Lecture 3

Chapter 5. Counting

5.1 The Basics of Counting

5.2 The Pigeonhole Principle

5.3 Permutations and Combinations

Review

- **<u>Sum Rule</u>:** If a task can be done in one of n_1 ways, or in one of n_2 ways, …, or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \le i < j \le m$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_m$.
- **<u>Product Rule</u>:** Suppose that a procedure can be broken down into a sequence of m successive tasks. If the task T_1 can be done in n_1 ways; the task T_2 can then be done in n_2 ways; ...; and the task T_m can be done in n_m ways, then there are $n_1 \cdot n_2 \cdots n_m$ ways to do the procedure.

The Product Rule: Example

- What is the value of k after the following code has been executed?

$k := 0$

for $i_1 := 1$ **to** n_1

  for $i_2 := 1$ **to** n_2

    …

    for $i_m := 1$ **to** n_m

      $k := k + 1$

$n_1 \cdot n_2 \cdots n_m$

The Product Rule: Example

- How many functions are there from a set with *m* elements to one with *n* elements?

$$n^m$$

- How many one-to-one functions are there from a set with *m* elements to one with *n* elements?

$$n \cdot (n-1)(n-2) \cdots (n-m+1)$$

- More examples in the textbook

IP Address Example

- In version 4 of the Internet Protocol (IPv4)
  - Internet address is a string of 32 bits
  - Network number (*netid*) + host number (*hostid*)
  - Valid computer addresses are in one of 3 types:
    - A ***class A*** IP address consists of 0, followed by a 7-bit "netid" $\neq \mathbf{1}^7$, and a 24-bit "hostid"
    - A ***class B*** address has 10, followed by a 14-bit netid and a 16-bit hostid.
    - A ***class C*** address has 110, followed by a 21-bit netid and an 8-bit hostid.
  - The 3 classes have distinct headers (0, 10, 110)
  - Hostids that are all 0s or all 1s are not allowed.

IP Address Example

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- How many valid computer addresses are there?

IP Address Solution

- # of addresses

  = (# class A) + (# class B) + (# class C)

  (by sum rule)
- # class A = (# valid netids)×(# valid hostids)

  (by product rule)
- # valid class A netids = $2^7 - 1 = 127$.
- # valid class A hostids = $2^{24} - 2 = 16{,}777{,}214$.
- Continuing in this fashion we find the answer is:

  3,737,091,842 (3.7 billion IP addresses)

Set Theoretic Version

- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then:
  - The ways to do either task 1 or 2 are $A \cup B$, and $|A \cup B| = |A| + |B|$
  - The ways to do both task 1 and 2 can be represented as $A \times B$, and $|A \times B| = |A| \cdot |B|$

Inclusion-Exclusion Principle

- Suppose that k out of m ways of doing task 1 <u>also</u> simultaneously accomplish task 2.
  - And there are also n ways of doing task 2.
- Then, the number of ways to accomplish "Do either task 1 or task 2" is $m + n - k$.
- Set theory: If A and B are not disjoint, then $|A \cup B| = |A| + |B| - |A \cap B|$.
  - If they are disjoint, this simplifies to $|A| + |B|$.

Inclusion-Exclusion Example

- Some hypothetical rules for passwords:
  - Passwords must be 2 characters long
  - Each character must be a letter a ~ z, a digit 0 ~ 9, or one of the 10 punctuation characters ! @ # $ % ^ & * ( )
  - Each password must contain *at least one* digit or punctuation character

Setup of Problem

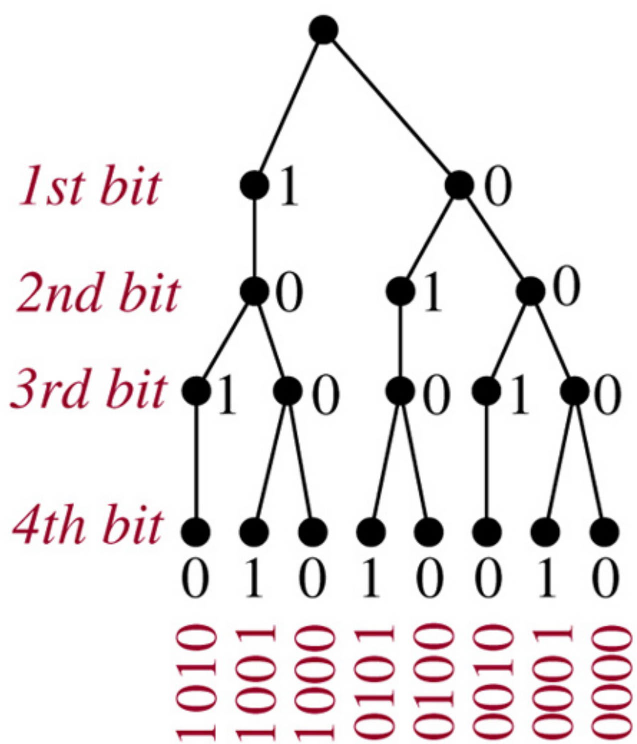
- A legal password has a digit or punctuation character in position 1 **or** position 2.
  - These cases overlap, so the principle applies.
- # of passwords with OK symbol in position #1 $= (10 + 10) \times (10 + 10 + 26) = 20 \times 46 = 920$
- # with OK symbol in pos. #2 $= 46 \times 20 = 920$
- # with OK symbol both places $= 20 \times 20 = 400$
- Answer: $920 + 920 - 400 = 1{,}440$

Tree Diagrams

- A tree diagram can be used in many different counting problems.
- To use trees in counting, we use a branch to represent each possible choice.
- We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

Tree Diagrams: Example

- How many bit strings of length four do not have two consecutive 1s?



The Pigeonhole Principle

- A.k.a. the "Dirichlet drawer principle" or the "Shoe Box Principle".
- If $k + 1$ or more objects are assigned to k places, then at least 1 place must be assigned 2 or more objects.
- In terms of the assignment function:
  - If $f: A \rightarrow B$ and $|A| \geq |B| + 1$, then some element of B has more than two preimages under f.
    - I.e., f is not one-to-one.

Pigeonhole Principle: Example

- There are 101 possible numeric grades (0% ~ 100%) rounded to the nearest integer.
  - Also, there are >101 students in a class.
- Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - i.e., the function from students to rounded grades is *not* a one-to-one function.

Another Example of P.P.

- 10 persons have first names as Alice, Bernare, and Charles, and last names as Lee, McDuff, and Ng. Show that at least two persons have the same first and last names.
- Solution:
  - 9 possible names for the 10 persons → 10 pigeons and 9 pigeonholes.
  - Assignment of names to people = assignment of pigeonholes to the pigeons
  - By the Pigeonhole Principle, some name (pigeonhole) is assigned to at least two persons (pigeons).

Generalized Pigeonhole Principle

- If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.
- *E.g.*, there are $N = 280$ students in a class. There are $k = 52$ weeks in the year.
  - Therefore, there must be at least 1 week during which at least $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the class have a birthday.

G.P.P. Example I

- Given: There are 280 students in a class.
  - Without knowing anybody's birthday, what is the largest value of *n* for which we can prove using the G.P.P. that at least *n* students must have been born in the same month?
- Answer:

$$\lceil 280/12 \rceil = \lceil 23.3 \rceil = 24$$

G.P.P. Example II

- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?
  - Phone #: NXX-NXX-XXXX
    - N: 2 ~ 9 and X: any digit
- Solution
  - NXX-XXXX: $(8 \cdot 10 \cdot 10) \cdot (10 \cdot 10 \cdot 10 \cdot 10) = 8$ million
  - By G.P.P. at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ phones have the identical numbers
  - Hence, at least 4 area codes are required