Vectors

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Objectives

After reading this chapter, you should be able to:

1. *define a vector,*
2. *add and subtract vectors,*
3. *find linear combinations of vectors and their relationship to a set of equations,*
4. *explain what it means to have a linearly independent set of vectors, and*
5. *find the rank of a set of vectors.*

What is a vector?

A vector is a collection of numbers in a definite order. If it is a collection of n numbers it is called a n-dimensional vector. So the vector $\vec{A}$ given by

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Is a n-dimensional column vector with n components, $a_1, a_2, \ldots\ldots, a_n$. The above is a column vector. A row vector [B] is of the form $\vec{B} = [b_1, b_2, \ldots., b_n]$ where $\vec{B}$ is a n-dimensional row vector with n components $b_1, b_2, \ldots., b_n$

Example 1

Give an example of a 3-dimensional column vector.

Solution

Assume a point in space is given by its (x,y,z) coordinates. Then if the value of $x = 3, y = 2, z = 5$ the column vector corresponding to the location of the points is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

When are two vectors equal?

Two vectors $\vec{A}$ and $\vec{B}$ are equal if they are of the same dimension and if their corresponding components are equal.

Given

$$\vec{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then $\vec{A} = \vec{B}$ if $a_i = b_i$, $i = 1,2,......,n$

Example 2

What are the values of the unknown components in $\vec{B}$ if

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} b_1 \\ 3 \\ 4 \\ b_4 \end{bmatrix}$$

and $\vec{A} = \vec{B}$

Example 2 (cont.)

Solution

$$b_1 = 2, b_4 = 1$$

How do you add two vectors?

Two vectors can be added only if they are of the same dimension and the addition is given by

$$[A]+[B]=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}+\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

Example 3

Add the two vectors

$$\vec{A} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\vec{B} = \begin{bmatrix} 5 \\ -2 \\ 3 \\ 7 \end{bmatrix}$$

Example 3 (cont.)

Solution

$$\vec{A}+\vec{B}=\begin{bmatrix}2\\3\\4\\1\end{bmatrix}+\begin{bmatrix}5\\-2\\3\\7\end{bmatrix}$$

$$=\begin{bmatrix}2+5\\3-2\\4+3\\1+7\end{bmatrix}$$

$$=\begin{bmatrix}7\\1\\7\\8\end{bmatrix}$$

Example 4

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A}_1 = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

where the rows represent the three brands of tires sold – Tirestone, Michigan and Copper respectively. In quarter 2, the sales are given by

$$\vec{A}_2 = \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix}$$

What is the total sale of each brand of tire in the first half of the year?

Example 4 (cont.)

Solution

The total sales would be given by

$$\vec{C} = \vec{A}_1 + \vec{A}_2$$

$$= \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 25+20 \\ 5+10 \\ 6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 45 \\ 15 \\ 12 \end{bmatrix}$$

So the number of Tirestone tires sold is 45, Michigan is 15 and Copper is 12 in the first half of the year.

What is a null vector?

A null vector is where all the components of the vector are zero.

Example 5

Give an example of a null vector or zero vector.

Solution

The vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Is an example of a zero or null vector

What is a unit vector?

A unit vector $\vec{U}$ is defined as

$$\vec{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

where

$$\sqrt{u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2} = 1$$

Example 6

Give examples of 3-dimensional unit column vectors.

Solution

Examples include

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ etc.}$$

How do you multiply a vector by a scalar?

If k is a scalar and $\vec{A}$ is a n-dimensional vector, then

$$k\vec{A} = k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$$

Example 7

What is $2\vec{A}$ if

$$\vec{A} = \begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

Example 7 (cont.)

Solution

$$2\vec{A} = 2\begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 25 \\ 2 \times 20 \\ 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 50 \\ 40 \\ 10 \end{bmatrix}$$

Example 8

A store sells three brands of tires: Tirestone, Michigan and Copper. In quarter 1, the sales are given by the column vector

$$\vec{A} = \begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

If the goal is to increase the sales of all tires by at least 25% in the next quarter, how many of each brand should be sold?

Example 8 (cont.)

Solution

Since the goal is to increase the sales by 25%, one would multiply the $\vec{A}$ vector by 1.25,

$$\vec{B} = 1.25\begin{bmatrix} 25 \\ 25 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 31.25 \\ 31.25 \\ 7.5 \end{bmatrix}$$

Since the number of tires must be an integer, we can say that the goal of sales is

$$\vec{B} = \begin{bmatrix} 32 \\ 32 \\ 8 \end{bmatrix}$$

Key Terms:

Vector
Addition of vectors
Rank
Dot Product
Subtraction of vectors
Unit vector
Scalar multiplication of vectors
Null vector