NUMERICAL FUNCTIONS

FUNCTIONS

Another important concept is that of *function*. The student will recall that a function f is a rule which assigns to each object x, also called *member* or *element*, of a set A an element y of a set B. To indicate this correspondence we write $y = f(x)$ where $f(x)$ is called the *value* of the function at x.

SPECIAL TYPES OF FUNCTIONS

1. Polynomials $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$. If $a_0 \neq 0$, n is called the *degree* of the polynomial. The polynomial equation $f(x) = 0$ has exactly n *roots* provided we

2. Exponential Functions $f(x) = a^x$. These functions obey the rules (*1*).

An important special case occurs where $a = e = 2.7182818 \cdots$.

The product $a \cdot a \ldots a$ of a real number a by itself p times is denoted by a^p, where p is called the *exponent* and a is called the *base*. The following rules hold:

1. $a^p \cdot a^q = a^{p+q}$
2. $\dfrac{a^p}{a^q} = a^{p-q}$
3. $(a^p)^r = a^{pr}$
4. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$

1.15. Evaluate each of the following:

(*a*) $\dfrac{3^4 \cdot 3^8}{3^{14}} = \dfrac{3^{4+8}}{3^{14}} = 3^{4+8-14} = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(*b*) $\sqrt{\dfrac{(5 \cdot 10^{-6})(4 \cdot 10^2)}{8 \cdot 10^5}} = \sqrt{\dfrac{5 \cdot 4}{8} \cdot \dfrac{10^{-6} \cdot 10^2}{10^5}} = \sqrt{2.5 \cdot 10^{-9}} = \sqrt{25 \cdot 10^{-10}} = 5 \cdot 10^{-5}$ or 0.00005

(*c*) $\log_{2/3}\left(\frac{27}{8}\right) = x$. Then $\left(\frac{2}{3}\right)^x = \frac{27}{8} = \left(\frac{3}{2}\right)^3 = \left(\frac{2}{3}\right)^{-3}$ or $x = -3$

(*d*) $(\log_a b)(\log_b a) = u$. Then $\log_a b = x, \log_b a = y$ assuming $a, b > 0$ and $a, b \neq 1$.

Then $a^x = b$, $b^y = a$ and $u = xy$.
Since $(a^x)^y = a^{xy} = b^y = a$ we have $a^{xy} = a^1$ or $xy = 1$ the required value.

3. Logarithmic Functions $f(x) = \log_a x$. These functions are *inverses* of the exponential functions, i.e. if $a^x = y$ then $x = \log_a y$ where a is called the *base* of the logarithm. Interchanging x and y gives $y = \log_a x$. If $a = e$, which is often called the *natural base* of logarithms, we denote $\log_e x$ by $\ln x$, called *the natural logarithm* of x. The fundamental rules satisfied by natural logarithms [or logarithms to any base] are

$$\ln(mn) = \ln m + \ln n, \quad \ln\frac{m}{n} = \ln m - \ln n, \quad \ln m^p = p \ln m \tag{4}$$

4. Trigonometric Functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$.

Some fundamental relationships among these functions are as follows.

(a) $$\sin x = \cos\left(\frac{\pi}{2} - x\right), \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right), \qquad \tan x = \frac{\sin x}{\cos x},$$
$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}$$

(b) $\sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \tan^2 x = 1, \quad \csc^2 x - \cot^2 x = 1$

(c) $\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$

(d) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

(e) $A \cos x + B \sin x = \sqrt{A^2 + B^2}\sin(x + \alpha)$ where $\tan \alpha = A/B$

The trigonometric functions are *periodic*. For example $\sin x$ and $\cos x$, shown in Fig. 1-2 and 1-3 respectively, have period 2π.

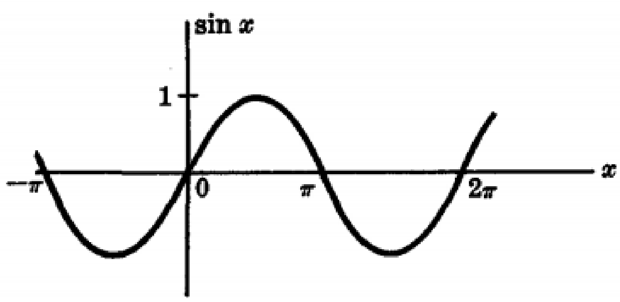

Fig. 1-2

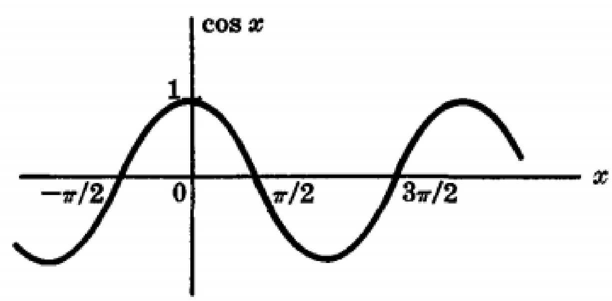

Fig. 1-3

5. Inverse Trigonometric Functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$. These are *inverses* of the trigonometric functions. For example if $\sin x = y$ then $x = \sin^{-1} y$, or on interchanging x and y, $y = \sin^{-1} x$.

6. Hyperbolic Functions. These are defined in terms of exponential functions as follows.

(a) $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$,

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$,

$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$, $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Some fundamental identities analogous to those for trigonometric functions are

(b) $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x + \tanh^2 x = 1$, $\coth^2 x - \operatorname{csch}^2 x = 1$

(c) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

The inverse hyperbolic functions, given by $\sinh^{-1} x$, $\cosh^{-1} x$, etc. can be expressed in terms of logarithms [see Problem 1.9, for example].

Example 1.4.2 Solve the following equation for x:

$$\log_3(x^4) + \log_3 x^3 - 2\log_3 x^{1/2} = 5.$$

Using logarithm properties, we get

$$\begin{aligned} 4\log_3 x + 3\log_3 x - \log_3 x &= 5 \\ 6\log_3 x &= 5 \\ \log_3 x &= \frac{5}{6} \\ x &= (3)^{5/6}. \end{aligned}$$

Example 1.4.3 Solve the following equation for x:

$$\frac{e^x}{1+e^x} = \frac{1}{3}.$$

On multiplying through, we get

$$3e^x = 1 + e^x \text{ or } 2e^x = 1, e^x = \frac{1}{2}$$
$$x = \ln(1/2) = -\ln(2).$$

Example 1.4.4 Prove that for all real x, $cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\ &= \frac{1}{4}[e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] \\ &= \frac{1}{4}[4] \\ &= 1 \end{aligned}$$

Example 1.4.5 Prove that

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(b) $\sinh 2x = 2 \sinh x \cosh y$.

Equation (b) follows from equation (a) by letting $x = y$. So, we work with equation (a).

(a)
$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^y + e^{-y}) \\
&\quad + \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^y - e^{-y}) \\
&= \frac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) \\
&\quad + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\
&= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)}] \\
&= \frac{1}{2}(e^{(x+y)} - e^{-(x+y)}) \\
&= \sinh(x+y).
\end{aligned}$$

Example 1.4.6 Find the inverses of the following functions:

(a) $\sinh x$ (b) $\cosh x$ (c) $\tanh x$

(a) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Then

$$\begin{aligned}
2e^x y = 2e^x \left(\frac{1}{2}(e^x - e^{-x})\right) &= e^{2x} - 1 \\
e^{2x} - 2ye^x - 1 &= 0 \\
(e^x)^2 - (2y)e^x - 1 &= 0 \\
e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} &= y \pm \sqrt{y^2 + 1}
\end{aligned}$$

Since $e^x > 0$ for all x, $e^x = y + \sqrt{1 + y^2}$.

On taking natural logarithms of both sides, we get

$$x = \ln(y + \sqrt{1 + y^2}).$$

The inverse function of $\sinh x$, denoted arcsinh x, is defined by

$$\boxed{\operatorname{arcsinh} x = \ln(x + \sqrt{1 + x^2})}$$

(b) As in part (a), we let $y = \cosh x$ and

$$\begin{aligned}
2e^x y &= 2e^x \cdot \frac{1}{2}(e^x + e^{-x}) = e^{2x} + 1 \\
e^{2x} &- (2y)e^x + 1 = 0 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}.
\end{aligned}$$

We observe that $\cosh x$ is an even function and hence it is not one-to-one. Since $\cosh(-x) = \cosh(x)$, we will solve for the larger x. On taking natural logarithms of both sides, we get

$$x_1 = \ln(y + \sqrt{y^2 - 1}) \text{ or } x_2 = \ln(y - \sqrt{y^2 - 1}).$$

We observe that

$$\begin{aligned}
x_2 = \ln(y - \sqrt{y^2 - 1}) &= \ln\left[\frac{(y - \sqrt{y^2 - 1})(y + \sqrt{y^2 - 1})}{y + \sqrt{y^2 - 1}}\right] \\
&= \ln\left(\frac{1}{y + \sqrt{y^2 - 1}}\right) \\
&= -\ln(y + \sqrt{y^2 - 1}) = -x_1.
\end{aligned}$$

Thus, we can define, as the principal branch,

$$\boxed{\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}),\ x \geq 1}$$

(c) We begin with $y = \tanh x$ and clear denominators to get

$$\begin{aligned}
y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} &&, \quad |y| < 1 \\
e^x[(e^x + e^{-x})y] &= e^x[(e^x - e^{-x})] &&, \quad |y| < 1 \\
(e^{2x} + 1)y &= e^{2x} - 1 &&, \quad |y| < 1 \\
e^{2x}(y - 1) &= -(1 + y) &&, \quad |y| < 1 \\
e^{2x} &= -\frac{(1 + y)}{y - 1} &&, \quad |y| < 1 \\
e^{2x} &= \frac{1 + y}{1 - y} &&, \quad |y| < 1 \\
2x &= \ln\left(\frac{1 + y}{1 - y}\right) &&, \quad |y| < 1 \\
x &= \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right) &&, |y| < 1.
\end{aligned}$$

Therefore, the inverse of the function $\tanh x$, denoted $\operatorname{arctanh} x$, is defined by

$$\boxed{\operatorname{arctanh}\,, x = \frac{1}{2}\ \ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1.}$$

1. Evaluate each of the following

(a) $\log_{10}(0.001)$ (b) $\log_2(1/64)$ (c) $\ln(e^{0.001})$

(d) $\log_{10}\left(\frac{(100)^{1/3}(0.01)^2}{(.0001)^{2/3}}\right)^{0.1}$ (e) $e^{\ln(e^{-2})}$

2. Prove each of the following identities

(a) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

(b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(c) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

3. Simplify the radical expression by using the given substitution.

(a) $\sqrt{a^2+x^2},\ x = a\sinh t$ (b) $\sqrt{x^2-a^2},\ x = a\cosh t$

(c) $\sqrt{a^2-x^2},\ x = a\tanh t$

4. Find the inverses of the following functions:

(a) $\coth x$ (b) sech x (c) csch x

5. If $\cosh x = \frac{3}{2}$, find $\sinh x$ and $\tanh x$.

6. Prove that $\sinh(3t) = 3\sinh t + 4\sinh^3 t$ (Hint: Expand $\sinh(2t+t)$.)

7. Sketch the graph of each of the following functions.

a) $y = 10^x$ b) $y = 2^x$ c) $y = 10^{-x}$ d) $y = 2^{-x}$

e) $y = e^x$ f) $y = e^{-x^2}$ g) $y = xe^{-x^2}$ i) $y = e^{-x}$

j) $y = \sinh x$ k) $y = \cosh x$ l) $y = \tanh x$ m) $y = \coth x$

n) $y = \operatorname{sech} x$ o) $y = \operatorname{csch} x$

8. Sketch the graph of each of the following functions.

a) $y = \log_{10} x$ b) $y = \log_2 x$ c) $y = \ln x$ d) $y = \log_3 x$

e) $y = \operatorname{arcsinh} x$ f) $y = \operatorname{arccosh} x$ g) $y = \operatorname{arctanh} x$

9. Compute the given logarithms in terms $\log_{10} 2$ and $\log_{10} 3$.

a) $\log_{10} 36$ b) $\log_{10}\left(\frac{27}{16}\right)$ c) $\log_{10}\left(\frac{20}{9}\right)$

d) $\log_{10}(600)$ e) $\log_{10}\left(\frac{30}{16}\right)$ f) $\log_{10}\left(\frac{6^{10}}{(20)^5}\right)$

10. Solve each of the following equations for the independent variable.

a) $\ln x - \ln(x+1) = \ln(4)$ b) $2\log_{10}(x-3) = \log_{10}(x+5) + \log_{10} 4$

c) $\log_{10} t^2 = (\log_{10} t)^2$ d) $e^{2x} - 4e^x + 3 = 0$

e) $e^x + 6e^{-x} = 5$ f) $2\sinh x + \cosh x = 4$

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(ii) by using **Osborne's rule**, which states: *'the six trigonometric ratios used in trigonometrical identities relating general angles may be replaced by their corresponding hyperbolic functions, but the sign of any direct or implied product of two sines must be changed'*.

Problem 13. Given $Ae^x + Be^{-x} \equiv 4\operatorname{ch} x - 5\operatorname{sh} x$, determine the values of A and B.

$$\begin{aligned} Ae^x + Be^{-x} &\equiv 4\operatorname{ch} x - 5\operatorname{sh} x \\ &= 4\left(\frac{e^x + e^{-x}}{2}\right) - 5\left(\frac{e^x - e^{-x}}{2}\right) \\ &= 2e^x + 2e^{-x} - \frac{5}{2}e^x + \frac{5}{2}e^{-x} \\ &= -\frac{1}{2}e^x + \frac{9}{2}e^{-x} \end{aligned}$$

Equating coefficients gives: $\boldsymbol{A = -\frac{1}{2}}$ **and** $\boldsymbol{B = 4\frac{1}{2}}$

Problem 14. If $4e^x - 3e^{-x} \equiv P\operatorname{sh} x + Q\operatorname{ch} x$, determine the values of P and Q.

DIFFERENTIATION FORMULAS

In the following u, v represent functions of x while a, c, p represent constants. We assume of course that the derivatives of u and v exist, i.e. u and v are *differentiable*.

1. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

2. $\frac{d}{dx}(cu) = c\frac{du}{dx}$

3. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$

5. $\frac{d}{dx}u^p = pu^{p-1}\frac{du}{dx}$

6. $\frac{d}{dx}(a^u) = a^u \ln a$

7. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

8. $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

9. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

10. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

11. $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$

12. $\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$

13. $\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$

14. $\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx}$

15. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

16. $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

17. $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

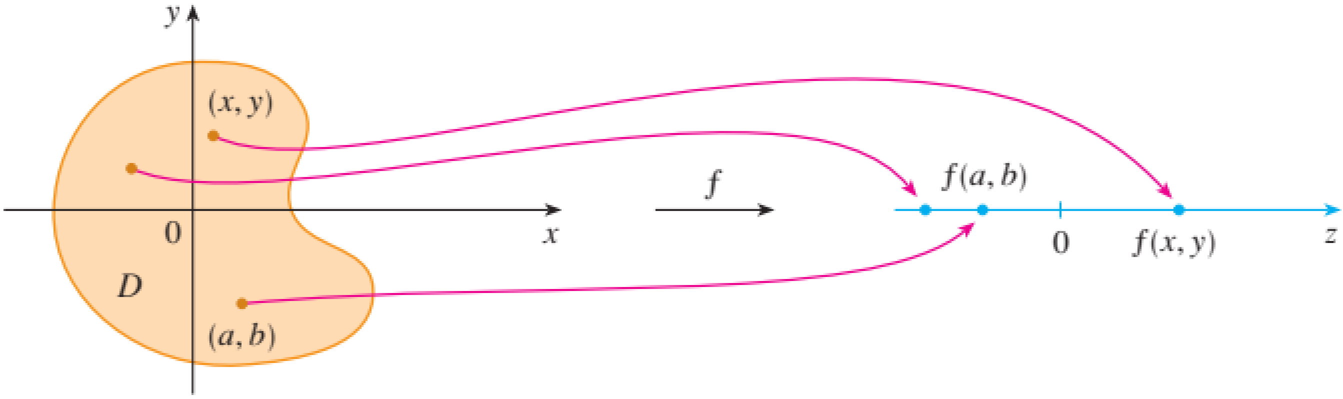
18. $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx}$

19. $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$

20. $\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$

FUNCTIONS OF SEVERAL VARIABLES

DEFINITION A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.



Domain = Set of values for which a function is defined.
Range = Set of possible values that the function can take or set of all images.
Value of a function or image of a point. E.g: If $f(x, y) = x^2 + 2y^3, then\ f(3, -1) = 3^2 + 2(-1)^2 = 7$

EXAMPLE. If $z = \sqrt{1-(x^2+y^2)}$, the domain for which z is real consists of the set of points (x, y) such that $x^2 + y^2 \leqq 1$, i.e., the set of points inside and on a circle in the xy plane having center at $(0, 0)$ and radius 1.
Range of z: Z has a least value of 0 when (x, y) lies on the circle and greatest value 1 at the centre (0,0). Thus

$$0 \leq Z \leq 1\ or\ Range(Z) = [0,1]$$

NB:$z = \sqrt{1-(x^2+y^2)} \Longrightarrow x^2 + y^2 + z^2 = 1^2, 0 \leq z \leq 1$, which is a hemisphere with centre (0,0,0) and radius 1

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center (0, 0) and radius 3. (See Figure 4.) The range of g is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\right\}$$

Since z is a positive square root, $z \geqslant 0$. Also

$$9 - x^2 - y^2 \leqslant 9 \quad \Rightarrow \quad \sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find the domain

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. ■

Exercise

6. Let $f(x, y) = \ln(x + y - 1)$.
(a) Evaluate $f(1, 1)$. (b) Evaluate $f(e, 1)$.
(c) Find and sketch the domain of f.
(d) Find the range of f.

7. Let $f(x, y) = x^2 e^{3xy}$.
(a) Evaluate $f(2, 0)$. (b) Find the domain of f.
(c) Find the range of f.

Limits

1 DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

NB: $\lim_{x \to a} f(x, b) \neq \lim_{y \to b} f(a, y)$ then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist. However, this is not sufficient to guarantee the existence of this limit.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

Example

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

TABLE 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

Properties of limits

variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\boxed{2} \qquad \lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

The Squeeze Theorem also holds.

NB: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to o} f(x,mx)$ where $m \in \mathbb{R}$

Example

Let $y = mx, m \in \mathbb{R}$ *then* $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x-y}{x+y}\right) = \lim\limits_{x\to o}\left(\frac{x-mx}{x+mx}\right) = \lim\limits_{x\to o}\left(\frac{1-m}{1+m}\right) = \frac{1-m}{1+m}$, which is not unique

Hence the limit does not exist.

Method 2

EXAMPLE. If $f(x, y) = \frac{x-y}{x+y}$, then $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{x-y}{x+y}\right) = \lim\limits_{x\to 0}(1) = 1$ and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \frac{x-y}{x+y}\right) = \lim\limits_{y\to 0}(-1) = -1$. Thus the iterated limits are not equal and so $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ cannot exist.

Continuity

$\boxed{4}$ **DEFINITION** A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Note that three conditions must be satisfied in order that $f(x, y)$ be continuous at (x_0, y_0).

1. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l$, i.e., the limit exists as $(x, y) \to (x_0, y_0)$
2. $f(x_0, y_0)$ must exist, i.e., $f(x, y)$ is defined at (x_0, y_0)
3. $l = f(x_0, y_0)$

If desired we can write this in the suggestive form $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = f(\lim\limits_{x\to x_0} x, \lim\limits_{y\to y_0} y)$.

EXAMPLE. If $f(x, y) = \begin{cases} 3xy & (x, y) \neq (1, 2) \\ 0 & (x, y) = (1, 2) \end{cases}$, then $\lim\limits_{(x,y)\to(1,2)} f(x, y) = 6 \neq f(1, 2)$. Hence, $f(x, y)$ is not continuous at $(1, 2)$. If we redefine the function so that $f(x, y) = 6$ for $(x, y) = (1, 2)$, then the function is continuous at $(1, 2)$.

If a function is not continuous at a point (x_0, y_0), it is said to be *discontinuous* at (x_0, y_0) which is then called a *point of discontinuity*. If, as in the above example, it is possible to redefine the value of a

EXAMPLE 6 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y)\to(0, 0)} g(x, y)$ does not exist (see Example 1). ■

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$. ■

Suppose that $\lim_{(x,y)\to(3,1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?

5–22 Find the limit, if it exists, or show that the limit does not exist.

5. $\displaystyle\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$

6. $\displaystyle\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x + y)$

7. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{4 - xy}{x^2 + 3y^2}$

8. $\displaystyle\lim_{(x,y)\to(1,0)} \ln\left(\frac{1 + y^2}{x^2 + xy}\right)$

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4 + 3y^4}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2 + y^2}$

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4 + y^4}$

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2}$

16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2 + 2y^2}$

17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$

19. $\displaystyle\lim_{(x,y,z)\to(3,0,1)} e^{-xy}\sin(\pi z/2)$

20. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$

21. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

22. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2}$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \dfrac{\sin(xy)}{e^x - y^2}$

30. $F(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$

31. $F(x, y) = \arctan\left(x + \sqrt{y}\right)$

32. $F(x, y) = e^{x^2 y} + \sqrt{x + y^2}$

33. $G(x, y) = \ln(x^2 + y^2 - 4)$

34. $G(x, y) = \tan^{-1}\left((x + y)^{-2}\right)$

35. $f(x, y, z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$

36. $f(x, y, z) = \sqrt{x + y + z}$

37. $f(x, y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

38. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

25–26 Find $h(x, y) = g(f(x, y))$ and the set on which h is continuous.

25. $g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$

26. $g(t) = t + \ln t, \quad f(x, y) = \dfrac{1 - xy}{1 + x^2 y^2}$

PARTIAL DIFFERENTIATION

The derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the partial derivative of the function with respect to the variable.

If $z = f(x, y)$ then

The partial derivative of f with respect to x is

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

The partial derivative of f with respect to y is

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE. If $f(x, y) = 2x^3 + 3xy^2$, then $f_x = \partial f/\partial x = 6x^2 + 3y^2$ and $f_y = \partial f/\partial y = 6xy$. Also, $f_x(1, 2) = 6(1)^2 + 3(2)^2 = 18$, $f_y(1, 2) = 6(1)(2) = 12$.

EXAMPLE 5 Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Higher Order Partial Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ and the order of differentiation is immaterial;

EXAMPLE. If $f(x, y) = 2x^3 + 3xy^2$ (see preceding example), then $f_{xx} = 12x$, $f_{yy} = 6x$, $f_{xy} = 6y = f_{yx}$. In such case $f_{xx}(1, 2) = 12$, $f_{yy}(1, 2) = 6$, $f_{xy}(1, 2) = f_{yx}(1, 2) = 12$.

EXAMPLE 6 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

V EXAMPLE 7 Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

V EXAMPLE 3 If $f(x, y) = \sin\left(\frac{x}{1 + y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1 + y}\right) = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{1}{1 + y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1 + y}\right) = -\cos\left(\frac{x}{1 + y}\right) \cdot \frac{x}{(1 + y)^2}$$

PARTIAL DIFFERENTIAL EQUATIONS

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is called **Laplace's equation** a

EXAMPLE 8 Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

SOLUTION

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

EXAMPLE 9 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \qquad u_{xx} = -\sin(x - at)$$

$$u_t = -a\cos(x - at) \qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation.

THE CHAIN RULE :

$\boxed{2}$ **THE CHAIN RULE (CASE 1)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

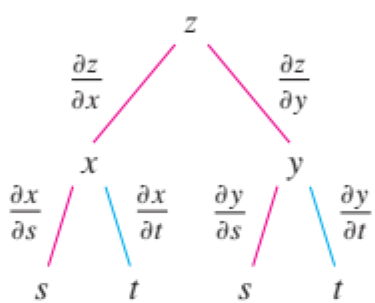
$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$ ■

$\boxed{3}$ **THE CHAIN RULE (CASE 2)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$$

V EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

SOLUTION With the help of the tree diagram in Figure 4, we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$ ■

DIFFERENTIALS

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

10
$$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

V EXAMPLE 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

SOLUTION

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

EXAMPLE 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

IMPLICIT DIFFERENTIATION

If $F(x, y) = 0$ then

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Similarly if $F(x, y, z) = 0$ then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, , $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$, etc

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Exercise

1–6 Use the Chain Rule to find dz/dt or dw/dt.

1. $z = x^2 + y^2 + xy, \quad x = \sin t, \quad y = e^t$

2. $z = \cos(x + 4y), \quad x = 5t^4, \quad y = 1/t$

3. $z = \sqrt{1 + x^2 + y^2}$, $x = \ln t$, $y = \cos t$

4. $z = \tan^{-1}(y/x)$, $x = e^t$, $y = 1 - e^{-t}$

$\boxed{5.}$ $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$

6. $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3$, $x = s\cos t$, $y = s\sin t$

8. $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$

9. $z = \sin\theta\cos\phi$, $\theta = st^2$, $\phi = s^2t$

10. $z = e^{x+2y}$, $x = s/t$, $y = t/s$

$\boxed{11.}$ $z = e^r\cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

$\boxed{3}$ **THEOREM** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?

Solution

Unit vector given by $\theta = \frac{\pi}{6}$ $is\ \hat{u} = cos\frac{\pi}{6}\hat{\imath} + sin\frac{\pi}{6}\hat{\jmath}$

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

$$D_{\mathbf{u}}f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle$$

and $$\nabla f(0, 1) = \langle 2, 0\rangle$$ ■

V EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\,\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\,\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}}f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\,\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$ ■

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

12
$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\,a + f_y(x, y, z)\,b + f_z(x, y, z)\,c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13
$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}}$$

V EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle \end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{2}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k} \right) \\ &= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

■

15 THEOREM Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}} f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle \\ &= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

■

TANGENT PLANES AND LINEAR APPROXIMATIONS

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Linear Approximation

In general, we know from (2) that an equation of the tangent plane to the graph of function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\boxed{3} \qquad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$\boxed{4} \qquad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Example

Find the linear approximation of $f(x,y) = xe^{xy}\ at\ (1,0)$. Hence, find an approximation for $f(1.1, -0.1)$

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Exercise

1. Given the surface $z = 4x^2 - y^2 + 2y\ and\ the\ point\ P(-1,2,4)$

a) Find the equation of the tangent plane and to the surface at P.

b) Hence, find the vector and cartesian equation of the normal to the surface at P.

c) Find the linear approximation of the function $f(x, y) = z\ at\ P$

2. Find the tangent plane and the cartesian equation of the normal to the surface $z = 2x^2 + y^2\ at\ (1,1,3)$

V EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

Exercise

1–6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 4x^2 - y^2 + 2y, \quad (-1, 2, 4)$

2. $z = 3(x - 1)^2 + 2(y + 3)^2 + 7, \quad (2, -2, 12)$

3. $z = \sqrt{xy}, \quad (1, 1, 1)$

4. $z = y \ln x, \quad (1, 4, 0)$

5. $z = y \cos(x - y), \quad (2, 2, 2)$

6. $z = e^{x^2 - y^2}, \quad (1, -1, 1)$

19. Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $f(1.95, 1.08)$.

20. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing f and the tangent plane.

17–18 Verify the linear approximation at $(0, 0)$.

17. $\dfrac{2x + 3}{4y + 1} \approx 3 + 2x - 12y$

18. $\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$

25–30 Find the differential of the function.

25. $z = x^3 \ln(y^2)$

26. $v = y \cos xy$

27. $m = p^5 q^3$

28. $T = \dfrac{v}{1 + uvw}$

29. $R = \alpha\beta^2 \cos \gamma$

30. $w = xye^{xz}$

31. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

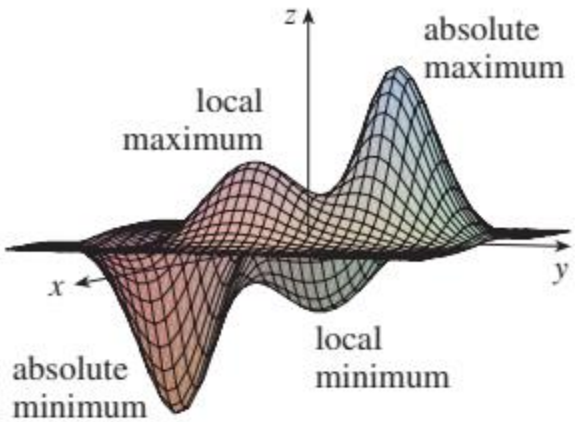
32. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

33. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

34. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

MAXIMUM AND MINIMUM VALUES

1 DEFINITION A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.



If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

2 THEOREM If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

V EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad\qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad\qquad f_{xy} = -4 \qquad\qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4. ■

Example 1.11 Find and classify the critical points of

$$f(x,y) = x^2y - x^2 - \frac{1}{3}y^3.$$

Solution: First finding the critical points

$$\begin{array}{lcl} f_x = 2xy - 2x, & & f_y = x^2 - y^2, \\ 2x(y-1) = 0, & \text{and} & x^2 - y^2 = 0, \\ x = 0 \quad \text{or} \quad y = 1 & \text{and} & x^2 = y^2. \end{array}$$

Therefore for $x = 0, \quad y^2 = 0 \Rightarrow y = 0$ and for $y = 1, \quad x^2 = 1 \Rightarrow x = \pm 1$.
Thus we have three critical points $(0,0),\ (1,1)$ and $(-1,1)$.
Checking for type of critical points

$$\begin{array}{l} f_{xx} = 2y - 2, \\ f_{yy} = -2y, \\ f_{xy} = 2x. \end{array} \quad \Rightarrow \mathbf{H} = \begin{bmatrix} 2y-2 & -2y \\ -2y & 2x \end{bmatrix}$$

At $(0,0)$,	$\det \mathbf{H} = (-2)(0) - 0^2 = 0$	therefore the test is inconclusive.
At $(1,1)$,	$\det \mathbf{H} = (0)(-2) - 2^2 = -4$	therefore (1,1) is a saddle point.
At $(-1,1)$,	$\det \mathbf{H} = (0)(-2) - (-2)^2 = -4$	therefore (-1,1) is a saddle point.

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$\boxed{4} \qquad 2x(10y - 5 - 2x^2) = 0$$

$$\boxed{5} \qquad 5x^2 - 4y - 4y^3 = 0$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$x^2 = 5y - 2.5 \tag{6}$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$4y^3 - 21y + 12.5 = 0 \tag{7}$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	-10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

Example 1.10 Find the critical point(s) of the function:

$$g(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y.$$

Solution: We need to solve

$$\begin{aligned} \frac{\partial g}{\partial x} &= 0 \\ 2x + 6y + 2 &= 0 \\ x + 3y &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \frac{\partial g}{\partial y} &= 0 \\ 8y + 6x - 4 &= 0 \\ 4y + 3x &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 3 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & -5 & 5 \end{array}\right].$$

Therefore

$$-5y = 5 \quad \Rightarrow \quad y = -1,$$

and

$$x + 3y = -1 \quad \Rightarrow \quad x = 2.$$

Therefore there is a critical point at $(2, -1)$.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad\qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad\qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad\qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad\qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad\qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f. ■

5–18 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$

6. $f(x, y) = x^3y + 12x^2 - 8y$

7. $f(x, y) = x^4 + y^4 - 4xy + 2$

8. $f(x, y) = e^{4y-x^2-y^2}$

9. $f(x, y) = (1 + xy)(x + y)$

10. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

11. $f(x, y) = x^3 - 12xy + 8y^3$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. ∎

Vector Operator

GRADIENT, DIVERGENCE, AND CURL

Consider the vector operator ∇ (*del*) defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Then if $\phi(x, y, z)$ and $\mathbf{A}(x, y, z)$ have continuous first partial derivatives

1. **Gradient.** The *gradient* of ϕ is defined by

$$\begin{aligned}\operatorname{grad}\phi = \nabla\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \\ &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\end{aligned}$$

2. **Divergence.** The *divergence* of $\mathbf{A}$ is defined by

$$\begin{aligned}\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

3. **Curl.** The *curl* of $\mathbf{A}$ is defined by

$$\begin{aligned}\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3\end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix}\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_2 & A_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ A_1 & A_2\end{vmatrix} + \mathbf{k}\begin{vmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ A_1 & A_2\end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\end{aligned}$$

7.32. If $\phi(x, y, z) = x^2yz$ and $\mathbf{A} = 3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k}$, find $\dfrac{\partial^2}{\partial y\,\partial z}(\phi\mathbf{A})$ at the point $(1, -2, -1)$.

$$\phi\mathbf{A} = (x^2yz)(3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k}) = 3x^4y^2z\mathbf{i} + x^2y^2z^3\mathbf{j} - x^3yz^2\mathbf{k}$$

$$\frac{\partial}{\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial z}(3x^4y^2z\mathbf{i} + x^2y^2z^3\mathbf{j} - x^3yz^2\mathbf{k}) = 3x^4y^2\mathbf{i} + 3x^2y^2z^2\mathbf{j} - 2x^3yz\mathbf{k}$$

$$\frac{\partial^2}{\partial y\,\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial y}(3x^4y^2\mathbf{i} + 3x^2y^2z^2\mathbf{j} - 2x^3yz\mathbf{k}) = 6x^4y\mathbf{i} + 6x^2yz^2\mathbf{j} - 2x^3z\mathbf{k}$$

If $x = 1$, $y = -2$, $z = -1$, this becomes $-12\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}$.

7.33. If $\mathbf{A} = x^2 \sin y\mathbf{i} + z^2 \cos y\mathbf{j} - xy^2\mathbf{k}$, find $d\mathbf{A}$.

Method 1:

$$\frac{\partial \mathbf{A}}{\partial x} = 2x\sin y\mathbf{i} - y^2\mathbf{k}, \quad \frac{\partial \mathbf{A}}{\partial y} = x^2\cos y\mathbf{i} - z^2\sin y\mathbf{j} - 2xy\mathbf{k}, \quad \frac{\partial \mathbf{A}}{\partial z} = 2z\cos y\mathbf{j}$$

$$\begin{aligned}
d\mathbf{A} &= \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz \\
&= (2x\sin y\mathbf{i} - y^2\mathbf{k})\,dx + (x^2\cos y\mathbf{i} - z^2\sin y\mathbf{j} - 2xy\mathbf{k})\,dy + (2z\cos y\mathbf{j})\,dz \\
&= (2x\sin y\,dx + x^2\cos y\,dy)\mathbf{i} + (2z\cos y\,dz - z^2\sin y\,dy)\mathbf{j} - (y^2\,dx + 2xy\,dy)\mathbf{k}
\end{aligned}$$

7.34. If $\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find (*a*) $\nabla\phi$, (*b*) $\nabla\cdot\mathbf{A}$, (*c*) $\nabla\times\mathbf{A}$, (*d*) $\operatorname{div}(\phi\mathbf{A})$, (*e*) $\operatorname{curl}(\phi\mathbf{A})$.

(*a*)
$$\begin{aligned}
\nabla\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \frac{\partial}{\partial y}(x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz^3)\mathbf{k} \\
&= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}
\end{aligned}$$

(*b*)
$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}) \\
&= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y
\end{aligned}$$

(*c*)
$$\begin{aligned}
\nabla\times\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2x^2y) - \frac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k} \\
&= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}
\end{aligned}$$

$$(d)\quad \text{div}\,(\phi\mathbf{A}) = \nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$
$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

$$(e)\quad \text{curl}\,(\phi\mathbf{A}) = \nabla\times(\phi\mathbf{A}) = \nabla\times(x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$
$$= (4x^4yz^3 - 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

7.37. Find a unit normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at the point $P(3, -1, 2)$.

By Problem 7.36, a vector normal to the surface is

$$\nabla(2x^2 + 4yz - 5z^2) = 4x\mathbf{i} + 4z\mathbf{j} + (4y - 10z)\mathbf{k} = 12\mathbf{i} + 8\mathbf{j} - 24\mathbf{k} \quad \text{at} \quad (3, -1, 2)$$

Then a unit normal to the surface at P is $\dfrac{12\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}}{\sqrt{(12)^2 + (8)^2 + (-24)^2}} = \dfrac{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}}{7}$.

Another unit normal to the surface at P is $-\dfrac{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}}{7}$.

7.38. If $\phi = 2x^2y - xz^3$, find (*a*) $\nabla\phi$ and (*b*) $\nabla^2\phi$.

$$(a)\quad \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (4xy - z^3)\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}$$

$$(b)\quad \nabla^2\phi = \text{Laplacian of } \phi = \nabla\cdot\nabla\phi = \frac{\partial}{\partial x}(4xy - z^3) + \frac{\partial}{\partial y}(2x^2) + \frac{\partial}{\partial z}(-3xz^2) = 4y - 6xz$$

Another method:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2}{\partial x^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial y^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial z^2}(2x^2y - xz^3)$$
$$= 4y - 6xz$$

7.77. If $\phi = xy + yz + zx$ and $\mathbf{A} = x^2y\mathbf{i} + y^2z\mathbf{j} + z^2x\mathbf{k}$, find (*a*) $\mathbf{A}\cdot\nabla\phi$, (*b*) $\phi\nabla\cdot\mathbf{A}$, and (*c*) $(\nabla\phi)\times\mathbf{A}$ at the point $(3, -1, 2)$. *Ans.* (*a*) 25, (*b*) 2, (*c*) $56\mathbf{i} - 30\mathbf{j} + 47\mathbf{k}$

7.81. Find a unit normal to the surface $x^2y - 2xz + 2y^2z^4 = 10$ at the point $(2, 1, -1)$.
Ans. $\pm(3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})/\sqrt{61}$

7.82. If $\mathbf{A} = 3xz^2\mathbf{i} - yz\mathbf{j} + (x + 2z)\mathbf{k}$, find curl curl $\mathbf{A}$. *Ans.* $-6x\mathbf{i} + (6z - 1)\mathbf{k}$

CONCEPT CHECK

1. (a) What is a function of two variables?
(b) Describe three methods for visualizing a function of two variables.

2. What is a function of three variables? How can you visualize such a function?

3. What does

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

mean? How can you show that such a limit does not exist?

4. (a) What does it mean to say that f is continuous at (a, b)?
(b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.
(b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret them as rates of change?
(c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y?

6. What does Clairaut's Theorem say?

7. How do you find a tangent plane to each of the following types of surfaces?
(a) A graph of a function of two variables, $z = f(x, y)$
(b) A level surface of a function of three variables, $F(x, y, z) = k$

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

9. (a) What does it mean to say that f is differentiable at (a, b)?
(b) How do you usually verify that f is differentiable?

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$. How do you interpret it as a rate? How do you interpret it geometrically?
(b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

14. (a) Define the gradient vector ∇f for a function f of two or three variables.
(b) Express $D_{\mathbf{u}} f$ in terms of ∇f.
(c) Explain the geometric significance of the gradient.

15. What do the following statements mean?
(a) f has a local maximum at (a, b).
(b) f has an absolute maximum at (a, b).
(c) f has a local minimum at (a, b).
(d) f has an absolute minimum at (a, b).
(e) f has a saddle point at (a, b).

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?
(b) What is a critical point of f?

17. State the Second Derivatives Test.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?
(b) State the Extreme Value Theorem for functions of two variables.
(c) How do you find the values that the Extreme Value Theorem guarantees?

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

EXERCISES

1–2 Find and sketch the domain of the function.

1. $f(x, y) = \ln(x + y + 1)$

2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

3–4 Sketch the graph of the function.

3. $f(x, y) = 1 - y^2$

4. $f(x, y) = x^2 + (y - 2)^2$

5–6 Sketch several level curves of the function.

5. $f(x, y) = \sqrt{4x^2 + y^2}$

6. $f(x, y) = e^x + y$

7. Make a rough sketch of a contour map for the function whose graph is shown.

9–10 Evaluate the limit or show that it does not exist.

9. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{2xy}{x^2 + 2y^2}$

10. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{2xy}{x^2 + 2y^2}$

11. A metal plate is situated in the xy-plane and occupies the rectangle $0 \leqslant x \leqslant 10$, $0 \leqslant y \leqslant 8$, where x and y are meas in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperat at equally spaced points were measured and recorded in th table.
(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?
(b) Estimate the value of $D_{\mathbf{u}} T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{\ }$ Interpret your result.
(c) Estimate the value of $T_{xy}(6, 4)$.

x \ y	0	2	4	6	8

12. Find a linear approximation to the temperature function $T(x, y)$ in Exercise 11 near the point (6, 4). Then use it to estimate the temperature at the point (5, 3.8).

13–17 Find the first partial derivatives.

13. $f(x, y) = \sqrt{2x + y^2}$

14. $u = e^{-r} \sin 2\theta$

15. $g(u, v) = u \tan^{-1} v$

16. $w = \dfrac{x}{y - z}$

17. $T(p, q, r) = p \ln(q + e^r)$

19–22 Find all second partial derivatives of f.

19. $f(x, y) = 4x^3 - xy^2$

20. $z = xe^{-2y}$

21. $f(x, y, z) = x^k y^l z^m$

22. $v = r \cos(s + 2t)$

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23. If $z = xy + xe^{y/x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy + z$.

24. If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}$$

25–29 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

25. $z = 3x^2 - y^2 + 2x, \quad (1, -2, 1)$

26. $z = e^x \cos y, \quad (0, 0, 1)$

27. $x^2 + 2y^2 - 3z^2 = 3, \quad (2, -1, 1)$

28. $xy + yz + zx = 3, \quad (1, 1, 1)$

29. $\sin(xyz) = x + 2y + 3z, \quad (2, -1, 0)$

41. If $z = f(u, v)$, where $u = xy$, $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = -4uv \frac{\partial^2 z}{\partial u \, \partial v} + 2v \frac{\partial z}{\partial v}$$

42. If $yz^4 + x^2z^3 = e^{xyz}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

43. Find the gradient of the function $f(x, y, z) = z^2 e^{x\sqrt{y}}$.

45–46 Find the directional derivative of f at the given point in the indicated direction.

45. $f(x, y) = 2\sqrt{x} - y^2$, $(1, 5)$,
in the direction toward the point $(4, 1)$

46. $f(x, y, z) = x^2y + x\sqrt{1 + z}$, $(1, 2, 3)$,
in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

STUDY QUESTIONS FROM SCHAUM'S SERIES

FUNCTIONS AND GRAPHS

6.1. If $f(x, y) = x^3 - 2xy + 3y^2$, find: (*a*) $f(-2, 3)$; (*b*) $f\left(\frac{1}{x}, \frac{2}{y}\right)$;

6.2. Give the domain of definition for which each of the following functions are defined and real, and indicate this domain graphically.

(*a*) $f(x, y) = \ln\{(16 - x^2 - y^2)(x^2 + y^2 - 4)\}$

(*b*) $f(x, y) = \sqrt{6 - (2x + 3y)}$

LIMITS AND CONTINUITY

6.4. Prove that $\lim_{\substack{x \to 1 \\ y \to 2}} (x^2 + 2y) = 5$.

Method 2, using theorems on limits.

$$\lim_{\substack{x \to 1 \\ y \to 2}} (x^2 + 2y) = \lim_{\substack{x \to 1 \\ y \to 2}} x^2 + \lim_{\substack{x \to 1 \\ y \to 2}} 2y = 1 + 4 = 5$$

6.5. Prove that $f(x, y) = x^2 + 2y$ is continuous at $(1, 2)$.

By Problem 6.4, $\lim_{\substack{x \to 1 \\ y \to 2}} f(x, y) = 5$. Also, $f(1, 2) = 1^2 + 2(2) = 5$.

Then $\lim_{\substack{x \to 1 \\ y \to 2}} f(x, y) = f(1, 2)$ and the function is continuous at $(1, 2)$.

6.6. Determine whether $f(x, y) = \begin{vmatrix} x^2 + 2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{vmatrix}$.

(*a*) has a limit as $x \to 1$ and $y \to 2$, (*b*) is continuous at $(1, 2)$.

(*a*) By Problem 6.4, it follows that $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x, y) = 5$, since the *limit* has nothing to do with the *value* at $(1, 2)$.

(*b*) Since $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x, y) = 5$ and $f(1, 2) = 0$, it follows that $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x, y) \neq f(1, 2)$. Hence, the function is *discontinuous* at $(1, 2)$.

Investigate the continuity of $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ at $(0, 0)$.

Let $x \to 0$ and $y \to 0$ in such a way that $y = mx$ (a line in the xy plane). Then along this line,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2}$$

Since the limit of the function depends on the manner of approach to $(0, 0)$ (i.e., the slope m of the line), the function cannot be continuous at $(0, 0)$.

Another method:

Since $\lim\limits_{x\to 0} \left\{ \lim\limits_{y\to 0} \dfrac{x^2 - y^2}{x^2 + y^2} \right\} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$ and $\lim\limits_{y=0} \left\{ \lim\limits_{x\to 0} \dfrac{x^2 - y^2}{x^2 + y^2} \right\} = -1$ are not equal, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ cannot exist. Hence, $f(x, y)$ cannot be continuous at $(0, 0)$.

PARTIAL DERIVATIVES

6.8. If $f(x, y) = 2x^2 - xy + y^2$, find (*a*) $\partial f/\partial x$, and (*b*) $\partial f/\partial y$

6.9. Let $f(x, y) = \begin{cases} xy/(x^2 + y^2) & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$. Prove that

that (*b*) $f(x, y)$ is discontinuous at $(0, 0)$.

(*b*) Let $(x, y) \to (0, 0)$ along the line $y = mx$ in the xy plane. Then $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y) = \lim\limits_{x\to 0} \dfrac{mx^2}{x^2 + m^2x^2} = \dfrac{m}{1 + m^2}$ so that the limit depends on m and hence on the approach and therefore does not exist. Hence, $f(x, y)$ is not continuous at $(0, 0)$.

6.10. If $\phi(x, y) = x^3y + e^{xy^2}$, find (*a*) ϕ_x, (*b*) ϕ_y, (*c*) ϕ_{xx}, (*d*) ϕ_{yy}, (*e*) ϕ_{xy}, (*f*) ϕ_{yx}.

$$(a)\quad \phi_x = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(x^3 y + e^{xy^2}) = 3x^2 y + e^{xy^2} \cdot y^2 = 3x^2 y + y^2 e^{xy^2}$$

$$(b)\quad \phi_y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^3 y + e^{xy^2}) = x^3 + e^{xy^2} \cdot 2xy = x^3 + 2xy\, e^{xy^2}$$

$$(c)\quad \phi_{xx} = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 y + y^2\, e^{xy^2}) = 6xy + y^2(e^{xy^2} \cdot y^2) = 6xy + y^4\, e^{xy^2}$$

$$\begin{aligned}(d)\quad \phi_{yy} = \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial}{\partial y}(x^3 + 2xy\, e^{xy^2}) = 0 + 2xy \cdot \frac{\partial}{\partial y}(e^{xy^2}) + e^{xy^2}\, \frac{\partial}{\partial y}(2xy) \\ &= 2xy \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2x = 4x^2 y^2 e^{xy^2} + 2x\, e^{xy^2}\end{aligned}$$

$$\begin{aligned}(e)\quad \phi_{xy} = \frac{\partial^2 \phi}{\partial y\, \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 y + y^2\, e^{xy^2}) = 3x^2 + y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y \\ &= 3x^2 + 2xy^3\, e^{xy^2} + 2y\, e^{xy^2}\end{aligned}$$

$$\begin{aligned}(f)\quad \phi_{yx} = \frac{\partial^2 \phi}{\partial x\, \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 + 2xy\, e^{xy^2}) = 3x^2 + 2xy \cdot e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 2y \\ &= 3x^2 + 2xy^3\, e^{xy^2} + 2y\, e^{xy^2}\end{aligned}$$

6.11. Show that $U(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies Laplace's partial differential equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0.$

We assume here that $(x, y, z) \neq (0, 0, 0)$. Then

$$\frac{\partial U}{\partial x} = -\tfrac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2x = -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= \frac{\partial}{\partial x}[-x(x^2 + y^2 + z^2)^{-3/2}] = (-x)[-\tfrac{3}{2}(x^2 + y^2 + z^2)^{-5/2} \cdot 2x] + (x^2 + y^2 + z^2)^{-3/2} \cdot (-1) \\ &= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\end{aligned}$$

Similarly $$\frac{\partial^2 U}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}, \qquad \frac{\partial^2 U}{\partial x^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Adding, $$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

6.12. If $z = x^2 \tan^{-1} \dfrac{y}{x}$, find $\dfrac{\partial^2 z}{\partial x\, \partial y}$ at $(1, 1)$.

$$\frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{1+(y/x)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = x^2 \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x^3}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right) = \frac{(x^2+y^2)(3x^2)-(x^3)(2x)}{(x^2+y^2)^2} = \frac{2 \cdot 3 - 1 \cdot 2}{2^2} = 1 \quad \text{at} \quad (1, 1)$$

The result can be written $z_{xy}(1, 1) = 1$.

Note: In this calculation we are using the fact that z_{xy} is continuous at $(1, 1)$ (see remark at the end of Problem 6.9).

6.15. If $z = f(x, y) = x^2 y - 3y$, find (*a*) Δz, (*b*) dz. (*c*) Determine Δz and dz if $x = 4$, $y = 3$, $\Delta x = -0.01$, $\Delta y = 0.02$. (*d*) How might you determine $f(5.12, 6.85)$ without direct computation?

Solution:

(*a*)
$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y)\\ &= \{(x+\Delta x)^2(y+\Delta y) - 3(y+\Delta y)\} - \{x^2 y - 3y\}\\ &= \underbrace{2xy\,\Delta x + (x^2-3)\Delta y}_{(A)} + \underbrace{(\Delta x)^2 y + 2x\,\Delta x\,\Delta y + (\Delta x)^2\,\Delta y}_{(B)}\end{aligned}$$

The sum (*A*) is the *principal part* of Δz and is the differential of z, i.e., dz. Thus,

(*b*) $dz = 2xy\,\Delta x + (x^2-3)\Delta y = 2xy\,dx + (x^2-3)\,dy$

Another method: $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = 2xy\,dx + (x^2-3)\,dy$

(*c*)
$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y) = f(4-0.01, 3+0.02) - f(4, 3)\\ &= \{(3.99)^2(3.02) - 3(3.02)\} - \{(4)^2(3) - 3(3)\} = 0.018702\\ dz &= 2xy\,dx + (x^2-3)\,dy = 2(4)(3)(-0.01) + (4^3-3)(0.02) = 0.02\end{aligned}$$

Note that in this case Δz and dz are approximately equal, because $\Delta x = dx$ and $\Delta y = dy$ are sufficiently small.

(*d*) We must find $f(x+\Delta x, y+\Delta y)$ when $x + \Delta x = 5.12$ and $y = \Delta y = 6.85$. We can accomplish this by choosing $x = 5$, $\Delta x = 0.12$, $y = 7$, $\delta y = -0.15$. Since Δx and Δy are small, we use the fact that $f(x+\Delta x, y+\Delta y) = f(x, y) + \Delta z$ is approximately equal to $f(x, y) + dz$, i.e., $z + dz$.

Now
$$\begin{aligned}z &= f(x, y) = f(5, 7) = (5)^2(7) - 3(7) = 154\\ dz &= 2xy\,dx + (x^2-3)\,dy = 2(5)(7)(0.12) + (5^2-3)(-0.15) = 5.1.\end{aligned}$$

Then the required value is $154 + 5.1 = 159.1$ approximately. The value obtained by direct computation is 159.01864.

(*a*) Let $U = x^2 e^{y/x}$. Find dU. (*b*) Show that $(3x^2y - 2y^2)\,dx + (x^3 -$
written as an exact differential of a function $\phi(x, y)$ and find this functior

(*a*) **Method 1:**

$$\frac{\partial U}{\partial x} = x^2 e^{y/x}\left(-\frac{y}{x^2}\right) + 2xe^{y/x}, \qquad \frac{\partial U}{\partial y} = x^2 e^{y/x}\left(\frac{1}{x}\right)$$

Then

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy = (2xe^{y/x} - ye^{y/x})\,dx + xe^{y/x}\,dy$$

Method 2:

$$\begin{aligned} dU &= x^2\, d(e^{y/x}) + e^{y/x}\, d(x^2) = x^2 e^{y/x}\, d(y/x) + 2xe^{y/x}\, dx \\ &= x^2 e^{y/x}\left(\frac{x\,dy - y\,dx}{x^2}\right) + 2xe^{y/x}\,dx = (2xe^{y/x} - ye^{y/x})\,dx + xe^{y/x}\,dy \end{aligned}$$

COMPOSITE FUNCTIONS

6.18. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, computer dz/dt at $t = \pi/2$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2 e^{xy^2})(-t\sin t + \cos t) + (2xye^{xy^2})(t\cos t + \sin t).$$

At $t = \pi/2, x = 0, y = \pi/2$. Then $\left.\dfrac{dz}{dt}\right|_{t=\pi/2} = (\pi^2/4)(-\pi/2) + (0)(1) = -\pi^3/8.$

Another method. Substitute x and y to obtain $z = e^{t^3 \sin^2 t \cos t}$ and then differentiate.

6.21. If $T = x^3 - xy + y^3$, $x = \rho\cos\phi$, $y = \rho\sin\phi$, find (*a*) $\partial T/\partial\rho$, (*b*) $\partial T/\partial\phi$.

$$\begin{aligned} \frac{\partial T}{\partial \rho} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \rho} = (3x^2 - y)(\cos\phi) + (3y^2 - x)(\sin\phi) \\ \frac{\partial T}{\partial \phi} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \phi} = (3x^2 - y)(-\rho\sin\phi) + (3y^2 - x)(\rho\cos\phi) \end{aligned}$$

This may also be worked by direct substitution of x and y in T.

6.24. Show that $z = f(x^2y)$, where f is differentiable, satisfies $x(\partial z/\partial x) = 2y(\partial z/\partial y)$.

Let $x^2y = u$. Then $z = f(u)$. Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u)\cdot 2xy, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u)\cdot x^2$$

Then $x\dfrac{\partial z}{\partial x} = f'(u)\cdot 2x^2y, \qquad 2y\dfrac{\partial z}{\partial y} = f'(u)\cdot 2x^2y$ and so $x\dfrac{\partial z}{\partial x} = 2y\dfrac{\partial z}{\partial y}.$

6.34. If $z^3 - xz - y = 0$, prove that $\dfrac{\partial^2 z}{\partial x\,\partial y} = -\dfrac{3z^2 + x}{(3z^2 - x)^3}$.

Differentiating with respect to x, keeping y constant and remembering that z is the dependent variable depending on the independent variables x and y, we find

$$3z^2 \frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z = 0 \qquad \text{and} \qquad (1) \quad \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

Differentiating with respect to y, keeping x constant, we find

$$3z^2 \frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0 \qquad \text{and} \qquad (2) \quad \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

Differentiating (*2*) with respect to x and using (*1*), we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{-1}{(3z^2 - x)^2}\left(6z\frac{\partial z}{\partial x} - 1\right) = \frac{1 - 6z[z/(3z^2 - x)]}{(3z^2 - x)^2} = -\frac{3z^2 + x}{(3z^2 - x)^3}$$

The result can also be obtained by differentiating (*1*) with respect to y and using (*2*).

GRADIENT, DIVERGENCE, AND CURL

Consider the vector operator ∇ (*del*) defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Then if $\phi(x, y, z)$ and $\mathbf{A}(x, y, z)$ have continuous first partial derivatives

1. **Gradient.** The *gradient* of ϕ is defined by

$$\operatorname{grad}\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$
$$= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

2. **Divergence.** The *divergence* of $\mathbf{A}$ is defined by

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$
$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

3. **Curl.** The *curl* of $\mathbf{A}$ is defined by

$$\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \mathbf{i}\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_2 & A_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ A_1 & A_2 \end{vmatrix}$$
$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

7.32. If $\phi(x, y, z) = x^2yz$ and $\mathbf{A} = 3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k}$, find $\dfrac{\partial^2}{\partial y\,\partial z}(\phi\mathbf{A})$ at the point $(1, -2, -1)$.

$$\phi\mathbf{A} = (x^2yz)(3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k}) = 3x^4y^2z\mathbf{i} + x^2y^2z^3\mathbf{j} - x^3yz^2\mathbf{k}$$

$$\frac{\partial}{\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial z}(3x^4y^2z\mathbf{i} + x^2y^2z^3\mathbf{j} - x^3yz^2\mathbf{k}) = 3x^4y^2\mathbf{i} + 3x^2y^2z^2\mathbf{j} - 2x^3yz\mathbf{k}$$

$$\frac{\partial^2}{\partial y\,\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial y}(3x^4y^2\mathbf{i} + 3x^2y^2z^2\mathbf{j} - 2x^3yz\mathbf{k}) = 6x^4y\mathbf{i} + 6x^2yz^2\mathbf{j} - 2x^3z\mathbf{k}$$

If $x = 1$, $y = -2$, $z = -1$, this becomes $-12\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}$.

7.33. If $\mathbf{A} = x^2 \sin y\mathbf{i} + z^2 \cos y\mathbf{j} - xy^2\mathbf{k}$, find $d\mathbf{A}$.

Method 1:

$$\frac{\partial\mathbf{A}}{\partial x} = 2x\sin y\mathbf{i} - y^2\mathbf{k}, \quad \frac{\partial\mathbf{A}}{\partial y} = x^2\cos y\mathbf{i} - z^2\sin y\mathbf{j} - 2xy\mathbf{k}, \quad \frac{\partial\mathbf{A}}{\partial z} = 2z\cos y\mathbf{j}$$

$$\begin{aligned} d\mathbf{A} &= \frac{\partial\mathbf{A}}{\partial x}dx + \frac{\partial\mathbf{A}}{\partial y}dy + \frac{\partial\mathbf{A}}{\partial z}dz \\ &= (2x\sin y\mathbf{i} - y^2\mathbf{k})\,dx + (x^2\cos y\mathbf{i} - z^2\sin y\mathbf{j} - 2xy\mathbf{k})\,dy + (2z\cos y\mathbf{j})\,dz \\ &= (2x\sin y\,dx + x^2\cos y\,dy)\mathbf{i} + (2z\cos y\,dz - z^2\sin y\,dy)\mathbf{j} - (y^2\,dx + 2xy\,dy)\mathbf{k} \end{aligned}$$

7.34. If $\phi = x^2yz^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find (*a*) $\nabla\phi$, (*b*) $\nabla \cdot \mathbf{A}$, (*c*) $\nabla \times \mathbf{A}$, (*d*) $\operatorname{div}(\phi\mathbf{A})$, (*e*) $\operatorname{curl}(\phi\mathbf{A})$.

(*a*) $$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \frac{\partial}{\partial x}(x^2yz^3)\mathbf{i} + \frac{\partial}{\partial y}(x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(x^2yz^3)\mathbf{k}$$
$$= 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$$

(*b*) $$\nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$
$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) = z - 2y$$

(*c*) $$\nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y^2 & 2x^2y \end{vmatrix}$$
$$= \left(\frac{\partial}{\partial y}(2x^2y) - \frac{\partial}{\partial z}(-y^2)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(2x^2y)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k}$$
$$= 2x^2\mathbf{i} + (x - 4xy)\mathbf{j}$$

(*d*) $$\operatorname{div}(\phi\mathbf{A}) = \nabla \cdot (\phi\mathbf{A}) = \nabla \cdot (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \frac{\partial}{\partial x}(x^3yz^4) + \frac{\partial}{\partial y}(-x^2y^3z^3) + \frac{\partial}{\partial z}(2x^4y^2z^3)$$
$$= 3x^2yz^4 - 3x^2y^2z^3 + 6x^4y^2z^2$$

(*e*) $$\operatorname{curl}(\phi\mathbf{A}) = \nabla \times (\phi\mathbf{A}) = \nabla \times (x^3yz^4\mathbf{i} - x^2y^3z^3\mathbf{j} + 2x^4y^2z^3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3yz^4 & -x^2y^3z^3 & 2x^4y^2z^3 \end{vmatrix}$$
$$= (4x^4yz^3 - 3x^2y^3z^2)\mathbf{i} + (4x^3yz^3 - 8x^3y^2z^3)\mathbf{j} - (2xy^3z^3 + x^3z^4)\mathbf{k}$$

7.37. Find a unit normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at the point $P(3, -1, 2)$.

By Problem 7.36, a vector normal to the surface is

$$\nabla(2x^2 + 4yz - 5z^2) = 4x\mathbf{i} + 4z\mathbf{j} + (4y - 10z)\mathbf{k} = 12\mathbf{i} + 8\mathbf{j} - 24\mathbf{k} \quad \text{at} \quad (3, -1, 2)$$

Then a unit normal to the surface at P is $\dfrac{12\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}}{\sqrt{(12)^2 + (8)^2 + (-24)^2}} = \dfrac{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}}{7}$.

Another unit normal to the surface at P is $-\dfrac{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}}{7}$.

7.38. If $\phi = 2x^2y - xz^3$, find (*a*) $\nabla\phi$ and (*b*) $\nabla^2\phi$.

(*a*) $$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (4xy - z^3)\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}$$

(*b*) $$\nabla^2\phi = \text{Laplacian of } \phi = \nabla \cdot \nabla\phi = \frac{\partial}{\partial x}(4xy - z^3) + \frac{\partial}{\partial y}(2x^2) + \frac{\partial}{\partial z}(-3xz^2) = 4y - 6xz$$

Another method:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2}{\partial x^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial y^2}(2x^2y - xz^3) + \frac{\partial^2}{\partial z^2}(2x^2y - xz^3)$$
$$= 4y - 6xz$$

7.77. If $\phi = xy + yz + zx$ and $\mathbf{A} = x^2y\mathbf{i} + y^2z\mathbf{j} + z^2x\mathbf{k}$, find (*a*) $\mathbf{A} \cdot \nabla\phi$, (*b*) $\phi\nabla \cdot \mathbf{A}$, and (*c*) $(\nabla\phi) \times \mathbf{A}$ at the point $(3, -1, 2)$. *Ans.* (*a*) 25, (*b*) 2, (*c*) $56\mathbf{i} - 30\mathbf{j} + 47\mathbf{k}$

7.81. Find a unit normal to the surface $x^2y - 2xz + 2y^2z^4 = 10$ at the point $(2, 1, -1)$.
Ans. $\pm(3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})/\sqrt{61}$

7.82. If $\mathbf{A} = 3xz^2\mathbf{i} - yz\mathbf{j} + (x + 2z)\mathbf{k}$, find curl curl $\mathbf{A}$. *Ans.* $-6x\mathbf{i} + (6z - 1)\mathbf{k}$