Lecture 2

Chapter 2. Basic Structures

2.3 Functions

Some Function Terminology

- If it is written that $f: A \rightarrow B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then we say:
  - A is the ***domain*** of f
  - B is the ***codomain*** of f
  - b is the ***image*** of a under f
    - a can not have more than 1 image
  - a is a ***pre-image*** of b under f
    - b may have more than 1 pre-image
  - The ***range*** $R \subseteq B$ of f is $R = \{b \mid \exists a\ f(a) = b\}$

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain: Example

- Suppose I declare that: "*f* is a function mapping students in this class to the set of grades {A, B, C, D, F}."
- At this point, you know *f* 's codomain is: {A, B, C, D, F}, and its range is unknown!
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A, B}, but its codomain is still {A, B, C, D, F}!

Function Operators

- $+$, $\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply two *real-valued functions* f,g: $\mathbf{R} \to \mathbf{R}$:
  - $(f + g)$: $\mathbf{R} \to \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
  - (fg): $\mathbf{R} \to \mathbf{R}$, where $(fg)(x) = f(x)g(x)$
- Example 6:
  Let f and g be functions from **R** to **R** such that $f(x) = x^2$ and $g(x) = x - x^2$.
  What are the functions $f + g$ and fg?

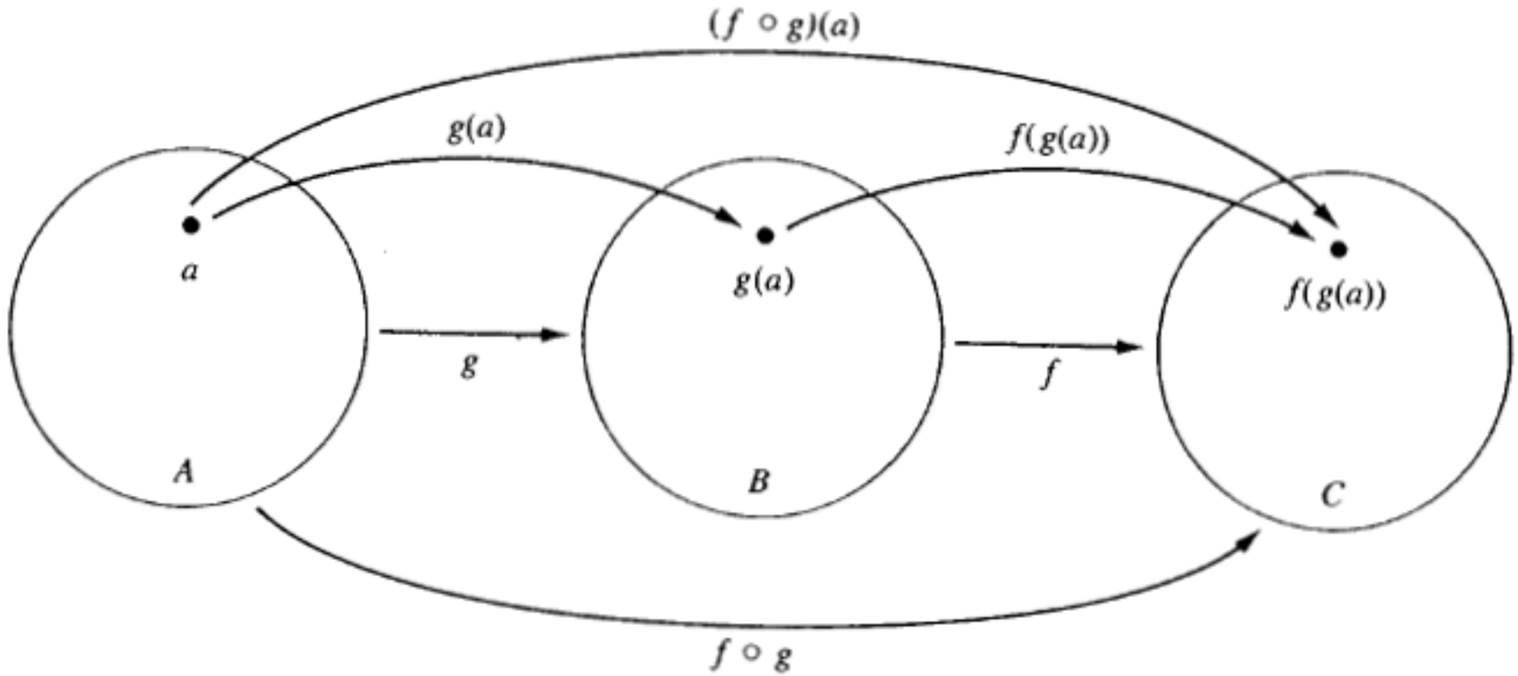
Function Composition Operator

Note the match here. It's necessary!

- For functions $g: A \rightarrow B$ and $f: B \rightarrow C$, there is a special operator called ***compose*** ("$\circ$").
  - It <u>composes</u> (creates) a new function from f and g by applying f to the result of applying g.
  - We say $(f \circ g): A \rightarrow C$, where $(f \circ g)(a) = f(g(a))$.
  - Note: $f \circ g$ cannot be defined unless range of g is a subset of the domain of f.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ is non-commuting. (Like Cartesian $\times$, but unlike $+$, $\wedge$, $\cup$) (Generally, $f \circ g \neq g \circ f$.)

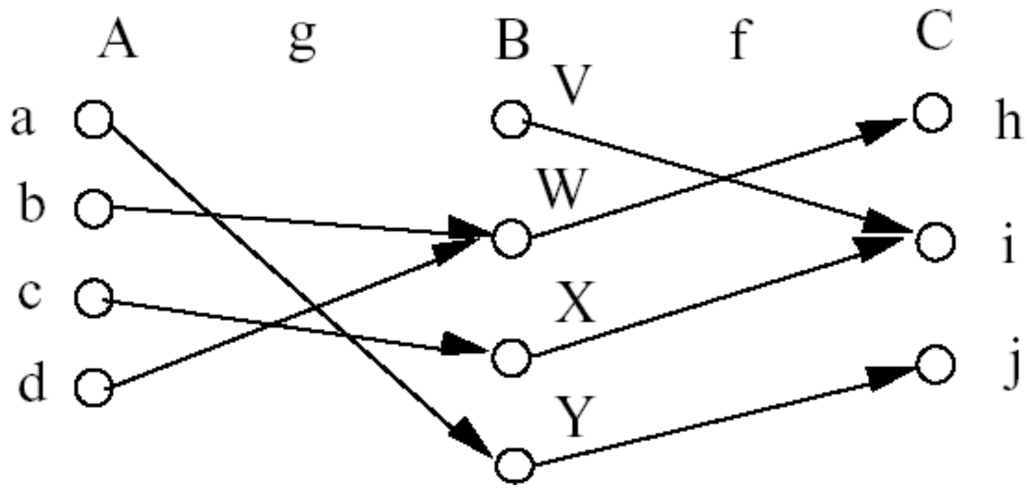
Function Composition Illustration

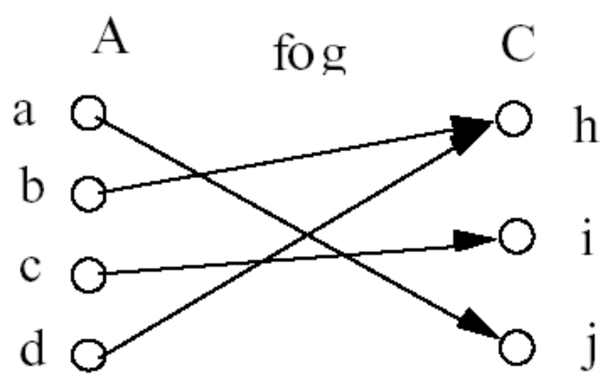
- $g: A \rightarrow B,\ f: B \rightarrow C$



Function Composition: Example

- g: $A \rightarrow B$, $f : B \rightarrow C$





Function Composition: Example

- Example 20: Let g: $\{a, b, c\} \to \{a, b, c\}$ such that
  $$g(a) = b,\ g(b) = c,\ g(c) = a.$$
  Let f : $\{a, b, c\} \to \{1, 2, 3\}$ such that
  $$f(a) = 3,\ f(b) = 2,\ f(c) = 1.$$
  What is the composition of f and g, and what is the composition of g and f ?
  - $f \circ g$: $\{a, b, c\} \to \{1, 2, 3\}$ such that
    $$(f \circ g)(a) = 2,\ (f \circ g)(b) = 1,\ (f \circ g)(c) = 3.$$
  - $g \circ f$ is not defined (why?)

Function Composition: Example

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is the composition of *f* and *g*, and what is the composition of *g* and *f* ?
  - $(f \circ g)(x) = f(g(x))$
    $= f(2x+1)$
    $= (2x+1)^2$
  - $(g \circ f)(x) = g(f(x))$
    $= g(x^2)$
    $= 2x^2 + 1$

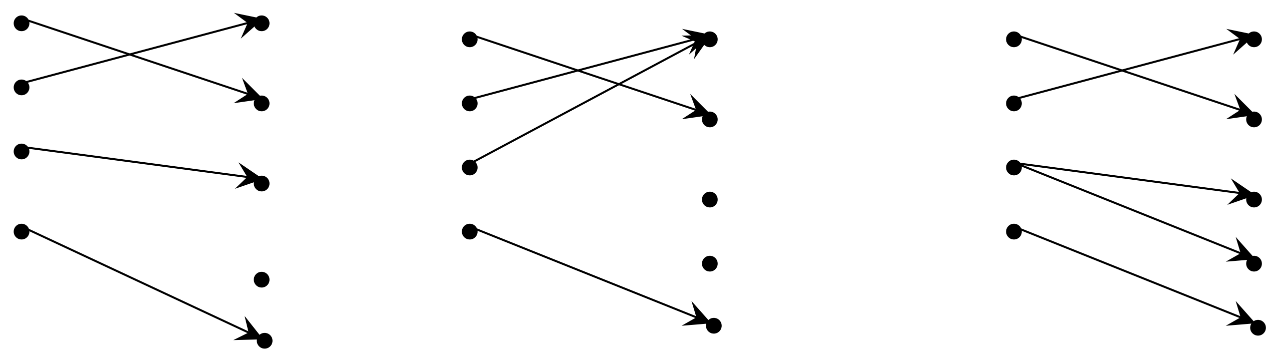
Note that $f \circ g \neq g \circ f$. $(4x^2+4x+1 \neq 2x^2+1)$

One-to-One Functions

- A function f is ***one-to-one*** (*1–1*), or ***injective***, or an ***injection***, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f (i.e. every element of its range has *only* 1 pre-image).
  - Formally, given $f : A \rightarrow B$,
    "f is injective": $\forall a,b\ (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a,b\ (a \neq b \rightarrow f(a) \neq f(b))$
- Only one element of the domain is mapped to any given one element of the range.
  - Domain & range have the same cardinality. What about codomain?

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one Not one-to-one Not even a function!

- <u>Example 8</u>:
  Is the function $f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ one-to-one?
- <u>Example 9</u>:
  Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ such that $f(x) = x^2$. Is f one-to-one?

Sufficient Conditions for 1–1ness

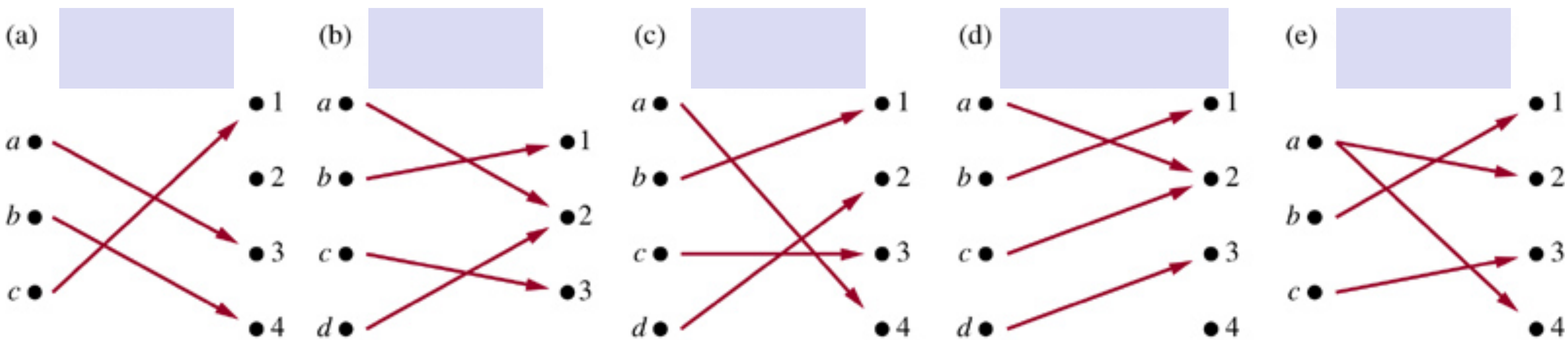
- For functions f over numbers, we say:
  - f is ***strictly*** (or ***monotonically***) ***increasing*** iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
  - f is ***strictly*** (or ***monotonically***) ***decreasing*** iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one.
  - *E.g.* x^3

Onto (Surjective) Functions

- A function $f : A \rightarrow B$ is ***onto*** or ***surjective*** or a ***surjection*** iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$ ($\forall b \in B, \exists a \in A: f(a) = b$) (i.e. its range is equal to its codomain).
- Think: An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



- Example13: Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

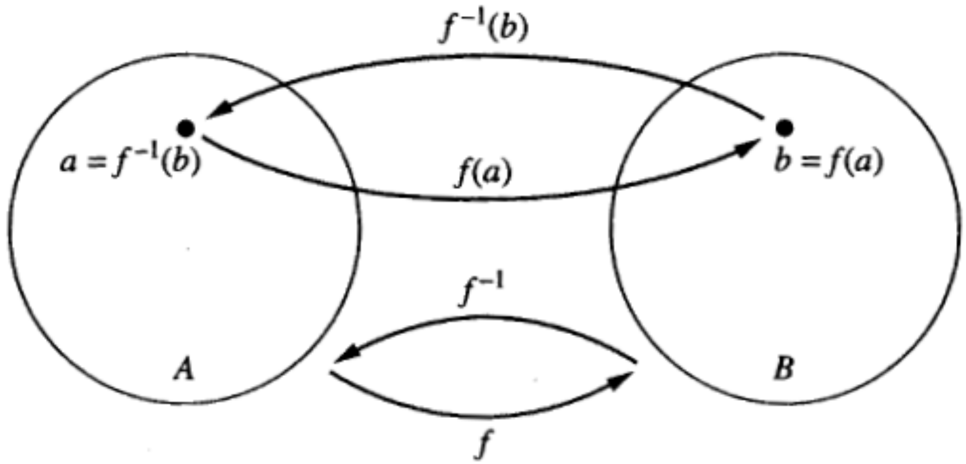
Bijections and Inverse Function

- A function f is said to be *a* ***one-to-one correspondence***, or *a* ***bijection***, or *reversible*, or *invertible*, iff it is both one-to-one and onto.

- Let $f : A \rightarrow B$ be a bijection.
  The ***inverse function*** of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.
  The inverse function of f is denoted by $f^{-1}: B \rightarrow A$.
  Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Function Illustration

- Let f: $A \rightarrow B$ be a bijection



- Example 16: Let f : {a, b, c} $\rightarrow$ {1, 2, 3} such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible, and if it is, what is its inverse? Yes. $f^{-1}(1) = c$, $f^{-1}(2) = a$, $f^{-1}(3) = b$
- Example 18: Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible? No. f is not a one-to-one function. So it's not invertible.