Applications of Taylor Series

Lecture Notes

These notes discuss three important applications of Taylor series:

1. Using Taylor series to find the sum of a series.
2. Using Taylor series to evaluate limits.
3. Using Taylor polynomials to approximate functions.

Evaluating Infinite Series

It is possible to use Taylor series to find the sums of many different infinite series. The following examples illustrate this idea.

EXAMPLE 1 Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

SOLUTION Recall the Taylor series for e^x:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x.$$

The sum of the given series can be obtained by substituting in $x = 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e.$$ ■

In the above example, note that we get a different series for every value of x that we plug in. For example,

$$1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots = e^2.$$

and

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}.$$

EXAMPLE 2 Find the sums of the following series:

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$ (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

SOLUTION

(a) Recall that

$$x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} - \cdots = \ln(1+x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2).$$

(b) Recall that

$$x - \frac{x}{3} + \frac{x}{5} - \frac{x}{7} + \frac{x}{9} - \cdots = \tan^{-1}(x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

This is known as the **Gregory-Leibniz formula** for π. ■

Limits Using Power Series

When taking a limit as $x \to 0$, you can often simplify things by substituting in a power series that you know.

EXAMPLE 3 Evaluate $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$.

SOLUTION We simply plug in the Taylor series for $\sin x$:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^3}$$

$$= \lim_{x\to 0} -\frac{1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + \cdots = -\frac{1}{3!} = -\frac{1}{6}$$ ■

EXAMPLE 4 Evaluate $\displaystyle\lim_{x\to 0}\frac{x^2e^x}{\cos x-1}$.

SOLUTION We simply plug in the Taylor series for e^x and $\cos x$:

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2e^x}{\cos x-1} &= \lim_{x\to 0}\frac{x^2\left(1+x+\frac{1}{2}x^2+\cdots\right)}{\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6\cdots\right)-1} \\
&= \lim_{x\to 0}\frac{x^2+x^3+\frac{1}{2}x^4+\cdots}{-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots} \\
&= \lim_{x\to 0}\frac{1+x+\frac{1}{2}x^2+\cdots}{-\frac{1}{2}+\frac{1}{24}x^4-\frac{1}{6!}x^2+\cdots} = \frac{1}{-1/2} = -2 \qquad \blacksquare
\end{aligned}$$

Sometimes a limit will involve a more complicated function, and you must determine the Taylor series:

EXAMPLE 5 Evaluate $\displaystyle\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}$.

SOLUTION Using the Taylor series formula, the first few terms of the Taylor series for $\ln(\cos x)$ are:

$$\ln(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots.$$

(Really, we only need that first term.) Therefore,

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to 0}\frac{-\frac{1}{2}x^2-\frac{1}{12}x^4+\cdots}{x^2} = \lim_{x\to 0}-\frac{1}{2}-\frac{1}{12}x^2+\cdots = -\frac{1}{2} \qquad \blacksquare$$

Limits as $x\to a$ can be obtained using a Taylor series centered at $x=a$:

EXAMPLE 6 Evaluate $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

SOLUTION Recall that

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

Plugging this in gives

$$\begin{aligned}\lim_{x\to 1}\frac{\ln x}{x-1} &= \lim_{x\to 1}\frac{(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots}{x-1}\\ &= \lim_{x\to 1}\left(1-\frac{1}{2}(x-1)+\frac{1}{3}(x-1)^2+\cdots\right) = 1\end{aligned}$$

■

Taylor Polynomials

A partial sum of a Taylor series is called a **Taylor polynomial**. For example, the Taylor polynomials for e^x are:

$$\begin{aligned}T_0(x) &= 1\\ T_1(x) &= 1+x\\ T_2(x) &= 1+x+\frac{1}{2}x^2\\ T_3(x) &= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3\\ &\vdots\end{aligned}$$

You can approximate any function $f(x)$ by its Taylor polynomial:

$$f(x) \approx T_n(x)$$

If you use the Taylor polynomial centered at a, then the approximation will be particularly good near $x = a$.

TAYLOR POLYNOMIALS

Let $f(x)$ be a function. The **Taylor polynomials** for $f(x)$ centered at $x = a$ are:

$$\begin{aligned}T_0(x) &= f(a)\\ T_1(x) &= f(a)+f'(a)(x-a)\\ T_2(x) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2\\ &\vdots\end{aligned}$$

You can approximate $f(x)$ using a Taylor polynomial.

Note that the 1st-degree Taylor polynomial is just the tangent line to $f(x)$ at $x = a$:

$$T_1(x) \;=\; f(a) \,+\, f'(a)\,(x-a)$$

This is often called the **linear approximation** to $f(x)$ near $x = a$, i.e. the tangent line to the graph. Taylor polynomials can be viewed as a generalization of linear approximations. In particular, the 2nd-degree Taylor polynomial is sometimes called the **quadratic approximation**, the 3rd-degree Taylor polynomial is the **cubic approximation**, and so on.

EXAMPLE 7

(a) Find the 5th-degree Taylor polynomial for $\sin x$.

(b) Use the answer from part (a) to approximate $\sin(0.3)$.

SOLUTION

(a) This is just all term terms of the Taylor series up to x^5:

$$T_5(x) \;=\; x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \;=\; x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

(b) $\sin(0.3) \;\approx\; T_5(0.3) \;=\; (0.3) - \dfrac{1}{6}(0.3)^3 + \dfrac{1}{120}(0.3)^5 \;=\; 0.295\,520\,25$ ■

EXERCISES

1–2 ■ Find the sum of the given series.

1. $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots$

2. $1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \cdots$

3–12 ■ Evaluate the following limits.

3. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

4. $\displaystyle\lim_{x\to 0} \frac{x}{e^{3x} - 1}$

5. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x^2)}{x^2}$

6. $\displaystyle\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

7. $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{x}$

8. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

9. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{\sin x}$

10. $\displaystyle\lim_{x\to 0} \frac{\tan^{-1}(x) - x}{\sin(x) - x}$

11. $\displaystyle\lim_{x\to \pi} \frac{1 + \cos x}{(x-\pi)^2}$

12. $\displaystyle\lim_{x\to 1} \frac{\ln x}{\sqrt{x} - 1}$

13. (a) Find the 3rd-degree Taylor polynomial for the function $f(x) = \ln x$ centered at $a = 1$.

(b) Use your answer from part (a) to approximate $\ln(1.15)$.

14. (a) Find the 4th-degree Taylor polynomial for e^{-x}.

(b) Use your answer from part (a) to approximate $e^{-0.3}$.

15. (a) Find the quadratic approximation for the function $f(x) = x^{3/2}$ centered at $a = 4$.

(b) Use your answer from part (a) to approximate $(4.2)^{3/2}$.

16. (a) Find the quadratic approximation for the function $f(x) = \sqrt[3]{x}$ centered at $a = 8$.

(b) Use your answer from part (a) to approximate $\sqrt[3]{8.6}$.