GATE 2017

ENGINEERING MATHEMATICS

Handwritten Notes

Common for

AE/AG/BT/CE/CH/EC/EE/IN/ME/MN/MT/PE/PI

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This book is intended to provide basic knowledge on Engineering Mathematics to the GATE aspirants. Even though the syllabus is same, the questions appearing in different papers follows different patterns. So the GATE aspirants are advised to go through their respective paper syllabus (available in official GATE website) and previous question papers to understand the depth of the subject required to prepare for GATE exam. Most of the solved problems in this material are the questions appeared in EC/EE/ME/CE/IN/PI papers. So the remaining branches students are advised to solve the previous problems from their respective papers. EE branch students are advised to prepare *transform theory,* which is not available in this book.

This book is still under preparation. If you find any mistakes in this, please inform me through an e-mail. Don't share this book without proper citation and credits.

Thank You

All The Best

SYLLABUS – GATE 2016

1. MECHANICAL ENGINEERING – ME
2. METALLURGICAL ENGINEERING - MT
3. PRODUCTION AND INDUSTRIAL ENGINEERING - PI

Linear Algebra: Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations: First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex variables: Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series. **(Except for MT paper)**

Probability and Statistics: Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step methods for differential equations.

CIVIL ENGINEERING - CE

Linear Algebra: Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus: Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE): First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of onedimensional diffusion equation; first and second order one-dimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics: Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods: Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's and Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

ELECTRONICS AND COMMUNICATION ENGINEERING – EC

Linear Algebra: Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations – existence and uniqueness.

Calculus: Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations: First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis: Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Probability and Statistics: Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

Numerical Methods: Solutions of non-linear algebraic equations, single and multi-step methods for differential equations, convergence criteria.

ELECTRICAL ENGINEERING - EE

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes's theorem, Gauss's theorem, Green's theorem.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's equation, Euler's equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables.

Complex variables: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal distribution, Binomial distribution, Correlation analysis, Regression analysis.

Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.

Transform Theory: Fourier Transform, Laplace Transform, z-Transform.

INSTRUMENTATION ENGINEERING - IN

Linear Algebra: Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus: Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Probability and Statistics: Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods: Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

1. PETROLEUM ENGINEERING – PE
2. CHEMICAL ENGINEERING - CH

Linear Algebra: Matrix algebra, Systems of linear equations, Eigen values and eigenvectors.

Calculus: Functions of single variable, Limit, continuity and differentiability, Taylor series, Mean value theorems, Evaluation of definite and improper integrals, Partial derivatives, Total derivative, Maxima and minima, Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Cauchy's and Euler's equations, Initial and boundary value problems, Laplace transforms, Solutions of one dimensional heat and wave equations and Laplace equation.

Complex variables: Complex number, polar form of complex number, triangle inequality.

Probability and Statistics: Definitions of probability and sampling theorems, Conditional probability, Mean, median, mode and standard deviation, Random variables, Poisson, Normal and Binomial distributions, Linear regression analysis.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations. Integration by trapezoidal and Simpson's rule. Single and multi-step methods for numerical solution of differential equations.

1. AGRICULTURE ENGINEERING – AG
2. BIOTECHNOLOGY – BT
3. MINING ENGINEERING - MN

Linear Algebra: Matrices and determinants, systems of linear equations, Eigen values and eigen vectors.

Calculus: Limit, continuity and differentiability; partial derivatives; maxima and minima; sequences and series; tests for convergence; Fourier series, Taylor series.

Vector Calculus: Gradient; divergence and curl; line; surface and volume integrals; Stokes, Gauss and Green's theorems. **(Except for BT paper)**

Differential Equations: Linear and non-linear first order Ordinary Differential Equations (ODE); Higher order linear ODEs with constant coefficients; Cauchy's and Euler's equations; Laplace transforms; Partial Differential Equations - Laplace, heat and wave equations.

Probability and Statistics: Mean, median, mode and standard deviation; random variables; Poisson, normal and binomial distributions; correlation and regression analysis; tests of significance, analysis of variance (ANOVA).

Numerical Methods: Solutions of linear and non-linear algebraic equations; numerical integration - trapezoidal and Simpson's rule; numerical solutions of ODE.

AEROSPACE ENGINEERING - AE

Linear Algebra: Vector algebra, Matrix algebra, systems of linear equations, rank of a matrix, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limits, continuity and differentiability, mean value theorem, chain rule, partial derivatives, maxima and minima, gradient, divergence and curl, directional derivatives. Integration, Line, surface and volume integrals. Theorems of Stokes, Gauss and Green.

Differential Equations: First order linear and nonlinear differential equations, higher order linear ODEs with constant coefficients. Partial differential equations and separation of variables methods.

LINEAR ALGEBRA

LINEAR ALGEBRA

Matrix: A matrix is a rectangular array of numbers or functions arranged in 'm' rows and 'n' columns such that each row has same no. of elements (n) and each column has same no. of elements (m).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

a_{ij} denotes element in i^{th} row and j^{th} column.

Row matrix: Matrix having single row. Ex: $[2 \ 4 \ 6 \ 8]$

Column matrix: Matrix having single column. Ex: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Square matrix: Matrix having equal no. of rows and columns.

Ex: $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

* The elements along diagnol of a square matrix (a_{ij}, $i=j$) are called leading or principle diagonal elements.

* The sum of diagonal elements of a square matrix A is called Trace of A.

Diagonal matrix: A square matrix, except the leading diagonal elements are equal to zero is called diagonal matrix.

Ex: $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scalar matrix: A diagonal matrix, whose leading diagonal elements are equal is called scalar matrix

Ex: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

unit matrix: A diagonal matrix, whose leading diagonal elements all equal to '1' is called unit matrix.

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$

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Null matrix: All the elements of matrix are zero Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Symmetic matrix: A square matrix with $a_{ij} = a_{ji}$ for all i and j.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Skew symmetric matrix: A square matrix with $a_{ij} = -a_{ji}$ for all i & j

Ex: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

* For a skew symmetrix matrix the leading diagonal elements all are equal to zero.

Matrix transpose: Interchanging of elements in rows with the corresponding elements in the columns. resulting matrix is denoted by 'A^T' or 'A''

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{m \times n} \Rightarrow A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{n \times m}$

* If $A = A^T$ then it is symmetric * $(A^T)^T = A$
* If $A = -A^T$ then it is skew-symmetric. * $(AB)^T = B^T \cdot A^T$

* Every given square matrix can be expressed as sum of symmetric & skew-symmetric matrices.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew symmetric}}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\frac{1}{2}(A + A^T) = \underbrace{\begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}}_{\text{Symmetric}}$ $\frac{1}{2}(A - A^T) = \underbrace{\begin{bmatrix} 0 & -1/2 \\ +1/2 & 0 \end{bmatrix}}_{\text{Skew-symmetric}}$

Triangular matrix:

* A square matrix, whose elements below the leading diagonal are zero is called upper triangular matrix.

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Ex: $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ → upper triangular

* A square matrix whose elements <u>above</u> the leading diagonal are equal to zero is called lower triangular matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ → Lower triangular

<u>Multiplication by scalar</u>: $k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$

<u>Multiplication of two matrices</u>:

$$[A]_{m\times n} \cdot [B]_{n\times p} = [AB]_{m\times p}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix} = \begin{bmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 \end{bmatrix}$$

* Multiplication is possible only if no. of columns in first matrix is equal to no. of rows in second matrix.
* $AB \neq BA$.
* $(AB)^T = B^T \cdot A^T$

<u>Determinant of matrix</u>:

* For 1×1 matrix, the number itself is determinent.
* For 2×2 matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinent is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

<u>Minor</u>: Minor of an element is the determinent obtained by deleting the row and column in which the element is present.

Ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ⇒ a_{11} minor is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

a_{23} minor is $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

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Cofactor: Co. factor of any element 'a_{ij}' in a matrix is equal to $(-1)^{i+j} \cdot m_{ij}$. where 'm_{ij}' is minor of 'a_{ij}'.

Ex: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then co.factor of b_3 i.e $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Determinant: Determinant of a matrix is defined as sum of product of elements of any row or column with corresponding co. factor.

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1$$

$$= a_1 (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

Tips for GATE:

* While calculating determinant, always select a row or column with more number of elements equal to '0'

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 6 \end{vmatrix} = 4[6-15] = -36$ [select 2nd row]

* Try to remember co. factor signs $(-1)^{i+j}$ for each element as

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}_{4\times4}$$

Properties of determinants:

1. Determinant remains unaltered by changing its rows into columns and columns into rows. $|A| = |A^T|$
2. If two parellel lines of determinant are interchanged, then the sign of determinant changes (same numerical value)
3. If two parellel lines are identical then det = 0
   If all the elements in a row or column are zeros then det = 0
4. * If each element of a row or column is multiplied by same factor, then determinant also multiplied by same factor.

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$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$|kA| = k^n |A|$

5. The determinant of upper or lower triangular matrix is equal to product of leading diagonal elements.

Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1[18-0] = 18$

$= \underbrace{1 \times 3 \times 6}_{\text{diagonal elements.}}$

6. If the elements of determinant Δ are function of 'x' and if 'K' parellel lines becomes equal when $x = a$ then $(x-a)^{K-1}$ is a factor of Δ.

Ex: $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(a-c)(b-c)$ by putting $a=b, a=c, b=c$.

**7. Product of determinants: *Det(AB)=det(A).det(B)*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}$$

8. $|\text{co. factor matrix of } A| = |A|^{n-1}$ n = order

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ co. factor matrix $= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$|A| = 4-6 = -2$ $|\text{co. factor matrix}| = 4-6 = -2$

9. If $|A| = 0$, then A is called Singular matrix *or non-invertible matrix*

10. If $|A| \neq 0$, then A is called non-singular matrix *or invertible matrix*

<u>Adjoint matrix</u>: Transpose of the co.factor matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ co. factor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\text{Adj}[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

* $A(\text{Adj } A) = (\det A) I$
* $\det(\text{Adj } A) = (\det A)^{n-1}$ $\quad n = \text{order}$

$|\text{adj} A| = |\,|A| \cdot A^{-1}|$
$= |A|^n |A^{-1}|$
$= |A|^n \cdot |A|^{-1} = |A|^{n-1}$

Inverse of square matrix: A matrix "B" is said to be inverse of a non-singular matrix "A" if $AB = BA = I$.

$$A^{-1} = \frac{\text{Adj}[A]}{|A|} \qquad A \cdot A^{-1} = I$$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$(\text{Adj}[A])^{-1} = \frac{A}{|A|}$

* $\text{adj}(\text{adj}[A]) = |A|^{n-2} \cdot A$

we know

$A \cdot \text{adj}[A] = |A| \cdot I$

replace A by adj[A]

$\text{adj} A \cdot [\text{adj}(\text{adj} A)] = |\text{Adj} A| \cdot I$

$A^{-1} \cdot |A| \, [\text{adj}(\text{adj} A)] = |\text{Adj} A| \cdot I$

$\text{adj}(\text{adj} A) = \frac{|A|^{n-1}}{|A|} \cdot A$

$= |A|^{n-2} \cdot A$

Properties:

* $(AB)^{-1} = B^{-1} \cdot A^{-1}$

* If $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$

* If a non-singular matrix A is symmetric then A^{-1} is also symmetric.

* Every odd order symmetric matrix is singular i.e $|A| = 0 \Rightarrow A^{-1}$ does not exist for that.

Problems

1. consider matrix $X_{4\times3}$, $Y_{4\times3}$, $P_{2\times3}$ then order of $[(P.(X^TY)^{-1})P^T]^T$

$$[(P_{2\times3} \cdot (X^T_{4\times3} \cdot Y_{4\times3})^{-1}) \cdot P^T_{2\times3}]^T = [(P_{2\times3} \cdot (X_{3\times4} \cdot Y_{4\times3})^{-1}) P_{3\times2}]^T$$

$$= [(P_{2\times3} \cdot (3\times3)^{-1} \cdot P_{3\times2}]^T$$

$$= [P_{2\times3} \cdot P_{3\times2}]^T = (2\times2)^T = (2\times2)$$

Tip for GATE:

Determinant calculation [for matrix with less no. of zero elements]

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (Applicable for 3x3)

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

2. The following represents equation of straight line $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$

The line passes through ?

a) (0, 0) b) (3, 4) c) (4, 3) d) (4, 4)

$$x(8) - 2(y) + 4(y-8) = 0$$

$$4x + y = 16$$ option b ✓

3. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then top row of A^{-1} is ?

Find the inverse of A

det(A)=1
cofactor Matrix=
Adj A=
Inverse of A=

$$|A| = 1 \qquad A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{(\text{co. factor})^T}{|A|}$$

co. factors of 1st column = $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

1st row of $A^{-1} = [5 \ -3 \ 1]$

4. If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 0.5 & a \\ 0 & b \end{bmatrix}$ then $a+b$?

$$A \cdot A^{-1} = I \Rightarrow \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$3b = 1 \Rightarrow b = 1/3$ & $2a = 0.1/3 \Rightarrow a = 1/60 \Rightarrow (a+b) = \frac{7}{20}$

5. If $A = (a_{ij})_{m \times n}$ such that $a_{ij} = i+j$, $\forall i, j$, then sum of the elements of A is

$A = (a_{ij})_{2\times 3} =$

$$A = \begin{bmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \vdots \\ m+1 & m+2 & m+3 & \cdots & m+n \end{bmatrix}_{m \times n}$$

$1\times m$, $2\times m$, $3\times m$ ... $n\times m$ → n terms

$\frac{m(m+1)}{2}$, $\frac{m(m+1)}{2}$, $\frac{m(m+1)}{2}$ ... $\frac{m(m+1)}{2}$ → n terms

$$\text{Sum of elements} = n \times \frac{m(m+1)}{2} + [m + 2m + 3m + \cdots + nm]$$

$$= \frac{mn(m+1)}{2} + m[1 + 2 + 3 + \cdots + n]$$

$$= \frac{mn(m+1)}{2} + \frac{mn(n+1)}{2}$$

$$= \frac{mn}{2}(m+n+2)$$

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6. If $A=(a_{ij})_{3\times3}$, $B=(b_{ij})_{3\times3}$ such that $b_{ij}=2^{i+j}\cdot a_{ij}\ \forall\ i,j$; If $|A|=2$ then $|B|=?$ (a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$$

$$|B| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix} = 2^2\cdot 2^3\cdot 2^4 \begin{vmatrix} a_{11} & 2a_{12} & 2^2a_{13} \\ a_{21} & 2a_{22} & 2^2a_{23} \\ a_{31} & 2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$= 2^9\cdot 2\cdot 2^2\cdot |A| = 2^{13}$$

7. If $A=(a_{ij})_{n\times n}$ such that $a_{ij}=i^2-j^2\ \forall\ i,j$. Then find sum of all the elements of A?

$[a_{ij}=i^n-j^n]$

$$A = \begin{bmatrix} 0 & -3 & -8 & \cdots & (1^2-n^2) \\ 3 & 0 & -5 & \cdots & (2^2-n^2) \\ 8 & 5 & 0 & \cdots & (3^2-n^2) \\ \vdots & \vdots & \vdots & & \\ (n^2-1^2) & (n^2-2^2) & (n^2-3^2) & \cdots & 0 \end{bmatrix}_{n\times n}$$

Above matrix is a skew symmetric matrix. It has all the diagonal elements equal to zero. When added up, non diagonal elements cancel out each other, resulting in final sum = 0. [always]

8. The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} = ?$

(or) Apply $R_2\to R_2-R_1$, $R_3\to R_3-R_1$ then find Δ with 3rd column

$$C_1\to C_1+C_2 \quad = \begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} + \begin{vmatrix} 2b & b & 1 \\ 2b & 1+b & 1 \\ 2b & 2b & 1 \end{vmatrix}$$

same lines

$$= 0 + 2b\begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

same lines

$$= 0$$

9. Find the determinant of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ac \\ 1/c & c & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} abc/a & a & bc \\ abc/b & b & ac \\ abc/c & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} bc & a & bc \\ ac & b & ac \\ ab & c & ab \end{vmatrix} = 0$$

← same lines

10. Find the value of $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

For similar type of problems

1. Apply $C_1 \to C_1 + C_2 + C_3 + C_4$
2. Apply $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$
3. Now find 'det' by taking 1st column.

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

$$= \begin{vmatrix} x+3a & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$ → upper triangular matrix

$$= (x+3a)(x-a)(x-a)(x-a) = (x+3a)(x-a)^3$$

11. $\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = (x+10).x.x.x = (x+10).x^3$

12. $\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = (1+a+b+c+d).1.1.1 = (1+a+b+c+d)$

13. If $A_{m\times n}$ and $B_{n\times p}$ are two matrices, then the no. of multiplications and additions in computing AB are?

To find each element of AB, we require 'n' multiplications and $(n-1)$ additions.

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There are 'mp' elements in matrix $[AB]_{m\times p}$. Therefore

Total no. of multiplications required $= mpn$

Total no. of additions required $= mp(n-1)$

14. The no. of different $n\times n$ symmetric matrices with each element being either '0' or '1' is

For symmetric matrices, If once we fix the values of elements along & above the diagonal then no need to fix the " below the diagonal.

$$\begin{array}{cl} \begin{bmatrix} \overset{2}{1/0} & \overset{2}{1/0} & \overset{2}{1/0} & \cdots & \overset{2}{1/0} \\ & 1/0 & 1/0 & \cdots & 1/0 \\ & & 1/0 & \cdots & 1/0 \\ & & & \cdots & 1/0 \end{bmatrix}_{n\times n} & \begin{array}{l} \to 2^n \text{ ways} \\ \to 2^{n-1} \text{ ways} \\ \to 2^{n-2} \text{ ways} \\ \vdots \\ \to 2^1 \text{ ways} \end{array} \end{array}$$

$\therefore$ Total no. of symmetric matrices $= 2^n \cdot 2^{n-1} \cdot 2^{n-2} \cdots 2^1$

$= 2^{1+2+3\cdots+n} = 2^{n(n+1)/2}$

15. If A, B, C, D be $n\times n$ matrices, each with non zero determinant. $ABCD = I$ then $B^{-1} = ?$

All are $n\times n$ matrices: $ABCD = I$

$A^{-1}.ABCD = A^{-1}.I \Rightarrow BCD = A^{-1}$

$B^{-1}.BCD = B^{-1}.A^{-1} \Rightarrow CD = B^{-1}.A^{-1}$

$CDA = B^{-1}.A^{-1}.A \Rightarrow CDA = B^{-1}$

Shortcut:

CEB D GAF

D^{-1}

$D^{-1} = GAFCEB$

Orthogonal matrix: A matrix is said to be orthogonal if $A.A^T = A^T.A = I$ $\Rightarrow |A| = \pm 1$

$A.A^T = I \Rightarrow A^{-1}.A.A^T = A^{-1}.I$

$\Rightarrow I.A^T = A^{-1}.I \Rightarrow \boxed{A^T = A^{-1}}$

If A is orthogonal matrix then A^{-1} & A^T are also orthogonal matrices.

Ex: For matrix $M = \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}$ such that $M^T = M^{-1}$. Find x?

Sol: $M.M^T = I \Rightarrow \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}\begin{pmatrix} 3/5 & x \\ 4/5 & 3/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow x = -\frac{4}{5}$

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* Determinant of odd order skew symmetric matrix is a perfect square.

Ex: $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} = 0+16 = 16 = (4)^2$

16. If x and y are two non-zero matrices of the same order such that $xy = (0)_{m\times n}$ then

(a) $|x| \neq 0$, $|y| = 0$ b) $|x| = 0$, $|y| \neq 0$

(c) $|x| \neq 0$, $|y| \neq 0$ ✓(d) $|x| = 0$, $|y| = 0$

$xy = 0 \Rightarrow |xy| = 0 \Rightarrow |x|.|y| = 0$

then either $|x| = 0$ or $|y| = 0$ or both $|x| = |y| = 0$

(i) If $|x| = 0$, $|y| \neq 0$ y^{-1} exist

$xy = 0 \Rightarrow xy.y^{-1} = 0 \Rightarrow xI = 0$

$\Rightarrow x = 0$ [Given x is non zero]

(ii) If $|y| = 0$, $|x| \neq 0$ x^{-1} exist $\Rightarrow y = 0$ [Given y is non zero]

$\therefore$ Both $|x| = |y| = 0$

EC-08 2 Marks

17. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ then $(AA^T)^{-1} = ?$

Given A is orthogonal $\Rightarrow A.A^T = I$

$\Rightarrow (A.A^T)^{-1} = I^{-1} = I$

18. ~~Find the inverse of~~ Given the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find A^T and AA^T

Hence deduce the inverse of A

Given Matrix is a orthogonal $\Rightarrow A^{-1} = A^T$.

19. If a_{ij} is defined by $a_{ij} = i^2 - j^2$ $\forall$ i, j where $1 \le i, j \le n$ then find A^{-1} for $n = 5$?

Given $A = (a_{ij})_{5\times5}$ $a_{ij} = i^2 - j^2 = -(j^2 - i^2) = -a_{ji}$

$\therefore$ A is skew symmetric and odd order

$\Rightarrow |A| = 0$ & A^{-1} does not exist

Rank of a Matrix: A matrix is said to be of rank r when

(i) it has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than r vanishes.

Rank of A is denoted by $\rho(A)$. Rank of null matrix = 0

Elementary transformations of a matrix:

1. Interchange of any rows / columns
2. Multiplication of any row / column by a non zero number.
3. Addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Elementary transformations do not change order or rank of the matrix.

Ex: Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 - R_1 \to R_2$
$R_3 - 2R_1 \to R_3$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A$ (say)

$|A| = 0 \Rightarrow$ rank $\neq 3$

2nd order minor $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} = -8 \neq 0$ $\therefore$ rank = 2

Gauss - Jordan Method to find Inverse:

1. Write two matrices A & I side by side
2. Reduce A to I by performing same row operations on both. Then other matrix represent A^{-1}.

Reducing A to I:

1. Using R_1, make 1st element in $R_2, R_3 \ldots$ to zero
2. Using R_2, make 2nd element in $R_3, R_4 \ldots$ to zero continue like this
3. Using R_n make last element in $R_{n-1}, R_{n-2} \ldots$ to zero
4. Using R_{n-1} make last-1 element in $R_{n-2}, R_{n-3} \ldots$ to zero
5. Proper constant multiplication make A as I

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Ex: Find inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_3$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\hookrightarrow A^{-1}$

<u>Normal form of a matrix</u>: Every non-zero matrix of rank 'r' can be reduced by sequence of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A.

For every matrix of rank 'r', there exist non-singular matrix P and Q such that $PAQ = I$.

Ex: For matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non singular matrix P & Q such that PAQ is in normal form. Hence find the rank of A.

Sol:

$$A = IAI$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row transformation $\to$ Same transformation in pre-factor matrix

Every column transformation $\to$ Same transformation in post factor matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

normal form $= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = 2$

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Row echelon form: A matrix 'A' is said to be in row echelon form if

(i) Zero rows should occupy last rows, if any

(ii) The no. of zeros before a non zero element of each row is less than the no. of such zero's before the non zero element of the next row.

Ex: $\begin{bmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* Rank of a matrix = No. of non-zero rows in row echelon form
* Non-zero rows are called linearly independent rows/vector.
* To reduce any matrix into row echelon form, we should use only row transformations.

Ex: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$ Find K if (i) Rank(A) = 2

(ii) Rank(A) = 3

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & K-1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$, $R_4 \to R_3 - R_2$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K+2 & 0 \end{bmatrix}$ $\Rightarrow$ For $\rho(A) = 2 \to K = -2$

For $\rho(A) = 3 \to K \neq -2$

* $\rho(O_{n\times n}) = 0$
* $\rho(I_{n\times n}) = n$
* $\rho\{adj(I_{n\times n})\} = n$
* $\rho(A) = \rho(A^T)$
* $\rho(A \pm B) \leq \rho(A) + \rho(B)$
* $\rho(A - B) \leq \rho(A) - \rho(B)$
* $\rho(AB) \geq \rho(A) + \rho(B) - n$ if A&B are $n\times n$ matrices
** $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
* If A is an $m\times n$ matrix, then $\rho(A) \leq \min(m,n)$

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* If $\rho(A_{n\times n}) = n$ then $\rho(\text{Adj}A) = n$
* If $\rho(A_{n\times n}) = n-1$ then $\rho(\text{Adj}A) = 1$
* If $\rho(A_{n\times n}) = n-2$ then $\rho(\text{Adj}\cdot A) = 0$
* Rank of non-singular matrix ($|A| \neq 0$) of order n is 'n'
* Rank of singular matrix ($|A| = 0$) of order n is $< n$
* If $A_{m\times 1}$, $B_{1\times n}$ are two non-zero matrices then $\rho(AB) = 1$
* Nullity of A = No. of columns of $A - \rho(A)$
* Rank of diagonal matrix = no. of non-zero elements in the diagonal.

2M **Prob:** If $A = (a_{ij})_{m\times n}$, such that $a_{ij} = i.j \ \forall\ i, j$ then $\rho(A) = ?$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \\ m & 2m & 3m & \cdots & mn \end{bmatrix}_{m\times n}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$\vdots$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow \text{Rank} = 1$$

Prob: If $X = (x_1\ x_2 \cdots x_n)^T$ is n-type non zero vector then
(i) $\rho(XX^T)$ (ii) $\rho(X^TX)$

Sol: $X_{n\times 1}$, $X^T_{1\times n}$ $\Rightarrow \rho(X.X^T) = \min\{\rho(X), \rho(X^T)\}$

$= \min\{1, 1\}$

$\rho(X.X^T) = 1$ [X is non zero]

Prob: The rank of 5×6 matrix Q is 4 then which of the following statement is true

a) Q will have 4 L.I rows & 4 L.I columns
b) Q will have 4 L.I rows & 5 L.I columns
c) QQ^T is invertible d) Q^TQ is invertible.

$\rho(5\times 6) = 4$ 4 non-zero rows & columns.

$Q_{5\times 6} \cdot Q^T_{6\times 5} = (QQ^T)_{5\times 5} \to$ Invertible

$|QQ^T|_{5\times 5} \neq 0 \to \rho(QQ^T) = 5$ (Not 4)

$Q^T_{6\times 5} \cdot Q_{5\times 6} = (Q^T.Q)_{6\times 6} \to$ Invertible

$|Q^T.Q|_{6\times 6} \neq 0 \to \rho(Q^T.Q) = 6$ (Not 4)

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Linearly dependent vectors: Two vectors x_1, x_2 are linearly dependent if one vector is expressed as multiple of other vector. x_1 & x_2 ⇒ same directional vectors

Ex: $x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$

$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$
[for more than 2 vectors]
Atleast one k, but not all equal to zero

$x_2 = 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3x_1$ or $x_1 = \frac{1}{3}x_2$

x_1, x_2 are linearly dependent.

Linearly independent vectors: Two vectors x_1, x_2 are Linearly independent if it is not possible to express one vector as a multiple of other vector.

$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$
All zeros

* If Rank of A = no. of given vectors or $|A| \neq 0$ the given vectors are said to be L.I
* If Rank of A < no. of given vectors or $|A| = 0$ the given vectors are said to be L.D
* If the given vectors are linearly dependent then anyone of the vector can be expressed as linear combination of other vectors.
* The set of unit vectors of Identity matrix are L.I

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0 \Rightarrow \text{L.I vectors}$$

* The set of vectors having atleast one zero vector are L.D.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

* **Dimension:** = No. of linearly independent vectors. = no. of non zero rows in R.E form = no. of non zero ~~rows~~ columns in C.E form
* **Basis:** = Set of linearly independent vectors = set of non zero rows in R.E form = set of non zero columns in C.E form

Ex: R.E form = $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 8 \\ 0 & 0 & -5 \end{bmatrix}$

Dimension = 3

Basis = $\{(1\ -2\ 4), (0\ 3\ 8), (0\ 0\ -5)\}$

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2M

Prob: If $a_1, a_2 \cdots, a_m$ are n-dimensional vectors with $m>n$, the vectors are L.D. The matrix Q is $a_1, a_2, \cdots a_m$ as columns. The rank of Q = ? (a) m (b) n (c) between m&n ✓d) $<m$

Sol:

$$Q = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ | & | & | & & | \end{bmatrix}_{n\times m} = (Q)_{n\times m}$$

$$\rho(Q_{n\times m}) \le \min(n,m)$$

$$\rho(Q_{n\times m}) \le m \begin{cases} = m \times (L.D) \\ < m \ ✓ \end{cases}$$

Prob: The nullity of $A = \begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix} = 1$ The value of $k = ?$

$$\text{nullity} = n(A) - \rho(A)$$

$$1 = 3 - \rho(A) \Rightarrow \rho(A) = 2 \Rightarrow |A| = 0$$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0 \Rightarrow k = -1$$

<u>Non Homogenious system of equations:</u> A system of equation of the form

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is called non homogeneous system of linear equations in 3 variables. These can be written in matrix form as $AX = B$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

<u>Augmented matrix:</u> Write elements of matrix B in A by adding as one more column at the end. Resulting matrix is called augmented matrix and is denoted by

$$[A \mid B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

Solution of linear system of equations: [In Exam use calculator]

1. Cramer's rule:

$$x = \frac{|A_1|}{|A|} \qquad y = \frac{|A_2|}{|A|} \qquad z = \frac{|A_3|}{|A|}$$

where $A_1 = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$ $\quad A_3 = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$

2. Matrix inversion Method:

$$AX = B \quad \Rightarrow \quad X = A^{-1} \cdot B$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times3}^{-1} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times1}$$

3. Reducing to echelon form:

Reduce the Augmented matrix to R.E form and write the equations again

Ex: Solve $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$?

Sol:

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$R_3 \to 7R_3 - 6R_2$
$R_4 \to 7R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$5z = 20 \Rightarrow z = 4$

$-7y + 6z = -8 \Rightarrow y = 4$

$x + 2y - z = 3 \Rightarrow x = -1$

*** Consistency:

1. If $\rho[A \mid B] \neq \rho[A]$ then the system of equations are inconsistent. i.e there is no solution.
2. If $\rho[A \mid B] = \rho[A] = n$ (no. of variables) then the equations are consistent and have unique solution.
3. If $\rho[A \mid B] = \rho[A] = r$ $(<n)$ then the equations are consistent and have infinite number of solutions with $(n-r)$ independent variables.

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Prob: For what values of λ & μ the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) No solution (ii) unique solution (iii) Infinite solutions.

Sol:

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) No solution $\lambda = 3, \mu \neq 10$

(ii) unique solution $\lambda \neq 3$

(iii) Infinite solutions $\lambda = 3, \mu = 10$

Prob: For what values of a & b the system of eq^{n}s have $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+az=b$ have (i) No solution (ii) unique (iii) Infinite.

Sol:

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$$

(i) No solution $a = 8, b \neq 15$

(ii) unique sol^n $a \neq 8$

(iii) Infinite sol^n $a = 8, b = 15$

Note: For the system $AX = B$, where $A_{m\times n}$, $X_{n\times 1}$, $B_{m\times 1}$

(i) If $m < n$ (no. of eq^{n}s < no. of variables) then there is no unique solution. It may have infinite or no solution.

(ii) If $m \geq n$ (no. of eq^{n}s $\geq$ no. of variables) then all the three conditions exists.

Prob: The system $x+2y+3z=8$ and $2x+4y+6z=10$ has

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 4 & 6 & 10 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

$\rho(A) \neq \rho(A \mid B)$ No solution

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Prob: If A is a 3x4 matrix and $Ax = B$ is an inconsistent system. Then the highest possible rank of A is

$$\rho(A) \le \min\{3,4\}$$

$$\rho(A) \le 3 \begin{cases} = 3 \times \text{ inconsistent} \\ = 2 \checkmark \text{ highest possible.} \end{cases}$$

Note: consider the system of equations $ax + by = e$, $cx + dy = f$

The above system of equations have

(1) No solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$ *parallel lines*

(2) Unique solution if $\frac{a}{c} \neq \frac{b}{d}$ *Intersecting lines*

(3) Infinite solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$ *Same line*

1M

Prob: The system of equations $4x + 2y = 7$, $2x + y = 6$ have

$$\frac{4}{2} = \frac{2}{1} \neq \frac{7}{6} \quad \therefore \text{ No solution}$$

Prob: $x + 2y = 5$, $2x + 3y = 9$ have

$$\frac{1}{2} \neq \frac{2}{3} \neq \frac{5}{9} \quad \therefore \text{ Unique solution}$$

2M

Prob: How many solutions does the following system of eq'ns have $-x + 5y = -1$ $x - y = 2$ $x + 3y = 3$

(a) Infinite (b) exactly 2 (c) Unique sol^n d) No sol^n

Sol: $[A|B] = \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$

$R_3 \to R_2 + R_1$, $R_3 \to R_3 + R_1$ $\sim \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$

$R_3 \to R_3 - 2R_2$ $\sim \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\rho(A) = \rho(A|B) =$ No. of unknowns $= 2$

$\therefore$ unique solution exists.

Prob: For the set of eq'ns $x_1 + 2x_2 + x_3 + x_4 = 2$, $3x_1 + 6x_2 + 3x_3 + 3x_4 = 6$ which of the following statement is true

1) There exist only trivial solution

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(2) NO solution (3) unique non trivial sol^n (4) Infinite no. of non trivial sol^n.

No. of equations (2) < no. of variables (n) = 4

Infinite solutions (non trivial)

<u>Homogenous system of equations:</u> A system of equations of the form $a_1x+b_1y+c_1z=0$

$a_2x+b_2y+c_2z=0$

$a_3x+b_3y+c_3z=0$. These system of eq^{n}s can be written in the matrix form as $AX=0$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These system of eq^{n}s are always consistent i.e $x=0$, $y=0$, $z=0$ always satisfy these equations.

(i) If Rank (A) = no. of variables (n) then system of equations will have zero solution or trivial solution or unique solution

(ii) If Rank (A) < n then the system of equations will have non trivial or non zero or Infinite solutions in (n-r) independent variables.

<u>Prob:</u> The homogenous system $x+y+z=0$, $(a+1)y+(a+1)z=0$, $(a^2-1)z=0$ has two linearly independent solutions then $a=?$

a) 1 b) −1 c) 0 d) 2

<u>Sol:</u>

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & a+1 & a+1 \\ 0 & 0 & a^2-1 \end{bmatrix}$$

$n-r=2$

$3-r=2 \Rightarrow r=1$

To make last rows $=0 \Rightarrow a=-1$

<u>Prob:</u> If the homogenous system $AX=0$ has co-efficient matrix of order $n\times n$ then the condition for existance of non trivial sol^n is

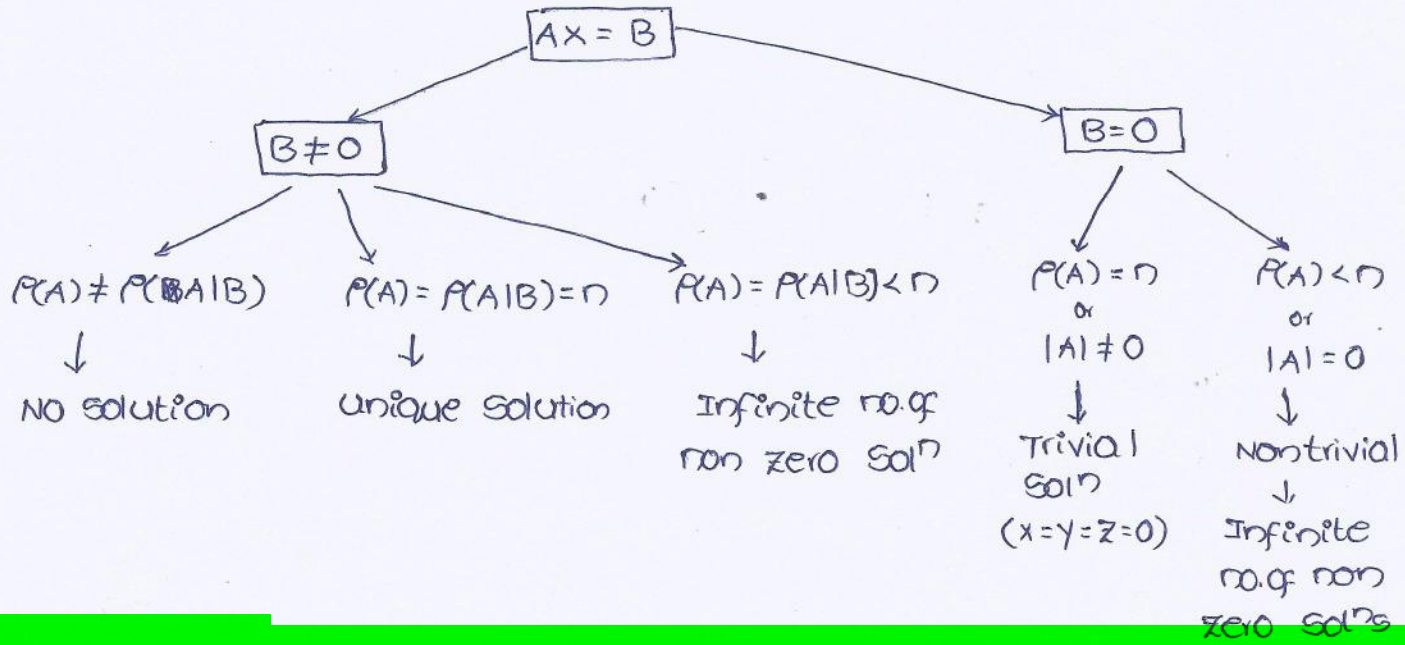
(i) $\rho(A)=n$ (ii) A is non singular (iii) $|A|\neq 0$ ✓(iv) A is singular

Note: 1. The necessary condition for existance of a non trivial sol^n for the homogenous system $Ax=0$ is (i) A is singular (ii) $|A|=0$ (iii) $\text{Rank}[A] < n$

2. For homogenous system $A_{m\times n} \cdot X_{n\times 1} = O_{m\times 1}$

(i) If $m<n$ (no. of eq^{n}s < no. of variables) always non trivial sol^n exists

(ii) If $m \geq n$ both the conditions may exist.



** Eigen values & Eigen vectors

The roots of the characterstic eq^n $|A-\lambda I|=0$ are called Eigen values of matrix A.

Ex: $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ $\quad |A-\lambda I|=0$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (5+\lambda)(2+\lambda)-4=0$$

$$\lambda = -1, -6.$$

Ex: $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

$$\Rightarrow \lambda = -2, 6, 3$$

* Eigen values are also called characterstic roots or latent roots or proper values. The set of eigen values of matrix A is called as spectrum of A.

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Properties:

* Sum of eigen values of A = Trace of A *(i.e. sum of entries in the leading diagonal)*
* Product of eigen values of A = Det of A
* A & A^T will have same eigen values
* The eigen values of symmetric matrices are always real.
* The " " of skew " " " " purely imaginary or zero's
* For an upper / Lower triangular matrix the eigen values are its principal diagnol elements.
* If A has eigen values $\lambda_1, \lambda_2, \lambda_3 ---$ then A^2 has eigen values $\lambda_1^2, \lambda_2^2, \lambda_3^2 ---$, A^{-1} has eigen values $1/\lambda_1, 1/\lambda_2, ---$
* If one of eigen values of A = 0, then A^{-1} doesn't exist. Since $|A| = 0$
* Eigen values of KA are $K\lambda_1, K\lambda_2 ---$

  " " " $A - KI$ are $\lambda_1 - K, \lambda_2 - K ---$

  " " " Adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, ---$
* Eigen values of scalar / Diagonal / matrix are diagonal elements.
* " " " Null matrix are all zeros
* " " " unity matrix are all 1's
* Eigen values of orthogonal matrix are unit Modulus i.e ± 1

Prob: $A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ Eigen values of A are

(i) $3, 3+5j, 6-j$ (ii) $-6+5j, 3-j, 3+j$ (iii) $3-j, 3+j, 5+j$ (iv) $3, -1+3j, -1-3j$.

cross check: Trace of A = $-1-1+3 = 1$

(iv) option Sum of ~~diagn~~ eigen values = $3-1+\cancel{3j}-1-\cancel{3j} = 1$

Prob: If $(-1+\sqrt{3})$ is eigen value of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ then other two eigen values are

(i) $(-1, -\sqrt{3}, 1)$ (ii) $(-1-\sqrt{3}, 2)$ (iii) $(-1+\sqrt{3}, 1)$ (iv) $(1+\sqrt{3}, -2\sqrt{3})$

cross check: (ii) $0 = -1+\sqrt{3}-1-\sqrt{3}+2 = 0$

Eigen vectors: The eigen vector corresponding to eigen value $\lambda = \lambda_r$ of the matrix A is the solution of homogenous system $[A - \lambda_r I]X = 0$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ Eigen values are 4, −1.

$\lambda = 4$: Eigen vector is $[A - 4I]X = 0$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} X = 0$$

$$R_2 \to R_2 + R_1 \sim \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = k \Rightarrow -3x + 2y = 0 \to x = 2k/3$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k/3 \\ k \end{bmatrix}$$ Infinite values. Ex: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\lambda = -1$: Eigen vector is $[A + I] = 0$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to 2R_2 - 3R_1 \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = l \Rightarrow 2x + 2y = 0 \Rightarrow x = -l$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l \\ l \end{bmatrix}$$

Note: 1. Eigen vectors corresponding to given eigen values are always infinite. in number

2. If $\lambda = \lambda_r$ is an eigen ~~vector~~ value of a matrix A then it always gives a non trivial solution for the homogenous system $[A - \lambda_r I]X = 0$

Note: $AX = \lambda X$

$AX = \lambda X I$

$(A - \lambda I)X = 0$ → Trivial solution $\Rightarrow |A - \lambda I| \neq 0$

→ $\underbrace{\text{Infinite solution}}_{\text{Non trivial solution.}} \Rightarrow |A - \lambda I| = 0$

* Eigen vectors corresponding to different eigen values of a real symmetric matrix are always orthogonal.

Ex: $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ Eigen vectors are $\underset{x_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{x_2}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$

$$x_1^T \cdot x_2 = [2 \quad 1] \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= [-2 + 2] = [0]_{(x_1)}$$

$\therefore x_1, x_2$ are Orthogonal vectors

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Prob: The vector $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ is an eigen vector of the matrix $A=\begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$. The eigen value corresponding to the eigen vector is ___

Sol: $(A-\lambda I)x=0$

$$\begin{bmatrix}-2-\lambda & 2 & -3\\2 & 1-\lambda & -6\\-1 & -2 & 0-\lambda\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Consider any row $\Rightarrow$ $-2-\lambda+4+3=0$ $\Rightarrow$ $\lambda=5$

Prob: The matrix $\begin{bmatrix}1 & 2 & 4\\3 & 0 & 6\\1 & 1 & P\end{bmatrix}$ has an eigen value $=3$. Sum of other two eigen values is ___

Sol: $1+0+P=3+\lambda_2+\lambda_3 \Rightarrow \lambda_2+\lambda_3=P-2$

Prob: Consider $A=\begin{bmatrix}2 & 3\\x & y\end{bmatrix}$. The eigen values of A are 4 & 8. The values of x, y are

$2+y=4+8$ $\qquad$ $\begin{vmatrix}2 & 3\\x & y\end{vmatrix}=4\times 8$

$y=12-2=10$ $\qquad$ $2(10)-3x=32 \Rightarrow x=-4$

Prob: The eigen values of 2×2 matrix A are given by -2 & -3 resp. The eigen values of $(x+I)^{-1}.(x+5I)$

Sol: $(x+I)^{-1}.(x+5I) \Rightarrow \frac{1}{(\lambda+1)}.(\lambda+5)$

$$=\frac{\lambda+1+4}{\lambda+1}=1+\frac{4}{\lambda+1}$$

$\lambda=-2 \Rightarrow 1+\frac{4}{-2+1}=-3$

$\lambda=-3 \Rightarrow 1+\frac{4}{-3+1}=-1$

Prob: The eigen vector of 3×3 matrix are $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}$ are orthogonal. What will be the 3rd orthogonal eigen vector.

Sol: $x_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\quad x_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\quad$ Let $x_3=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Here 3 vectors are orthogonal

$x_1^T x_3=0$ $\qquad$ $x_2^T x_3=0$

$\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\qquad$ $\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$x+z=0 \rightarrow (i)$ $\qquad$ $x-z=0 \rightarrow (ii)$

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add (i) & (ii) $\quad 2x = 0$

$x = 0$

put in (i) $\quad 0 + z = 0 \Rightarrow z = 0$

Zero vector can't be a eigen vector $y \neq 0 \quad \therefore y = c$ say. $(\neq 0)$

The third eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Prob: The eigen vector of 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are given by $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ b \end{bmatrix}$, what is $(a+b)$?

sol: $\lambda = 1, 2 \qquad$ put $\lambda = 1$

$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1\begin{bmatrix} 1 \\ a \end{bmatrix} \Rightarrow 1 + 2a = 1 \Rightarrow a = 0$

put $\lambda = 2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2\begin{bmatrix} 1 \\ b \end{bmatrix} \Rightarrow 1 + 2b = 2 \Rightarrow b = 1/2$

$a + b = 0 + 1/2 = 1/2.$

Prob: which of the following is an eigen vector of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

sol: $[A - 2I]x = 0$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \Rightarrow \underbrace{\begin{matrix} y = 0 \\ z = 0 \end{matrix}}_{\text{fix}} \quad$ Let $x = k \quad \therefore$ vector $= \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Prob: which of the following is an eigen vector of matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

sol: ✓a) $[1 \; 1 \; 0]^T$ b) $[1 \; 0 \; 1]^T$ c) $[0 \; 1 \; 1]^T$ d) $[1 \; 1 \; 1]^T$

Prob: which of the following is an eigen vector of $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$

a) $[-1 \; 1]^T$ ✓b) $[3 \; -1]^T$ c) $[1 \; -1]^T$ d) $[-2 \; 1]^T$

$\lambda = 1, -2$

$\lambda = 1 \Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad -x - 3y = 0$

Let $y = k \Rightarrow x = -3k$

$\therefore$ vector $= \begin{bmatrix} -3k \\ k \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for $k = -1$

Note: $|A - \lambda I| = \lambda^2 - Tr(A).\lambda + \det(A)$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $Tr(A) = a + d$

$\det(A) = ad - bc.$

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Cayley - Hamilton theorem: Every square matrix satisfies it's own characterstic equation ie if the characterstic equation of the nth order square matrix A is

$$|A-\lambda I| = (-1)^n\lambda^n + K_1\lambda^{n-1} + \cdots + K_n = 0$$

then $$(-1)^n A^n + K_1 A^{n-1} + \cdots + K_n = 0$$

Prob: If $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ then $A^{-1} = ?$

Sol: $A^2 + 7A + 6I = 0 \Rightarrow A + 7I + 6A^{-1} = 0$

$$\Rightarrow A^{-1} = \frac{1}{6}(-A - 7I)$$

Prob: If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ then $A^8 =$

Sol: $\lambda^2 + 0\lambda - 5 = 0 \Rightarrow \lambda^2 = 5I$

$A^2 = 5I \Rightarrow A^8 = 625I = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

Prob: If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $A^{-1} =$

Sol: $\lambda = 1, 1, 1 \Rightarrow (\lambda - 1)^3 = 0$

$(A-1)^3 = 0$

$A^3 - 1 + 3A - 3A^2 = 0$

$A^2 - A^{-1} + 3I - 3A = 0 \Rightarrow A^{-1} = A^2 - 3A + 3I$

* By cayley - hamilton theorem : $A^2 - A[Tr(A)] + |A| \cdot I = 0$

* For 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ characterstic equation is

$$\lambda^3 - \lambda^2[Tr(A)] + \lambda\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right\} - |A| = 0$$

Method to find higher powers of A:

Let $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - (-3)\lambda + 2 = 0$

$\Rightarrow \lambda = -1, -2$

If λ values are not repeated then $\lambda^n = a\lambda + b \rightarrow (1)$

for $\lambda = -1$ $\quad (-1)^n = -a + b \rightarrow (2)$

for $\lambda = -2$ $\quad (-2)^n = -2a + b \rightarrow (3)$

put (3) in (2) $\Rightarrow$ [illegible]

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$\therefore b = (-1)^n + a$

$= (-1)^n + (-1)^n - (-2)^n = 2(-1)^n - (-2)^n$

put a & b in eq.(1) $\Rightarrow \lambda^n = [(-1)^n - (-2)^n]\lambda + [2(-1)^n - (-2)^n]$

by C-H theorem $A^n = [(-1)^n - (-2)^n]A + [2(-1)^n - (-2)^n]I$

Ex: put $n=9$ $A^9 = 511A + 510I$

<u>Properties:</u>

Idompotent $A^2 = A$

Involutory $A^2 = I$

Nilpotent $A^n = 0$
n- index

Hermetian $A = \bar{A}^T = A^\theta = A^*$

skew hermetian $A = \bar{A}^T = -A^\theta = -A^*$

unitary $U \cdot U^\theta = I \Rightarrow U^{-1} = U^\theta$

prob: The matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Nilpotent

prob: The matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow$ Involutory

prob: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

$A^2 = I \Rightarrow$ Involutory $\Rightarrow A \cdot A = I \rightarrow A \cdot A^T = I$

$A^T = A \Rightarrow$ Symmetric $\Rightarrow$ Orthogonal.

CALCULUS

CALCULUS

Basics (pre-calculus):

Logorithms: If $\log_b a = m$ then $a = b^m$

1. $\log_c(ab) = \log_c a + \log_c b$
2. $\log_c(a/b) = \log_c a - \log_c b$
3. $\log_{b^n} a^m = \frac{m}{n}\log_b a$
4. $\log_b a = \frac{1}{\log_a b} = \log_c a \cdot \log_b c = \frac{\log_c a}{\log_c b}$
5. $a^x = e^{\log_e a^x} = e^{x\log_e a}$ $\qquad \log_e \to \ln$
6. $\log_a a = 1$, $\log_a 1 = 0$, $\log_c(1/a) = -\log_c a$

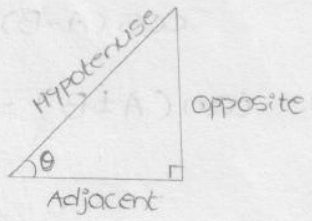
Ex: $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = 2.322$

GATE IN(08): The expression $e^{-\ln x}$ for $x>0$ is equal to

sol: $e^{-\ln x} = e^{-\log_e x}$

$= e^{\log_e(1/x)} = 1/x = x^{-1}$

Trignometry:

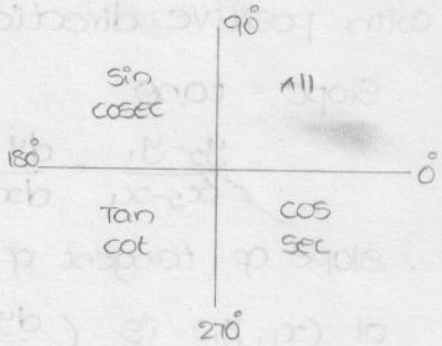


$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\operatorname{cosec}\theta}$

$\cos\theta = \frac{\text{Adjacent}}{\text{hypotenuse}} = \frac{1}{\sec\theta}$

$\tan\theta = \frac{\text{opposite}}{\text{Adjacent}} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$

	0°	30°	45°	60°	90°
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞



(i) If the angle is $(90\pm\theta)$, $(270\pm\theta)$ i.e about vertical line, then co-function of initial function will apply.

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(ii) If the angle is $(180 \pm \theta)$, $(360 \pm \theta)$ i.e about horizontal line then same function will apply.

(iii) The positivity or negativity of the resulting function depends on the quadrant in which <u>initial function</u> lies.

Ex: $\sin(270+\theta) = -\cos\theta$ $\qquad$ $\cot(270-\theta) = \tan\theta$

$\cos(90+\theta) = -\sin\theta$ $\qquad$ $\sec(180+\theta) = -\sec\theta$.

Formulas:

1. $\sin^2\theta + \cos^2\theta = 1$; $\operatorname{cosec}^2\theta - \cot^2\theta = 1$; $1 + \tan^2\theta = \sec^2\theta$.

2. $\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$

put $B = A \Rightarrow \sin 2A = 2\sin A \cos A$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

3. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

put $B = A$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A = 2\cos^2 A - 1$

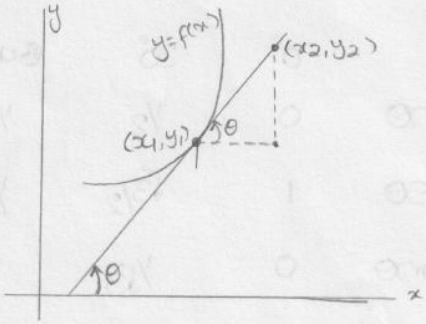
$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

4. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

put $B = A \Rightarrow \tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Slope of a line: It is the tangent of angle made by the line with positive direction of x-axis.



Slope $= \tan\theta$

$= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{dy}{dx}$

slope of tangent of curve $y = f(x)$ at (x_1, y_1) is $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$

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Equation of straight line:

1. point slope form: point is (x_1, y_1) & slope m

$$y - y_1 = m(x - x_1)$$

2. Two point form:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

3. slope intercept form: slope is 'm', y-intercept is 'c'

$$y = mx + c$$

4. Intercept form: 'x' intercept is 'a', 'y' intercept is 'b'

$$\frac{x}{a} + \frac{y}{b} = 1$$

5. General equation of straight line $ax + by + c = 0$

$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$

slope $= -a/b$ y-intercept $-c/b$

$$ax + by = -c$$

$$\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1 \Rightarrow \text{x- intercept} = -c/a$$

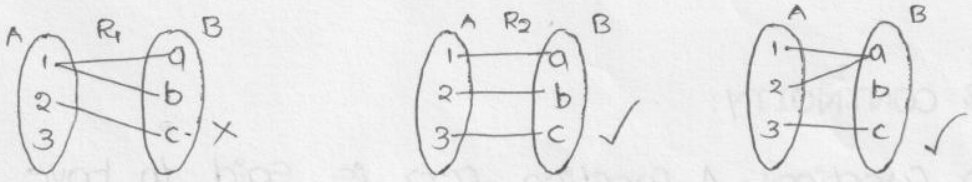
Function: A function from set A to set B is a relation from A to B satisfying the condition

"To each element in A, there exist a unique element in B"

i.e (i) every element in A must be associated

(ii) It must be associated with a unique element in B.

Ex:



Explicit function: $z = f(x_1, x_2, \ldots x_n)$

Dependent variable

Independent variable

Implicit function: $\phi(z, x_1, x_2, \ldots x_n) = C$

composit function: If $z = f(x, y)$ where $x = \phi(t)$ & $y = \psi(t)$ i.e

z is a function of some function.

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Some Special functions:

1. Even function: $f(-x) = f(x)$ Ex: x^2, $\cos x$ ...

2. odd function: $f(-x) = -f(x)$ Ex: x, $\sin x$ ...

3. Modulus function:

$$f(x) = |x| = \begin{cases} x & ; x>0 \\ -x & ; x<0 \\ 0 & ; x=0 \end{cases}$$

GATE-99: $f(x) = e^x$ Neither even nor odd

4. Step / Greatest integer function:

$$f(x) = [x] = n \in \mathbb{Z} \quad \text{where } n \le x < n+1$$

Ex: $[7.2] = 7$; $[7.999] = 7$; $[-1.2] = -2$

Symmetric properties of the curve: Let $f(x,y) = c$ be the eqn of a curve

(i) If $f(x,y)$ contains only even powers of x i.e $f(-x,y) = f(x,y)$ then it is symmetric about y-axis

(ii) If $f(x,y)$ contains only even powers of y i.e $f(x,-y) = f(x,y)$ then it is symmetric about x-axis

(iii) If $f(x,y) = f(y,x)$ then it is symmetric about $y=x$.

GATE (97): The curve given by the equation $x^2+y^2 = 3axy$ is

(a) symmetrical about x-axis (b) y-axis (c) line $y=x$

(d) tangential to $x=y=a/3$

Sol: $f(x,y) = f(y,x) \Rightarrow x^2+y^2 = 3axy$

$\therefore$ symmetrical about line $y=x$.

LIMITS & CONTINUITY:

Limit of a function: A function $f(x)$ is said to have limit value 'l' as x tend to 'a' if

$$\lim_{x\to a} f(x) = l$$

Left limit: when $x<a$, $x\to a$ $\quad \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

Right limit: when $x>a$, $x\to a$ $\quad \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

A limit exists if LHL = RHL

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Ex: If $f(x) = \begin{cases} 2x+3 & \text{for } x \geq 2 \\ x+9 & \text{for } x < 2 \end{cases}$ then $\underset{x \to 2}{lt}\, f(x) = ?$

Sol: $\underset{x \to 2^+}{lt}\, 2x+3 = 7 \neq \underset{x \to 2^-}{lt}\, x+9 = 11$

$\therefore$ Limit doesnot exist.

Indeterminate form: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$

Standard limits:

(i) $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

(ii) $\underset{x \to 0}{lt}\, \frac{e^{mx}-1}{x} = e$

(iii) $\underset{x \to 0}{lt}\, \frac{a^x - 1}{x} = \log_e a$

(iv) $\underset{x \to 0}{lt}\, [1+ax]^{1/x} = e^a$

(v) $\underset{x \to 0}{lt}\, \left[1 + \frac{a}{x}\right]^x = e^a$

(vi) $\underset{x \to 0}{lt}\, \frac{\sin x}{x} = 1$ or $\underset{x \to 0}{lt}\, \frac{\tan x}{x} = 1$

(vii) $\underset{x \to \infty}{lt}\, \frac{\sin x}{x} = 0$

(viii) $\underset{x \to 0}{lt}\, \frac{\sin mx}{x} = m$

(ix) $\underset{x \to 0}{lt}\, \left[\frac{a^x + b^x}{2}\right]^{1/x} = \sqrt{ab}$

(x) $\underset{x \to 0}{lt}\, [\cos x + a \sin bx]^{1/x} = e^{ab}$

(xi) $\underset{x \to 0}{lt}\, \left[\frac{1 - \cos ax}{x^2}\right] = \frac{a^2}{2}$

(xii) $\underset{x \to \infty}{lt}\, \frac{\log_e x}{x} = 0$

Limit properties:

1. $\underset{x \to a}{lt}\, [c\, f(x)] = c \underset{x \to a}{lt}\, f(x)$

2. $\underset{x \to a}{lt}\, [f(x) \pm g(x)] = \underset{x \to a}{lt}\, f(x) \pm \underset{x \to a}{lt}\, g(x)$

3. $\underset{x \to a}{lt}\, [f(x) \cdot g(x)] = \underset{x \to a}{lt}\, f(x) \cdot \underset{x \to a}{lt}\, g(x)$

4. $\underset{x \to a}{lt}\, \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{lt}\, f(x)}{\underset{x \to a}{lt}\, g(x)}$ provided $\underset{x \to a}{lt}\, g(x) \neq 0$

5. $\underset{x \to a}{lt}\, [f(x)]^n = \left[\underset{x \to a}{lt}\, f(x)\right]^n$ where n is a real number.

6. $\underset{x \to a}{lt}\, \sqrt[n]{f(x)} = \sqrt[n]{\underset{x \to a}{lt}\, f(x)}$

7. $\underset{x \to a}{lt}\, c = c$; c is a real number

8. If $P(x)$ is a polynomial then $\underset{x \to a}{lt}\, P(x) = P(a)$

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(GATE-95): $\lim_{x\to 0} x\sin(1/x) = \underline{\quad}$

sol: Let $x = t \Rightarrow 1/x = 1/t$, $t \to \infty$

$$\lim_{x\to 0} x\sin(1/x) = \lim_{t\to\infty} \frac{\sin t}{t} = 0$$

(GATE-99): $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} =$

sol: $$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = \frac{1}{\sqrt{1+0}} = 1$$

(GATE-01 IN): $\lim_{x\to \pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4}$

sol: Let $x - \pi/4 = t \Rightarrow t \to 0$

$$\lim_{x\to\pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4} = \lim_{t\to 0} \frac{\sin 2t}{t} = 2$$

(GATE-02 CE): $\lim_{n\to\infty} n^{1/n} =$

sol: Let $y = \lim_{n\to\infty} n^{1/n}$ Apply log on b.s

$$\log_e y = \lim_{n\to\infty} \log_e n^{1/n} = \lim_{n\to\infty} \frac{1}{n}\log_e n = 0$$

$$\log_e y = 0 \Rightarrow y = e^0 = 1$$

(GATE-03): $\lim_{x\to 0} \frac{\sin^2 x}{x} = \underline{0}$

sol: $$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \times \sin x$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \sin x$$

$$= 1 \times 0 = 0$$

(GATE-04): $\lim_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} =$

sol: $$\lim_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} = \lim_{x\to 0} \frac{x^2(x+1)}{x^2(2x-7)}$$

$$= \lim_{x\to 0} \frac{x+1}{2x-7} = \frac{1}{-7} = -\frac{1}{7}$$

(GATE-07 ME): $\lim_{x\to 0} \frac{e^x - (1+x+x^2/2)}{x^3}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

by substituting we will get

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$$\lim_{x\to 0} \frac{\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\infty}{x^3} = \lim_{x\to 0}\left(\frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\cdots+\infty\right)$$

$$= \frac{1}{3!}+0 = \frac{1}{6}$$

(GATE-07 EC): $\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\theta}$ is

Sol: $\lim_{x\to 0} \frac{\sin mx}{x} = m \Rightarrow \lim_{\theta\to 0} \frac{\sin(\frac{1}{2}\cdot\theta)}{\theta} = \frac{1}{2} = 0.5$

(GATE-08 CS): $\lim_{x\to\infty} \frac{x-\sin x}{x+\cos x} =$

Sol: $= \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = \frac{1-\lim_{x\to\infty}\frac{\sin x}{x}}{1+\lim_{x\to\infty}\frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$

(GATE-08 ME): $\lim_{x\to 8} \frac{x^{1/3}-2}{x-8}$

Sol: $= \lim_{x\to 8} \frac{x^{1/3}-8^{1/3}}{x-8}$ $\qquad \lim_{x\to a}\frac{x^n-a^n}{x-a} = n\cdot a^{n-1}$

$$= \frac{1}{3}\cdot 8^{\frac{1}{3}-1} = \frac{1}{3}\times 8^{-2/3} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}$$

(GATE-08 PI): $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} =$

Sol: $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} = \lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{1}{e^x}$

$$= 1\times\frac{1}{e^0} = 1\times 1 = 1$$

(GATE-10 CS): $\lim_{n\to\infty}(1-1/n)^{2n} =$

Sol: $\lim_{n\to\infty}(1-1/n)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{-1}{n}\right)^n\right]^2$

$$= \left[\lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n\right]^2 = (e^{-1})^2 = e^{-2}$$

(GATE-14 CE): $\lim_{x\to\infty} \frac{x+\sin x}{x} =$

Sol: $\lim_{x\to\infty}\frac{x}{x} + \underbrace{\lim_{x\to\infty}\frac{\sin x}{x}}_{0} = \lim_{x\to\infty} 1+0 = 1+0 = 1$

Pdo $\lim_{x\to 0}\frac{|x|}{x} = ?$

Sol: $LHL = \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{h\to 0}\frac{|-h|}{-h} = -1$

$RHL = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{h\to 0}\frac{|h|}{h} = 1$

$LHL \neq RHL \quad \therefore$ limit doesn't exist.

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Continuity of $f(x)$: A function $f(x)$ is said to be continous at $x=a$, when $\lim_{x\to a} f(x) = f(a)$

Ex: $f(x) = \begin{cases} 2x+3 & \text{for } x>2 \\ x+5 & \text{for } x<2 \\ 15 & \text{for } x=2 \end{cases}$

$$\lim_{x\to 2^+} f(x) = 7 = \lim_{x\to 2^-} f(x) = 7 \neq f(2) = 15$$

Limit exists but not continous.

(GATE-97): If $y=|x|$ for $x<0$ and $y=x$ for $x\geq 0$ then

✓(a) $\frac{dy}{dx}$ is discontinous at $x=0$ (b) y is discontinous at $x=0$

(c) y is not defined at $x=0$ (d) Both y & dy/dx are discontinous at $x=0$

Sol: $y=|x|$ is continous everywhere

$\frac{dy}{dx} = \frac{|x|}{x}$ is not continous at $x=0$ because $\lim_{x\to 0} \frac{|x|}{x} =$ not exist

$\therefore$ Ans: (a)

(GATE-2013 CS): which of the following function is continous at $x=3$

✓(a) $f(x) = \begin{cases} 2 & \text{if } x=3 \\ x-1 & \text{if } x>3 \\ \frac{x+3}{3} & \text{if } x<3 \end{cases}$ (b) $f(x) = \begin{cases} 4 & \text{if } x=3 \\ 8-x & \text{if } x\neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3 & \text{if } x\leq 3 \\ x-4 & \text{if } x>3 \end{cases}$ (d) $f(x) = \frac{1}{x^3-27}$ if $x\neq 3$

Sol: (a) $\lim_{x\to 3^+} f(x) = 3-1 = 2$ $\lim_{x\to 3^-} f(x) = \frac{3+3}{3} = 2$ $f(3) = 2$

$\therefore$ $f(x)$ is continous at $x=3$

Ans: (a)

Note: If a function is continous at $x=a$ then limit of the function also exists at $x=a$ and is equal to $f(a)$.

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Differentiability: A function $f(x)$ is said to be differentiable at $x=a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. and is denoted $f'(a)$

In general the derivative function is given by

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Ex: If $f(x) = x^2$ then $f'(x) = 2x$ & $f'(x) = 2a$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} x+a = 2a$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{x^2+\Delta x^2+2\cdot x\cdot\Delta x - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(\Delta x+2x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \Delta x + 2x = 0+2x = 2x$$

* Left hand derivative: $LHD = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}$

Right hand derivative: $RHD = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Necessary condition for function to be differential is

$$LHD = RHD$$

* Every differentiable function is a continous function. But every continous function is not differentiable

Note:

(i) $|x|$ is not differentiable at $x=0$

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$LHD \neq RHD \Rightarrow |x|$ not differentiable.

(ii) $|x-a|$ is not differentiable at $x=a$

(iii) $|ax+b|$ is not differentiable at $x=-b/a$

(GATE-95): The function $f(x) = |x+1|$ on the interval $[-2,0]$

Sol: $|x+a|$ is continous everywhere.

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$|x+a|$ is not differentiable at $x=-a$.

$\therefore$ $|x+1|$ is not differentiable at $x=-1 \in [-2,0]$

Ans: continous, but not differentiable.

(GATE-07 IN): The function $f(x) = |x|^3$, at $x=0$ is = (where x is real)

(a) continous but not differentiable

(b) once differentiable but not twice

✓(c) twice " " thrice (d) thrice differentiable.

Sol:

$$f(x) = |x|^3 = \begin{cases} x^3 & ; x>0 \\ -x^3 & ; x<0 \\ 0 & ; x=0 \end{cases}$$

$f(0)=0$, $LHL = RHL = 0 \Rightarrow$ continous.

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0} \frac{|-h|^3-0}{-h} = \lim_{h\to 0} \frac{-h^3}{-h} = \lim_{h\to 0} h^2 = 0$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^3-0}{-h} = \lim_{h\to 0} -h^2 = 0$$

$LHD = RHD \Rightarrow$ differentiable.

$$f'(x) = \begin{cases} 3x^2 & ; x>0 \\ -3x^2 & ; x<0 \\ 0 & ; x=0 \end{cases} \qquad f''(x) = \begin{cases} 6x & ; x>0 \\ -6x & ; x<0 \\ 0 & ; x=0 \end{cases} \qquad f'''(x) = \begin{cases} 6 & ; x>0 \\ -6 & ; x<0 \\ \text{Not diff'be} & ; x=0 \end{cases}$$

$\therefore$ $f(x)$ is twice differentiable but not thrice.

(GATE-10 ME): The function $y = |2-3x|$

Sol: $y = |2-3x|$ continous $\forall x \in R$

$y = |2-3x|$ not differentiable at $x = 2/3$

$2-3x=0$
$x = 2/3$

Derivatives of Some functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	$\to nx^{n-1}$	e^x	$\to e^x$	$\sec x$	$\to \sec x \cdot \tan x$
$\sqrt{x}$	$\to \frac{1}{2\sqrt{x}}$	$\log_e x$	$\to 1/x$	$\operatorname{cosec} x$	$\to -\operatorname{cosec} x \cdot \cot x$
$\frac{1}{x^n}$	$\to \frac{-n}{x^{n+1}}$	$\sin x$	$\to \cos x$	$\sin^{-1}x$ (or) $-\cos^{-1}x$	$\to \frac{1}{\sqrt{1-x^2}}$
$\frac{1}{\sqrt{x}}$	$\to \frac{-1}{2x\sqrt{x}}$	$\cos x$	$\to -\sin x$	$\tan^{-1}x$ (or) $-\cot^{-1}x$	$\to \frac{1}{1+x^2}$
a^x	$\to a^x \log_e a$	$\tan x$	$\to \sec^2 x$	$\sec^{-1}x$ (or) $-\operatorname{cosec}^{-1}x$	$\to \frac{1}{x\sqrt{x^2-1}}$
		$\cot x$	$\to -\operatorname{cosec}^2 x$		

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Note: (1) $(uv)' = uv' + u'v$

(2) $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

(3) $\frac{d}{dx}[f(g(x))] = f'(g(x)).g'(x)$

(4) Leibniz formula: $(f.g)^n = {}^nC_0 f^n.g + {}^nC_1 f^{n-1}.g' + \dots + {}^nC_n f.g^n$

where 'n' is order of differentiation.

L'Hospital rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

Note: 1. In case of other indeterminate forms $0.\infty$, 0^0, ∞^0, 1^∞, $\infty-\infty$, first we have to reduce it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use this rule

2. For the forms 0^0, ∞^0, 1^∞ take logarithm and then take limits

3. $0^\infty = 0$, $\infty^\infty = \infty$, $\infty.\infty = \infty$, $\infty+\infty = \infty$, $\infty^{-\infty} = 0$ are not indeterminate

Ex: $\lim_{x\to 0} \frac{\ln(1+x)}{\sin x} = \frac{0}{0}$ form

Apply L'H rule $= \lim_{x\to 0} \frac{1/(1+x)}{\cos x} = \frac{1}{1} = 1$

Ex: $\lim_{x\to 0} x^x = 0^0$ form Let $y = x^x$

Apply logarithm first $\ln y = \ln x^x = x\ln x$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} x.\ln x$$

$$\ln\left[\lim_{x\to 0} y\right] = \lim_{x\to 0} \frac{\ln x}{1/x} = \frac{\infty}{\infty} \text{ Apply L'H}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} x = 0$$

$$\lim_{x\to 0} y = \lim_{x\to 0} x^x = e^0 = 1$$

(GATE-93 ME): $\lim_{x\to 0} \frac{x(e^x-1) + 2(\cos x - 1)}{x(1-\cos x)}$ $\frac{0}{0}$

Sol: using L'H rule

$$\lim_{x\to 0} \frac{(e^x-1) + x.e^x + 2(-\sin x)}{(1-\cos x) + x(\sin x)} \quad \frac{0}{0}$$

Again apply L'H rule.

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$$= \lim_{x\to 0} \frac{e^x + e^x + xe^x - 2\cos x}{\sin x + \sin x + x\cos x} \qquad \frac{0}{0}$$

Again apply L'H rule

$$= \lim_{x\to 0} \frac{e^x + e^x + e^x + xe^x + 2\sin x}{\cos x + \cos x + \cos x - x\sin x} = \frac{1+1+1+0}{1+1+1-0} = 1$$

(GATE-95 CS): $\lim_{x\to\infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2} \qquad \frac{\infty}{\infty}$

Sol: Apply L' Hospital

$$\lim_{x\to\infty} \frac{3x^2 + \sin x}{2x + 2\sin x \cdot \cos x} = \lim_{x\to\infty} \frac{3x + \frac{\sin x}{x}}{2 + \frac{\sin 2x}{x}} = \frac{3\times\infty + 0}{2+0} = \infty$$

(GATE-99 IN): $\lim_{x\to 0} \frac{1}{10} \cdot \frac{1 - e^{-j5x}}{1 - e^{-jx}} \qquad \frac{0}{0}$

Sol: Apply L' Hospital

$$\lim_{x\to 0} \frac{1}{10} \cdot \frac{-(-5j)e^{-j5x}}{-(-j)\cdot e^{-jx}} = \frac{5j}{10j} = 1/2$$

(GATE-99): $\lim_{x\to a} (x-a)^{x-a} \qquad 0^0$

Sol: Let $y = (x-a)^{(x-a)} \Rightarrow \log y = (x-a)\log(x-a)$

$$\lim_{x\to a} \log y = \lim_{x\to a} [(x-a)\log(x-a)]$$

$$\log\left[\lim_{x\to a} y\right] = \lim_{x\to a} \frac{\log(x-a)}{1/(x-a)} \qquad \frac{\infty}{\infty}$$

Apply L'H rule $= \lim_{x\to a} \frac{1/x-a}{-1/(x-a)^2} = \lim_{x\to a} -(x-a) = 0$

$$\lim_{x\to a} y = \lim_{x\to a} (x-a)^{(x-a)} = e^0 = 1$$

(GATE-07 PI): $\lim_{x\to \pi/4} \frac{\cos x - \sin x}{x - \pi/4} \qquad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x\to\pi/4} \frac{-\sin x - \sin x}{1} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(GATE-12 ME, PI): $\lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right) \qquad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x\to 0} \frac{\sin x}{2x} \qquad \frac{0}{0}$

Again apply L'H $= \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

(GATE-14 ME): $\lim_{x\to 0} \frac{x - \sin x}{1 - \cos x} \qquad \frac{0}{0}$

Sol: Apply L'H rule

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$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \quad \frac{0}{0}$$

again apply L'H rule $= \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

(GATE-14 ME): $\lim_{x\to 0} \left(\frac{e^{2x}-1}{\sin(4x)} \right) \quad \frac{0}{0}$

Sol: Apply L'H $= \lim_{x\to 0} \frac{2.e^{2x}}{4.\cos 4x} = \frac{2}{4} = 1/2$

**(GATE-14 CE): $\lim_{a\to 0} \frac{x^a - 1}{a} = \frac{0}{0}$

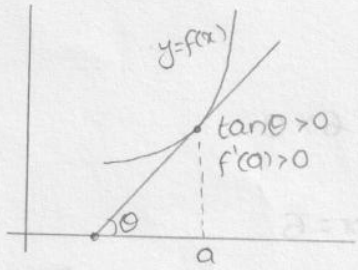
Sol: Apply L'H differentiate w.r.to a

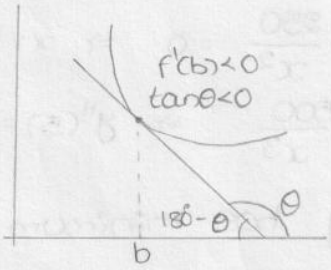
$$= \lim_{a\to 0} \frac{x^a.\log x - 0}{1} = x^0.\log x = \log x.$$

Increasing (or) decreasing nature of function:

A function $y=f(x)$ will have increasing nature at $x=a$, if $f'(a)>0$

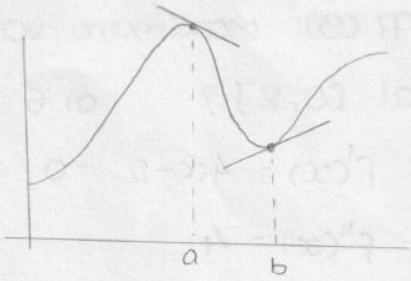
A function $y=f(x)$ will have decreasing nature at $x=b$ if $f'(b)<0$





Local

Maxima and minima:



To obtain the maxima or minima

1. Find $f'(x)$ and solve the equation $f'(x)=0$ to obtain the <u>stationary points</u> say $x=a, b, c$.
2. Find $f''(x)$
3. If $f''(a)<0$, then at $x=a$, $f(x)$ will have maximum value

   If $f''(b)>0$, then at $x=b$, $f(x)$ will have minimum value

   If $f''(c)=0$, then at $x=c$, $f(x)$ won't have minimum/maximum. The point is called <u>saddle point</u>.

Absolute maxima/minima: If $f(x)$ defined in interval $[a,b]$ then

Absolute minimum value = min $\{f(a), f(b)$, all local minimum values$\}$

Absolute maximum value = max $\{f(a), f(b)$, all local maximum values$\}$

If stationary points are out of given interval then don't consider them

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Ex: The function $f(x) = x^3 - 9x^2 + 24x - 12$ has

Sol: $f'(x) = 3x^2 - 18x + 24 = 0$ $\qquad f''(x) = 6x - 18$

$x = 2, 4$ $\qquad f''(2) = -6 < 0$

$f''(4) = 6 > 0$

$\therefore f(x)$ has Max. at $x = 2$, Min at $x = 4$.

(GATE-07 EC)

Ex: The maximum value of $f(x) = x^2 - x - 2$ in the interval $[-4, 4]$ is

✓(a) 18 (b) 10 (c) -2.25 (d) indeterminate

Sol: $f'(x) = 2x - 1 = 0 \Rightarrow x = 1/2$

$f''(x) = 2$ can't find

$f(-4) = 18$ $\quad f(4) = 10$ $\quad f(1/2) = -2.25$

$\therefore f(x)$ has maximum at $x = -4$.

(GATE-94): The function $y = x^2 + \frac{250}{x}$ at $x = 5$ attains

Sol: $y' = 2x - \frac{250}{x^2} = 0 \Rightarrow x = 5$

$y'' = 2 + \frac{500}{x^3} \Rightarrow y''(5) = 2 + 4 = 6 > 0$

$\therefore y$ has minimum at $x = 6$

(GATE-97 CS): Maximum value of function $f(x) = 2x^2 - 2x + 6$ in the interval $[0, 2]$?

a) 6 ✓b) 10 c) 12 d) 5.5 ↘ $f(1/2)$

Sol: $f'(x) = 4x - 2 = 0 \Rightarrow x = 1/2$

$f''(x) = 4$

$f(0) = 0 + 6 = 6$ $\qquad f(2) = 8 - 4 + 6 = 10$

$\therefore$ Maximum value $= 10$ in $[0, 2]$

(GATE-99 CE): Number of inflection points for the curve $y = 2x^4 + x$

Sol: $\frac{d^2y}{dx} = 0 \Rightarrow 8 \times 3 \times x^2 = 0$

$\Rightarrow x = 0 \Rightarrow y = 0$

$\therefore (0,0)$ is the only inflection point.

(GATE-05 EE): For the function $f(x) = x^2 e^{-x}$, the max. occurs when $x = ?$

Sol: $f'(x) = e^{-x}[-x^2 + 2x] = 0 \Rightarrow x = 0, 2$ $\qquad f''(0) = 2 > 0$

$f''(x) = -e^{-x}[-x^2 + 2x] + e^{-x}[-2x + 2]$ $\qquad f''(2) = -2e^{-2} < 0$

$\therefore f(x)$ has max. at $x = 2$

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(GATE-07 EE): The function $f(x) = (x^2-4)^2$ has (x is real number)

(a) only one minimum (b) only two minima (c) Three minima

(d) Three maxima.

Sol: $f'(x) = 2(x^2-4)\cdot 2x = 0 \Rightarrow x = \pm 2, 0$

$f''(x) = 4(3x^2-4)$

$f''(0) = -16 < 0$ $\quad f''(2) = 32 > 0$ $\quad f''(-2) = 32 > 0$

$\therefore f(x)$ has minimum at $x = \pm 2$

(GATE-08 CS): The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is

Sol: $f'(x) = 12x^3 - 48x^2 + 48x = 0 \Rightarrow x = 0, 2, 2$

$f''(x) = 36x^2 - 96x + 48$

$f''(0) = 48 > 0$ $\quad f''(2) = 0$ No extrema

$\therefore f(x)$ has <u>only one</u> extrema (minimum) at $x = 0$

(GATE-08 EC): For real values of 'x', the minimum value of function $f(x) = e^x + e^{-x}$ is

Sol: $f(x) = e^x + e^{-x}$ where $x \in R$

$f'(x) = e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = 0$

$f''(x) = e^x + e^{-x} \Rightarrow f''(0) = 1 + 1 = 2 > 0$

$\therefore f(x)$ has Minimum at $x = 0$, Minimum value $f(0) = 2$

(GATE-10 EC): If $e^y = x^{1/x}$ then y has a

Sol: $e^y = x^{1/x} \Rightarrow y = \log x^{1/x} = \dfrac{\log x}{x}$

$y' = \dfrac{1 - \log x}{x^2} = 0 \Rightarrow x = e$

$y'' = \dfrac{x^2(-1/x) - (1 - \log x)x}{x^4}$

$y''(e) = -e/e^4 < 0 \Rightarrow y$ has maximum at $x = e$

(GATE-12 EC, EE, IN): The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ Similar Ques. in GATE-14 EC

Sol: $f'(x) = 3x^2 - 18x + 24 = 0 \Rightarrow x = 2, 4$

$f''(x) = 6x - 18$ $\quad f''(2) = -6 < 0$ $\quad f''(4) = 6 > 0$

↳ local maximum at $x = 2$

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Local maximum $= f(2) = 25$

In the interval $[1,6]$ $\quad f(1) = 21 \quad f(6) = 41$

$\therefore$ Absolute maximum $= \max\{f(1), f(6), \text{local max.}\}$

$= \max\{21, 41, 25\} = 41.$

(GATE-14 EC): For $0 \le t < \infty$ the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

Sol: $f'(t) = (-e^{-t} + 4e^{-2t}) = 0 \Rightarrow e^{-t} = 1/4$

$\Rightarrow t = \log_e 4$

$f''(t) = (e^{-t} - 8e^{-2t})$

$f''(\log_e 4) = e^{-\log_e 4} - 8e^{-2\log_e 4} = \frac{1}{4} - 8 \cdot \frac{1}{16} = -1/4 < 0$

$\therefore f(t)$ has maximum value at $t = \log_e 4$

(GATE-14 EC): The max. value of function $f(x) = \ln(1+x) - x$ (where $x > -1$) occurs at $x =$

Sol: $f'(x) = \frac{1}{1+x} - 1 = 0 \Rightarrow x = 0$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1 < 0$

$\therefore f(x)$ has maximum at $x = 0$

(GATE-14 EE): The max. value of the function $f(x) = xe^{-x}$ in the interval $(0, \infty)$ is

Sol: $f'(x) = (-x \cdot e^{-x} + e^{-x}) = 0 \Rightarrow x = 1$

At $x = 1$, $f''(x) < 0$

$\therefore$ Maximum exists at $x = 1$ and $f(1) = 1 \cdot e^{-1} = 1/e$

(GATE-14 EE): Minimum of the real valued function $f(x) = (x-1)^{2/3}$ occurs at $x =$

Sol: $f(x) = \left[(x-1)^{1/3}\right]^2 \quad \therefore f(x) \ge 0$ always

The minimum value of $f(x)$, $f(x) = 0$

$(x-1)^{2/3} = 0 \Rightarrow x = 1$

(GATE-14 EE): The minimum value of $f(x) = x^3 - 3x^2 - 24x + 100$ in interval $[-3, 3]$ is

Sol: $f'(x) = 3x^2 - 6x - 24 = 0 \Rightarrow x = -2, 4$

but $x = 4 \notin [-3, 3]$

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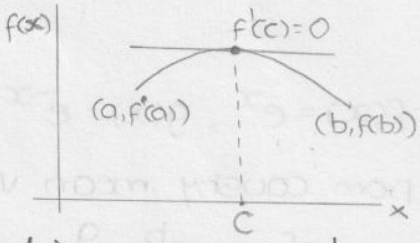
$f''(x) = 6x-6 \Rightarrow f''(-2) = -18 < 0$ we get maximum

$\therefore$ Absolute minimum $= \min\{f(-3), f(3)\}$ $\quad f(-3) = 118 \quad f(3) = 28$

$= \min\{118, 28\} = 28$

MEAN VALUE THEOREMS:

Rolle's theorem: If $f(x)$ is a function defined in an interval $[a,b]$ such that

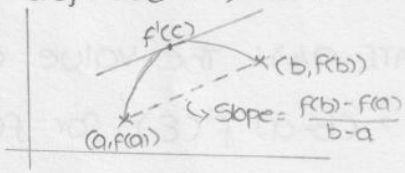


(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ exists in (a,b)

(3) $f(a) = f(b)$ then there exist $c \in (a,b)$ such that $f'(c) = 0$.

Langranges theorem: If $f(x)$ is a function defined in an interval $[a,b]$ such that



(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ is exists in (a,b) then there exist $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \min_{a \le x \le b} f'(x) \le \frac{f(b)-f(a)}{b-a} \le \max_{a \le x \le b} f'(x)$$

cauchy's mean value theorem: If $f(x), g(x)$ are two functions defined in an interval such that

(1) $f(x), g(x)$ are continous in $[a,b]$

(2) $f'(x), g'(x)$ exists in (a,b)

(3) $g'(x) \neq 0 \ \forall \ x \in (a,b)$ then there exist $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

prob: The value 'c' of mean value theorem for $f(x) = x^2 - 5x + 6$ in the interval $[2,3]$ is

Sol: $f(2) = 4-10+6 = 0$

$f(3) = 9-15+6 = 0$

From rolle's theorem $\exists \ c \in (2,3)$ such that $f'(c) = 0$

$\Rightarrow 2c - 5 = 0 \Rightarrow c = 5/2$

prob: $f(x) = x^3 - 4x^2 + 4x$ in the interval $[0,2]$ is

$f(0) = 0 \qquad f(2) = 8-16+8 = 0$

$\therefore$ From rolle's theorem $\exists \ c \in (0,2)$ such that $f'(c) = 0$

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$3c^2 - 8c + 4 = 0 \Rightarrow c = 2, 2/3$ $\quad c = 2 \notin (0,2)$

$c = 2/3 \in (0,2)$

prob: $f(x) = 1 + x^2$ in $[1, 2]$ is

$f(1) = 2 \qquad f(2) = 5 \qquad f(a) \neq f(b)$

from langranges theorem $\quad f'(c) = \frac{5-2}{1} = 3$

$2c = 3 \Rightarrow c = 3/2 \in (1,2)$

prob: $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$

sol: From cauchy mean value theorem

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \Rightarrow -e^{2c} = -e^{a+b}$$

$$\Rightarrow 2c = a + b \Rightarrow c = \frac{a+b}{2} \in (a,b)$$

(GATE-94): The value of ε in the mean value theorem of $f(b) - f(a) = (b-a) f'(\varepsilon)$ for $f(x) = Ax^2 + Bx + C$ in (a,b) is

sol: $f'(\varepsilon) = \frac{f(b) - f(a)}{b-a}$

$$2A\varepsilon + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b-a}$$

By comparing

$2A\varepsilon + B = A(b+a) + B$. $\qquad 2\varepsilon = a + b \Rightarrow \varepsilon = \frac{a+b}{2}$ **

(GATE-95): If $f(0) = 2$ and $f'(x) = \frac{1}{5-x^2}$, then the lower and upper bounds of $f(1)$ estimated by the mean value theorem are

sol: Let $f(x)$ be defined in $[0,1]$ by lagrange's M.V.T

$\exists\, c \in (0,1)$ such that $f'(c) = \frac{f(1) - f(0)}{1-0}$

$$\frac{1}{5-c^2} = \frac{f(1) - 2}{1}$$

We know that $\quad \text{Min}\{f'(x)\} < f'(c) < \text{Max}\{f'(x)\} \qquad c \in (0,1)$

$$\frac{1}{5} < f(1) - 2 < \frac{1}{4} \Rightarrow 2.2 < f(1) < 2.25$$

TAYLOR SERIES: The Taylor series of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Maclaurin's Series: put $a = 0$

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

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Important expansions:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

4. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

5. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

6. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

prob: In the taylor series of $f(x) = e^x$ about $x = 2$ the co.eff of $(x-2)^4$ is

sol: $= \frac{1}{4!} f^{IV}(2) = \frac{1}{4!} e^x \Big|_{x=2} = \frac{e^2}{24}$

prob: In the taylor series of $f(x) = e^x + \sin x$ about $x - \pi$ the co.eff of $(x-\pi)^2$ is

sol: $= \frac{1}{2!} f''(\pi) = \frac{1}{2!}\left[e^x - \sin x\right]\Big|_{x=\pi} = \frac{e^\pi}{2}$

prob: The linear approximation for e^{-x} around $x = 2$ is

sol: $f(x) = f(2) + (x-2) f'(2)$

$= e^{-2} + (x-2)(-e^{-2}) = e^{-2}(3-x)$

(GATE - 00 CE): The Taylor series expansion of $\sin x$ about $x = \pi/6$ is

sol: $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$

$f(x) = \sin x$ and $a = \pi/6$ $\Rightarrow \sin(\pi/6) = 1/2 = f(a)$

$f'(x) = \cos x \Rightarrow f'(a) = \cos \pi/6 = \sqrt{3}/2$

$f''(x) = -\sin x \Rightarrow f''(a) = -\sin \pi/6 = -1/2$

$\sin x = \frac{1}{2} + (x - \pi/6)\sqrt{3}/2 + \frac{(x-\pi/6)^2}{2!}\left(-\frac{1}{2}\right) + \cdots$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 + \cdots$

(GATE - 01 CE): Limit of the following series as x approaches $\pi/2$ is $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x$

sol: $\lim_{x \to \pi/2} \sin x = \sin \pi/2 = 1$

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(GATE-09 EC): The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x=\pi$ is given by

Sol: $$\frac{\sin x}{x-\pi} = \frac{1}{x-\pi}\left[\sin\pi + (x-\pi)(\cos\pi) + \frac{(x-\pi)^2}{2!}(-\sin\pi) + \cdots\right]$$

$$= \frac{1}{x-\pi}\left[0 + (x-\pi)(-1) + \frac{(x-\pi)^2}{2!}(0) + \frac{(x-\pi)^3}{3!} - \cdots\right]$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!}$$

(or)

put $x-\pi = t \Rightarrow x = \pi + t$

$$f = \frac{\sin(\pi+t)}{t} = -\frac{\sin t}{t} \quad \text{about } t=0$$

$$= -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} \cdots\right] = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} \cdots$$

(GATE-14 EC): The Taylor series expansion of $3\sin x + 2\cos x$ is

Sol: $$3\sin x + 2\cos x = 3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] + 2\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= 2 + 3x - x^2 - \frac{x^3}{2} + \cdots$$

PARTIAL DIFFERENTIATION:

If $f(x,y)$ is a function of two variables (x,y) then

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Ex: $f(x) = x^3 + 3xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 \qquad \frac{\partial f}{\partial y} = 3x(2y) = 6xy$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial f^2}{\partial y^2} = 6x$$

$$\frac{\partial^2 f}{\partial y \cdot \partial x} = 6y \qquad \frac{\partial^2 f}{\partial x \cdot \partial y} = 6y$$

Euler's theorem: If $f(x,y)$ is a homogenous function of degree 'n' then

$$x \cdot \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \cdot \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Homogeneous function: A function $f(x,y)$ is said to be a homogeneous function of degree 'n' if

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$f(x,y) = x^n \cdot \phi(y/x)$ (or) $y^n \psi(x/y)$.

Ex: 1. $f(x,y) = x^3 + 3xy^2 = x^3\left[1+3\left(y/x\right)^2\right]$ deg = 3

2. $f(x,y) = \log_e y - \log_e x = x^0 \log_e(y/x)$ deg = 0

3. $f(x,y) = y\sin(y/x) = y\sin\left(\frac{1}{x/y}\right)$ deg = 1

4. $f(x,y) = \sin\left(\frac{x^4+y^4}{x-y}\right)$ Not homogeneous

Note: If $u(x,y) = f(x,y) + g(x,y) + h(x,y)$, where f, g & h are homogenous function with degree m, n & p respectively then

(a) $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = mf + ng + ph$

(b) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$

Note: If $f(u)$ is homogenous function in two variables x & y with degree 'n', then

(a) $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = F(u)$.

(b) $x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = F(u)[F'(u)-1]$

Total differentiation: If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$, then the total derivative of 'z' w.r.to 't' is

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Total differential co.efficient $dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$

Note: 1. If $f(x,y) = c$ is an implicit function then $\frac{dy}{dx} = -\frac{f_x}{f_y}$

(2) If $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Pb: If $z = e^x \sin y$, where $x = \log t$ & $y = t^2$ then $\frac{dz}{dt} = ?$

Sol: $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$ $x = \log t$, $t = e^x$

$= e^x \cdot \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$

$= \frac{e^x}{t}(\sin y + 2t^2 \cos y) = \sin y + 2y\cos y$

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prob: The total derivative of x^2y w.r.to 'x', where x & y are connected by the relation $x^2+xy+y^2=1$ is

sol: Let $u=x^2y$ $\Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

$$= 2xy + x^2\, dy/dx$$

Given relation $x^2+xy+y^2=1 = f(x,y)$ say

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{2x+y}{x+2y}$$

$$\therefore \frac{du}{dx} = 2xy + x^2\left(-\frac{2x+y}{x+2y}\right) = 2xy - x^2\left(\frac{2x+y}{x+2y}\right)$$

prob: If $u = f(x+cy)+g(x-cy)$ then $\frac{u_{xx}}{u_{yy}} =$

sol: Let $r = x+cy$ $\quad s = x-cy$

$$\frac{\partial r}{\partial x} = 1 \quad \frac{\partial r}{\partial y} = c \quad \frac{\partial s}{\partial x} = 1 \quad \frac{\partial s}{\partial y} = -c$$

$$u = f(r)+g(s) \Rightarrow u_x = f'(r)\cdot\frac{\partial r}{\partial x} + g'(s)\cdot\frac{\partial s}{\partial x} = f'(r)+g'(s)$$

$$u_{xx} = f''(r)+g''(s)$$

$$u_y = f'(r)\cdot\frac{\partial r}{\partial y} + g'(s)\cdot\frac{\partial s}{\partial y} = f'(r)\cdot c + g'(s)(-c)$$

$$\therefore u_{yy} = f''(r)c^2 + g''(s)\cdot c^2$$

$$\frac{u_{xx}}{u_{yy}} = \frac{1}{c^2} = c^{-2}$$

prob: If $u = \frac{x^2y}{x^{5/2}+y^{5/2}}$ then $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} =$

sol: $$u = \frac{x^2y}{x^{5/2}+y^{5/2}} = \frac{x^2y}{x^2[x^{1/2}+y^{1/2}]} = \frac{y}{x^{1/2}+y^{1/2}}$$

$$= \frac{y}{y^{1/2}\left[\left(\frac{x}{y}\right)^{1/2}+1\right]} = y^{1/2}\cdot\frac{1}{\left[1+(x/y)^{1/2}\right]}$$

$$\therefore n = 1/2$$

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u = \frac{1}{2}\left(\frac{1}{2}-1\right)u$$

$$= -\frac{1}{4}u.$$

prob: If $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}}\right]$ then $xu_x + yu_y =$

sol: $$\operatorname{cosec} u = \frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}} \qquad n = \frac{1}{4}-\frac{1}{5} = \frac{1}{20}$$

$$xu_x + yu_y = n\frac{f(u)}{f'(u)} = \frac{1}{20}\times\frac{\operatorname{cosec} u}{-\operatorname{cosec} u\cdot\cot u} = -\frac{1}{20}\tan u.$$

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(GATE-08 ME): Let $f = y^x$. what is $\frac{\partial^2 f}{\partial x \cdot \partial y}$ at $x=2$, $y=1$?

Sol: $f = y^x \Rightarrow \frac{\partial f}{\partial y} = x \cdot y^{x-1}$ (x const)

$$\frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} = y^{x-1} + x \cdot y^{x-1} \log y \cdot (1) \quad (y \text{ const})$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\substack{x=2 \\ y=1}} = y^{x-1} + x \cdot y^{x-1} \log y = 1^1 + 2 \cdot 1^1 \log 1 = 1$$

Maxima & Minima for function of two variables:

Let $z = f(x,y)$

Consider $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \cdot \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Method: (i) Find p, q, r, s, t.

(ii) Equate p & q to zero for obtaining the stationary points

(iii) At each stationary point find r, s, t.

(a) If $rt - s^2 > 0$, $r > 0$ then the function $f(x,y)$ has a minima at the stationary point

(b) If $rt - s^2 > 0$, $r < 0$ then $f(x,y)$ has a maximum at the stationary point

(c) If $rt - s^2 < 0$, then $f(x,y)$ has no extreme at that point and is known as saddle point.

Constrained Maxima & minima:

Lagranges method of undetermined multipliers:

Let $f(x,y,z)$ & $\phi(x,y,z) = C$ then $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) = 0$

$$\left.\begin{aligned} F_x = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \\ F_y = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \\ F_z = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \text{Lagrange's eq's}$$

λ - lagranges multiplier.

(GATE-93 ME): The function $f(x,y) = x^2 y - 3xy + 2y + x$ has

Sol: $\frac{\partial f}{\partial x} = 2xy - 3y + 1 = 0$ $\qquad$ $\frac{\partial f}{\partial y} = x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

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at $x = 1 \Rightarrow 2(1)y - 3y + 1 = 0 \Rightarrow y = 1 \Rightarrow (1,1)$

at $x = 2 \Rightarrow 2(2)y - 3y + 1 = 0 \Rightarrow y = -1 \Rightarrow (2,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 2y \Rightarrow \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = 2 \qquad \frac{\partial f^2}{\partial x \cdot \partial y} = 2x - 3$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\left[\frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2}\right] - \left[\frac{\partial^2 f}{\partial x \cdot \partial y}\right]^2 < 0 \text{ at both } (1,1) \,\&\, (2,-1)$$

$\therefore$ NO extremum

(GATE-02): The function $f(x,y) = 2x^2 + 2xy - y^3$ has stationary points

Sol: $\frac{\partial f}{\partial x} = 4x + 2y = 0 \qquad \frac{\partial f}{\partial y} = 2x - 3y^2 = 0$

$\Rightarrow 2x = -y \qquad \Rightarrow -y - 3y^2 = 0 \Rightarrow y = 0, -1/3$

$\therefore x = 0, 1/6$

$\therefore (0,0), (1/6, -1/3)$ are the stationary points

(GATE-07 PI): For the function $f(x,y) = x^2 - y^2$ defined on R^2, the point $(0,0)$ is

Sol: $\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x = 0 \qquad \frac{\partial f}{\partial y} = -2y = 0 \Rightarrow y = 0$

$(x,y) = (0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = -2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \cdot \partial y}\right)^2 = 2(-2) = -4 < 0$$

$\therefore$ Neither maxima nor minima

prob: If $u = \log_e\left(\frac{x^4 + y^4}{x + y}\right)$ then $x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} =$

Sol: $e^u = \frac{x^4 + y^4}{x + y} = P$ say $\qquad n = 3$

$$x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y} = nP \Rightarrow x \cdot e^u \cdot \frac{\partial u}{\partial x} + y \cdot e^u \cdot \frac{\partial u}{\partial y} = ne^u$$

$$\Rightarrow x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3$$

Prob: The minimum value of $x^2 + y^2 + z^2$ where $x + y + z = 1$ is

Sol: $f = x^2 + y^2 + z^2 \qquad \phi = x + y + z - 1 \qquad x + y + z = 1$

$F = f + \lambda\phi \qquad x = 1/3 \quad y = 1/3 \quad z = 1/3$

$$\left.\begin{aligned} f_x &= 2x + \lambda = 0 \Rightarrow \lambda = -2x \\ f_y &= 2y + \lambda = 0 \Rightarrow \lambda = -2y \\ f_z &= 2z + \lambda = 0 \Rightarrow \lambda = -2z \end{aligned}\right\} \Rightarrow x = y = z$$

$f_{min} = x^2 + x^2 + x^2$

$= 3x^2 = 3(1/3) = 1$

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Integration: If $\frac{d}{dx} f(x) = F(x)$ then $\int F(x) = f(x) + C$

$f(x)$	$\int f(x)$
x^n	$\frac{x^{n+1}}{n+1}$
$1/x$	$\log_e x$
$1/x^2$	$-1/x$
$\sqrt{x}$	$\frac{2}{3} x\sqrt{x}$
$1/\sqrt{x}$	$2\sqrt{x}$
e^x	e^x
a^x	$a^x/\log_e a$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x \cdot \tan x$	$\sec x$
$\operatorname{cosec} x \cdot \cot x$	$-\operatorname{cosec} x$

$f(x)$	$\int f(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$ (or) $-\cos^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$ (or) $-\cot^{-1} x$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x$ (or) $-\operatorname{cosec}^{-1} x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(x/a)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}(x/a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}(x/a)$

$* \int \frac{f'(x)}{f(x)} dx = \log_e |f(x)|$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\log_e \cos x = \log_e |\sec x|$

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log_e |\sin x|$

$\int \sec x \, dx = \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx = \log_e |\sec x + \tan x|$

$\int \operatorname{cosec} x \, dx = \log_e |\operatorname{cosec} x - \cot x|$

Integration by parts:

ILATE

$\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx \right) dx$

$= uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Ex: $\int \log_e x \, dx = \log_e x \int dx - \int \frac{1}{x} \cdot x \, dx = x(\log_e x - 1)$

$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x = e^x (x^2 - 2x + 2)$

$\int x e^x dx = x e^x - e^x = e^x (x - 1)$

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$$\int \frac{1}{1-\cos x}\,dx = \int \frac{(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx$$

$$= \int \frac{(1+\cos x)}{1-\cos^2 x}\,dx = \int \frac{1+\cos x}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x\,dx + \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\,dx$$

$$= -\cot x + \int \cot x \cdot \operatorname{cosec} x\,dx = -\cot x - \operatorname{cosec} x$$

prob: $\int \frac{x^4}{x^2+1}\,dx = \int \frac{x^4-1+1}{x^2+1}\,dx$

$$= \int \frac{(x^2+1)(x^2-1)+1}{x^2+1}\,dx = \int (x^2-1)\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x$$

prob: $\int \frac{\sin^8 x}{\cos^{10} x}\,dx = \int \tan^8 x \cdot \sec^2 x\,dx$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int t^8\,dt = \frac{t^9}{9} = \frac{\tan^9 x}{9}$$

prob: $\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx$

$$= \int \frac{(1-\cos x)\cdot(1+\cos x)}{1+\cos x}\,dx = \int (1-\cos x)\,dx$$

$$= x - \sin x$$

prob: $\int \frac{x^2-a^2}{x^2+a^2}\,dx = \int \frac{x^2+a^2-2a^2}{x^2+a^2}\,dx$

$$= \int dx - 2a^2 \int \frac{1}{x^2+a^2}\,dx = x - 2a^2 \cdot \frac{1}{a} \cdot \tan^{-1}(x/a)$$

$$= x - 2a\tan^{-1}(x/a)$$

prob: $\int \frac{x}{\sqrt{1-x^2}}\,dx =$

Let $1-x^2 = t \Rightarrow -2x\,dx = dt \Rightarrow x\,dx = \frac{-dt}{2}$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{t}}\cdot\left(\frac{-dt}{2}\right) = \frac{-1}{2}\int \frac{1}{\sqrt{t}}\,dt$$

$$= -\frac{1}{2}\cdot 2\sqrt{t} = -\sqrt{t} = -\sqrt{1-x^2}$$

prob: $\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$

Let $x+1 = t \Rightarrow dx = dt$

$$= \int \frac{1}{t^2+1}\,dt = \tan^{-1} t = \tan^{-1}(1+x)$$

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DEFINITE INTEGRALS: If $\int f(x)\,dx = F(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

Note: $\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\cdot\frac{dv}{dx} - f(u)\cdot\frac{du}{dx}$

properties of definite integral:

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) If $c \in (a,b)$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(iii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(iv) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

(v) $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ If } f(x) \text{ is even} \\ 0 & ; \text{ If } f(x) \text{ is odd} \end{cases}$

(vi) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$

(vii) $\int_0^a x\cdot f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$; If $f(a-x) = f(x)$

(viii) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; If $f(x+a) = f(x)$

(ix) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\times\frac{2}{3}\text{ (or) }\frac{1}{2}\right]K$

where $K = \begin{cases} 1 & ; \text{ if } n \text{ is } \cancel{\text{even}} \text{ odd} \\ \pi/2 & ; \text{ if } n \text{ is } \cancel{\text{odd}} \text{ even} \end{cases}$

(x) $\int_0^{\pi/2} \sin^m x\cdot\cos^n x\,dx = \left\{\frac{[(m-1)(m-3)\cdots 2 \text{ (or) } 1][(n-1)(n-3)\cdots 2 \text{ (or) } 1]}{[(m+n)(m+n-2)(m+n-4)\cdots]}\right\}\cdot K$

where $K = \begin{cases} \pi/2 & ; \text{ when } m \,\&\, n \text{ are even} \\ 1 & ; \text{ otherwise} \end{cases}$

Prob: $\int_{-2\pi}^{2\pi} \cos^6 x\,dx$

Sol: $f(x) = \cos^6 x$ is a even function $\int_{-2\pi}^{2\pi} \cos^6 x\,dx = 2\int_0^{2\pi} \cos^6 x\,dx$

$f(2\pi - x) = [\cos(2\pi - x)]^6 = \cos^6 x$

$\int_{-2\pi}^{2\pi} \cos^6 x\,dx = 2\int_0^{2\pi} \cos^6 x\,dx = 2\cdot 2\int_0^{\pi} \cos^6 x\,dx$

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$$= 4 \cdot 2 \int_0^{\pi/2} \cos^6 x \, dx \qquad \cos^6(\pi - x) = [-\cos x]^6 = \cos^6 x$$

$$= 8 \times \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{4}$$

prob: $\int_1^3 x\sqrt{x^2-1}\, dx$

Let $x^2 - 1 = t^2$ $\qquad x = 1 \Rightarrow t = 0$

$2x\,dx = 2t\,dt$ $\qquad x = 3 \Rightarrow 3^2 - 1 = t^2$

$x\,dx = t\,dt$ $\qquad t = \sqrt{8}$

$$= \int_0^{\sqrt{8}} t \cdot t\,dt = \int_0^{\sqrt{8}} t^2\,dt$$

$$= \frac{t^3}{3}\Big|_0^{\sqrt{8}} = \frac{(\sqrt{8})^3}{3} = \frac{8\sqrt{8}}{3}$$

prob: $\int_0^1 x \cdot \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

Let $\sin^{-1} x = t \Rightarrow \cancel{\sin x = t}$ $\qquad x = \sin t$ $\qquad x = 0 \Rightarrow t = \sin^{-1} 0 = 0$

$dx = \cos t\,dt$ $\qquad x = 1 \Rightarrow t = \sin^{-1} 1 = \pi/2$

$$= \int_0^{\pi/2} \sin t \cdot \frac{t}{\cos t} \cdot \cos t\,dt$$

$$= \int_0^{\pi/2} t \sin t\,dt = \left[t(-\cos t) - 1(-\sin t)\right]_0^{\pi/2}$$

$$= \left[-t\cos t + \sin t\right]_0^{\pi/2} = [1 - 0] = 1$$

prob: $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^1 \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\, dx$

$$= \int_0^1 \frac{(1-x)}{\sqrt{1-x^2}}\, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx - \frac{1}{2}\int_0^1 \frac{2x}{\sqrt{1-x^2}}\, dx$$

$$= \left[\sin^{-1} x + \sqrt{1-x^2}\right]_0^1 = \sin^{-1} 1 - \sqrt{1-0} = \frac{\pi}{2} - 1$$

prob: $\int_0^2 |1-x|\, dx = \int_0^1 (1-x)\,dx - \int_1^2 (1-x)\,dx$

$(1-x) > 0$ for $0 < x < 1$

$(1-x) < 0$ for $1 < x < 2$

$$= \left[x - \frac{x^2}{2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_1^2$$

$$= \left[1 - \frac{1}{2}\right] - \left[2 - \frac{4}{2} - 1 + \frac{1}{2}\right] = 1$$

prob: $\int_{-\pi}^{\pi} (|x| + \sin x)\, dx = \int_{-\pi}^{\pi} |x|\,dx + \int_{-\pi}^{\pi} \sin x\,dx$

$|x|$ even

$\sin x$ odd

$$= 2\int_0^{\pi} |x|\,dx + 0$$

$$= 2\left(x^2/2\right)_0^{\pi} = \pi^2$$

prob: $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\, dx$

Sol: Let $f(x) = \tan x \Rightarrow f(0 + \pi/2 - x) = \tan\left(\frac{\pi}{2} - x\right) = \cot x$

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\, dx = \int_0^{\pi/2} \frac{f(x)}{f(x) + f(0 + \pi/2 - x)} = \frac{\pi/2 - 0}{2}$$

$$= \pi/4$$

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Prob: $\int_0^n [x]\,dx =$

Sol: $I = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \cdots$

$= (x]_1^2 + 2(x)_2^3 + 3(x)_3^4 + \cdots$

$= (2-1) + 2(3-1) + 3(4-3) + 4(5-4) + \cdots$

$= 1+2+3+4\cdots = \frac{n(n+1)}{2}$

Prob: $\int_0^{\pi/2} \log(\tan x)\,dx$ is

Sol: $\int_0^{\pi/2} \log(\tan x)\,dx = I$ say $\qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$I = \int_0^{\pi/2} \log[\tan(\pi/2 - x)]\,dx = \int_0^{\pi/2} \log(\cot x)\,dx$

$I + I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\,dx$

$2I = \int_0^{\pi/2} \log(\tan x \cdot \cot x)\,dx \qquad \tan x \cdot \cot x = 1$

$2I = \int_0^{\pi/2} \log(1)\,dx = 0 \Rightarrow I = 0$

Prob: $\int_{-\pi}^{\pi} \log\frac{(1+\sin x)}{(1-\sin x)}\,dx = I$

$f(-x) = \log\left(\frac{1-\sin x}{1+\sin x}\right) = -\log\left(\frac{1+\sin x}{1-\sin x}\right) = -f(x)$

odd function $\therefore I = 0$

(GATE-97): If $\phi(x) = \int_0^{x^2} \sqrt{t}\,dt$ then $\frac{d\phi}{dx} =$

Sol: $\phi = \frac{2}{3} t\sqrt{t}\Big|_0^{x^2} = \frac{2}{3} x^2\sqrt{x^2} - 0 = \frac{2}{3}x^3$

$\frac{d\phi}{dx} = \frac{2}{3} \cdot 3 \cdot x^2 = 2x^2$

(GATE-01): The value of the integral $I = \int_0^{\pi/4} \cos^2 x\,dx$

Sol: $I = \int_0^{\pi/4} \cos^2 x\,dx = \int_0^{\pi/4} \left[\frac{1+\cos 2x}{2}\right] dx$

$= \frac{x}{2}\Big|_0^{\pi/4} + \frac{\sin 2x}{4}\Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}\sin(\pi/2)$

$= \frac{\pi}{8} + \frac{1}{4}$

(GATE-02): $\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{1+\cos x}\,dx = I$

$f(-x) = -f(x) \Rightarrow$ odd function $\therefore I = 0$

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(GATE-05 IN): $\int_{-1}^{1} \frac{1}{x^2} dx$

$= 2\int_0^1 \frac{1}{x^2} dx$

$= 2\cdot\left(-\frac{1}{x}\right)_0^1 = \infty$

Sol: $= \int_{-1}^{0} \frac{1}{x^2} dx + \int_0^{-1} \frac{1}{x^2} dx = \left(\frac{x^{-1}}{-1}\right)_{-1}^{0} + \left(\frac{x^{-1}}{-1}\right)_0^{-1}$

$= (\infty - 1) + (1 - \infty) = \infty$

(GATE-08 PI): $\int_{-\pi/2}^{\pi/2} (x\cos x) dx$

Sol: $f(x) = x\cos x \Rightarrow f(-x) = -x\cos(-x) = -x\cdot\cos x = -f(x)$

$\Rightarrow$ odd function $\therefore I = 0$

(GATE-10 EE): $P = \int_0^1 x e^x dx$

Sol: $\int_0^1 x e^x dx = [x e^x - e^x]_0^1 = (e - e) + (e^0 - 0) = 1$

(GATE-11 CS): $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx = I$

Sol $I = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}} dx = \int_0^{\pi/2} e^{2ix} dx = \left[\frac{e^{2ix}}{2i}\right]_{\pi/2}^{0}$

$= \frac{e^{i\pi}}{2i} - \frac{1}{2i} = \frac{\cos\pi + i(0)}{2i} - \frac{1}{2i} = -\frac{1}{2i} - \frac{1}{2i} = -\frac{2}{2i}$

$= -\frac{1}{i} = i$

(GATE-13 ME): $\int_1^e \sqrt{x}\cdot\ln(x)\cdot dx$

$\int uv = uv - \int v du$

$u = \ln x$

Sol: $= \left[\left(\ln x\cdot\frac{x^{3/2}}{3/2}\right)_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^{3/2}}{3/2} dx\right]$

$= \frac{2}{3}\cdot e^{3/2} - 0 - \int_1^e \frac{2}{3}\cdot x^{1/2} dx$

$= \frac{2}{3}e^{3/2} - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right)_1^e = \frac{2}{3}e^{3/2} - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{9}e^{3/2} + \frac{4}{9}$

(GATE-13 CE): $\int_0^{\pi/6} \cos^4 3\theta\cdot\sin^3 6\theta\cdot d\theta$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/6 \Rightarrow t = \pi/2$

Sol: Let $3\theta = t \Rightarrow \theta = t/3 \Rightarrow d\theta = dt/3$

$I = \int_0^{\pi/2} \cos^4 t\cdot\sin^3 2t\cdot dt/3$

$= \frac{1}{3}\int_0^{\pi/2} \cos^4 t\cdot(2\sin t\cdot\cos t)^3 dt$

$= \frac{8}{3}\int_0^{\pi/2} \cos^7 t\cdot\sin^3 t\, dt = \frac{8}{3}\left[\frac{(6\times4\times2)(2)}{10\times8\times6\times4\times2}\right] = 1/15$

(GATE-14 ME): $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2 + \cos(x-1)} dx$

Sol: $f(2-x) = \frac{(2-x-1)^2\sin(2-x-1)}{(2-x-1)^2 + \cos(2-x-1)} = \frac{(x-1)^2\cdot(-\sin(x-1))}{(x-1)^2 + \cos(x-1)} = -f(x)$

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According to $\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$

$\therefore I = 0$

Improper Integrals:

First kind: $\int_a^b f(x)\,dx$ if $a=-\infty$ (or) $b=\infty$ (or) both

i.e $\int_{-\infty}^{b} f(x)\,dx$ or $\int_a^{\infty} f(x)\,dx$ (or) $\int_{-\infty}^{\infty} f(x)\,dx$

Second kind: $\int_a^b f(x)\,dx$ if a & b are finite but $f(x)$ is infinite for some $x \in [a,b]$

Ex: $\int_{-1}^{1} \log(1+x)\,dx$; $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$; $\int_0^3 \frac{1}{x^2-5x+4}\,dx$

Convergence of an improper integral:

(i) If $\int_a^b f(x)\,dx$ = finite, then it is a convergent improper integral

(ii) If $\int_a^b f(x)\,dx$ = Infinite, then it is a divergent improper integral

prob: $\int_0^{\infty} \frac{1}{a^2+x^2}\,dx$

Sol: $I = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^{\infty} = \frac{1}{a}\left[\tan^{-1}\infty - \tan^{-1}0\right]$

$= \frac{1}{a}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2a}$ finite (Convergent)

prob: $\int_{-\infty}^{0} e^{ax}\cos px\,dx$

Sol: $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

$I = \left[\frac{e^{ax}}{a^2+p^2}(a\cos px + p\sin px)\right]_{-\infty}^{0} = \frac{a}{a^2+p^2} - 0 = \frac{a}{a^2+p^2}$ finite (Convergent)

prob: $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$

Sol: $I = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \underbrace{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx}_{\text{even}} + \underbrace{\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx}_{\text{odd}}$

$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + 0$

$= 2\sin^{-1}x\Big|_0^1 = 2\sin^{-1}1 = 2 \cdot \pi/2 = \pi$

finite

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comparision test: For 1st kind of improper integrals

(a) Let $0 \le f(x) \le g(x)$ then

(i) $\int_a^b f(x)dx$ converges if $\int_a^b g(x)$ is ~~div~~ conv ergent.

(ii) $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)$ is divergent.

(b) Limit form: Let $f(x)$ & $g(x)$ be two positive functions such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ (non-zero finite) then $\int_a^b f(x)dx$ & $\int_a^b g(x)dx$ both are converge or diverge together.

prob: $\int_1^\infty e^{-x^2}dx$

sol: $e^{x^2} \ge e^x \ \forall\ x \ge 1 \Rightarrow e^{-x^2} \le e^{-x} \ \forall\ x \le 1$

$\int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty = 1$ $\therefore$ convergent

$\therefore \int_1^\infty e^{-x^2}dx$ is also convergent.

prob: $\int_2^\infty \frac{1}{\log x}dx$

sol: $\log x < x \ \forall\ x \ge 2 \Rightarrow \frac{1}{\log x} > \frac{1}{x} \ \forall\ x \ge 2$

$\int_2^\infty \frac{1}{x}dx = \log x\Big|_2^\infty$ = infinite $\Rightarrow$ divergent

$\therefore \int_2^\infty \frac{1}{\log x}dx$ is also divergent.

Comparison test: For 2nd kind of improper integrals

Limit form: Let $f(x)$ & $g(x)$ be two positive functions suchthat

(i) 'a' is a point of discontinuity and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l_1$ (non zero finite)

(ii) 'b' is a point of discontinuity and $\lim_{x\to b^-} \frac{f(x)}{g(x)} = l_2$ (non zero finite)

then $\int_a^b f(x)dx$ & $\int_a^b g(x)dx$ both converge (or) diverge together.

prob: $\int_1^2 \frac{\sqrt{x}}{\log x}dx$

$\frac{1}{\log x} > \frac{1}{x} \ \forall\ x > 1 \Rightarrow \frac{\sqrt{x}}{\log x} > \frac{1}{\sqrt{x}} \ \forall\ x > 1$

$\int_1^2 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^2 = 2\sqrt{2} - 2$ = finite $\Rightarrow$ convergent

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$\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ may / may not be convergent

2nd Method: $f(x) = \frac{\sqrt{x}}{\log x}$ Let $g(x) = \frac{1}{x\log x}$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1\sqrt{1} = 1$$

Let $\log x = t$ $\quad x=1 \Rightarrow t=0$, $x=2 \Rightarrow t=\log 2$

$\frac{1}{x}dx = dt$

$$\int_1^2 \frac{1}{x\log x}\,dx = \int_0^{\log 2} \frac{1}{t}\cdot dt$$

$$= \log t \Big|_0^{\log 2} = \text{Infinite (diverge)}$$

$\therefore \int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ is also divergent.

(GATE-00): $\lim_{a\to\infty} \int_1^a x^{-4}\,dx$

(a) diverges ✓(b) converges to $1/3$ c) converges to $-1/a^3$ d) converges to 0

Sol: $\lim_{a\to\infty} \int_1^a x^{-4}\,dx = \lim_{a\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^a = \lim_{a\to\infty} \left[\frac{a^{-3}}{-3} - \frac{1}{-3}\right]$

$$= \lim_{a\to\infty} -\frac{1}{3a^3} + \frac{1}{3} = 0 + \frac{1}{3} = 1/3$$

(GATE-05): $\int_1^\infty x^{-3}\,dx$

$$= \frac{x^{-2}}{-2}\Big|_1^\infty = -\frac{1}{2}\left[\frac{1}{\infty} - 1\right] = 1/2$$

(GATE-08 ME): Which of the following integrals is unbounded?

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^\infty \frac{1}{1+x^2}\,dx$ (c) $\int_0^\infty x e^{-x}\,dx$ ✓(d) $\int_0^1 \frac{1}{1-x}\,dx$

Sol: (a) $\int_0^{\pi/4} \tan x\,dx = \log \sec x\Big]_0^{\pi/4} = \log\sqrt{2} - \log 1 = \log\sqrt{2}$

(b) $\int_0^\infty \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^\infty = \tan^{-1}\infty - \tan^{-1}0 = \pi/2 - 0 = \pi/2$

(c) $\int_0^\infty x e^{-x}\,dx = \left[x\cdot\frac{e^{-x}}{-1} - \frac{e^{-x}}{(-1)^2}\right]_0^\infty = 0 - 0 + \frac{1}{(-1)^2} = 1$

(d) $\int_0^1 \frac{1}{1-x}\,dx = \left[\frac{\log(1-x)}{-1}\right]_0^1 = \log 0 + \log 1 = \infty + 0 = \infty$ $\Rightarrow$ unbounded

(GATE-10 ME): $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ $\Rightarrow$ even function

$$= 2\int_0^\infty \frac{1}{1+x^2}\,dx = 2\cdot\pi/2 = \pi$$

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Gamma function: $\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1} dx \quad (n>0)$.

Note: (i) $\Gamma 1 = 1$

(ii) $\Gamma 1/2 = \sqrt{\pi}$

(iii) $\Gamma(n+1) = n\Gamma n \quad \forall n>0$

(iv) $\Gamma(n+1) = n! \quad \forall n \in Z^+$

(v) $\int_0^\infty e^{-ax} \cdot x^{n-1} dx = \frac{\Gamma n}{a^n}$

Prob: $\int_0^\infty e^{-x^2} dx =$

Sol Let $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{1}{2} t^{-1/2} dt$

$$I = \int_0^\infty e^{-t} \cdot \frac{1}{2} t^{1/2} dt = \frac{1}{2} \int_0^\infty e^{-t} \cdot t^{1/2 - 1} dt = \frac{1}{2} \Gamma 1/2 = \frac{\sqrt{\pi}}{2}$$

Prob: $\int_{-\infty}^\infty e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

Prob: $\int_0^1 (x \log x)^4 dx =$

$x=0 \Rightarrow t = -\infty$
$x=1 \Rightarrow t=0$

Sol: Let $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$$I = \int_{-\infty}^0 [e^{-t} \cdot (-t)]^4 \cdot (-e^{-t} dt) = \int_0^\infty e^{-5t} \cdot t^{5-1} dt$$

$$= \frac{\Gamma 5}{5^5} = \frac{4!}{5^5}$$

$\log_a 0 = \begin{cases} \infty ; & a<1 \\ -\infty ; & a>1 \end{cases}$

Beta function: $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad (m>0,\ n>0)$

Note: (i) $\beta(m,n) = \beta(n,m)$

(ii) $\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}} dx$

(iv) $\beta(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

i.e $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$(p>-1/2,\ q>-1/2)$

Prob: $\int_0^2 x^7 (16 - x^4)^{10} dx =$

$x=0 \Rightarrow t=0$
$x=2 \Rightarrow 16 = 16t$
$t=1$

Sol^n: Let $x^4 = 16t \Rightarrow 4x^3 dx = 16\,dt \Rightarrow x^3 dx = 4\,dt$

$$I = \int_0^1 16t(16-16t)^{10} 4\,dt = 16^{11} \cdot 4 \int_0^1 t(1-t)^{10} dt$$

$$= 4 \times 16^{11} \times \beta(2,11)$$

$$= 4 \times 16^{11} \times \frac{\Gamma 2 \cdot \Gamma 11}{\Gamma 13} = 4 \times 16^{11} \times \frac{1 \times 10!}{12!}$$

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prob: $\int_0^\infty \left(\frac{x}{1+x^2}\right)^3 dx$

sol: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/2} \left(\frac{\tan\theta}{\sec^2\theta}\right)^3 \sec^2\theta \, d\theta = \int_0^{\pi/2} \frac{\tan^3\theta}{\sec^4\theta} \cdot d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cdot \cos\theta \cdot d\theta = \frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right) = \frac{1}{2}\beta(2,1)$$

$$= \frac{1}{2} \times \frac{\Gamma 2 \cdot \Gamma 1}{\Gamma 3} = \frac{1}{2} \times \frac{1 \times 1}{2} = \frac{1}{4}$$

(GATE-94 ME): $\int_0^\infty e^{-y^3} \cdot y^{1/2} dy$

sol: put $y^3 = t \Rightarrow 3y^2 dy = dt$

$\Rightarrow t^{1/3} = y \qquad y^{1/2} dy = \frac{1}{3} y^{-3/2} dt = \frac{1}{3} t^{-1/2} dt$

$$\int_0^\infty e^{-t} \cdot \frac{1}{3} t^{-1/2} dt = \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1} dt = \frac{1}{3} \Gamma(1/2) = \frac{\sqrt{\pi}}{3}$$

(GATE-10 PI): $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} dx$ even function

sol: put $x^2/2 = t \Rightarrow x\,dx = dt$

$\Rightarrow \sqrt{2t} \cdot dx = dt \Rightarrow dx = \frac{1}{\sqrt{2}} t^{-1/2} \cdot dt$

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-t} \cdot \frac{1}{\sqrt{2}} t^{-1/2} dt = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

MULTIPLE INTEGRALS:

double integral: An integral of the form $\boxed{\int_c^d \boxed{\int_a^b f(x)dx} \, dy}$ is called double integral. To evaluate this integral we have to follow the following steps.

1. First we evaluate the inner box & then the outer box.
2. While integrating w.r.to one variable we treat the other variable as constant
3. If the limits are constant limits then we need to follow the order in which dx & dy are present.
4. If the limits are variable limits then we need to follow the order in which the limits are present. and they are limits for that variable which is not present in the limits.

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prob: $\int_{x=0}^{1}\int_{y=0}^{x^2}(x+y)\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} x\cdot dy + \int_{y=0}^{x^2} y\,dy\right]dx$

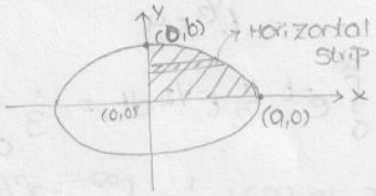
$$= \int_{x=0}^{1}\left[x\cdot(y)_0^{x^2} + \left(\frac{y^2}{2}\right)_0^{x^2}\right]dx$$

$$= \int_{x=0}^{1}\left[x^3 + \frac{x^4}{2}\right]dx = \left[\frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{1}{4}+\frac{1}{10}$$

prob: $\int_0^4\int_0^{x^2} e^{y/x}\cdot dy\cdot dx = \int_0^4\left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$

$$= \int_0^4 [xe^x - x]\,dx = (xe^x - e^x)_0^4 - \left(\frac{x^2}{2}\right)_0^4 = 3e^4 - 7$$

prob: The value of $\iint_R xy\,dx\,dy$, where R is a region in the +ve quadrant at the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is



Sol: Consider horizontal strip

$x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$

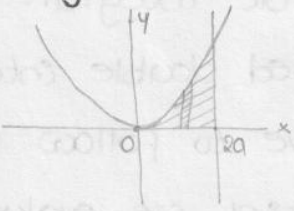
$y=0$ to $y=b$

$$\iint_R xy\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\cdot dx\cdot dy$$

$$= \int_{y=0}^{b} y\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}}\cdot dy = \int_{y=0}^{b}\frac{y}{2}\left[\frac{a^2}{b^2}(b^2-y^2)-a^2\right]dy$$

$$= +\frac{a^2}{2}\int_{y=0}^{b}(y-y^3)\,dy = +\frac{a^2}{2}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^b = +\frac{a^2}{2}\left[\frac{b^2}{4}\right] = \frac{a^2b^2}{8}$$

prob: $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis $x=2a$, $x^2=4ay$ $(a>0)$



Sol: Consider vertical strip

$y=0$ to $y=x^2/4a$, $x=0$ to $2a$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dx\cdot dy$$

$$= \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_{x=0}^{2a} x\cdot\frac{x^4}{32a^2}\,dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx = \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{32a^2}\cdot\frac{64a^6}{6} = \frac{a^4}{3}$$

Note: $\iint_R dx\,dy$ always represents area bounded by R

$\iiint_V dx\,dy\,dz$ always represents volume bounded by V.

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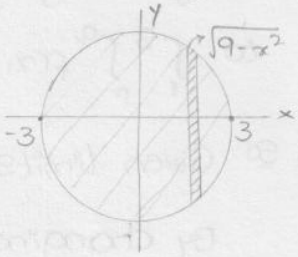
Prob: $\iint_R dx\,dy$ where R is region bounded by the circle $x^2+y^2=9$



Sol: Consider vertical strip

$$y=-\sqrt{9-x^2} \text{ to } \sqrt{9-x^2}, \quad x=-3 \text{ to } 3$$

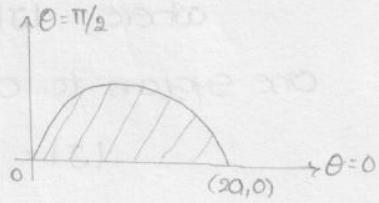
$$\iint_R dx\,dy = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,.\,dx$$

$$= \int_{-3}^{3} (y)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx$$

$$= \int_{-3}^{3} 2.\sqrt{9-x^2}\,dx = 2.2\int_0^3 \sqrt{9-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 = 9\pi$$

Prob: The value of $\iint_R r^2\sin\theta\,dr\,d\theta$, where 'r' is the region bounded by the semicircle $r=2a\cos\theta$ above initial line is



Sol: $r=0$ to $2a\cos\theta$, $\theta=0$ to $\pi/2$

$$\iint_R r^2\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\sin\theta\int_{r=0}^{2a\cos\theta} r^2\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\sin\theta\left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{8a^3\cos^3\theta.\sin\theta}{3}\,d\theta$$

Let $\cos\theta = t \Rightarrow -\sin\theta.d\theta = dt$

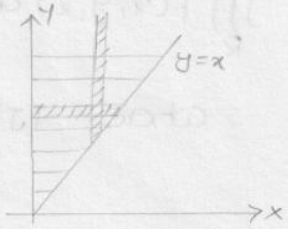
$\theta=0 \Rightarrow t=1$
$\theta=\pi/2 \Rightarrow t=0$

$$= \int_1^0 \frac{8a^3}{3}t^3(-dt) = \frac{8a^3}{3}\int_0^1 t^3\,dt$$

$$= \frac{8a^3}{3}\left(\frac{t^4}{4}\right)_0^1 = \frac{8a^3}{3}.\frac{1}{4} = 2a^3/3$$

Change of order of integration:

Prob: $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$



Sol: Given limits are $y=x$ to $y=\infty$
$x=0$ to $x=\infty$

Horizontal strip $y=0$ to $y=\infty$, $x=0$ to y

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{\infty}\left[\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\right]dy = \int_0^\infty e^{-y}\,dy$$

$$= -e^{-y} = [0+1] = 1$$

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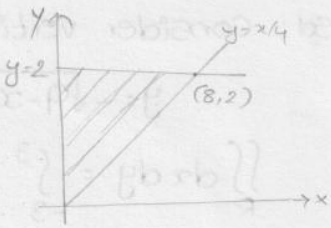
(GATE05)

prob: By changing the order of integration $\int_0^8 \int_{x/4}^{2} f(x,y)\,dy\,dx$ leads to $\int_r^s \int_p^q f(x,y)\,dx\,dy$ then q is



Sol: Given limits $y = x/4$ to 2, $x = 0$ to 8

By changing limits, $y = 0$ to 2, $x = 0$ to $4y$

$\therefore q = 4y$

prob: Evaluate $\int_0^2 \int_0^x \int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$

Sol: $I = \int_0^2 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx = \int_0^2 \left[\frac{(x+y)^2}{4}\right]_0^x dx = 2$

Change of variables: In a double integral, if $x = f(u,v)$ & $y = g(u,v)$ then $\iint_R \phi(x,y) = \iint_R \phi(f,g)\,|J|\,du\,dv = \iint_R \phi(u,v)\,|J|\,du\,dv$

where, $|J| \rightarrow$ Jacobian of transformation used to transform one system to other.

$$|J| = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Cartesian form $(x,y) \longrightarrow$ polar form (r,θ):

$x = r\cos\theta \qquad y = r\sin\theta$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \iint_R \phi(x,y)\,dx\,dy = \iint_R \psi(r,\theta)\,r\,dr\,d\theta$$

$\rightarrow$ In a triple integral if $x = f(u,v,\omega)$, $y = g(u,v,\omega)$, $z = h(u,v,\omega)$ then $\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R \psi(u,v,\omega)\,|J|\,du\,dv\,d\omega$

where $$|J| = J\left(\frac{x,y,z}{u,v,\omega}\right) = \frac{\partial(x,y,z)}{\partial(u,v,\omega)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \omega} \end{vmatrix}$$

Cartesian (x,y,z) to cylindrical polar form: (r,θ,z)

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

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$$\iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \varphi(r,\theta,z)\, r\,dr\,d\theta\,dz$$

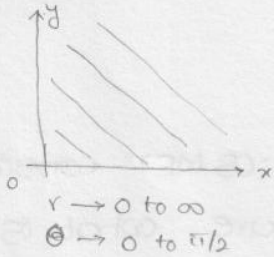
Cartesian to spherical polar form:
(r,θ,ϕ)

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$|J| = \begin{vmatrix} \sin\theta\cdot\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\cdot\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \varphi(r,\theta,\phi)\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

(IN 07)
prob: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$



sol: Let $x = r\cos\theta$, $y = r\sin\theta$ then $|J| = r$

& $x^2+y^2 = r^2$

$$\int_0^\infty\int_0^{\pi/2} e^{-r^2}\cdot r\cdot dr\,d\theta$$

Let $r^2 = t \Rightarrow r\,dr = dt/2$

$$= \int_0^{\pi/2}\int_0^\infty e^{-t}\cdot dt/2\cdot d\theta = \int_0^{\pi/2}\frac{1}{2}\left[-e^{-t}\right]_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \pi/4$$

(GATE 05)
Prob: By a change of variable $x(u,v) = uv$ & $y(u,v) = v/u$ in double integration, the integrand $f(x,y)$ changes to $f(uv, v/u)\,\phi(u,v)$ then $\phi(u,v)$ is

sol: $\phi(u,v) = |J| = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & 1/u \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$

(GATE-95): By reversing the order of integration $\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$ may be represented as

(a) $\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$ (b) $\int_0^2\int_y^{\sqrt{y}} f(x,y)\,dx\,dy$ ✓(c) $\int_0^4\int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$ (d) $\int_{x^2}^{2x}\int_0^2 f(x,y)\,dy\,dx$

sol: $y = x^2$ to $y = 2x$ $\qquad$ $x = 0$ to $x = 2$

$x = \sqrt{y} \qquad x = y/2 \qquad\qquad y = 0^2 = 0 \qquad y = 2^2 = 4$

$= 2(0) = 0 \qquad = 2(2) = 4$

$$\therefore \int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx = \int_{y=0}^{4}\int_{x=y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

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(GATE-04): The volume of an object expressed in spherical co-ordinate is given by $v = \int_0^{2\pi}\int_0^{\pi/3}\int_1^1 r^2 \sin\phi \, dr\, d\phi\, d\theta$. The value of the integral

Sol: $v = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_0^1 \sin\phi \, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}(-\cos\phi)_0^{\pi/3} d\theta$

$= \frac{1}{3}\times\frac{1}{2}\times 2\pi = \pi/3$

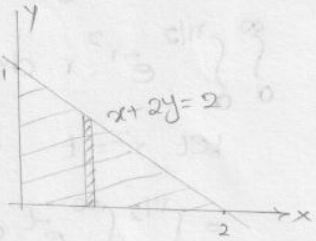
(GATE-08 CS): $\int_0^3\int_0^x (6-x-y)\,dx\,dy$

Sol: $\int_0^3 [6y - xy - y^2/2]_0^x \, dx = \int_0^3 (6x - x^2 - x^2/2)\,dx$

$= \int_0^3 (6x - 3x^2/2)\,dx = 6\left(\frac{x^2}{2}\right)_0^3 - \frac{3}{2}\left(\frac{x^3}{3}\right)_0^3$

$= 6\times\frac{9}{2} - \frac{3}{2}\times\frac{27}{3} = 27 - 13.5 = 13.5$

(GATE-08 ME): consider shaded triangular region P shown in the figure. What is $\iint_P xy\,dx\,dy$?



$x+2y=2$
$\Downarrow$
$y = \frac{2-x}{2}$

Sol: consider vertical strip

$y = 0$ to $\frac{2-x}{2}$

$x = 0$ to 2

$I = \int_0^2\int_0^{\frac{2-x}{2}} xy\,dy\,dx = \int_0^2 x\left(\frac{y^2}{2}\right)_0^{\frac{2-x}{2}} dx$

$= \int_0^2 \frac{x}{2}\left(\frac{2-x}{2}\right)^2 dx = 1/6$

(GATE-14 EC): The volume under the surface $z(x,y) = x+y$ and above the triangle in the xy plane defined by $\{0 \le y \le x$ and $0 \le x \le 12\}$ is

Sol: volume $= \int_0^{12}\int_0^x z\,dy\,dx = \int_0^{12}\int_0^x (x+y)\,dy\,dx$

$= \int_0^{12}(xy + y^2/2)_0^x\,dx = \int_0^{12}(x^2 + x^2/2)\,dx$

$= \frac{3}{2}\left(x^3/3\right)_0^{12} = 864$

(GATE-14 EE): To evaluate the double integral $\int_0^8\int_{y/2}^{(y/2+1)}\left(\frac{2x-y}{2}\right)dx\,dy$ we make the substitution $u = \left(\frac{2x-y}{2}\right)$ and $v = y/2$. The integral will reduce to

Sol: $v = y/2 \Rightarrow dv = \frac{1}{2}dy \Rightarrow dy = 2dv$

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$u = \left(\frac{2x-y}{2}\right) \Rightarrow du = dx$

As $x: \frac{y}{2} \to \frac{y}{2}+1 \Rightarrow u: 0 \to 1$ and $y: 0 \to 8 \Rightarrow v: 0 \to 4$

$\therefore$ integral ~~becomes~~ become $\int_0^4 \int_0^1 2u\,du\,dv$

(GATE-14 ME): $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$

sol: $\int_0^2 \int_0^x e^{x+y}\,dy\,dx = \int_0^2 e^x (e^y)_0^x\,dx = \int_0^2 e^x(e^x-1)\,dx$

$= \int_0^2 (e^{2x}-e^x)\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^2 = \frac{(e^2-1)^2}{2}$

Lengths, Areas & volumes:

1. Length of an arc of a curve $y = f(x)$ between the lines $x = x_1$ & $x = x_2$ is $l = \int_{x_1}^{x_2} \sqrt{1+(dy/dx)^2}\,dx$
2. Length of an arc of a curve $x = f(t)$ & $y = g(t)$ between $t = t_1$ to $t = t_2$ is $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(dy/dt\right)^2}$
3. Area of the region bounded by the curve $y = f(x)$ & $y = g(x)$ between $x = x_1$ & $x = x_2$ is $A = \int_{x_1}^{x_2} [g(x) - f(x)]\,dx = \int_{x_1}^{x_2} \int_{f(x)}^{g(x)} dy\,dx$
4. The volume generated by revolving $y = f(x)$ between $x = x_1$ & $x = x_2$ about x-axis is

$$v = \int_{x_1}^{x_2} \pi y^2\,dx$$

About y-axis is $v = \int_{y_1}^{y_2} \pi x^2\,dy$

5. In polar form: (i) about initial line (i.e $\theta = 0^\circ$) $v = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\,d\theta$

(ii) about line $\theta = \pi/2$ $v = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta\,d\theta$

(GATE 08 ME)

prob: The length of the curve $y = \frac{2}{3}x^{3/2}$ b/w $x = 0$ & 1 is

sol: $\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2} \cdot x^{\frac{3}{2}-1} = x^{1/2}$

$l = \int_{x_1}^{x_2} \sqrt{1+(dy/dx)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = 1.22$

prob: The length of curve $x = \cos^3\theta$ $y = \sin^3\theta$ b/w $\theta = 0$ & $\pi/2$ is

sol: $l = \int_0^{\pi/2} \sqrt{(3\cos^2\theta(-\sin\theta))^2 + (3\sin^2\theta \cdot \cos\theta)^2}\,d\theta = \int_0^{\pi/2} 3\sin\theta\cos\theta\,d\theta$

$= 3/2.$

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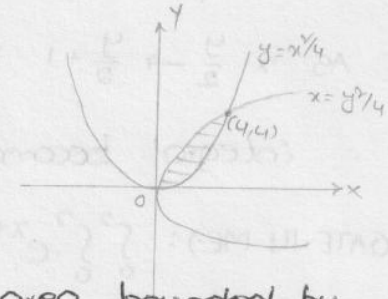
(GATE 09 ME)

Prob: The area between the curve $y^2 = 4x$ & $x^2 = 4y$ is

Sol: $A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{1}{4} \cdot \frac{x^3}{3}\right]_0^4$

$= \left[\frac{4}{3} \times 8 - \frac{1}{4} \times \frac{64}{3}\right] = 16/3$

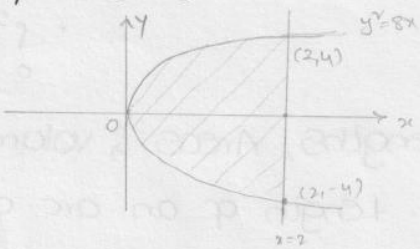


(GATE-94)

Prob. The volume generated by revolving the area bounded by Parabola $y^2 = 8x$ and st. line $x = 2$ about y-axis is

Sol: $V = \int_{y_1}^{y_2} \pi x^2 dy = \int_{-4}^{4} \pi \cdot \frac{y^4}{64} dy$

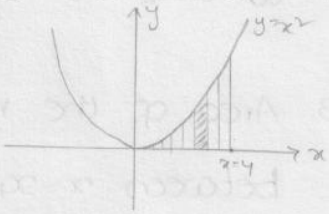
$= \frac{\pi}{64}\left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$



Total volume $= \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi (2)^2 dy = 32\pi$

Remaining volume $= 32\pi - 32\pi/5 = 128\pi/5$

(GATE-97): Area bounded by the curve $y = x^2$ and the lines $x = 4$ and $y = 0$ is given by



Sol: Consider vertical strip $y = 0$ to $y = x^2$

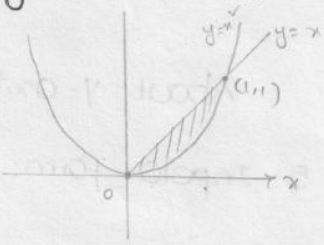
$x = 0$ to $x = 4$

$\int_{x=0}^{4}\int_{y=0}^{x^2} dx\,dy = \int_{x=0}^{4} (y)_0^{x^2} dx$

$= \int_0^4 x^2 dx = \left.\frac{x^3}{3}\right|_0^4 = \frac{4^3}{3} = 64/3$

(GATE-12 ME, PI)

(GATE-04): The area enclosed b/w the parabola $y = x^2$ and the straight line $y = x$ is



Sol: Area $= \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = 1/6$

(GATE-10 ME): The parabolic arc $y = \sqrt{x}$ $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

Sol: Volume $= \int_{x_1}^{x_2} \pi y^2 dx = \int_1^2 \pi (\sqrt{x})^2 dx = \pi\left(\frac{x^2}{2}\right)_1^2 = 3\pi/2$

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VECTOR CALCULUS

Vector calculus

vector differentials	vector integrals
1. Gradiant	1. Line integral
2. Divergence	2. Surface integral
3. curl	3. volume integral

Scalar function: A scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in space. Its value is real and depends on the point $p(x,y,z)$ in space, but not on any particular co-ordinate system being used.

Ex: $x^2+y^2+z^2=C$

vector function: A function $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ defined at each point $p \in D$ is called a vector function.

Ex: $\bar{v} = 3x^2yz\,\bar{i} + 3xy^2z\,\bar{j} + 3xyz^2\,\bar{k}$

vector differential operator: It is denoted by '∇' (del)

In 2-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y}$

In 3-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{(Scalar)}$$

★ The equation $\nabla^2\phi = 0$ is called laplace equation. Any function which satisfies laplace eq^n is called <u>Harmonic</u> function

vector operations: $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$$\bar{a} \pm \bar{b} = (a_1 \pm b_1)\bar{i} + (a_2 \pm b_2)\bar{j} + (a_3 \pm b_3)\bar{k}$$

$$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{scalar product}^{**}$$

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$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ vector product**

→ Magnitude of $\bar{a}$ ⇒ $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

→ unit vector along the direction of $\bar{a}$ = $\dfrac{\bar{a}}{|\bar{a}|}$

→ Angle between two vectors $\bar{a}$ & $\bar{b}$ ⇒ $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

→ $[\bar{a}\ \bar{b}\ \bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

→ $\bar{r} = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$ → position vector

$\dfrac{d\bar{r}}{dt}$ → velocity / Tangent vector

$\dfrac{d^2\bar{r}}{dt}$ → Accelaration vector

unit tangent vector = $\dfrac{(d\bar{r}/dt)}{|d\bar{r}/dt|}$

Gradient of a scalar function: Let $\phi(x, y, z)$ is a scalar function then

$$\text{grad}\,\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

* Gradient of a scalar point function is a vector function
* Gradient gives normal to the Surface
* unit normal = $\dfrac{\nabla\phi}{|\nabla\phi|}$

prob: Gradient of $\phi = 3x^2y - y^3z^2$ at $(-1, -1, 2)$ is?

Sol: $\nabla\phi = \bar{i}[6xy] + \bar{j}[3x^2 - 3y^2z^2] + \bar{k}[-2y^3z]$

$\nabla\phi|_{(1,-1,2)} = -6\bar{i} - 9\bar{j} + 4\bar{k}$ (Normal)

unit normal = $\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{-6\bar{i} - 9\bar{j} + 4\bar{k}}{\sqrt{36+81+16}} = \dfrac{1}{\sqrt{133}}(-6\bar{i} - 9\bar{j} + 4\bar{k})$

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prob: $U = x+y+z$, $V = x^2+y^2+z^2$, $W = xy+yz+zx$. Find $[\nabla U \ \nabla V \ \nabla W]$

Sol: $\nabla U = \bar{i} + \bar{j} + \bar{k}$

$\nabla V = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$\nabla W = (y+z)\bar{i} + (x+z)\bar{j} + (x+y)\bar{k}$

$$[\nabla U \ \nabla V \ \nabla W] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

* Angle between two <u>curves</u> is the angle between their <u>tangents</u> at the common point
* Angle between two <u>surfaces</u> is the angle between their <u>normals</u> at the common point

prob: Find angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2,-1,+2)$

Sol: $\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ $\qquad$ $\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - \bar{k}$

$= 4\bar{i} - 2\bar{j} + 4\bar{k}$ $\qquad$ $= 4\bar{i} - 2\bar{j} - \bar{k}$

$\cos\theta$ = Angle b/w normals $= \dfrac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \dfrac{8}{3\sqrt{21}}$

Properties of Gradient:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(C_1 f + C_2 g) = C_1\nabla f + C_2\nabla g$; C_1, C_2 arbitrary constants.
3. $\nabla(fg) = f\cdot\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\cdot\nabla f - f\nabla g}{g^2}$, $g \neq 0$

→ Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$

$r^2 = x^2+y^2+z^2$

$2r\,\partial r = 2x\,\partial x \quad \Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

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1. Grad r $= \nabla r = \frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}$

$$= \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r} = \frac{\bar{r}}{r}$$

2. Grad $(1/r) = \nabla(1/r)$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\bar{i} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\bar{j} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}\bar{i} - \frac{1}{r^2}\cdot\frac{y}{r}\bar{j} - \frac{1}{r^2}\cdot\frac{z}{r}\cdot\bar{k} = -\frac{\bar{r}}{r^3}$$

3. Grad $(r^2) = \nabla r^2$

$$= \frac{\partial}{\partial x}(r^2)\bar{i} + \frac{\partial}{\partial y}(r^2)\bar{j} + \frac{\partial}{\partial z}(r^2)\bar{k}$$

$$= 2r\cdot\frac{\partial r}{\partial x}\bar{i} + 2r\cdot\frac{\partial r}{\partial y}\bar{j} + 2r\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= 2r\cdot\frac{x}{r}\bar{i} + 2r\cdot\frac{x}{r}\bar{j} + 2r\cdot\frac{x}{r}\bar{k} = 2\bar{r}$$

4. Grad $(\log r) = \nabla(\log r)$

$$= \frac{\partial}{\partial x}(\log r)\bar{i} + \frac{\partial}{\partial y}(\log r)\bar{j} + \frac{\partial}{\partial z}(\log r)\bar{k}$$

$$= \frac{1}{r}\cdot\frac{\partial r}{\partial x}\bar{i} + \frac{1}{r}\cdot\frac{\partial r}{\partial y}\bar{j} + \frac{1}{r}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= \frac{1}{r}\cdot\frac{x}{r}\bar{i} + \frac{1}{r}\cdot\frac{y}{r}\bar{j} + \frac{1}{r}\cdot\frac{z}{r}\cdot\bar{k} = \frac{\bar{r}}{r^2}$$

5. Grad $(r^n) = \nabla(r^n)$

$$= \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}\cdot\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}\cdot\frac{\partial r}{\partial y}\cdot\bar{j} + nr^{n-1}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= nr^{n-1}\cdot\frac{x}{r}\bar{i} + nr^{n-1}\cdot\frac{y}{r}\bar{j} + nr^{n-1}\cdot\frac{z}{r}\bar{k} = nr^{n-2}\cdot\bar{r}$$

6. Grad $(e^{r^2}) = \nabla(e^{r^2})$

$$= e^{r^2}\cdot 2r\left[\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right]$$

$$= e^{r^2}\cdot 2r\left[\frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}\right] = 2e^{r^2}\bar{r}$$

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Directional derivative: D.D of the surface ϕ in the direction of $\bar{r}$ is given by

$$D.D = \nabla\phi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\bar{r}}{|\bar{r}|}$$

→ DD is maximum in the direction of $\nabla\phi$ (normal)

→ The maximum value of DD is $|\nabla\phi|$

prob: Find the DD of $\phi = 2xy + z^2$ in the direction of $\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1,-1,3)$

sol:

$$\nabla\phi = \bar{i}(2y) + \bar{j}(2x) + \bar{k}(2z)$$

$$\nabla\phi|_{(1,-1,3)} = -2\bar{i} + 2\bar{j} + 6\bar{k}$$

$$\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k} \Rightarrow \hat{n} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{9}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$DD = \nabla\phi \cdot \hat{n} = \frac{-2+4+12}{3} = 14/3$$

prob: Find D.D of $\phi = xyz^2 + xz$ at the point $(1,1,1)$ in the direction of normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

sol:

$$\nabla\phi = (yz^2 + z)\bar{i} + (xz^2)\bar{j} + (2xyz + x)\bar{k}$$

$$\nabla\phi|_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

$$\nabla S = (3y^2)\bar{i} + (6xy+1)\bar{j} + (-1)\bar{k}$$

$$\nabla S|_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{11}}$$

$$DD = \nabla\phi \cdot \hat{n} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Divergence of vector: Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ be a vector point function then

$$\text{div}(\bar{v}) = \nabla \cdot \bar{v}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(v_1\bar{i} + v_2\bar{j} + v_3\bar{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

* Divergence of vector point function is a scalar point function.

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* Divergence gives rate at which fluid flows out of a unit volume
* If divergence of $\bar{v} = 0$ then $\bar{v}$ is called solenoidal vector.

Curl of vector: Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ be a vector point function then

$$\text{curl}(\bar{v}) = \nabla \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

* curl of vector point function is again a vector point function
* curl gives rotation of the fluid
* If $\text{curl}(\bar{v}) = \bar{0}$ then $\bar{v}$ is called irrotational vector.
* If $\bar{v}$ is linear velocity then angular velocity $\bar{\omega} = \frac{1}{2}[\text{curl}\,\bar{v}]$

prob: If $\bar{v} = x^2yz\,\bar{i} + xy^2z\,\bar{j} + xyz^2\,\bar{k}$. Find div $\bar{v}$ & curl $\bar{v}$ at $(1,-1,1)$

Sol: $\text{div}\,\bar{v} = \nabla \cdot \bar{v} = 2xyz + 2xyz + 2xyz = 6xyz\Big|_{(1,-1,1)} = -6$

$$\text{curl}\,\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = \bar{i}[xz^2 - xy^2] - \bar{j}[yz^2 - x^2y] + \bar{k}[y^2z - x^2z]$$
$$= \bar{0}$$

Properties: (vector Identity)

GATE 06 1. $\text{curl}(\text{grad}\,\phi) = \bar{0}$

2. $\text{div}(\text{curl}\,\bar{v}) = 0$

3. $\text{div}(\nabla\phi) = \nabla^2\phi$

4. $\text{div}[\phi\bar{v}] = \phi\,\text{div}(\bar{v}) + \nabla\phi \cdot \bar{v}$

GATE-05 5. $\text{curl}[\phi\bar{v}] = \phi\,\text{curl}(\bar{v}) + \nabla\phi \times \bar{v}$

GATE-05 6. $\text{curl}[\text{curl}\,\bar{v}] = \nabla(\text{div}\,\bar{v}) - \nabla^2\bar{v}$

Prob: If $\bar{F} = \nabla(2x^3y^2 \cdot z^4)$; Find div $\bar{F}$; curl $\bar{F}$

Sol: $\text{div}\,\bar{F} = \text{div}[\nabla(2x^3y^2z^4)]$

$= \nabla^2(2x^3y^2z^4)$ from ③

$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\text{curl}\,\bar{F} = \text{curl}[\nabla(2x^3y^2z^4)]$

$= \bar{0}$ from ①

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prob: Find a, b, c of $\bar{F} = (x+2y+az)\bar{i} + (bx-3y+z)\bar{j} + (4x+cy+2z)\bar{k}$ if $\bar{F}$ is irrotational.

Sol: $$\text{curl}(\bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y+z & 4x+cy+2z \end{vmatrix}$$

$$= \bar{i}[c-1] - \bar{j}(4-a) + \bar{k}(b-2)$$

If $\bar{F}$ is irrotational then $\text{curl}(\bar{F}) = \bar{0} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$$\therefore a = 4, \; b = 2, \; c = 1$$

prob: Find 'a' if $\bar{F} = 2xy\,\bar{i} + 3x^2y\,\bar{j} - 3ayz\,\bar{k}$ is solenoidal at (1,2,3).

Sol: $$\text{div}(\bar{F}) = (2y + 3x^2 - 3ay)\Big|_{(1,2,3)} = 0$$

$$4 + 3 - 6a = 0$$

$$a = 7/6$$

(GATE-93): If the linear velocity $\bar{v}$ is given by $\bar{v} = x^2y\,\bar{i} + xyz\,\bar{j} - yz^2\,\bar{k}$ then the angular velocity $\bar{w}$ at the point (1,1,-1) is

Sol: $$\bar{w} = \frac{1}{2}\text{curl}\,\bar{v} = \frac{1}{2}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -yz^2 \end{vmatrix}$$

$$= \frac{1}{2}[\bar{i}(-z^2 - xy) - \bar{j}(0-0) + \bar{k}(yz - x^2)]$$

At (1, 1, -1) $$= \frac{1}{2}[-2\bar{i} - 2\bar{k}] = -(\bar{i} + \bar{k})$$

(GATE-94): The directional derivative of $f(x,y) = 2x^2 + 3y^2 + z^2$ at point P(2,1,3) in the direction of the vector $a = \bar{i} - 2\bar{k}$ is

Sol: $$\nabla f = 4x\,\bar{i} + 6y\,\bar{j} + 2z\,\bar{k}$$

$$\nabla f\big|_{(2,1,3)} = 8\bar{i} + 6\bar{j} + 6\bar{k}$$

$$\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{k})$$

$$D.D = \nabla f \cdot \hat{n} = \frac{8-12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

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(GATE-95): The derivative of $f(x,y)$ at point $(1,2)$ in the direction of vector $\bar{i}+\bar{j}$ is $2\sqrt{2}$ and in the direction of the vector $-2\bar{j}$ is -3. Then the derivative of $f(x,y)$ in the direction $-\bar{i}-2\bar{j}$ is

Sol: Let $\bar{a} = \bar{i}+\bar{j}$

$$(\nabla f).\frac{\bar{a}}{|\bar{a}|} = 2\sqrt{2} \quad \Rightarrow \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right).\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right) = 2\sqrt{2}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4 \rightarrow (1)$$

Let $\bar{b} = -2\bar{j}$

$$(\nabla f).\frac{\bar{b}}{|\bar{b}|} = -3 \quad \Rightarrow \frac{-2\,\partial f/\partial y}{2} = -3 \quad \Rightarrow \frac{\partial f}{\partial y} = 3 \rightarrow (2)$$

from (1) $\frac{\partial f}{\partial x} = 1$

$\therefore$ D.D in the direction $-\bar{i}-2\bar{j}$ is $\left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right).\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right)$

$$= (\bar{i}+3\bar{j}).\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}$$

(GATE 96): The directional derivative of the function $f(x,y,z) = x+y$ at the point $P(1,1,0)$ along the direction $\bar{i}+\bar{j}$ is

Sol: $$D.D = \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)_{(1,1,0)} \cdot \left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right)$$

$$= \frac{1+1}{\sqrt{2}} = \sqrt{2}$$

(GATE 99): For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal fluid, $\nabla^2\phi$ should be equal to zero. In that case, the value of 'a' has to be

Sol: $$\phi = ax^2y - y^3$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial z} = 0$$

$$ay - 3y = 0 \quad \Rightarrow a = 3$$

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(GATE-02): The directional derivative of the following function at (1,2) in the direction of $(4\bar{i}+3\bar{j})$ is : $f(x) = x^2+y^2$

Sol: $DD = (\nabla f)_{(1,2)} \cdot \hat{n}$

$$= (2x\,\bar{i} + 2y\,\bar{j})_{(1,2)} \cdot \frac{4\bar{i}+3\bar{j}}{\sqrt{25}} = \frac{8+12}{5} = 4$$

(GATE-03): The vector feild $F = x\bar{i} - y\bar{j}$ is

(a) divergence free, but not irrotational

(b) irrotational, but not divergence free

✓(c) divergence free and irrotational

(d) neither divergence free nor irrotational

Sol: $\text{div}\,\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 1-1 = 0$ (divergent free)

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \bar{0}$$ Irrotational

(GATE-05 EE): For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1,3) is

Sol: $\text{grad}\,u = (\nabla u)_{(1,3)} = (x\bar{i} + \frac{2}{3}y\bar{j})_{(1,3)}$

$$= \bar{i} + 2\bar{j}$$

$$|\nabla u| = \sqrt{1+4} = \sqrt{5}$$

(GATE-05 IN): A scalar field is given by $f = x^{2/3} + y^{2/3}$, where x and y are cartesian co.ordinates. The derivative of 'f' along the line $y=x$ directed away from the origin at point (8,8) is

Sol: <u>unit vector</u> along $y=x$ is

$$\hat{a} = \cos\frac{\pi}{4}\bar{i} + \sin\frac{\pi}{4}\bar{j} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$$

$$(\nabla f)_{(8,8)} = \left(\frac{2}{3}x^{-1/3}\bar{i} + \frac{2}{3}y^{-1/3}\bar{j}\right)_{(8,8)}$$

$$= \tfrac{1}{3}\bar{i} + \tfrac{1}{3}\bar{j}$$

$$D.D = (\nabla f)\cdot\bar{a} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \sqrt{2}/3$$

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(GATE-06 CE): The D.D of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(1,2,3)$ in the direction of the vector $\bar{a} = \bar{i} - 2\bar{k}$ is

Sol: $(\nabla f)_{(1,2,3)} = (4x\bar{i} + 6y\bar{j} + 2z\bar{k})_{(1,2,3)}$

$= 4\bar{i} + 12\bar{j} + 6\bar{k}$

$$DD = (\nabla f).\frac{\bar{a}}{|\bar{a}|} = (4\bar{i} + 12\bar{j} + 6\bar{k}).\left(\frac{\bar{i} - 2\bar{k}}{\sqrt{5}}\right)$$

$$= \frac{4-12}{\sqrt{5}} = -8/\sqrt{5}$$

(GATE-07 CE): The velocity vector is given as $\bar{v} = 5xy\,\bar{i} + 2y^2\bar{j} + 3yz^2\bar{k}$ The divergence of this velocity vector at $(1,1,1)$ is

Sol: $\text{div}\,\bar{v} = 5y + 4y + 6yz$

At $(1,1,1)$ $\text{div}\,\bar{v} = 5+4+6 = 15$

(GATE-07 PI): The angle between two planar vector $\bar{a} = \frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ and $\bar{b} = -\frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ is

Sol:
$$\cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{-3/4 + 1/4}{\sqrt{\frac{3}{4}+\frac{1}{4}}.\sqrt{\frac{3}{4}+1/4}} = -1/2$$

$$\theta = \cos^{-1}(-1/2) = 120^\circ$$

(GATE-07 EE): Divergence of the vector field $V(x,y,z) = -(x\cos xy + y)\bar{i} + (y\cos xy)\bar{j} + [(\sin z^2) + x^2 + y^2]\bar{k}$ is

Sol:
$$\text{div}\,\bar{v} = \frac{\partial}{\partial x}[-(x\cos xy + y)] + \frac{\partial}{\partial y}(y\cos xy) + \frac{\partial}{\partial z}[(\sin z^2) + x^2 + y^2]$$

$$= 2z\cos z^2$$

(GATE-08 ME): The divergence of the vector field $(x-y)\bar{i} + (y-x)\bar{j} + (x+y+z)\bar{k}$ is

Sol: $\text{div}\,\bar{v} = 1+1+1 = 3$

(GATE-08 ME): The D.D of scalar function $f(x,y,z) = x^2 + 2y^2 + z$ at the point $P = (1,1,2)$ in the direction of vector $\bar{a} = 3\bar{i} - 4\bar{j}$ is

Sol:
$$DD = (2x\bar{i} + 4y\bar{j} + \bar{k})_{(1,1,2)} \cdot \left(\frac{3\bar{i} - 4\bar{j}}{5}\right)$$

$$= \frac{6-16}{5} = -2$$

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(GATE-09 CE): For a scalar function $f(x,y,z) = x^2 + 3y^2 + 2z^2$ the gradient at the point $P(1,2,-1)$ is

Sol: $\text{Grad } f = 2x\,\bar{i} + 6y\,\bar{j} + 4z\,\bar{k}$

At $(1,2,-1)$ $\quad \nabla f = 2\bar{i} + 12\bar{j} - 4\bar{k}$

(GATE-09 CE): For a scalar function $f(x,y,z) = x^2 + 3y^2 + 2z^2$ the directional derivative at $P(1,2,-1)$ in the direction of $\bar{i} - \bar{j} + 2\bar{k}$ is

Sol: $(\nabla f).\hat{n} = \dfrac{2-12-8}{\sqrt{6}} = \dfrac{-18}{\sqrt{6}} = -3\sqrt{6}$

(GATE-09 IN): A sphere of unit radius is centered at the origin. The unit normal at the point (x,y,z) on the surface of the sphere is the vector

a) (x,y,z) $\quad$ b) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\quad$ c) $\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{3}}\right)$ $\quad$ d) $\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, \frac{z}{\sqrt{2}}\right)$

Sol: Equation of sphere $x^2 + y^2 + z^2 = 1$

Normal to the surface $= \nabla\phi = 2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}$

$$\text{unit normal} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\,\bar{i} + y\,\bar{j} + z\,\bar{k}}{\underbrace{\sqrt{x^2 + y^2 + z^2}}_{=1}} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$$

(GATE-09 ME): The divergence of vector field. $3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\,\hat{k}$ at point $(1,1,1)$ is equal to

Sol: $\text{div } \bar{f} = 3z + 2x - 2yz$

$\text{div } \bar{f}\,|_{(1,1,1)} = 3 + 2 - 2 = 3$

(GATE-10 EE): Divergence of the 3-dimensional radial vector field $\bar{r}$ is

Sol: $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$

$\text{div } \bar{r} = 1 + 1 + 1 = 3$

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(GATE-11 CE): If $\bar{a}$ and $\bar{b}$ are two arbitrary vectors with magnitudes a and b respectively. Then $|\bar{a}\times\bar{b}|^2$ will be equal to

Sol: $|\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2\sin^2(\bar{a},\bar{b})$

$$= a^2b^2[1-\cos^2(\bar{a},\bar{b})]$$

$$= a^2b^2\left[1-\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2|\bar{b}|^2}\right] = a^2b^2\left[\frac{a^2b^2-\bar{a}\cdot\bar{b}}{a^2b^2}\right]$$

$$= a^2b^2-\bar{a}\cdot\bar{b}$$

(GATE-11 PI): If A(0,4,3), B(0,0,0), and C(3,0,4) are three points defined in x,y,z co-ordinate system then which one of the following vectors is perpendicular to both $\bar{AB}$ & $\bar{BC}$ vectors

a) $16\bar{i}+9\bar{j}-12\bar{k}$ b) $16\bar{i}-9\bar{j}+12\bar{k}$ c) $16\bar{i}-9\bar{j}-12\bar{k}$ d) $16\bar{i}+9\bar{j}+12\bar{k}$

Sol: $\bar{BA} = \bar{OA}-\bar{OB} = 4\bar{j}+3\bar{k}$

$\bar{BC} = \bar{OC}-\bar{OB} = 3\bar{i}+4\bar{k}$

vector perpendicular to $\bar{BA}$ & $\bar{BC}$ $= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 4 & 3 \\ 3 & 0 & 4 \end{vmatrix} = \bar{BA}\times\bar{BC}$

$$= 16\bar{i}+9\bar{j}-12\bar{k}$$

(GATE-11 PI): If $T(x,y,z) = x^2+y^2+2z^2$ defines the temperature at any location (x,y,z) then the magnitude of the temperature gradient at point P(1,1,1) is

Sol: $\nabla T = 2x\bar{i}+2y\bar{j}+4z\bar{k}$

$(\nabla T)_{(1,1,1)} = 2\bar{i}+2\bar{j}+4\bar{k} \Rightarrow |\nabla T| = \sqrt{4+4+16} = 2\sqrt{6}$

(GATE-11 EE): The two vectors [1 1 1] and [1 a a^2] where $a = \frac{-1}{2}+j\frac{\sqrt{3}}{2}$ are

✓a) orthogonal b) orthonormal c) parellel d) collinear

Sol: Let $\bar{p} = [1\ 1\ 1] = \bar{i}+\bar{j}+\bar{k}$

$$\bar{q} = [1\ a\ a^2] = \bar{i}+\left(\frac{-1}{2}+j\frac{\sqrt{3}}{2}\right)\bar{j}+\left(\frac{-1}{2}-j\frac{\sqrt{3}}{2}\right)\bar{k}$$

$$\bar{p}\cdot\bar{q} = 1-\frac{1}{2}+j\frac{\sqrt{3}}{2}-\frac{1}{2}-j\frac{\sqrt{3}}{2} = 0$$

Cube roots of unity are $1, \omega, \omega^2$

$1+\omega+\omega^2=0$

Both vectors are orthogonal

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(GATE-12 EC, EE, IN): The direction of vector A is radially outward from the origin, with $|A| = Kr^n$ where $r^2 = x^2+y^2+z^2$ and K is cons The value of n for which $\nabla \cdot \bar{A} = 0$

Sol: Given $|\bar{A}| = Kr^n$

unit radial vector in the direction of A is $\frac{\bar{r}}{r}$

$$\therefore \text{ vector } \bar{A} = Kr^n \cdot \frac{\bar{r}}{r} = Kr^{n-1}\bar{r}$$

Given $\nabla \cdot \bar{A} = 0$

$$K\,\nabla \cdot (r^{n-1}\bar{r}) = 0$$

$$\nabla \cdot [r^{n-1}x\,\bar{i} + r^{n-1}y\,\bar{j} + r^{n-1}z\,\bar{k}] = 0$$

$$\frac{\partial}{\partial x}(r^{n-1} \cdot x) + \frac{\partial}{\partial y}(r^{n-1}y) + \frac{\partial}{\partial y}(r^{n-1}z) = 0$$

$$\left[r^{n-1} + x(n-1)r^{n-2} \cdot \frac{\partial r}{\partial x}\right] + \left[r^{n-1} + y(n-1)r^{n-2} \cdot \frac{\partial r}{\partial y}\right] + [\cdots] = 0$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \; 3r^{n-1} + (n-1)r^{n-3}[x^2+y^2+z^2] = 0$$

$$3r^{n-1} + (n-1)r^{n-3} \cdot r^2 = 0$$

$$3r^{n-1} + (n-1)r^{n-1} = 0$$

$$3+n-1 = 0$$

$$n = -2$$

(GATE-12 ME, PI): For the spherical surface $x^2+y^2+z^2=1$, the unit outward normal vector at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ is given by

Sol: $$(\nabla\phi)_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)} = (2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k})_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)}$$

$$= \sqrt{2}\,\bar{i} + \sqrt{2}\,\bar{j} \quad \text{(Normal)}$$

$$\text{unit normal} = \frac{\sqrt{2}\,\bar{i} + \sqrt{2}\,\bar{j}}{\sqrt{2+2}} = \frac{\bar{i}}{\sqrt{2}} + \frac{\bar{j}}{\sqrt{2}}$$

(GATE-13 EC): The divergence of the vector field $\bar{A} = x\,\bar{a}_x + y\,\bar{a}_y + z\bar{a}$ is

Sol: $$\text{div } \bar{A} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

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(GATE-13 EE): The curl of gradient of scalar field defined by $v = 2x^2y + 3y^2z + 4z^2x$ is

Sol: $\text{curl}(\text{grad } v) = \bar{0}$

(GATE-14 ECE): If $\bar{r} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$ and $|\bar{r}| = r$, then $\text{div}(r^2\nabla(\ln r)) =$

Sol: $\nabla[f(r)] = f'(r).\frac{\bar{r}}{r}$

$$\nabla[\log r] = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$

$$\text{div}[r^2(\nabla(\log r))] = \text{div}(\bar{r}) = 1+1+1 = 3$$

(GATE-14 EC): The magnitude of the gradient for the function $f(x,y,z) = x^2+3y^2+z^3$ at the point $(1,1,1)$ is

Sol: $\nabla f = 2x\bar{i} + 6y\bar{j} + 3z^2\bar{k}$

$$(\nabla f)_{(1,1,1)} = 2\bar{i} + 6\bar{j} + 3\bar{k}$$

$$|\nabla f| = \sqrt{4+36+9} = 7$$

(GATE-14 EE): Let $\nabla.(fv) = x^2y + y^2z + z^2x$, where f and v are scalar and vector feilds respectively. If $v = y\bar{i} + z\bar{j} + x\bar{k}$ then $v.(\nabla f)$ is

Sol: $\nabla.(fv) = f(\nabla.v) + v.(\nabla f)$

$$v = y\bar{i} + z\bar{j} + x\bar{k} \Rightarrow \nabla.v = 0+0+0 = 0$$

$$\therefore \nabla.(fv) = v.(\nabla f) = x^2y + y^2z + z^2x$$

(GATE-14 CE): A particle moves along a curve whose parametric equations are: $x = t^3+2t$, $y = -3e^{-2t}$, $z = 2\sin 5t$, where x, y & z show variations of distance covered by particle (in cm) with time t (in s). The magnitude of the accelaration of particle (in cm/s^2) at $t=0$ is

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = (t^3+2t)\bar{i} + (-3e^{-2t})\bar{j} + (2\sin 5t)\bar{k}$

$$\text{velocity} = \frac{d\bar{r}}{dt} = (3t^2+2)\bar{i} + (6e^{-2t})\bar{j} + 10\cos 5t\ \bar{k}$$

$$\text{Acceleration} = \frac{d^2\bar{r}}{dt^2} = 6t\,\bar{i} - 12e^{-2t}\,\bar{j} - 50\sin 5t\ \bar{k}$$

At $t=0$ $\frac{d^2r}{dt^2} = -12\bar{j}$

$$\therefore \text{Magnitude} = |-12\bar{j}| = 12$$

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Line integrals: Integration along a curve is called line integral and is denoted by $\int_C \bar{F}\cdot d\bar{r}$

$$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_C F_1\,dx + \int_C F_2\,dy + \int_C F_3\,dz$$

prob: $\bar{F} = (5xy - 6x^2)\bar{i} + (4y - 6x)\bar{j}$ then $\int_C \bar{F}\cdot d\bar{r}$ along the curve $y = x^3 : (1,1) \rightarrow (2,8)$

Sol: $\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)\,dx + \int_C (4y - 6x)\,dy$

$$y = x^3 \Rightarrow dy = 3x^2\,dx$$

$$= \int_1^2 (5x^4 - 6x^2)\,dx + \int_1^2 (4x^3 - 6x)\,3x^2\,dx$$

$$= 76.5$$

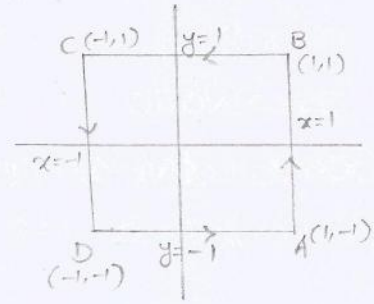
prob: $\bar{F} = x^2\bar{i} + xy\bar{j}$. Find $\int_C \bar{F}\cdot d\bar{r}$ along $y = x : (0,0) \rightarrow (1,1)$

Sol: $\int_C \bar{F}\cdot d\bar{r} = \int_C x^2\,dx + \int_C xy\,dy$

$$y = x \Rightarrow dy = dx$$

$$= \int_0^1 x^2\,dx + \int_0^1 x^2\,dx = 2\cdot\frac{x^3}{3}\Big|_0^1 = 2/3$$

prob: $\oint_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$ where 'c' is a square bounded by $x = \pm 1$, $y = \pm 1$.



Sol: Along AB: $x = 1 \Rightarrow dx = 0$

$$\int_{-1}^{1} (1 + y^2)\,dy = \left[y + y^3/3\right]_{-1}^{1} = 8/3$$

Along BC: $y = 1 \Rightarrow dy = 0$

$$\int_1^{-1} (x^2 + x)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^{-1} = -2/3$$

Along CD: $x = -1 \Rightarrow dx = 0$

$$\int_1^{-1} (1 + y^2)\,dy = -8/3$$

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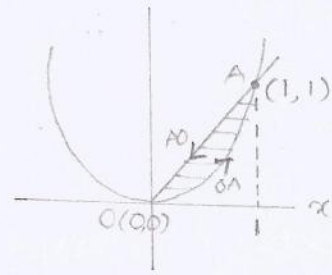
Along DA : $y=-1$

$$\int_{-1}^{1}(x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1} = 2/3$$

$$\therefore \oint_C (x^2+xy)dx+(x^2+y^2)dy = \frac{8}{3}+\frac{2}{3}-\frac{8}{3}-\frac{2}{3} = 0$$

prob: $\oint_C (xy+y^2)dx + x^2dy$: 'c' is bounded by $y=x$: $y=x^2$

Sol: Along OA: $y=x^2 \Rightarrow dy = 2x\cdot dx$



$$\int_{OA}(x^3+x^2)dx + \int_{OA} 2x^3dx = \int_0^1 (x^4+3x^3)dx$$
$$= 19/20$$

Along AO: $y=x \Rightarrow dy=dx$

$$\int_{AO}(x^2+x^2)dx + \int_{AO} x^2dx = 3\int_1^0 x^2dx = x^3\Big|_1^0 = -1$$

$$\therefore \oint_C (xy+y^2)dx + x^2dy = \frac{19}{20}-1 = -1/20$$

Greens theorem: Let M & N are functions of x, y defined in a region with a simple closed curve 'c' then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

[Relation b/w line & double integration]

Stokes theorem: Let 'S' be a <u>open</u> surface bounded by a simple closed curve 'c' then GATE 13EC

GATE 09EC 13EC $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$

GATE 05EE [Relation b/w line & surface integration]

Gauss-divergence theorem: Let 'S' be a <u>closed</u> surface then

09EE

$$\oiint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

[Relation b/w surface & volume integration]

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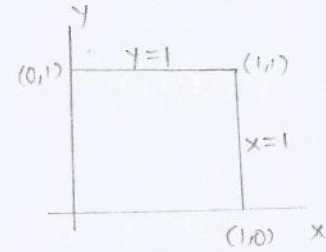
(GATE-05): value of the integral $\oint_c xy\,dy - y^2 dx$, where c is the square cut from the first quadrant by the line $x=1$ and $y=1$ will be (use Green's theorem to change the line integral into double integral)



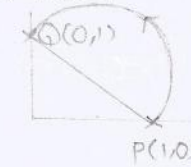
Sol: $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here $M = -y^2$ $N = +xy$

$$\oint_c xy\,dy - y^2 dx = \int_{x=0}^{1}\int_{y=0}^{1} (y + 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} 3x\,dx = 3/2$$

(GATE-08 EE): Consider points P and Q in xy-plane with P(1,0) and Q(0,1). The line integral $2\int_P^Q (x\,dx + y\,dy)$ along the semi circle with the line segment PQ as diameter



Sol: $M = 2x$ $N = 2y$

$\frac{\partial M}{\partial y} = 0$ $\frac{\partial N}{\partial x} = 0$

$\therefore$ From greens theorem $2\oint_P^Q (x\,dx + y\,dy) = 0$

(GATE-08 PI): If $\bar{r}$ is position vector of any point on a closed surface S that encloses the volume V then $\iint_S \bar{r}\cdot d\bar{s}$ is equal to

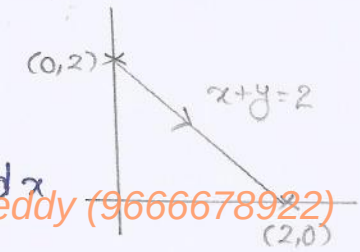
Sol: From gauss-divergence theorem

$$\iint_S \bar{r}\cdot d\bar{s} = \iiint_V (\nabla\cdot\bar{r})\,dv \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$= \iiint_V (1+1+1)\,dv = 3V$$

(GATE-09 EE): $F(x,y) = (x^2+xy)\hat{a}_x + (y^2+xy)\hat{a}_y$. Its line integral over the straight line from $(x,y) = (0,2)$ to $(2,0)$ evaluates to

Sol: Given straight line is $x+y=2$



$y = 2 - x \Rightarrow dy = -dx$

$$\int \bar{F}\cdot d\bar{r} = \int_0^2 [x^2 + x(2-x)]\,dx \;\; \int_0^2 [(2-x)^2 + x(2-x)]\,dx$$

$= 0$

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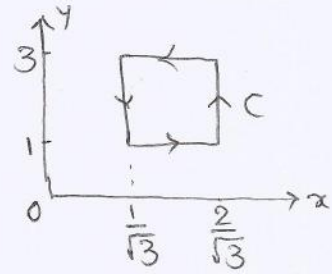
(GATE-09 PI): The line integral of the vector function $\bar{F} = 2x\hat{i} + x^2\hat{j}$ along the x-axis from $x=1$ to $x=2$ is

Sol: x-axis $\Rightarrow y=0 \Rightarrow dy=0$

$$\int \bar{F}\cdot d\bar{r} = \int_1^2 2x\,dx + \int_1^2 x^2 dy$$

$$= \int_1^2 2x\,dx = 4-1 = 3$$

(GATE-10 EC): If $\bar{A} = xy\,\hat{a}_x + x^2\hat{a}_y$ then $\oint \bar{A}\cdot d\bar{r}$ over the path shown in the figure is



Sol: Apply Green's theorem

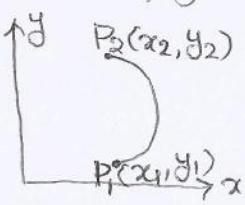
$$M = xy \qquad N = x^2$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\oint \bar{A}\cdot d\bar{r} = \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} \int_{y=1}^{3} (2x - x)\,dy\,dx$$

$$= \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} x\cdot(y)_1^3\,dx = 2\cdot\frac{x^2}{2}\Big|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{4}{3} - \frac{1}{3} = 1$$

(GATE-11 PI): The line integral $\int_{P_1}^{P_2} (y\,dx + x\,dy)$ from $P(x_1, y_1)$ to $P_2(x_2, y_2)$ along the semi circle P_1P_2 as shown in figure



Sol: $$\int_{P_1}^{P_2} y\,dx + x\,dy = \int_{P_1}^{P_2} d(xy)$$

$$= (xy)_{P_1}^{P_2} = x_2y_2 - x_1y_1$$

(GATE-11 EC): Consider a closed surface 'S' surrounding a volume V. If $\bar{r}$ is the position vector of a point inside 'S' with $\hat{n}$ the unit normal on 'S', the value of $\oiint 5\bar{r}\cdot\hat{n}\,ds$ is

Sol: From gauss-divergence theorem

$$\oiint \bar{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$\Rightarrow \oiint 5\bar{r}\cdot\hat{n}\,ds = 5\iiint \text{div}\,\bar{r}\,dv \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$= 5\iiint 3\,dv$$

$$= 15V$$

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(GATE 13 ME): The following surface integral is to be evaluated over a sphere for the given steady velocity vector field $F = x\bar{i} + y\bar{j} + z\bar{k}$ defined w.r.to a cartesian co-ordinate system having i, j, k as unit base vectors. $\iint_S \frac{1}{4}(F \cdot n)\, dA$ where S is sphere, $x^2+y^2+z^2=1$ and n is the outward unit normal vector to the sphere. The value of surface integral is

Sol: $$\iint_S \frac{1}{4} \bar{F} \cdot \hat{n}\, dA = \frac{1}{4} \iiint_V \nabla \cdot \bar{F}\, dv$$

$$= \frac{1}{4} \iiint 3\, dv = \frac{3}{4} (\text{volume of sphere})$$

$$= \frac{3}{4} \times \frac{4}{3} \pi r^3 = \pi r^3 = \pi$$

$x^2+y^2+z^2=1$
$\Rightarrow r=1$

(GATE-13 EE): Given a vector field $\bar{F} = y^2 x\, \hat{a}_x - yz\, \hat{a}_y - x^2\, \hat{a}_z$, the line integral $\int F \cdot dl$ evaluated along a segment on the x-axis from $x=1$ to $x=2$ is

Sol: x-axis $\Rightarrow$ $y=0,\ z=0$

$dy=0,\ dz=0$

$$\therefore \int \bar{F} \cdot dl = \int_1^2 y^2 x\, dx = 0 \quad [\because y=0]$$

(GATE-14 EE): The line integral of function $F = yz\,\bar{i}$ in the counter clockwise direction, along the circle $x^2+y^2=1$ at $z=1$ is

Sol: $$\int_C \bar{F} \cdot d\bar{l} = \int_C yz\, dx$$

$$= \int (yz\,dx - 0\,dy)$$

Apply greens theorem $M = yz$ $N = 0$

$$\frac{\partial M}{\partial y} = z \qquad \frac{\partial N}{\partial x} = 0$$

at $z=1$ $\frac{\partial M}{\partial y} = 1$

$$= \iint_S (0-1)\, dx \cdot dy = -\iint_S dx \cdot dy = -S = -\pi r^2 \quad [\because r=1]$$

$$= -\pi$$

(GATE-14 ME): The integral $\oint_C (y\,dx - x\,dy)$ is evaluated along the circle $x^2+y^2 = 1/4$ traversed in counter clockwise direction. The integral is equal to

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Sol: Apply green's theorem

$$\oint_C (y\,dx - x\,dy) = \iint_R (-1-1)\,dx\,dy$$

$$= -2\,(\text{Areal of given circle})$$

$$= -2 \cdot \pi \left(\frac{1}{2}\right)^2 = -\pi/2$$

$x^2 + y^2 = (1/2)^2$

$r = 1/2$

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DIFFERENTIAL EQUATIONS

Differential Equations

Differential Equation: Equations involving differential co.efficients are called differential equations.

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{\partial y}{\partial x}$ ... are called differential co.efficients

Ordinary differential equation: The differential equation in which all the differential co.efficient have reference to a single independent variable.

Ex: 1. $\frac{d^2y}{dt^2} + n^2x = 0$

2. $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$

3. $e^x dx + e^y dy = 0$

partial differential equation: The differential equation in which there are two or more independent variables and partial differential ~~di~~ coefficients with respect to any of them.

Ex: 1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

2. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

Order of D.E: It is the order of the highest differential co.efficient that occurs in the equation.

Degree of D.E: It is the power of highest differential co.efficient provided the equation is free from fractional powers.

Ex:	Order	Degree	
1. $[1+(dy/dx)^2]^{3/2} = (d^2y/dx^2)$	2	2	[PI-05]
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$	2	1	
3. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$	2	1	
4. $\left(\frac{dy}{dx}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^{2/3}$	2	4	

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Formation of D.E: Differential equations are formed by the elemination of orbitrary constants / functions from a relation in the variables and constants. From D.E involving 'n' arbitrary constants we get the D.E of nth order.

Prob: Form the D.E by eliminating the arbitrary constants present in $y = ax + bx^2$

$$y = ax + bx^2 \rightarrow (1)$$

d.w.r.to x $\quad y' = a + 2bx \rightarrow (2)$

d.w.r.to 'x' $\quad y'' = 2b$

$$b = y''/2 \rightarrow (3)$$

from (2) & (3) $\quad y' = a + xy''$

$$a = y' - xy'' \rightarrow (4)$$

from (1) & (4) $\quad y = (y' - xy'')x + \frac{y''}{2}x^2$

$$x^2y'' + 2xy' - 2x^2y'' = 2y$$

$$x^2y'' - 2xy' + 2y = 0$$

$$x^2\frac{d^2y}{dx} - 2x\frac{dy}{dx} + 2y = 0$$

prob: $\quad y = a + bx$

$$y = a + bx \rightarrow (1)$$

d.w.r.to x $\quad y' = b \rightarrow (2)$

d.w.r.to x $\quad y'' = 0$

$$\frac{d^2y}{dx^2} = 0$$

prob: $\quad y = a\cos x + b\sin x \rightarrow (1)$

$$y' = -a\sin x + b\cos x \rightarrow (2)$$

$$y'' = -a\cos x - b\sin x \rightarrow (3)$$

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from (1) & (3) $y'' = -y$

$$y'' + y = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

prob: $y = ae^{x} + be^{2x}$

$$= y + be^{2x}$$

$$y' = ae^{x} + 2be^{2x} \longrightarrow (1)$$

$$y'' = y' + 2be^{2x}$$

$$be^{2x} = \frac{y'' - y'}{2} \longrightarrow (2)$$

from (1) & (2)

$$y' = y + \frac{y'' - y'}{2}$$

$$3y' = 2y + y'' \Rightarrow y'' + 2y - 3y' = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

prob: $(x-a)^2 + (y-b)^2 = 1 \longrightarrow (1)$

D.w.r.to 'x' $\quad 2(x-a) + 2(y-b)y' = 0 \longrightarrow (2)$

$$(x-a) + (y-b)y' = 0$$

D.w.r.to 'x' $\quad 1 + (y-b)y'' + (y')^2 = 0$

$$y - b = -\frac{[1+(y')^2]}{y''} \longrightarrow (3)$$

from (2) & (3) $\quad x - a = \frac{[1+(y')^2]}{y''} y' \longrightarrow (4)$

from (1) & (4) $\quad \left[\frac{(1+(y')^2)y'}{y''}\right]^2 + \left[-\frac{(1+(y')^2)}{y''}\right]^2 = 1$

$$[1+(y')^2]^2 [(y')^2 + 1] = (y'')^2$$

$$[1+(y')^2]^3 = (y'')^2$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left[\frac{d^2y}{dx}\right]^2$$

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Solution of D.E:

An Equation in terms of the variables which satisfies the given differential equation is called solution of D.E

Ex: For the D.E $\frac{d^2y}{dx^2}+y=0$, the solution is

$$y = a\cos x + b\sin x$$

also $y = 2\cos x$, $\cos x + \sin x$, $\sin 3x$ are also solutions.

The solution of D.E is of two types

General solution: The solution of D.E which contains as many arbitrary constants as that of the order of D.E is called the general solution.

Particular solution: A solution obtained from ~~particular~~ general solution by giving particular values to arbitrary cons-tants are called particular solutions.

D.E of 1st order and 1st degree

The general form of D.E of 1st order and 1st degree is $\frac{dy}{dx} = f(x,y)$. Depending on the function $f(x,y)$ these equations are divided into 4-types

1. variable seperable form D.E
2. Homogeneous D.E
3. Linear D.E
4. Exact D.E

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1. variable seperable form: If in an equation it is possible to collect all functions of 'x' and 'dx' on one side and all the functions of 'y' and 'dy' on the other side, then the variables are said to be seperable. Thus the general form of such an equation is

$$\psi(y)\,dy = \phi(x)\,dx$$

I.O.B.S $\quad \int \psi(y)\,dy = \int Q(x)\,dx + C$

Prob: $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

I.O.B.S $\quad \log_e y = \log_e x + C \qquad$ Let $C = \log_e K$

$$= \log_e x + \log_e K$$

$$\log_e y = \log_e xK$$

$y = xK$ is the solution.

Prob: $\dfrac{dy}{dx} = 1 + x + y + xy$

$$\frac{dy}{dx} = 1 + x + y(1+x) = (1+x)(1+y)$$

$$\frac{1}{1+y}\,dy = (1+x)\,dx$$

I.O.B.S $\quad \log_e(1+y) = x + \dfrac{x^2}{2} + C$

Prob: $y - x\dfrac{dy}{dx} = a\left[y^2 + \dfrac{dy}{dx}\right]$

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\frac{1}{y-ay^2}\,dy = \frac{1}{x+a}\,dx$$

$$\frac{1}{y(1-ay)}\,dy = \frac{1}{x+a}\,dx$$

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$$\left[\frac{a}{1-ay}+\frac{1}{y}\right]dy=\frac{1}{x+a}dx$$

I.O.B.S $\quad -\log_e|1-ay|+\log_e|y|=\log_e|x+a|+c \qquad c=\log_e K$

$$\log_e\frac{|y|}{|1-ay|}=\log_e|x+a|K$$

$$\therefore \quad \frac{|y|}{|1-ay|}=|x+a|K$$

$$\frac{y}{(1-ay)}=(x+a)K$$

Prob: $(x+y+1)^2\frac{dy}{dx}=1 \longrightarrow (1)$

put $\quad x+y+1=t$ say

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

from (1) $\quad t^2\left(\frac{dt}{dx}-1\right)=1$

$$\frac{dt}{dx}=\frac{1+t^2}{t^2} \Rightarrow \frac{t^2}{1+t^2}dt=dx$$

$$\Rightarrow \left[1-\frac{1}{1+t^2}\right]dt=dx$$

I.O.B.S

$$t-\tan^{-1}t=x+c$$

$$(x+y+1)-\tan^{-1}(x+y+1)=x+c$$

Prob: $\frac{dy}{dx}=e^{3x-2y}+x^2e^{-2y}$

$$\frac{dy}{dx}=e^{3x}\cdot e^{-2y}+x^2\cdot e^{-2y}$$

$$=(x^2+e^{3x})\cdot e^{-2y}$$

$$e^{2y}dy=(x^2+e^{3x})dx$$

I.O.B.S $\quad \frac{e^{2y}}{2}=\frac{x^3}{3}+\frac{e^{3x}}{3}+c$

$$3e^{2y}=2(e^{3x}+x^3)+6c$$

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2. Homogeneous D.E: D.E's of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where f_1, f_2 are homogeneous functions of same degree are called homogeneous D.E's

To solve these equations we take substitution

$y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(or)

$x = vy$ so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$

Later, seperate the variables and integrate.

Prob: solve $(x^2 - y^2)dx - xy\,dy = 0$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$ Homogeneous in x & y

put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x^2v} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\frac{v}{1-2v^2}\,dv = \frac{dx}{x}$$

I.O.B.S $$-\frac{1}{4}\int \frac{4v}{1-2v^2}\,dv = \int \frac{1}{x}dx + c$$

$$-\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$4\log x + \log(1-2v^2) = -4c$$

$$x^4 \cdot (1-2v^2) = e^{-4c}$$

$$x^4\left(1 - 2\frac{y^2}{x^2}\right) = K$$

$$x^2(x^2 - 2y^2) = K$$

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Prob: Solve $(e^{x/y}+1)dx + e^{x/y}(1-x/y)dy = 0$

$$\frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{e^{x/y}+1}$$ is a homogeneous eqn

put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{e^{v}+1}$$

$$y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{e^{v}+1} - v = -\frac{v+e^{v}}{1+e^{v}}$$

$$-\frac{dy}{y} = \frac{1+e^{v}}{v+e^{v}}dv$$

I.O.B.S $-\log y = \log_e(v+e^{v}) + c$

$$\log y + \log(v+e^{v}) = -c$$

$$y(v+e^{v}) = e^{-c} = K \text{ say}$$

$$y\left(\frac{x}{y} + e^{x/y}\right) = K$$

$$x + ye^{x/y} = K$$

Prob: Solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx - x\sec^2(y/x)dy = 0$

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2\left(\frac{y}{x}\right)$$

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$$

$$x\frac{dv}{dx} = -\tan v\cdot\cos^2 v = -\frac{\tan v}{\sec^2 v}$$

$$\frac{\sec^2 v}{\tan v}dv = -\frac{1}{x}dx$$

$$\log(\tan v) = -\log x + c$$

$$x\tan v = K$$

$$x\tan(y/x) = K$$

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Equations reducible to homogeneous form:

The equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ can be reduced to homogeneous form as follows

Case I: when $\frac{a}{a'} \neq \frac{b}{b'}$

put $x = X + h$ $\quad y = Y + k$

$dx = dX \quad dy = dY$

$$\frac{dY}{dX} = \frac{aX + bY + (ah+bk+c)}{a'X + b'Y + (a'h + b'k + c')} \rightarrow (1)$$

solve $\left.\begin{array}{l} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{array}\right\}$ To get h & k

So that (1) becomes

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$ is a homogenous in X & Y

put $Y = vX$ or $X = vY$

solve it

case II: when $\frac{a}{a'} = \frac{b}{b'}$

Let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ say

$$\therefore \frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c'} \rightarrow (1)$$

put $ax+by = t$

D.w.r.to 'x' $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

from (1) $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{mt+c'}$

separate variables 'x' and 't'

solve it.

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3. Linear D.E: A D.E is said to be linear if the dependant variable and it's differential co.efficients occur only in the first degree and not multiplied together.

Prob: Which of the following is a linear D.E?

(a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$ (b) $\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y=0$

(c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=0$ ✓(d) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

The standard form of a 1st order linear D.E (Leibnitz linear Equation) is (CE-97)

$$\frac{dy}{dx}+Py=Q$$ where P, Q are function of 'x' or 'constant'

It's solution is $y(I.F)=\int Q(I.F)dx+C$

where $I.F=e^{\int P dx}$

Prob: solve $(x+1)\frac{dy}{dx}-y=e^{3x}(1+x)^2$

$$\frac{dy}{dx}-\frac{1}{(x+1)}y=e^{3x}(1+x)$$

$$P=\frac{-1}{x+1} \qquad Q=e^{3x}(1+x)$$

$$I.F=e^{\int\frac{-1}{x+1}dx}=e^{-\log_e(1+x)}=\frac{1}{1+x}$$

Solution is $y\cdot\frac{1}{x+1}=\int\frac{e^{3x}(1+x)}{1+x}dx+C$

$$\frac{y}{x+1}=\frac{e^{3x}}{3}+C$$

Prob: Solve $y(\log y)dx+(x-\log y)dy=0$

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

$$P=\frac{1/y}{\log y} \qquad Q=\frac{1}{y}$$

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$$I.F = e^{\int \frac{1/y}{\log y} dy} = e^{\log(\log y)} = \log y$$

solution is $x \log y = \int \frac{1}{y} \log y \, dy + c$

$$x \log y = \frac{1}{2} (\log y)^2 + c$$

$$x = \frac{1}{2} \log y + c.(\log y)^{-1}$$

Prob: $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \tan x . \sec^2 x$$

$$I.F = e^{\int \sec^2 x \, dx} = e^{\tan x}$$

solution is $y(e^{\tan x}) = \int \tan x . \sec^2 x . e^{\tan x} dx + c$

$$= \int t e^t dt + c \qquad \tan x = t$$

$$= te^t - e^t + c \qquad \sec^2 dx = dt$$

$$y(e^{\tan x}) = e^{\tan x}(\tan x - 1) + c$$

Bernoulli's equation:

The equation $\frac{dy}{dx} + Py = Qy^n$ where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called bernoulli's equation.

Divide b.s by $y^n \Rightarrow y^{-n} \frac{dy}{dx} + Py^{1-n} = Q \rightarrow (1)$

Let $y^{1-n} = z \Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{dz}{dx}$$

from (1) $\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q$

$$\frac{dz}{dx} + P(1-n) z = Q(1-n) \qquad \text{(CE-05)}$$

which is a Linear D.E in 'z'

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Prob: solve $x\frac{dy}{dx}+y=x^3y^6$

$$y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2 \rightarrow (1)$$

let $y^{-5}=z \Rightarrow -5y^{-6}\frac{dy}{dz}=\frac{dz}{dx}$

from (1) $-\frac{1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2$

$$\frac{dz}{dx}-\frac{5}{x}\cdot z=-5x^2 \quad \text{Leibnitz eqn}$$

$$I.F=e^{-\int\frac{5}{x}dx}=e^{-5\log x}=x^{-5}$$

solution is $z\cdot e^{-5}=\int(-5x^2)\cdot x^{-5}dx+C$

$$z\cdot e^{-5}=-5\cdot\frac{x^{-3+1}}{-3+1}+C$$

$$y^{-5}\cdot e^{-5}=-5\,\frac{x^{-2}}{-2}+C$$

4. Exact D.E: D.E's which can be expressable as an exact differential of some function of x, y are called exact D.E

Ex: $ydx+xdy=0 \Rightarrow d(xy)=0$

$$\Rightarrow xy=c$$

$\frac{1}{y}dx-\frac{x}{y^2}dy=0 \Rightarrow d\left(\frac{x}{y}\right)=0$

$$\Rightarrow \frac{x}{y}=c$$

In general, for an equation of the form $Mdx+Ndy=0$ to be exact, the required condition is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. If it is satisfied then the solution of the equation is

$$\int_{y\ \text{const}} Mdx+\int(\text{terms of } N \text{ independent of } x)\,dy=c$$

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prob: solve $y\sin 2x\,dx - (1+y^2+\cos^2 x)dy = 0$

$M = y\sin 2x \qquad N = -(1+y^2+\cos^2 x)$

$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = 2\cos x \cdot \sin x = \sin 2x$

solution is $\int_{y\ \text{const}} y\sin 2x + \int -(1+y^2)\,dy + c$

$$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c$$

prob: $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$\frac{\partial M}{\partial y} = -4x - 4y \qquad \frac{\partial N}{\partial y} = -4x - 4y$

solution is $\int_{y\ \text{const}} (x^2 - 4xy - 2y^2)\,dx + \int y^2\,dy = 0$

$$\frac{x^3}{3} - 4y\cdot\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = c$$

$$x^3 - 6x^2y - 6xy^2 + y^3 = k$$

prob: $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$\frac{\partial M}{\partial y} = 2x - \sec^2 y + 1 \qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$

$= 2x - \tan^2 y$

solution is $\int_{y\ \text{const}} (2xy + y - \tan y)\,dx + \int \sec^2 y\,dx = 0$

$$2y\cdot\frac{x^2}{2} + xy - x\cdot\tan y + \tan y = c$$

Previous problems

prob: (GATE 94): The necessary & sufficient conditions for the D.E of the form $M(x,y)\,dx + N(x,y)\,dy$ to be exact is

Sol: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

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CE-99: If C is a constant then the solution of $\frac{dy}{dx} = 1+y^2$ is

Sol:

$$\frac{1}{1+y^2}\,dy = dx$$

I.O.B.S $\tan^{-1}y = x + C$

$$y = \tan(x+C)$$

IN-08: Particular soln $y = \tan(x+3)$

ME-03: The solution of D.E $\frac{dy}{dx} + y^2 = 0$ is

Sol:

$$\frac{1}{y^2}\,dy = -dx$$

I.O.B.S $-\frac{1}{y} = -x + C$

$$\frac{1}{y} = x - C \Rightarrow y = \frac{1}{x+K} \quad (K = -C)$$

CE-04: Bio transformation of an organic compound having concentration (x) can be modeled using an ordinary differential eqn $\frac{dx}{dt} + Kx^2 = 0$, where K is the reaction constant. If $x = a$ at $t = 0$ then solution of the eqn is

Sol:

$$\frac{dx}{dt} = -Kx^2$$

$$-\frac{dx}{x^2} = K\,dt$$

I.O.B.S $+\frac{1}{x} = +Kt + C \rightarrow (1)$

Given $x = a$ at $t = 0 \Rightarrow \frac{1}{a} = C$

$\therefore$ soln is $\frac{1}{x} = Kt + \frac{1}{a}$

EE-05: The solution of 1st order D.E $\dot{x}(t) = -3x(t)$, $x(0) = x_0$ is

EC-08: $x(0) = 2$

Sol:

$$\frac{dx}{dt} = -3x$$

$$\frac{1}{x}\,dx = -3\,dt$$

I.O.B.S $\log x = -3t + C$

$$x = e^{-3t}\,e^{C} \rightarrow (1)$$

Given $x(0) = x_0$

from (1) $x_0 = e^0 \cdot e^C$

$e^C = x_0$

$\therefore$ Solution is $x = e^{-3t} \cdot x_0$

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$$I.F = e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$$

Solution is $y \cdot x = \int x^3 \cdot x \, dx + C$

$$y \cdot x = \frac{x^5}{5} + C$$

Given $y(1) = \frac{6}{5} \Rightarrow \frac{6}{5} = \frac{1}{5} + C \Rightarrow C = 1$

$\therefore$ Solution is $xy = \frac{x^5}{5} + 1$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

PI-10: The solution of the D.E $\frac{dy}{dx} - y^2 = 1$ satisfying the condition $y(0) = 1$ is

Sol:

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{1+y^2} dy = \int dx + c$$

$$\tan^{-1} y = x + C$$

Given $y(0) = 1 \Rightarrow \tan^{-1}(1) = 0 + C \Rightarrow C = \pi/4$

$\therefore$ solution is $\tan^{-1} y = x + \pi/4$

$$y = \tan(x + \pi/4)$$

PI-10: Which of the following D.E has a solution given by the function $y = 5 \sin(3x + \pi/3)$

Sol:

$$y = 5 \sin 3x \cdot \cos \pi/3 + 5 \cos 3x \cdot \sin \pi/3$$

$$= \underbrace{5 \cos \pi/3}_{a} \cdot \sin 3x + \underbrace{5 \sin \pi/3}_{b} \cos 3x$$

$y = a \sin 3x + b \cos 3x$ a & b are arbitrary constants

$y' = 3a \cos 3x - 3b \sin 3x$

$y'' = -9a \sin 3x - 9b \cos 3x$

$= -9y$

$$y'' + 9y = 0$$

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(06 - ME): The solution of D.E $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

Sol:

$$P = 2x \qquad Q = e^{-x^2}$$

$$I.F = e^{\int P dx} = e^{x^2}$$

Solution is $y(e^{x^2}) = \int e^{-x^2} \cdot e^{x^2} dx + C$

$$y \cdot e^{x^2} = x + C$$

Given $y(0) = 1 \Rightarrow 1 \cdot e^0 = 0 + C \Rightarrow C = 1$

$$\therefore y e^{x^2} = x + 1$$

$$y = (x+1) e^{-x^2}$$

(CE-07): The solution of D.E $\frac{dy}{dx} = x^2 y$ with the condition that $y = 1$ at $x = 0$ is

Sol:

$$\frac{1}{y} dy = x^2 dx$$

I.O.B.S $\quad \log y = \frac{x^3}{3} + C$

$$y = e^{x^3/3} \cdot e^C$$

Given $y = 1$ at $x = 0 \Rightarrow 1 = e^0 \cdot e^C \Rightarrow e^C = 1$

$\therefore$ solution is $y = e^{x^3/3}$

(CE-09): solution of D.E $3y \frac{dy}{dx} + 2x = 0$ represents a family of

Sol:

$$3y\, dy + 2x\, dx = 0$$

I.O.B.S $\quad 3\frac{y^2}{2} + 2\frac{x^2}{2} = C$

$$\frac{x^2}{1} + \frac{y^2}{(2/3)} = C$$

Represents a family of ellipses

(ME-09): The solution of $x\frac{dy}{dx} + y = x^4$ with $y(1) = \frac{6}{5}$ is

Sol:

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = x^3$$

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9 (IN-10): consider the differential equation $\frac{dy}{dx} + y = e^x$ with $y(0) = 1$. Then the value of $y(1)$ is

Sol: $I.F = e^{\int dx} = e^x$

Solution is $y \cdot e^x = \int e^x \cdot e^x \, dx + C$

$$y e^x = \frac{e^{2x}}{2} + C$$

Given $y(0) = 1 \Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$$\therefore y e^x = \frac{e^{2x}}{2} + \frac{1}{2}$$

$$y = \frac{e^x}{2} + \frac{1}{2} e^{-x} = \frac{1}{2}(e^x + e^{-x})$$

$$y(1) = \frac{e^1 + e^{-1}}{2}$$

(EE-11): With K as constant, the possible solution for the 1st order D.E $\frac{dy}{dx} = e^{-3x}$ is

Sol: $dy = e^{-3x} dx$

I.O.B.S $y = \frac{e^{-3x}}{-3} + K$

(EC-11): The solution of the D.E $\frac{dy}{dx} = Ky$, $y(0) = C$ is

Sol: $\frac{1}{y} dy = K dx$

$$\log y = Kx + C_1$$

$$y = e^{Kx} \cdot e^{C_1}$$

Given $y(0) = C \Rightarrow C = e^{C_1}$

$$\therefore y = e^{Kx} \cdot C$$

(ME-11): Conside the D.E, $\frac{dy}{dx} = (1+y^2)x$. The general solution with constant 'c' is

Sol: $\frac{1}{1+y^2} dy = x \, dx$

I.O.B.S $\tan^{-1} y = \frac{x^2}{2} + C \Rightarrow y = \tan\left(\frac{x^2}{2} + C\right)$

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(CE-11): The solution of D.E $\frac{dy}{dx}+\frac{y}{x}=x$ with condition that $y=1$ at $x=1$ is

Sol: $I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

solution is $y.x = \int x.x\,dx + C$

$$xy = \frac{x^3}{3}+C$$

Given $y=1$ at $x=1$

$$1 = \frac{1}{3}+C \Rightarrow C=\frac{2}{3}$$

$$\therefore xy = \frac{x^3}{3}+\frac{2}{3} \Rightarrow y = \frac{x^2}{3}+\frac{2}{3x}$$

EC,EE, IN-12: With initial condition $x(1)=0.5$, the solution of the D.E $t\frac{dx}{dt}+x=t$ is

Sol: $\frac{dx}{dt}+\frac{x}{t}=1$

$I.F = e^{\int \frac{1}{t}dt} = e^{\log_e t} = t$

solution is $x.t = \int t.1\,dt + C$

$$xt = \frac{t^2}{2}+C$$

Given $x(1)=0.5 \Rightarrow 0.5(1) = \frac{1}{2}(1)+C \Rightarrow C=0$

$$\therefore xt = \frac{t^2}{2} \Rightarrow x = \frac{t}{2}$$

(CE-12): The solution of O.D.E $\frac{dy}{dx}+2y=0$ for the boundary condition, $y=5$ at $x=1$ is

Sol: $I.F = e^{\int P dx} = e^{2x}$

Sol^n is $y.e^{2x} = \int 0\,dx + C$

$$y = c.e^{-2x}$$

Given $y=5$ at $x=1 \Rightarrow 5 = ce^{-2}$

$$c = \frac{5}{e^2}$$

$$y = \frac{5}{e^2}e^{-2x}$$

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(10) (EC-13): A system described by a linear, constant co.efficient ordinary, first ODE has an exact solution given by $y(t)$ for $t>0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t>0$, we need to

Sol: Solution is $y(t) = \frac{1}{I.F}\int \underbrace{x(t)}_{\text{forcing fun}}(I.F)dt + \underbrace{y(0)}_{\text{Initial cond}}$

To get $-2y(t)$ multiply on b.s with '-2'

$$-2y(t) = \frac{1}{I.F}\int(-2x(t))(I.F)dt + (-2y(0))$$

$\therefore$ Forcing function changes to $-2x(t)$

Initial condition " " $-2y(0)$

(ME-14): The general solution of the D.E $\frac{dy}{dx} = \cos(x+y)$, with 'c' as a constant is

Sol: Let $x+y = t \Rightarrow 1+\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} - 1 = \cos t$$

$$\frac{dt}{dx} = 1+\cos t = 2\cos^2\frac{t}{2}$$

$$\frac{1}{2}\sec^2\frac{t}{2}\,dt = dx$$

$$\tan\left(\frac{t}{2}\right) = x+c \Rightarrow \tan\left(\frac{x+y}{2}\right) = x+c$$

(ME-14): The solution of the initial value problem $\frac{dy}{dx} = -2xy$ $y(0)=2$ is

Sol:

$$\frac{1}{y}dy = -2x\,dx$$

$$\log y = -x^2+c$$

$$y = e^{-x^2}\cdot e^c$$

Given $y(0) = 2 \Rightarrow 2 = e^c$

$$y = 2e^{-x^2}$$

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Higher order linear differential equation:

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of nth order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = x$$

where $P_1, P_2, \ldots, P_n$ are functions of x or constants

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = x \rightarrow (1)$$

where $K_1, K_2, \ldots K_n$ are constants.

operator D:

Denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, $\frac{d^3}{dx^3} = D^3, \ldots$

from (1) $D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \cdots + K_n y = x$

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = x$$

$$f(D) \cdot y = x \rightarrow (2)$$

where $f(D) = D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n$

i.e a polynomial in D

The solution of eq (2) is in the form

y = complementing function + particular integral

$= C.F + P.I$

Complementary function: It is the solution of a differential equation of the form $f(D) y = 0$

particular solution: It is the one particular solution of the given equation

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Rules for finding complementing function:

In the D.E $f(D)Y=0$ the equation $f(D)=0$ is called auxilary equation (A.E). Depending on the roots of A.E, we have following cases for complementing function.

case I: When roots of the A.E are real and distinct say $m_1, m_2, m_3, \ldots, m_n$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

case II: When roots of the A.E are real and repeated say $m_1, m_1, m_1, m_4, m_5, \ldots, m_n$

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$$

case III: When roots of the A.E are complex and distinct say $\alpha + i\beta, m_3, m_4, \ldots, m_n$

$$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

case IV: When roots of A.E are complex and repeated say $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \ldots, m_n$

$$C.F = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

prob: Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Sol: A.E $\Rightarrow D^2 + 5D + 6 = 0$

$$(D+2)(D+3) = 0$$

$$D = -2, -3$$

C.F is $y = c_1 e^{-2x} + c_2 e^{-3x}$

prob: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Sol: A.E $\Rightarrow D^2 + 2D + 1 = 0$

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$$(D+1)^2 = 0 \Rightarrow D = -1, -1.$$

$$C.F = (C_1 + C_2 x)e^{-x}$$

prob: solve D.E $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$

Sol: $D^2 + D + 1 = 0 \Rightarrow D = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$$C.F = e^{\frac{-1}{2}x}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$$

prob: solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = 0$

Sol: A.E $D^2 + 2D + 3 = 0 \Rightarrow D = -1 \pm \sqrt{2}i$

$$C.F\ x = e^{-t}\left[C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t\right]$$

prob: solve $\frac{d^3y}{dx^3} + y = 0$

Sol: A.E $D^3 + 1 = 0 \Rightarrow (D+1)(D^2 - D + 1) = 0$

$$D = -1, \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$C.F = e^{\frac{-1}{2}x}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + C_3 e^{-x}$$

prob: solve $\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} - 6x = 0$

Sol: A.E $D^3 - 6D^2 + 11D - 6 = 0$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(D-1)(D^2 - 5D + 6) = 0$$

$$D = 1, 2, 3$$

C.F $\quad x = C_1 e^{t} + C_2 e^{2t} + C_3 e^{3t}$

Inverse operator: An operator $\frac{1}{f(D)}$ which is operating with $f(D)\,y$ gives 'y'. Here $f(D)$ & $\frac{1}{f(D)}$ are called inverse operators to each other.

For the D.E $f(D)\cdot y = x$, the particular Integral given by $P.I = \frac{1}{f(D)}x$

depending on the function 'x' we have the following cases.

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Rules for particular Integration (P.I):

Case I: When $x = e^{ax}$

$$P.I = \frac{1}{f(D)} \cdot e^{ax} = \frac{1}{f(a)} e^{ax}$$ [Replace D by a if $f(a) \neq 0$]

When $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(D)} e^{ax}$$

$$= x \cdot \frac{1}{f'(a)} \cdot e^{ax} \quad \text{if } f'(a) \neq 0$$

If $f'(a) = 0$, then

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax} \quad \text{if } f''(a) \neq 0$$

⋮

prob: Find the P.I of $(D^2 + 3D + 2) y = e^{2x}$

Sol:

$$P.I = \frac{1}{D^2 + 3D + 2} e^{2x}$$

$$= \frac{1}{4 + 6 + 2} e^{2x} = \frac{1}{12} e^{2x}$$

prob: Find the P.I of $(D^2 + 3D + 2) y = e^{-2x}$

Sol:

$$P.I = \frac{1}{D^2 + 3D + 2} e^{-2x} \qquad f(-2) = 4 - 6 + 2 = 0$$

$$= \frac{x}{2D + 3} e^{-2x} = \frac{x}{-4 + 3} e^{-2x}$$

$$= -x e^{-2x}$$

prob: Find the P.I of $(D^2 + 2D + 1) y = e^{-x}$

Sol:

$$P.I = \frac{1}{D^2 + 2D + 1} e^{-x} \qquad f(-1) = 1 - 2 + 1 = 0$$

$$= \frac{x}{2D + 2} e^{-x} = \frac{x^2}{2} e^{-x} \qquad f'(-1) = -2 + 2 = 0$$

Note: 1. $x = \sinh x = \dfrac{e^x - e^{-x}}{2}$

2. $x = k = k \cdot e^{0x}$ 3. $x = a^x = e^{\log_e a^x} = e^{x \log a}$

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Case II: When $x = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{1}{f(D^2)} \sin(ax+b)$$

Replace D^2 by $-a^2$

$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b) \quad \text{if } f(-a^2) \neq 0$$

If $f(-a^2) = 0$

$$\frac{1}{f(D^2)} \sin(ax+b) = x \cdot \frac{1}{f'(-a^2)} \sin(ax+b) \quad \text{if } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$

$$\frac{1}{f(D^2)} \sin(ax+b) = x^2 \frac{1}{f''(-a^2)} \sin(ax+b) \quad \text{if } f''(-a^2) \neq 0$$

$$\vdots$$

Prob: Find the P.I of $(D^2+1)y = \sin(x+2)$

Sol:

$$P.I = \frac{1}{D^2+1} \sin(x+2) \qquad f(-1) = 0$$

$$= \frac{x}{2D} \sin(x+2)$$

$$= \frac{x \cdot D}{2D^2} \sin(x+2) = -\frac{x}{2} D[\sin(x+2)]$$

$$= -\frac{x}{2} \cos(x+2)$$

Prob: Find the P.I of $(D^2+3D+2)y = \cos(2x-1)$

Sol:

$$P.I = \frac{1}{D^2+3D+2} \cos(2x-1)$$

$$= \frac{1}{-4+3D+2} \cos(2x-1)$$

$$= \frac{1}{3D-2} \cos(2x-1)$$

$$= \frac{3D+2}{(3D-2)(3D+2)} \cos(2x-1)$$

$$= \frac{3D+2}{9D^2-4} \cos(2x-1) = \frac{3D+2}{9(-4)-4} \cos(2x-1)$$

$$= \frac{-1}{40}\left[-6\sin(2x-1) + 2\cos(2x-1)\right]$$

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Note: 1. When $x = \sin^2 x = \frac{1-\cos 2x}{2}$

2. When $x = \sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Case III: When $x = x^n$

$$P.I = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$$

Expand $[f(D)]^{-1}$ in increasing powers of 'D' and operate on x^n.

Prob: Find the P.I of $(D^2+2D+1)y = (x^2+x)$

Sol: $$P.I = \frac{1}{D^2+2D+1}(x^2+x)$$

$$= \frac{1}{(1+D)^2}(x^2+x) = (1+D)^{-1}[x^2+x]$$

$$= [1-2D+3D^2-4D^3+\dots][x^2+x]$$

$$= x^2+x-2(2x+1)+3(2) = x^2-3x+4.$$

Prob: Find P.I of $(D^2+2D+3)y = x$

Sol: $$P.I = \frac{1}{D^2+2D+3} x$$

$$= \frac{1/3}{1+\frac{2}{3}D+D^2/3} x = \frac{1}{3}\left[1+\frac{2}{3}D+\frac{D^2}{3}\right]^{-1} x$$

$$= \frac{1}{3}\left[1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)+\left(\frac{2D}{3}+\frac{D^2}{3}\right)^2-\dots\right]x$$

$$= \frac{1}{3}\left[x-\frac{2}{3}\right]$$

Prob: Find the P.I of $(D^2+2D)y = x$

Sol: $$P.I = \frac{1}{D^2+2D} x$$

$$= \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x$$

$$= \frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{2}-\right] x$$

$$= \left[\frac{1}{2D}-\frac{1}{4}+\frac{D}{4}\right] x \quad \text{[high order terms vanishes]}$$

$$= \frac{1}{2}\frac{x^2}{2}-\frac{x}{4}+\frac{1}{4} = \frac{x^2-x+1}{4}$$

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Case IV: When $X = e^{ax} \cdot V$, V being a function of 'x'

$$P.I = \frac{1}{f(D)}(e^{ax} \cdot V) = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

Prob: Find P.I of $(D^2 - 2D + 4)y = e^x \cos x$

Sol: $$P.I = \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x = e^x \frac{1}{-1+3} \cos x = \frac{e^x \cos x}{2}$$

Case V: When X is any other function of x.

$$P.I = \frac{1}{f(D)} X$$

If $f(D) = (D - m_1)(D - m_2) \ldots$, Resolve into partial fractions

$$P.I = \left[\frac{A_1}{(D - m_1)} + \frac{A_2}{(D - m_2} + \ldots\right] X$$

$$\boxed{\text{Complete solution} = C.F + P.I}$$

Equations reducible to linear equation with constant coefficients:

1. Cauchy's homogeneous linear D.E: An equation of the form $x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} x \frac{dy}{dx} + k_n y = X$ is called Cauchy's linear D.E. To convert this equation into linear equation with constant co-efficients.

Put $\log x = t \Rightarrow x = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \Rightarrow x \cdot \frac{dy}{dx} = Dy \qquad \text{if } D = \frac{d}{dt}$$

$$x^2 \frac{dy}{dx} = D(D-1)y, \quad x^3 \frac{dy}{dx} = D(D-1)(D-2)y \ldots$$

By substituting these values this equation reduces to the form $f(D) \cdot y = T$

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(14) Prob: Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol: This is cauchy's Linear D.E

Put $x = e^t \Rightarrow t = \log x$

$D(D-1)y - Dy + y = t$

$\Rightarrow [D-1]^2 y = t$

C.F: A.E $\Rightarrow (D-1)^2 = 0 \Rightarrow D = 1, 1$

$C.F = (C_1 + C_2 t)e^t$

P.I: $y = \frac{1}{(D-1)^2} t = (1-D)^{-2} t$

$= [1 + 2D + 3D^2 + \cdots] t = t + 2$

Complete solution $y = (C_1 + C_2 t)e^t + t + 2$

$t = \log x \Rightarrow \quad y = (C_1 + C_2 \log x)x + \log x + 2$

2. ~~Lagrange's~~ Legendre's linear D.E: An equation of the form

$$(ax+b)^n \frac{d^ny}{dx^n} + K_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + K_{n-1}(ax+b)\frac{dy}{dx} + K_n y = X$$

Put $\log_e(ax+b) = t \Rightarrow ax+b = e^t$

$(ax+b)\frac{dy}{dx} = aDy$

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y \qquad D = \frac{d}{dt}$

$\vdots$

Then the equation reduces to the form $f(D).y = T$

Initial value problem: A D.E along with the condition at only one ~~variable~~ value of the variable is called an Initial value problem.

Ex: $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 1$, $y'(0) = -1$

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Boundary value problem: A D.E along with conditions at two or more points of the independent variable is called a boundary value problem.

For I.V.P & B.V.P, the conditions are given to find the values of the arbitrary constants present in the general solution.

Previous problems

(ME-93): The differential $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y = 0$ is

(a) linear ✓(b) non-linear (c) homogeneous (d) of degree 2

(94): The differential equation $\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} + Ky = 0$ is

✓(a) Linear of fourth order

(b) Linear and fourth degree

(c) Non-linear of fourth order

(d) Non-homogeneous

(PI-94): Solve for y if $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with $y(0) = 1$ and $y'(0) = -2$

Sol:

$$D^2 + 2D + 1 = 0$$

$$(D+1)^2 = 0 \Rightarrow D = -1, -1$$

$\therefore$ Solution is $y = (C_1 + C_2 t)e^{-t} \longrightarrow (1)$

Given $y(0) = 1 \Rightarrow 1 = (C_1 + 0)e^{-0}$

$$\Rightarrow C_1 = 1 \longrightarrow (2)$$

$$y' = (C_1 + C_2 t)(-e^{-t}) + e^{-t} \cdot C_2$$

Given $y'(0) = -2 \Rightarrow -2 = -C_1 \cdot e^0 + e^0 C_2$

$$\Rightarrow -2 = -C_1 + C_2 \Rightarrow C_2 = -1$$

$\therefore$ The general solution is $y = (1-t)e^{-t}$

[RHS in question $= 0 \Rightarrow$ P.I $= 0$]

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(15)

(EC-94): $y = e^{-2x}$ is a solution of D.E $y'' + y' - 2y = 0$

✓(a) True (b) False

Sol:

$$D^2 + D - 2 = 0$$

$$D = 1, -2$$

$$C.F = C_1 e^{x} + C_2 e^{-2x}$$

$\therefore e^{-2x}$ is a solution

(ME-94): Solve for y if $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with $y(0) = 1$ and $y'(0) = 2$

Sol:

$$(D^2 + 2D + 1)\, y = 0$$

$$D = -1, -1$$

Solution is $y = (C_1 + C_2 t)e^{-t}$

Given $y(0) = 1 \Rightarrow 1 = (C_1)e^0 = C_1$

$y'(0) = 2 \Rightarrow y' = -(C_1 + C_2 t)e^{-t} + e^{-t} \cdot C_2$

$$2 = -(C_1)e^0 + e^0 C_2$$

$$2 = -C_1 + C_2 \Rightarrow C_2 = 3$$

$\therefore$ Solution is $y = (1 + 3t)e^{-t}$

(95): The D.E $y'' + (x^3 \sin x)^5 y' + y = \cos x^3$ is

(a) homogeneous (b) non linear

✓(c) 2nd order linear (d) non-homogeneous with constant co-efficient

(ME-95): The solution to the D.E $f''(x) + 4f'(x) + 4f(x) = 0$

Sol:

$$D^2 + 4D + 4 = 0$$

$$D = -2, -2$$

Solution is $y = (C_1 + C_2 x)e^{-2x}$

$$= C_1 e^{-2x} + C_2 x e^{-2x}$$

$f_1(x) = e^{-2x}$, $f_2(x) = x e^{-2x}$ are two solutions (as per options)

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(1995): The solution of a D.E $y''+3y'+2y=0$ is

Sol: $D^2+3D+2=0$

$D=-1,-2$

$\therefore$ solution is $y=C_1e^{-x}+C_2e^{-2x}$

(1996): solve $\frac{d^4v}{dx^4}+4\lambda^4 v=1+x+x^2$

Sol: $D^4+4\lambda^4=0$

$(D^2+2\lambda^2)^2-4D^2\lambda^2=0$

$(D^2+2\lambda^2-2D\lambda)(D^2+2\lambda^2+2D\lambda)=0$

$D=-\lambda\pm\lambda i,\ \lambda\pm\lambda i$

$C.F = e^{-\lambda x}[C_1\cos\lambda x+C_2\sin\lambda x]+e^{\lambda x}[C_3\cos\lambda x+C_4\sin\lambda x]$

$P.I = \left(\frac{1}{D^4+4\lambda^4}\right)(1+x+x^2)$

$=\frac{1}{4\lambda^4}\left[1+\frac{D^4}{4\lambda^4}\right]^{-1}(1+x+x^2)$

$=\frac{1}{4\lambda^4}\left[1-\frac{D^4}{4\lambda^4}\right](1+x+x^2)=\frac{1+x+x^2}{4\lambda^4}$

Complete solution is $y=C.F+P.I$

(ME-96): The particular solution for the D.E $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=5\cos x$ is

Sol: $P.I=\frac{1}{D^2+3D+2}5\cos x$

$=\frac{1}{3D+1}5\cos x$

$=\frac{3D-1}{(3D+1)(3D-1)}5\cos x=\frac{3D-1}{9D^2-1}5\cos x$

$=\frac{3D-1}{-9-1}5\cos x$

$=-\frac{1}{2}(3D-1)\cos x=1.5\sin x+0.5\cos x$

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(16) (CE-98): solve $\frac{d^4y}{dx^4} - y = 15\cos 2x$.

Sol: $D^4 - 1 = 0 \Rightarrow D = \pm 1, \pm i$

C.F $\Rightarrow$ $y = C_1 e^{x} + C_2 e^{-x} + e^{0x}[C_3 \cos x + C_4 \sin x]$

$$P.I = \frac{1}{D^4 - 1} 15 \cdot \cos 2x$$

$$= \frac{1}{(D^2)^2 - 1} 15 \cdot \cos 2x = \frac{1}{(-4)^2 - 1} 15 \cos 2x$$

$$= \cos 2x$$

$\therefore$ complete solution $y = C.F + P.I$

(1998): The general solution of the D.E $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol: This is cauchy's homogeneous linear D.E

put $x = e^t \Rightarrow t = \log x$

$\therefore$ $D(D-1) - D + 1 = 0$

$D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$

solution is $y = (A + Bt)e^t$

$= (A + B\log x)x$

$= Ax + Bx\log x$

(1998): The radial displacement in a rotating disc is governed by the D.E $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} = 8x$ where u is the displacement and x is the radius. If $u=0$ at $x=0$ and $u=2$ at $x=1$, calculate the displacement $x = 1/2$

Sol:

$$x^2 u'' + x u' - u = 8x^3$$

put $x = e^t \Rightarrow t = \log x$

C.F $\Rightarrow$ $D(D-1) + D - 1 = 0$

$D^2 - 1 = 0 \Rightarrow D = \pm 1$

$$u_c = C_1 e^{t} + C_2 e^{-t} = C_1 x + \frac{C_2}{x}$$

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P.I $\Rightarrow$ $u_p = \frac{1}{D^2-1} 8x^3$

$= -[1-D^2]^{-1} 8x^3$

$= -[1+D^2+D^4+\cdots] 8x^3$

$= -8x^3 - 48x$

Complete solution is $u = c_1 x + \frac{c_2}{x} - 8x^3 - 48x$

Given $u=0$ at $x=0$ $\Rightarrow 0 = ?$ [data wrong]

(1999): The equation $\frac{d^2y}{dx^2} + (x^2+4x)\frac{dy}{dx} + y = x^8 - 8$ is a

Sol: Non-homogeneous linear ordinary D.E

(2000): Find the solution of the D.E $\frac{d^2y}{dt^2} + \lambda^2 y = \cos(\omega t + K)$ with initial conditions $y(0)=0$, $\frac{dy(0)}{dt} = 0$. Here $\lambda, \omega, \& K$ are constants.

Sol: C.F $D^2 + \lambda^2 = 0 \Rightarrow D = \pm \lambda i$

$y_c = c_1 \cos\lambda t + c_2 \sin\lambda t$

P.I $y_p = \frac{1}{D^2+\lambda^2} \cos(\omega t + K)$

$= \frac{1}{-\omega^2+\lambda^2} \cos(\omega t + K)$

$\therefore y = y_c + y_p = c_1 \cos\lambda t + c_2 \sin\lambda t + \frac{1}{\lambda^2-\omega^2} \cos(\omega t + K)$

Given $y(0) = 0$

$0 = c_1 + 0 + \frac{1}{\lambda^2-\omega^2} \cdot \cos K \Rightarrow c_1 = \frac{\cos K}{\omega^2-\lambda^2}$

$\frac{dy}{dt} = -c_1 \sin\lambda t \cdot \lambda + c_2 \cdot \lambda \cos\lambda t + \frac{-\omega}{\lambda^2-\omega^2} \sin(\omega t + K)$

Given $\frac{dy(0)}{dt} = 0$

$0 = 0 + c_2 \lambda - \frac{\omega}{\omega^2-\lambda^2} \cdot \sin K$

$c_2 = \frac{\omega \sin K}{\lambda(\omega^2-\lambda^2)}$

$y = \frac{\cos K}{\omega^2-\lambda^2} \cos\lambda t + \frac{\omega \sin K}{\lambda(\omega^2-\lambda^2)} \sin\lambda t + \frac{1}{\lambda^2-\omega^2} \cos(\omega t + K)$

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(17) (CE-01): The solution for the following D.E with boundary conditions $y(0)=2$ and $y'(1)=-3$ is, where $\frac{d^2y}{dx^2}=3x-2$

Sol: C.F $\Rightarrow$ $D^2=0$

$D=0,0$

$$y_c=(C_1+C_2x)e^0=C_1+C_2x$$

P.I $\Rightarrow$ $y_p=\frac{1}{D^2}(3x-2)$

$$=\frac{1}{D}\left[3\frac{x^2}{2}-2x+C_3\right]$$

$$=3\cdot\frac{x^3}{2\times3}-2\frac{x^2}{2}+C_3x+C_4$$

$$=\frac{x^3}{2}-x^2+C_3x+C_4$$

$$y=y_c+y_p=C_1+C_2x+C_3x+C_4-x^2+\frac{x^3}{2}$$

Given $y(0)=2$ $\Rightarrow$ $2=C_1+C_4 \rightarrow (1)$

$y'=C_2+C_3-x^2+\frac{x^3}{2}$ $\Rightarrow$ $-3=C_2+C_3-\frac{1}{2}$

$\Rightarrow C_2+C_3=-5/2 \rightarrow (2)$

$$\therefore\ y=2-\frac{5}{2}x+x^2+\frac{x^3}{2}$$

(2001): Solve the D.E $\frac{d^2y}{dx^2}+y=x$ with the following conditions (i) at $x=0$, $y=1$ (ii) at $x=0$, $y'=1$

Sol: C.F $D^2+1=0$ $\Rightarrow$ $D=\pm i$

$$y_c=C_1\cos x+C_2\sin x$$

P.I $y_p=\frac{1}{D^2+1}x=[1+D^2]^{-1}x$

$$=[1-D^2+D^4-\cdots]x$$

$$=x$$

Complete solution $y=x+C_1\cos x+C_2\sin x$

(i) At $x=0$, $y=1$ $\Rightarrow$ $1=C_1+C_2(0)$ $\Rightarrow$ $C_1=1$

(ii) At $x=0$, $y'=1$ $y'=1+C_1(-\sin x)+C_2\cos x$

$1=1+0+C_2$ $\Rightarrow$ $C_2=0$

$$\therefore\ y=x+\cos x$$

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(IN-05): The general solution of D.E $(D^2-4D+4)y=0$ is of the form

Sol: $D^2-4D+4=0 \Rightarrow D=2,2$

$$y=(C_1+C_2x)e^{2x}$$

(EE-05): The solution of the first order differential eq^n $\dot{x}(t)=-3x(t)$, (Completed)

(EE-05): For the equation $\ddot{x}(t)+3\dot{x}(t)+2x(t)=5$, then Solution $x(t)$ approaches the following values as $t\to\infty$

Sol: C.F $D^2+3D+2=0$

$$D=-1,-2$$

$$x_c=C_1e^{-t}+C_2e^{-2t}$$

P.I $x_p=\dfrac{1}{D^2+3D+2}\cdot 5$

$$=\frac{1}{D^2+3D+2}5\cdot e^{0t}=\frac{1}{0+0+2}\cdot 5=\frac{5}{2}$$

Complete solution $x=x_c+x_p$

$$=C_1e^{-t}+C_2e^{-2t}+\frac{5}{2}$$

As $t\to\infty$ $x=C_1e^{-\infty}+C_2e^{-2\infty}+\frac{5}{2}$

$$=5/2$$

(CE-05): The solution $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+17y=0$; $y(0)=1$, $\left(\dfrac{dy}{dx}\right)_{x=\pi/4}=0$ in the range $0<x<\pi/4$ is given by

Sol: C.F $D^2+2D+17=0 \Rightarrow D=-1\pm 4i$

Sol^n is $y=e^{-x}[C_1\cos 4x+C_2\sin 4x]$

$$\frac{dy}{dx}=e^{-x}[-4C_1\sin 4x+4C_2\cos 4x]$$
$$+[C_1\cos 4x+C_2\sin 4x](-e^{-x})$$

Given $y(0)=1$

$$1=e^0[C_1] \Rightarrow C_1=1$$

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$$0 = e^{-\pi/4}\left[-4C_1(0)+4C_2\right]+(-e^{-\pi/4})[C_1]$$

$$0 = 1-4C_2 \Rightarrow C_2 = 1/4$$

solution is $y = e^{-x}\left[\cos 4x + \frac{1}{4}\sin 4x\right]$

(ME-05): The complete solution of O.D.E $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$ $y = C_1e^{-x}+C_2e^{-3x}$ then p and q are

Sol: $m_1 = -1 \quad m_2 = -3$

$$(D+1)(D+3)=0$$

$$D^2+4D+3=0 \Rightarrow p=4,\ q=3$$

(ME-05): (above Question). Find solution of D.E $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(q+1)y=0$?

Sol: $D^2+4D+4=0 \Rightarrow (D+2)^2=0$

$$D=-2,-2$$

$$y=(C_1+C_2x)e^{-2x}$$

(EC-05): A solution of the D.E $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6x=0$ is given by

Sol: $D^2-5D+6=0 \Rightarrow D=2,3$

$$y = C_1e^{2x}+C_2e^{3x}$$

~~(CE-06): The solution of D.E~~ $\cancel{x^2\frac{dy}{dx}}$

(IN-06): For the initial value problem $y''+2y'+101y=(10.4)e^x$, $y(0)=1.1$ and $y(0)=-0.9$. Find solution?

Sol: C.F $\quad D^2+2D+101=0$

$$D=-1\pm 10i$$

$$y_c = e^{-x}\left[C_1\cos 10x + C_2\sin 10x\right]$$

P.I $\quad y_p = \frac{1}{D^2+2D+10}(10.4e^x)$

$$= \frac{1}{104}(10.4e^x) = 0.1e^x$$

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$$y = e^{-x}[C_1\cos 10x + C_2 \sin 10x] + 0.1e^{x}$$

$$y' = e^{-x}[-10C_1 \sin 10x + 10C_2 \cos 10x] + [C_1\cos 10x + C_2 \sin 10x](-e^{-x}) + 0.1e^{x}$$

$y(0) = 0.1$ & $y'(0) = -0.9$

$\Rightarrow C_1 = 1,\ C_2 = 0$

$\therefore y = e^{-x}\cos 10x + 0.1e^{x}$

(EC-06): For the D.E $\frac{d^2y}{dx^2} + k^2 y = 0$, the boundary conditions are (i) $y=0$ for $x=0$ and (ii) $y=0$ for $x=a$. The form of non zero solution of y are

(a) $y = \sum_m A_m \sin\left(\frac{m\pi x}{a}\right)$ (b) $y = \sum_m A_m \cos\left(\frac{m\pi x}{a}\right)$

(c) $y = \sum_m A_m x^{\pi m/a}$ (d) $y = \sum_m A_m e^{-\frac{m\pi x}{a}}$

Sol:

$$(D^2 + k^2)y = 0$$

$$D^2 + k^2 = 0 \Rightarrow D = \pm ki$$

$$y = C_1 \cos(kx) + C_2 \sin kx$$

Given $y=0$ at $x=0 \Rightarrow y = 0 = C_1$

$y = 0$ for $x = a \Rightarrow 0 = C_2 \sin ka$

for non-zero solution $C_2 \neq 0 \Rightarrow \sin ka = 0$

$ka = n\pi \quad n \in \mathbb{Z}$

$k = \frac{n\pi}{a},\ n \in \mathbb{Z}$

$\therefore y = C_2 \sin\left(\frac{n\pi x}{a}\right)$

Linear combination of the above is also a solution

(ME-06): For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the P.I is

Sol: $\text{P.I} = \frac{3e^{2x}}{D^2+4D+3} = \frac{3e^{2x}}{4+8+3} = \frac{e^{2x}}{5}$

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(EC-07): The solution of the D.E $K^2\frac{d^2y}{dx^2} = y - y_2$ under the boundary conditions (i) $y = y_1$ at $x=0$ and (ii) $y = y_2$ at $x=\infty$ where K, y_1 and y_2 are constants is

Sol:

$$K^2D^2y - y = -y_2$$

$$D^2y - \frac{1}{K^2}y = -\frac{y_2}{K^2}$$

$$\left(D^2 - \frac{1}{K^2}\right)y = -\frac{y_2}{K^2}$$

C.F $\Rightarrow$ $D^2 - \frac{1}{K^2} = 0 \Rightarrow D = \pm\frac{1}{K}$

$$y_c = C_1e^{-x/K} + C_2e^{x/K}$$

P.I $\Rightarrow$ $y_p = \frac{1}{D^2 - \frac{1}{K^2}}\left(\frac{-y_2}{K^2}\right)e^{0x}$

$$= K^2 \cdot \frac{y_2}{K^2} = y_2$$

$\therefore$ complete solution $y = C_1e^{-x/K} + C_2e^{x/K} + y_2$

Given $y = y_1$ at $x=0$ $\Rightarrow$ $y_1 = C_1 + C_2 + y_2$

Given $y = y_2$ at $x=\infty$ $\Rightarrow$ $y_2 = C_1(0) + C_2(\infty) + y_2$

$$C_2 = \frac{0}{\infty} = 0$$

$$C_1 = y_1 - y_2$$

$$\therefore y = (y_1 - y_2)e^{-x/K} + y_2$$

(ME-08): Given that $x'' + 3x = 0$ and $x(0) = 1$, $x'(0) = 1$ what is $x(1)$?

Sol:

$$D^2 + 3 = 0 \Rightarrow D = \pm\sqrt{3}i$$

$$x = C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t$$

Given $x(0) = 1$ $\Rightarrow$ $1 = C_1$

Given $x'(0) = 1$ $\Rightarrow$ $x' = -\sqrt{3}C_1\sin\sqrt{3}t + \sqrt{3}C_2\cos\sqrt{3}t$

$$1 = \sqrt{3}C_2 \Rightarrow C_2 = 1/\sqrt{3}$$

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$$x = \cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t$$

$$\therefore x(1) = \cos\sqrt{3} + \frac{1}{\sqrt{3}}\sin\sqrt{3} = 0.4096$$

(ME-08): It is given that $y'' + 2y' + y = 0$, $y(0) = 0$ & $y(1) = 0$ what is $y(0.5)$?

Sol:

$$D^2 + 2D + 1 = 0 \Rightarrow (D+1)^2 = 0$$

$$D = -1, -1$$

$$y = (C_1 + C_2 x)e^{-x}$$

Given $y(0) = 0 \Rightarrow 0 = C_1$

$y(1) = 0 \Rightarrow 0 = C_1 + C_2 e^{-1} \Rightarrow C_2 = 0$

$$y = 0 \Rightarrow y(0.5) = 0$$

(PI-08): The solutions of the D.E $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ are

Sol:

$$D^2 + 2D + 2 = 0$$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y = e^{-x}(C_1 \cos x + C_2 \sin x)$$

(or)

$$y = C_1 e^{-(1+i)x} + C_2 e^{-(1-i)x}$$

PI (-09): The homogeneous part of the D.E $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r$ (p, q, r are constants) has real distinct roots if

Sol:

$$D^2 + PD + q = 0$$

$$D = \frac{-P \pm \sqrt{P^2 - 4q}}{2}$$

Roots are real and distanct if $P^2 - 4q > 0$

(PI-09): The solution of the D.E $\frac{d^2y}{dx^2} = 0$ with boundary conditions (i) $\frac{dy}{dx} = 1$ at $x = 0$ (ii) $\frac{dy}{dx} = 1$ at $x = 1$ is

Sol:

$$D^2 = 0 \Rightarrow D = 0, 0$$

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$$y = (C_1 + C_2 x)e^0 = C_1 + C_2 x$$

$$y' = 0 + C_2 = C_2$$

$$y'(0) = 1 \Rightarrow 1 = C_2$$

$$\therefore y = C_1 + x$$

(EE-10): For the D.E $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left(\frac{dx}{dt}\right)_{t=0} = 0$ the solution

Sol:

$$D^2 + 6D + 8 = 0$$

$$D = -2, -4$$

$$\therefore x = C_1 e^{-2t} + C_2 e^{-4t}$$

Given $x(0) = 1 \Rightarrow 1 = C_1 + C_2 \rightarrow (1)$

$x'(0) = 0 \Rightarrow x' = -2C_1 e^{-2t} - 4C_2 e^{-4t}$

$$0 = -2C_1 - 4C_2 \rightarrow (2)$$

from (1) & (2) $C_1 = 2$, $C_2 = -1$

$\therefore$ solution $x = 2e^{-2t} - e^{-4t}$

(EC-10): The function $n(x)$ satisfy the d.E $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where L is a constant. The boundary conditions are $n(0) = K$, $n(\infty) = 0$. The solution to this equation is

Sol:

$$D^2 - \frac{1}{L^2} = 0 \Rightarrow D = \pm\frac{1}{L}$$

$$n(x) = C_1 e^{\frac{1}{L}x} + C_2 e^{-\frac{1}{L}x}$$

Given $n(0) = K \Rightarrow K = C_1 + C_2$

$n(\infty) = 0 \Rightarrow 0 = C_1(\infty) + C_2(0)$

$$C_1 = 0$$

$$\therefore C_2 = K$$

$$\therefore n(x) = Ke^{-\frac{x}{L}}$$

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(CE-10): The solution to the O.D.E $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ is

Sol:
$$D^2+D-6=0$$
$$(D+3)(D-2)=0 \Rightarrow D=-3,2$$
$$y=c_1e^{-3x}+c_2e^{2x}$$

(IN-11): Consider the D.E $y''+2y'+y=0$ with boundary conditions $y(0)=1$ & $y(1)=0$. The value of $y(2)$ is

Sol:
$$D^2+2D+1=0 \Rightarrow D=-1,-1$$
$$y=(c_1+c_2x)e^{-x}$$

Given $y(0)=1 \Rightarrow 1=c_1$

$y(1)=0 \Rightarrow 0=(c_1+c_2)e^{-1} \Rightarrow c_1+c_2=0$

$c_2=-1$

$$\therefore y=(1-x)e^{-x}$$
$$y(2)=(1-2)e^{-2}=-e^{-2}$$

(PI-11): The solution of D.E $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=9x+6$ with c_1 and c_2 as constants is

Sol: C.F $\Rightarrow$ $D^2+6D+9=0 \Rightarrow D=-3,-3$
$$y_c=(c_1+c_2x)e^{-3x}$$

P.I $\Rightarrow$ $y_p=\frac{1}{D^2+6D+9}(9x+6)$
$$=\frac{1}{(D+3)^2}(9x+6)=\frac{1}{9}\left[1+\frac{D}{3}\right]^{-2}(9x+6)$$
$$=\frac{1}{9}\left[1-2\frac{D}{3}+3\left(\frac{D^2}{9}\right)-\cdots\right](9x+6)$$
$$=\frac{1}{9}\left[(9x+6)-\frac{2}{3}(9)\right]=x+\frac{2}{3}-\frac{2}{3}$$
$$=x$$

$$\therefore y=y_c+y_p$$
$$=(c_1+c_2x)e^{-3x}+x$$

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(21) (ME, PI-12): Consider the D.E $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ with the boundary conditions of $y(0)=0$ and $y(1)=1$. The complete solution of D.E is

Sol: Let $x=e^t \Rightarrow t=\log x$

$$[D(D-1)+D-4]=0$$

$$D^2-4=0 \Rightarrow D=\pm 2$$

$$\therefore y=c_1e^{2t}+c_2e^{-2t}=c_1x^2+c_2x^{-2}$$

Given $y(0)=1 \Rightarrow 1=0+c_2(\infty) \Rightarrow c_2=0$

$y(1)=1 \Rightarrow 1=c_1+c_2 \Rightarrow c_1=1$

$$\therefore y=x^2$$

(IN-2012): The maximum value of the solution $y(t)$ of the D.E $\ddot{y}(t)+y(t)=0$ with initial condition $\dot{y}(0)=1$ and $y(0)=1$ for $t\geq 0$ is

Sol: $D^2+1=0 \Rightarrow D=\pm i$

$$y=c_1\cos t+c_2\sin t$$

Given $y(0)=1 \Rightarrow 1=c_1$

$\dot{y}(0)=1 \Rightarrow y'=-c_1\sin t+c_2\cos t$

$1=c_2$

$$\therefore y=\cos t+\sin t$$

Maximum at $\dot{y}=0$

$\Rightarrow -\sin t+\cos t=0$

$\sin t=\cos t$

$\Rightarrow t=\pi/4$

$\ddot{y}=-\cos t-\sin t$

$\ddot{y}\big|_{t=\frac{\pi}{4}}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}$

$=-\sqrt{2}<0$

Maximum value $y_{max}=\cos\pi/4+\sin\pi/4$

$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

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(ME-13): The solution to the D.E $\frac{d^2u}{dx^2} - K\frac{du}{dx} = 0$ where 'K' is a constant subjected to the boundary conditions $u(0)=0$ and $u(L)=U$ is

Sol:

$$\cancel{D^2 - K = 0 \Rightarrow D = \pm\sqrt{K}} \quad D^2 - KD = 0$$

$$\cancel{u = C_1\cos\sqrt{K}\,x + C_2\sin\sqrt{K}\,x}$$

$$D(D-K) = 0 \Rightarrow D = 0, K$$

$$\therefore u = C_1 e^{0} + C_2 e^{Kx}$$

Given $u(0)=0 \Rightarrow 0 = C_1 + C_2 \rightarrow (1)$

$u(L) = U \Rightarrow U = C_1 + C_2 e^{KL} \rightarrow (2)$

$(2)-(1) \Rightarrow U = C_2(e^{KL}-1) \Rightarrow C_2 = \frac{U}{e^{KL}-1}$

$$C_1 = \frac{U}{1-e^{KL}}$$

$$\therefore u(x) = \frac{U}{1-e^{KL}} + \frac{U}{e^{KL}-1}\cdot e^{Kx}$$

(EC-14): If the characterstic equation of D.E $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots, then the values of α are

Sol: $D^2 + 2\alpha D + 1 = 0$

$$D = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4}}{2}$$

For equal roots $(2\alpha)^2 - 4 = 0 \Rightarrow \alpha^2 = 1$

$\alpha = \pm 1$

(EC-14): If a and b are constants, the most general solution of the D.E $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$ is

Sol: $D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1$

$$x = (a + bt)e^{-t}$$

(EC-14): With initial values $y(0) = y'(0) = 1$, the solution of the D.E $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x=1$ is

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(22) Sol: $(D+2)^2=0 \Rightarrow D=-2,-2$.

$$\therefore y=(C_1+C_2x)e^{-2x}$$

Given $y(0)=1 \Rightarrow 1=C_1$

$y'(0)=1 \Rightarrow y'=-C_2\,2xe^{-2x}+C_2e^{-2x}-2C_1e^{-2x}$

$$1=-2+C_2(0+1)$$

$$C_2=3$$

$$\therefore y=(1+3x)e^{-2x}$$

At $x=1$ $\quad y=4e^{-2}=0.541$

(EE-14): The solution for the D.E $\frac{d^2x}{dt^2}=-9x$, with initial conditions $x(0)=1$ and $\left.\frac{dx}{dt}\right|_{t=0}=1$ is

Sol:

$$\frac{d^2x}{dt^2}+9x=0$$

$$D^2+9=0 \Rightarrow D=\pm 3i$$

$$x=C_1\cos 3t+C_2\sin 3t$$

Given

$x(0)=1$ $\quad 1=C_1$

$x'(0)=1$ $\quad x'=-3C_1\sin 3t+3C_2\cos 3t$

$$1=3C_2 \Rightarrow C_2=1/3$$

$$\therefore x=\cos 3t+\frac{1}{3}\sin 3t$$

(EE-14): Consider the D.E $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$. Find ~~the~~ a Solution?

Sol: put $x=e^t \Rightarrow t=\log x$

$$D(D-1)+D-1=0$$

$$D^2-1=0 \Rightarrow D=\pm 1$$

$$y=C_1e^{-t}+C_2e^{-t}$$

$$y=C_1x+C_2\frac{1}{x}$$

According to options $y=\frac{1}{x}$ is a Solution $(C_1=0, C_2=1)$

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(ME-14): If $y = f(x)$ is the solution of $\frac{d^2y}{dx^2} = 0$ with the boundary conditions $y = 5$ at $x = 0$, and $\frac{dy}{dx} = 2$ at $x = 10$, $f(15) =$ ___

Sol:

$$D^2 = 0 \Rightarrow D = 0, 0$$

$$y = (C_1 + C_2 x)e^{0x} = C_1 + C_2 x$$

Given $y = 5$ at $x = 0 \Rightarrow 5 = C_1$

$\frac{dy}{dx} = 2$ at $x = 10 \Rightarrow y' = C_2 = 2$

$$\therefore y = 5 + 2x = f(x)$$

$$f(15) = 5 + 30 = 35$$

(CE-14): Water is flowing at a steady rate through a homogeneous and saturated horizontal soil strip of 10m length. The strip is being subjected to a constant water head (H) of 5m at the begining and 1m at the end. If the governing equation of flow in the soil strip is $\frac{d^2H}{dx^2} = 0$ (where x is distance along soil strip), the value of H (in m) at the middle of the strip is

Sol:

$$\frac{d^2H}{dx^2} = 0 \Rightarrow D^2 = 0$$

$$H = (C_1 + C_2 x)e^{0.x}$$

$$= C_1 + C_2 x$$

Given $H = 5m$ at $x = 0$ $\quad 5 = C_1$

$H = 1m$ at $x = 10$ $\quad 10 = C_1 + 10C_2 \Rightarrow C_2 = \frac{-2}{5}$

$$\therefore H = 5 - \frac{2}{5}x$$

At middle of the strip i.e at $x = 5$

$$H = 5 - \frac{2}{5}.5 = 5 - 2 = 3$$

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Partial Differential Equations

Notation: $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial x . \partial y} = s$ $\frac{\partial^2 z}{\partial y^2} = t$

Formation of P.D.E: unlike the case of ODE, which arise from the elimination of arbitrary constants; the PDE can be formed by elimination of arbitrary constants or arbitrary functions.

1. If equation involves n arbitrary constants and n independent variable then we can obtain a PDE of only first order
2. If an equation involves n arbitrary functions then we can obtain a PDE of nth order
3. If the no. of arbitrary constants are more than the no. of independent variables then we will obtain a PDE of higher order.

Prob: PDE corresponding to $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow (1)$

Sol:

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \qquad 2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$p = \frac{x}{a^2} \qquad q = y/b^2$$

from (1) $2z = px + qy$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Prob: The PDE of $z = f(x^2 - y^2)$

Sol:

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\,2x = p$$

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) = q$$

$$\frac{p}{q} = \frac{-x}{y}$$

$$py + qx = 0$$

$$\frac{\partial z}{\partial x}y + \frac{\partial z}{\partial y}x = 0$$

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Prob: The PDE of $z = f(x+ay) + g(x-ay)$

$$\frac{\partial z}{\partial x} = p = f'(x+ay) + g'(x-ay)$$

$$\frac{\partial z}{\partial y} = q = f'(x+ay).a + g'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x+ay) + g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = t = f''(x+ay).a^2 + g''(x+ay).a^2$$

$$= a^2[f''(x+ay) + g''(x+ay)]$$

$$= a^2 r$$

$$\frac{\partial^2 z}{\partial y^2} - a^2 \frac{\partial z}{\partial x^2} = 0$$

Solving PDE:

First order Linear PDE: A Linear PDE of first order, commonly known as Lagrange's linear equation is of the form

$$Pp + Qq = R \qquad P, Q \text{ and } R \text{ are functions of } x, y, z.$$

This equation is called quasi-linear equation.

* When P, Q and R are independent of 'z' then it is linear equation.

To solve this equation

(i) form the subsidary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) Solve these simultaneous equation, which gives $u = a$ and $v = b$ as solutions.

(iii) Write the complete solution as $\phi(u, v) = 0$ (or) $u = f(v)$

Prob: Solve $\frac{y^2 z}{x} p + xzq = y^2 \Rightarrow y^2 z p + x^2 z q = y^2 x$

Sol: Subsidary equation $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$

From the first two factors $x^2 z\, dx = y^2 z\, dy$

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$$x^2 dx = y^2 dy$$

I.O.B.S $\quad \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow x^3 - y^3 = a$ (const.)

From first & third fractions $\quad x\,dx = z\,dz$

I.O.B.S $\quad \frac{x^2}{2} = \frac{z^2}{2} + c_2$

$$x^2 - z^2 = b \quad \text{(const.)}$$

$\therefore$ complete solution is $\quad x^3 - y^3 = f(x^2 - z^2)$

Non-linear equations of first order: Those equations in which p and q occur other than in the first degree are called non-linear PDE of first order. There are four standard forms of these equations

Form I: Equations of the type $f(p, q) = 0$

It's complete solution is $z = ax + by + c$. where a and b are connected by the relation $f(a, b) = 0$.

From $f(a, b) = 0$, express $b = \phi(a)$

$\therefore$ The required solution is $z = ax + \phi(a)y + c$. where a and c are arbitrary constants.

prob: Solve $\sqrt{p} + \sqrt{q} = 1$

Sol:

$$f(p, q) = \sqrt{p} + \sqrt{q} - 1 = 0$$

$$f(a, b) = \sqrt{a} + \sqrt{b} - 1 = 0 \Rightarrow \sqrt{b} = 1 - \sqrt{a}$$

$$\Rightarrow b = (1 - \sqrt{a})^2$$

The solution is $\quad z = ax + by + c$

$$= ax + (1 - \sqrt{a})^2 y + c$$

Form II: Equations of the type $f(z, p, q) = 0$

To solve this

(i) assum $u = x + ay$ and substitute $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$ in the given equation

(ii) Solve the resulting ODE in z and u

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(iii) replace u by $x+ay$

Prob: Solve $p(1+q)=qz$

Sol:
$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=az\frac{dz}{du}$$

$$1+a\frac{dz}{du}=az \Rightarrow \frac{a}{az-1}dz=du$$

$$\int\frac{a}{az-1}dz=\int du+c$$

$$\log(az-1)=u+c$$

$$=x+ay+c$$

Form III: Equations of the type $f(x,p)=g(y,q)$

Assum $f(x,p)=g(y,q)=a$

$f(x,p)=a$ $\qquad$ $g(y,q)=a$

Solve for p $\qquad$ Solve for q

$p=\phi(x)$ $\qquad$ $q=\psi(y)$

Since $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$dz=p\,dx+q\,dy$$

$$dz=\phi(x)dx+\psi(y)dy$$

I.O.B.S

$$z=\int\phi(x)dx+\int\psi(y)dy+c$$

Prob: Solve $p^2+q^2=x+y$

$$p^2-x=-q^2+y=a \text{ say}$$

$p^2-x=a$ $\qquad$ $-q^2+y=a$

$p=\sqrt{a+x}$ $\qquad$ $q=\sqrt{y-a}$

$$dz=\sqrt{a+x}\,dx+\sqrt{y-a}\,dy$$

$$z=\frac{2}{3}(a+x)^{3/2}+\frac{2}{3}(y-a)^{3/2}+b$$

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form IV: $z = px + qy + f(p,q)$.

It's complete solution is $z = ax + by + f(a,b)$

prob: Solve $z = px + qy + pq$ (CE-2010)

Solution is $z = ax + by + ab$

Higher order PDE:

Homogeneous linear equations with constant co-efficients:

An equation of the form $\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y)$ in which k's are constants, is called a homogeneous linear partial D.E of nth order with constant co-efficients. It is called homogeneous because all terms contain derivatives of same order.

Denote $\frac{\partial z}{\partial x} = D$ and $\frac{\partial z}{\partial y} = D'$ so that

$$[D^n + k_1 D^{n-1} \cdot D' + \cdots + k_n (D')^n] z = F(x,y)$$

$$f(DD')z = F(x,y)$$

Solution is $z = C.F + P.I$

Rules for C.F: For the P.D.E $f(DD')z = 0$, $f(\lambda = 0)$ is the Auxilary Equation. Where $\lambda = D/D'$.

Depending on roots of A.E, we have the following cases

(i) case-1: When the roots are different say $m_1, m_2, m_3 \cdots$

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \cdots$$

(ii) case-2: When the roots are repeated say $m, m, m \cdots$

$$C.F = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \cdots$$

Prob: Solve $\frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \cdot \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 0$

Sol:

$$[D^2 - 3DD' + 2(D')^2] z = 0$$

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A.E is $\lambda^2-3\lambda+2=0 \Rightarrow \lambda=1,2$

$$z=f_1(y+x)+f_2(y+2x)$$

prob: solve $\frac{\partial^2 z}{\partial x^2}-a^2\frac{\partial^2 z}{\partial y^2}=0$

Sol $[D^2-a^2(D')^2]z=0$

A.E is $\lambda^2-a^2=0 \Rightarrow \lambda=\pm a$

$$z=f_1(y+ax)+f_2(y-ax)$$

prob: solve $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\cdot\partial y}+\frac{\partial^2 z}{\partial y^2}=0$

Sol: $[D^2+2DD'+(D')^2]z=0$

A.E is $\lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1$

$$z=f_1(y-x)+x f_2(y-x)$$

Rules for P.I:

From the symbolic form $P.I=\frac{1}{f(D,D')}F(x,y)$

(i) When $F(x,y)=e^{ax+by}$

$$P.I=\frac{1}{f(D,D')}e^{ax+by}$$

put $D=a$ and $D'=b$

(ii) When $F(x,y)=\sin(mx+ny)$ (or) $\cos(mx+ny)$

$$P.I=\frac{1}{f(D^2,DD',D'^2)}\sin \text{ or } \cos(mx+ny)$$

put $D^2=-m^2$, $DD'=-mn$, $(D')^2=-n^2$

(iii) When $F(x,y)=x^m y^n$

$$P.I=\frac{1}{f(DD')}x^m y^n=[f(DD')]^{-1}x^m y^n$$

Expand in ascending powers of D'/D & operate on $x^m y^n$

(iv) When $F(x,y)$ = Any function of x & y

$$P.I=\frac{1}{f(D,D')}F(x,y)$$

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Resolve $1/f(D, D')$ into fractions and operate each fraction on $F(x, y)$ such that $\frac{1}{D-mD'} \cdot F(x, y)$.

Replace $y = c - mx$

$$\therefore \frac{1}{D-mD'} \cdot F(x, y) = \int F(x, c-mx)\,dx$$

After integration replace $c = y + mx$

Classification of PDE: A PDE of the form

$$A \cdot \frac{\partial^2 z}{\partial x^2} + B \frac{\partial^2 z}{\partial x \cdot \partial y} + C \frac{\partial^2 z}{\partial y^2} + F(x, y, z, p, q) = 0$$ is said to be

(i) elliptic if $B^2 - 4AC < 0$

(ii) parabola if $B^2 - 4AC = 0$

(iii) Hyperbola if $B^2 - 4AC > 0$

Prob: $x^2 \frac{\partial z}{\partial x^2} + (1-y^2) \frac{\partial^2 z}{\partial y^2} = 0$ where $-\infty < x < \infty$ $x \neq 0$, $-1 < y < 1$ is

Sol: $B^2 - 4AC = -4x^2(1-y^2) < 0$

$\therefore$ ellipse

Prob: $(1+x^2) r + (5+2x^2) s + (4+x^2) t = 0$

Sol: $B^2 - 4AC = 9 > 0$ $\therefore$ Hyperbola

Prob: $r + 4s + 4t = 0$

Sol: $B^2 - 4AC = 16 - 16 = 0$ $\therefore$ parabola

previous problems

(IN-13): The type of partial D.E $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

Sol: We can write it as [ME-96 Similar]

$$1 \cdot \frac{\partial^2 f}{\partial x^2} + 0 \cdot \frac{\partial^2 f}{\partial x \cdot \partial t} + 0 \cdot \frac{\partial^2 f}{\partial t^2} + \left(-\frac{\partial f}{\partial t}\right) = 0$$

$$B^2 - 4AC = 0$$

$\therefore$ parabolic

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(ME-07): The PDE $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} = 0$ has degree & order

(a) 1, 2 (b) 1, 1 (c) 2, 1 (d) 2, 2

(ME-10): The blasius equation $\frac{d^3f}{d\eta^3} + \frac{f}{2}\frac{d^2f}{d\eta^2} = 0$ is a

Sol: 3rd order non linear ordinary differential eqn

(ME-13): The PDE $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ is a

Sol: 2nd order non linear

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COMPLEX VARIABLES

Complex variables

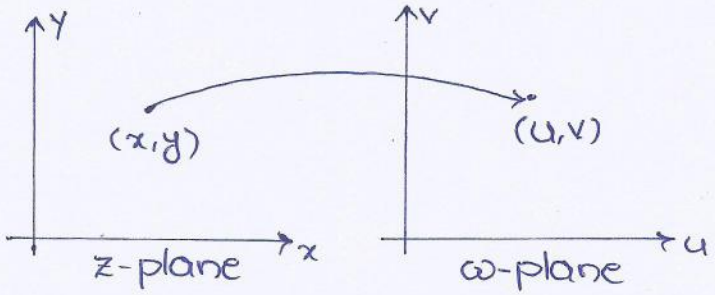
A variable of the form $z = x+iy$ where $i^2=-1$ and x, y are real is called complex variable. By considering $x = r\cos\theta$, $y = r\sin\theta$ we have

$z = r[\cos\theta + i\sin\theta] = re^{i\theta}$ is called polar form or Mod amplitude form. where $r = |z| = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

We can associate a complex valued function to the complex variable z then we can write

$$\omega = f(z) = u(x,y) + i\,v(x,y)$$

To represent this complex function we require two planes namely z-plane & ω-plane. For each point (x,y) in z-plane there exist a point (u,v) in ω-plane with the function $\omega = f(z)$. The point (u,v) is called image of (x,y)



Every complex function $f(z)$ can always be expressable in the form $\omega = f(z) = u(x,y) + i\,v(x,y)$ where $u(x,y)$ is called real part of $f(z)$ and $v(x,y)$ is called imaginary part of $f(z)$.

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$f(z) = e^z = e^{x+iy} = e^x\cos y + i\,e^x\sin y$

$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$f(z) = \log_e z = \log_e(x+iy) = \log_e(re^{i\theta})$

$= \log_e r + i\theta.\log_e e = \log_e\sqrt{x^2+y^2} + i.\tan^{-1}(y/x)$

$= \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}(y/x)$

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prob: The mod amplitude form of $\frac{1+i}{1-i}$ is

sol: $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{2i}{2} = i$

$r = \sqrt{0^2+1^2} = 1 \qquad \theta = \tan^{-1}(1/0) = \pi/2$

$\therefore$ Mod amplitude form $= 1e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$

prob: If $\alpha + i\beta = \frac{1}{a+ib}$ then $(\alpha^2+\beta^2)(a^2+b^2) =$

sol: $\alpha + i\beta = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

$\therefore \alpha = \frac{a}{a^2+b^2} \qquad \beta = \frac{-b}{a^2+b^2}$

$\alpha^2+\beta^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{1}{(a^2+b^2)}$

$(a^2+b^2)(\alpha^2+\beta^2) = \underline{1}$

12 EC, EE, IN
07 IN
(GATE 96 ME)

prob: The value of $i^i =$

sol: $i^i = e^{\log_e i^i} = e^{i\log_e i} = e^{i\left[\frac{1}{2}\log_e 1 + i\tan^{-1}(1/0)\right]}$

$= e^{i \cdot i\pi/2} = e^{-\pi/2}$

$* \log_e i = i\pi/2 \Rightarrow i = e^{i\pi/2}$

prob: The value of $\sqrt{2i}$

sol: $\sqrt{2i} = \sqrt{2}.\sqrt{i} = \sqrt{2}.e^{\log_e\sqrt{i}} = \sqrt{2}.e^{\frac{1}{2}\log_e i}$

$= \sqrt{2}e^{\frac{1}{2}(i\pi/2)} = \sqrt{2}\,e^{i\pi/4} = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1+i$

$* \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$* \; e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \cdots$

$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$ [Maclaurin's expansion]

$= \cos y + i\sin y$ [Euler's formula]

$* \; e^{2n\pi i} = \cos(2n\pi i) + i\sin(2n\pi i) = 1$

$\Rightarrow e^z = e^{z+2n\pi i}$, n is any integer

$\therefore e^z$ is periodic with period $2\pi i$ [GATE 97 CE]

$* \; e^{-iy} = \cos y - i\sin y$

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* $\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ $\qquad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$

$\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ $\qquad \sinh z = \frac{1}{2}(e^{z}-e^{-z})$

* $\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) = \frac{1}{2}[e^{i(x+iy)}+e^{-i(x+iy)}]$

$= \frac{1}{2}[\cos x(e^{-y}+e^{y}) + i\sin x(e^{-y}-e^{y})]$

$= \cos x.\cosh y - i\sin x.\sinh y$

* $\sin z = \sin x.\cosh y + i\cos x.\sinh y$

* $\sin(iz) = \frac{1}{2i}[e^{i(iz)}-e^{-i(iz)}] = \frac{1}{2i}[e^{-z}-e^{z}]$

$= \frac{i}{2}[e^{z}-e^{-z}] = i\sinh z$

$\cos(iz) = \frac{1}{2}[e^{i(iz)}+e^{-i(iz)}] = \frac{1}{2}[e^{-z}+e^{z}] = \cosh z$

* $\sinh z = \frac{1}{i}\sin(iz) = -i\sin(i(x+iy))$

$= i\sin(y-ix) = \sinh x.\cos y + i\cosh x.\sin y$

$\cosh z = \cosh x.\cos y + i\sinh x.\sin y$

* In general $\ln z = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x) + i2n\pi$. The principal value is obtained when $n=0$

prob: Find the general and principle values of $\log(1+\sqrt{3}i)$

sol: $|z| = \sqrt{1+3} = 2$ $\qquad Arg(z) = \tan^{-1}(\sqrt{3}) = \pi/3$

$\log(1+\sqrt{3}i) = \log 2 + i(\frac{\pi}{3}+2n\pi)$, n any integer

principal value $Log(1+\sqrt{3}i) = \log 2 + i\pi/3$ $\quad (n=0)$

prob: Find general and principal values of $(-i)^i$

sol: $(-i)^i = e^{i\log(-i)} = e^{i[\log 1 + i(-\frac{\pi}{2}+2n\pi)]}$

$= e^{\frac{\pi}{2}-2n\pi}$

principal value $= e^{\pi/2}$

Properties of Moduli:

- $|z_1+z_2| \le |z_1|+|z_2|$

  $|z_1-z_2| \le |z_1+|z_2|$

  $|z_1+z_2| \ge |z_1|-|z_2|$

  $|z_1-z_2| \ge |z_1|-|z_2|$

- $|z_1 z_2 z_3 \cdots z_n| = |z_1|.|z_2|.\ldots|z_n|$

  $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

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(GATE 94 IN): The real part of the complex number $z = x + iy$ is given by (a) $Re(z) = z - z^*$ (b) $Re(z) = \frac{z - z^*}{2}$ (c) $= \frac{z + z^*}{2}$ (d) $= z + z^*$

Sol: $z = x + iy \Rightarrow z^* = x - iy$

$$Re(z) = x = \frac{z + z^*}{2}$$

(GATE 97 IN): The complex number $z = x + iy$ which satisfy the equation $|z+1| = 1$ lie on

(a) circle with (1,0) as center and radius 1

(b) circle with (-1,0) as center and radius 1

(c) y-axis (d) x-axis

Sol: $|z+1| = 1 \Rightarrow |x + iy + 1| = 1$

$$(x+1)^2 + y^2 = 1$$

Equation of circle with (x_0, y_0) as center and r as radius is $(x - x_0)^2 + (y - y_0)^2 = r^2$

$\therefore |z+1| = 1$ is a circle with center (-1,0) and radius = 1

(GATE 2005 CE): Which one of the following is not true for the complex numbers z_1 and z_2?

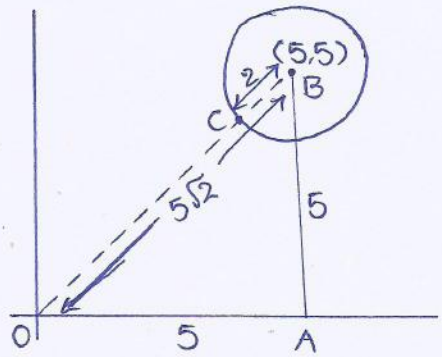
(a) $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$ (b) $|z_1 + z_2| \le |z_1| + |z_2|$ (c) $|z_1 + z_2| \le ||z_1| - |z_2||$

(d) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$|z_1 + z_2| \ge |z_1| - |z_2|$

(GATE 2005 IN): Consider the circle $|z - 5 - 5i| = 2$ in the complex number plane (x, y) with $z = x + iy$. The minimum ~~number~~ distance from origin to circle is?

Sol:



$OB^2 = OA^2 + AB^2 = 50$

$OB = 5\sqrt{2}$

$OC = OB - CB = 5\sqrt{2} - 2$

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(GATE-06 EC): For the function of a complex variable $\omega = \ln z$ (where $\omega = u + iv$ and $z = x + iy$) the $u =$ constant lines get mapped in the z-plane as

Sol: $\omega = \ln z = \ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)$

$$u = \frac{1}{2}\ln(x^2+y^2) = C \quad \text{(constant)}$$

$$\ln(x^2+y^2) = 2C$$

$$x^2+y^2 = e^{2C} \Rightarrow x^2+y^2 = (e^C)^2$$

Represent concentric circles with radius e^C.

(GATE-07 PI): If a complex number $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ then z^4 is

Sol: $z = \frac{\sqrt{3}}{2} + i\frac{1}{2} \Rightarrow iz = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \omega$ say

[ω is a root of $x^3 = 1$]

$$i^4 z^4 = \omega^4 = \omega^3 . \omega \quad [\because \omega^3 = 1]$$

$$z^4 = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(GATE-08 PI): The value of the expression $\frac{-5+i10}{3+4i}$

Sol: $\frac{(-5+i10)}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{25+50i}{25} = 1+2i$

(GATE-08 EC): The equation $\sin(z) = 10$ has

(a) no real or complex solution (b) Exactly two distinct complex sol's

(c) a unique solution ✓(d) an infinite number of complex sol's

Sol: $\sin z = 10 \Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 10$

$$\Rightarrow (e^{iz}) - \frac{1}{e^{iz}} = 20i$$

$$\Rightarrow (e^{iz})^2 - 20i(e^{iz}) - 1 = 0$$

$$e^{iz} = \frac{-(-20i) \pm \sqrt{-400+4}}{2} = 10i \pm 3\sqrt{11}\,i = i[10 \pm 3\sqrt{11}]$$

$$iz = \log_e i[10 \pm 3\sqrt{11}]$$

$$= \log_e i + \log_e(10 \pm 3\sqrt{11})$$

$$iz = \log 1 + i\left(\frac{\pi}{2} \pm 2n\pi\right) + \log(10 \pm 3\sqrt{11})$$

$$z = \frac{\pi}{2} \pm 2n\pi - i\log(10 \pm 3\sqrt{11})$$

$\therefore$ z has infinite no of complex sol's

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(GATE-2013 EE): Square roots of $-i$, where $i=\sqrt{-1}$ are

(a) $i, -i$ ✓(b) $\cos(-\pi/4)+i\sin(-\pi/4)$, $\cos(3\pi/4)+i\sin(3\pi/4)$

(c) $\cos(\pi/4)+i\sin(3\pi/4)$, $\cos(3\pi/4)+i\sin(\pi/4)$

(d) $\cos(3\pi/4)+i\sin(-3\pi/4)$, $\cos(-3\pi/4)+i\sin(3\pi/4)$

Sol: $i=e^{i\pi/2} \Rightarrow -i=\frac{1}{i}=e^{-i\pi/2}$

$\sqrt{-i}=(-i)^{1/2}=e^{-i\pi/4}$

and $-i=i^3=e^{i3\pi/2} \Rightarrow \sqrt{-i}=e^{i3\pi/4}$

$\therefore$ square roots of $-i$ are $e^{-i\pi/4}$, $e^{i3\pi/4}$

(GATE-14 EE): All the values of the multi valued complex function 1^i, where $i=\sqrt{-1}$ are.

(a) purely imaginary ✓(b) real & non negative (c) on the unit circle

(d) equal in real and imaginary parts.

Sol: $1^i=(\cos 2n\pi+i\sin 2n\pi)^i$

$=(e^{i2n\pi})^i=e^{-2n\pi}=\frac{1}{e^{2n\pi}}$ It is real & non-negative.

(GATE-14 ME): The argument of complex number $\frac{1+i}{1-i}$

Sol: $\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{2i}{2}=i=0+1.i$

Argument $=\tan^{-1}(1/0)=\tan^{-1}(\infty)=\pi/2$

Analytic function: A function $f(z)$ is said to be analytic in a region R of z-plane if the derivative of $f(z)$ exists at each and every point in that region.

Necessary and sufficient condition: For a function $f(z)=u+iv$ to be analytic in the region 'R'

(i) Partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist in 'R'

(ii) They should satisfy $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

(or)

$$\underbrace{u_x=v_y,\quad u_y=-v_x}_{\text{Cauchy Reiman or C-R equations}}$$

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prob: $f(z) = z^2$.

$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y \qquad u_y = -v_x$

$\therefore f(z) = z^2$ is analytic throughout the z-plane.

prob: $f(z) = \bar{z}$

$\bar{z} = x - iy \Rightarrow u = x \qquad v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$

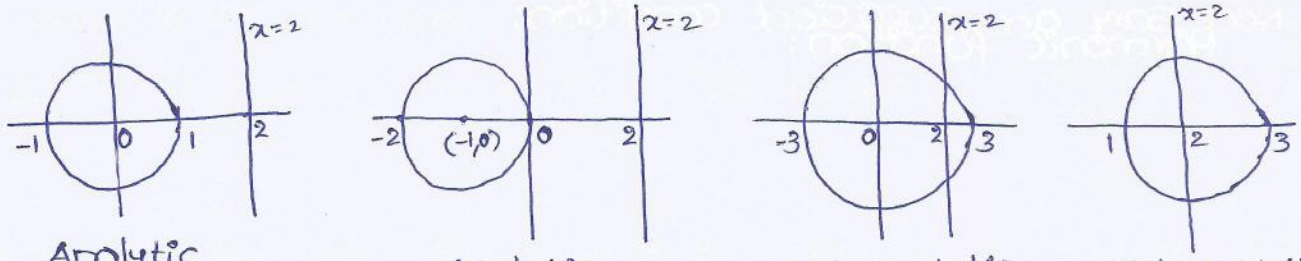
$\therefore f(z) = \bar{z}$ is not analytic throughout the z-plane.

Note: 1. Any function which involves $\bar{z}$ is always not analytic throughout the z-plane.

2. Every polynomial in z is always analytic throughout the z-plane.

Ex: $f(x) = u + iv$ suppose if $u_x = \frac{1}{x-2}$, $u_y = 1$, $v_x = -1$, $v_y = \frac{1}{x-2}$

This function is analytic in the regions which does not contain $x=2$ like $|z|=1$, $|z+1|=1$ and this function is not analytic in the regions which contains $x=2$ like $|z|=3$, $|z-2|=1$



Analytic Analytic Not analytic Not analytic

Entire function (or) Holomorphic (or) Regular function:

A function which is analytic in the entire z-plane is called entire function.

Ex: Every polynomial in z.

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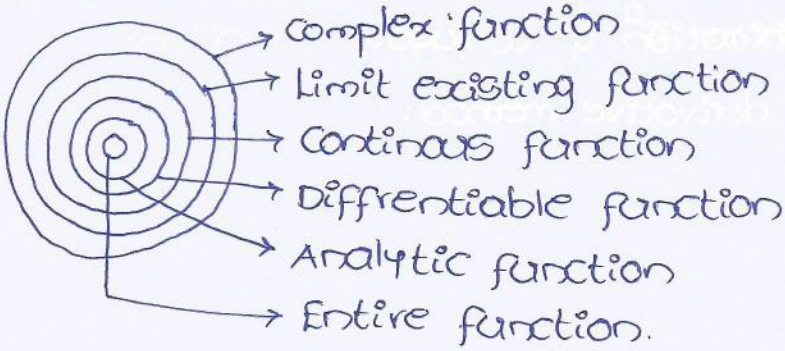

C-R equations in polar form:

$$f(z) = u + iv \Rightarrow f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

P.d.w.r. to 'r' $\quad f'(re^{i\theta}) \times e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (1)$

P.d.w.r. to 'θ' $\quad f'(re^{i\theta}) \times r \times i \times e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \rightarrow (2)$

from (1) & (2) $\quad ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$$-r.\frac{\partial v}{\partial r} + i.r.\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i.\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r.\frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r.\frac{\partial u}{\partial r}$$

$$\Rightarrow \quad v_r = -\frac{1}{r}u_\theta \qquad u_r = \frac{1}{r}v_\theta$$

Properties of analytic function: If $f(z) = u + iv$ is analytic function then

1. $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal to each other

   Ex: $f(z) = z^2 = x^2 - y^2 + i\,2xy$

   $x^2 - y^2 = c_1$ & $2xy = c_2$ are $\perp$'ar to each other.

2. If $u(x,y)$ is a harmonic function then $v(x,y)$ is also a harmonic function.

   Harmonic function: A function $H(x,y)$ which satisfies the laplace equation $\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$ is called harmonic function.

$$u_x = v_y \qquad u_y = -v_x$$

$$u_{xx} = v_{xy} \qquad u_{yy} = -v_{yx}$$

$$\left.\begin{aligned} u_{xx} + u_{yy} &= 0 \\ \text{Illy} \quad v_{xx} + v_{yy} &= 0 \end{aligned}\right\}$$ If $f(z) = u + iv$ is analytic then u & v satisfies laplace equation

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Determination of Conjugate function:

Total derivative method: Let the real part of $u(x,y)$ of an analytic function $f(z) = u+iv$ be given. Then to find $v(x,y)$ we proceed as follows. The total derivative of v is given by

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$$

By using C-R eq^ns $\quad dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$

$$v = \underbrace{\int(-u_y)\,dx}_{**y \text{ const}} + \underbrace{\int u_x\cdot dy}_{***\text{terms independent of } 'x'} + C$$

Similarly, If $v(x,y)$ is given. Then to find $u(x,y)$.

$$du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy = v_y\,dx - v_x\,dy \quad [\text{using C-R}$$

$$u = \int v_y\,dx - \int v_x\,dy + C$$

Milne-Thompson method:

1. If $u(x,y)$ is given

   Take $f'(z) = u_x - iu_y$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

2. If $v(x,y)$ is given

   Take $f'(z) = v_y + iv_x$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

(GATE-05 PI): The function $\omega = u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ is not analytic at the point

✓(a) (0,0) (b) (0,1) (c) (1,0) (d) (2,∞)

Sol: $\omega = \log(x+iy)$ log is not defined at origin

(GATE-07 CE): potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function ψ with the condition $\psi = 0$ at $x=0, y=0$?

Sol: Given $\phi(x,y) = x^2 - y^2$

Take $d\psi = \psi_x\,dx + \psi_y\cdot dy$

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$$= -\phi_y \cdot dx + \phi_x \cdot dy$$

$$= -(-2y)\,dx + 2x \cdot (dy)$$

$$\psi = 2xy + K$$

At $x=0,\ y=0 \Rightarrow \psi = 2xy + K = 0 \Rightarrow K = 0$

$\therefore$ stream function is $\psi(x,y) = 2xy$

(GATE - 09 ME): An analytic function of a complex variable $z = x+iy$ is expressed as $f(z) = u(x,y) + i\,v(x,y)$ where $i = \sqrt{-1}$. If $u = xy$ then expression for v should be

Sol: Given $u = xy$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy$$

$$dv = -x\,dx + y\,dy$$

$$v = -\frac{x^2}{2} + \frac{y^2}{2} + K = \frac{y^2 - x^2}{2} + K$$

(GATE - 2010 PI): If $f(x+iy) = x^3 - 3xy^2 + i\phi(x,y)$ where $i = \sqrt{-1}$ and $f(x+iy)$ is an analytic function then $\phi(x,y)$ is

Sol: Given $\psi(x,y) = x^3 - 3xy^2$

$$d\phi = \phi_x dx + \phi_y dy = -\psi_y dx + \psi_x dy$$

$$d\phi = -(0 - 6xy)\,dx + (3x^2 - 3y^2)\,dy$$

$$\phi = \frac{6x^2y}{2} + \left(-3 \cdot \frac{y^3}{3}\right) + K$$

$$= 3x^2y - y^3 + K$$

(GATE - 11 CE): For an analytic function $f(x+iy) = u(x,y) + i\,v(x,y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be constant is

Sol: Given $u = 3x^2 - 3y^2$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy$$

$$dv = -(0 - 6y)\,dx + 6x\,dy$$

$$v = 6xy + K$$

(GATE - 14 EC): The real part of an analytic function $f(z)$ where $z = x+iy$ is given by $e^{-y}\cos x$. The imaginary part of $f(z)$ is

Sol: Given $u = e^{-y}\cos x$

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$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$$
$$= -u_y\,dx + u_x\cdot dy$$
$$= e^{-y}\cos x\,dx - e^{-y}\sin x\,dy$$
$$v = e^{-y}\sin x + C$$

(GATE-14 ME): An analytic function of a complex variable $z = x+iy$ is expressed as $f(z) = u(x,y) + i\,v(x,y)$, where $i=\sqrt{-1}$. if $u(x,y) = 2xy$ then $v(x,y)$ must be

Sol: Given $u = 2xy$

$$dv = v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy$$
$$= -2x\,dx + 2y\,dy$$
$$v = -x^2 + y^2 + K$$

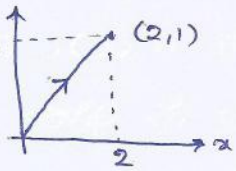
(GATE-14 ME): $u(x,y) = x^2 - y^2$. $v(x,y) = ?$

Sol: $dv = -u_y\,dx + u_x\,dy = 2y\,dx + 2x\,dy$

$$v = 2xy + C$$

Complex integration: An integral of the form $\int_C f(z)\,dz$ where $dz = dx + i\,dy$ is called a complex integral. This integral value depends not only on $f(z)$ but also on the curve 'C'.

Prob: The value of $\int_0^{2+i} \bar{z}\,dz$ along (i) $y = x/2$ (ii) The real axis to 2 and then vertically to $2+i$.



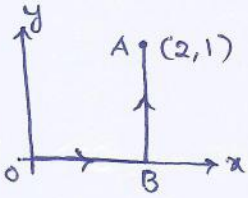
Sol: (i) Along $y = x/2 \Rightarrow dy = \frac{dx}{2}$

$$\int_0^{2+i} (x - iy)(dx + i\,dy) = \int_0^2 \left(x - i\frac{x}{2}\right)\left(dx + i\frac{dx}{2}\right)$$ (or convert to y)

$$= \left(1 - i/2\right)\left(1 + i/2\right)\int_0^2 x\cdot dx$$
$$= \left(1 + 1/4\right)\left(x^2/2\right)_0^2 = 5/2$$

(ii) Along OB: $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2



$$\int_{OB} \bar{z}\,dz = \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

Along BA: $x = 2 \Rightarrow dx = 0$, y varies from 0 to 1

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$$\int_{BA} \bar{z}\, dz = \int_0^1 (2 - iy)(i\,dy) = i(2 - i/2) = 2i + 1/2$$

$$\therefore \int_{OBA} \bar{z}\, dz = 2 + \frac{1}{2} + 2i = \frac{5}{2} + 2i$$

prob: The value of $\int_{1-i}^{2+i} x\, dz$ along $x = t+1,\ y = 2t^2 - 1$ is

$x = t+1 \Rightarrow dx = dt$ $\qquad$ $x = 1 \Rightarrow t = 0$

$y = 2t^2 - 1 \Rightarrow dy = 4t\, dt$ $\qquad$ $x = 2 \Rightarrow t = 1$

$$\therefore \int_0^1 (t+1)(dt + i\,4t\,dt) = \int_0^1 (t+1)(1 + i4t)\,dt$$

$$= \frac{t^2}{2} + 4i\frac{t^3}{3} + t + 4i\frac{t^2}{2} = \frac{3}{2} + \frac{10}{3}i$$

Cauchy's Integral theorem:

If $f(z)$ is analytic inside and on a closed curve 'c' then $\oint_C f(z)\,dz = 0$

Ex: $\oint_{|z|=1} (z^2 + 3z + 5)\,dz = 0$ $\qquad$ [polynomial in z is analytic everywhere]

$\oint_{|z|=1} \frac{z^2+3z+5}{z-2}\,dz = 0$ $\qquad$ [Analytic inside $|z|=1$, but not at $z=2$]

$\oint_{|z|=3} \frac{z^2+3z+5}{z-2}\,dz$ $\qquad$ C.I.T is not applicable.

Cauchy's integral formula:

If $f(z)$ is analytic inside and on a closed curve 'c' and $z = a$ is a point inside 'c' then

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Ex: $\oint_{|z|=3} \frac{z^2+3z+5}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i\,[4+6+5]$

$= 30\pi i$

Generalised Cauchy's integral formula:

If $f(z)$ is analytic inside and on a closed curve 'c' and $z = a$ is a point inside 'c' then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i\, \frac{f^n(a)}{n!}$$

Ex: $\oint_{|z|=3} \frac{z^2+3z+5}{(z-2)^2}\,dz = 2\pi i\, \frac{f'(2)}{1!} = 2\pi i\,[4+3]$

$= 14\pi i$

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Generalized cauchy's integral formula: If $f(z)$ is analytic inside and on a closed curve 'c' and $z=a$ is a point inside 'c' then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i\,\frac{f^{n}(a)}{n!}$$

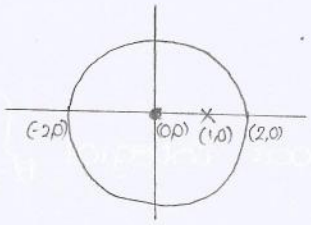
Ex: $\displaystyle\oint_{|z|=3} \frac{z^2+3z+5}{(z-2)^2}\,dz = 2\pi i\,\frac{f'(2)}{1!} = 14\pi i$

Procedure to follow:

1. Draw the given curve 'c'.
2. Factorize the denominator of given function and mark the roots
   (i) If all the roots are out side of curve 'c' then according to cauchy's integral theorem given integral = 0
   (ii) If some roots are inside 'c' then
   (a) write down numerator and the factors of denominator which corre-sponds to roots outside the curve 'c' as $f(z)$ (if any)
   (b) Depending on inside roots whether it is a simple root or multiple root apply cauchy's integral formula or generalized cauchy's integral formula.

prob: $\displaystyle\oint \frac{z^2-z+1}{z-1}\,dz$ where C is (i) $|z|=2$ (ii) $|z|=1/2$

(i) C: $|z|=2$



Root: $z=1$

$\Rightarrow (1,0)$

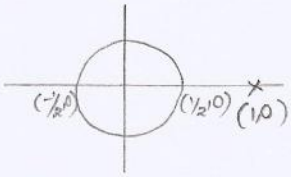
$z=1$ lies <u>inside</u> given C: $|z|=2$

$\therefore$ Let $f(z) = z^2-z+1$ (only numerator)

$$\oint \frac{z^2-z+1}{z-1}\,dz = 2\pi i\, f(1) \qquad (z=1 \text{ is a simple root})$$

$$= 2\pi i\,[1-1+1] = 2\pi i$$

(ii) C: $|z|=1/2$



$z=1$ lies <u>outside</u> of C: $|z|=1/2$

$\therefore \displaystyle\oint \frac{z^2-z+1}{z-1}\,dz = 0$

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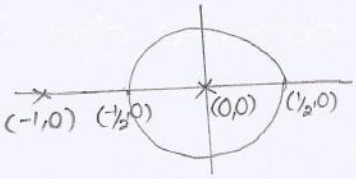
prob: $\oint_C \frac{2z+1}{z^2+z}\,dz$ where C is $|z| = 1/2$.

Sol:



Denominator: $z^2+z = z(z+1)$

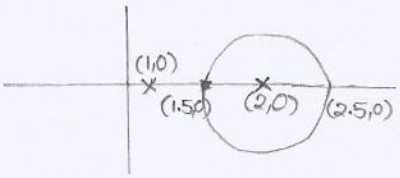
Roots: $z=0$, $z=-1$

$z=0$ lies inside and $z=-1$ lies outside of 'c'

$$\therefore f(z) = \frac{2z+1}{z+1}$$

$$\therefore \oint_C \frac{2z+1}{z^2+z}\,dz = \oint_C \frac{f(z)}{z}\,dz = 2\pi i\, f(0) = 2\pi i\left[\frac{1}{1}\right] = 2\pi i$$

prob: $\oint_C \frac{z}{(z-1)(z-2)^2}\,dz$ where C is $|z-2| = 1/2$



Denominator: $(z-1)(z-2)^2$

Roots: $z=1$ (Simple), $z=2$ (multiple)

$z=1$ lies outside and $z=2$ lies inside

$$\therefore f(z) = \frac{z}{z-1}$$

$$\oint_C \frac{z}{(z-1)(z-2)^2}\,dz = \oint_C \frac{f(z)}{(z-2)^2}\,dz = 2\pi i\,\frac{f'(2)}{1!}$$

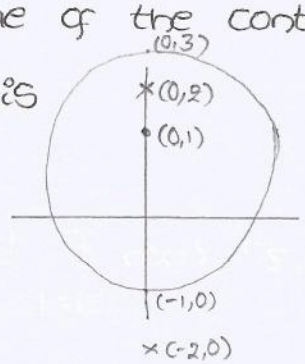
$$= 2\pi i\left[\frac{(z-1)-z}{(z-1)^2}\right]_{z=2} = -2\pi i$$

prob: $\oint_C \frac{2z+1}{(2z-1)^2}\,dz$ where C is $|z| = 1$

$$= \frac{1}{4}\oint_C \frac{2z+1}{(z-1/2)^2} = \frac{1}{4}\cdot 2\pi i\,\frac{f'(1/2)}{1!} \qquad f(z) = 2z+1$$

$$= \frac{1}{4}\cdot 2\pi i \times 2 = \pi i$$

(GATE-2006 EC): The value of the contour integral $\int_{|z-j|=2} \frac{1}{z^2+4}\,dz$ in the positive sense is



Sol: $C: |z-j| = 2$

$z^2+4 = (z+2i)(z-2i)$

$z = -2i$ outside, $z = 2i$ inside

$$\Rightarrow f(z) = \frac{1}{(z+2i)}$$

$$\therefore \int_{|z-j|=2} \frac{1}{z^2+4}\,dz = \int_{|z-j|=2} \frac{f(z)}{(z-2i)}\,dz = 2\pi i\, f(2i)$$

$$= 2\pi i \times \frac{1}{4i} = \pi/2$$

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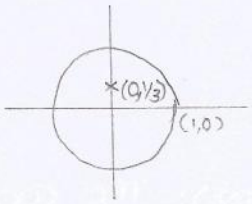
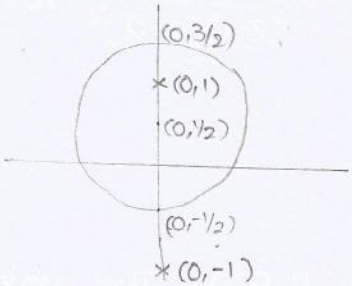
(GATE 06 CE): Using cauchy's integral theorem, the value of the integral (integration being taken in contour clockwise direction) $\oint_C \frac{z^3-6}{3z-i}\,dz$ is where C is $|z|=1$



Sol: $$\oint_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\oint_C \frac{z^3-6}{z-i/3}\,dz$$

$$= \frac{1}{3}\cdot 2\pi i\cdot\left[\left(\tfrac{i}{3}\right)^3-6\right] = \frac{2\pi}{81} - 4\pi i$$

(GATE EC 07): The value of $\oint_C \frac{1}{z^2+1}\,dz$ where C is the contour $|z-i/2|=$ is



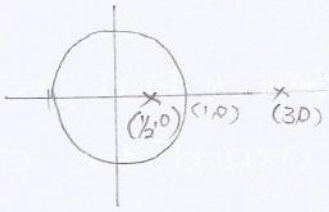
Sol: $$\oint_C \frac{1}{z^2+1}\,dz = \oint_C \frac{1/z+i}{z-i}\,dz = 2\pi i\cdot\frac{1}{2i} = \pi$$

(GATE ME EC 08): The integral $\oint f(z)\,dz$ evaluated around the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$

Sol: C: $|z|=1$, $z=0$ is inside 'C'

$$\therefore \oint_C \frac{\cos z}{z}\,dz = 2\pi i\cos(0) = 2\pi i$$

(GATE CE 09): The value of integral $\oint_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}\,dz$ where c is a closed curve given by $|z|=1$ is

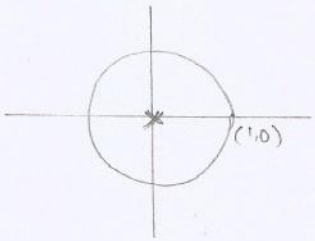


Sol: $$\oint_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}\,dz = \frac{1}{2}\oint_C \frac{\cos(2\pi z)}{(z-1/2)(z-3)}\,dz$$

$$= \frac{1}{2}\oint_C \frac{\cos(2\pi z)/(z-3)}{(z-1/2)}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i\cdot\frac{\cos(\pi)}{(1/2-3)} = \frac{2\pi i}{5}$$

(GATE-09 EC): If $f(z) = C_0 + C_1 z^{-1}$ then $\oint_{|z|=1} \frac{1+f(z)}{z}\,dz$ is



Sol: $f(z) = C_0 + C_1 z^{-1} = C_0 + \frac{C_1}{z}$

$$\oint_{|z|=1} \frac{1+C_0+\frac{C_1}{z}}{z}\,dz = \oint_{|z|=1} \frac{z+zC_0+C_1}{z^2}\,dz$$

$$= \frac{2\pi i}{1!}\left[1+C_0\right]_{z=0} = 2\pi i(1+C_0)$$

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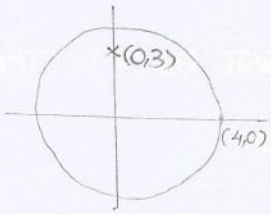
(GATE-09 IN): The value of $\oint_C \frac{\sin z}{z}\,dz$, where the contour of the integration is a simple closed curve around the origin is

Sol: $z=0$ is inside the curve surrounds it.

$$\therefore \oint_C \frac{\sin z}{z}\,dz = 2\pi i \sin(0) = 0$$

(GATE-10 IN): The contour C in the adjoining figure is described by $x^2+y^2=16$. Then the value of $\oint_C \frac{z^2+8}{(0.5)z-(1.5)j}\,dz$

Sol: Given $x^2+y^2=16 \Rightarrow |z|=4$

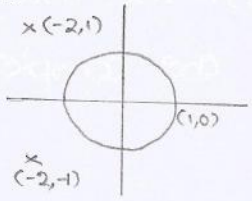


$$\oint_C \frac{z^2+8}{\frac{1}{2}z-\frac{3}{2}j}\,dz = 2\oint_C \frac{z^2+8}{z-3j}\,dz$$

$$= 2\left[2\pi j\left((3j)^2+8\right)\right] \qquad (z=3j \text{ lies inside } |z|=4)$$

$$= -4\pi j$$

(GATE-11 EC): The value of integral $\oint_C \frac{-3z+4}{z^2+4z+5}\,dz$, when C is the circle $|z|=1$ is given by



Sol: $z^2+4z+5 = (z+2-i)(z+2+i)$

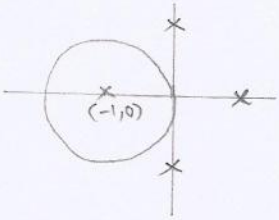
$z=-2+i \qquad z=-2-i$

Both roots lies outside $C: |z|=1$

$$\therefore \oint_C \frac{-3z+4}{z^2+4z+5}\,dz = 0$$

(GATE-11 PI): The value of $\oint_C \frac{z^2}{z^4-1}\,dz$ using cauchy's integral around the circle $|z+1|=1$ where $z=x+iy$ is

Sol: $$\oint_C \frac{z^2}{z^4-1}\,dz = \oint_C \frac{z^2}{(z^2-1)(z^2+1)}\,dz$$



$$= \frac{1}{2}\oint_C \left[\frac{1}{z^2-1}+\frac{1}{z^2+1}\right]dz$$

$$= \frac{1}{2}\oint_C \frac{1}{z^2-1}\,dz + \frac{1}{2}\oint_C \frac{1}{z^2+1}\,dz$$

$$= \frac{1}{2}\oint_C \frac{1}{(z-1)(z+1)}\,dz + \frac{1}{2}\oint_C \frac{1}{(z+i)(z-i)}\,dz$$

Only $z=-1$ lies inside the circle $|z+1|=1$

$$\oint_C \frac{z^2}{z^4-1}\,dz = \frac{1}{2}\cdot 2\pi i\left[\frac{1}{-1-1}\right] + \cancel{\frac{1}{2}}\ 0$$

$$= -\frac{\pi i}{2}$$

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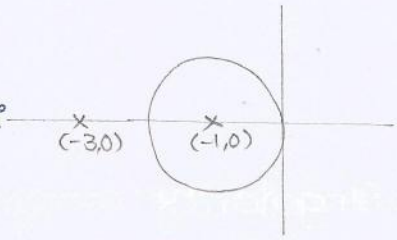
(GATE-12 EC, EE, IN): Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z+1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)\,dz$ is



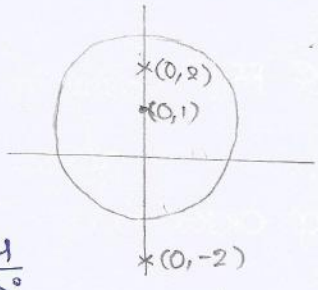
Sol: $z=-1$ lies inside & $z=-3$ lies outside of given $C: |z+1|=1$

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \frac{1}{2\pi j}\oint_C \frac{1}{z+1}\,dz - \frac{1}{2\pi j}\oint_C \frac{2}{z+3}\,dz$$

$$= \frac{1}{2\pi j}\cdot 2\pi j\,(1)_{z=-1} - \frac{1}{2\pi j}(0)$$

$$= 1.$$

(GATE-13 EE): $\oint \frac{z^2-4}{z^2+4}\,dz$ evaluated anticlockwise around the circular $|z-i| = 2$, where $i=\sqrt{-1}$ is



Sol: $z^2+4 = (z+2i)(z-2i)$

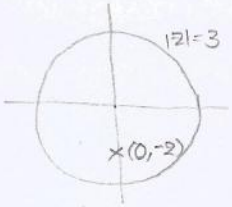
$z=-2i \qquad z=2i$

$z=-2i$ lies outside $\Rightarrow f(z) = \frac{z^2-4}{z+2i}$

$$\oint \frac{z^2-4}{z^2+4}\,dz = \oint \frac{f(z)}{z-2i}\,dz$$

$$= 2\pi i\cdot f(2i) = -4\pi$$

(GATE-14 EC): C is a closed path in the z-plane given by $|z|=3$. The value of the integral $\oint_C \frac{z^2-z+4j}{z+2j}\,dz$ is

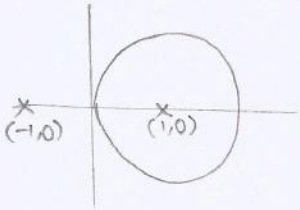


Sol: $z=-2j$ lies inside $|z|=3$

$$\oint_C \frac{z^2-z+4j}{z+2j}\,dz = 2\pi j\left[(-2j)^2-(-2j)+4j\right]$$

$$= -4\pi(3+2j)$$

(GATE-14 EE): Integration of the complex function $f(z) = \frac{z^2}{z^2-1}$, in the counterclockwise direction, around $|z-1|=1$, is



Sol: $z^2-1 = (z+1)(z-1)$

$z=-1$ lies outside of $C: |z-1|=1$

$$\therefore F(z) = \frac{z^2}{(z+1)}$$

$$\oint_C f(z) = \oint_C \frac{F(z)}{(z-1)} = 2\pi i\,F(1) = 2\pi i\cdot\frac{1}{2} = \pi i$$

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Residue theorem: If $f(z)$ is analytic inside and on a closed curve 'c' except at a finite number of singularities inside 'c' then

$$\oint_c f(z)\,dz = 2\pi i\,[\text{Sum of residues over the singularities}]$$

Singularity: A point where $f(z)$ fails to be analytic is called a singularity of $f(z)$

Ex: $\frac{1}{(z-3)(z+7)}$ then $z=3$, $z=-7$ are singularities of $f(z)$

Singularities are of 4-types:

1. Simple pole
2. pole of order 'n'
3. Essential singularity
4. Removable singularity.

Simple pole: $f(z) = \frac{1}{(z-3)(z+7)}$. Here $z=3,-7$ are simple poles of $f(z)$

pole of order 'n': $f(z) = \frac{1}{(z+2)^2(z-5)}$. Here $z=-2$ is a pole of order '2' and $z=5$ is a pole of order '1' or simple pole.

Essential singularity: $f(z) = \sin\left(\frac{1}{z-1}\right) = \frac{1}{z-1} - \frac{1}{(z-1)^3 3!} + \frac{1}{(z-1)^5 5!} - \cdots$

Here $z=1$ is essential singularity of $f(z)$.

Removable singularity: $f(z) = \frac{z^2-3z+2}{(z-1)(z+2)} = \frac{z-2}{z+2}$

Here $z=1$ is a removable singularity. and $z=-2$ is a simple pole.

Calculation of residue:

1. If $z=a$ is a simple pole of $f(z)$ then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} (z-a) f(z)$$

2. If $z=a$ is a pole of order 'n' of $f(z)$ then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left\{(z-a)^n f(z)\right\}$$

* Residue method is used to find $\oint_c f(z)\,dz$ when there are 2 or more poles of $f(z)$ exists inside the region 'c'.

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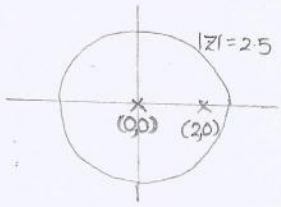
procedure to find $\oint_C f(z)\,dz$ using residue method:

1. Remove removable singularities of $f(z)$ if any.
2. Draw the region 'c' and mark the poles of $f(z)$
3. Residue corresponds to poles outside 'c' is $=0$
4. Find the residues corresponds to poles inside 'c' using above formula.

$$\oint_C f(z)\,dz = 2\pi i\,[\text{Sum of residues}]$$

prob: $\oint_C \frac{z^2+1}{z^2-2z}\,dz$ where $C: |z| = 2.5$



$$\oint_C \frac{z^2+1}{z^2-2z}\,dz = \oint_C \frac{z^2+1}{z(z-2)}\,dz$$

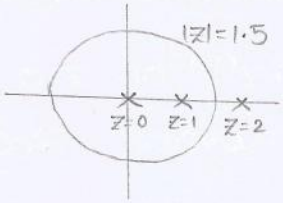
$z=0,\ z=2$ are simple poles & lies inside $|z| = 2.5$

$$\text{Res } f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{z^2+1}{\cancel{z}(z-2)} = \frac{0+1}{-2} = -1/2$$

$$\text{Res } f(z)\big|_{z=2} = \lim_{z\to 2} \cancel{(z-2)}\,\frac{z^2+1}{z\cancel{(z-2)}} = \frac{4+1}{2} = 5/2$$

$$\oint_C f(z)\,dz = 2\pi i\left[-\frac{1}{2}+\frac{5}{2}\right] = 4\pi i$$

prob: $\oint_C \frac{1-2z}{z(z-1)(z-2)}\,dz$ where 'c' is $|z| = 1.5$



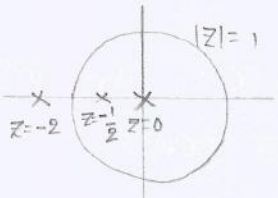
Sol: $z=0,1$ lies inside $|z| = 1.5$ & are simple poles

$$\text{Res } f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{1-2z}{\cancel{z}(z-1)(z-2)} = \frac{1}{2}$$

$$\text{Res } f(z)\big|_{z=1} = \lim_{z\to 1} \cancel{(z-1)}\cdot\frac{1-2z}{z\cancel{(z-1)}(z-2)} = \frac{-1}{-1} = 1$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\left[\frac{1}{2}+1\right] = 3\pi i$$

prob: $\oint_C \frac{2z-1}{z(z+2)(2z+1)}\,dz$ where 'c' is $|z| = 1$



Sol: $$\text{Res } f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{2z-1}{\cancel{z}(z+2)(2z+1)} = -1/2$$

** $$\text{Res } f(z)\big|_{z=-1/2} = \lim_{z\to -1/2} \cancel{(z+1/2)}\cdot\frac{2z-1}{z(z+2)\cdot 2\cancel{(z+1/2)}}$$

$$= 4/3$$

$z=0,\ z=-1/2$ lies inside

$$\oint_C f(z)\,dz = 2\pi i\left(-1/2+4/3\right) = \frac{10\pi i}{6}$$

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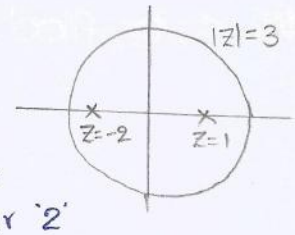
prob: $\oint_c \frac{z^2}{(z-1)^2(z+2)} dz$ where 'c' is $|z|=3$



sol: Both $z=1$, $z=-2$ lies inside $|z|=3$

$z=-2$ is a simple pole & $z=1$ is pole of order '2'

$$\text{Res } f(z)\Big|_{z=1} = \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}\left\{(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z+2)}\right\}$$

$$= \lim_{z\to 1} \frac{d}{dz}\left\{\frac{z^2}{z+2}\right\}$$

$$= \lim_{z\to 1} \frac{(z+2)2z - z^2}{(z+2)^2} = \frac{6-1}{9} = 5/9$$

$$\text{Res } f(z)\Big|_{z=-2} = \lim_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$$\oint_c f(z)\,dz = 2\pi i\left[\frac{5}{9}+\frac{4}{9}\right] = 2\pi i$$

(GATE 08 EC): The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z=2$

sol: $$\text{Res } f(z)\Big|_{z=2} = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left\{(z-2)^2 \cdot \frac{1}{(z+2)^2(z-2)^2}\right\}$$

$$= \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left\{\frac{1}{(z+2)^2}\right\}$$

$$= \lim_{z\to 2}\frac{-2}{(z+2)^3} = -1/32$$

(GATE 08 EE): Given $x(z) = \frac{z}{(z-a)^2}$ with $|z|>a$, the residue of $x(z)\,z^{n-1}$ at $z=a$ for $n\geq 0$ will be

sol: Let $f(z) = x(z).z^{n-1} = \frac{z^n}{(z-a)^2}$

$z=a$ is a pole of order '2'

$$\text{Res } f(z)\Big|_{z=a} = \lim_{z\to a}\left[\frac{d}{dz}\left\{(z-a)^2 \cdot \frac{z^n}{(z-a)^2}\right\}\right]$$

$$= \lim_{z\to a}\left[\frac{d}{dz} z^n\right]$$

$$= \lim_{z\to a} n z^{n-1} = n a^{n-1}$$

(GATE-10 EC): The residues of a complex function $x(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles

sol: All the poles $z=0,1,2$ are simple poles

$\text{Res } x(z)\Big|_{z=0} = 1/2$ $\quad$ $\text{Res } x(z)\Big|_{z=1} = 1$ $\quad$ $\text{Res } x(z)\Big|_{z=2} = -3/2$

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Taylor series of $f(z)$: Taylor series of $f(z)$ about $z=a$ is given by $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$

* Consist of only positive powers of $(z-a)$

Laurent's series: Laurent's series of $f(z)$ about $z=a$ is

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z-a)^n$$

* It consists of both positive & negative powers of $(z-a)$

prob: write $\frac{1}{(z-1)(z+3)}$ in powers of $(z-1)$

sol:
$$\frac{1}{(z-1)(z+3)} = \frac{1}{(z-1)(z-1+4)}$$
$$= \frac{1}{4(z-1)\left(1+\frac{z-1}{4}\right)}$$
$$= \frac{1}{4(z-1)}\left[1+\left(\frac{z-1}{4}\right)\right]^{-1}$$
$$= \frac{1}{4(z-1)}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right]$$
$$= \frac{1}{4(z-1)} - \frac{1}{4^2} + \frac{1}{4^3}(z-1) - \frac{1}{4^4}(z-2)^2 + \cdots$$

Residue: In laurent's series expansion of $f(z)$ about $z=a$ ~~if there is only one term with negative power of $(z-a)$ then $z=a$ is~~ the co.efficient of $(z-a)^{-1}$ is called the "residue"

* In laurent's series expansion of $f(z)$ about $z=a$

1. If there is only one term with negative power of $(z-a)$ then $z=a$ is called a <u>Simple pole</u>
3. If there are infinite number of terms with negative power of $(z-a)$ then $z=a$ is called an <u>essential singular</u>
2. If there are 'n' terms with negative power of $(z-a)$ then $z=a$ is called <u>pole of order 'n'</u>
4. If there are no terms with negative power of $(z-a)$ then $z=a$ is called <u>Removable singularity</u>

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prob: $\frac{1}{(z-1)(z+3)} = \boxed{\frac{1}{4}}(z-1)^{-1} - \frac{1}{4^2} + \frac{1}{4^3}(z-1) - \frac{1}{4^4}(z-1)^2 + \cdots$

$$\text{Residue} = \frac{1}{(z+3)}\Big|_{z=1} = \frac{1}{4}$$

prob: Expand $\frac{1}{(z-1)^2(z+3)}$ interms of $(z-1)$ & find residue at $z=1$

sol: $\frac{1}{(z-1)^2(z+3)} = \frac{1}{4(z-1)^2}\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right]$

$$= \frac{1}{4}(z-1)^{-2} - \boxed{\frac{1}{4^2}}(z-1)^{-1} + \frac{1}{4^3} - \frac{1}{4^4}(z-1) + \cdots$$

$$\text{Residue}\Big|_{z=1} = \lim_{z\to 1} \frac{d}{dz}\left\{\frac{1}{z+3}\right\}$$

$$= \lim_{z\to 1} \frac{-1}{(z+3)^2} = -\frac{1}{4^2}$$

prob: $\oint_{|z|=3} (z-2)\cos\left(\frac{1}{z-2}\right) dz$

sol: $(z-2)\cos\left(\frac{1}{z-2}\right) = (z-2)\left[1 - \frac{1}{2!}\frac{1}{(z-2)^2} + \frac{1}{4!}\cdot\frac{1}{(z-2)^4} - \cdots\right]$

$$= (z-2) - \boxed{\frac{1}{2!}}\cdot\frac{1}{(z-2)} + \frac{1}{4!}\cdot\frac{1}{(z-2)^3} - \cdots$$

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$$

prob: $\oint_C z^2 e^{1/z} dz$ where C is $|z| = 1$

$$z^2 e^{1/z} = z^2\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2\cdot\frac{1}{2!} + \cdots\right]$$

$$= z^2 + z + \frac{1}{2!} + \boxed{\frac{1}{3!}}\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^2} + \cdots$$

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{1}{3!}\right) = \pi i/3$$

prob: $\oint_C \frac{e^{-z}}{z^2} dz$ where C is $|z| = 1$

cauchy integral formula: $2\pi i\,\frac{f'(0)}{1!} = 2\pi i\,\frac{(-e^{-z})_{z=0}}{1} = -2\pi i$

cauchy residue theorem: $\text{Res}\, f(z)\big|_{z=0} = \lim_{z\to 0}\frac{d}{dz}(e^{-z}) = \lim_{z\to 0} -e^{-z} = -1$

$$\oint_C f(z)\,dz = 2\pi i(-1) = -2\pi i$$

Laurentz series: $\frac{e^{-z}}{z^2} = \frac{1}{z^2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right]$

$$\oint f(z)\,dz = 2\pi i(-1) = -2\pi i$$

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(GATE-07 IN): For the function $\frac{\sin z}{z^3}$ of a complex variable z, the point $z=0$ is

Sol: $\frac{\sin z}{z^3} = \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$

$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$

Highest negative power is $\frac{1}{z^2}$. $\therefore$ $z=0$ is pole of order '2'

(GATE-11 IN): The contour integral $\int_C e^{1/z}\,dz$ with C as the counter clockwise unit circle in the z-plane is equal to

Sol: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$

$\text{Res } f(z)\big|_{z=0} = 1$

$\therefore \oint_C e^{1/z}\,dz = 2\pi i(1) = 2\pi i = 2\pi\sqrt{-1}$

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PROBABILITY AND STATISTICS

PROBABILITY & STATISTICS

Random experiment: An ~~outcom~~ experiment whose output cannot be predicted but it is one among the several possibilities.

Ex: Tossing coin, rolling dyes.

Event: The outcome of the random experiment is called event. and the chance of this event is $P(E) = \frac{\text{favourable cases}}{\text{Total no. of cases}}$

and $0 \leq P(E) \leq 1$

permutations & combinations:

Factorial: Let n be positive integer, $n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times$

and $0! = 1$

permutations: The different arrangements of a given number of things by taking some or all at a time

Ex: All permutations made with letters a, b, c by taking two at a time are (ab, ba, ac, ca, bc, cb)

Ex: All permutations made with letters a, b, c by taking all at a time are (abc, acb, bac, bca, cab, cba)

Number of permutations: Number of permutations of n things, taken 'r' at a time is given by

$$^nP_r = n.(n-1).(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Ex: $^6P_2 = 6 \times 5 = 30$ $\quad$ $^7P_3 = 7 \times 6 \times 5 = 210$

* Number of all permutations of n things, taken all at a time $= n!$
* If there are 'n' subjects of which P_1 are alike of one kind; P_2 are alike of another kind; and so on P_r are alike of rth kind such that $(P_1 + P_2 + P_3 + \ldots + P_r) = n$

No. of permutations of these 'n' objects $= \frac{n!}{P_1!\, P_2! \ldots P_r!}$

Combinations: Each of different groups / selections which can be formed by taking some or all of a number of objects.

Ex: select 2 out of 3 boys A, B, C. Then possible selections are AB, BC, CA. (AB and BA represent same selection)

<u>Number of combinations</u>: The number of combinations of n things, taken r at a time is $^nC_r = \dfrac{n!}{r!\,(n-r)!}$

<u>Note</u>: $^nC_n = 1$ and $^nC_0 = 1$, $^nC_r = {}^nC_{n-r}$, $^nC_r \cdot r! = {}^nP_r$

Ex: $^{11}C_4 = \dfrac{11\times10\times9\times8}{4\times3\times2\times1} = 330$

$^{16}C_{13} = {}^{16}C_{16-13} = {}^{16}C_3 = \dfrac{16\times15\times14}{3\times2\times1} = 560.$

Ex: The no. of shake hands out of 10 friends if each shakehand with other $= {}^{10}C_2$ $(A \rightleftharpoons B)$

Ex: The no. of e-mails out of 10 friends if each giving e-mail to other $= {}^{10}P_2$ $(A \rightarrow B,\ A \leftarrow B)$

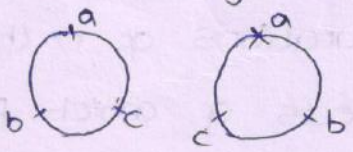
* selection of r things out of n things = no. of selections in which one particular thing is already there + no. of selections in which that particular thing is not there.

$$^nC_r = {}^{(n-1)}C_{(r-1)} + {}^{(n-1)}C_r$$

Ex: $^6C_3 = {}^5C_2 + {}^5C_3 = 20$

<u>Circular arrangements</u>: The no. of ways to arrange 'n' things in a circular order is given by $(n-1)!$

Ex: The no. of ways to arrange a, b, c in a circular order is 2 ways they are



In case of circle there is no starting & ending points. once we fix one of the place, as e then the remaining n-1 places can be arranged themselves like in a row in $(n-1)!$ ways.

Arrangements

	In a row					In a circular order	
	6 boys & 6 Girls	6 boys & 5 girls	6 boys & 4 girls	6A's & 6B's	6A's & 6 Girls	6 boys & 6 girls	6 boys & 5 girls
1. Total ways	$12!$	$11!$	$10!$	$\frac{12!}{6!\,6!}$	$\frac{12!}{6!}$	$11!$	$10!$
2. All girls (B's) must be together	$7!\,6!$	$7!\,5!$	$7!\,4!$	$\frac{7!}{6!} = 7$	$\frac{7!}{6!} \times 6! = 7!$	$6!\,6!$	$6!\,5!$
3. No two girls (B's) must be together	$6!\,[{}^7C_6 \times 6!]$	$6!\,[{}^7C_5 \times 5!]$	$6!\,[{}^7C_4 \times 4!]$	$\frac{6!}{6!} \times {}^7C_6$	$\frac{6!}{6!} \times {}^7C_6 \times 6!$	$5!\,6!$	$5!\,[{}^6C_5 \times 5!]$
4. All girls (B's) not be together	$12! - 7!\,6!$	$11! - 7!\,5!$	$10! - 7!\,4!$	$\frac{12!}{6!\,6!} - 7$	$\frac{12!}{6!} - 7!$	$11! - 6!\,6!$	$10! - 6!\,5!$
5. No two boys (A's) & no two girls (B's) must be together	$6!\,6! + 6!\,6!$	$6!\,5!$	0	$1 \times 1 + 1 \times 1$	$1 \times 6! + 1 \times 6!$	$5!\,6!$	0

Note: If difference b/w no. of boys & girls is 2 or more then no 2 boys & no 2 girls sit together in a <u>row</u> is not possible.

Note: If difference b/w no. of boys & girls is 1 or more then no 2 boys & no 2 girls sit together in a <u>circle</u> is not possible

Elementary event : outcome of a random. experiment

Sample space: set of all possible outcomes. denoted by S.

Ex: Tossing a coin $S = \{H, T\}$

Rolling a die $S = \{1, 2, 3, 4, 5, 6\}$

The event represented by set 'S' is a sure event so, it always occurs.

Impossible event: If there is no outcome of a experiment it is called impossible event. and denoted by ϕ.

Equally likely event: If all the elementary events of 'S' have the same chance of occuring, then the events are said to be equally likely events.

Mutually exclusive Events: Two events cannot occur together

Ex: Let 2 dice are thrown.

E_1: sum of the numbers on the dice ≤ 4 $= \{2, 3, 4\}$

E_2: sum of the numbers on the dice > 9 $= \{10, 11, 12\}$

sample space $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$E_1 \cap E_2 = \phi$

Mutually exhaustive events:

If $E_1 \cup E_2 \cup E_3 \ldots \cup E_m = S$ then the events are exhaustive.

* If both $E_1 \cup E_2 \cup E_3 \ldots \cup E_m = S$ and $E_i \cap E_j = \phi$ $i \neq j$ occur then the events are called mutually exclusive & exhaustive.

Combination of events:

$E_1 \cup E_2$ = E_1 or E_2 or both occur

$E_1 \cap E_2$ = E_1 and E_2 both occur

E_1^c or $\bar{E}_1$ = set of elements of S which do not belong to E_1

$E_1 - E_2$ = Event E_1 but not E_2

* $A \cap \bar{A} = \phi$, A and $\bar{A}$ are mutually exclusive.

Probability of an event:

$$P(A) = P = \frac{\text{number of elementary events favourable to } A}{\text{Total number of equally likely elementary events}}$$

* $0 \le P(A) \le 1$
* $P(\phi) = 0$ and $P(S) = 1$
* Law of addition: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

* If A & B are mutually exclusive then $A \cap B = \phi$

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

* $P(\text{not } A) = P(\bar{A}) = 1 - P(A)$

Tossing coins:

1 coin $\rightarrow$ 2 ways $\rightarrow \{T, H\}$

2 coins $\rightarrow 2^2$ ways $\rightarrow \{TT, TH, HT, HH\}$

$\vdots$

n coins $\rightarrow 2^n$ coins

Prob: Two coins are tossed together what is the probability of getting

(i) Same face: $\frac{2}{2^2}$ $\{HH, TT\}$

(ii) More heads than tails: $\frac{1}{2^2}$ $\{HH\}$

Prob: If 3 coins tossed

(i) $\frac{2}{2^3}$ $\{HHH, TTT\}$

(ii) $\frac{{}^3C_2 + {}^3C_3}{2^3}$ $\{HHT, HTH, THH, HHH\}$

H	T		
0	3	×	
1	2	×	
2	1	✓	$\rightarrow {}^3C_2$
3	0	✓	$\rightarrow {}^3C_3$

Prob: If 4 coins tossed

(i) $\frac{2}{2^4}$ $\{HHHH, TTTT\}$

(ii) $\frac{{}^4C_3 + {}^4C_4}{2^4}$

H	T		
0	4		
1	3		
2	2		
3	1	✓	$\rightarrow {}^4C_3$
4	0	✓	$\rightarrow {}^4C_4$

prob: A coin is tossed until it shows the same face in two consecutive throws. what is the probability of success by tossing a coin not more than 4 times.

Sol: Success at 2nd time HH, TT $\rightarrow$ 2 ways

" at 3rd time HTT, THH $\rightarrow$ 2 ways

" at 4th time THTT, HTHH $\rightarrow$ 2 ways

$$\text{Probability} = \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4} = \frac{7}{8}$$

prob: Not more than 3 times

Sol: $$\text{probability} = \frac{2}{2^2} + \frac{2}{2^3} = \frac{3}{4}$$

prob: A & B simultaneously toss a coin till one of them gets head. If A starts the game then what is the probability for

(i) B to win (ii) A to win

Sol: (i) A started $\rightarrow$ B win

TH TTTH TTTTTH $\qquad a + ar + ar^2 + \cdots = \frac{a}{1-r}$ if $|r|<1$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1/4}{1-1/4} = \frac{1}{3}$$

(ii) A started $\rightarrow$ A win

H TTH TTTTH

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{1/2}{1-1/4} = 2/3$$

prob: A coin is tossed until which shows head. what is the probability of success by tossing a coin an odd no. of times

Sol: H TTH TTTTH

$$1/2 + 1/2^3 + 1/2^5 + \cdots = \frac{1/2}{1-1/4} = 2/3$$

prob: An even no. of times

Sol: TH TTTH

$$1/2^2 + 1/2^4 + 1/2^6 + \cdots = \frac{1/4}{1-1/4} = 1/3$$

Rolling dice: 1 die - 6 ways $\{1, 2, 3, 4, 5, 6\}$

2 dice - 6^2 ways $\{11, 12, 13, 14, 15, 16, 21, 22, 23 \ldots\}$

3 dice - 6^3 ways

Prob: Two dice are rolled together. what is the probability of getting

(i) Same face $6/6^2 = 1/6$ $\{11, 22, 33, 44, 55, 66\}$

(ii) Sum as 10 $3/6^2$ $\{6.4, 46, 55\}$

(iii) Sum morethan 10 $3/6^2$ $\{56, 65, 66\}$

(iv) First number more than second num. $15/6^2$

Note: When two dice are rolled then $A = B \rightarrow 6$ ways

$A > B \rightarrow 15$ ways

$B > A \rightarrow 15$ ways.

prob: 3 dice rolled

(i) same face $\frac{6}{6^3} = 1/36$

(ii) Sum as morethan 16 $= \frac{3+1}{6^3}$ $\{(6,6,5), (5,6,6), (6,5,6), (6,6,6)\}$

prob: 4 dice rolled

(i) Same face $= \frac{6}{6^4}$

(ii) Sum as 20 $= \frac{35}{6^4}$

5 5 5 5 $\rightarrow$ 1 way

5 5 6 4 $\rightarrow \frac{4!}{2!\,2!} = 12$

6 6 5 3 $\rightarrow \frac{4!}{2!} = 12$

6 6 4 4 $\rightarrow \frac{4!}{2!\,2!} = 6$

6 6 6 2 $\rightarrow \frac{4!}{3!} = 4$

35 ways

(iii) Sum as 21 $= \frac{20}{6^4}$

6 5 5 5 $\rightarrow 4$

6 6 5 4 $\rightarrow 12$

6 6 6 3 $\rightarrow 4$

20

Prob: A & B simultaneously roll dice till one of them gets the number '4'. If A starts the game, what is the probability for

(i) B to win

$$\frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \cdots = \frac{5/36}{1 - \frac{25}{36}} = \frac{5}{11}$$

(A lost, B won, A lost, B lost, A lost, B won)

(ii) A to win $1 - 5/11 = 6/11$

prob: There are two biased dice of which 1st dice shows an even number twice as frequently as odd number, 2nd dice shows 5, thrice as frequently as any other number. If these dice are rolled together what is the probabity of getting

(i) Sum as 10 (ii) Sum as morethan 10

	dice1	dice2
1	k	l
2	2k	l
3	k	l
4	2k	l
5	k	3l
6	2k	l
	$9k = 1$	$8l$
	$k = 1/9$	$l = 1/8$

(i) Sum as 10

(5,5), (6,4), (4,6)

$$\frac{1}{9} \times \frac{3}{8} + \frac{2}{9} \times 1/8 + \frac{2}{9} \times 1/8$$

(ii) Sum more than 10

(5,6), (6,5), (6,6)

$$\frac{1}{9} \times \frac{1}{8} + \frac{2}{9} \times \frac{3}{8} + \frac{2}{9} \times 1/8 =$$

Pack of cards:

52
- 26 black
  - 13 spades ♠
    - Honours (4): A, J, Q, K — face cards
    - Number (9): 2 to 10
  - 13 clubs ♣
- 26 Red
  - 13 diamonds ◇
  - 13 hearts ♡

Suits

prob: Two cards are drawn at random from a pack of cards both must belong to same suit & both must belong to different suits.

sol: (i) $\frac{4C_1 \cdot 13C_2}{52C_2}$ (ii) $\frac{4C_2 \cdot 13C_1 \cdot 13C_1}{52C_2}$

prob: Three cards

sol: (i) $\frac{4C_1 \cdot 13C_3}{52C_3}$ (ii) $\frac{4C_3 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1}{52C_3}$

prob: Four cards

sol: (i) $\frac{4C_1 \cdot 13C_4}{52C_4}$ (ii) $\frac{4C_4 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1}{52C_4}$

Prob: A card is drawn at random from a pack of cards. What is the probability that it is (i) King (ii) King or red (iii) King or Queen (iv) King or face card (v) King and Queen.

Sol: (i) $4/52$

(ii) $P(K \cup R) = P(K) + P(R) - P(K \cap R)$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

(iii) $P(K \cup Q) = P(K) + P(Q)$ K, Q are mutually exclusive

$$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$$

(iv) $P(K \cup F) = P(K) + P(F) - P(K \cap F)$

$$= \frac{4}{52} + \frac{12}{52} - \frac{4}{52} = 12/52$$

(v) $P(K \cap Q) = 0$

Prob: A card is drawn from a pack of 52 cards. Find the probability that (i) neither diamond nor a face (ii) neither 10 nor a King comes.

Sol: (i) $P(D^c \cap F^c) = 1 - P(D \cup F) = 1 - [P(D) + P(F) - P(D \cap F)]$

$$= 1 - \left[\frac{13}{52} + \frac{12}{52} - \frac{3}{52}\right]$$

(ii) $P(10^c \cap K^c) = 1 - P(10 \cup K) = 1 - [P(10) + P(K) - P(10 \cap K)]$

$$= 1 - \left[\frac{4}{52} + \frac{4}{52} - 0\right] = \frac{44}{52}$$

Leap year concept:

Leap year	Non leap year
366 days	365 days
52 weeks + 2 days	52 weeks + 1 day
↓	↓
SM, MT, TW, WT, TF, FS, SS	S, M, T, W, T, F, S.

Prob: The probability for a leap year selected at random to have (i) 53 sundays (ii) 52 sundays is

Sol: (i) $2/7$ (ii) $5/7$.

prob: probability for non-leap year to have (i) 53 sundays (ii) 52 sundays.

sol: (i) $1/7$ (ii) $6/7$

Multiplication theorem / conditional probability:

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$$

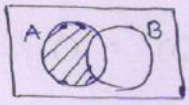
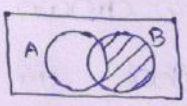
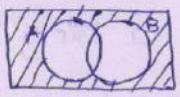
(conditional probability)

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$$

Independent events: Two events are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

6. $P(A^c \cup B^c) = P((A \cap B)^c)$
$= 1 - P(A \cap B)$

* If A & B are two events then

1. $P(A) + P(A^c) = 1$
2. $P(A \cap B^c) = P(A) - P(A \cap B)$ only 'A' 
3. $P(A^c \cap B) = P(B) - P(A \cap B)$ only 'B' 
4. $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$ Neither A nor B 
5. $P(A^c/B) = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)}$

$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A/B)$$

prob: Two cards are drawn one after the other <u>without</u> replacement what is the probability that 1st is king and 2nd is queen

sol: $\frac{4}{52} \times \frac{4}{51}$

prob: 1st card is known to be king, what is the probability for 2nd card is queen

$4/51$

prob: If it is <u>with</u> replacement then what is the probability that 1st is king and 2nd is queen is

$\frac{4}{52} \times \frac{4}{52}$

Total probability theorem: Let $B_1, B_2, \ldots B_n$ be a set of mutually exclusive and exhaustive events of the sample space 'S' with $P(B_k) \neq 0$, $k = 1, 2, \ldots n$ and let A be any event of 'S' Then

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

prob: A factory has four independent units A, B, C, D which produce 40%, 30%, 20%, 10% of identical items respectively. The percentage of defective items produced by these units are 2%, 1%, 0.5%, 0.25% respectively. If an item is selected at random, find probably that the item is defective.

Sol: M : defective item

$$P(M) = P(A) \cdot P(M/A) + P(B) \cdot P(M/B) + P(C) \cdot P(M/C) + P(D) \cdot P(M/D)$$

$$= 0.4(0.02) + 0.3(0.01) + 0.2(0.5) + 0.1(0.25) = 0.01225$$

prob: Box A contains 6 red and 4 blue balls, box B contains 3 red and 7 blue balls. Two balls are drawn at random from box A and placed in box B. Now a ball is drawn at random from Box B. Find the probability that it is blue ball.

Sol:

Balls taken from A	probability	Box B contains	probability of drawing blue ball
2 red	$\frac{6C_2}{10C_2} = 1/3$	5 red, 7 blue	$7C_1/12C_1 = 7/12$
1 red, 1 blue	$\frac{6C_1 \cdot 4C_1}{10C_2} = \frac{8}{15}$	4 red, 8 blue	$\frac{8C_1}{12C_1} = 2/3$
2 blue	$\frac{4C_2}{10C_2} = \frac{2}{15}$	3 red, 9 blue	$\frac{9C_1}{12C_1} = 3/4$

$$P(E) = P(B_1) \cdot P(E/B_1) + P(B_2) P(E/B_2) + P(B_3) \cdot P(E/B_3)$$

$$= \frac{1}{3} \cdot \frac{7}{12} + \frac{8}{15} \cdot \frac{2}{3} + \frac{2}{15} \cdot \frac{3}{4} = 0.65$$

****Baye's theorem: (probability of causes)**

$$P(B_k/A) = \frac{P(B_k) \cdot P(A/B_k)}{\sum_{k=1}^{n} P(B_k) \cdot P(A/B_k)}$$

Taking ball from a bag / box:

prob: There are 2 bags of which 1st bag contains 5 red & 4 green balls, 2nd bag contain 6 red & 7 green balls. A ball is drawn at random from one of the bags

(i) What is the probability that it is a red ball

$$\frac{1}{2} \times \frac{5}{9} + \frac{1}{2} \times \frac{6}{13}$$

(ii) It is found to be ~~round~~ red. What is the probability that it is obtained from 2nd bag

$$= \frac{1/2 \times 6/13}{1/2 \times 5/9 + 1/2 \times 6/13} =$$

(iii) If chance of selection of 1st bag is 1/3 & 2nd bag is 2/3 then

$$= \frac{2/3 \times 6/13}{1/3 \times 5/9 + 2/3 \times 6/13}$$

prob: A bag A contains 8 white and 4 black balls. A second bag B contains 5 white and 6 black balls. One ball is drawn at random from bag A and is placed in Bag B. Now, a ball is drawn at random from bag B. It is found that this ball is white. Find the probability that black ball has been transfered from bag A.

Sol.

B_1: Transfer white ball

$$P(B_1) = \frac{8}{12} = \frac{2}{3}$$

Now bag B contains 6 white and 6 black balls

B_2: Transfer ~~white~~ black ball

$$P(B_2) = \frac{4}{12} = \frac{1}{3}$$

Now bag B contains 5 white & 7 black balls

E: Drawing white ball from Bag B

$$P(E/B_1) = \frac{6}{12} = \frac{1}{2}$$

$$P(E/B_2) = \frac{5}{12}$$

probability that black ball is transfered

$$P(B_2/E) = \frac{P(B_2) \cdot P(E/B_2)}{P(E)} = \frac{5/36}{17/36}$$

$$= \frac{5}{17}$$

**

Prob: The probability for a man to know the answer for a question in an exam is 1/3. If he don't know the answer, then he guess the answer. The probability for the guessed answer to be correct is 1/4. He answered one question and is found to be correct. What is probability that he knows the answer.

Sol: E_1 : Knows answer $\quad P(E_1) = 1/3$

E_2 : guess answer (Don't know answer) $\quad P(E_2) = 1 - 1/3 = 2/3$

A : Answer is correct

$P(A/E_1)$: Probability for known answer to be correct (Not given)

** Assume $P(A/E_1) = 1$

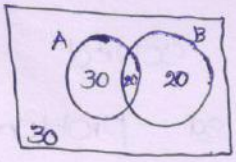
$P(A/E_2)$: probability for guessed answer to be correct $= 1/4$

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{1/3 \cdot 1}{1/3 \cdot 1 + 2/3 \cdot 1/4} = 2/3$$

Venn diagrams:

Prob: In a town 50% people read newspaper A, 40% read B, 20% read both A and B. A person is selected at random what is the probability that for him to read



(i) Atlease one = only one + Both

$$= \left(\frac{30}{100} + \frac{20}{100}\right) + \frac{20}{100} = \frac{70}{100}$$

(ii) Atmost one = No one + only one

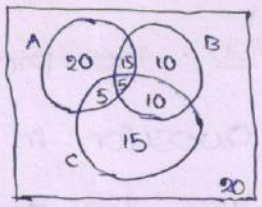
$$= \frac{30}{100} + \left(\frac{30}{100} + \frac{20}{100}\right) = \frac{80}{100}$$

(iii) Exactly one $= \frac{30}{100} + \frac{20}{100} = \frac{50}{100}$

(iv) None $= \frac{30}{100}$

Prob: In a town 45% read A, 40% read B, 35% read C, 20% read A&B 15% read B&C, 10% read A&C, 5% read A&B&C. A person

selected at random. What is the prob. that



(i) Atleast one = 80/100

(ii) Atmost one = $\frac{20}{100} + \frac{20}{100} + \frac{10}{100} + \frac{15}{100} = \frac{65}{100}$

(iii) Exactly one = $\frac{20}{100} + \frac{10}{100} + \frac{15}{100} = \frac{45}{100}$

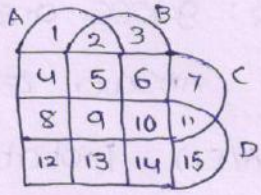
(iv) None = $\frac{20}{100}$

(v) Atleast two = two + Three = $\frac{5}{100} + \frac{15}{100} + \frac{10}{100} + \frac{5}{100} = \frac{35}{100}$

(vi) Atmost two = None + one + two = $\frac{20}{100} + \frac{20}{100} + \frac{15}{100} + \frac{10}{100} + \frac{15}{100} + \frac{10}{100} + \frac{5}{100} = \frac{95}{100}$

(vii) Exactly two = $\frac{30}{100}$

Note: If 4 variables then venn diagram



Atleast one to happen:

prob: A room has 3 bulb holders. A bag contains 15 bulbs of which 5 are fused. 3 bulbs selected at random to fit into holders. What is the probability that rom get lighted

sol: Room get lighted = Atleast one light should glow

= 1 − No light should glow

$= 1 - \frac{5C_3}{15C_3}$

prob: The probability for A to solve a problem is 1/3. that of B to solv is 1/7. What is the probability that problem will be solved

sol: problem solved = Atleast one should solve

= 1 − No one should solve

$= 1 - \frac{2}{3} \cdot \frac{2}{7} = \frac{21-4}{21} = 17/21$

prob: The probability for a man to hit a target is 1/3. What is the probability for him to hit the target atleast once out of 5 chances.

sol: probability to hit atleast once = 1 − Not hitting in 5 chances

$= 1 - \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 1 - \left(\frac{2}{3}\right)^5$

Miscellaneous problems:

1. prob: A number is selected at random from 1st 200 natural numbers. Find the probability that it is divible by 6 or 8.

Sol: $\frac{33}{200} + \frac{25}{250} - \frac{8}{200} = \frac{1}{4}$

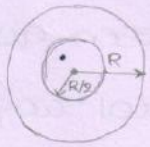
$6)\,200\,(33$, 198, 2

$8)\,200\,(25$, 200, 0

$2\,|\,8, 6$
$4, 3$
L.C.M = 24

$24)\,200\,(8$, 192, 8

2. prob: A point is selected at random inside circle. What is the probability that the point is closer to the center than to it's circumference.



Sol: For the point is to be closer to center, it should inside the circle with radius R/2

$\therefore$ probability $= \frac{\pi(R/2)^2}{\pi R^2} = \frac{1}{4}$

3. Prob: In a certain college 4% men & 1% women are taller than 1.8m, furthermore 60% of students are women. A student is selected at random & is taller than 1.8m. What is the probability that the student is a women?

Sol: E_1 : Student is men

E_2 : Student is women

A : Taller than 1.8m

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{\frac{60}{100} \times \frac{1}{100}}{\frac{40}{100} \times \frac{4}{100} + \frac{60}{100} \times \frac{1}{100}} = 3/11$$

4. prob: A coin is weighted so that $P(H) = 2/3$, $P(T) = 1/3$ is tossed. If head appears then a number is selected at random from 1 to 9. If tail appears a number is " " " " 1 to 5. Find the probabity of getting an even number.

$$= \underbrace{\frac{2}{3}}_{P(H)} \times \underbrace{\frac{4}{9}}_{\text{Even number from 1-9}} + \underbrace{\frac{1}{3}}_{P(T)} \times \underbrace{\frac{2}{5}}_{\text{Even number from 1-5}}$$

5. prob: A determinant is selected at random from set of all determinants of order 2×2 whose entries are either zero or one. only. What is the probability that the choosen determinant is

positive?

sol: $\begin{bmatrix} - & - \\ - & - \end{bmatrix} = 2\times2\times2\times2 = 16$ ways.

$\underbrace{ad}_{\text{Equal to '1'}} - \underbrace{bc}_{\text{Equal to '0'}} > 0$ only possibility $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 3$

probability $= 3/16$

6. Prob: A quadratic eqn of the form $ax^2+bx+c=0$ is selected at random from all possibles. whose co.eff's a, b, c are distinct and are selected from the set $\{1,2,3,4,6,8,9\}$. What is the probability that choosen quatratic equation to have equal roots?

sol: Total ways $= {}^7P_3$

Equal roots $\Rightarrow b^2 = 4ac$ $\rightarrow 6^2 = 4.1.9$, $\rightarrow 6^2 = 4.9.1$

$$\text{probability} = \frac{2}{{}^7P_3}$$

Note: If $P(E) = m/n$

odds in favour of E is $m:(n-m)$

odds against of E is $(n-m):m$

7. Prob: The odds are 2 to 5 in favour of A to solve a problem & 3 to 7 in against B to solve the problem. What is the probability for problem to be solved?

sol: $m : n-m = 2:5$ $\qquad$ $n-m : m = 3:7$

$m=2,\ n=7$ $\qquad$ $m=7,\ n=10$

$P(A) = 2/7$ $\qquad$ $P(B) = 7/10$

probability for problem to be solved $= 1 - 5/7 \cdot 3/7 = 55/70$

RANDOM VARIABLE:

A variable whose value cannot be predicted but it is one among the several possibilities is called a Random variable (R.V)

Ex: No. of Heads turns up when tossing a coin 3 times is a R.V. Here R.V takes values 0, 1, 2, 3

We can distribute probability for R.V as below.

x	0	1	2	3
$P(x=x_i)$	1/8	3/8	3/8	1/8

1-Dimensional R.V: Assigning a real value to each outcome of an experiment is called 1 Dimensional R.V. The corresponding data is known as univariate data.

2-Dimensional R.V: connecting 2 outcomes at a time to the one real value provided that, those outcomes are drawn from same sample space. The corresponding data is known as bi-variate data.

Discrete Random variable: If a random variable x takes a finite number or countably infinite number of values, then x is called a discrete R.V. We denote the possible values taken by x as $x_1, x_2, \ldots x_n$.

Let $P(x=x_i)=P_i$ then P_i is called the probability function if it satisfies the following conditions

(i) $P_i \geq 0$ for all i

(ii) $\sum_i P_i = 1$

The collection of pairs (x_i, P_i), $i=1,2,3\ldots$

x	x_1	x_2	x_3	----
$P(x=x_i)$	P_1	P_2	P_3	---

is called the probability distribution or discrete probability distribution or probability mass function

cumulative distribution function: The function $F(x)$ defined by $F(x) = P(X \le x)$

$= \sum_{x_i \le x} p(x_i)$ is called cumulative distribution function.

We can write $p(x_i) = P(x = x_i) = F(x_i) - F(x_{i-1})$

properties:
1. $0 \le F(x) \le 1$
2. If $x_1 < x_2$ then $F(x_1) \le F(x_2)$
3. $P(a \le X \le b) = F(b) - F(a)$

Expectation: The expectation of a discrete R.V 'x' is defined as

$$E(x) = \sum_{i=0}^{n} x_i \cdot p(x_i)$$

Mean $\mu_x = E(x) = \sum_{i=0}^{n} x_i p(x_i) = \sum_i x_i p_i$

variance $\sigma_x^2 = E[(x-\mu_x)^2] = \sum_i (x_i - \mu_x)^2 p_i$

$= E(x^2) - [E(x)]^2$

$= \sum_i x_i^2 p_i - \mu_x^2$

Prob: A R.V x has following probability distribution

x	0	1	2	3	4
$P(x)$	c	$2c$	$2c$	c^2	$5c^2$

Find c? Evaluate $P(x<3)$, $P(0<x<4)$. Determine distribution function of x. Find the mean and variance of x.

sol:

$\sum_{x=0}^{4} P(x) = 1 \Rightarrow c + 2c + 3c + c^2 + 5c^2 = 1$

$\Rightarrow 6c^2 + 5c - 1 = 0$

$\Rightarrow c = 1/6, -1$ ↳ not possible, because $p(x) \ge 0$

$\therefore c = 1/6$

$P(x<3) = P(x=0) + P(x=1) + P(x=2)$

$= c + 2c + 2c = 5c = 5/6$

$P(0<x<4) = P(x=1) + P(x=2) + P(x=3)$

$= 2c + 2c + c^2 = 25/36$

Probability distribution and distribution function are

x	0	1	2	3	4
$p(x)$	$1/6$	$2/6$	$2/6$	$1/36$	$5/36$
$F(x)$	$1/6$	$3/6$	$5/6$	$31/36$	1

Mean $\mu_x = \Sigma x_i p_i = 0 + \frac{2}{6} + \frac{4}{6} + \frac{3}{36} + \frac{20}{36} = \frac{59}{36}$

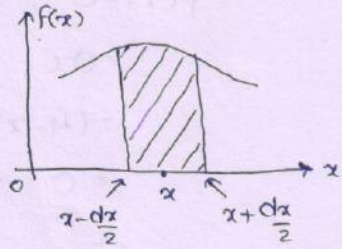
variance $\sigma_x^2 = E(x^2) - [E(x)]^2$

$$= \left[0\left(\frac{1}{6}\right) + 1\left(\frac{2}{6}\right) + 4\left(\frac{2}{6}\right) + 9\left(\frac{1}{36}\right) + 16\left(\frac{5}{36}\right)\right] - \left(\frac{59}{36}\right)^2 = 1.4529$$

<u>Continous</u> <u>Random</u> <u>variable</u>: A R.V x is said to be continous R.V if it takes all possible values in a given interval I_x.

<u>probability</u> <u>density</u> <u>function</u>: consider a small interval $(x - \frac{dx}{2}, x + \frac{dx}{2})$ of length dx about the point 'x'. Let $f(x)$ be a continous function so that $f(x)dx$ represents probability that x falls in this interval. That is

$$P\left(x - \frac{dx}{2} \le x \le x + \frac{dx}{2}\right) = f(x)\,dx$$



The function $f(x)\,dx$ represents area under the curve $f(x)$. The function $f(x)$ is called probability density function (pdf) of x.

properties: (i) $f(x) \ge 0$ for all $x \in I_x$

(ii) $\int_{I_x} f(x)\,dx = 1$

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

Now $P(x = a) = P(a \le x \le a) = \int_a^a f(x)\,dx = 0$

$$\therefore P(a \le x \le b) = P(a < x \le b) = P(a \le x < b) = P(a < x < b)$$

Mean $= \int_{I_x} x f(x)\,dx = \mu_x$

variance $= \sigma_x^2 = \int_{I_x} (x - \mu_x)^2 f(x)\,dx$

Cumulative density function (cdf):

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

For the interval $I_x = [a, b]$

$$\text{Mean} = \mu = \int_a^b x f(x)\,dx$$

$$\text{Variance} = \sigma^2 = \int_a^b x^2 f(x)\,dx - (\mu)^2$$

Properties: (i) $0 \leq F(x) \leq 1$, $-\infty < x < \infty$

(ii) $F(x)$ is a non decreasing function i.e if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$

(iii) $F(\infty) = 1$ and $F(-\infty) = 0$

(iv) $F(x)$ may have countable number of discontinuities

(v) $f(x) = F'(x)$ at all points where $F(x)$ is differentiable.

prob: If the density function of a continous R.V 'X' is given by

$f(x) = 0$, $x < 0$

$= ax$, $0 \leq x \leq 2$

$= (4-x)a$, $2 \leq x \leq 4$

$= 0$, $x > 4$

(i) Find a?

(ii) Find cdf of X

(iii) Find $P(X > 2.5)$

sol: (i) $\int_0^4 f(x)\,dx = 1$

$$\int_0^2 ax\,dx + \int_2^4 a(4-x)\,dx = 1 \Rightarrow 2a + 2a = 1 \Rightarrow a = 1/4$$

(ii) $x < 0$: $F(x) = 0$

$0 \leq x \leq 2$: $F(x) = \int_0^x \frac{x}{4}\,dx = \frac{x^2}{8}$

$2 \leq x \leq 4$: $F(x) = \int_0^2 \frac{x}{4}\,dx + \int_2^x \frac{1}{4}(4-x)\,dx = \frac{1}{8}(8x - x^2 - 8)$

$x > 4$: $F(x) = 1$

(iii) $P(X > 2.5) = \int_{2.5}^4 \frac{1}{4}(4-x)\,dx$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_{2.5}^4 = 9/32$$

prob: If the probability density function of a continous R.V is given by $f(x) = e^x$, $0 \leq x < \infty$. Find mean and variance?

Sol: Mean $= \mu = \int_0^\infty x \cdot e^{-x} dx = -[(x+1)e^{-x}]_0^\infty = 1$

Variance $= \int_0^\infty x^2 e^{-x} dx - (\mu)^2$

$= (-x^2 e^{-x})_0^\infty + 2\int_0^\infty x e^{-x} dx - 1 = 2 - 1 = 1$

Properties of Expectation:

1. If X is a R.V and 'a' is a constant, then $E(ax) = aE(x)$
2. If X & Y are R.V's then $E(X+Y) = E(X) + E(Y)$

   $E(X-Y) = E(X) - E(Y)$

   $E(X \cdot Y) = E(X) \cdot E(Y/X) = E(Y) \cdot E(X/Y)$
3. If X & Y are independent R.V's $E(X \cdot Y) = E(X) \cdot E(Y)$
4. If $Y = aX + b$ where a and b are constants. Then $E(Y) = aE(X) + b$
5. E(constant) = constant

Properties of variance:

1. If X is a R.V and 'a' is a constant

   $V(aX) = a^2 V(X)$

   $V(-Y) = (-1)^2 V(Y) = V(Y)$
2. If X and Y are independent R.V's $V(x \pm y) = V(x) + V(y)$
3. If a and b are constant and X and Y are independent R.V's

   $V(aX - bY) = a^2 V(X) + b^2 V(Y)$

   $V\left(\frac{X}{a} - \frac{Y}{b}\right) = \frac{1}{a^2} V(X) + \frac{1}{b^2} \cdot V(Y)$
4. If $Y = aX + b \Rightarrow V(Y) = V(aX+b)$

   $= a^2 V(X) + 0$ $[\because V(\text{const}) = 0]$

   $= a^2 \cdot V(X)$
5. If X & Y two R.V's (dependent R.V's)

   $V(X+Y) = V(X) + V(Y) + 2\,\text{cov}(X,Y)$

   $\downarrow$

   Covariance of (X,Y)

   $\text{Cov}(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

   $\rightarrow \text{cov}(X,X) = E(X^2) - (E(X))^2 = V(X)$

   $\rightarrow \text{cov}(a,b) =$ ~~cov(ab) = Cov~~

   $= E(ab) - E(a) \cdot E(b) = ab - ab = 0$

Note: The mean and variance for the sum of the numbers on the dice is $E(x) = \frac{7n}{2}$ $V(x) = \frac{35}{12}n$ where 'n' is number of dice rolled.

Prob: 3 unbiased dice are thrown. Find the mean and variance for the sum of the numbers on them?

$$E(x) = \frac{7n}{2} = \frac{7\times3}{2} = \frac{21}{2}$$

$$V(x) = \frac{35}{12}n = \frac{35\times3}{12} = 35/4$$

prob: Two unbiased dice are rolled. Find the expectation for sum 7 on them?

$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

Sol: x is assuming only 7.

$$E(\text{Sum } 7) = 7.P(7) = 7\times\frac{6}{36} = 7/6$$

prob: A player tosses 3 coins. He gains 500 rupees if '3' heads occur. 300 rupees if 2 heads occur, 100 rupees if only 1 head occur. On the otherhand he looses 1500 rs if 3 tails occur. Find value of the game?

Sol: x: No. of heads possibility

x	3	2	1	0
P(x)	1/8	3/8	3/8	1/8

VALUE OF GAME = GAIN - LOSE = Gain × it's prob. - Lose × it's prob.

$$E(x) = 500\times\frac{1}{8} + 300\times3/8 + 100\times3/8 - 1500\times1/8$$

$$= \frac{200}{8} = 25$$

Note: If Game is said to be fair, the expected value of game is said to be 0. (No loss, No gain)

prob: If 'x' is a continous R.V, and $f(x) = Kx^2e^{-x}$, $0<x<\infty$

(i) Find value of K (ii) Mean and variance.

Sol:

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Kx^2e^{-x}dx = 1$$

Gamma function $\Gamma_n = \int_0^{\infty} e^{-x}.x^{n-1}dx$; $n>0$

$$\Gamma_n = (n-1)\Gamma(n-1) = (n-1)!$$ $[\Gamma 1 = 1, \Gamma 0 = \text{Not defined}]$

$K\Gamma 3 = 1 \Rightarrow K \cdot 2! = 1 \Rightarrow K = 1/2$

(ii) Mean $E(x) = \int_0^\infty x \cdot f(x)\,dx$

$$= \frac{1}{2}\int_0^\infty x \cdot x^2 e^{-x}\,dx = \frac{1}{2}\int_0^\infty x^3 e^{-x}\,dx$$

$$= \frac{1}{2}\Gamma(4) = \frac{1}{2} \times 3! = 3.$$

variance $E(x^2) - [E(x)]^2 = \frac{1}{2}\int_0^\infty x^2 \cdot x^2 e^{-x}\,dx - (3)^2$

$$= \frac{1}{2}\int_0^\infty x^4 e^{-x}\,dx - 9$$

$$= \frac{1}{2}\Gamma 5 - 9 = \frac{1}{2} \times 4! - 9 = 3$$

Prob: If $f(x) = |x|$ $-1 < x < 1$. Find variance?

Sol: $E(x) = \int_{-1}^{1} x\, f(x)\,dx = \int_{-1}^{1} x\,|x|\,dx$

$$= -\int_{-1}^{0} x^2\,dx + \int_0^1 x^2\,dx = 0$$

odd fun × ~~odd~~ even fun = odd fun

$\int_{-a}^{a}$ odd fun $dx = 0$

$E(x^2) = \int_{-1}^{1} x^2 \cdot f(x)\,dx = \int_{-1}^{1} x^2\,|x|\,dx$

$$= 2 \times \int_0^1 x^3\,dx$$

$$= 2 \times \left(\frac{x^4}{4}\right)_0^1 = 2 \times 1/4 = 1/2$$

even × even = even

$\int_{-a}^{a}$ even $= 2\int_0^a$ even

variance $= E(x^2) - [E(x)]^2 = 1/2$

2-Dimensional Random variable:

Joint probability function or mass function of (x, y):

If (x, y) be a 2D-R.V then

$$P(x = x_i, y = y_i) = P(x = x_i \cap y = y_i) = P(x_i, y_j) = P_{ij}$$

P_{ij} is called joint probability function of (x, y)

(i) $P_{ij} \geq 0$, for all i, j

(ii) $\sum_i \sum_j P_{ij} = 1$

Marginal probability function:

Marginal prob. function of x, $P(x = x_i) = \sum_{j=1}^{n} P(x_i, y_j)$

Marginal prob. function of y; $P(y = y_i) = \sum_{i=1}^{m} P(x_i, y_j)$

Conditional probability function:

conditional probability function of x given $y = y_j$ is

$$P(x = x_i / y = y_j) = \frac{P(x = x_i \cap y = y_j)}{P(y = y_j)} = \frac{P_{ij}}{P(y = y_j)}$$

conditional probability function of y given $x = x_i$ is

$$P(y = y_j / x = x_i) = \frac{P[y = y_j \cap x = x_i]}{P(x = x_i)} = \frac{P_{ij}}{P(x = x_i)}$$

Two R.V'S x and y are said to be independent, if

$$P(x = x_i, y = y_j) = P(x = x_i) \cdot P(y = y_j)$$

Cumulative distribution function:

The CDF of (x, y) is defined by $F(x, y) = P(x \le x \text{ and } y \le y)$

$$= \sum_i \sum_j P_{ij}, \; y_j \le y, \; x_i \le x.$$

properties:

(i) $F(-\infty, y) = 0$, $F(x, -\infty) = 0$, $F(-\infty, \infty) = 1$

(ii) $F(x, y)$ is a monotonic non decreasing function.

(iii) $P(a < x < b, y \le y) = F(b, y) - F(a, y)$

(iv) $P(x \le x, c < y < d) = F(x, d) - F(x, c)$

(v) At all points of continuity of $f(x, y)$, $\frac{\partial^2 F}{\partial x \cdot \partial y} = f(x, y)$

continous:

Joint probability density function:

If $D \in R$ then $P((x, y) \in D) = \iint_D f(x, y)\, dx\, dy$

$$P(a \le x \le b, c \le y \le d) = \int_c^d \int_a^b f(x, y)\, dx \cdot dy$$

Properties: (i) $f(x, y) \ge 0$

(ii) $\iint_R f(x, y)\, dx\, dy = 1$

* cdf of (x, y) is $F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\, dx \cdot dy$

* Marginal density of x, $f_x(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

* Marginal density of y, $f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

Marginal distribution functions are $F_X(x) = \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f(x,y)\,dy \right] dx$

$$F_Y(y) = \int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f(x,y)\,dx \right] dy$$

$$\therefore f_X(x) = \frac{dF_X(x)}{dx} \qquad f_Y(y) = \frac{dF_Y(y)}{dy}$$

$\Rightarrow P(a \le x \le b) = \int_a^b f_X(x)\,dx$

$P(c \le y \le d) = \int_c^d f_Y(y)\,dy.$

$\Rightarrow$ conditional density of x given $y = f(x/y) = \dfrac{f(x,y)}{f_Y(y)}$

conditional density of y given $x = f(y/x) = \dfrac{f(x,y)}{f_X(x)}$

$\Rightarrow$ Two continous random variables are independent if $f(x,y) = f_X(x) f_Y(y)$.

<u>prob</u>: The Joint pdf of the R.V x and y is given by $f(x,y) = K(xy+y^2)$ $0 \le x \le 1$, $0 \le y \le 2$. Find (i) $P(>1)$ (ii) $P(x > 1/2, y < 1)$ (iii) $P(x+y \le 1)$

<u>sol</u>: (i) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) = 1$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{2} K(xy+y^2)\,dy\,dx = 1$$

$$\Rightarrow K \int_{x=0}^{1} \left[x \cdot \left(\frac{y^2}{2}\right)_0^2 + \left(\frac{y^3}{3}\right)_0^2 \right] dx = 1 \qquad \Rightarrow K = 3/11$$

$$P(y>1) = \frac{3}{11} \int_{x=0}^{1} \int_{y=1}^{2} (xy+y^2)\,dy\,dx = \frac{111}{132}$$

(ii) $P(x > 1/2, y < 1) = \dfrac{3}{11} \int_{x=1/2}^{1} \int_{y=0}^{1} (xy+y^2)\,dy\,dx = \dfrac{17}{176}$

(iii) $P(x+y \le 1) = \dfrac{3}{11} \int_{x=0}^{1} \int_{y=0}^{1-x} (xy+y^2)\,dy\,dx$

$$= \frac{3}{11} \int_{x=0}^{1} \left[x \left[\frac{y^2}{2} \right]_0^{1-x} + \left[\frac{y^3}{3} \right]_0^{1-x} \right] dx = \frac{3}{88}$$

Functions of Random variables:

Let x be a R.V defined on the sample space S and let g be a function suchthat $y = g(x)$ is also a R.V defined on S. Then pdf of y is given by $f_Y(y) = \left| \dfrac{dx}{dy} \right| f_X(x)$

<u>prob</u>: A continous R.V x has pdf $f(x) = x/2 \quad 0 < x < 2$
$= 0$ elsewhere.

Find the pdf of $y = 3x+2$

Sol: $y = 3x+2 \Rightarrow x = \frac{y-2}{3}$ $\quad 0<x<2 \Rightarrow 2<y<8$

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$$

$$f_Y(y) = \frac{1}{3}\cdot\frac{x}{2} = \frac{1}{18}(y-2) \qquad 2<y<8$$

↳ function of y

Transformation of 2-D R.V's:

U, V defined by transformations $u = u(x,y)$ and $v = v(x,y)$ then joint pdf $g_{UV}(u,v)$ of transformed variables U and V is given by

$$g_{UV}(u,v) = |J|\, f_{XY}(x,y)$$

where $|J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ = Jacobian

Prob: The pdf of R.V's (X,Y) is given by $f(x,y) = \frac{1}{4}e^{-\left(\frac{x+y}{2}\right)}$, $x>0$ $y>0$
$= 0$ elsewhere

Find the distribution of $(X-Y)/4$.

Sol: transformations $u = \frac{1}{4}(x-y)$, $v = y$

$\Rightarrow x = 4u+v$, $y = v$

$y>0 \Rightarrow v>0$

$x>0 \Rightarrow 4u+v>0$

$v > -4u$ if $u<0$

$v>0$ if $u \geq 0$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 4$$

$$g_U(u,v) = |J|\, f_{XY}(x,y)$$

$$= \frac{4}{4}\cdot e^{-(4u+2v)/2} = e^{-(2u+v)}, \quad -\infty<u<\infty$$

for $u<0$: $f_U(u) = \int_{-4u}^{\infty} e^{-(2u+v)}\,dv = e^{-2u}\cdot(e^{4u}) = e^{2u}$

for $v>0$: $f_U(u) = \int_0^{\infty} e^{-(2u+v)}\,dv = e^{-2u}$

$\therefore f_U(u) = e^{-2|u|} \quad -\infty<u<\infty$

*** PROBABILITY DISTRIBUTIONS:

Binomial or bernoullies distribution: A R.V X whose probability distribution is given by the formula $P(X=r) = {}^nC_r\, p^r q^{n-r}$ then x is called binomial R.V. Here n denotes no. of times experiment is conducted.

p = probability of success

q = probability of failure $= 1-p$

For Binomial distribution

Mean $E(x) = np$ $\qquad E[x^2] = n(n-1)p^2 + np$

$var(x) = np(p-1) = npq$

central moment (μ_k): $\mu_k = E\{x - E(x)\}^k$

For the binomial distribution $B(n,p)$, we have $x = r$, $E(x) = np$, $q = 1-p$

$$\mu_k = \sum_{r=0}^{n} (r-np)^k \, {}^nC_r \, p^r q^{n-r}$$

$$\mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\,\mu_{k-1}\right]$$

$\rightarrow \mu_0 = 1,\ \mu_1 = 0 \Rightarrow \mu_2 = npq$ = variance

$\mu_3 = npq(q-p)$ = 3rd central moment

Skewness: $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ Skewness defined interms of 'μ_3'

$\Gamma = \sqrt{\beta_1}$ Γ is also a measure of skewness.

Note: If $\mu_3 = 0 \Rightarrow \beta_1 = 0$ Then curve is symmetry

If $\mu_3 = -ve \Rightarrow$ Then curve is negatively skewed.

If $\mu_3 = +ve \Rightarrow$ Then curve is positively skewed.

* Let x be a binomial variate. Then the <u>moment generating function</u> (MGF) of x is defined by

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{n} e^{tr} \, {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} {}^nC_r (pe^t)^r q^{n-r}$$

$$= (q + pe^t)^n$$

Ex: When a coin is tossed 3 times. x denotes the R.V no. of heads turned up. Then here $p = q = 1/2$, $n = 3$ and

$P(x=0) = {}^3C_0 (1/2)^0 (1/2)^3 = 1/8$

$P(x=1) = {}^3C_1 (1/2)^1 (1/2)^2 = 3/8$ ---

x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8

Mean $= \Sigma x_i p_i = \dfrac{3+6+3}{8} = 3/2 = np$

* $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \dfrac{(1-2p)^2}{npq}$ $\qquad \Gamma_1 = \dfrac{1-2p}{\sqrt{npq}}$

If $p = 1/2 \longrightarrow$ Symmetry

$p < 1/2 \longrightarrow$ positively skewed $\qquad p > 1/2 \rightarrow$ Negatively skewed.

prob: For a binomial distribution, mean is 10 & variance is 5. How many times the experiment is conducted?

Sol: $\frac{npq}{np} = \frac{5}{10} \Rightarrow q = 1/2 \Rightarrow p = 1/2$

$\Rightarrow n = 20$

prob: Find the probability of getting a exactly 2 in 3 times with a pair of dice.

Sol: $n = 3, \quad x = 2$ $\quad \{(5,4), (4,5), (6,3), (3,6)\}$

$P(\text{Sum}=9) = \frac{4}{36} = 1/9 \qquad q = 1 - 1/9 = 8/9$

required prob. $= nC_x p^x q^{n-x}$

$= 3C_2 (1/9)^2 (8/9)^1 = \frac{8}{243}$

prob: The probability of man hitting the target is 1/3.

(i) If he fires 5 times, what is the probability of his hitting the target atleast twice

(ii) How many times must he fire so that probability of his hitting the target atleast once is more than 90%.

Sol: $n = 5, \; p = 1/3, \; q = 2/3$

(i) $P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=1) + P(x=0)]$

$= 1 - [(2/3)^5 + 5C_1 (1/3)(2/3)^4] = \frac{131}{243}$

(ii) $P(x \geq 1) > 90\% \Rightarrow 1 - P(x=0) > 0.9$

$\Rightarrow P(x=0) < 0.1$

$\Rightarrow q^n < 0.1 \Rightarrow (2/3)^n < 0.1 \Rightarrow n =$

prob: 2 dice are rolled 120 times. Find the average no. of times in which the number on the first die exceeds no. on 2nd.

Sol: $n = 120 \qquad p = 15/36$

$A = B \rightarrow 6$ ways
$A > B \rightarrow 15$ "
$A < B \rightarrow 15$ "

Average = Mean $= np = 120 \times \frac{15}{36} = 50$

prob: If x is a R.V and $E(x) = 4$, $V(x) = 4/3$

Find ~~V(x) = 4/3~~ (i) $P(x \leq 2)$ (ii) Comment on β_1

Sol: $E(x) = np = 4 \qquad V(x) = npq = 4/3$

$$\frac{npq}{np} = \frac{4/3}{4} \Rightarrow q = 1/3 \quad , \; p = 2/3$$

$$n = 4 \times 3/2 = 6$$

$$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= \left(\frac{1}{3}\right)^6 + 6C_1\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + 6C_2\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = \frac{73}{729}$$

(ii) Since $p = \frac{2}{3} > 1/2$ Negatively skewed.

<u>poisson's distribution</u>: A R.V 'x' whose probability distribution is given by $P(x=r) = \frac{e^{-\lambda}\cdot\lambda^r}{r!}$. Then x is called poisson R.V and the distribution is called poisson distribution.

$$P(x; \lambda > 0) = P(r) = \begin{cases} \frac{e^{-\lambda}\cdot\lambda^r}{r!} & , \; \lambda > 0 \\ & \; 0 \le r < \infty \\ 0 & \text{otherwise} \end{cases}$$

* Mean = variance = $\lambda = np$
* It is used when observations are HIGH and SUCCESS probability is low. i.e $n \to \infty$ and $p \to 0$
* $E(x) = \text{mean} = \lambda$

  $V(x) = \mu_2 = \lambda$

  $\mu_3 = \lambda$

  $\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$

  Since $\lambda > 0$, poisson distribution is always <u>positively skewed</u>.
* MGF $M_x(t) = E[e^{t\lambda}] = e^{\lambda(e^t - 1)}$

<u>Prob</u>: A product is supposed to contain 2% defective items. What is the probability in a sample of 50 items to contain (i) Exactly 2 (ii) less than 2 defective items.

sol: using Bi-nomial dist^n

$p = 2/100 = 1/50 \quad n = 50$

$P(x=2) = 50C_2(1/50)^2(49/50)^{48}$

Not possible

using poisson's dist^n

$\lambda = np = 50 \times 1/50 = 1$

$P(x=2) = \frac{e^{-1}\cdot 1^2}{2!} = 1/2e$

$P(x<2) = 1/e + 1/e = 2/e$

<u>prob</u>: A telephone switch board receives 20 calls on an average during an hour. Find the probability for a period of 5 mins (i) NO calls (ii) Exactly 3 (iii) Atleast 2 calls received.

Sol: For 60 min $\longrightarrow$ 20 calls (avg)

5 min $\longrightarrow \frac{20}{60} \times 5 = 1.65$ calls (avg) $= \lambda$

(i) $P(x=0) = \frac{e^{-1.65}(1.65)}{0!} = e^{-1.65}$

(ii) $P(x=3) = \frac{e^{-1.65}(1.65)^3}{3!}$

(iii) $P(x \geq 2) = 1 - P(x<2)$

$= 1 - [P(x=0) + P(x=1)] = \left[1 - \left[e^{-1.65} + \frac{1.65 e^{-1.65}}{1!}\right]\right]$

Prob: For a Poisson distribution with R.V as x, $P(x=1) = P(x=2)$

Then $P(x=4) =$

Sol: $e^{-\lambda} \cdot \lambda^1 = \frac{e^{-\lambda} \cdot \lambda^2}{2} \Rightarrow \lambda = 2$

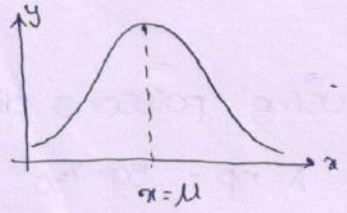
$P(x=4) = \frac{e^{-2} \cdot 2^4}{4!} = \frac{4}{6e^2}$

Normal distribution (Gaussian distribution): A continous R.V 'x' whose probability density function is given by

$$N(x; \mu, \sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < \infty \\ & -\infty < \mu < \infty \\ & 0 < \sigma < \infty \\ 0, & \text{otherwise} \end{cases}$$

The mean of normal distribution is μ & variance is σ^2 and also $\int_{-\infty}^{\infty} f(x)dx = 1$.

By taking values of x & $f(x)$ in xy plane we can get the graph of normal dist^n as below.



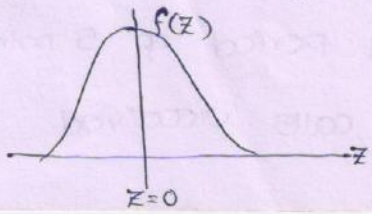
To calculate $P(a<x<b)$ for the normal dist^n we have to calculate

$$\int_a^b f(x)\,dx = \int_a^b e^{-\alpha x^2} dx$$

Which cannot be evaluated analytically.

Standard Normal distribution:

Let $z = \frac{x-\mu}{\sigma} \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}$ $\quad -3 \leq z \leq 3$



Mean = 0, variance = 1

prob: The avg. height of a student in a class is 175cm with a standard deviation of 10cm. A student is selected at random from that class. What is the prob. for him to have height b/w (i) B/w 170 to 180cm (ii) more than 180.. Given $P(0<Z<0.5) = 0.1915$

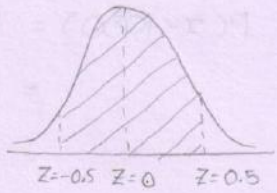
Sol: $\mu = 175cm$ $\sigma = 10cm$

(i) $P(170 < x < 180)$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-175}{10}$$

At $x = 170 \Rightarrow Z = -5/10 = -0.5$

At $x = 180 \Rightarrow Z = 5/10 = 0.5$

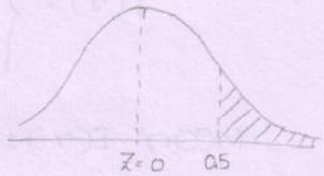


$P(170<x<180) = P(-0.5<Z<0.5)$

$= 2\times P(0<Z<0.5) = 2\times 0.1915 = 0.383$

(ii) $P(x>180) = P(Z>0.5)$

$= \frac{1}{2} - P(0<Z<0.5) = \frac{1}{2} - 0.1915 = 0.3185$



prob: The resistance of resistors is observed to be normal with mean resistance of 100Ω, SD of 2Ω. What % of resistors will have resistance (i) b/w 98Ω to 102Ω (ii) more than 102Ω. given $P(-\infty<Z<1) = 0.8413$

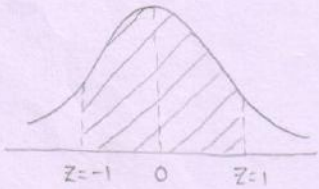
Sol: $\mu = 100$ $\sigma = 2$

(i) $P(98 < x < 102)$

$$Z = \frac{x-100}{2}$$

$x = 98 \Rightarrow Z = -1$

$x = 102 \Rightarrow Z = 1$



$P(98<x<102) = P(-1<Z<1) = 2\times P(0<Z<1)$

$= 2\times[0.8413 - 0.5] = 0.6826 = 68.26\%$

(ii) $P(x>102) = P(Z>1)$

$= 1 - 0.8413 = 0.15987 = 15.97\%$

prob: The avg. life of an electric ~~field~~ bulb of a company is 2040 hrs with a S.D of 60 hrs. A bulb of this company is purchased. What is the prob. that the bulb is likely to burn for

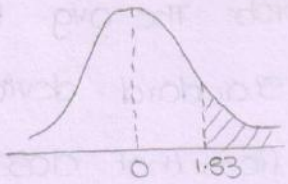
(i) More than 2150 hrs (ii) Less than 1950 hrs. Given $P(0<Z<1.83) = 0.4664$

$P(0<Z<1.5) < 0.4332$.

Sol: $\mu = 2040 \quad \sigma = 60$

(i) $P(x > 2150)$

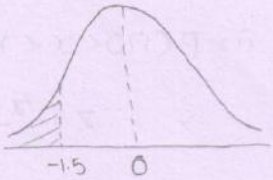
$$z = \frac{x - 2040}{60} \qquad x = 2150 \Rightarrow z = \frac{110}{60} = 11/6 = 1.83$$



$$P(x > 2150) = P(z > 1.83)$$
$$= 0.5 - P(0 < z < 1.83) = 0.5 - 0.4664 = 0.0336$$

(ii) $P(x < 1950)$

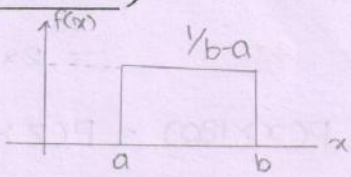
$$z = \frac{1950 - 2040}{60} = -1.5$$



$$P(x < 1950) = P(z < -1.5)$$
$$= 0.5 - P(-1.5 < z < 0)$$
$$= 0.5 - P(0 < z < 1.5) = 0.5 - 0.4332 = 0.0668$$

Uniform distribution (Rectangular distribution):

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ & a < b \\ 0 & \text{otherwise} \end{cases}$$



Mean $E(x) = \frac{a+b}{2}$

variance $V(x) = \frac{(b-a)^2}{12}$

* If X is uniform R.V in the interval $-a < x < a$ and it's density function is

$$f(x) = \frac{1}{2a} \qquad \text{Mean} = 0$$
$$\text{variance} = a^2/3$$

STATISTICS: collection of data, analysis of data & interpretation of data.

Types of data: (i) Grouped and ungrouped

(ii) Closed and open.

Grouped data: If data is in the form of class intervals and frequency then the data is known as grouped data or distributing the frequencies to their corresponding class intervals, then the data is known as frequency distribution.

ungrouped data: If data contains only observations, without any class intervals then the data is known as ungrouped data. or raw data.

closed data: If the class intervals are in a continous form without any discontinuity, then the data is known as closed data otherwise open data.

Mean (Average):

$$\bar{x}_{UGD} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{x}_{GD} = \frac{\sum_{i=1}^{n} f_i x_i}{N}$$

n: no. of observations

N: Sum of frequencies

x: mid point, $\frac{UL+LL}{2}$

F: Frequencies

Median:

1. If 'n' is odd, the middle observation is the median
2. If 'n' is even, average between middle observations

provided that

(i) Data is rearranged either in increasing / decreasing order

(ii) No. of observations above the middle is equal to the number of observations below.

$$Md = l + \frac{\left(\frac{N}{2} - m\right) \times c}{f}$$

l = lower limit for Ideal class

f = frequency of Ideal class

c = class length

m = cumulative frequency

Mode: The most frequently repeated observation is known as Mode.

For grouped data $M_0 = l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) C$ $\quad \Delta_1 = f - f_1$, $\Delta_2 = f - f_2$

Relation: Mode = 3×median − 2× mode

pr.do: Raw data 2, 4, 2, 3, 5, 2, 6, 5, 2, 7

sol: $\text{Mean} = \frac{\text{Total sum of observations}}{\text{No. of observations}} = \frac{38}{10} = 3.8$

Ascending order $\underline{2, 2, 2, 2}, \underbrace{3, 4,}\ \underline{5.5, 6, 7}$

$\text{Median} = \frac{3+4}{2} = \frac{7}{2} = 3.5$

Mode = Most repeated value = 2

pr.do: Grouped data:

C.I	0-10	10-20	20-30	30-40	40-50	50-60
frequency	3	4	12	24	13	4

sol:

C.I	freq f_i	Mid x_i	cum. freq
0-10	3	5	3
10-20	4	15	7
20-30	12 f_1	25	19 m
Ideal class → 30-40	24 f	35	43
40-50	13 f_2	45	56
50-60	4	55	60 N
	60		

x_i = Mid value of class

$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{2020}{60} = 33.66$

$\text{Median} = l + \frac{\left(\frac{N}{2} - m\right)}{f} \times C$

$= 30 + \frac{(30-19)}{24} \times 10 = 34.58$

$\text{Mode} = l + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times C$ $\quad \Delta_1 = f - f_1 = 12$, $\Delta_2 = f - f_2 = 11$

$= 30 + \frac{12}{12+11} \times 10 = 35.21$

Range: Maximum − minimum

Greatest value − Least value

Standard deviation: $\sqrt{\text{variance}} = (S.D)$

$\text{variance} = (S.D)^2 = \sigma_x^2$

$$\sigma_x^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - (\bar{x})^2$$

* variance is the sum of squares of deviation from mean.

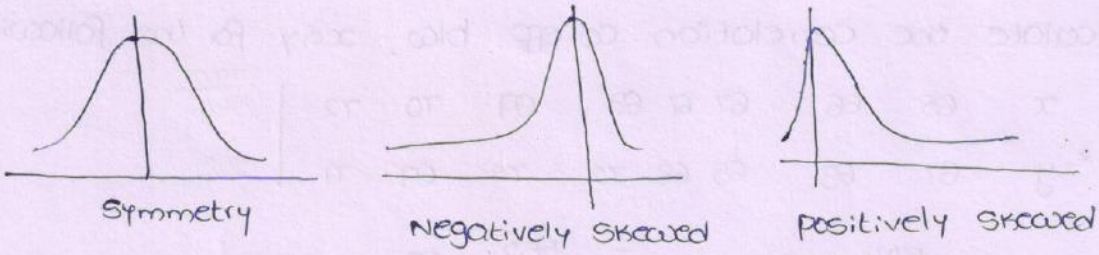
For grouped data

$$\sigma_x^2 = \frac{1}{N}\Sigma f_i x_i^2 - \left(\frac{1}{N}\Sigma f_i x_i\right)^2$$

$$\sigma_x^2 = \frac{1}{N}\Sigma f_i (x_i - \bar{x})^2$$

co.efficient of variation $C.V = \frac{\text{standard deviation}}{\text{Mean}} \times 100$

Skewness: "Lack of Symmetry."



Pearson's co.efficient of skewness:

$$S_{KP} = \frac{\text{Mean} - \text{Mode}}{\sigma} = \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

practical limit of SKP $-3 \le S_{KP} \le 3$

For symmetry $S_{KP} = 0$

Symmetry	Negative skewness	positive skewness
Mode = Median = Mean	Mode > Median > Mean	Mode < Median < Mean

CORRELATION & REGRESSION:

The degree of relation b/w the two variables is known as correlation.

Positive correlation: If the changes in the both the variables are in the same direction (increasing or decreasing) Then those variables are known as positively Correlated variables.

Negative correlation: If the changes in the one variable is affecting the changes of other variables in reverse direction, then those variables are known as negatively correlated.

correlation co.efficient b/w x, y is given by $r = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y}$

$$r = \frac{\frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}\sqrt{\frac{1}{n}\Sigma(y_i - \bar{y})^2}} = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}$$

$x = x_i - \bar{x}$

$y = y_i - \bar{y}$

* $-1 \le r \le 1$
* If $0 < r \le 1$ then x & y are positively correlated
* If $-1 \le r < 0$ then x & y are negatively correlated
* If $r = 0$ then there is no correlation b/w x & y
* If x & y are independent R.V's then co.variance $= 0$ i.e $Cov(x,y) = 0$

$\Rightarrow r(x,y) = 0$. But converse is not true.

Prob: calculate the correlation co.eff b/w x & y for the following data.

									Total
x	65	66	67	67	68	69	70	72	544
y	67	68	65	68	72	72	69	71	552

Sol: $\bar{x} = \frac{544}{8} = 68$ $\quad \bar{y} = \frac{552}{8} = 69$

									Σ
$X = x - \bar{x}$	-3	-2	-1	-1	0	1	2	4	
$Y = y - \bar{y}$	-2	-1	-4	-1	3	3	0	2	
XY	6	2	4	1	0	3	0	8	24
X^2	9	4	1	1	0	1	4	16	36
Y^2	9	1	16	1	9	9	0	4	44

$$r = \frac{24}{\sqrt{36}\sqrt{44}} = 0.6$$

$\therefore$ x & y are positively correlated.

Regression: The linear relationship b/w 2D random variable is known as regression.

Lines of regression: y on x $\quad (y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$

x on y $\quad (x - \bar{x}) = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$

Regression co.efficient y on x, $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

Regression co.efficient x on y, $b_{xy} = r\frac{\sigma_x}{\sigma_y}$

$$b_{xy} \cdot b_{yx} = r^2$$

Correlation co.eff. $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$ (Geometric mean)

* Both regression co.eff. must have same sign.

i.e if both +ve $\Rightarrow$ r is +ve

if both -ve $\Rightarrow$ r is -ve

* Two regression lines are passing through the point $(\bar{x}, \bar{y})$

* If $b_{xy} = b_{xy}$ $\Rightarrow r\frac{\sigma_x}{\sigma_y} = r\frac{\sigma_y}{\sigma_x}$ $\Rightarrow \sigma_y^2 = \sigma_x^2$

$\therefore$ variances are also equal.

* Angle b/w regression lines $\theta = \tan^{-1}\left(\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \cdot \sigma_y}{(\sigma_x)^2 + (\sigma_y)^2}\right)$

If $r = 0$ $\Rightarrow \theta = \pi/2$ $\Rightarrow \perp^{lar}$

$r = 1$ $\Rightarrow \theta = 0$ or π $\Rightarrow$ coinside (perfect correlation)

prob: The regression equation are $3x + 2y = 1$; $2x + 4y = 0$

(i) Find r? (ii) $\bar{x}, \bar{y}$

Sol: which ever the co.eff (magnitude) in the expression is higher then it is the DEPENDENT VARIABLE.

x on y	y on x	$r = \sqrt{b_{xy} \cdot b_{yx}}$
$3x + 2y = 1$	$2x + 4y = 0$	$= \sqrt{-2/3 \times -1/2}$
$x = \frac{1}{3} - \frac{2}{3}y$	$y = -\frac{2}{4}x$	$= -1/\sqrt{3}$ (both -ve)
$b_{xy} = -2/3$	$b_{yx} = -1/2$	

(ii) Means $\bar{x}, \bar{y}$ also ~~passing~~ on the lines.

$$6\bar{x} + 4\bar{y} = 2$$
$$2\bar{x} + 4\bar{y} = 0$$
$$4\bar{x} = 2 \Rightarrow \bar{x} = 1/2 \qquad \bar{y} = -1/4$$

Note: $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$$= \frac{cov(x,y)}{\sigma_x \cdot \sigma_y} \cdot \frac{\sigma_y}{\sigma_x} = \frac{cov(x,y)}{\sigma_x^2} = \frac{cov(x,y)}{var(x)}$$

similarly $b_{xy} = \frac{cov(x,y)}{var(y)}$

NUMERICAL METHODS

NUMERICAL METHODS

To find solutions to

1. Algebraic and Transcendental equations
2. System of linear equations
3. Integration of a function
4. Differential equations.

① Solution of Algebraic and transcendental equations:

Algebraic equations: polynomials of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

Transcendental equations: combination of polynomial, exponential and trigonometric functions.

Ex: $\tan x = x$, $xe^x - 1 = 0$, $\cos x - xe^x = 0$

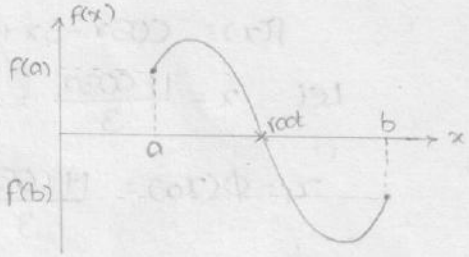
Intermediate value theorem: If $f(x)$ is a continous function defined on $[a,b]$ and $f(a)$, $f(b)$ are having opposite signs, then there exist atleast one root of $f(x)$ in $[a,b]$

Ex: $f(x) = x^3 + x - 1$

$f(0) = -1$ (-ve)

$f(1) = 1+1-1$ (+ve)

$\therefore$ one root in the interval $[0,1]$



Iterative Method (Successive approximation):

1. Express $f(x)=0$ as $x = \phi(x)$
2. choose an initial approximation x_0 to the root in the given interval
3. Find the next approximations using $x_1 = \phi(x_0)$
   $x_2 = \phi(x_1)$
   $x_3 = \phi(x_2)$ ---
4. Stop the iteration if two consecutive iterations gives the same approximation.

** The sequence of approximations converges to the root if $|\phi'(x)| < 1$

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Ex: $x^3+x-1=0$: $[a,b]=[0,1]$

$x = 1-x^3$	$x = \sqrt[3]{1-x}$	$x(x^2+1)=1$
$\lvert\phi'(x)\rvert = \lvert -3x^2\rvert$	$\lvert\phi'(x)\rvert = \left\lvert \frac{1}{3}(1-x)^{-2/3}(-1)\right\rvert$	$x = \frac{1}{x^2+1}$
$= 3x^2$	$\lvert\phi'(x)\rvert = \frac{1/3}{(1-x)^{2/3}}$	$\lvert\phi'(x)\rvert = \frac{-1}{(1+x^2)^2}\cdot 2x$
$\lvert\phi'(0)\rvert = 0 < 1$	$\lvert\phi'(0)\rvert = 1/3 < 1$	$\lvert\phi'(0)\rvert = 0 < 1$
$\lvert\phi'(1)\rvert = 3 > 1$	$\lvert\phi'(1)\rvert = \infty$	$\lvert\phi'(1)\rvert = 1/2 < 1$
×	×	✓

$\therefore$ Let $\phi(x) = \frac{1}{1+x^2}$ and Let $x_0 = 0$

$x_1 = \phi(x_0) = \frac{1}{1+0^2} = 1$

$x_2 = \phi(x_1) = \frac{1}{1+1^2} = 0.5$

$x_3 = \phi(x_2) = \frac{1}{1+(0.5)^2} = 0.8$

$x_4 = \phi(x_3) = \frac{1}{1+(0.8)^2} = 0.6097$

$\vdots$

using calc: $\frac{1}{1+(Ans)^2}$ press "="

$x_{18} = 0.6822$

$x_{19} = 0.6824$

$x_{20} = 0.6822$

$x_{21} = 0.6824$

$x_{22} = 0.6822$

Ex: $\cos x = 3x-1$

$f(x) = \cos x - 3x + 1$: $[0,1]$

Let $x = \frac{1+\cos x}{3} = \phi(x)$ & Let $x_0 = 0$

$x_1 = \phi(x_0) = \frac{1+\cos 0}{3} = 0.6666$

$\vdots$

$x_5 = 0.6071$, $x_6 = 0.6071$

using calc: $\frac{1+\cos(Ans)}{3}$ press "="

Bi-Section Method: Suppose that a root of $f(x)=0$ lies in the interval $I_0 = (a_0, b_0)$ i.e $f(a_0)\times f(b_0) < 0$ (opposite signs)

1. Bi-Sect the interval to obtain $c_1 = \frac{a_0+b_0}{2}$
2. Root lies in the interval $I_1 = (a_0, c_1)$ if $f(a_0)\times f(c_1) < 0$

   Root lies in the interval $I_2 = (c_1, b_0)$ otherwise.
3. Repeat above procedure for 'n' times, finally the root is given by Midpoint of the interval

Ex: x^3+x-1 : $[0,1]$

$c_1 = \frac{0+1}{2} = 0.5$

$f(c_1) = (0.5)^3 + (0.5) - 1 = -0.37$ (-ve)

$f(a_0)\times f(c_1) = (-1)\times(-0.37) = 0.37 > 0$

$\therefore$ New interval $= (c_1, b_0) = (0.5, 1)$

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$c_2 = \frac{0.5+1}{2} = 0.75$

$f(0.5) \times f(0.75) = -0.064 < 0$ $\therefore$ New interval = $[0.5, 0.75]$

⋮

$c_{12} = 0.68225$ $\therefore$ New interval = $[0.6822, 0.6825]$

Regula falsi Method / Method of false position:

1. Assume two initial approximations x_0, x_1 such that $f(x_0) \times f(x_1) < 0$
2. Find next approximation using $x_{k+1} = \frac{x_{k-1} \cdot f_k - x_k f_{k-1}}{f_k - f_{k-1}}$
3. Root lies in the interval (x_0, x_2) if $f(x_0) \times f(x_2) < 1$
   Root lies in the interval (x_2, x_1) otherwise.
4. Continue the process untill desired accuracy is achieved.

If $f(x)$ in the interval $[a, b]$, for simplicity select $x_0 = a$, $x_1 = b$

then $x_{n+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

Ex: $x^3 + x - 1 = 0 : [0 \ 1]$

$x_0 = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = 0.5$ $x_1 = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)} = 0.6363$

$f(x_0) = -0.37$ (-ve) ⋮

New interval = $[0.5, 1]$

Another way: Fix one end of interval (say b) $f(b) > 0$ and apply iteration formula to find new value for other end of the interval. After desired iterations, other end of interval itself gives the root.

Secant Method / Chord Method:

1. It is same as regula falsi method. The only difference is, while selecting initial approximation, we don't consider the condition $f(x_0) \times f(x_1) < 0$
2. So, this method may or may not converge
3. If it converge, it converges faster than regula falsi method.

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Newton raphson Method: The iterative formula in Newton raphson method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

→ Intially assume x_0

→ Newton raphson method fails if $f'(x) = 0$

→ If $f'(x)$ is very large then root can be find very rapidly

(99 CS) → condition for the convergence of newton raphson method is $|f(x).f''(x)| < |f'(x)|^2$

Ex: $x^3 + x - 1 = 0$: $[0, 1]$

Sol: $f(x) = x^3 + x - 1 \Rightarrow f'(x) = 3x^2 + 1$

Let $x_0 = 0.5$

$$x_1 = 0.5 - \frac{(0.5)^2 + (0.5) - 1}{3(0.5)^2 + 1} = 0.714$$

$x_2 = 0.6831$ $\quad x_3 = 0.6823$ $\quad x_4 = 0.6823$

(95 CS)

pdo: Find iterative formula for $\sqrt{N}$ using Newton Raphson method?

Sol $x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

$f(x_n) = x_n^2 - N \Rightarrow f'(x_n) = 2x_n$

From N.R Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left[\frac{N}{x_n} + x_n\right]$$

(96-CS)

pdo: Iterative formula for $\sqrt[3]{N}$

$x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

$f(x_n) = x_n^3 - N \Rightarrow f'(x_n) = 3x_n^2$

From N.R Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

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(GATE-95): Let $f(x) = x - \cos x$. Using NR method find x_{n+1} using x_n

Sol: $f(x) = x - \cos x \Rightarrow f'(x) = 1 + \sin x$

$$x_{n+1} = x_n - \frac{[x_n - \cos x_n]}{1 + \sin x_n}$$

(GATE-97 CS): N.R method is used to find the root of the equation $x^2 - 2 = 0$. If the iterations are started from -1, then the iteration will

a) converges to -1 b) converges to $\sqrt{2}$ c) converges to $-\sqrt{2}$ ✓ d) not converge

Sol: from N.R method $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$

Iteration started from $-1 \Rightarrow x_0 = -1$

$$\therefore x_1 = \frac{x_0^2 + 2}{2x_0} = -1.5 \qquad x_3 = \frac{x_2^2 + 2}{2x_2} = -1.4141$$

$$x_2 = \frac{x_1^2 + 2}{2x_1} = -1.4166 \qquad x_4 = \frac{x_3^2 + 2}{2x_3} = -1.4141$$

$\therefore$ Iterations converge to $-\sqrt{2}$

(GATE-05 CE): Given $a > 0$, we wish to calculate its reciprocal value $\frac{1}{a}$ by using NR method for $f(x) = 0$. For $a = 7$ and starting with $x_0 = 0.2$ the first two iterations will be

Sol: $x = \frac{1}{a} \Rightarrow x - \frac{1}{a} = 0$ (or) $\frac{1}{x} - a = 0$

$$f(x) = \frac{1}{x} - a \Rightarrow f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{\left(-\frac{1}{x_n^2}\right)} = 2x_n - a x_n^2$$

Given $a = 7$, $x_0 = 0.2$

$$x_1 = 2(0.2) - 7(0.2)^2 = 0.12$$

$$x_2 = 2(0.12) - 7(0.12)^2 = 0.1392$$

(GATE-05 ME): Starting from $x_0 = 1$, one step of NR method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value x_1 as

$$f'(x) = 3x^2 + 3$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1) + 3(1) - 7}{3(1) + 3} = 1 + \frac{1}{2} = 1.5$$

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(GATE-05 PI): The real root of the equation $xe^x = 2$ is evaluated using NR method. If the first approximation of the value of x is 0.8679, the 2nd approximation of x, correct to 3 decimal places is

Sol: $f(x) = xe^x - 2 \Rightarrow f'(x) = xe^x + e^x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{(0.8679)e^{0.8679} - 2}{(0.8679)e^{0.8679} + e^{0.8679}}$$

$$= 0.853$$

(GATE-07 CE): The following equation need to be numerically solved using NR method $x^3 + 4x - 9 = 0$. The iterative equation for this purpose is

Sol: $f(x) = x^3 + 4x - 9 \Rightarrow f'(x) = 3x^2 + 4$

$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$

$$= \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{3x_k^2 + 4} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

(GATE-07 EC): The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using NR method. If $x = 2$ taken as the initial approx. of the solution then the next approx. using this method will be

Sol: $f'(x) = 3x^2 - 2x + 4$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8 - 4 + 8 + 4}{12 - 4 + 4} = 4/3$$

(GATE-08 EE): Equation $e^x - 1 = 0$ is required to be solved using NR method with an initial guess $x_0 = -1$. Then after one step of NR method estimate x_1 of the solution will be given

Sol: $f(x) = e^x - 1 \Rightarrow f'(x) = e^x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{e^{-1} - 1}{e^{-1}} = 0.71828$$

(GATE-08 EC): The recursion relation to solve $x = e^{-x}$ using NR method is

Sol: $f(x) = x - e^{-x} \Rightarrow f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{(1 + x_n)e^{-x_n}}{(1 + e^{-x_n})}$$

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(GATE-10 CS): NR method is used to compute a root of the eqn $x^2-13=0$ with 3.5 as the initial value. The approximation after one iteration is

Sol: $x^2-13=0 \Rightarrow x=\sqrt{13}$ $\quad x_0=3.5$

$$x_1=\frac{1}{2}\left[x_0+\frac{13}{x_0}\right]=\frac{1}{2}\left[3.5+\frac{13}{3.5}\right]=3.607$$

(GATE-11 EC): A numerical solution of the equation $f(x)=x+\sqrt{x}-3=0$ can be obtained using NR method. If the starting value is $x=2$ for the iteration then the value of x that is to be used in next step is

$x_0=2$

Sol: $f(x)=x+\sqrt{x}-3 \Rightarrow f'(x)=1+\frac{1}{2\sqrt{x}}$

$$x_1=2-\frac{2+\sqrt{2}-3}{1+1/2\sqrt{2}}=1.6939$$

(GATE-13 EE): When the NR method is applied to solve the equation $f(x)=x^3+2x-1=0$, the solution at the end of 1st iteration with the initial value as $x_0=1.2$ is

Sol: $f'(x)=3x^2+2$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1.2-\frac{(1.2)^3+2(1.2)-1}{3(1.2)^2+2}=0.705$$

(GATE-14 EE): The function $f(x)=e^x-1$ is to be solved using NR method. If the initial value of x_0 is taken 1.0, then the absolute error observed at 2nd iteration is

Sol: $f'(x)=e^x$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{(e-1)}{e}=1/e$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{1}{e}-\frac{(e^{1/e}-1)}{e^{1/e}}=0.06$$

(GATE-14 ME): The real root of the equation $5x-2\cos x=0$ (upto two decimal places) is Ans: 0.54

Sol: $f'(x)=5+2\sin x$

Let $x_0=1$

By NR method $x_1=1-\frac{5-2\cos 1}{5+2\sin 1}=0.5631$

$x_2=0.5426$

$x_3=0.5426$

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(GATE-14 PI): If the equation $\sin x = x^2$ is solved by Newton Raphson's method with the initial guess of $x=1$, then the value of x after 2 iterations would be

Sol: $f(x) = x^2 - \sin x \Rightarrow f'(x) = 2x - \cos x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \left(\frac{1-\sin 1}{2-\cos 1}\right) = 0.8915$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \simeq 0.73$$

(GATE-14 CS): In NR method, an initial guess of $x_0 = 2$ is made the sequence $x_0, x_1, x_2 \ldots$ is obtained for the function $0.75x^3 - 2x^2 - 2x + 4 = 0$. Consider the statements

(I) $x_3 = 0$ (II) The method converge to solution in a finite number of iterations. Which of the following is true?

a) only I (b) only II (c) Both I and II (d) Neither I nor II

Sol:

$$f(x) = 0.75x^3 - 2x^2 - 2x + 4$$

$$f'(x) = 2.25x^2 - 4x - 2 \qquad x_0 = 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 \quad , \quad x_3 = 0 \ , \ x_4 = 2 \ , \ x_5 = 0 \ldots$$

$\therefore$ Iterations will not converge.

System of Non linear equations: NR method can be extend to find the roots of system of non linear equations. Consider $f(x,y) = 0$, $g(x,y) = 0$.

The approximation of roots $x_{k+1} = x_k + \Delta x$ $\quad y_{k+1} = y_k + \Delta y$ $\quad k = 0, 1, 2 \ldots$

where $$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

J is called Jacobian matrix $\quad J_k = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_k, y_k)}$

→ A necessary and sufficient condition for convergence is $\rho(J_k^{-1}) < 1$. where ρ denotes the spectral radius (largest eigen value in magnitude.

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Method	Order convergence	
Bi section	Linear convergence	1
Regula falsi	Linear convergence	1
Secant method	Super linear convergence	1.62
Newton Raphson	Quadratic convergence	2

Note: Let us consider n^{th} degree polynomial $f(x)=0$

(i) No. of positive real roots of $f(x) \le$ No. of sign changes in $f(x)=0$

(ii) No. of negative real roots of $f(x) \le$ No. of sign changes in $f(-x)=0$

(iii) No. of complex roots $= n -$ (no. of +ve + no. of -ve roots)

(GATE-07 IN): The polynomial $P(x) = x^5+x+2$ has

(i) all real roots (ii) 3 real & 2 complex (iii) 1 real & 4 complex (iv) all complex

Sol: $P(x)=0$ doesn't have any sign changes. No. of positive roots $=0$

$P(-x) = -x^5-x+2$. one sign change. No. of negative roots $=1$

$\therefore$ complex roots $= 5-1=4$

Note: If $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are n roots of equations $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$

(i) $\sum_{i=1}^{n} \alpha_i = -\frac{a_1}{a_0}$ (ii) $\alpha_1.\alpha_2.\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$

(GATE-08 IN): It is known that two roots of the non-linear equation $x^3-6x^2+11x-6=0$ are 1 and 3. The third root will be

Sol: $\alpha_1+\alpha_2+\alpha_3 = -\frac{a_1}{a_0} = 6$ (or) $\alpha_1\alpha_2\alpha_3 = (-1)^3 \cdot \frac{-6}{1}$

$4+\alpha_3 = 6$ $\qquad$ $3.\alpha_3 = 6$

$\alpha_3 = 2$ $\qquad$ $\alpha_3 = 6/3 = 2$

② Solution to system of linear equations:

(i) Gauss elimination method: discussed in "Matrix algebra" topic.

(ii) LU decomposition (Method of factorization)

or

Do-little method:

Let us consider system of linear eq's in matrix form $AX=B$

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Decompose A into $A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore$ The system of equations $LUX = B$

Let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

solve to find y_1, y_2, y_3.

from $UX = Y \Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Solve to find x_1, x_2, x_3.

CROUT'S method: The Lower and upper triangular matrix in Crout's method are

$$A = LU$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

③. Solution to Integration of function:

The area bounded by curve $f(x)$ is denoted by $\int_a^b f(x)\,dx$. Divide $[a, b]$ into "n" equal sub intervals ~~very~~ where length of each interval is "h" (step size)

$a = x_0$

$x_1 = x_0 + h$

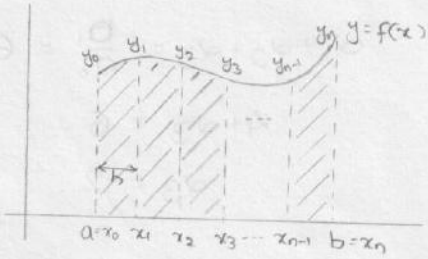
$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 3h$

$\vdots$

$x_n = x_0 + nh \Rightarrow b = a + nh$

$\Rightarrow h = \frac{b-a}{n}$



(i) Trapezoidal rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

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(ii) Simpson's $\frac{1}{3}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\dots+y_{n-2})+4(y_1+y_3+y_5+\dots+y_{n-1})\right]$$

(iii) Simpson $\frac{3}{8}$th rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+\dots+y_{n-3})+3(y_1+y_2+y_4+y_5+\dots+y_{n-1})\right]$$

* In trapezoidal rule error is of order h^2
* In Simpson's $1/3$rd rule error is of order h^4
* In Simpson's $3/8$th rule error is of order h^5
* Simpson's $1/3$rd rule is applicable if number of intervals are EVEN
* Simpson's $3/8$th rule is applicable if number of intervals are multiples of 8.
* Trapezoidal rule is applicable for any number of intervals.
* Simpson's rule for integration gives exact results when $f(x)$ is a polynomial of degree '2'.
* Trapezoidal rule for integration gives exact results when $f(x)$ is a polynomial of degree ≤ 1 (0 or 1)

(GATE-07 ME): $\int_0^{2\pi} \sin x\,dx$ is evaluated by T.R rule with eight equal intervals. (with 5 significant digits)

Sol: $n=8$ $\quad h = 2\pi/8 = \pi/4$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$y=\sin x$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0

T.R rule $\int_0^{2\pi} \sin x \cdot dx = \frac{\pi/4}{2}\left[(0+0)+2\left(\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}\right)\right]$

$= 0.00000$

(GATE-10 ME):

x	0	60	120	180	240	300	360
y	0	1068	-323	0	323	-355	0

Evaluate $\int_0^{2\pi} y\,dx$ using Simpson's rule.

a) 542 b) 995 c) 1444 d) 1986

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x values are in degrees. and limits of integration are interms of π $\Rightarrow$ $h = 60^\circ = \pi/3$

By simson's $1/3$ rule $\int_0^{2\pi} y\,dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_3) + 4(y_1+y_2+y_4+y_5)\right]$

$= 995$

(GATE-II ME): The integral $\int_1^3 \frac{1}{x}\,dx$ when evaluated using $1/3$rd rule on two equal intervals each of length 1.

Sol: $h = \frac{b-a}{n} = \frac{3-1}{2} = 1$

x	1	2	3
$y = 1/x$	1 (y_0)	$1/2$ (y_1)	$1/3$ (y_2)

By Simpson's $1/3$rd rule

$$\int_1^3 \frac{1}{x}\,dx = \frac{1}{3}\left[(1+1/3) + 2(0) + 4(1/2)\right] = 1.111$$

Note: Error in Trapezoidal rule $= -\left(\frac{b-a}{12}\right)h^2 \max[f''(x)]$

Error in Simpson's $\frac{1}{3}$ rule $= -\frac{(b-a)}{180}\cdot h^4 \max[f^{IV}(x)]$

Error in simpson's $\frac{3}{8}$ rule $= -\frac{3}{80}h^5 \max[f^{IV}(x)]$

Note: If Error tolerance (ϵ) is prescribed then the no. of sub-intervals required to achieve that accuracy are given by

In trapezoidal rule : $N^2 \geq \frac{(b-a)^3}{12\epsilon}\cdot \max[f''(x)]$

In Simpson's $1/3$ rule : $N^4 \geq \frac{(b-a)^5}{2880\epsilon}\max[f^{IV}(x)]$

pr.b. The minimum number of equal length sub intervals needed to approximate $\int_1^2 xe^x\,dx$ to an accuracy of atleast $1/3\times10^{-6}$ using Trapezoidal rule.

(a) $1000\cdot e$ (b) 1000 (c) $100e$ (d) 100

Sol: $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x$

$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$

e^x is increasing and x is increasing in $[1,2]$

$\therefore \max[f''(x)] = (2e^x + xe^x)\big|_{x=2} = 4e^2$

Accuracy $\geq \frac{1}{3}\times10^{-6} \Rightarrow \epsilon \leq 1/3\times10^{-6}$

$\therefore N^2 \geq \frac{(b-a)^3}{12\epsilon}\cdot\max[f''(x)] \Rightarrow N^2 \geq \frac{(1)^3}{12\times\frac{1}{3}\times10^{-6}}\times4e^2 \Rightarrow N \geq 1000e$

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Solution of ordinary differential equations:

Single step method: If we use only the values y_n, $f(x_n, y_n)$ to obtain the solution value y_{n+1} at the next nodal point $x = x_{n+1}$ then the method is called explicit single step method.

$$y_{n+1} = y_n + \phi(x_n, y_n, h)$$

If y_{n+1} and $f(x_{n+1}, y_{n+1})$ are also used, then the method is called implicit S.S method.

$$y_{n+1} = y_n + \phi(x_n, x_{n+1}, y_n, y_{n+1}, h)$$

1. Taylor series method:

$$y(x) = y(x_n) + (x - x_n)y'(x_n) + \cdots + \frac{(x-x_n)^p}{p!} y^{(p)}(x_n) + \cdots$$

Substitute $x = x_{n+1}$ $\Rightarrow$ $x_{n+1} - x_n = h$

$$y(x_{n+1}) = y(x_n) + h \cdot y'(x_n) + \cdots + \frac{h^p}{p!} y^{(p)}(x_n) + \cdots$$

2. Euler's method (RK first order method): If $dy/dx = f(x, y)$ & $y(x_0) = y_0$ then

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$
$$y_2 = y_1 + h \cdot f(x_1, y_1)$$
$$\vdots$$

3. Euler's backward method:

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

The non linear equation for y_{n+1} can be solved by NR method.

4. Modified Euler method / Euler - Cauchy method / Heun's method / RK 2nd order Method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$
$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf_n)\right]$$

where $f_n = f(x_n, y_n)$

Let $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_{n+1}, y_n + k_1)$

$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

also denoted as

$y_1^P = y_0 + hf(x_0, y_0)$

$y_1^C = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^P)\right]$

also called predictor - corrector formula or improved Euler cauchy formula.

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5. Runge's Method (RK method of order 3):

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$K' = hf(x_0 + h,\ y_0 + K_1)$$

$$K_3 = hf(x_0 + h,\ y_0 + K')$$

$$K = \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$y_1 = y_0 + K$$

6. Runge-Kutta method (RK method of order 4):

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$\Delta y = 1/6\,[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_1 = y_0 + \Delta y$$

(GATE-93): Given the differential equation $y' = x - y$ with initial condition $y(0) = 0$. The value of $y(0.1)$ calculated numerically upto the third place of decimal by the 2nd order RK method with step size $h = 0.1$ is

Sol: Given $y' = x - y \Rightarrow \frac{dy}{dx} = y' = f(x, y)$

Given $y(0) = 0 \Rightarrow x_0 = 0,\ y_0 = 0$ $\quad h = 0.1$

From RK 2nd order method

$$K_1 = hf(x_0, y_0) \qquad x_1 = x_0 + h$$

$$= 0.1\,[x_0 - y_0] = 0 \qquad = 0 + 0.1 = 0.1$$

$$K_2 = hf(x_1, y_0 + K_1)$$

$$= (0.1)f(0.1, 0) = (0.1)[0.1 - 0] = 0.01$$

$$y(0.1) = y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 0 + \frac{1}{2}(0 + 0.01) = \frac{1}{200} = 0.005$$

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(GATE-10 EC): consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with initial condition $y(0)=0$. Using Euler's first order method with a step size of 0.1 then the value of $y(0.3)$ is

Sol: $f(x,y) = \frac{dy}{dx} = y+x$ $\quad h=0.1$

$y(0)=0 \Rightarrow x_0=0,\ y_0=0$

$\therefore x_1 = x_0+h = 0.1 \qquad x_2 = x_0+2h = 0.2 \qquad x_3 = x_0+3h = 0.3$

$y_1 = y_0 + h f(x_0, y_0) = y_0 + h[x_0+y_0] = 0$

$y_2 = y_1 + h f(x_1, y_1) = 0 + 0.1[0.1+0] = 0.01$

$y_3 = y_2 + h f(x_2, y_2) = 0.01 + 0.1[0.2+0.01] = 0.031$

$\therefore y(0.3) = 0.031$

(GATE-13 IN): while numerically solving the D.E $\frac{dy}{dx} + 2xy^2 = 0$ $y(0)=1$ using Euler's predictor corrector (improved Euler cauchy) method with step size of 0.2, the value of y after the first step is

Sol: $\frac{dy}{dx} = -2xy^2 = f(x,y) \qquad f(0,1)=0$

$y_1^P = y_0 + h f(x_0, y_0) = 1 + 0.2(0) = 1$

$y_1^C = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^P)] = 1 + \frac{0.2}{2}(0-0.4) = 0.96$

(GATE-14 ME): consider an O.D.E $\frac{dx}{dt} = 4t+4$. If $x=x_0$ at $t=0$, the increment in x calculated using RK 4th order multi-step method with a step size of $\Delta t = 0.2$ is

Sol $\frac{dx}{dt} = 4t+4 = f(t,x) \qquad$ at $t=0 \quad x=x_0=0$

$h=0.2$

By RK fourth order method

$K_1 = hf(t_0, x_0) = 0.2\times 4 = 0.8 \qquad K_2 = hf\left(t_0+\frac{h}{2}, x_0+\frac{K_1}{2}\right) = 0.88$

$K_3 = hf\left(t_0+\frac{h}{2}, x_0+\frac{K_2}{2}\right) = 0.88 \qquad K_4 = hf(t_0+h, x_0+K_3) = 0.96$

$K = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] = 0.88$

$x_1 = x_0 + K = 0 + 0.88 = 0.88$

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Multistep methods: [NO question in GATE till now]

1. Adams-Bashforth methods
2. Adams-moultons methods
3. Milne-Simsons methods

CURVE FITTING:

1. $y = ax + b$

Normal equations are $\Sigma y = a\Sigma x + nb \rightarrow (1)$

$\Sigma xy = a\Sigma x^2 + b\Sigma x \rightarrow (2)$

solving 1 & 2 we get a, b

2. $y = ax^2 + bx + c$

Normal equations are $\Sigma y = a\Sigma x^2 + b\Sigma x + c$

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$

$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

3. $y = a \cdot x^b \Rightarrow \log y = \log a + b\log x$

$Y = A + bX$ where $X = \log x$, $A = \log a$, $Y = \log y$

Normal equations are $\left.\begin{array}{l}\Sigma Y = nA + b\Sigma X \\ \Sigma XY = A\Sigma X + b\Sigma X^2\end{array}\right\}$ Solving gives A, b

$a = \text{antilog}(A) = 10^A$

4. $y = a \cdot b^x \Rightarrow \log y = \log a + x\log b$

$Y = A + xB$ where $Y = \log y$, $A = \log a$, $B = \log b$

Normal equations are $\left.\begin{array}{l}\Sigma Y = B\Sigma x + nA \\ \Sigma xY = B\Sigma x^2 + A\Sigma x\end{array}\right\}$ Solving gives A & B

$a = 10^A$; $b = 10^B$

(GATE-08): Three values of x and y are to be fitted in a straight line in the form $y = a + bx$ by the method of least squares. Given $\Sigma x = 6$, $\Sigma y = 21$, $\Sigma x^2 = 14$, $\Sigma xy = 46$, the values of a & b are respectively.

Sol: $n = 3$ $\quad \Sigma y = na + b\Sigma x \Rightarrow 21 = 3a + 6b$

$\Sigma xy = a\Sigma x + b\Sigma x^2 \Rightarrow 46 = 6a + 14b$

solving $\Downarrow$

$a = 3,\ b = 2$

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