Lecture 3

Chapter 4. Induction and Recursion

4.3 Recursive Definitions and Structural Induction

Recursive Definitions

- In induction, we *prove* all members of an infinite set satisfy some predicate *P* by:
  - proving the truth of the predicate for larger members in terms of that of smaller members.
- In ***recursive definitions***, we similarly *define* a function, a predicate, a set, or a more complex structure over an infinite domain (universe of discourse) by:
  - defining the function, predicate value, set membership, or structure of larger elements in terms of those of smaller ones.

Recursion

- ***Recursion*** is the general term for the practice of defining an object in terms of *itself*
  - or of part of itself.
  - This may seem circular, but it isn't necessarily.
- An inductive proof establishes the truth of $P(k+1)$ *recursively* in terms of $P(k)$.
- There are also recursive *algorithms*, *definitions*, *functions*, *sequences*, *sets*, and other structures.

Recursively Defined Functions

- Simplest case: One way to define a function f:$\mathbf{N}\rightarrow S$ (for any set S) or series $a_n = f(n)$ is to:
  - Define $f(0)$
  - For $n > 0$, define $f(n)$ in terms of $f(0),\ldots,f(n-1)$
- **Example**: Define the series $a_n = 2^n$ where n is a nonnegative integer recursively:
  - a_n looks like $2^0, 2^1, 2^2, 2^3,\ldots$
  - Let $a_0 = 1$
  - For $n > 0$, let $a_n = 2\cdot a_{n-1}$

Another Example

- Suppose we define $f(n)$ for all $n \in \mathbf{N}$ recursively by:
  - Let $f(0) = 3$
  - For all $n > 0$, let $f(n) = 2 \cdot f(n-1) + 3$

- What are the values of the following?
  - $f(1) =$ $2 \cdot f(0) + 3 = 2 \cdot 3 + 3 = 9$
  - $f(2) =$ $2 \cdot f(1) + 3 = 2 \cdot 9 + 3 = 21$
  - $f(3) =$ $2 \cdot f(2) + 3 = 2 \cdot 21 + 3 = 45$
  - $f(4) =$ $2 \cdot f(3) + 3 = 2 \cdot 45 + 3 = 93$

Recursive Definition of Factorial

- Give an inductive (recursive) definition of the factorial function,

  $$F(n) = n! = \prod_{1 \le i \le n} i = 1 \cdot 2 \cdots n$$

  - Basis step: $F(1) = 1$
  - Recursive step: $F(n) = n \cdot F(n-1)$ for $n > 1$
    - $F(2) = 2 \cdot F(1) = 2 \cdot 1 = 2$
    - $F(3) = 3 \cdot F(2) = 3 \cdot \{2 \cdot F(1)\} = 3 \cdot 2 \cdot 1 = 6$
    - $F(4) = 4 \cdot F(3) = 4 \cdot \{3 \cdot F(2)\} = 4 \cdot \{3 \cdot 2 \cdot F(1)\}$

      $= 4 \cdot 3 \cdot 2 \cdot 1 = 24$

The Fibonacci Numbers

- The ***Fibonacci numbers*** $f_{n\geq 0}$ is a famous series defined by:

$$f_0 = 0, \quad f_1 = 1, \quad f_{n\geq 2} = f_{n-1} + f_{n-2}$$