Lecture 2

Chapter 2. Basic Structures

2.2 Set Operations

Set Identities

- Identity: $A \cup \varnothing = A = A \cap U$
- Domination: $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A$, $A \cap A = A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$,
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- Complement: $A \cup \overline{A} = U$, $A \cap \overline{A} = \varnothing$

DeMorgan's Law for Sets

- Exactly analogous to (and provable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

- To prove statements about sets, of the form $E_1 = E_2$ (where the *E*s are set expressions), here are three useful techniques:

1. Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
2. Use set builder notation & logical equivalences.
3. Use a *membership table*.
4. Use a Venn diagram.

Method 1: Mutual Subsets

Example: Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
  - Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
  - We know that $x \in A$, and either $x \in B$ or $x \in C$.
    - Case 1: $x \in A$ and $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
    - Case 2: $x \in A$ and $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
- Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. (Try it!)

Method 2: Set Builder Notation & Logical Equivalence

- Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin (A \cap B)\} && \text{def. of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{def. of “does not belong”} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{def. of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{De Morgan’s law (logic)} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{def. of “does not belong”} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{def. of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{def. of union} \\
&= \overline{A} \cup \overline{B} && \text{by set builder notation}
\end{aligned}
$$

Method 3: Membership Tables

- Analog to truth tables in propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

- Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
1	1	1	0	0
1	0	1	1	1
0	1	1	0	0
0	0	0	0	0

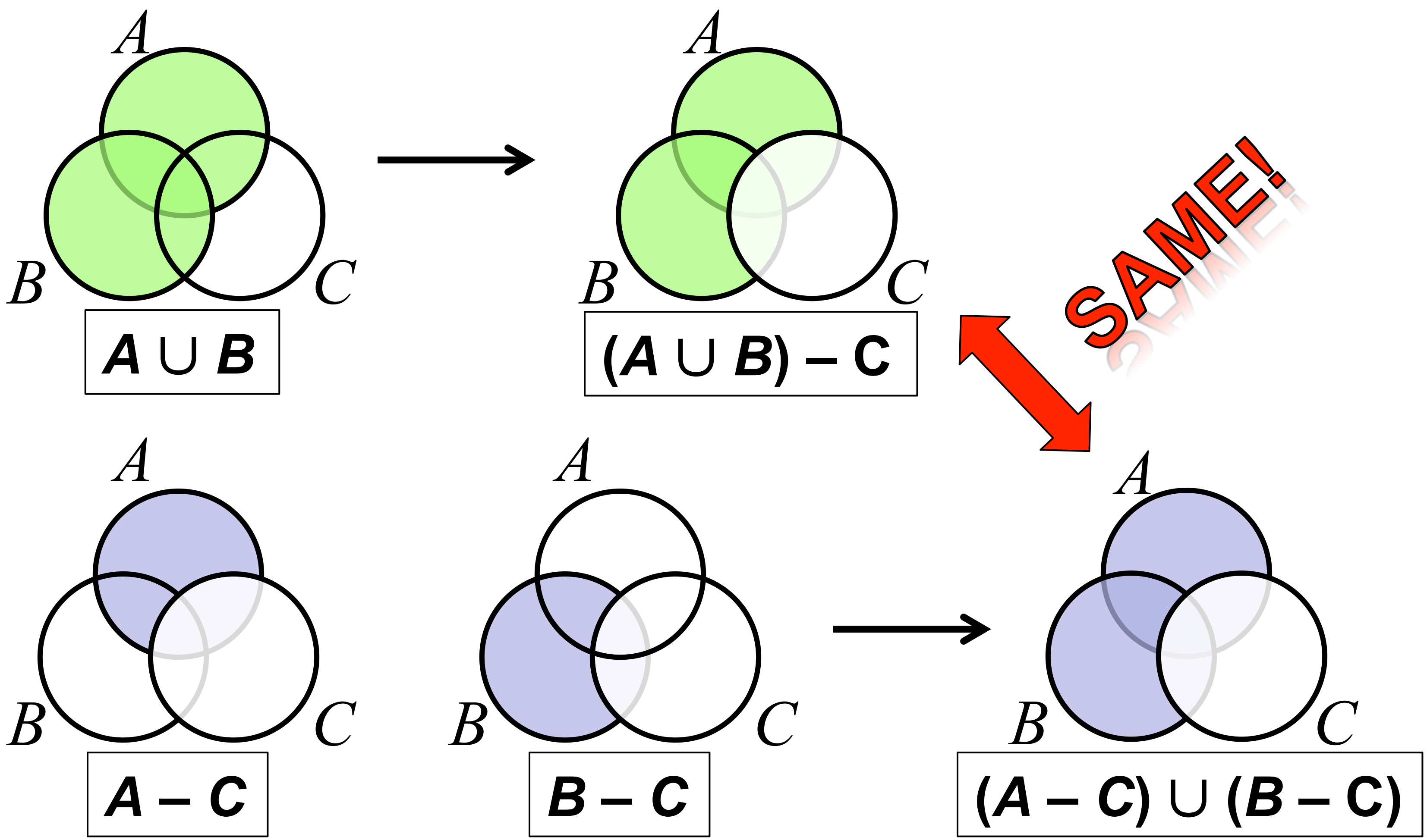
Membership Table Exercise

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A B C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
1 1 1	1	0	0	0	0
1 1 0	1	1	1	1	1
1 0 1	1	0	0	0	0
1 0 0	1	1	1	0	1
0 1 1	1	0	0	0	0
0 1 0	1	1	0	1	1
0 0 1	0	0	0	0	0
0 0 0	0	0	0	0	0

Method 4: Venn Diagram

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$.



Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on pairs of sets A and B to operating on sequences of sets $A_1, \ldots, A_n$, or even on sets of sets, $X = \{A \mid P(A)\}$.

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union:
  $A_1 \cup A_2 \cup \ldots \cup A_n = ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- More generally, union of the sets A_i for $i \in I$:
  $$\bigcup_{i \in I} A_i$$
- For infinite number of sets: $\bigcup_{i=1}^{\infty} A_i$

Generalized Union Examples

- Let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$
$$= \{1,2,3,\ldots\} \cup \{2,3,4,\ldots\} \cup \cdots \cup \{n, n+1, n+2,\ldots\}$$
$$= \{1,2,3,\ldots\}$$

- Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$
$$= \{1\} \cup \{1,2\} \cup \{1,2,3\} \cup \cdots$$
$$= \{1,2,3,\ldots\} = \mathbf{Z}^{+}$$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection:
  $A_1 \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- Generally, intersection of sets A_i for $i \in I$: $\bigcap_{i \in I} A_i$
- For infinite number of sets: $\bigcap_{i=1}^{\infty} A_i$

Generalized Intersection Examples

- Let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$$
$$= \{1,2,3,\ldots\} \cap \{2,3,4,\ldots\} \cap \cdots \cap \{n, n+1, n+2,\ldots\}$$
$$= \{n, n+1, n+2,\ldots\}$$

- Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots$$
$$= \{1\} \cap \{1,2\} \cap \{1,2,3\} \cap \cdots$$
$$= \{1\}$$

Bit String Representation of Sets

- A frequent theme of this course are methods of *representing* one discrete structure using another discrete structure of a different type.

- For an enumerable universal set U with ordering $x_1, x_2, x_3, \ldots$, we can represent a finite set $S \subseteq U$ as the finite bit string B = $b_1 b_2 \ldots b_n$ where $b_i = 1$ if $x_i \in S$ and $b_i = 0$ if $x_i \notin S$.

- *E.g.* $U = \mathbf{N}$, $S = \{2,3,5,7,11\}$, B = 0011 0101 0001.

- In this representation, the set operators "$\cup$", "$\cap$", "$\overline{}$" are implemented directly by bitwise OR, AND, NOT!

Examples of Sets as Bit Strings

- Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order, then

  $S_1 = \{1, 2, 3, 4, 5\} \Rightarrow B_1 = 11\ 1110\ 0000$

  $S_2 = \{1, 3, 5, 7, 9\} \Rightarrow B_2 = 10\ 1010\ 1010$

  - $S_1 \cup S_2 = \{1, 2, 3, 4, 5, 7, 9\}$

    $\Rightarrow$ bit string = 11 1110 1010 $= \boldsymbol{B_1 \vee B_2}$

  - $S_1 \cap S_2 = \{1, 3, 5\}$

    $\Rightarrow$ bit string = 10 1010 0000 $= \boldsymbol{B_1 \wedge B_2}$

  - $\overline{S_1} = \{6, 7, 8, 9, 10\}$

    $\Rightarrow$ bit string = 00 0001 1111 $= \boldsymbol{\neg B_1}$