A Project Report

On

# THEORY OF ELASTICITY AND WAVE PROPAGATION

BY

**INDRANIL BHAUMIK**

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Under the supervision of

**DR. SUMIT KUMAR VISHWAKARMA**

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**Birla Institute of Technology and Science-Pilani,**

**Hyderabad Campus**

Certificate

This is to certify that the project report entitled "THEORY OF ELASTICITY AND WAVE PROPAGATION" submitted by Mr. INDRANIL BHAUMIK (ID No. 2014B4A70924H) in fulfillment of the requirements of the course **MATH F266** Study Oriented Project Course, embodies the work done by him under my supervision and guidance.

**Date:**

**(Dr. Sumit Kumar Vishwakarma)**

BITS- Pilani, Hyderabad Campus

# ABSTRACT

**Seismic waves** are waves of energy that travel through the Earth's layers, and are a result of earthquakes, volcanic eruptions, magma movement, large landslides and large man-made explosions that give out low-frequency acoustic energy. There are several different kinds of seismic waves, and they all move in different ways. The two main types of waves are **body waves** and **surface waves**. Earthquakes radiate seismic energy as both body and surface waves. Seismic waves are **elastic** in nature. Earthly material must behave elastically in order to transmit them to next segment. In this report we shall understand the behaviour of wave propagation with varying elastic properties of earth’s layers, using **D’Alembert’s law** and **Stress-Strain theory**. We shall obtain differential equations of the phase propagation and applying given conditions, we will see the relation between the elastic properties of the waves (by varying different constants) and phase velocity of wave or propagation velocity.

# CONTENTS

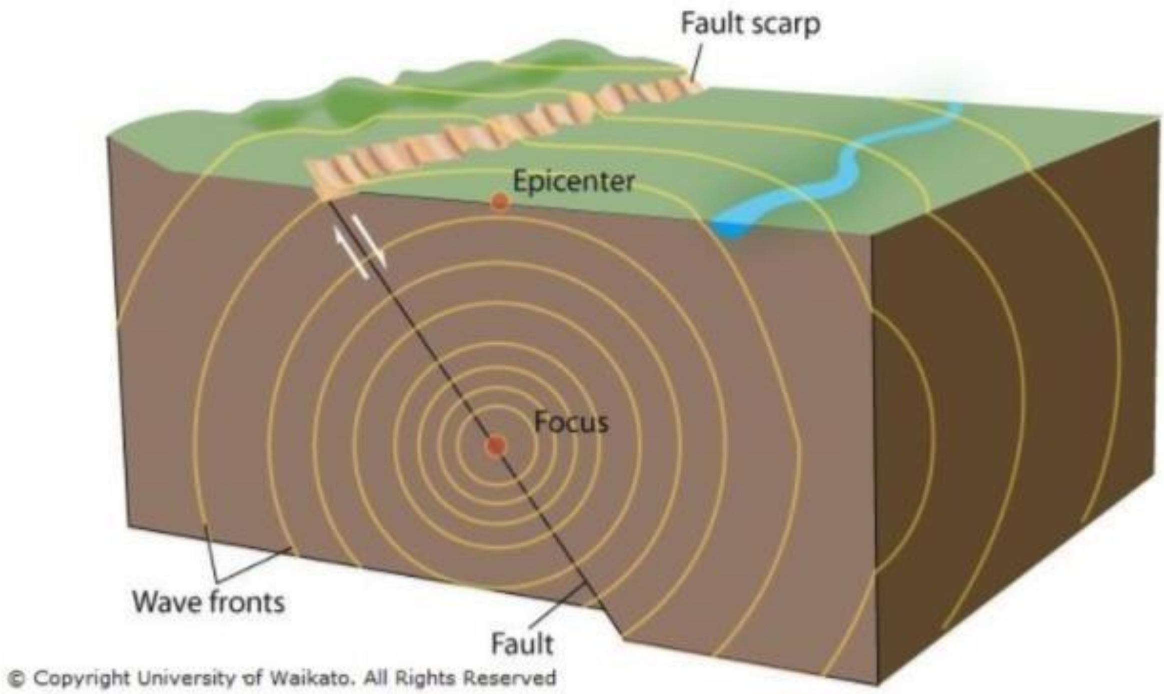
# INTRODUCTION

Seismology is the scientific study of earthquakes and the propagation of elastic waves through the Earth or through other planet-like bodies. The field also includes studies of earthquake environmental effects such as tsunamis as well as diverse seismic sources such as volcanic, tectonic, oceanic, atmospheric, and artificial processes such as explosions. A related field that uses geology to infer information regarding past earthquakes is paleoseismology. A recording of earth motion as a function of time is called a seismogram.

Most of what we know today of the structure and physical properties of our planet Earth, from its uppermost crust down to its centre, results from the analysis of ***Seismic waves*** generated by more or less localized natural or man-made sources such as earthquakes or explosions.

When an earthquake occurs, the shockwaves of released energy that shake the Earth and temporarily turn soft deposits, such as clay, into jelly (liquefaction) are called seismic waves, from the Greek ‘seismos’ meaning ‘earthquake’. Seismic waves are usually generated by movements of the Earth’s tectonic plates but may also be caused by explosions, volcanoes and landslides. As the waves travel through different densities and stiffness, the waves can be refracted and reflected. Because of the different behaviour of waves in different materials, seismologists can deduce the type of material the waves are travelling through.

The results can provide a snapshot of the Earth’s internal structure and help us to locate and understand fault planes and the stresses and strains acting on them.




There are several different kinds of seismic waves, and they all move in different ways. The two main types of waves are **body waves** and **surface waves**. Body waves can travel through the earth's inner layers, as the name suggests, while surface waves can only move along the surface of the planet. Earthquakes radiate seismic energy as both **body** and **surface waves**.

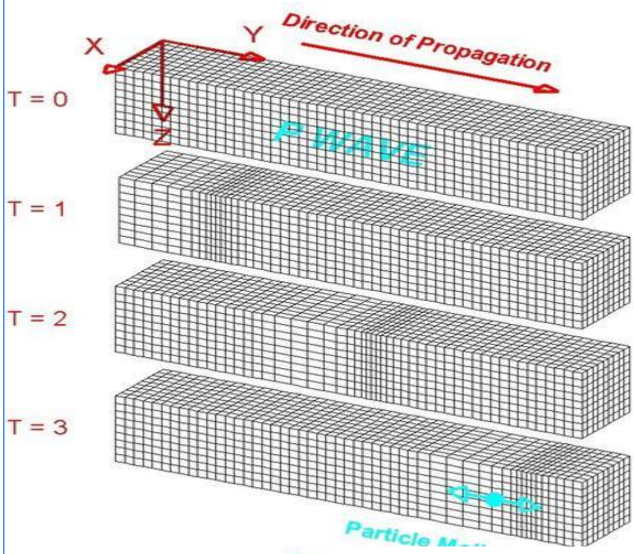
## <u>Body waves</u>:

Traveling through the interior of the earth, **body waves** arrive before the surface waves emitted by an earthquake. These waves are of a higher frequency than surface waves.

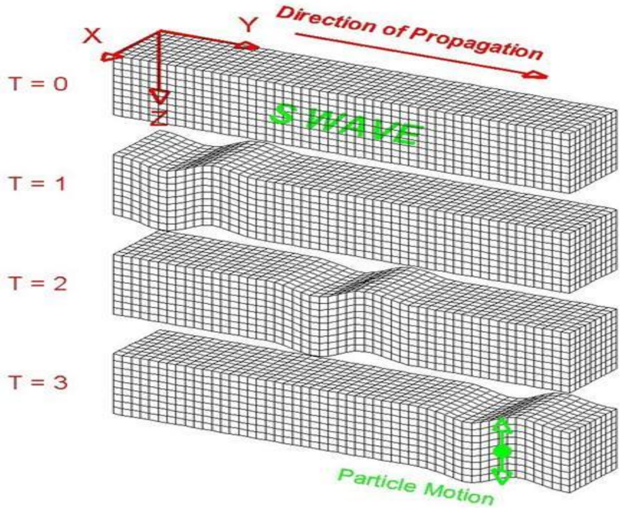
## <u>Surface waves</u>:

Travelling only through the crust, **surface waves** are of a lower frequency than body waves, and are easily distinguished on a seismogram as a result. Though they arrive after body waves, it is surface waves that are almost entirely responsible for the damage and destruction associated with earthquakes. This damage and the strength of the surface waves are reduced in deeper earthquakes.

# BODY WAVES

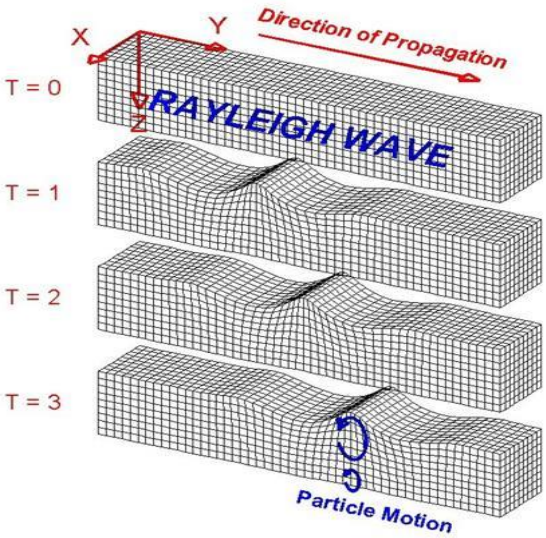



The first kind of body wave is the **P wave** or **primary wave**. This is the fastest kind of seismic wave, and, consequently, the first to '*arrive*' at a seismic station. The P wave can move through solid rock and fluids, like water or the liquid layers of the earth. P waves are also known as **compressional waves**, because of the pushing and pulling they do. Subjected to a P wave, particles move in the same direction that the wave is moving in, which is the direction that the energy is travelling in, and is sometimes called the 'direction of wave propagation'.




The second type of body wave is the **S** i.e. **secondary wave**, which is the second wave you feel in an earthquake. An S wave is slower than a P wave and can only move through solid rock, not through any liquid medium. It is this property of S waves that led seismologists to conclude that the Earth's **outer core** is a liquid. S waves move rock particles up and down, or side-to-side--perpendicular to the direction that the wave is travelling in (the direction of wave propagation).

# <u>SURFACE WAVES</u>:




The first kind of surface wave is the **Rayleigh wave**, named for John William Strutt, Lord Rayleigh. A Rayleigh wave rolls along the ground just like a wave rolls across a lake or an ocean. Because it rolls, it moves the ground up and down, and side-to-side in the same direction that the wave is moving. Most of the shaking felt from an earthquake is due to the Rayleigh wave, which can be much larger than the other waves

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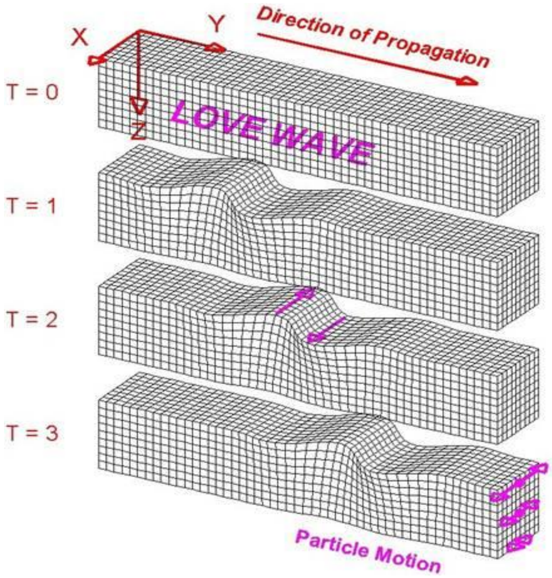



The other kind of surface wave is called a **Love wave**, named after A.E.H. Love, a British mathematician. It's the fastest surface wave and moves the ground from side-to-side. Confined to the surface of the crust, Love waves produce entirely horizontal motion. And this will remain as our prime focus.

# <u>LITERATURE SURVEY</u>

In the course of undertanding the behaviour of general Siesmic waves or Love waves in focus, we will be needing certain prerequisites. This includes the current knowledge including **substantive** findings, as well as **theoretical** and **methodological** contributions for us to understand different solutions and the process of obtaining them.

- D'Alembert's Law;
- Variable Seperable Method for Partial Differential Equations;
- Stress-Strain Theroy;

- Disperssion Equation of a progressive Love wave.

## D'Alembert'sLaw:

The method of D'Alembert provides a solution to the one-dimensional **wave equation**

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad (1)$$

that models vibrations of a string.

The general solution can be obtained by introducing new variables $\xi = x - c\,t$ and $\eta = x + c\,t$, applying the **chain rule** to obtain

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\\ &= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\\ &= -c\,\frac{\partial}{\partial \xi} + c\,\frac{\partial}{\partial \eta}.\end{aligned}$$

Using (4) and (5) to compute the left and right sides of (3) then gives

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right) && (6)\\ &= \frac{\partial^2 y}{\partial \xi^2} + 2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta} + \frac{\partial^2 y}{\partial \eta^2} && (7)\\ \frac{\partial^2 y}{\partial t^2} &= \left(-c\,\frac{\partial}{\partial \xi} + c\,\frac{\partial}{\partial \eta}\right)\left(-c\,\frac{\partial y}{\partial \xi} + c\,\frac{\partial y}{\partial \eta}\right) && (8)\\ &= c^2\,\frac{\partial^2 y}{\partial \xi^2} - 2\,c^2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta} + c^2\,\frac{\partial^2 y}{\partial \eta^2}, \quad \text{respectively.} && (9)\end{aligned}$$

So plugging in and expanding then gives

$$\frac{\partial^2 y}{\partial \xi\,\partial \eta} = 0. \qquad (10)$$

This partial differential equation $y\,(x,\,t)$ has general solution

$$\begin{aligned}&= f\,(\xi) + g\,(\eta) && (11)\\ &= f\,(x - c\,t) + g\,(x + c\,t), && (12)\end{aligned}$$

Where, $f$ and $g$ are **arbitrary functions**, with $f$ representing a **right-traveling** wave and $g$ a **left-traveling** wave.

## Variable-Seperable Method:

The one-dimensional wave equation can be solved by **Separation of Variables** using a trial solution

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}.},$$

$$\psi(x, t) = X(x)\, T(t). \tag{1}$$

This gives

$$T\frac{d^2 X}{d x^2} = \frac{1}{v^2} X \frac{d^2 T}{d t^2} \tag{2}$$

$$\frac{1}{X}\frac{d^2 X}{d x^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{d t^2} = -k^2. \tag{3}$$

So the solution for '$X$' is

$$X(x) = C\cos(k x) + D\sin(k x). \tag{4}$$

Rewriting (3) gives

$$\frac{1}{T}\frac{d^2 T}{d t^2} = -v^2 k^2 \equiv -\omega^2, \tag{5}$$

so the solution for '$T$' is

$$T(t) = E\cos(\omega t) + F\sin(\omega t), \tag{6}$$

where $v \equiv \omega / k$. Applying the boundary conditions $\psi(0, t) = \psi(L, t) = 0$ to (4) gives

$$C = 0 \quad k L = m\pi, \tag{7}$$

where '$m$' is an **integer**. Plugging (4) (6) and (7) back in for $\psi$ in (1) gives, for a particular value of '$m$',

$$\psi_m(x, t) = [E_m \sin(\omega_m t) + F_m \cos(\omega_m t)]\, D_m \sin\left(\frac{m\pi x}{L}\right) \tag{8}$$

$$\equiv [A_m \cos(\omega_m t) + B_m \sin(\omega_m t)] \sin\left(\frac{m\pi x}{L}\right). \tag{9}$$

The initial condition $\dot{\psi}(x, 0) = 0$ then gives $B_m = 0$, so (9) becomes

$$\boxed{\psi_m(x,t) = A_m \cos(\omega_m t)\sin\left(\frac{m\pi x}{L}\right).} \tag{10}$$

The general solution is a sum over all possible values of ‘m’, so

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \cos(\omega_m t)\sin\left(\frac{m\pi x}{L}\right). \tag{11}$$

Using **orthogonality** of sines again,

$$\int_0^L \sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \tfrac{1}{2}L\,\delta_{lm}, \tag{12}$$

where $\delta_{lm}$ is the **Kronecker delta** defined by

$$\delta_{mn} \equiv \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}, \tag{13}$$

gives

$$\int_0^L \psi(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{l=1}^{\infty} A_l \sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \tag{14}$$

$$= \sum_{l=1}^{\infty} A_l\,\tfrac{1}{2}L\,\delta_{lm} \tag{15}$$

$$= \tfrac{1}{2}L A_m, \tag{16}$$

So we have

$$A_m = \frac{2}{L}\int_0^L \psi(x,0)\sin\left(\frac{m\pi x}{L}\right)dx. \tag{17}$$

The computation of $A_m$s for specific initial distortions is derived in the **Fourier sine series**. We already have found that $B_m = 0$, so the equation of motion for the string (10), with

$$\omega_m \equiv v\,k_m = \frac{v\,m\pi}{L}, \tag{18}$$

is

$$\boxed{\psi(x,t) = \sum_{m=1}^{\infty} A_m \cos\left(\frac{v\,m\pi t}{L}\right)\sin\left(\frac{m\pi x}{L}\right),}$$

Where the **$A_m$** are **coefficients**.

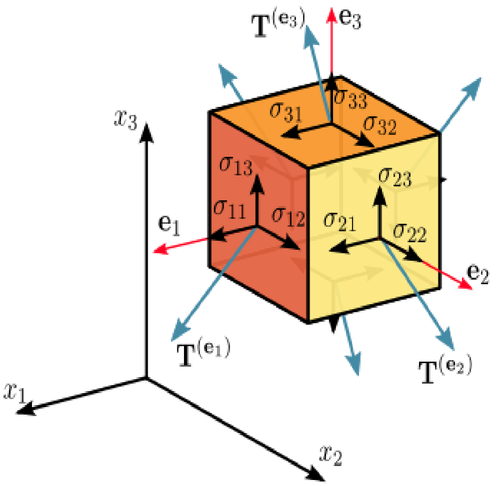
# STRESS-STRAIN THEORY:

Earth material must behave elastically to transmit Seismic Waves. The degree of elasticity/plasticity of real Earth material depends mainly on the strain rate, i.e., on the length of time it takes to achieve a certain amount of distortion. Elastic material resists or reacts different to stresses depending on the type of deformation. It can be quantified by various elastic moduli:

- The bulk modulus κ
- The shear modulus μ
- The Young's modulus E
- The Poisson's ratio

**Traction**, is the force used to generate motion between a body and a tangential surface, through the use of dry friction, though the use of shear force of the surface is also commonly used. **Traction** can also refer to the maximum tractive force between a body and a surface, as limited by available friction; when this is the case, traction is often expressed as the ratio of the maximum tractive force to the normal force and is termed the coefficient of traction (similar to coefficient of friction).

$$t(\hat{n}) = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$




## Stress-Tensor

In continuum mechanics, the Cauchy stress tensor **σ**, true stress tensor, or simply called the stress tensor is a second order tensor named after Augustin-Louis Cauchy. The tensor consists of nine components $\boldsymbol{\sigma_{ij}}$ that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. The tensor relates a unit-length direction vector **n** to the stress vector $\mathbf{T}^{(\mathbf{n})}$ across an imaginary surface perpendicular to **n**:

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i .$$

where,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Now for the above tensor, we can apply **Taylor's expansion**, for initial conditions. As we know Tylor's Expansion is given as:

$f(x_o + h) = f(x_o) + hf'(x_o) + h^2 f''(x_o) + \ldots$

And hence, applying it to the tensor we obtain,

$$u(r) = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = u(r_o) + \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Where u represents the displacement in a particular direction,
$u_x$: Displacement along positive X-axis,
$u_y$: Displacement along positive Y-axis,
$u_z$: Displacement along positive z-axis.

Which can be rewritten as,

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_{x_o} \\ u_{y_o} \\ u_{z_o} \end{pmatrix} + \begin{pmatrix} \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}y + \frac{\partial u_x}{\partial z}dz \\ \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_z}{\partial z}dz \\ \frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz \end{pmatrix}$$

Now, tensor J is a combination of both axial strain '$\epsilon$'and Rotational component of plane 'Ω', can refer fig XXXX

J= $\epsilon$ (Axial Strain) + Ω(Rotational Component

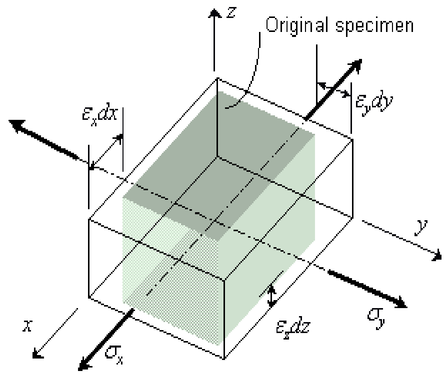
$$= \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) & 0 \end{pmatrix}$$

## Generalised Hooke's Law

The generalized Hooke's Law can be used to predict the deformations caused in a given material by an arbitrary combination of stresses. The linear relationship between stress and strain applies for. The generalized Hooke's Law also reveals that strain can exist without stress.

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \qquad \varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$




Where, **E** is the Young's Modulus and **n** is the Poisson Ratio. For example, if the member is experiencing a load in positive y-direction (which in turn causes a stress in the y-direction), the Hooke's Law shows that strain in the x-direction does not equal to zero. This is because as material is being pulled outward by the y-plane, the material in the x-plane moves inward to fill in the space once occupied, just like an elastic band becomes thinner as you try to pull it apart. In this situation, the x-plane does not have any external force acting on them but they experience a change in length. Therefore, it is valid to say that strain exist without stress in the x-plane.

$$\tau_{ij} = C_{ijkl} e_{kl} \tag{1}$$

**Note:** i, j, k, l has 3 values each.
Hence, $C_{ijkl}$ has a maximum of $3^4$ i.e. **81** values.

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

In isotropic conditions, there are 2 constants, **λ** and **μ**.

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \tag{2}$$

Note: $\delta_{ij}=0$ , $i \neq j$ and 1 , $i = j$

So,

$$\boxed{C_{1111} = \lambda + 2\mu;\;\; C_{1122} = \lambda;\;\; C_{1121} = 0}$$

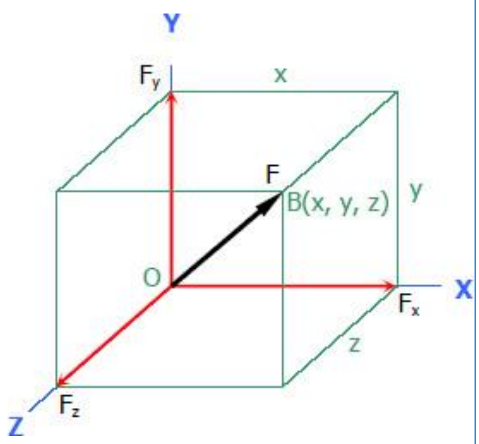
Putting (2) into (1),

$$\tau_{ij} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})]\, e_{kl}$$

If k=l, then,

$$\tau_{ij} = [\lambda\delta_{ij}\delta_{kk} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jk})]\, e_{kk}$$

$$= \lambda\delta_{ij}e_{kk} + 2\mu\, e_{ij}$$




Now for calculating the net force along the axis, we can use newton’s 2nd law,

$$t(\hat{n}) = \tau.\hat{n}$$

$$F = t(\hat{n}_1)dx_2\,dx_3 \quad = \begin{pmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{pmatrix} dx_2 dx_3$$

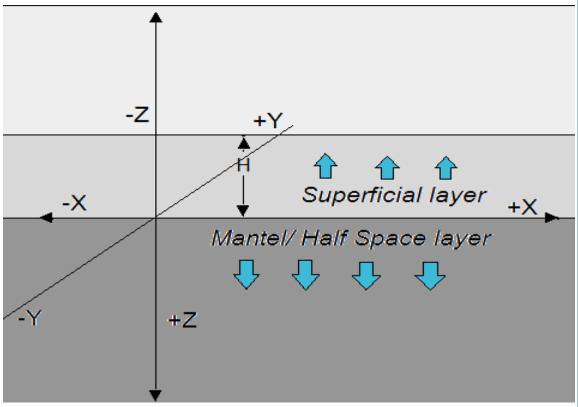
Hence the net force along X-axis and Y-axis is:

$$F_x = \frac{d}{dx_1}\begin{pmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{pmatrix} dx_2 dx_3 dx_1 \quad F_y = \frac{d}{dx_2}\begin{pmatrix} \tau_{12} \\ \tau_{22} \\ \tau_{32} \end{pmatrix} dx_1 dx_3 dx_2$$

## Dispersion Equation

Let us understand the behaviour of Love waves, in different parts of earth. Let us call the earth crust as surface and in a coordinate system it shall be Z=0 plane, for very small part of earth crust. Below the surface let us take, Z as positive and above, Z as negetive. For:

$$\text{Along Z} : \begin{cases} \text{if } Z > 0, \text{ mantel part or half space} \\ \quad \text{if } Z < 0, \text{ superficial layer.} \end{cases}$$




Where the half space extents till infinite as we are considering very small part of earth crust. But, the superficial layer had a finite width of (let's say) '**H**'. At Z=0 these two layers meet, so accordingly wave motion and the shear stree at the surface will be same for both the layers, which, further we will see that forms a boundary condition.

Equation of motion is given as

$$(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u = \rho.\left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$(\lambda + \mu)\frac{\partial \Delta}{\partial y} + \mu \nabla^2 v = \rho.\left(\frac{\partial^2 v}{\partial t^2}\right)$$

$$(\lambda + \mu)\frac{\partial \Delta}{\partial z} + \mu \nabla^2 w = \rho.\left(\frac{\partial^2 w}{\partial t^2}\right)$$

**u** :displacement along x
**v** : displacement along y
**w** : displacement along z

With $\boldsymbol{\lambda}, \boldsymbol{\mu}$ $and$ $\boldsymbol{\rho}$are elastic coefficients of the material the wave is propagating.

Taking the case of Love wave:

$$\mu\nabla^2 v = \rho.\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\rho}{\mu}.\frac{\partial^2 v}{\partial t^2}$$

This can be solved using variable separable, since 'V' is a function of *'x', 'z'* and 't'. As the wave is propagating along positive x-axis, we can take the function of 'x' and 't' as implicit i.e. **X(x,t)** and z-component as $\mathbf{V_1(z)}$

**X**(x,t) = $e^{ik(x-ct)}$

Assuming Solution: $v = v_0(z)e^{ik(x-ct)}$

Therefore the obtained ODE will be:

$$\boxed{v_1'' - v_1 k^2\left(1 - \frac{\rho}{\mu}c^2\right) = 0}$$

Let the solutions obtained be,$v_1 and\ v_2$. Where $v_1$ represents for superficial layer, and $v_2$ represents for mantel.

**Boundary Conditions:**

At the crust, i.e. z=0, wave propagation will be same for both the layers, therefore:

1. At $z = 0,\qquad v_1 = v_2$

Similarly at z=0 shear stress for both the layers should be same:

2. At $z = 0, (\tau_{yz})_1 = (\tau_{yz})_2$

And since superficial layer is finite and stress of love waves damps along z, away from z=0, which implies:

3. At $z = -H, (\tau_{yz})_1 = 0$

**Solution of ODE:**

$$v = c_1 e^{k\left(1-\frac{\rho}{\mu}c^2\right)^{\frac{1}{2}}z} + c_2 e^{-k\left(1-\left(\frac{\rho}{k}\right).c^2\right)^{\left(\frac{1}{2}\right)}z}$$

Therefore we get the solutions as:

$$v_1 = c_1 e^{k\left(1-\frac{\rho_1}{\mu_1}c^2\right)^{\frac{1}{2}}z} + c_2 e^{-k\left(1-\left(\frac{\rho_1}{k}\right).c^2\right)^{\left(\frac{1}{2}\right)}z}$$

$$v_2 = c_1 e^{k\left(1-\frac{\rho_2}{\mu_2}c^2\right)^{\frac{1}{2}}z} + c_2 e^{-k\left(1-\left(\frac{\rho_2}{k}\right).c^2\right)^{\left(\frac{1}{2}\right)}z}$$

Applying Boundary conditions :

1. $[v_1(z)]_{z=0} = c_1 + c_2 \quad , \quad [v_2(z)_{z=0}] = c_3 + c_4$

   ⇨ $c_1 + c_2 = c_3 + c_4$

2. $\tau_{xy} = \mu[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}]$

   ⇨ $\tau_{xy} = \mu(\frac{\partial v}{\partial z})$

   ⇨ $\left(\tau_{xy}\right)_1 = \mu_1(c_1 k\left(1 - \frac{\rho_1 c^2}{\mu_1}\right)^{\frac{1}{2}} e^{k\left(1-\frac{\rho_1 c^2}{\mu_1}\right)^{\frac{1}{2}} z} - c_2 k(1 - \frac{\rho_1 c^2}{\mu_1})$

   ⇨ $\left(\tau_{xy}\right)_1 = \mu_1(c_1 k\left(1 - \frac{\rho_1 c^2}{\mu_1}\right)^{\frac{1}{2}} e^{k\left(1-\frac{\rho_2 c^2}{\mu_2}\right)^{\frac{1}{2}} z} - c_2 k(1 - \frac{2\rho_2 c^2}{\mu_2})$

   ⇨ $\left(1 - \frac{\rho_1}{\mu_1} c^2\right)^{\frac{1}{2}} (\mu_1)(c_1 - c_2) = \left(1 - \frac{\rho_2}{\mu_2} c^2\right)^{\frac{1}{2}} (\mu_2)(c_3 - c_4)$ - (ii)

3. $\left[\left(\tau_{xy}\right)_1\right]_{z=-H} = 0$

   ⇨ $\mu_1 k\left(1 - \frac{\rho_1}{\mu_1}.c^2\right)^{\frac{1}{2}} (c_1 e^{-k\left(1-\frac{\rho_1}{\mu_1}c^2\right)^{\left(\frac{1}{2}\right)} H} - c_2 e^{k\left(1-\frac{\rho_1}{\mu_1}c^2\right)^{\left(\frac{1}{2}\right)} H} = 0$

   ⇨ $c_1 e^{-\phi} = c_2 e\hat{}\phi$

   ⇨ $c_1 = c_2 e^{2\phi}$

$$where\ \phi = k\left(1 - \left(\frac{\rho_1}{\mu_1}\right) c^2\right)^{\frac{1}{2}} H$$

In zone 2 : $c_3$ should be equal to 0 because as $z \to \infty$ , $v \to 0$

Therefore we get the new boundary conditions as:

1. $c_1 + c_2 = c_4$
2. $(c_1 - c_2)\mu_1\left(1 - \frac{\rho_1}{\mu_1}.c^2\right) = c_4.\mu_2(\frac{\rho_2}{\mu_2} c^2 - 1)$
3. $c_1 = c_2.e\hat{}(2k\left(1 - \frac{\rho_1}{\mu_1}.c^2\right)^{\frac{1}{2}} H$

Thus we get the solution of the above system of equation as:

Let

$$M = \begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ \frac{\mu_1 \phi_1}{H} & -\frac{\mu_1 \phi_1}{H} & \frac{\mu_2 \phi_2}{H} \\ e^{-\phi} & -e^{\phi} & 0 \end{bmatrix}$$

Now for non-trivial solutions of the above system of equations,we get :

$$\det(M) = 0$$

⇨ $e^{\phi}.\frac{\mu_2}{KH}.\phi_2 + e^{-\phi}.\frac{\mu_2}{KH}.\phi_2 + (e^{\phi} - e^{-\phi})\frac{\mu_1}{KH}.\phi_1 = 0$

$\Rightarrow \; (e^{\phi}+e^{-\phi}).\phi_2.\frac{\mu_2}{H}+\frac{\mu_1}{H}.\phi_1(e^{\phi}-e^{-\phi})=0$

Taking $\theta = i\phi = k.\left(\frac{\rho_1}{\mu_1}.e^2-1\right)^{\frac{1}{2}}.H$

$\Rightarrow \; \frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}=\frac{\mu_1\phi_1}{\mu_2\phi_2}$

or $\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{\frac{\mu_2}{\mu_1}\left(1-\frac{\rho_2}{\mu_2}c^2\right)^{\frac{1}{2}}}{\left(1-\frac{\rho_1}{\mu_1}c^2\right)^{\frac{1}{2}}}$

$\Rightarrow \; \frac{e^{i\theta}-e^{-i\theta}}{i\,(e^{i\theta}+e^{-i\theta})}=\frac{\frac{\mu_2}{\mu_1}\left(1-\frac{\rho_2}{\mu_2}c^2\right)^{\frac{1}{2}}}{\left(\frac{\rho_1}{\mu_1}c^2-1\right)^{\frac{1}{2}}}$

$\Rightarrow \; \tan\theta = \frac{\mu_2}{\mu_1}.\left(\frac{1-\frac{\rho_2}{\mu_2}.c^2}{c^2\frac{\rho_1}{\mu_1}-1}\right)^{\frac{1}{2}}$

Therefore,

$$\boxed{\tan\left(HK\left(c^2\frac{\rho_1}{\mu_1}-1\right)^{\frac{1}{2}}\right)=\frac{\mu_2}{\mu_1}.\left(\frac{1-\frac{\rho_2}{\mu_2}.c^2}{c^2\frac{\rho_1}{\mu_1}-1}\right)^{\frac{1}{2}}}$$

## **Dispersion Equation of Love Wave**

# WAVE PROPOGATION

As seen in ‘Dispersion Equation’ sub-topic, now we shall find actual dispersion eqaution by including more practical assumptions. We shall consider both the layers with different elastic properties and with different varying style along different directions, and with more appropriate equation of motion.

## Wave propagation in Superficial layer

Equation of motion will be as:

$$N\frac{\partial^2 V}{\partial x^2}+\frac{\partial}{\partial z}(L\frac{\partial V}{\partial z})=\rho\frac{\partial^2 V}{\partial t^2}$$

And taking the material properties as:

$$\mu = \begin{cases} N, & along\ x \\ L, & along\ z \end{cases}$$

Where,

$N = N_0 (1+mz)^2$

$L = L_0 (1+mz)^2$

And

$$\rho = \rho_0 (1+mz)^2$$

This can be solved using variable separable, since 'V' is a function of *'x', 'z'* and't'. As the wave is propagating along positive x-axis, we can take the function of 'x' and 't' as implicit i.e**X(x,t)** and z-component as $\mathbf{V_1(z)}$

$$X(x,t) = e^{ik(x-ct)}$$

Now we substitute: $V(z) = V_1(z) e^{ik(x-ct)}$

Hence we can rewrite the equation as a function of 'z' as:

$$N\left(-k^2 V_1(z)\right) + L_0 \{(1+mz)^2 V_1''(z) + V_1(z) 2m(1+mz)\} = -k^2 c^2 \rho V_1'(z)$$

On further solving we get,

$$\boxed{V_1''(z) + V_1'(z)(2m/1+mz) + V_1(z)[\{k^2(\rho_0 c^2 - N_0)\}/L_0] = 0}$$

Now take

$$\mathbf{P(z)} = 2m/1+mz,\ \mathbf{Q(z)} = k^2(\rho_0 c^2 - N_0)\}/L_0,$$

Substitute P(z) and Q(z) in such a way that V'(z) term is removed,

$$y''(z) + [Q(z) - 0.5P'(z) - 0.25P^2(z)]y(z) = 0$$

Where $V_1(z) = ye^{-0.5\int P dz}$

Therefore, $Q(z) - 0.5P'(z) - 0.25P^2(z) = \{k^2(\rho_0 c^2 - N_0)\}/L_0$

obtained equation is:

$$\boxed{y''(z) + y(z)[\{k^2(\rho_0 c^2 - N_0)\}/L_0] = 0}$$

On solving the above equation,

Implies, $\boxed{y = c_1 e^{i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z} + c_2 e^{-i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z}}$

Where $e^{-0.5\int P dz} = 1/(1+mz)$

Hence

$$V_1(z) = 1/(1+mz)[c_1 e^{i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z} + c_2 e^{-i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z}]$$

And since,

$$V(z) = V_1(z)e^{ik(x-ct)}$$

Therefore wave propagation in the superficial layer is given by,

$$\boxed{V(z) = 1/(1+mz)[c_1 e^{i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}z} + c_2 e^{-i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}z}]e^{ik(x-ct)}}$$

## Wave propagation in Mantle/ Half-Space layer

In the half-space, z extends till infinity. And the wave motion-Equation is given by:

$$\frac{\partial S_{21}}{\partial x} + \frac{\partial S_{23}}{\partial z} - \frac{p}{2}\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2(\rho v)}{\partial t^2}$$

Where

$$S_{21} = 2\mu e_{yx}$$
$$S_{23} = 2\mu e_{yz}$$

while '**p**' is initial stress,

and material properties are $\mu = \mu_0(1-az)$, $\rho = \rho_0(1+bz)$

In the wave motion equation, **S21** and **S23** are shear stress form the stress tensor, hence

$$S_{21} = 2\mu\left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] = \mu\frac{\partial v}{\partial x}$$
$$S_{23} = 2\mu\left[\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] = \mu\frac{\partial v}{\partial z}$$

Upon substitution,

$$\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial v}{\partial z}\right] - \frac{p}{2}\left[\frac{\partial^2 v}{\partial x^2}\right] = \frac{\partial^2(\rho v)}{\partial t^2}$$

Applying similar solving technique as in **superficial layer**, we take

$$v(x,z,t) = v(z)e^{ik(x-ct)}$$

And following are the derivatives to be substituted,

$$\frac{\partial v}{\partial x} = (ik)v(z)e^{ik(x-ct)} \qquad \frac{\partial^2 v}{\partial x^2} = (ik)^2 v(z)e^{ik(x-ct)}$$

$$\frac{\partial v}{\partial t} = (-ick)v(z)e^{ik(x-ct)} \qquad \frac{\partial^2 v}{\partial t^2} = (ick)^2 v(z)e^{ik(x-ct)}$$

$$\frac{\partial v}{\partial z} = v^{'}(z)e^{ik(x-ct)} \qquad \frac{\partial^2 v}{\partial z^2} = v^{''}(z)e^{ik(x-ct)}$$

Substituting the above expressions in the equation of motion,

$$\frac{\partial}{\partial z}\left[\mu(ik)v(z)e^{ik(x-ct)}\right] + \frac{\partial}{\partial z}\left[\mu v^{'}(z)e^{ik(x-ct)}\right] - \frac{p}{2}(ik)^2 v(z)e^{ik(x-ct)} = \rho(ick)^2 v(z)e^{ik(x-ct)}$$

$$\mu(ik)^2 v(z)e^{ik(x-ct)} + (\mu v'' + \mu^{'} v^{'})e^{ik(x-ct)} - \frac{p}{2}(ik)^2 v(z)e^{ik(x-ct)} = \rho(ick)^2 v(z)e^{ik(x-ct)}$$

Taking $e^{ik(x-ct)} \neq 0$ and substituting for $\mu$ and $\rho$,

$$v''(z) - \frac{a}{1-az}v'(z) + \left[(\rho_0(1+bz)c^2 + \frac{p}{2} - \mu_0(1-az)\right]\frac{k^2}{\mu_0(1-az)}v(z) = 0$$

$$v''(z) - \frac{a}{1-az}v'(z) + [(\rho_0 c^2 + \frac{p}{2} - \mu_0)\frac{k^2}{\mu_0} + (\rho_0 c^2 b + \mu_0 a)\frac{k^2}{\mu_0}]\frac{v(z)}{1-az} = 0$$

And the obtained ordinary differential equation of V w.r.t ‘z’ is:

$$\boxed{v''(z) - \frac{a}{1-az}v'(z) + \frac{\alpha + \beta z}{1-az}v(z) = 0}$$

Where,

$$\alpha = (\rho_0 c^2 + \frac{p}{2} - \mu_0)\frac{k^2}{\mu_0}$$

$$\beta = (\rho_0 c^2 b + \mu_0 a)\frac{k^2}{\mu_0}$$

The solution of the equation is obtained using MATLAB

```
>> sym z;
>> s=dsolve('D2v(z)-(a/(1-a*z))*Dv(z)+(alp+bet*z)/(1-a*z)*v(z)=0','z');
>> s

s =

(C4*whittakerM(-((a/bet)^(1/2)*(bet + a*alp))/(2*a^2), 0, (2*bet*z*(a/bet)^(1/2))/a - (2*bet*(a/bet)^(1/2))/a^2))/(a*z - 1)^(1/2) +

(C5*whittakerW(-((a/bet)^(1/2)*(bet + a*alp))/(2*a^2), 0, (2*bet*z*(a/bet)^(1/2))/a - (2*bet*(a/bet)^(1/2))/a^2))/(a*z - 1)^(1/2)
```

## Boundary Conditions

Since

$$\boxed{V_1(z) = 1/(1+mz)[c_1 e^{i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z} + c_2 e^{-i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}\,z}]e^{ik(x-ct)}}$$

1. At the crust, i.e. z=0, wave propagation will be same for both the layers.

$V_1=V_2$

$$\boxed{C_1+C_2= C_4\ WhittakerW(B,0,C)}$$

2. At z= -H, shear stress vanishes.

$$\left(\tau_{yz}\right)_1=0$$

$$\mu\left[\frac{-m}{(1-mH)^2}\left(C_1e^{\sqrt{A}H}+C_2e^{-\sqrt{-A}H}\right)+\frac{1}{(1-mH)}\left(C_1e^{\sqrt{A}H}\sqrt{A}-C_2e^{-\sqrt{-A}H}\sqrt{A}\right)\right]=0$$

$$C_1\left[e^{-\sqrt{A}H}\left(\sqrt{A}(1-mH)-m\right)\right]-C_2\left[e^{\sqrt{A}H}\left(\sqrt{A}(1-mH)+m\right)\right]=0$$

*where*

$$A=\frac{k^2}{L_0}\left(N_0-\rho_0c^2\right)$$

3. At z=0 shear stress for both the layers should be same.

$$\left(\tau_{yz}\right)_1=\left(\tau_{yz}\right)_2$$

$$P_1C_1+P_2C_2+P_4C_4=0$$

$$P_1=\frac{\mu_0}{\mu_1}\left(\sqrt{A}-m\right)$$

$$P_2=-\frac{\mu_0}{\mu_1}\left(\sqrt{A}+m\right)$$

$$P_4=\frac{a}{2}WhittakerW(B,0,C)+2k\left[\frac{WhittakerW(B,0,C)}{8\sqrt{1-\frac{b\rho_1c^2}{a\mu_1}}}\left(\frac{P}{\mu_1}-4+2\frac{(a-b)\rho_1c^2}{a\mu_1}\right)+\frac{a}{2k}WhittakerW(B+1,0,C)\right]$$

*where*

$$B=\frac{k}{4a}\sqrt{\frac{1}{1-\frac{b\rho_1c^2}{a\mu_1}}}\left(\frac{P}{\mu_1}+2\frac{(a-b)\rho_1c^2}{a\mu_1}\right)$$

$$C=\frac{2k}{a}\sqrt{1-\frac{b\rho_1c^2}{a\mu_1}}z-\frac{2k}{a\sqrt{az-1}}\sqrt{1-\frac{b\rho_1c^2}{a\mu_1}}$$

## Non Dimensional Coefficients

$$a_{11} = 1$$
$$a_{12} = 1$$
$$a_{13} = -Whitta\ker W(B,0,C)$$
$$a_{21} = e^{-ix\sqrt{y-\frac{N_0}{L_0}}}\left(\frac{-mh}{x} + i\sqrt{y-\frac{N_0}{L_0}}(1-mh)\right)$$
$$a_{22} = e^{ix\sqrt{y-\frac{N_0}{L_0}}}\left(\frac{-mh}{x} - i\sqrt{y-\frac{N_0}{L_0}}(1-mh)\right)$$
$$a_{23} = 0$$
$$a_{31} = i\sqrt{y-\frac{N_0}{L_0}} - \frac{mh}{x}$$
$$a_{32} = i\sqrt{y-\frac{N_0}{L_0}} + \frac{mh}{x}$$
$$a_{33} = -\frac{\mu_0}{L_0}\left(\frac{aH}{2x} \times Whitta\ker W(B,0,C)\right) + Whitta\ker W^{'}(B,0,C)$$

$$Whitta\ker W(B,0,C) = e^{-\frac{C}{2}} C^B \left[1 - \frac{(B-0.5)^2}{1!C} + \frac{(B-0.5)^2(B-1.5)^2}{2!C^2}\right]$$

$$B = \frac{x}{4aH}\sqrt{\frac{1}{1-\frac{b}{a}y^2\frac{c_0^{\,2}}{c_1^{\,2}}}}\left(\frac{P}{\mu_1} + 2\left(1-\frac{b}{a}\right)y^2\left(\frac{c_0^{\,2}}{c_1^{\,2}}\right)\right)$$
$$C = \frac{2k}{a}\sqrt{1-\frac{b\rho_1 c^2}{a\mu_1}}\,z - \frac{2k}{a\sqrt{az-1}}\sqrt{1-\frac{b\rho_1 c^2}{a\mu_1}}$$

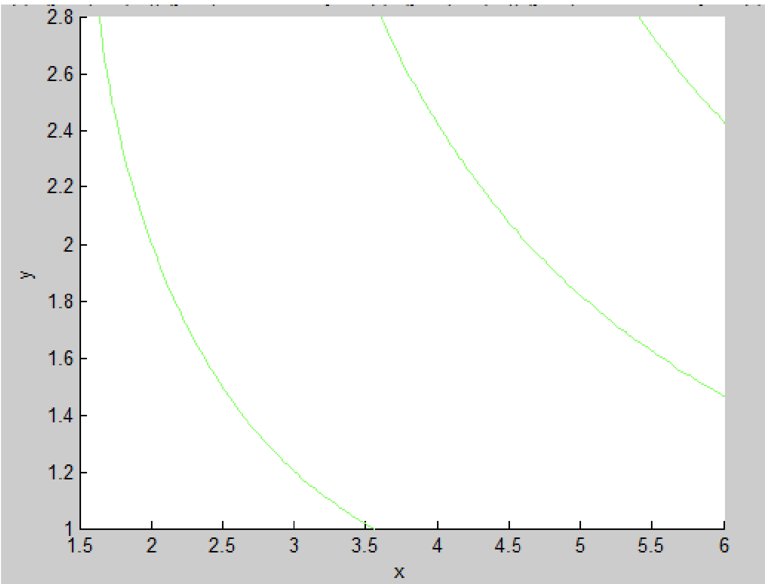
$$AA = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The dispersion equation is obtained by taking, Det(AA) = 0.

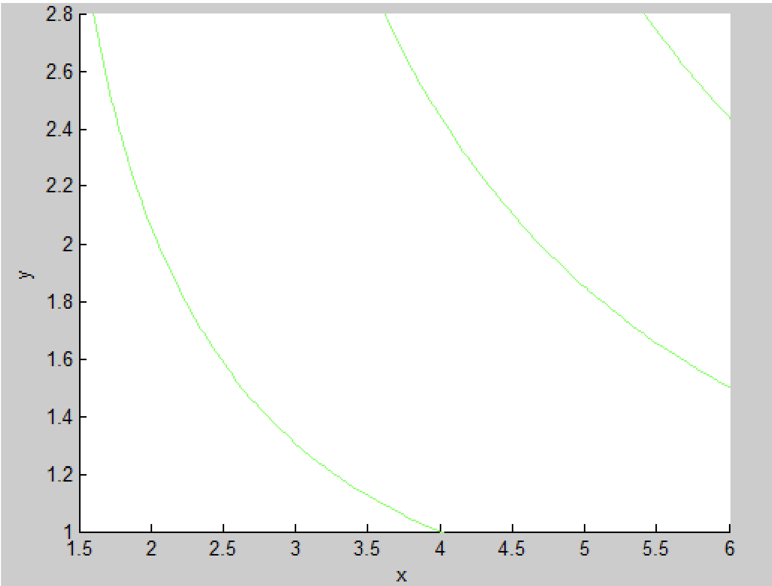
# MATLAB Code

```
syms x y
N0_L0=3;
m_H=4;
mu0_L0=3;
c0_c1=2
a_H=2
b_H=4
P_mu=2
a11=i*sqrt(y-N0_L0)-m_H/x;
a12=-(i*sqrt(y-N0_L0)+m_H/x)

z=(-2*x)*(1/a_H)*(b_H/a_H)*y*(c0_c1)+2*x*(1/a_H);
R=((1-(b_H/a_H)*y*c0_c1)^(-1/2))*(0.25*P_mu*(x/a_H)+0.5*y*(c0_c1)*(x/a_H)-0.5*(b_H/a_H)*(x/a_H)*y*(c0_c1));
WW=(exp(-z/2))*(z^R)*(1-(1/z)*(0.5-R)^2+((0.5-R)^2)*((1.5-R)^2)*(1/2)*(1/z^2))
DWW=z^R*exp(-z/2)*((R - 1/2)^2/z^2 - ((R - 1/2)^2*(R - 3/2)^2)/z^3) - (z^R*exp(-z/2)*(((R - 1/2)^2*(R - 3/2)^2)/(2*z^2) -
(R - 1/2)^2/z + 1))/2 + R*z^(R - 1)*exp(-z/2)*(((R - 1/2)^2*(R - 3/2)^2)/(2*z^2) - (R - 1/2)^2/z + 1);
a13=-(mu0_L0)*((1/2)*(a_H/x)*WW+DWW)

a21=(i*sqrt(y-N0_L0)-i*m_H*sqrt(y-N0_L0)-m_H/x)*exp(-i*x*sqrt(y-N0_L0));
a22=-(i*sqrt(y-N0_L0)-i*m_H*sqrt(y-N0_L0)+m_H/x)*exp(i*x*sqrt(y-N0_L0));
a23=0;

a31=1;
a32=1;
a33=WW;

AA=[a11 a12 a13; a21 a22 a23; a31 a32 a33]


exp1=real(det(AA))


hold on
N0_L0=3;
m_H=8;
mu0_L0=3;
c0_c1=2
a_H=2
b_H=4
P_mu=2
a11=i*sqrt(y-N0_L0)-m_H/x;
a12=-(i*sqrt(y-N0_L0)+m_H/x)

z=(-2*x)*(1/a_H)*(b_H/a_H)*y*(c0_c1)+2*x*(1/a_H);
R=((1-(b_H/a_H)*y*c0_c1)^(-1/2))*(0.25*P_mu*(x/a_H)+0.5*y*(c0_c1)*(x/a_H)-0.5*(b_H/a_H)*(x/a_H)*y*(c0_c1));
WW=(exp(-z/2))*(z^R)*(1-(1/z)*(0.5-R)^2+((0.5-R)^2)*((1.5-R)^2)*(1/2)*(1/z^2))  %WhittakerW expansion at z=0
DWW=z^R*exp(-z/2)*((R - 1/2)^2/z^2 - ((R - 1/2)^2*(R - 3/2)^2)/z^3) - (z^R*exp(-z/2)*(((R - 1/2)^2*(R - 3/2)^2)/(2*z^2) -
(R - 1/2)^2/z + 1))/2 + R*z^(R - 1)*exp(-z/2)*(((R - 1/2)^2*(R - 3/2)^2)/(2*z^2) - (R - 1/2)^2/z + 1);  %Differentiation of
whittakerW at z=0
a13=-(mu0_L0)*((1/2)*(a_H/x)*WW+DWW)

a21=(i*sqrt(y-N0_L0)-i*m_H*sqrt(y-N0_L0)-m_H/x)*exp(-i*x*sqrt(y-N0_L0));
a22=-(i*sqrt(y-N0_L0)-i*m_H*sqrt(y-N0_L0)+m_H/x)*exp(i*x*sqrt(y-N0_L0));
a23=0;

a31=1;
a32=1;
a33=WW;

AA=[a11 a12 a13; a21 a22 a23; a31 a32 a33]
exp2=real(det(AA))


axis([1.5,6,1,2.8]);


ezplot(exp1);
hold on
ezplot(exp2);
```

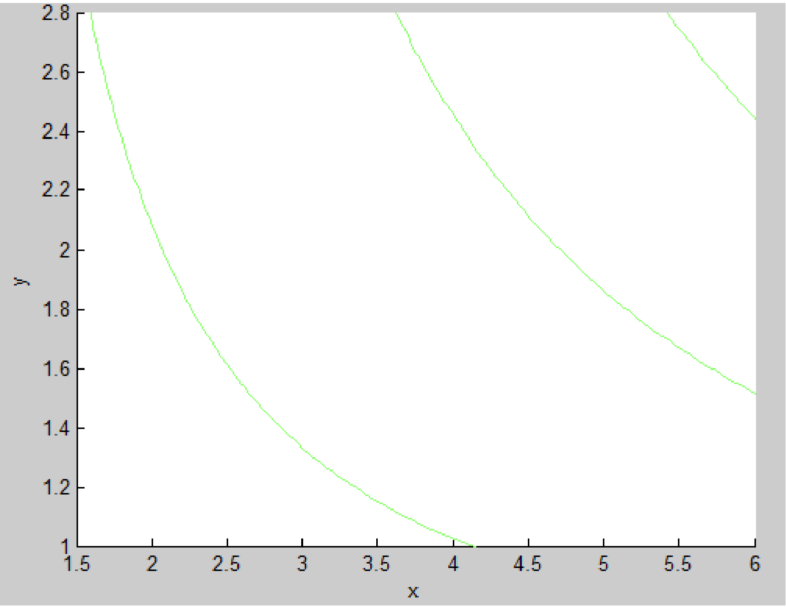
## Graphs for different values of mH (x= kH , y= $\frac{c}{c_0}$ where $c_0 = \sqrt{\frac{L_0}{\rho_0}}$ )


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
y
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x

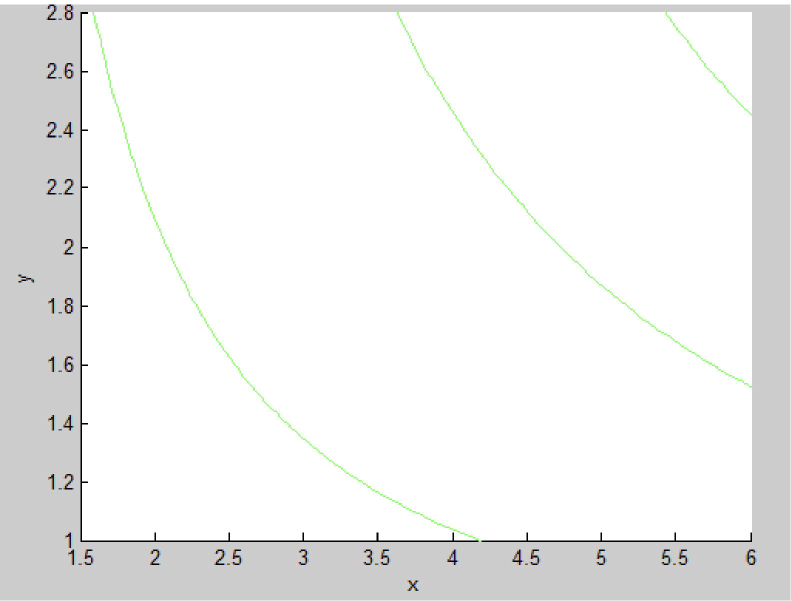
**mH = 4**


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
y
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x

**mH = 6**


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
y
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x

**mH = 8**


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
y
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
x

**mH = 10**

# Conclusion

- In dispersion equation under literature survey we saw that phase velocity is directly propotional to density while inversly propotional to dynamic viscocity.
- Displacement function or wave propogation in superficial layer is given as:

$$V(z) = 1/(1+mz)[c_1 e^{i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}z} + c_2 e^{-i\sqrt{\{k^2(\rho_0 c^2 - N_0)\}/L_0}z}]e^{ik(x-ct)}$$

- Ordinary differential Equation of wave propagation in terms of ‘z’ is given as:

$$v''(z) - \frac{a}{1-az}v'(z) + \frac{\alpha+\beta z}{1-az}v(z) = 0$$

$$\alpha = (\rho_0 c^2 + \frac{p}{2} - \mu_0)\frac{k^2}{\mu_0}$$

$$\beta = (\rho_0 c^2 b + \mu_0 a)\frac{k^2}{\mu_0}$$

- The dispersion equation is obtained using MATLAB by applying boundary conditions to the solutions of wave equation in superficial layer and half space.
- Graphs for different values of mH have been plotted, using kH on X-axis and $\frac{c}{c_0}$ on Y-axis.
- As the values of mH increases the graph shifts towards right away from the origin, however the change in the shift is very small as the value of mH increases.

# References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