A Project Report

On

FRACTIONAL DIFFERENTIAL EQUATIONS

BY

INDRANIL BHAUMIK

2014B4A70924H

Under the supervision of

DR. JAGAN MOHAN JONNALAGADDA

SUBMITTED IN FULLFILLMENT OF THE REQUIREMENTS OF

MATH F366: LAB ORIENTED PROJECT (LOP)

BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI (RAJASTHAN)

HYDERABAD CAMPUS

(JAN- APR 2018)

ACKNOWLEDGMENTS

'No writer exists in vacuum, no writer's work is untouched in one way or the other through family, friends or others'. Therefore I, Indranil Bhaumik would like to express my sincere gratitude towards my institute and my faculty in-charge Dr. Jagan Mohan Jonnalagadda for the progress and completion of my study project on "Fractional Differential Equations"**.**

I would even like to thank my project-mates whose constant help and support as partners in the project helped me to meet the deadlines and expectations of the project.

Birla Institute of Technology and Science-Pilani,

Hyderabad Campus

Certificate

This is to certify that the project report entitled "FRACTIONAL DIFFERENTIAL EQUATIONS" submitted by Mr. INDRANIL BHAUMIK (ID No. 2014B4A70924H) in fulfillment of the requirements of the course **MATH F366** Lab Oriented Project Course, embodies the work done by him under my supervision and guidance.

Date: **(Dr. Jagan Mohan Jonnalagadda)**

BITS- Pilani, Hyderabad Campus

ABSTRACT

We are well aware of differential equations with integral order. The generalization of the same is called “Fractional Differential Equations”. Such equations have applications in many fields which are evaluated using various transform methods. Earlier derivatives like Riemann-Liouville & Caputo didn’t satisfy basic properties of ordinary derivatives and hence a new definition called “Conformable Fractional Derivatives” was devised. This project is aimed at learning these fractional derivatives, attempting to solve some equations, applying transforms and finding various properties of these derivatives.

CONTENTS

INTRODUCTION

While still be a lesser-explored field, the interest in Fractional derivatives is increasing due to its huge potential in solving problems in physics, engineering & control systems. Generally, it’s difficult to solve such problems directly and hence, transform methods are applied to convert them into more usual forms, say ordinary derivatives. Due to the various irregularities in the definitions of Riemann-Liouville & Caputo definitions of fractional derivatives where they don’t satisfy some basic properties that ordinary derivatives do, hence, Conformable derivatives were devised for uniformity and is the main focus of this project where we’ll learn more about its properties and look for possible solutions. We’ll also try to solve a few boundary-value problems and hence find the Green’s function.

PREVIOUS DEFINITIONS

Riemann-Liouville Fractional Derivative:

The Riemann-Liouville fractional derivative of $f(x)$ of order $\alpha, n - 1 < \alpha < n$,

$n \in \mathbb{N}$ is defined by:

$$D_a^{\alpha} f(x) = D^n \left(D_a^{-(n-\alpha)} f(x) \right)$$

$$= \frac{d^n}{dx^n} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha+1} f(t).dt$$

Caputo Fractional Derivative:

Let $n \in \mathbb{N}$, $\alpha \in [n - 1, n)$. Let the function $f(x)$ be an n-times differentiable function. Then the representation of the Caputo α derivative:

$${}^{c}D_a^{\alpha} f(x) = D_a^{-(n-\alpha)}\ {}^{c}D_a^{n} f(x)$$

where $$D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^{\alpha} \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

is the Riemann-Liouville α integral.

Shortcomings:

1. The Riemann-Liouville derivative does not satisfy D_a^{α} =1

   (D_a^{α} =1 for the Caupto derivative) , if α is not a natural number.

2. All fractional derivatives do not satisfy the Known product rule:

$$D_a^{\alpha}(fg) = fD_a^{\alpha}(f) + gD_a^{\alpha}(f)$$

3. All fractional derivatives do not satisfy the known quotient rule:

$$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}$$

4. All fractional derivatives do not satisfy the chain rule:

$$D_a^{\alpha}(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)g^{(\alpha)}(t)$$

5. All fractional derivatives don't satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$, in general.
6. The Caputo definition assumes that the function f is differentiable.

CONFORMABLE DERIVATIVES:

To overcome these shortcomings, a new definition was used.

Definition: Given a function : f: $[0,\infty) \to \mathbb{R}$. Then the (conformable fractional derivative) of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

For all t >0, $\alpha \in (0.1)$, if f is α-differentiable in some $(0,\alpha).\alpha > 0$ and, $\lim_{t\to 0^+} f(\alpha)(t)$ exists, then define$f(\alpha)(0) = \lim_{t\to 0^+} f(\alpha)(t)$.

We sometimes, write $f(\alpha)(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we say f is α-differentiable.

Theorem: If a function f: $[0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$. $\alpha \in$ (0.1] then f is continuous at t_0.

Proof:

Because $f(\alpha)$is differentiable at $x = t_0$, we know that

$$f^{(\alpha)}(t_0) = \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon t_0{}^{1-\alpha}) - f(t_0)}{\varepsilon} \text{ exists.}$$

If we next assume that $x \neq t0$ we can write the following

$$f(t_0 + \varepsilon t_0{}^{1-\alpha}) - f(t_0) = \frac{f(t_0 + \varepsilon t_0{}^{1-\alpha}) - f(t_0)}{\varepsilon} \varepsilon$$

Then some basic properties of limits give us

$$\lim_{\varepsilon \to 0}\left(f(t_0 + \varepsilon {t_0}^{1-\alpha}) - f(t_0)\right) = \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon {t_0}^{1-\alpha}) - f(t_0)}{\varepsilon} . \lim_{\varepsilon \to 0} \varepsilon$$

$$\lim_{\varepsilon \to 0}\left(f(t_0 + \varepsilon {t_0}^{1-\alpha}) - f(t_0)\right) = f'(t_0).0$$

$Let\ h = \varepsilon {t_0}^{1-\alpha}$. Then,

$$lim_{h \to 0} f(t_0 + h) = f(t_0)$$

Hence, f is continuous at t_0.

Theorem: Let $\alpha \in (0.1]$ and f, g be α-differentiable at a point $t > 0$ .Then:

1. $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}$, where f is differentiable.
2. $T_\alpha (af+bg) = a\, T_\alpha(f) + b\, T_\alpha(g)$, for all $a,b \in \mathbb{R}$
3. $T_\alpha (t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$
4. $T_\alpha (\lambda)=0$, for all constant functions $f(t) = \lambda$
5. $T_\alpha (fg) = f\, T_\alpha(g) + g\, T_\alpha(f)$
6. $T_\alpha (f/g) = [g\, T_\alpha(f) - f\, T_\alpha(g)]/ g^2$

Theorem: Rolle's Theorem for Conformable Fractional Differentiable Functions

Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies

i. f is continuous on $[a,b]$,
ii. f is α-differentiable for some $\alpha \in (0,1)$
iii. $f(a) = f(b)$.

Then, there exists $c \in (a,b)$, such that $f(\alpha)(c) = 0$.

Proof:

Since f is continuous on $[a,b]$, and $f(a) = f(b)$, there is $c \in (a,b)$, which is a point of local extrema.

With no loss of generality, assume c is a point of local minimum.

So, $f^{(\alpha)}(c) = \lim_{\varepsilon\to 0^+} \frac{f(c+\varepsilon c^{1-\alpha})-f(c)}{\varepsilon} = \lim_{\varepsilon\to 0^-} \frac{f(c+\varepsilon c^{1-\alpha})-f(c)}{\varepsilon}$,

But the first limit is non-negative, and the second limit is non-positive. Hence,

$f^{(\alpha)}(c) = 0.$

Theorem: Mean Value Theorem for Conformable Fractional Differentiable Functions

Let a > 0 and f: $[a, b] \to \mathbb{R}$ be a given function that satisfies:

i) f is continuous on $[a, b]$.

ii) f is α-differentiable for some $\alpha \in (0, 1)$.

Then, there exists $c \in (a, b)$, such that

$$f^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}$$

Proof: The equation of the secant through $(a, f(a))$ and $(b,f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right)$$

Which we can write as

$$y = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right) + f(a)$$

Let $g(x) = f(x) - \left[\frac{f(b)-f(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}\left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right) + f(a)\right].$

Note that g (a) = g(b)=0, g is continuous on $[a,b]$ and differentiable on (a,b). So by Rolle's theorem there are c in (a,b) such that $g^{(\alpha)}(c) = 0$. But,

$$g^{(\alpha)}(x) = f^{(\alpha)}(x) - \left[\frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\right]$$

So,

$$g^{(\alpha)}(c) = f^{(\alpha)}(c) - \left[\frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\right] = 0$$

$$f^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}$$

Let us now solve one fractional differential equation according to conformable definitions:

Given:

$$y^{(1/2)} + y = x^3 + 3x^{5/2}, y(0) = 0$$

To find :

$$y_h \text{ of } y^{1/2} + y = 0$$

We use:

$$y_h = e^{r\sqrt{x}}$$

Now, we have :

$$y^{(1/2)} + y = 0$$

$$\frac{r}{2}e^{r\sqrt{x}} + e^{r\sqrt{x}} = 0$$

$$e^{r\sqrt{x}}\left(\frac{r}{2}+1\right)=0$$

$$\frac{r}{2}+1=0 \qquad\qquad r=-2$$

$$y_h=e^{-2\sqrt{x}}$$

And simply the particular solution is $y_p = x^3$.

And by plugging the initial condition $y_p = x^3$, then A = 0.

$$\therefore y=y_h+y_p=e^{-2\sqrt{x}}+x^3$$

Solution of Conformable Fractional Ordinary Differential Equations via Differential Transform Method

The differential transform method (DTM) is one of the numerical methods that is used for finding the solution of differential equations.

Definition: Assume $f(t)$ is infinitely α-differentiable function for some $\alpha \in (0,1]$. Conformable fractional differential transform of $f(t)$ is defined as

$$F_\alpha(k) = \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{t_0} f\right)^{(k)}(t)\right]_{t=t_0},$$

where $(T_\alpha^{t_0} f)^{(k)}(t)$ denotes the application of the fractional derivative k times.

Definition: Let $F_\alpha(k)$ be the conformable fractional differential transform of $f(t)$. Inverse conformable fractional differential transform of $F(k)$ is defined as

$$f(t) = \sum_{k=0}^{\infty} F_\alpha(k)(t-t_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{t_0} f\right)^{(k)}(t)\right]_{t=t_0}(t-t_0)^{\alpha k}.$$

CFDT of initial conditions for integer order derivatives are defined as

$$F_\alpha(k) = \begin{cases} if\ \alpha k \in Z^+ & \frac{1}{(\alpha k)!}\left[\frac{d^{\alpha k} f(t)}{dt^{\alpha k}}\right]_{t=t_0} \\ if\ \alpha k \notin Z^+ & 0 \end{cases} \text{ for } k = 0,1,2,\ldots,\left(\frac{n}{\alpha}-1\right),$$

where n is the order of conformable fractional ordinary differential equation (CFODE).

Theorem:

If $f(t) = T_\alpha^{t_0}\left(u(t)\right)$, then $F_\alpha(k) = \alpha(k+1)U_\alpha(k+1)$.

Proof: Let CFDT of $u(t)$ is as following

$$U_\alpha(k) = \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{t_0} u\right)^{(k)}(t)\right]_{t=t_0}$$

For $f(t) = T_\alpha^{t_0}\left(u(t)\right)$,

$$F_\alpha(k) = \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{t_0}(T_\alpha^{t_0}u)\right)^{(k)}(t)\right]_{t=t_0} = \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{t_0}u\right)^{(k+1)}(t)\right]_{t=t_0}$$

$$= \alpha(k+1)\frac{1}{\alpha^{k+1}(k+1)!}\left[\left(T_\alpha^{t_0}u\right)^{(k+1)}(t)\right]_{t=t_0} = \alpha(k+1)U_\alpha(k+1)$$

Now, let us find the solution of a conformable fractional ordinary differential equations by the help of CFDTM:

We'll take the equation:

$y^{(\alpha)} + y = 0, y(0) = 1$ for $\alpha \in (0,1]$

Exact solution of the above equation is:

$y(t) = e^{-\frac{1}{\alpha}t^{\alpha}}$.

Using the theorem mentioned above, we can rewrite the equation as:

$\alpha(k+1)Y_\alpha(k+1) + Y_\alpha(k) = 0, Y_\alpha(0) = 1.$

Hence, the recurrence relation is obtained as:

$$Y_\alpha(k+1) = -\frac{1}{\alpha(k+1)}Y_\alpha(k), Y_\alpha(0) = 1.$$

For $k = 0,1,2,\dots,n,$

$$Y_\alpha(1) = -\frac{1}{\alpha}Y_\alpha(0) = -\frac{1}{\alpha}$$
$$Y_\alpha(2) = -\frac{1}{2\alpha}Y_\alpha(1) = \frac{1}{2!\,\alpha^2}$$
$$Y_\alpha(3) = -\frac{1}{3\alpha}Y_\alpha(2) = -\frac{1}{3!\,\alpha^3}$$
$$\vdots$$
$$Y_\alpha(n) = \frac{(-1)^n}{n!\,\alpha^n}$$

is obtained. Hence, the solution is:

$y(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\alpha^n}t^{n\alpha} = e^{-\frac{1}{\alpha}t^{\alpha}}$.

which is similar to the exact solution.

CONFORMABLE FRACTIONAL LAPLACE TRANSFORM

We've already given the definition of Conformable derivative earlier in the report. Since, different notations will be used in this section hence, it's mentioned below for convenience.

Conformable Fractional Derivative. For any function $\phi : [0,\infty) \rightarrow \Re$ the conformable fractional derivative of order ϖ is given as

$$\mathfrak{T}_{\varpi}\phi(\tau) = \lim_{\hbar\to 0} \frac{\phi\left(\tau + \hbar\tau^{1-\varpi}\right) - \phi(\tau)}{\hbar}$$

for all $\tau > 0$, $\varpi \in (0, 1)$. In addition, if ϕ is $(\lceil\varpi\rceil + 1)$-differentiable and continuous at $\tau > 0$, then for $\varpi \in (\lceil\varpi\rceil, \lceil\varpi\rceil + 1]$,

$$\mathfrak{T}_{\varpi}\phi(\tau) = \lim_{\hbar\to 0} \frac{\phi^{(\lceil\varpi\rceil-1)}\left(\tau + \hbar\tau^{(\lceil\varpi\rceil-1)}\right) - \phi^{(\lceil\varpi\rceil-1)}(\tau)}{\hbar}$$

Theorem: Let $\varpi \in (0, 1]$ and ϕ, ψ be ϖ-differentiable at a point $\tau > 0$, then

1. $T_{\tilde{\omega}}(a\phi + b\psi) = aT_{\varpi}(\phi) + bT_{\tilde{\omega}}(\psi)$, $\forall a, b \in \Re$.
2. $T_{\tilde{\omega}}(\tau^{p}) = p\tau^{p-\tilde{\omega}}$, $\forall p \in \Re$
3. $T_{\tilde{\omega}}(\gamma) = 0$, for all constant function $\phi(\tau) = \gamma$
4. $T_{\tilde{\omega}}(\phi\psi) = \phi\, T_{\tilde{\omega}}(\psi) + \psi\, T_{\tilde{\omega}}(\phi)$.
5. $T_{\tilde{\omega}}(\phi/\psi) = (\psi\, T_{\tilde{\omega}}(\phi) - \phi\, T_{\tilde{\omega}}(\psi))/\psi^{2}$.
6. If ϕ is differentiable, then $T_{\tilde{\omega}}\phi(\tau) = \tau^{1-\tilde{\omega}} d\phi/d\tau$. Analogously, if ϕ is $(\lceil\varpi\rceil + 1)$- differentiable, then $T_{\tilde{\omega}}\phi(\tau) = \tau^{[\tilde{\omega}]-\tilde{\omega}} - \phi^{([\tilde{\omega}])}(\tau)$.

Conformable Fractional Integral:

Let ϕ be a continuous function in $[a, b]$, then conformable fractional integral of order $\varpi \in (0, 1)$ is given as

$$\mathfrak{I}_{\varpi}^{a}\phi(\tau) = \mathfrak{I}_{1}^{a}\left(\tau^{\varpi-1}\phi(\tau)\right) = \int_{a}^{\tau} x^{\varpi-1}\phi(x)\,dx$$

Theorem: For any continuous function ϕ in the domain of $I_{\tilde{\omega}}$, for $\tau \geq a$.

$$\mathfrak{T}_{\varpi}\mathfrak{I}_{\varpi}^{a}\phi(\tau) = \phi(\tau)$$

Properties of Conformable Fractional Laplace Transform:

Definition: Let $\phi : [\tau_0,\infty) \rightarrow \Re$ be a real valued function with $\tau 0 \in \Re$, then the conformable fractional Laplace transform (CFLT) of order $0 < \varpi \leq 1$ is defined as

$$\begin{aligned}\mathfrak{L}_{\varpi}^{\tau_0}\{\phi(\tau)\}(\varepsilon) = \Phi_{\varpi}^{\tau_0}(\varepsilon) &= \int_{\tau_0}^{\infty} \exp\left(-\varepsilon\frac{(\tau-\tau_0)^{\varpi}}{\varpi}\right)\phi(\tau)\, d\varpi(\tau,\tau_0)\\ &= \int_{\tau_0}^{\infty} \exp\left(-\varepsilon\frac{(\tau-\tau_0)^{\varpi}}{\varpi}\right)\phi(\tau)(\tau-\tau_0)^{\varpi-1}\, d\tau\end{aligned}$$

Theorem: Conformable Fractional Derivatives Let $\phi : [p_0,\infty) \rightarrow \Re$ be a continuous and differentiable real valued function with $p0 \in \Re$, then the conformable fractional Laplace transform (CFLT) $L_{\tilde{\omega}}{}^{p0}$ of $T_{\tilde{\omega}}{}^{p0}\, \phi(\tau)$ is defined as

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon) = \varepsilon\Phi_{\varpi}^{\tau_0}(\varepsilon) - \phi(p_0)$$

Proof: From the definition we have

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon) = \int_{p_0}^{\infty} \exp\left(-\varepsilon\frac{(\tau-p_0)^{\varpi}}{\varpi}\right)\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\, d\varpi(\tau,\tau_0)$$

Following properties and expanding the usual integration by part we reach at

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon) = -\phi(p_0) + \varepsilon\int_{p_0}^{\infty} \exp\left(-\varepsilon\frac{(\tau-p_0)^{\varpi}}{\varpi}\right)\phi(\tau)(\tau-p_0)^{\varpi-1}\, d\tau$$

Hence,

$$\mathfrak{L}_{\varpi}^{p_0}\{\mathfrak{T}_{\varpi}^{p_0}\phi(\tau)\}(\varepsilon) = \varepsilon\Phi_{\varpi}^{\tau_0}(\varepsilon) - \phi(p_0)$$

Let us take an example:

Consider a periodic function,

$$\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)=\begin{cases}3\dfrac{\tau^{\varpi}}{\varpi}, & 0<\tau<\dfrac{2^{\varpi}}{\varpi}\\[2ex] 6, & \dfrac{2^{\varpi}}{\varpi}<\tau<\dfrac{4^{\varpi}}{\varpi}\end{cases}$$

with the period 4. Then, we have the CFLT as

$$\mathbf{\Psi}_{\varpi}^{0}(\varepsilon)=\frac{\mu(\varepsilon)}{1-\exp\left(-\varepsilon 4^{\varpi}/\varpi\right)}$$

where μ(ε) can be attained as

$$\mu(\varepsilon)=\int_0^{4^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\Psi\left(\frac{\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau$$

$$=\int_0^{2^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)\left(\frac{3\tau^{\varpi}}{\varpi}\right)\tau^{\varpi-1}d\tau+\int_{2^{\varpi}/\varpi}^{4^{\varpi}/\varpi}\exp\left(-\varepsilon\frac{\tau^{\varpi}}{\varpi}\right)(6)\,\tau^{\varpi-1}d\tau$$

$$=3\varpi^{1+\varpi}\left(1-2\exp\left(-\varepsilon 4^{\varpi^2}/\varpi^{1+\varpi}\right)\right)+\exp\left(-\varepsilon 2^{\varpi^2}/\varpi^{1+\varpi}\right)\left(-3\varepsilon 2^{\varpi}+\varpi^{1+\varpi}(-3+6\varepsilon)\right)$$

LAPLACE TRANSFORM OF CONFORMABLE DERIVATIVE OF A FUNCTION

Let f(t) be the given function. To solve:

$$L\left\{T_\alpha f(t)\right\} = \int_0^\infty e^{-st} \lim_{\xi\to\infty}\left(\frac{f\left(t+\xi t^{1-\alpha}\right)-f(t)}{\xi}\right) dt \quad \text{---- (1)}$$

Let us simplify the expression inside limits,

$$X(t) = \lim_{\xi\to\infty}\left(\frac{f\left(t+\xi t^{1-\alpha}\right)-f(t)}{\xi}\right)$$

$$\xi t^{1-\alpha} = h \to \xi = h t^{\alpha-1}, \text{ as } \xi\to 0,\ h\to 0$$

Using these,

$$X(t) = \lim_{h\to\infty}\left(\frac{f(t+h)-f(t)}{h}\right)\cdot t^{1-\alpha}$$

$$X(t) = f'(t)\cdot t^{1-\alpha}$$

Substituting in (1);

$$I(s) = L\left\{T_\alpha f(t)\right\} = \int_0^\infty e^{-st} f'(t)\cdot t^{1-\alpha} dt$$

$$I(s) = t^{1-\alpha} \int_0^\infty e^{-st} f'(t) dt -$$
$$(1-\alpha)\int_0^\infty t^{-\alpha}\left\{\int_0^\infty e^{-st} f'(t) dt\right\}$$

Now put

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

Then,

$$\int_0^\infty e^{-st} f'(t)dt = F'(s) + sF(s)$$

Resubstituting the above equation in (1) we get

$$I(s) = t^{1-\alpha}\left(F'(s) + sF(s)\right)$$
$$- (1-\alpha)\int_0^\infty t^{\alpha}\left(F'(s) + sF(s)\right)dt$$

Boundary-Value Problems

We’ve solved two BVP’s containing fractional derivatives and hence we’re also stating the Green’s function.

1. y(a) + y(b)= 0 = y’(a) + y’(b)….(Anti-periodic conditions)
2. u(a) – u’(a) = 0 = u(b) + u’(b)…(Robin’s conditions)

Initial Equation:

$$y(t)=c_1+c_2(t-a)+\int_a^t (t-s)(s-a)^{\alpha-2}h(s)ds$$

$$y^{'}(t)=c_2+\int_a^t (s-a)^{\alpha-2}h(s)ds$$

Solutions:

Assumption: a=0 and b=T.

1. Anti-periodic conditions:

Substituting for a & b in the initial equation, we get

$$2c_1+c_2(b-a)+\int_a^b (b-s)(s-a)^{\alpha-2}h(s)ds$$

$$2c_2+\int_a^b (b-a)^{\alpha-2}h(s)ds=0$$

Solving for the constants,

$$c_1=\tfrac{1}{4}(b-a)^{\alpha-1}\int_a^b (1-(b-s)(s-a)^{\alpha-2})h(s)ds$$

$$c_2=-\tfrac{1}{2}\int_a^b (b-a)^{\alpha-2}h(s)ds$$

Substituting c_1 & c_2 in the equation,

$$y(t)=\int_0^t \tfrac{1}{4}(T)^{\alpha-1}(1-(T-s)(s)^{\alpha-2})h(s)ds-\tfrac{1}{2}(t)(T)^{\alpha-2}h(s)ds+(t-s)s^{\alpha-2}h(s)ds$$

$$+\int_t^T \tfrac{1}{4}(T)^{\alpha-1}(1-(T-s)(s)^{\alpha-2})h(s)ds - \tfrac{1}{2}(t)(T)^{\alpha-2}h(s)ds$$

Dividing the integral into two parts from 0 to t and then from t to T, we get the Green's function as follows:

$$G_1(t,s) = \tfrac{1}{4}(T)^{\alpha-1}(1-(T-s)(s)^{\alpha-2})h(s)ds - \tfrac{1}{2}(t)(T)^{\alpha-2}$$

$$G_2(t,s) = \tfrac{1}{4}(T)^{\alpha-1}(1-(T-s)(s)^{\alpha-2})h(s)ds - \tfrac{1}{2}(t)(T)^{\alpha-2}h(s)ds + (t-s)s^{\alpha-2}$$

2. Robin's conditions:

Here, substituting in the 1st initial equation, we can directly obtain

$$c_1 - c_2 = 0$$

$$\Rightarrow c_1 = c_2$$

Using the 2nd equation,

$$u(b) - u'(b) = c_1 + c_2(b-a) + \int_a^b (b-s)(s-a)^{\alpha-2}h(s)ds$$

$$+c_2 + \int_a^b (s-a)^{\alpha-2}h(s)ds$$

$$y(b) + y'(b) = 0$$

$$= c_1 + c_2 T + \int_0^T (t-s)(s-a)^{\alpha-2}h(s)ds$$

$$+c_2 + \int_0^T s^{\alpha-2}h(s)ds$$

$$= c_1 + c_2(T+1) + \int_0^T (T-s+1)s^{\alpha-2}h(s)ds$$

$$= c_2(T+2) + \int_0^T (T-s+1)s^{\alpha-2}h(s)ds$$

$$[Since, c_1 = c_2]$$

$$So, c_2(T+2) = -\int_0^T (T-s+1)s^{\alpha-2}h(s)ds$$

$$Thus, c_1 = c_2 = -\frac{\int_0^T (T-s+1)s^{\alpha-2}h(s)ds}{(T+2)}$$

So, now substituting the constants in y(t) we get,

$$y(t) = (t-a+1)*[\frac{-\int_0^T (T-s+1)s^{\alpha-2}h(s)ds}{(T+2)}] + \int_a^t (t-s)(s-a)^{\alpha-2}h(s)ds$$

$$y(t) = \tfrac{-(t+1)}{(T+2)}\int_0^T (T-s+1)s^{\alpha-2}h(s)ds + \int_0^t (t-s)(s-a)^{\alpha-2}h(s)ds$$

$$= \tfrac{-1}{(T+2)}[\int_0^t (T-s+1)(t+1)s^{\alpha-2}h(s)ds + \int_t^T (T-s+1)(t+1)(s)^{\alpha-2}h(s)ds]$$

$$y(t) = \int_0^t (\frac{-(T-s+1)(t+1)}{(T+2)} + (t-s))s^{\alpha-2}h(s)ds - \int_t^T \frac{-(T-s+1)(t+1)}{(T+2)}s^{\alpha-2}h(s)ds$$

$$y(t) = \int_0^t \frac{(t-T-s-1)}{(T+2)}s^{\alpha-2}h(s)ds - \int_t^T \frac{-(T-s+1)(t+1)}{(T+2)}s^{\alpha-2}h(s)ds$$

Thus, we get the Green's function as:

$$G_1(t,s) = \frac{-(T-s+1)(t+1)}{(T+2)}s^{\alpha-2}$$

$$G_2(t,s) = \frac{(t-T-s-1)}{(T+2)}s^{\alpha-2}$$

CONCLUSION

The report illustrates the vast potential of fractional derivatives. As we've seen, the basic properties of ordinary differential equations are satisfied by the conformable derivatives. With these definitions, we believe that we can derive further properties and hence can list out the general rules for some known functions which can be applied while solving such equations and can actually be applied to solve problems from various fields by approximating them as fractional derivative problems. This work is for our future consideration.

REFERENCES:

1. **Y. Cenesiz and A. Kurt (2016):** New Fractional Complex Transform for Conformable Fractional Partial Differential Equations
2. **Emrah Unal, Ahmet Gokdogan:** Solution of Conformable Fractional Ordinary Differential Equations via Differential Transform Method
3. **Najeeb Alam Khan, Oyoon Abdul Razzaq, Muhammad Ayaz (2018):** Some Properties and Applications of Conformable Fractional Laplace Transform (CFLT)
4. **Roshdi R Khalil, M. Al Horani H Horani, Abdelrahman Yousef, M. Sababheh (2014):** A new definition of Fractional Derivative
5. **Shadi Ahmad Al-Tarawneh (2016):** Solving Fractional Differential Equations By using Conformable Fractional Derivatives Definitions