Second Order Equations
(Mass-Damper-Spring System)

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solve $\ddot{x} + x = 0$

Stability of $\dot{x} = Ax$

A is stable when all eig(A) have negative real parts

(Why?)

$e^{a+bi} =$ ...

Repeated Roots: Solve $\ddot{x} - 2\dot{x} + x = 0$

* $e^{At} = e^{It + At - It} = \cdots$

Matrix Exponentials

$$e^{A} = Pe^{[\lambda]}P^{-1}$$

Solve $\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x$

Matrix Exponentials

$e^A e^B = e^{A+B}$ holds if AB=BA

Singular Value Decomposition (SVD)

$$A = U \Sigma V^{\top}$$

$V^{\top}V = U^{\top}U = I$

V consists of eigenvectors of $A^{\top}A$

U consists of eigenvectors of $AA^{\top}$

Σ is related to eigenvalues of $A^{\top}A$

Visualization: SVD

```
A = [1 0 1; -1 -2 0; 0 1 -1];
[U, Sigma, V] = svd(A)
```

```
U =

   -0.1200   -0.8097    0.5744
    0.9018    0.1531    0.4042
   -0.4153    0.5665    0.7118


Sigma =

    2.4605         0         0
         0    1.6996         0
         0         0    0.2391


V =

   -0.4153   -0.5665    0.7118
   -0.9018    0.1531   -0.4042
    0.1200   -0.8097   -0.5744
```

Visualization: SVD

```
A = [1 0 1; -1 -2 0; 0 1 -1];
[U, Sigma, V] = svd(A)
```

```
U*Sigma*V'

ans =

    1.0000   -0.0000    1.0000
   -1.0000   -2.0000    0.0000
   -0.0000    1.0000   -1.0000
```

```
U =

   -0.1200   -0.8097    0.5744
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```

If largest eigenvalue of A is σ_1, prove

$$\|Ax\| \leq \sigma_1 \|x\|$$

Visualization: Matrix as linear transformation

Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

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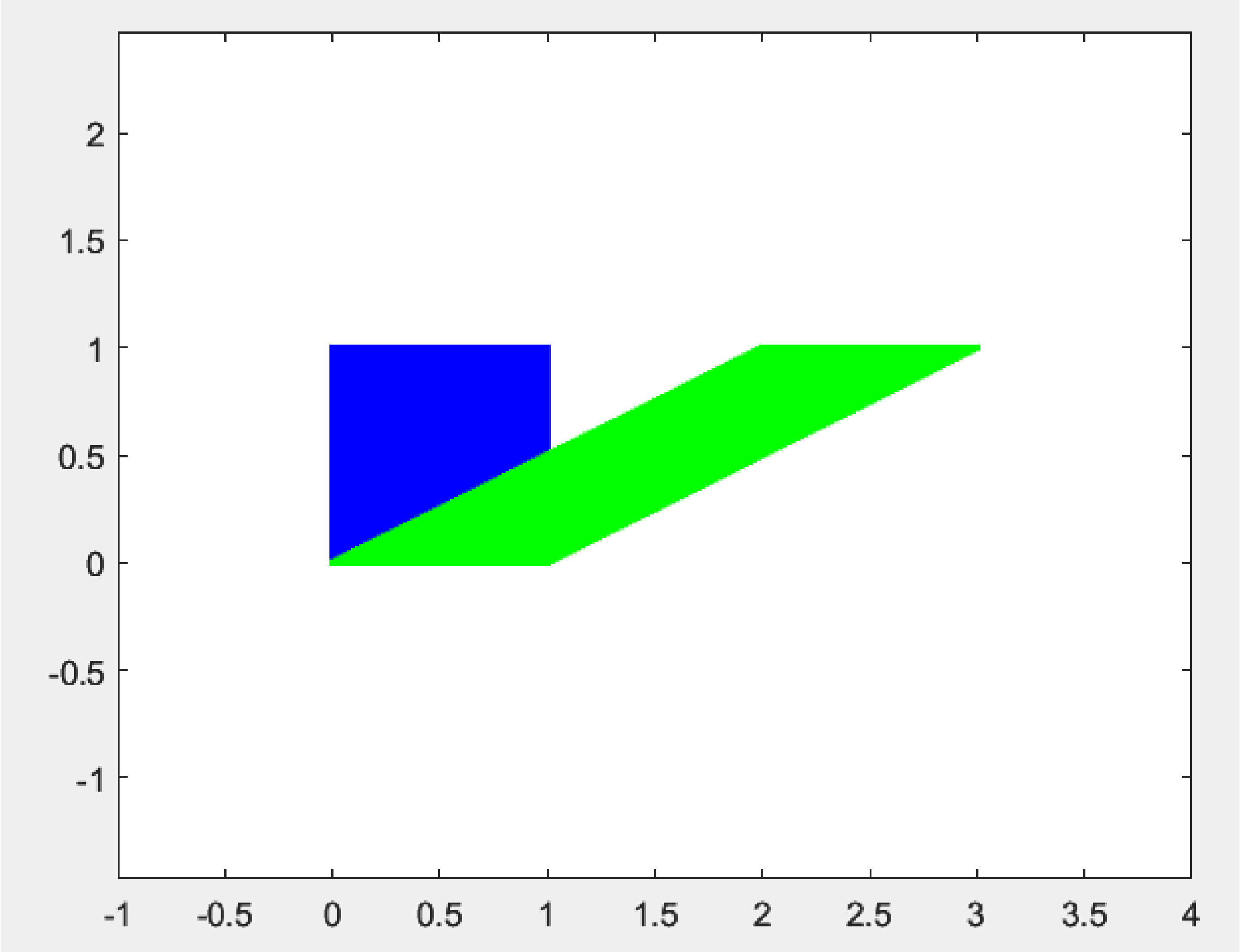

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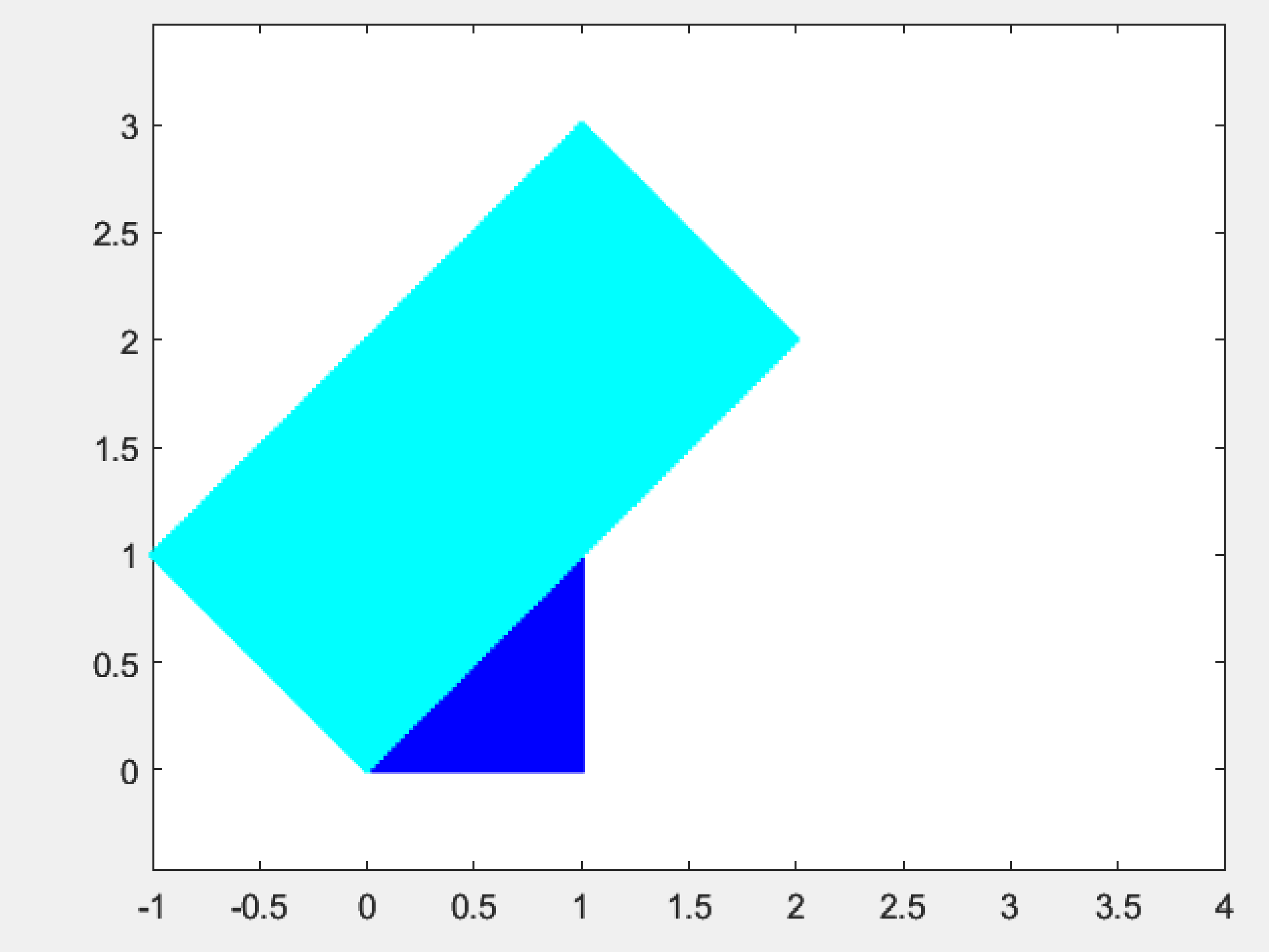

Visualization: Matrix as linear transformation

Consider $A = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$

```
[evec, eval] = eig(A)
```

```
evec =

   -0.9628   -0.4896
    0.2703   -0.8719


eval =

   -1.5616         0
         0    2.5616
```

Visualization: Matrix as linear transformation

Consider $A = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$

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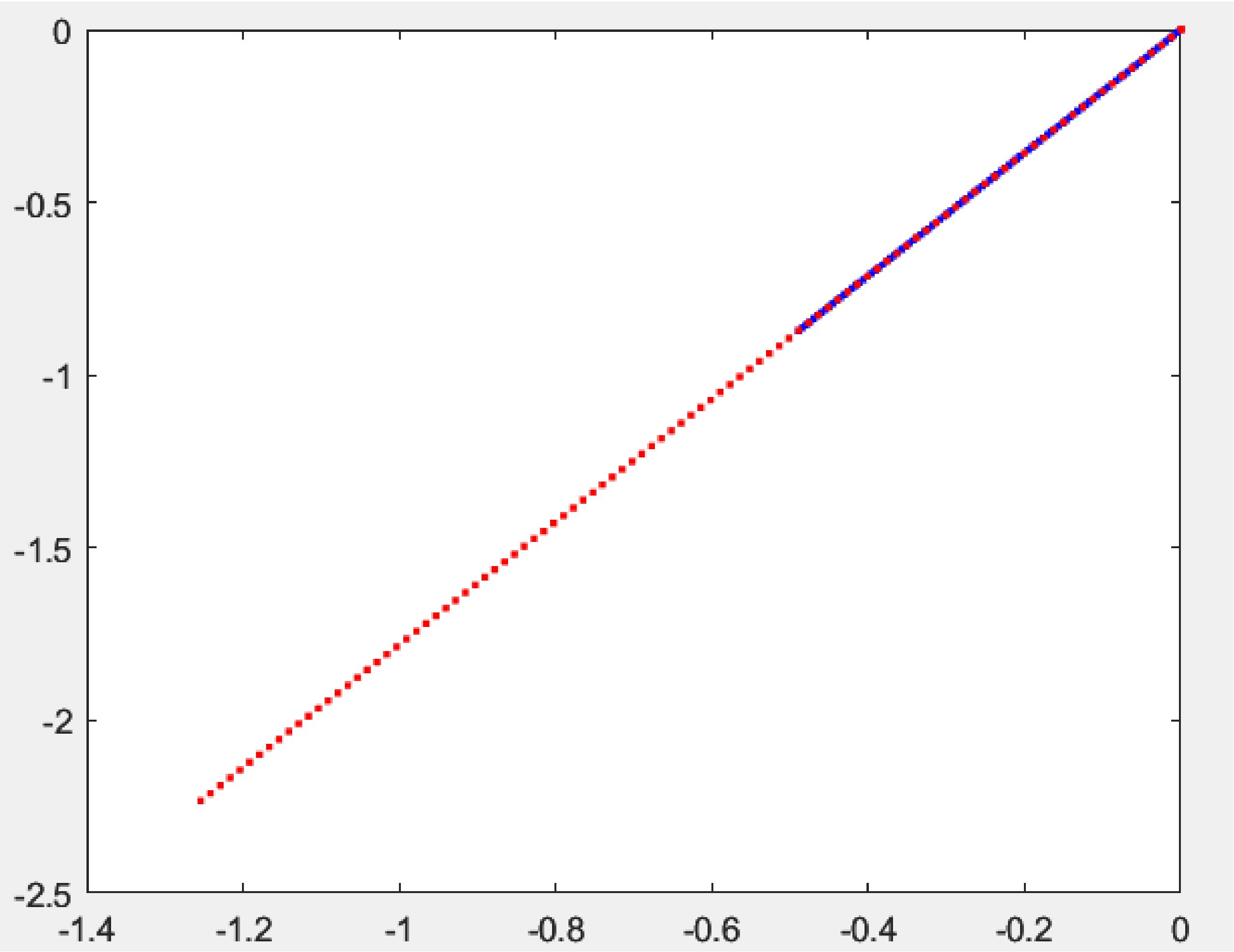

Visualization: Linear Systems

Consider $A = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}$

$$\dot{x} = Ax$$

Initial condition: (1,1)

```
evec =

   0.8165 + 0.0000i   0.8165 + 0.0000i
   0.2041 - 0.5401i   0.2041 + 0.5401i


eval =

  -1.5000 + 1.3229i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 - 1.3229i
```

Visualization: Linear Systems

Consider $A = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}$