性自由粒子的拉格朗日量为

$$L(\mathbf{v})=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}. \tag{3}$$

粒子的广义动量为

$$\mathbf{p}=\frac{\partial L}{\partial \mathbf{v}}, \tag{4}$$

由 (2.17) 式——

$$\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}=0, \tag{5}$$

此处相应有

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}}&=0,\\ 0-\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}&=0.\end{aligned} \tag{6}$$

粒子的运动方程为

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}=0. \tag{7}$$

导出其四维形式的运动方程:

由 (2.21) 即本文中式 (1) 有

$$
\begin{aligned}
S &= -mc\int \mathrm{d}s \\
&= -mc\int \sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu},
\end{aligned} \tag{8}
$$

对 x_μ 做变分,

$$
\begin{aligned}
\delta S &= -mc\int \delta\sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu} \\
&= -mc\int \delta\sqrt{\eta^{\mu\nu}\mathrm{d}x_\mu \mathrm{d}x_\mu} \\
&= -mc\int \frac{\eta^{\mu\nu}\mathrm{d}\delta x_\mu \mathrm{d}x_\mu + \eta^{\mu\nu}\mathrm{d}x_\mu \mathrm{d}\delta x_\mu}{2\mathrm{d}s} \\
&= -mc\int \frac{\eta^{\mu\nu}\mathrm{d}\delta x_\mu \mathrm{d}x_\mu}{\mathrm{d}s} \\
&= -mc\int \frac{\mathrm{d}\delta x_\mu \mathrm{d}x^\mu}{\mathrm{d}s} \\
&= -mc\int \frac{\mathrm{d}\delta x_\mu \mathrm{d}x^\mu}{\mathrm{d}s} \\
&= -mc\int \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)\mathrm{d}(\delta x_\mu).
\end{aligned} \tag{9}
$$

取边界 a, b 进行分部积分,

$$
\delta S = -mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta x_\mu\Big|_a^b + mc\int_a^b (\delta x_\mu)\mathrm{d}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right). \tag{10}
$$

参考书中第 24 页式 (2.16) 下面对 式 (2.16) 的处理, 在端点处要求 $\delta x_\mu(a) = \delta x_\mu(b) = 0$, 因此上式第一式的贡献是 0.

$$
\delta S = mc\int_a^b \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2}\delta x_\mu \mathrm{d}s, \tag{11}
$$

由于 δx_μ 是任意的变分, 因此要使 $\delta S = 0$ 需有

$$
\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} = 0, \text{for } \mu = 0, 1, 2, 3 \tag{12}
$$

验证它们 (式 (7), 式 (12)) 都给出同样的运动模式:

式 (7) 意味着

$$
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0. \tag{13}
$$

式 (12) 意味着

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \text{Constant}, \text{for } \mu = 0, 1, 2, 3 \tag{14}
$$

也即

$$
\frac{\mathrm{d}s}{\mathrm{d}x^\mu} = \frac{\sqrt{\sum_\mu \mathrm{d}x_\mu \mathrm{d}x^\mu}}{\mathrm{d}x^\mu} = \text{Constant}, \text{for } \mu = 0, 1, 2, 3 \tag{15}
$$

考虑 $\mu=0$, $x^0=ct$, 此时

$$x_0=\eta_{00}x^0=x^0=ct, \tag{16}$$

考虑 $\mu=i$, $i=1,2,3$, 此时

$$x_i=\eta_{ii}x^i=-x^i,\ \text{for}\, i=1,2,3, \tag{17}$$

式 (15) 展开为

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}x^\mu}&=\frac{\sqrt{\mathrm{d}x^0\mathrm{d}x^0-\mathrm{d}x^1\mathrm{d}x^1-\mathrm{d}x^2\mathrm{d}x^2-\mathrm{d}x^3\mathrm{d}x^3}}{\mathrm{d}x^\mu}\\&=\frac{\sqrt{(\mathrm{d}ct)^2-(\mathrm{d}x)^2-(\mathrm{d}y)^2-(\mathrm{d}z)^2}}{\mathrm{d}x^\mu}\end{aligned} \tag{18}$$

考虑 $\mu=0$,

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}ct}&=\frac{\sqrt{(\mathrm{d}ct)^2-(\mathrm{d}x)^2-(\mathrm{d}y)^2-(\mathrm{d}z)^2}}{\mathrm{d}ct}\\&=\frac{1}{c}\sqrt{c^2-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\end{aligned} \tag{19}$$

因此公式 (15) 意味着

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=\text{Contsant},\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=\text{Contsant},\\ \dfrac{\mathrm{d}z}{\mathrm{d}t}=\text{Contsant},\end{cases} \tag{20}$$

也即

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}&=\text{Constant},\\ \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}&=0.\end{aligned} \tag{21}$$

因此两式给出相同的运动模式.

另: 引入固有时 τ,

$$\mathrm{d}s=c\mathrm{d}\tau. \tag{22}$$

(即高显《经典力学》中的 $|\mathrm{d}s|$). 式 (12)

$$\begin{aligned}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2}&=0,\\ \frac{1}{c^2}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}&=0,\\ \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}&=0.\end{aligned} \tag{23}$$

也表示匀速直线运动.

自由相对论性粒子的四动量

> 验证一个自由的相对论性粒子的四动量形式，即 (2.26) 和 (2.27) 式.

验证 (2.26) 式和 (2.27) 式，(2.26) 式如下，

$$p^\mu = mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \left(\frac{E}{c}, \mathbf{p}\right). \tag{24}$$

(2.27) 式如下

$$E = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}. \tag{25}$$

(2.26) 式从对式 (10) 中的边界项的系数得到 $p^\mu = mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$，空间分量即需验证如此得到的动量与

$$\begin{aligned}\mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} \\ &= \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\end{aligned} \tag{26}$$

相同．空间分量即 $\mu = 1,2,3$ 时

$$\begin{aligned}p^\mu &= mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \\ &= mc\frac{\mathrm{d}x^\mu}{\sqrt{(\mathrm{d}ct)^2-(\mathrm{d}x)^2-(\mathrm{d}y)^2-(\mathrm{d}z)^2}} \\ &= mc\frac{\mathrm{d}x^\mu/\mathrm{d}t}{\sqrt{c^2-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}} \\ &= m\frac{v^\mu}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}.\end{aligned} \tag{27}$$

时间分量即 $\mu = 0$ 时，

$$\begin{aligned}p^0 &= mc\frac{\mathrm{d}x^0}{\mathrm{d}s} \\ &= mc\frac{\mathrm{d}x^0}{\sqrt{(\mathrm{d}ct)^2-(\mathrm{d}x)^2-(\mathrm{d}y)^2-(\mathrm{d}z)^2}} \\ &= mc\frac{\mathrm{d}ct/\mathrm{d}t}{\sqrt{c^2-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}} \\ &= \frac{mc}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}.\end{aligned} \tag{28}$$

由于 $\frac{E}{c} = p^0$，因此

$$E = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}. \tag{29}$$

即 (2.27) 式得证. 至此 (2.26) 式时间分量空间分量也得证.

相对论性粒子与标量场的相互作用

> 仿照相对论性自由粒子运动方程的推导,给出一个相对论性粒子与标量外场相互作用的运动方程 (2.31),并根据拉格朗日量 (2.32) 推导等价的三维形式的粒子运动方程.

仿照相对论性自由粒子运动方程的推导,给出一个相对论性粒子与标量外场相互作用的运动方程 (2.31).

$$\begin{aligned}
\delta S &= -mc\int \delta\left(\mathrm{d}s e^{\Phi(x)}\right) \\
&= -mc\int\left(\delta\left(\sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu}\right)e^{\Phi(x)} + \sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu}e^{\Phi(x)}\frac{\partial\Phi(x)}{\partial x_\mu}\delta x_\mu\right) \\
&= -mc\int\left(\underbrace{e^{\Phi(x)}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\mathrm{d}\delta x_\mu}_{\text{分部积分去掉边界项}} + e^{\Phi(x)}\frac{\partial\Phi(x)}{\partial x_\mu}\delta x_\mu \mathrm{d}s\right) \\
&= mc\int\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\Phi(x)}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)\delta x_\mu \mathrm{d}s - e^{\Phi(x)}\frac{\partial\Phi(x)}{\partial x_\mu}\delta x_\mu \mathrm{d}s\right) \\
&= mc\int\left(e^{\Phi(x)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\Phi(x)\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}s} + e^{\Phi(x)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) - e^{\Phi(x)}\frac{\partial\Phi(x)}{\partial x_\mu}\right)\delta x_\mu \mathrm{d}s \\
&= mc\int e^{\Phi(x)}\left(\left(\frac{\partial}{\partial s}\Phi(x) + \frac{\partial}{\partial x^\nu}\Phi(x)\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}s} + \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} - \frac{\partial\Phi(x)}{\partial x_\mu}\right)\delta x_\mu \mathrm{d}s \\
&= mc\int e^{\Phi(x)}\left(\frac{\partial\Phi(x)}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\mu}{\mathrm{d}s} + \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} - \frac{\partial\Phi(x)}{\partial x_\mu}\right)\delta x_\mu \mathrm{d}s \\
&= 0
\end{aligned} \tag{30}$$

因此

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\partial\Phi}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \frac{\partial\Phi(x)}{\partial x_\mu} \tag{31}$$

根据拉格朗日量 (2.32) 推导等价的三维形式的粒子运动方程:

系统的拉格朗日量 (2.32) 为

$$L = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}e^{\Phi(\mathbf{x},t)}, \tag{32}$$

广义动量为

$$\begin{aligned}
\mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} \\
&= \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}e^{\Phi(\mathbf{x},t)},
\end{aligned} \tag{33}$$

拉格朗日方程有

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}}=0,\\
\frac{\partial L}{\partial \mathbf{x}}-\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}=0.
\end{aligned}
\tag{34}
$$

运动方程为

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}e^{\Phi(\mathbf{x},t)}\right)=\frac{\partial L}{\partial \mathbf{x}}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right)e^{\Phi(\mathbf{x},t)}+\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{\mathrm{d}}{\mathrm{d}t}e^{\Phi(\mathbf{x},t)}=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\frac{\partial e^{\Phi(\mathbf{x},t)}}{\partial \mathbf{x}}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right)e^{\Phi(\mathbf{x},t)}+\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}e^{\Phi(\mathbf{x},t)}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathbf{x},t)=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}e^{\Phi(\mathbf{x},t)}\frac{\partial \Phi(\mathbf{x},t)}{\partial \mathbf{x}}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right)+\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathbf{x},t)=-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\frac{\partial \Phi(\mathbf{x},t)}{\partial \mathbf{x}}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right)+\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathbf{x},t)+mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\frac{\partial \Phi(\mathbf{x},t)}{\partial \mathbf{x}}=0
\end{aligned}
\tag{35}
$$

为参考高显《经典力学》(4.41) 以核对上式结果是否正确. 记 $p_i=\frac{mv_i}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$, 上式写成分量形式为

$$
\dot{p}_i+p_i\dot{\Phi}+mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\frac{\partial \Phi}{\partial x_i}=0,
\tag{36}
$$

与高显《经典力学》(4.41) 一致.

6. 相对论性粒子与矢量场的相互作用之四维协变形式

> 仿照相对论性自由粒子运动方程的推导，给出一个相对论性粒子与矢量外场 $A(x)$ 相互作用的四维协变形式的运动方程 (2.37）并验证四维场强张量的表达式 (2.38).

根据题意即从 (2.33) 式得到 (2.37) 式. 作用量为 (2.33) 式如下

$$
S=-mc\int \mathrm{d}s-\frac{e}{c}\int A_\mu(x)\mathrm{d}x^\mu.
\tag{37}
$$

将上式第一项记为 S_{free}, 第二项记为 S_{int}.

$$
\begin{aligned}
\delta S_{\text{int}} &= -\frac{e}{c}\int \delta\left(A_\mu \mathrm{d}x^\mu\right) \\
&= -\frac{e}{c}\int \left[(\delta A_\mu)\mathrm{d}x^\mu + A_\mu \delta\left(\mathrm{d}x^\mu\right)\right] \\
&= -\frac{e}{c}\int \left[\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} + \underbrace{A_\mu \frac{\mathrm{d}\left(\delta x^\mu\right)}{\mathrm{d}s}}_{\text{分部积分去掉边界项}}\right]\mathrm{d}s \\
&= -\frac{e}{c}\int \left[\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{\mathrm{d}A_\mu}{\mathrm{d}s}\delta x^\mu\right]\mathrm{d}s \\
&= -\frac{e}{c}\int \left[\frac{\partial A_\mu}{\partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta x^\nu - \frac{\partial A_\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\delta x^\mu\right]\mathrm{d}s \\
&= -\frac{e}{c}\int \underbrace{\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right)}_{F_{\mu\nu}}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\delta x^\mu \mathrm{d}s
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\delta S_{\text{free}} &= mc\int_a^b \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2}\delta x^\mu \mathrm{d}s \\
\delta S_{\text{int}} &= -\frac{e}{c}\int F_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\delta x^\mu \mathrm{d}s \\
\delta S = 0 &\Longleftrightarrow mc\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} - \frac{e}{c}F_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0
\end{aligned} \tag{39}
$$

相对论性粒子与矢量场的相互作用之三维形式

> 给出一个相对论性粒子与矢量外场 $A(x)$ 相互作用的三维形式的运动方程 (2.39) 并验证其中的电场强度 E 和磁感应强度 B 的确由 (2.40) 式给出.

由作用量相应地有

$$
\begin{aligned}
L_{\text{free}} &= -mc^2\sqrt{1-\mathbf{v}^2/c^2}, \\
L_{\text{int}} &= -e\Phi + \frac{e}{c}\mathbf{v}\cdot\mathbf{A},
\end{aligned} \tag{40}
$$

广义动量为

$$
\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = \gamma m\mathbf{v} + \frac{e}{c}\mathbf{P}. \tag{41}
$$

欧拉-拉格朗日方程为

$$
\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\partial L}{\partial x}. \tag{42}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \frac{e}{c}\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \frac{e}{c}\left(\frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}\right) \\
\frac{\partial L}{\partial \boldsymbol{x}} &= \boldsymbol{\nabla}\left(-e\Phi + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right) = -e\boldsymbol{\nabla}\Phi + \frac{e}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= -e\boldsymbol{\nabla}\Phi + \frac{e}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{e}{c}\left(\frac{\partial\boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}\right) \\
&= \left(-e\boldsymbol{\nabla}\Phi - \frac{e}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + \frac{e}{c}(\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}) \\
&= \left(-e\boldsymbol{\nabla}\Phi - \frac{e}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) + \frac{e}{c}(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})) \\
&= e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v}\times\boldsymbol{B}
\end{aligned} \tag{44}
$$

其中

$$
\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t},\ \boldsymbol{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{45}
$$

前面用到

$$
\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} \tag{46}
$$

下面证明上式.
首先证明三重矢积

$$
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{47}
$$

借助 Levi-Civita 符号，任意两向量叉乘为$\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk}\mathbf{a}_i\mathbf{b}_j\mathbf{e}_k$，

$$
\begin{aligned}
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= \varepsilon_{ijk}A_i(\varepsilon_{lmn}B_lC_m\boldsymbol{e}_n)_j\boldsymbol{e}_k \\
&= \varepsilon_{ijk}\varepsilon_{lmj}A_iB_lC_m\boldsymbol{e}_k \\
&= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})A_iB_lC_m\boldsymbol{e}_k \\
&= \delta_{kl}\delta_{im}A_iB_lC_m\boldsymbol{e}_k - \delta_{km}\delta_{il}A_iB_lC_m\boldsymbol{e}_k \\
&= A_iB_kC_i\boldsymbol{e}_k - A_iB_iC_k\boldsymbol{e}_k \\
&= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}
\end{aligned} \tag{48}
$$

下面借助三重矢积证明式 (46).
以下标 c 标记 $\boldsymbol{\nabla}$ 算符不作用在 $\mathbf{v}$ 上面.

$$
\begin{aligned}
\mathrm{LHS} &= \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) \\
&= \mathbf{v}_c\times(\boldsymbol{\nabla}\times\mathbf{A}) \\
&= \boldsymbol{\nabla}(\boldsymbol{v}_c\cdot\boldsymbol{A}) - (\boldsymbol{v}_c\cdot\boldsymbol{\nabla})\boldsymbol{A}
\end{aligned} \tag{49}
$$

$\mathbf{v}$ 是独立变量, $\boldsymbol{\nabla}$ 算符不作用在 $\mathbf{v}$ 上.

$$
\begin{aligned}
\mathrm{RHS} &= \boldsymbol{\nabla}(\boldsymbol{v}_c\cdot\boldsymbol{A}) - (\boldsymbol{v}_c\cdot\boldsymbol{\nabla})\boldsymbol{A} \\
&= \mathrm{LHS}
\end{aligned} \tag{50}
$$