Discrete Structure for Computer Science

based on slides by Dr. Baek and Dr. Still
Originals by Dr. M. P. Frank and Dr. J.L. Gross
Provided by McGraw-Hill

Lecture 3

Chapter 4. Induction and Recursion

4.1 Mathematical Induction

Mathematical Induction

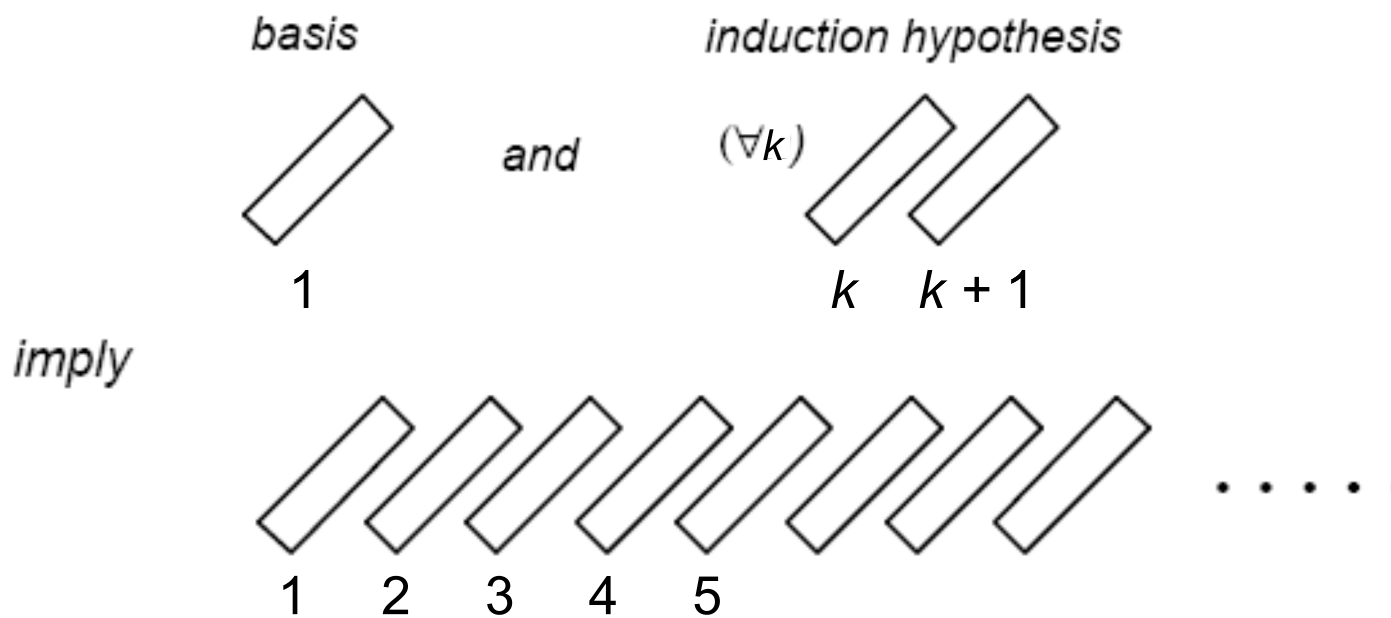
- A powerful, rigorous technique for proving that a statement $P(n)$ is true for ***every*** positive integers n, no matter how large.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

$$\begin{array}{l} P(1) \\ \forall k \geq 1\ [P(k) \rightarrow P(k+1)] \\ \hline \therefore \forall n \geq 1\ P(n) \end{array}$$

"The First Principle of Mathematical Induction"

The "Domino Effect"

- **Premise #1:** Domino #1 falls.
- **Premise #2:** For every $k \in \mathbf{Z}^+$, if domino #k falls, then so does domino #k+1.
- **Conclusion:** All of the dominoes fall down!



Note: this works even if there are infinitely many dominoes!

Mathematical Induction Recap.

- **PRINCIPLE OF MATHEMATICAL INDUCTION:**

  To prove that a statement $P(n)$ is true for all positive integers n, we complete two steps:

  - **BASIS STEP**: Verify that $P(1)$ is true
  - **INDUCTIVE STEP**: Show that the conditional statement $\underbrace{P(k)}_{\text{Inductive Hypothesis}} \rightarrow P(k+1)$ is true for all positive integers k

Validity of Induction

Proof: that $\forall n \geq 1\ P(n)$ is a valid consequent:

Given any $k \geq 1$, the 2nd premise

$\forall k \geq 1\ (P(k) \rightarrow P(k+1))$ trivially implies that

$(P(1) \rightarrow P(2)) \wedge (P(2) \rightarrow P(3)) \wedge \ldots \wedge (P(n-1) \rightarrow P(n))$.

Repeatedly applying the hypothetical syllogism rule to adjacent implications in this list $n - 1$ times then gives us $P(1) \rightarrow P(n)$; which together with $P(1)$ (premise #1) and *modus ponens* gives us $P(n)$.

Thus $\forall n \geq 1\ P(n)$. ■

Outline of an Inductive Proof

- Let us say we want to prove $\forall n \in \mathbf{Z}^+ \; P(n)$.
  - Do the ***base case*** (or ***basis step***):
    Prove $P(1)$.
  - Do the ***inductive step***:
    Prove $\forall k \in \mathbf{Z}^+ \; P(k) \rightarrow P(k+1)$.
    - *E.g.* you could use a direct proof, as follows:
    - Let $k \in \mathbf{Z}^+$, assume $P(k)$. (*inductive hypothesis*)
    - Now, under this assumption, prove $P(k+1)$.
  - The inductive inference rule then gives us
    $\forall n \in \mathbf{Z}^+ \; P(n)$.

Induction Example

- Show that, for $n \geq 1$

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

- Proof by induction
  - $P(n)$: the sum of the first n positive integers is $n(n+1)/2$, i.e. $P(n)$ is
  - ***Basis step***: Let $n = 1$. The sum of the first positive integer is 1, i.e. $P(1)$ is true.

$$1 = \frac{1(1+1)}{2}$$

Example *(cont.)*

- ***Inductive step***: Prove $\forall k \geq 1: P(k) \rightarrow P(k+1)$.
  - Inductive Hypothesis, $P(k)$:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

  - Let $k \geq 1$, assume $P(k)$, and prove $P(k+1)$, i.e.

$$\underbrace{1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}}_{P(k+1)}$$

This is what you have to prove

Example *(cont.)*

By inductive hypothesis P(k)

- ***Inductive step*** continues…

$$1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$=\frac{k^2+3k+2}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

P(k+1)

- Therefore, by the principle of mathematical induction $P(n)$ is true for all integers n with $n \geq 1$