AIMS | African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

Module: Plane Geometry and Solid Figures

Topic: Differential Equations:

Lesson: Solution to First order differential equations separable variable

Class: U6

By: NGETI AKAMA RODRIGUE

Motivation

This chapter introduces first order differential equations. Differential Equations and their solutions are of great importance in many different areas of engineering, sciences and technology.

Some of the areas of the applications of differential equations are: Radioactive decay of elements; Motion of a body; Population models; Prediction of weather, Prediction of stock etc.

Here we are going to deal with equations that relate an unknown function and one or more of its derivatives to an independent variable.

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Lesson Objectives

At the end of this lesson, you should be able to:

- Identify a differential equation
- State the order and degree of a differential equation
- Distinguish between a particular solution and a general solution
- Solve first order separable differential equations.

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

Pre-requisite knowledge

You can carry out:

- Operations in algebra
- Carry out integration of functions using the various techniques

AIMS | African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

Verification of pre-requisite

Question: Compute the following integrals

1. $\int \left(\frac{2x+3}{x^2-4}\right)dx$

2. $\int \left(\frac{3}{x^2+9}\right)dx$

3. $\int \frac{4e^x}{1+e^x}\,dx$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Solution

1. $\int\left(\frac{2x+3}{x^2-4}\right)$dx $= \int\left(\frac{2x}{x^2-4}\right)$dx + $\int\left(\frac{3}{x^2-4}\right)$dx

$=\ln|x^2-4| - \frac{3}{2}tanh^{-1}\left(\frac{x}{2}\right) + k$

$= ln|x^2-4| - \frac{3}{4}ln\left(\frac{2+x}{2-x}\right) + k$

But $\int\left(\frac{f\prime(x)}{f(x)}\right)dx = ln|f(x)| + c$ and $\int\frac{1}{a^2-x^2}dx = \frac{1}{a}tanh^{-1}\left(\frac{x}{a}\right) + p$

$\int\left(\frac{2x+3}{x^2-4}\right)$dx $= ln|x^2-4| - \frac{3}{4}ln\left(\frac{2+x}{2-x}\right) + k\left(tanh^{-1}x = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)\right)$

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

2. $\int \left(\frac{3}{x^2+9}\right)$dx = 3 $\left(\frac{1}{3} tan^{-1}\left(\frac{x}{3}\right)\right) + k$

Using $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + k$

$\therefore \int \left(\frac{3}{x^2+9}\right)$dx $= tan^{-1}\left(\frac{x}{3}\right) + k$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

3. $\int \frac{4e^x}{1+e^x} dx = 4\int \frac{e^x}{1+e^x}$

[Recall $\int \frac{f'(x)}{f(x)} dx = log f(x)$]

$\Longrightarrow 4\int \frac{e^x}{1+e^x} = 4\log(1+e^x) + c$

$\therefore \int \frac{4e^x}{1+e^x} dx = 4\log(1+e^x) + c$

AIMS | African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

PROBLEM SITUATION

It was found out that the rate at which the HIV virus grows in a patient is directly proportional to the initial amount P present in the patient, at time t. However, if the patient starts taking the Anti retroviral drug, the virus dies at the rate μPt where μ is a positive constant .

How can the doctor determine the maximum amount of virus present in the patient as time goes on?

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

Activities

i. Show that at time t the rate of growth of the virus in the patient's body is governed by the differential equation

$$\frac{dP}{dt} = (k - \mu t)P,$$

where k is another positive constant and t is time in days

When $t = 0, \frac{dP}{dt} = 2P$ and when $t = 1, \frac{dP}{dt} = \frac{19}{10}P$

ii. Show that $\frac{dP}{dt} = \frac{1}{10}(20 - t)P$

AIMS | African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

iii. Show that the differential equation above can also be written as $\frac{dP}{P} = \frac{1}{10}(20 - t)dt$

iv. Carry out the integration $\int \frac{1}{P} dP = \int \frac{1}{10}(20 - t)dt$. Using $P(0) = 1000$

Show that $P(t) = 1000e^{\frac{1}{20}(40t - t^2)}$

v. Find the time required for *P(t)* to be maximum.

vi. Hence, determine the maximum value of *P(t).*

vii. What conclusion can you make about the maximum number of virus present in the patient?

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Solution to activities

i. $\frac{dP}{dt} \propto P \implies \frac{dP}{dt} = kP \text{ where } k \text{ is a positive constant.}$

After the drug is administered, then $\frac{dP}{dt} = kP - \mu tp$

$\text{where } \mu \text{ and } k \text{ are positive constants (since } P \text{ decreases at the rate } \mu tp\text{)}$

Hence $\frac{dP}{dt} = (k - \mu t)p$

ii. $t = 0, \frac{dP}{dt} = 2P \implies 2P = kP \text{ hence } k = 2$

t = 1, $\frac{dP}{dt} = \frac{19}{10}P \implies \frac{19}{10}P = (2 - \mu)P \text{ hence}, \mu = \frac{1}{10}$

Therefore $\frac{dP}{dt} = \frac{1}{10}(20 - t)P$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

iii. $\frac{dP}{dt} = \frac{1}{10}(20 - t)P \implies \frac{dP}{P} = \frac{1}{10}(20 - t)dt$

iv. $\int \frac{1}{P} dP = \int \frac{1}{10}(20 - t)dt$ (integrating both sides gives)

$lnP = \frac{1}{10}\left(20t - \frac{t^2}{2}\right) + k$ but P(0) = 1000 (i.e P= 1000 when t = 0)

Thus, k = ln1000 $\therefore P = e^{\frac{1}{20}(40t - t^2) + \ln 1000}$

Hence P(t) = $1000e^{\frac{1}{20}(40t - t^2) + \ln 1000}$

v. The maximum number P occurs when $\frac{dP(t)}{dt} = 0$

But $\frac{dP}{dt} = 100(20 - t)e^{\frac{1}{20}(40t - t^2)}$, hence, $100(20 - t)e^{\frac{1}{20}(40t - t^2)} = 0$

$\implies$ t = 20. the maximum will occur at 20 days

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

vi. When $t = 20, P(20) = 1000e^{20}$

vi. Therefore the maximum number of viruses present in the patient after 20 days is $P = 1000e^{20}$

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

DIFFERENTIAL EQUATIONS

A **differential equation** (D.E) is an equation involving an unknown function and one or more of its derivatives to an independent variable $y = f(x)$. Differential equations are classified according to the highest derivative which occurs in the differential equation called **order** .

Examples i. $\frac{dy}{dx} = 7x$ -- **First order DE.**

ii. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$ **Second order DE**

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

The **degree** of a differential equation is the highest power of the highest derivative which the equation contains after simplification.

Example:

$\left(\frac{d^2y}{dx^2}\right)^3 - 3\left(\frac{dy}{dx}\right)^5 + y = 0$ is a second order differential equation of degree three.

This is got from the term $\left(\frac{d^2y}{dx^2}\right)^3$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

A differential equation is called an ordinary differential equation, if it has ordinary derivatives such as:

$$2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - y = 0$$

When a differential equation is given, together with sufficient information (data), it is possible by integration to determine unknown constants so as to obtain the original function. This process is called '**Solving the Differential Equation**'.

A solution to a differential equation is a function between variables (independent and dependent), which is free of derivatives of any order, and which satisfies the Differential Equation identically.

This solution is not necessarily unique since the derivative of a constant is zero. The set of all solutions of a DE is call the **General solution.** This general solution is usually with one or more constant(s).

The solution without constant(s) is called a **Particular solution**

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Additional information called **boundary conditions** (initial conditions) are used to find particular solutions from the general solution.

Example:

$y = 3x + c$ is the general solution of the differential equation $\frac{dy}{dx} = 3$. Given the boundary conditions $y = 2$ when $x = 1$ produces the particular solution $y = 3x - 1$.

Remark: The independent variable for the above differential equations is x while y is the dependent variable.

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

First Order Differential Equations

This is a differential equation with order 1

i. **First order separable differential equations**

This is a differential equation of the form:

$$\frac{dy}{dx} = f(x).\, g(y)$$

which can be separated as

$$\frac{1}{g(y)}\, dy = f(x)\, dx$$

AIMS | African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

This differential equation can be solved by integrating both sides

Solving this when the boundary conditions (initial conditions) are given gives the particular solution of the different equation

Examples:

1. Find the solution of each of the following differential equations

a) Find the general solution of the DE $\frac{dy}{dx} = 2sinx + 3cosx$

b) Find the general solution to the DE $y\frac{dy}{dx} = x(1 + y^2)$

c) Find the solution to the DE $x(x + 1)\frac{dy}{dx} = y(y + 1)$ given that $y = 1$ when $x = 2$. Find the value of y when $x = 1$.

d) Solve, the DE $x(1 - y)\frac{dy}{dx} = -2x, given\ that\ y = 2\ when\ x = e,$ $hence, show\ that\ y = ln\left(\frac{x^2y}{2}\right)$

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Solutions

a) Find the general solution of the DE $\frac{dy}{dx} = 2sinx + 3cosx.$

Solution:

$$\frac{dy}{dx} = 2sinx + 3cosx$$

$$\Rightarrow \int dy = \int (2sinx3cosx)dx \Rightarrow y = -2cosx + 3sinx + c$$

b) $y\frac{dy}{dx} = x(1+y^2) \Rightarrow \frac{dy}{dx} = x\frac{(1+y^2)}{y}$

$$\Rightarrow \frac{y}{1+y^2}dy = xdx$$

$$\Rightarrow \frac{1}{2}\int \frac{2y}{1+y^2}dy = \int x\,dx$$

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Solution to b) cont.

$$\Rightarrow \frac{1}{2}\ln(1+y^2) = \frac{x^2}{2} + k$$

$$\Rightarrow \ln(1+y^2) = x^2 + 2k$$

$$\Rightarrow y^2 = e^{x^2+2k} - 1$$

$$\Rightarrow y^2 = e^{x^2} . e^{2k} - 1$$

$\Rightarrow y = \pm\sqrt{ce^{x^2} - 1}$ where c$= e^{2k}$ which is a constant

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

c) $x(x+1)\frac{dy}{dx} = y(y+1)$ given that $y = 1$ when $x = 2$. Find the value of y when $x = 1$.

Solution:

$$x(x+1)\frac{dy}{dx} = y(y+1) \Longrightarrow \frac{1}{y(y+1)}dy = \frac{1}{x(x+1)}dx$$

$$\Longrightarrow \int \frac{1}{y(y+1)}dy = \int \frac{x}{x(x+1)}dx$$

But $\frac{1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$ $\left(same\ in\ \frac{1}{x(x+1)}\right)$

$\Longrightarrow 1 = A(y+1) + B(y). Solving\ gives\ A = 1\ and\ B = -1$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

$\int \frac{1}{y(y+1)} dy = \int \frac{x}{x(x+1)} dx$ becomes

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$= ln|y| - ln|y+1| = ln|x| - ln|x+1| + k$

$= ln\left|\frac{y}{y+1}\right| = ln\left|\frac{x}{x+1}\right| + k$

Given that y = 1 when x = 2

$$\Rightarrow ln\left(\frac{1}{2}\right) = ln\left(\frac{2}{3}\right) + k \Rightarrow k = ln\frac{3}{4}$$

Thus $ln\left|\frac{y}{y+1}\right| = ln\left|\frac{x}{x+1}\right| + k$ becomes $ln\left|\frac{y}{y+1}\right| = ln\left|\frac{x}{x+1}\right| + ln\frac{3}{4}$

$\Rightarrow \frac{y}{y+1} = \frac{3}{4}\left(\frac{x}{x+1}\right)$ is the solution to the DE.

When x = 1, $\frac{y}{y+1} = \frac{3}{4}\left(\frac{x}{x+1}\right)$ will be $\frac{y}{y+1} = \frac{3}{4}\left(\frac{1}{2}\right) \therefore y = \frac{3}{4}$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

d) Solve, the DE $x(1-y)\frac{dy}{dx} = -2x, given\ that\ y = 2\ when\ x = e,\ \ hence, show\ that\ y = ln\left(\frac{x^2y}{2}\right)$

Solution:

Separating the variables will give $\frac{y-1}{2y}dy = \frac{1}{x}dx$

$\int\left(\frac{1}{2} - \frac{1}{2y}\right)dy = \int\frac{1}{x}dx \Longrightarrow \frac{1}{2}y - \frac{1}{2}lny = lnx + k \Longrightarrow y - lny = lnx^2 + c$

When y = 2, x = e $\Longrightarrow 2 - ln2 = 2 + c \Longrightarrow c = -ln2$

Thus $y - lny = \ln x^2 - ln2 \Longrightarrow y = ln\left(\frac{x^2y}{2}\right)$

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Homework

1. Solve in the form y = f(x), the DE $(1 - x^2)\frac{dy}{dx} - 2xy = 0,$

   $given\ that\ y = 1 when\ x = 0.$

2. Solve for y if $\frac{dy}{dx} = x^2y^2\ and\ y = 3, when\ x = 1$

3. Solve the differential equation: $x^2\frac{dy}{dx} = 4x + xy$

4. Find in the form $y = f(x)$, the solution of the differential equation

   $(1 + x^2)\frac{dy}{dx} = 4x(1 + y)$ given that $y = 1$ when $x = 1$. Show that for

   $x > 0$ the gradient of the curve is always positive

AIMS African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

Solution to number 1

1. $(1 - x^2)\frac{dy}{dx} - 2xy = 0,$

$\Longrightarrow (1 - x^2)\frac{dy}{dx} = 2xy \Longrightarrow \frac{dy}{y} = \frac{2x}{1 - x^2}$dx

$\Longrightarrow \int \frac{dy}{y} = \int \frac{2x}{1 - x^2} dx$

$\Longrightarrow lny = -ln|1 - x^2| + k$

y = 1 when x = 0 $\Longrightarrow$ k = 0

Thus $lny = -ln|1 - x^2|$

$lny = ln\left|\frac{1}{1 - x^2}\right|$ or $y = \frac{1}{1 - x^2}$

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

Solution to number 2

2. Solve for y if $\frac{dy}{dx} = x^2y^2\ and\ y = 3, when\ x = 1$

$\frac{dy}{dx} = x^2y^2 \Longrightarrow\ y^{-2}dy =\ x^2dx$

$\Longrightarrow\ \int y^{-2}dy =\ \int x^2dx$

$$= -y^{-1} = \frac{x^3}{3} + c$$

$\Longrightarrow \boldsymbol{-\frac{1}{y} = \frac{x^3}{3} + c}$ This is the general solution

Initial conditions are y = 3 when x = 1

$\Longrightarrow -\frac{1}{3} = \frac{1}{3} + c \therefore -\frac{2}{3} = c$. Substituting value of c in the general solution gives:

$\Longrightarrow \boldsymbol{-\frac{1}{y} = \frac{x^3}{3} - \frac{2}{3}}$ This is the particular solution.

AIMS | African Institute for Mathematical Sciences TEACHER TRAINING PROGRAM CAMEROON

References:

1. Advanced Level Pure Mathematics Made Easy, first Edition (2017), Ewane Roland Alunge,
2. https://www.toppr.com/guides/maths/differential-equations/general-and-particular-solutions-of-a-differential-equation/

African Institute for Mathematical Sciences
TEACHER TRAINING PROGRAM CAMEROON

If you can not so the home work exercises, go over the lesson again.

GOOD LUCK IN YOUR STUDIES