Chapter 1. The Foundations

1.3 Predicates and Quantifiers

Previously…

- In predicate logic, a ***predicate*** is modeled as a ***proposional function*** $\boldsymbol{P(\cdot)}$ from subjects to propositions.
  - $P(x)$: "x is a prime number" (x: any subject)
  - $P(3)$: "3 is a prime number." (proposition!)
- Propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take
  - $P(x,y,z)$: "***x*** gave ***y*** the grade ***z***"
  - P(Mike,Mary,A): "**Mike** gave **Mary** the grade **A**."

Universe of Discourse (U.D.)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* objects at once.
- e.g., let $P(x)$ = "$x + 1 > x$". We can then say, "For ***any*** number x, $P(x)$ is true" instead of $(\mathbf{0} + 1 > \mathbf{0}) \wedge (\mathbf{1} + 1 > \mathbf{1}) \wedge (\mathbf{2} + 1 > \mathbf{2}) \wedge \ldots$

- The collection of values that a variable x can take is called x's ***universe of discourse*** or the ***domain of discourse*** (often just referred to as the ***domain***).

Quantifier Expressions

- ***Quantifiers*** provide a notation that allows us to *quantify* (count) *how many* objects in the universe of discourse satisfy the given predicate.
- "∀" is the FOR∀LL or ***universal*** quantifier.
  $\forall x\, P(x)$ means *for all* x in the domain, $P(x)$.
- "∃" is the ∃XISTS or ***existential*** quantifier.
  $\exists x\, P(x)$ means there *exists* an x in the domain (that is, 1 or more) such that $P(x)$.

The Universal Quantifier $\forall$

- ***$\forall x\, P(x)$: For all x in the domain, P(x).***
- $\forall x\, P(x)$ is
  - *true* if $P(x)$ is true for every x in D (*D: domain of discourse*)
  - *false* if $P(x)$ is false for at least one x in D
    - For every real number x, $x^2 \geq 0$ TRUE
    - For every real number x, $x^2 - 1 > 0$ FALSE
- A ***counterexample*** to the statement $\forall x\, P(x)$ is a value x in the domain D that makes $P(x)$ false
- What is the truth value of $\forall x\, P(x)$ when the domain is empty? TRUE

The Universal Quantifier $\forall$

- If all the elements in the domain can be listed as x_1, $x_2, \ldots, x_n$ then, $\forall x\, P(x)$ is the same as the conjunction:

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

- Example: Let the domain of x be parking spaces at UH. Let $P(x)$ be the statement "x is full." Then the ***universal quantification*** of $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at UH are full."
  - or "Every parking space at UH is full."
  - or "For each parking space at UH, that space is full."

The Existential Quantifier $\exists$

- ***$\exists x\, P(x)$: There exists an x in the domain (that is, 1 or more) such that $P(x)$.***
- $\exists x\, P(x)$ is
  - *true* if $P(x)$ is true for at least one x in the domain
  - *false* if $P(x)$ is false for every x in the domain
- What is the truth value of $\exists x\, P(x)$ when the domain is empty? FALSE
- If all the elements in the domain can be listed as $x_1, x_2, \ldots, x_n$ then, $\exists x\, P(x)$ is the same as the disjunction:

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

The Existential Quantifier $\exists$

- Example:
  Let the domain of x be parking spaces at UH.
  Let $P(x)$ be the statement "x is full."
  Then the ***existential quantification*** of $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking spaces at UH are full."
  - or "There is a parking space at UH that is full."
  - or "At least one parking space at UH is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a ***free variable x*** (meaning, x is undefined).
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and ***binds*** one or more of those variables, to produce an expression having one or more ***bound variables***.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable y, and one bound variable x. [Which is which?]
- "$P(x)$, where $x = 3$" is another way to bind x.
- An expression with <u>zero</u> free variables is a bona-fide (actual) proposition.
- An expression with <u>one or more</u> free variables is not a proposition:

  e.g. $\forall x\, P(x,y) = Q(y)$

Quantifiers with Restricted Domains

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for
    "For all x that are greater than zero, $P(x)$."
    $= \forall x\,(x > 0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for
    "There is an x greater than zero such that $P(x)$."
    $= \exists x\,(x > 0 \wedge P(x))$

Translating from English

- Express the statement "*Every student in this class has studied calculus*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has studied calculus."
  - If domain for x consists of the students in this class, then
  - it can be translated as $\forall x\, C(x)$

or

  - If domain for x consists of all people
  - Let $S(x)$ be the predicate: "x is in this class"
  - Translation: $\forall x\, (S(x) \rightarrow C(x))$

Translating from English

- Express the statement "*Some students in this class has visited Mexico*" using predicates and quantifiers.
  - Let $M(x)$ be the statement: "x has visited Mexico"
  - If domain for x consists of the students in this class, then
  - it can be translated as $\exists x\, M(x)$

  or
  - If domain for x consists of all people
  - Let $S(x)$ be the statement: "x is in this class"
  - Then, the translation is $\exists x\, (S(x) \wedge M(x))$

Translating from English

- Express the statement "*Every student in this class has visited either Canada or Mexico*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has visited Canada" and $M(x)$ be the statement: "x has visited Mexico"
  - If domain for x consists of the students in this class, then
  - it can be translated as $\forall x\,(C(x) \vee M(x))$

Negations of Quantifiers

- $\forall x\, P(x)$: "Every student in the class has taken a course in calculus" ($P(x)$: "x has taken a course in calculus")
  - "Not every student in the class … calculus"

    $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

- Consider $\exists x\, P(x)$: "There is a student in the class who has taken a course in calculus"
  - "There is no student in the class who has taken a course in calculus"

    $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negations of Quantifiers

- Definitions of quantifiers: If the domain = $\{a, b, c, \ldots\}$
  - $\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge \cdots$
  - $\exists x\, P(x) \equiv P(a) \vee P(b) \vee P(c) \vee \cdots$
- From those, we can prove the laws:
  - $$\begin{aligned} \neg\forall x\, P(x) &\equiv \neg(P(a) \wedge P(b) \wedge P(c) \wedge \cdots) \\ &\equiv \neg P(a) \vee \neg P(b) \vee \neg P(c) \vee \cdots \\ &\equiv \exists x\, \neg P(x) \end{aligned}$$
  - $$\begin{aligned} \neg\exists x\, P(x) &\equiv \neg(P(a) \vee P(b) \vee P(c) \vee \cdots) \\ &\equiv \neg P(a) \wedge \neg P(b) \wedge \neg P(c) \wedge \cdots \\ &\equiv \forall x\, \neg P(x) \end{aligned}$$
- Which *propositional* equivalence law was used to prove this?

DeMorgan's

Negations of Quantifiers

Theorem:

- **Generalized De Morgan's laws for logic**

1. $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

2. $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

Negations: Examples

- What are the negations of the statements $\forall x\,(x^2 > x)$ and $\exists x\,(x^2 = 2)$?
  - $\neg\forall x\,(x^2 > x) \equiv \exists x\,\neg(x^2 > x) \equiv \exists x\,(x^2 \leq x)$
  - $\neg\exists x\,(x^2 = 2) \equiv \forall x\,\neg(x^2 = 2) \equiv \forall x\,(x^2 \neq 2)$

- Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.
  - $\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x\,\neg(P(x) \rightarrow Q(x))$

$$\equiv \exists x\,\neg(\neg P(x) \vee Q(x))$$

$$\equiv \exists x\,(P(x) \wedge \neg Q(x))$$

Summary

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.