Chapter 1. The Foundations

1.1 Propositional Logic

Review: The Implication Operator

- The conditional statement (a.k.a. ***implication***) $p \rightarrow q$ states that p implies q.
- *I.e.*, If p is true, then q is true; but if p is not true, then q could be either true or false.
- *E.g.*, let p = "You study hard."
  q = "You will get a good grade."
  $p \rightarrow q$ = "If you study hard, then you will get a good grade." (else, it could go either way)
  - p: *hypothesis* or *antecedent* or *premise*
  - q: *conclusion* or *consequence*

Review: Implication Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	**F**
F	T	T
F	F	T

The <u>only</u> False case!

- $p \rightarrow q$ is **false** <u>only</u> when p is true but q is **not** true.
- $p \rightarrow q$ does **not** require that p or q **<u>are ever true</u>**!
  - *E.g.* "(1=0) $\rightarrow$ pigs can fly" is TRUE!

Examples of Implications

- "If this lecture ever ends, then the sun will rise tomorrow." *True* or *False*? (T → T)
- "If 1+1=6, then Obama is president." *True* or *False*? (F → T)
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*? (F → F)
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*? (T → F)

English Phrases Meaning $p \to q$

- "p implies q"
- "if p, then q"
- "if p, q"
- "when p, q"
- "whenever p, q"
- "q if p"
- "q when p"
- "q whenever p"
- "p only if q"
- "p is sufficient for q"
- "q is necessary for p"
- "q follows from p"
- "q is implied by p"

We will see some equivalent logic expressions later.

Converse, Inverse, Contrapositive

- Some terminology, for an implication $p \rightarrow q$:
- Its ***converse*** is: $q \rightarrow p$.
- Its ***inverse*** is: $\neg p \rightarrow \neg q$.
- Its ***contrapositive***: $\neg q \rightarrow \neg p$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which?

Contrapositive

Examples

- p: Today is Easter
  q: Tomorrow is Monday

- $p \to q$:
  If today is Easter then tomorrow is Monday.
- ***Converse***: $q \to p$
  If tomorrow is Monday then today is Easter.
- ***Inverse***: $\neg p \to \neg q$
  If today is not Easter then tomorrow is not Monday.
- ***Contrapositive***: $\neg q \to \neg p$
  If tomorrow is not Monday then today is not Easter.

The Biconditional Operator

- The ***biconditional*** statement $p \leftrightarrow q$ states that p ***if and only if*** (*iff*) q.
- p = "It is below freezing."
  q = "It is snowing."
  $p \leftrightarrow q$ = "It is below freezing if and only if it is snowing."
  or
  = "That it is below freezing is necessary and sufficient for it to be snowing"

Biconditional Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- p is necessary and sufficient for q
- If p then q, and conversely
- p iff q

- $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.
- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- $p \leftrightarrow q$ does **not** imply that p and q are true.
- Note this truth table is the exact **opposite** of $\oplus$'s! Thus, $p \leftrightarrow q$ means $\neg(p \oplus q)$.

Boolean Operations Summary

- Conjunction: $p \wedge q$, (read p and q), "discrete math is a required course and I am a computer science major".
- Disjunction: , $p \vee q$, (read p or q), "discrete math is a required course or I am a computer science major".
- Exclusive or: $p \oplus q$, "discrete math is a required course or I am a computer science major but not both".
- Implication: $p \rightarrow q$, "if discrete math is a required course then I am a computer science major".
- Biconditional: $p \leftrightarrow q$, "discrete math is a required course if and only if I am a computer science major".

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators. What are they? Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

- For an implication $p \rightarrow q$
- Its ***converse*** is: $q \rightarrow p$
- Its ***inverse*** is: $\neg p \rightarrow \neg q$
- Its ***contrapositive***: $\neg q \rightarrow \neg p$

Compound Propositions

- A ***propositional variable*** is a variable such as p, q, r (possibly subscripted, e.g. p_j) over the Boolean domain.
- An ***atomic proposition*** is either Boolean constant or a propositional variable: e.g. T, F, p
- A ***compound proposition*** is derived from atomic propositions by application of propositional operators: e.g. $\neg p$, $p \vee q$, $(p \vee \neg q) \rightarrow q$
- Precedence of logical operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- Precedence also can be indicated by parentheses.
  - e.g. $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$

An Exercise

- Any compound proposition can be evaluated by a truth table
- $(p \vee \neg q) \rightarrow q$

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Translating English Sentences

- Let p = "It rained last night",
  q = "The sprinklers came on last night,"
  r = "The lawn was wet this morning."

  Translate each of the following into English:

  $\neg p$ = "It didn't rain last night."

  $r \wedge \neg p$ = "The lawn was wet this morning, and it didn't rain last night."

  $\neg r \vee p \vee q$ = "The lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

Another Example

- Find the converse of the following statement.
  - "Raining tomorrow is a sufficient condition for my not going to town."
- **Step 1**: Assign propositional variables to component propositions.
  - p: It will rain tomorrow
  - q: I will not go to town
- **Step 2**: Symbolize the assertion: $p \rightarrow q$
- **Step 3**: Symbolize the converse: $q \rightarrow p$
- **Step 4**: Convert the symbols back into words.
  - "If I don't go to town then it will rain tomorrow" or
  - "Raining tomorrow is a *necessary condition* for my not going to town."

Logic and Bit Operations

- A ***bit*** is a binary (base 2) digit: 0 or 1.
- Bits may be used to represent truth values.
  - By convention:
    0 represents "False"; 1 represents "True".
- A ***bit string of length n*** is an ordered sequence of $n \geq 0$ bits.
- By convention, bit strings are (sometimes) written left to right:
  - e.g. the "first" bit of the bit string "1001101010" is 1.
  - What is the length of the above bit string?

Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.

- Example:

01 1011 0110	
11 0001 1101	
11 1011 1111	Bit-wise OR
01 0001 0100	Bit-wise AND
10 1010 1011	Bit-wise XOR

End of 1.1

You have learned about:

- Propositions: what they are
- Propositional logic operators'
  - symbolic notations, truth tables, English equivalents, logical meaning
- Atomic vs. compound propositions
- Bits, bit strings, and bit operations

- Next section:
  - Propositional equivalences
  - Equivalence laws
  - Proving propositional equivalences