Equivalence Proof

Proposition: In language THREE the sentence $E + E$ is semantically equivalent to $2 * E$ for any subexpression $E \in L(\text{THREE})$.

(Observe that $E + E \neq 2 * E$ because this is just syntax, that is, the abstract syntax trees of these two expressions do not look the same, but we expect that semantically they express the same thing.)

Proof: From elementary algebra we know that given any integer value i we have $i + i = 2 \times i$. We will use this identity in our proof.

Assume that $\langle E, C\rangle \rightarrow e$, that is, we assume that our subexpression E evaluates to some integer value e in the context of the binding environment C. So,

$$\frac{\langle \texttt{var}(x), C\rangle \rightarrow lookup(x, C) = k \qquad \langle \texttt{var}(x), C\rangle \rightarrow lookup(x, C) = k}{\langle \texttt{plus}(\texttt{var}(x), \texttt{var}(x)), C\rangle \rightarrow k + k = 2 \times k}$$

which follows from our algebraic identity above. Now,

$$\frac{\langle 2, C\rangle \rightarrow 2 \qquad \langle \texttt{var}(x), C\rangle \rightarrow lookup(x, C) = k}{\langle \texttt{times}(2, \texttt{var}(x)), C\rangle \rightarrow 2 \times k}$$

This concludes our proof. □