Discrete Structure for Computer Science

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Provided by McGraw-Hill

Lecture 2

Chapter 2. Basic Structures

2.1 Sets

2.2 Set Operations

Sets so far…

- Today
  - $\in$ relational operator, and the empty set $\varnothing$
  - Venn diagrams
  - Set relations $=$, $\subseteq$, $\subseteq$, $\subset$, $\supset$, $\not\subset$, etc.
  - Cardinality $|S|$ of a set S
  - Power sets $P(S)$
  - Cartesian product $S \times T$
  - Set operators: $\cup$, $\cap$, -

Basic Set Relations: Member of

- $x \in S$ ("x is in S") is the proposition that object x is an *∈lement* or *member* of set S.
  - *e.g.* $3 \in \mathbf{N}$,

    $a \in \{x \mid x \text{ is a letter of the alphabet}\}$
  - Can define set equality in terms of $\in$ relation:

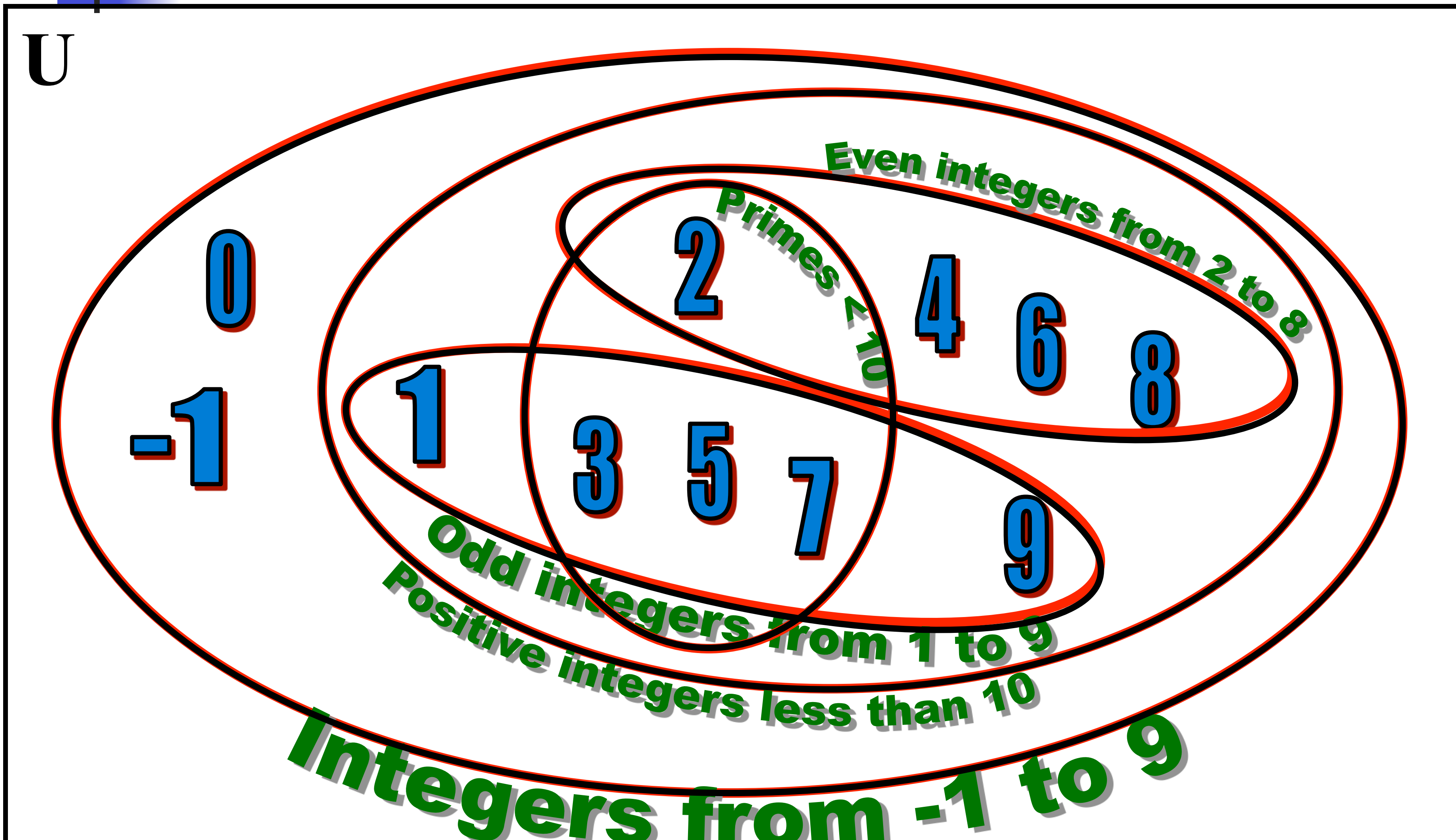
    $\forall S,T\colon S = T \leftrightarrow [\forall x\ (x \in S \leftrightarrow x \in T)]$

    "Two sets are equal iff they have all the same members."
- $x \notin S \equiv \neg(x \in S)$ "x is not in S"

The Empty Set

- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\ \} = \{x \mid \textbf{False}\}$
- No matter the domain of discourse, we have the axiom $\neg\exists x\colon x\in\varnothing$.
- $\{\ \} \neq \{\varnothing\} = \{\ \{\ \}\ \}$
  - $\{\varnothing\}$ it isn't empty because it has $\varnothing$ as a member!

Venn Diagrams



Subset and Superset

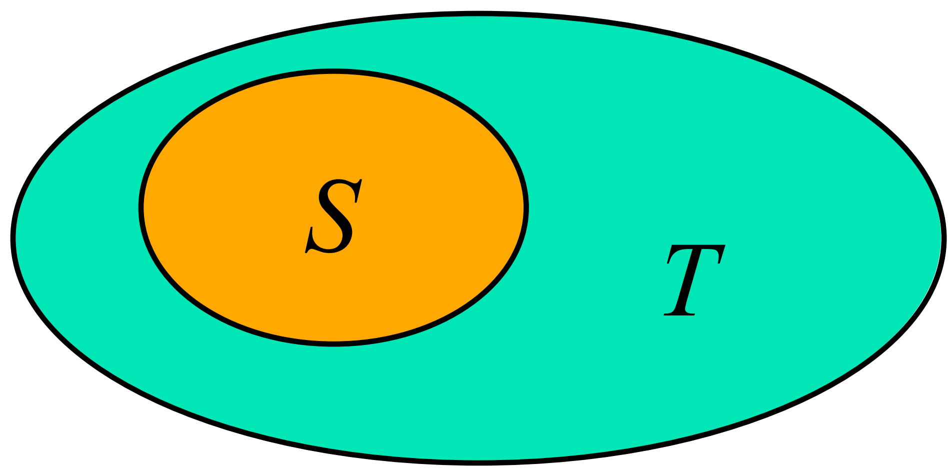
- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
- $S \subseteq T \equiv \forall x\,(x \in S \rightarrow x \in T)$
- $\varnothing \subseteq S$, $S \subseteq S$
- $S \subseteq T$ ("S is a superset of T") means $T \subseteq S$
- Note $(S = T) \equiv (S \subseteq T \wedge T \subseteq S)$

  $\equiv \forall x(x \in S \rightarrow x \in T) \wedge \forall x(x \in T \rightarrow x \in S)$

  $\equiv \forall x(x \in S \leftrightarrow x \in T)$
- $S \nsubseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ but $T \not\subseteq S$. Similar for $S \supset T$.
- Example:
  $\{1, 2\} \subset \{1, 2, 3\}$



Venn Diagram of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- Example:

  Let $S = \{x \mid x \subseteq \{1, 2, 3\}\}$

  then $S = \{ \emptyset,$

  $\{1\}, \{2\}, \{3\},$

  $\{1, 2\}, \{1, 3\}, \{2, 3\},$

  $\{1, 2, 3\} \}$

- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality and Finiteness

- $|S|$ (read "the *cardinality* of S") is a measure of how many <u>different</u> elements S has.
- *E.g.*, $|\varnothing| = 0$, $|\{1, 2, 3\}| = 3$, $|\{a, b\}| = 2$,

  $|\{\{1, 2, 3\}, \{4, 5\}\}| =$ 2

- If $|S| \in \mathbf{N}$, then we say S is *finite*. Otherwise, we say S is *infinite*.
- What are some infinite sets we've seen?

  N, Z, Q, R

The *Power Set* Operation

- The ***power set*** $P(S)$ of a set S is the set of all subsets of S. $P(S) = \{x \mid x \subseteq S\}$.
- Examples
  - $P(\{a, b\}) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}$
  - $S = \{0, 1, 2\}$

    $P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
  - $P(\emptyset) = \{\emptyset\}$
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- Note that for finite S, $|P(S)| = 2^{|S|}$.
- It turns out $\forall S\ (|P(S)| > |S|)$, *e.g.* $|P(\mathbf{N})| > |\mathbf{N}|$.

Ordered *n*-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence* or *list of length n* is written $(a_1, a_2, \ldots, a_n)$. Its *first* element is a_1, its second element is a_2, *etc.*
- Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.

Contrast with sets' {}

- Empty sequence, singlets, pairs, triples, quadruples, quintuples, …, *n*-tuples.

Cartesian Products of Sets

- For sets A and B, their ***Cartesian product*** denoted by $\boldsymbol{A \times B}$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

  $$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}.$$

  - *E.g.* $\{a, b\} \times \{1, 2\}$

    $= \{ (a, 1), (a, 2), (b, 1), (b, 2) \}$
- Note that for finite A, B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is ***not*** commutative: *i.e.*, $\neg \forall A,B\ (A \times B = B \times A)$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$

  $= \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$

The Union Operator

- For sets *A* and *B*, their ***union*** $A\cup B$ is the set containing all elements that are either in *A*, **or** ("v") in *B* (or, of course, in both).
- Formally, $\forall A,B: A\cup B = \{x \mid x\in A \vee x\in B\}$.
- Note that $A\cup B$ is a **superset** of both *A* and *B* (in fact, it is the smallest such superset):

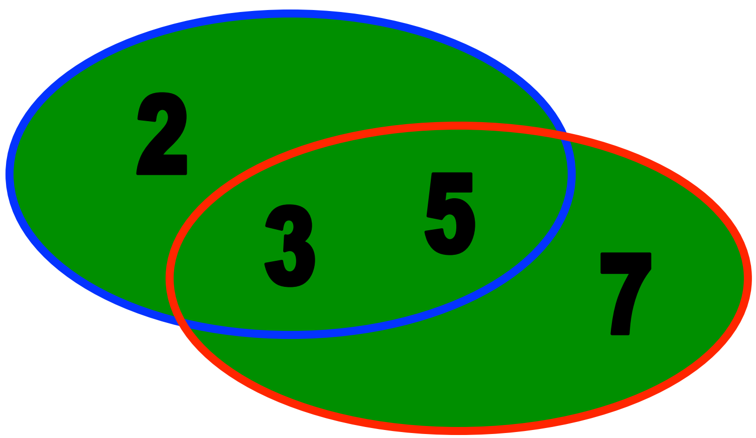
$$\forall A,B: (\mathrm{A} \subseteq \mathrm{A}\cup\mathrm{B}) \wedge (\mathrm{B} \subseteq A\cup\mathrm{B})$$

Union Examples

- $\{a, b, c\} \cup \{2, 3\} = \{a, b, c, 2, 3\}$
- $\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\}$

  $= \{2, 3, 5, 7\}$

Required Form



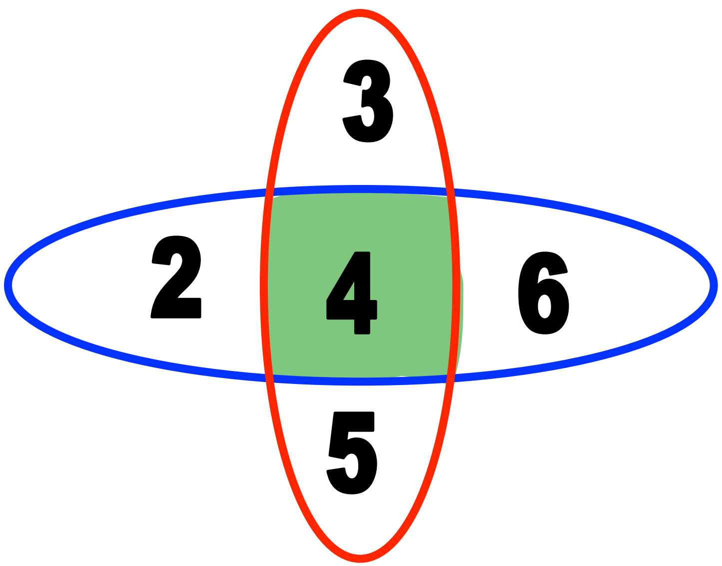
The Intersection Operator

- For sets A and B, their ***intersection*** $A\cap B$ is the set containing all elements that are simultaneously in A **and** (“$\wedge$”) in B.
- Formally, $\forall A,B: A\cap B = \{x \mid x\in A \wedge x\in B\}$.
- Note that $A\cap B$ is a **subset** of both A and B (in fact it is the largest such subset):

$$\forall A,B: (A\cap B \subseteq A) \wedge (A\cap B \subseteq B)$$

Intersection Examples

- $\{a, b, c\} \cap \{2, 3\} = \underline{\ \emptyset\ }$
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \underline{\ \{4\}\ }$



Think "The **intersection** of University Ave. and Dole St. is just that part of the road surface that lies on *both* streets."

Disjointedness

- Two sets *A*, *B* are called ***disjoint*** (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)
- Example: the set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?
  $|A \cup B| = |A| + |B| - |A \cap B|$

- Example: How many students in the class major in Computer Science or Mathematics?
  - Consider set $E = C \cup M$,
    $C = \{s \mid s$ is a Computer Science major$\}$
    $M = \{s \mid s$ is a Mathematics major$\}$

  - Some students are joint majors!
    $|E| = |C \cup M| = |C| + |M| - |C \cap M|$

Set Difference

- For sets A and B, the ***difference of A and B***, written $A - B$, is the set of all elements that are in A but not B.

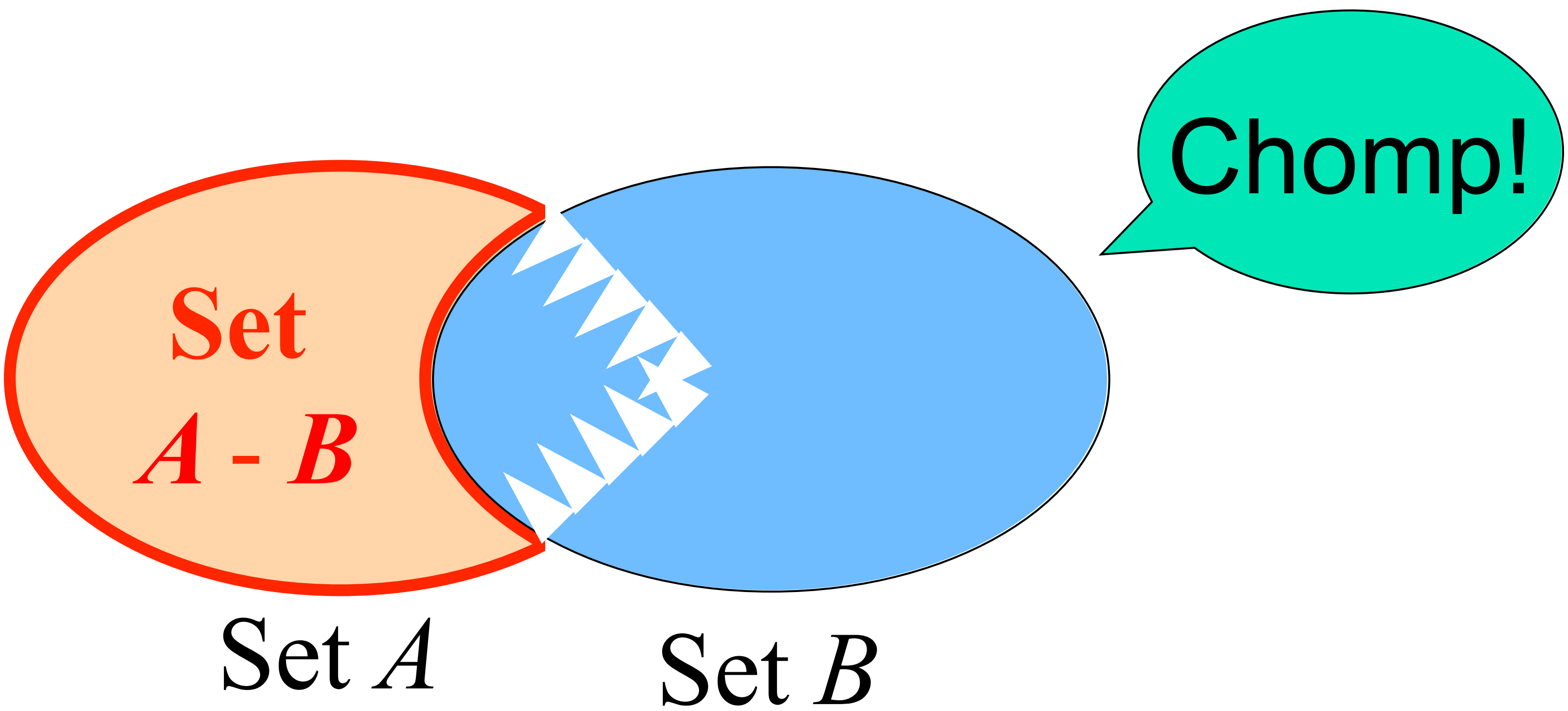
- Formally:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$
$$= \{x \mid \neg(x \in A \rightarrow x \in B)\}$$

- Also called:
  The ***complement of B with respect to A***.

Set Difference: Venn Diagram

- $A - B$
  is what's left after B "takes a bite out of A"



Set Difference Examples

- $\{1, \cancel{2}, \cancel{3}, 4, \cancel{5}, 6\} - \{2, 3, 5, 7, 9, 11\} =$

  $\{1, 4, 6\}$

- **Z** - **N** = {… , −1, 0, 1, 2, … } - {0, 1, … }

  = {*x* | *x* is an integer but not a natural #}

  = {… , −3, −2, −1}

  = {*x* | *x* is a negative integer}

Set Complements

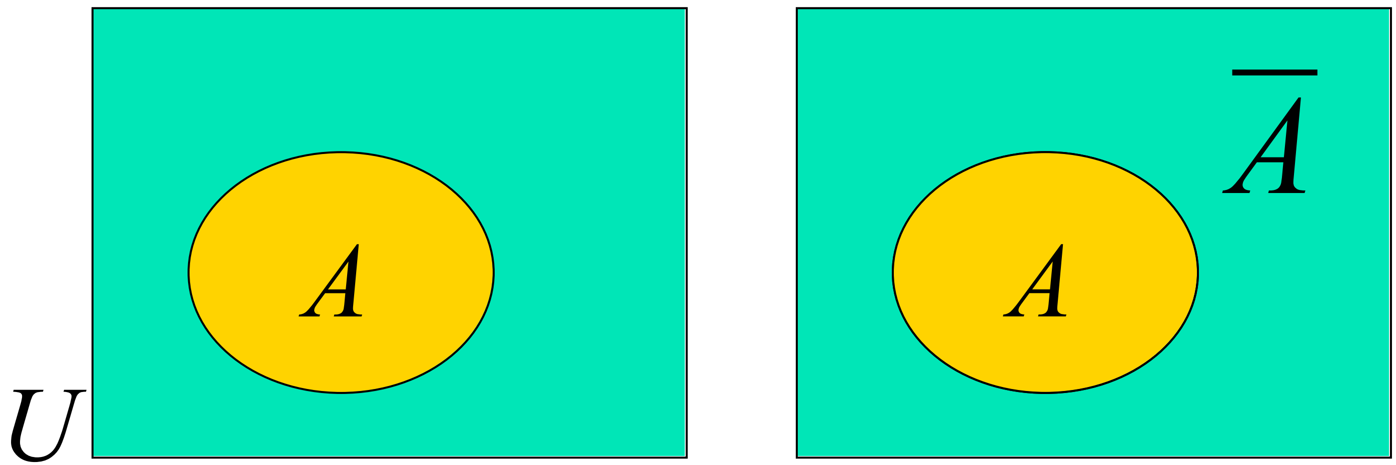
- The *universe of discourse* (or the *domain*) can itself be considered a set, call it *U*.
- When the context clearly defines *U*, we say that for any set $A \subseteq U$, the ***complement*** of *A*, written as $\overline{A}$, is the complement of *A* with respect to *U*, *i.e.*, it is *U* - *A.*
- *E.g.*, If *U* = **N**,

$$\overline{\{3,5\}} = \{0,1,2,4,6,7,...\}$$

More on Set Complements

- An equivalent definition, when *U* is obvious:

$$\overline{A} = \{x \mid x \notin A\}$$



Interval Notation

- $a, b \in \mathbf{R}$, and $a < b$ then
  - $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$
  - $[a, b] = \{x \in \mathbf{R} \mid a \le x \le b\}$
  - $(a, b] = \{x \in \mathbf{R} \mid a < x \le b\}$
  - $(-\infty, b] = \{x \in \mathbf{R} \mid x \le b\}$
  - $[a, \infty) = \{x \in \mathbf{R} \mid a \le x\}$
  - $(a, \infty) = \{x \in \mathbf{R} \mid a < x\}$