Lecture 3

Chapter 5. Counting

5.3 Permutations and Combinations

5.4 Binomial Coefficients

Permutations

- A ***permutation*** of a set S of distinct elements is an **ordered** sequence that contains each element in S exactly once.
  - E.g. $\{A, B, C\} \rightarrow$ six permutations:

    ABC, ACB, BAC, BCA, CAB, CBA
- An **ordered** arrangement of r distinct elements of S is called an ***r-permutation** of S*.
- The number of r-permutations of a set with $n = |S|$ elements is

$$\boxed{P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}, \quad 0 \le r \le n.}$$

- $P(n,n) = n!/(n-n)! = n!/0! = n!$ (Note: $0! = 1$)

Permutation Examples

- **Example**: Let $S = \{1, 2, 3\}$.
  - The arrangement 3, 1, 2 is a permutation of S ($3! = 6$ ways)
  - The arrangement 3, 2 is a 2-permutation of S ($3 \cdot 2 = 3!/1! = 6$ ways)
- **Example**: There is an armed nuclear bomb planted in your city, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?

$P(10,3) = 10 \times 9 \times 8 = 720$,
so there is a 1 in 720 chance that you'll survive!

More Permutation Examples

- **Example 6**: Suppose that a sales woman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

> First city is determined, and the remaining seven can be ordered arbitrarily: $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

- **Example 7**: How many permutations of the letters ABCDEFGH contain the string ABC?

> ABC must occur as a block, i.e. consider it as one object
> Then, it'll be the number of permutations of six objects (ABC, D, E, F, G, H), which is $6! = 720$

Combinations

- An ***r-combination*** of elements of a set S is an **unordered** selection of r elements from the set. Thus, an r-combination is simply a subset $T \subseteq S$ with r members, $|T| = r$.
- **Example**: S = {1, 2, 3, 4}, then {1, 3, 4} is a 3-combination from S
- **Example**: How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.
  - Notation: $C(n, r)$ or $\binom{n}{r}$, where $n = 52$ and $r = 7$

Combinations

- The number of *r*-combinations of a set with $n = |S|$ elements is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

- Note that $C(n, r) = C(n, n-r)$
  - Because choosing the *r* members of *T* is the same thing as choosing the $(n - r)$ non-members of *T*.

$$C(n,n-r) = \binom{n}{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}$$

Combination Example I

- How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.

- Answer:

$C(52, 7) = P(52, 7) / P(7, 7) = 52! / (7!\cdot 45!)$

$= (52\cdot \cancel{51}\cdot \cancel{50}\cdot \cancel{49}\cdot \cancel{48}\cdot 47\cdot 46) / (\cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)$

$17 \quad 10 \quad 7 \quad \cancel{8}$

$\cancel{2}$

$52\cdot 17\cdot 10\cdot 7\cdot 47\cdot 46 = 133{,}784{,}560$

Combination Example II

- $C(4, 3) = 4$, since, for example, the 3-combinations of a set {1, 2, 3, 4} are {1, 2, 3}, {1, 3, 4}, {2, 3, 4}, {1, 2, 4}.
  - $C(4, 3) = P(4, 3) / P(3, 3) = 4! / (3! \times 1!)$
    $= (4 \times 3 \times 2) / (3 \times 2 \times 1) = 4$
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?
  - $C(6, 3) = 6! / (3! \times 3!)$
    $= (6 \times 5 \times 4) / (3 \times 2 \times 1) = 20$
  - There are 20 different groups to be picked

Combination Example III

- A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?
  - $C(8, 6) \times C(7, 5)$

$$= \{P(8, 6) / P(6, 6)\} \times \{P(7, 5) / P(5, 5)\}$$

$$= \{8!/ (2! \times 6!)\} \times \{7!/ (2! \times 5!)\}$$

$$= \{(8 \times 7) / 2!\} \times \{(7 \times 6) / 2!\}$$

$$= 28 \times 21$$

$$= 588$$