The Optimality of Upgrade Pricing

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These slides at: `bit.ly/upgradeprices`
Utilities: `github.com/indraos/multi-goods-helpers`

Introduction

- Selling a base product and upgrades that cannot be purchased separately is frequently observed
  and an, arguably, simple mechanism
- Upgrade Pricing (UP): Sale of inclusion-ordered bundles

  This presentation: When is Upgrade Pricing optimal?
- We will study robust—w.r.t. type distribution for fixed support—optimality
- Several other approaches exist:
  - Demand Profiles: Wilson 1993
  - Monge-Kantorovich Duality: Daskalakis, Deckelbaum, and Tzamos 2017, Kash and Frongillo 2016
  - Lagrangian Duality: Cai, Devanur, and Weinberg 2016, Carroll 2017, Haghpanah and Hartline 2020 ▸ More

When is Grand Bundling Optimal?

Theorem (Haghpanah and Hartline 2020)

Grand bundling is robustly optimal for any distribution iff the type support is a subset of a line through the origin or a -45-degree line.

- Mixed bundling often dominates separate pricing McAfee, McMillan, and Whinston 1989a

Hypothesis:
UP is robustly optimal for a larger class of type supports

Model

- Monopolist sells d goods, zero costs
- Additive buyer utility $u((q,t);\theta) = \sum_{j=1}^{d} \theta^j q^j - t$
- n types $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\} \subseteq \mathbb{R}^d$ ▸ Beyond Finite Support ▸ Beyond Paths
- Buyer type $\theta \sim F \in \Delta(\Theta)$, probability mass function f
- By revelation principle, buyer can design direct mechanism $(q_1, t_1), (q_2, t_2), \ldots, (q_n, t_n) \in [0,1]^d \times \mathbb{R}_+$
- Designer wishes to maximize revenue $\sum_{\theta \in \Theta} f_i t_i$
- A mechanism is upgrade pricing if $\{q_1, q_2, \ldots, q_n\}$ can be (totally) ordered in inclusion/component-wise order

Informal Description of of Results

Theorem (Regularity, informal)

If F is "regular" and "weakly monotone", then UP is optimal.

Theorem (Ironing, informal)

If $\operatorname{supp} F$ *has "monotone marginal rate of substitution",* F *is "weakly monotone", and additional technical conditions on* F *hold, then UP is optimal.* ▸ More on Necessity

Theorem (Separate Sales and Upgrades, informal)

If types are monotone with respect to component-wise partial order, then upgrade pricing implementability is equivalent to implementation via separate pricing.

Section 1

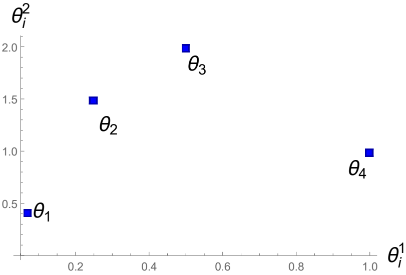
Optimality of Upgrade Pricing with Regular Distributions

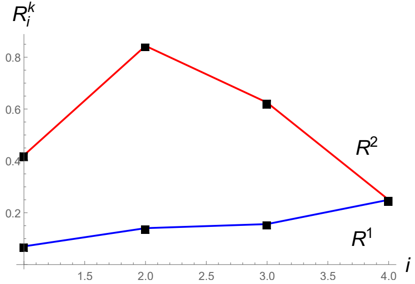
Towards Optimality for Regular Distributions

Definition

$$R_i^j = (1 - F_i)\theta_i^j$$

is the pseudo-revenue from item j and type θ_i.





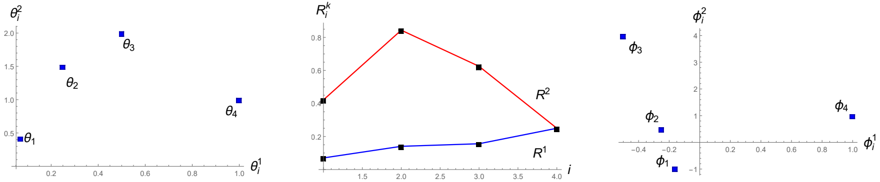
Regularity

Definition

F is regular if $i \mapsto R_i^j$ is single-peaked for all goods $j \in [d]$.

- Equivalent view: Slopes cross zero only once

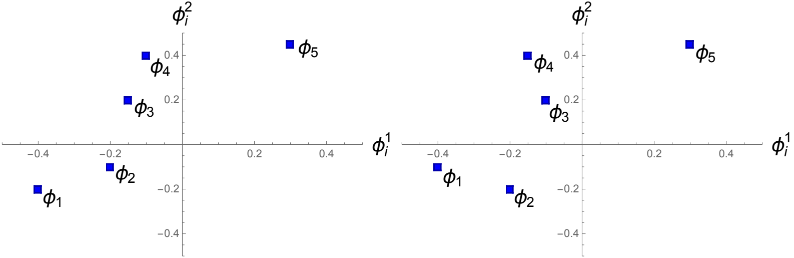
$$\frac{R_i^k - R_{i+1}^k}{f_i} = \frac{\theta_i^k(1-F_i) - \theta_{i+1}^k(1-F_{i+1})}{f_i}$$
$$= \theta_i^k - \frac{1-F_{i-1}}{f_i}(\theta_{i+1}^k - \theta_i^k) =: \phi_i^k$$



Weak Monotonicity

Definition

F is weakly monotone if $\theta_i^j \leq \theta_{i'}^j$ for any $i \leq \arg\max_i R_i^j \leq i'$, $j \in [d]$



Regularity Theorem

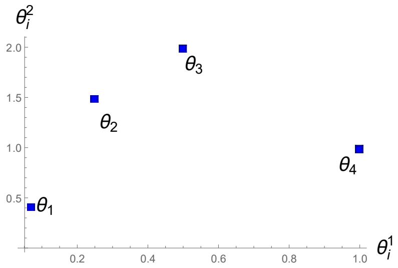
Theorem (Regularity)

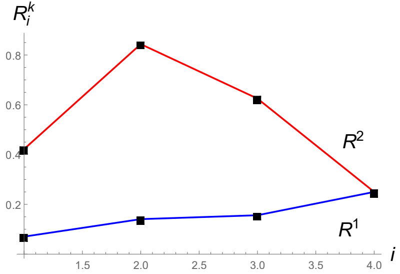
If F is regular and weakly monotone, then UP is optimal. In particular, the following is an allocation of an optimal mechanism:

$$q_i^j = \mathbb{1}_{i \geq \arg\max_i R_i^j}. \tag{1}$$

What We Capture, What We Do Not

We Capture





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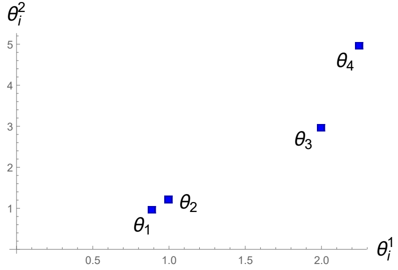
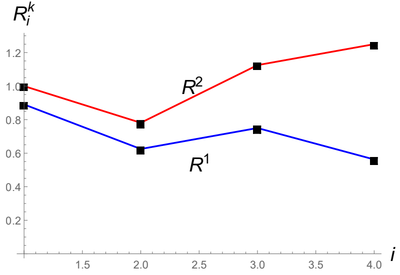




Proof Strategy

$$q_i^j = \mathbb{1}_{i \geq \arg\max_i R_i^j}. \tag{!}$$

Proof Strategy.

- Observe that (!) is upgrade pricing ✓
- Write down a dual to the monopolist's problem
- Propose a dual certificate of optimality for (!)

□

Duality

- Introduce dual variables λ_{ij}, $i \in [n]$, $j \in \{0\} \cup [n]$;
- λ_{ij} corresponds to IC($i \to j$), λ_{i0} corresponds to IR(i)
- Define virtual value $\phi_i^\lambda = \theta_i - \sum_{j=1}^n \lambda_{ji}(\theta_j - \theta_i) \in \mathbb{R}^d$

Lemma (Duality, compare Cai, Devanur, and Weinberg 2016)

A mechanism $(q_i, t_i)_{i \in \{0\} \cup [n]}$ maximizes revenue if and only if there are multipliers $\lambda_{ji} \geq 0$, $j \in [n]$, $i \in \{0\} \cup [n]$ such that

Virtual Welfare Maximization $(q_i)_{i \in [n]}$ *optimizes*

$$\max_{(q_i)_{i \in [n]} \in [0,1]^n} \sum_{i=1}^n f_i \langle q_i \cdot \phi_i^\lambda \rangle$$

Feasibility of Flow $f_i = \sum_{j=0}^n \lambda_{ij} - \sum_{j=1}^n \lambda_{ji}$ *for all* $i \in [n]$

Compl. Slackness $\lambda_{ji}(\langle q_j, \theta_j \rangle - t_j - \langle q_i, \theta_j \rangle - t_i) = 0$ *for* $j \in [n], i \in \{0\} \cup [n]$

Implementability *There are transfers t s.th. (q, t) is implementable*

Lagrangian and Feasibility Relation to Grand Bundling Optimality

Virtual Values for Regular Distributions

- Virtual values depend on dual variables λ_{ij}
- $\lambda_{ji} = \mathbb{1}_{j=i+1}(1 - F_i)$ gives virtual values

$$\phi_i^{\lambda} := \theta_i - \frac{1 - F_{i-1}}{f_i}(\theta_{i+1} - \theta_i) = \phi_i.$$

Optimality for Regular Distributions.

- Check that $\lambda_{ji} = \mathbb{1}_{j=i+1}(1 - F_i)$ is a dual certificate

Virtual Welfare Maximization ✓

Feasibility of Flow ✓

Complementary Slackness ✓

Implementability ✓

More formal Duality Lemma □

Section 2

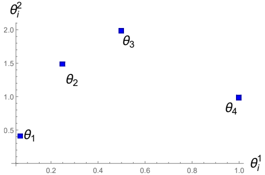
Optimality with Ironing

Definition

We say that a type space Θ has monotone marginal rates of substitution if for any $i, j \in [n]$, $l, k \in [d]$

$$i \leq j \text{ and } k \leq l \implies \frac{\theta_i^k}{\theta_i^l} \leq \frac{\theta_j^k}{\theta_j^l}$$

▸ Relation to Ratio Monotonicity



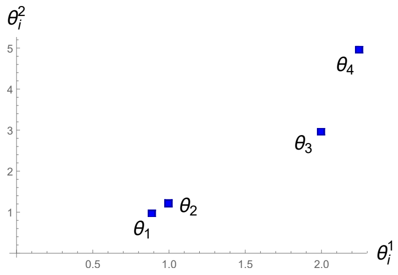
Theorem (With Ironing)

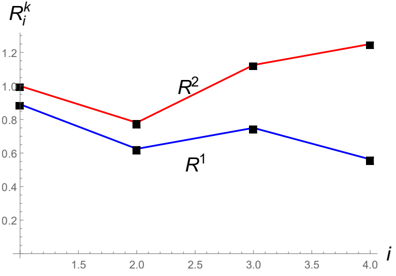
If F has monotone marginal rates of substitution and has weakly monotone types, and is ▸ *Mostly Regular, then UP is optimal with allocation*

$$q^j(\theta_i) = 1_{i \geq \arg\max_i R_i^j}. \tag{2}$$

What We Capture, What We Do Not

We Capture





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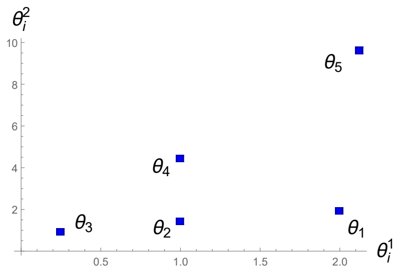
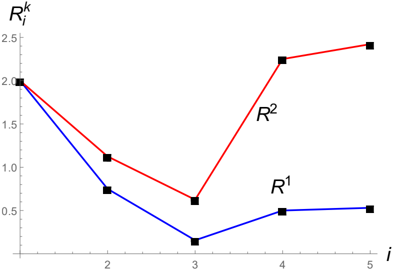




Entangled Virtual Values

- Wanted: λ_{ij}; we think of it as ironing virtual values
- Challenge: Virtual values for different goods are entangled

Lemma (Ordered Slopes)

If F has monotone marginal rate of substitution and λ is downward, then for $1 \leq k \leq l \leq d$,

$$\frac{\phi_i^k}{\theta_i^k} \leq \frac{\phi_i^l}{\theta_i^l}.$$

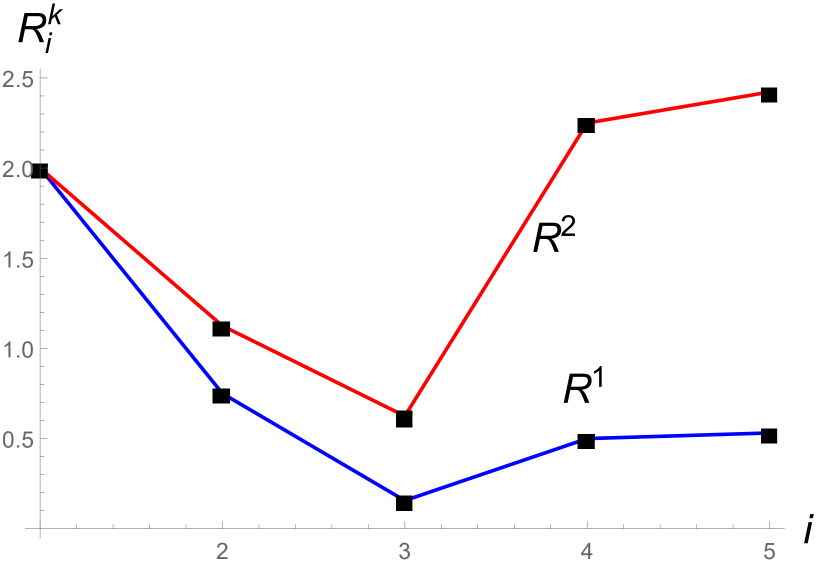
Proof.

$$\frac{\phi_i^{\lambda,k}}{\theta_i^k} = \frac{\theta_i^k - \sum_{j=1}^n \lambda_{ji}(\theta_j^k - \theta_i^k)}{\theta_i^k} = 1 + \sum_{j=i}^n \lambda_{ji} - \sum_{j=i}^n \lambda_{ji} \frac{\theta_j^k}{\theta_i^k}$$

$$\leq 1 + \sum_{j=i}^n \lambda_{ji} - \sum_{j=i}^n \lambda_{ji} \frac{\theta_j^l}{\theta_i^l} = \frac{\theta_i^l - \sum_{j=1}^n \lambda_{ji}(\theta_j^l - \theta_i^l)}{\theta_i^l} = \frac{\phi_i^{\lambda,l}}{\theta_i^l}.$$

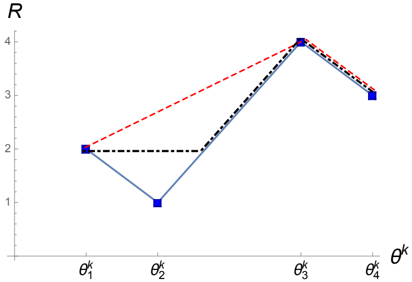
$\square$

Ordered Slopes



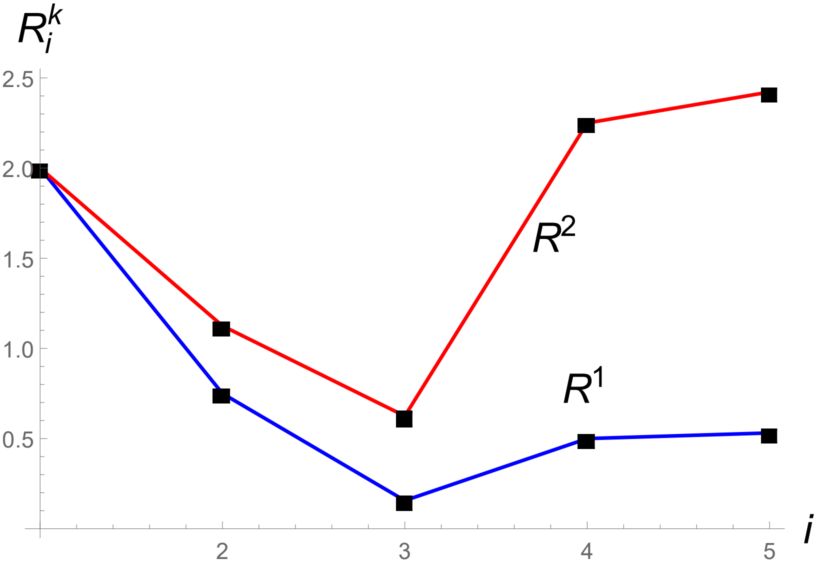
Idea of the Ironing

- We know that as long as λ is downward that the signs of virtual values are ordered
- Idea: Iron intervals to zero virtual value: dimension-wise quasi-concave closure

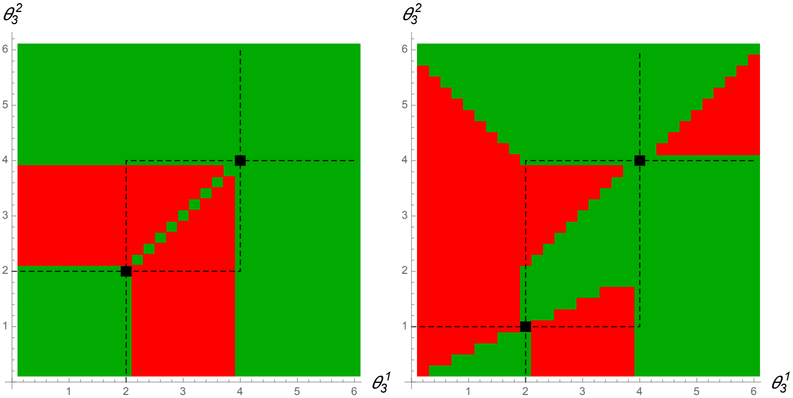


- Inside of ironing intervals no big problem
- Main care needed at the boundary of the interval.

Ordered Slopes



The Power of Upgrade Pricing Beyond Grand Bundling



- Upgrade Pricing is more powerful than Grand Bundling.

‣ For a Fixed Distribution

Section 3

Upgrade Pricing and Separate Pricing

Upgrade Pricing and Separate Pricing

- UP generalizes Grand Bundling when types are MRS, but not on a line through the origin
- As we show: Upgrade Pricing and Separate Pricing are equally powerful for a larger class of type supports
- Types are monotone if $\theta_i^j \leq \theta_{i+1}^j$ for any $i \in [n-1], j \in [d]$

Definition

A mechanism is separate pricing if it has a representation

$$q_i^k = \begin{cases} 1 & \theta_i^k \geq p^k \\ 0 & \text{else,} \end{cases} \qquad t_i = \sum_{k=1}^{d} p_k 1_{q_i^k = 1}.$$

for some $p^k \in \mathbb{R}_+$, $k \in [d]$.

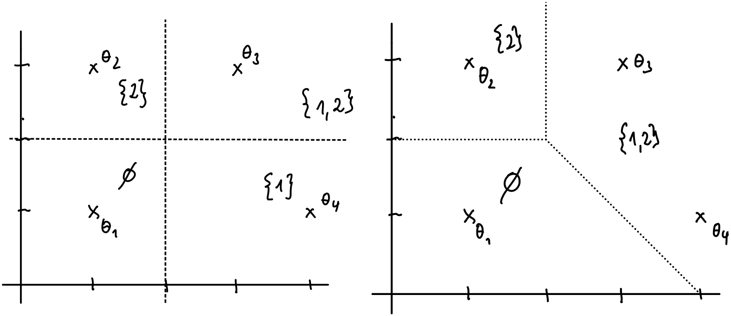
Proof

Theorem (Upgrade Pricing Redundancy)

If Θ is monotone, then the outcome of any upgrade pricing mechanism can be implemented via separate pricing, and vice versa. Otherwise, neither implication needs to hold.

Proof.

- UP $\rightarrow$ SP: Sell products at the price of upgrades
- SP $\rightarrow$ UP: Sell upgrades at price of products



□

Section 4

Epilogue

Conclusion

- Showed that Upgrade Pricing is more powerful than Grand Bundling with a robust optimality lense
- Proposed a multi-dimensional ironing targeting the dimension-wise quasi-concave closure, which allows to certify optimality of mechanisms

Future Work:

- Ironing result without technical conditions
- Formulations for
  - continuous
  - stochastically ordered

  distributions
- Extension to partial bundling in bundles

More

◂ Back

- Add-On Pricing Ellison 2005
- Upgrade pricing with vertical heterogeneity Johnson and Myatt 2003
- Mixed Bundling dominates separate pricing McAfee, McMillan, and Whinston 1989b

Beyond Finite Support

Back

As a generalization of Madarász and Prat 2017, we get the following meta-theorem:

Theorem

Assume that we can show optimality for a class of distributions F with finite support that is contained in a compact interval that satisfies some property P. If the set of finitely supported distributions with property P is dense (with respect to the Wasserstein metric) in the class of continuous distributions with this property, then the property holds also for continuous distributions.

Beyond Paths

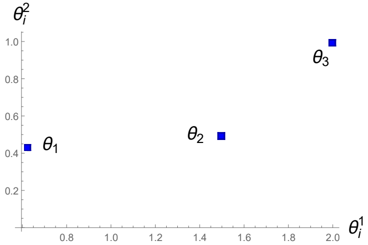
Back

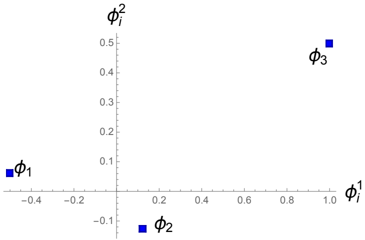
- Haghpanah and Hartline 2020 and unpublished work from Skrzypacz and Yang use a decomposition introduced in Strassen 1965
- It relates mixtures on paths to stochastic orders
- This allows to lift total orders on types to stochastic orders
- Our technical conditions do not work well with Strassen-type theorems

More on Necessity

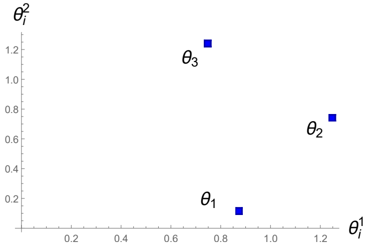
Back

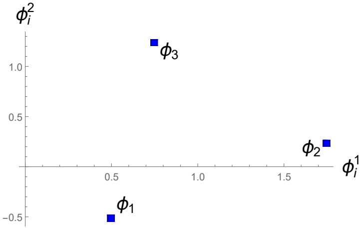
- Monotonicity without MRS: Strict Upgrade Pricing





- MRS without Monotonicity: No Upgrade Pricing





Lagrangian and Feasibility

Back

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{n} f_i t_i + \sum_{j=1}^{n}\sum_{i=0}^{n} \lambda_{ji}(\langle q_j, \theta_j\rangle - t_j - \langle q_i, \theta_j\rangle - t_i) \\
&= \sum_{i=1}^{n} t_i \left(f_i - \sum_{j=0}^{n} \lambda_{ij} + \sum_{j=1}^{n} \lambda_{ji} \right) + \sum_{j=1}^{n}\sum_{i=0}^{n} \lambda_{ji}\langle q_j, \theta_j\rangle - \lambda_{ji}\langle q_i, \theta_j\rangle \\
&= \sum_{j=1}^{n}\sum_{i=0}^{n} \lambda_{ji}\langle q_j, \theta_j\rangle - \sum_{j=1}^{n}\sum_{i=0}^{n} \lambda_{ji}\langle q_i, \theta_j\rangle \\
&= \sum_{j=1}^{n} \left(\left(\sum_{i=1}^{n} \lambda_{ij} - \sum_{i=0}^{n} \lambda_{ji} \right) \langle q_j, \theta_j\rangle - \sum_{i=0}^{n} \lambda_{ji}(\langle q_i, \theta_j\rangle - \langle q_j, \theta_j\rangle) \right) \\
&= \sum_{j=1}^{n} \left(f_j\langle q_j, \theta_j\rangle - \sum_{i=0}^{n} \lambda_{ji}(\langle q_i, \theta_j\rangle - \langle q_j, \theta_j\rangle) \right) = \sum_{j=1}^{n} f_j\langle q_j, \phi_j\rangle.
\end{aligned}
$$

Relation to Grand Bundling Optimality

Back

- In Haghpanah and Hartline 2020, the same theorem is presented in a single-dimensional version, by considering $\phi_i^{\lambda'} = \langle \phi_i^{\lambda}, \mathbb{1} \rangle$
- Our analysis that treats dimensions separately, does not allow for this simplification

More formal

Back

- Implementability is direct from weak monotonicity
- Feasibility of flow is by definition.
- Complementary slackness follows as only local downward IC constraints have non-zero dual variables
- Virtual Welfare Maximization: By single-peakedness and the fact that virtual values are derivatives of pseudo-revenues (from the right), we get that an allocation allocating items right of the maximum of the revenue curve maximizes virtual welfare.

Duality Lemma

Back

Lemma (Duality, compare Cai, Devanur, and Weinberg 2016)

A mechanism $(q_i, t_i)_{i \in \{0\} \cup [n]}$ *maximizes revenue if and only if there are multipliers* $\lambda_{ji} \geq 0$, $j \in [n]$, $i \in \{0\} \cup [n]$ *such that*

Virtual Welfare Maximization $(q_i)_{i \in [n]}$ *optimizes*
$$\max_{(q_i)_{i \in [n]} \in [0,1]^n} \sum_{i=1}^{n} f_i \langle q_i \cdot \phi_i^\lambda \rangle$$

Feasibility of Flow $f_i = \sum_{j=0}^{n} \lambda_{ij} - \sum_{j=1}^{n} \lambda_{ji}$ *for all* $i \in [n]$

Compl. Slackness $\lambda_{ji}(\langle q_j, \theta_j \rangle - t_j - \langle q_i, \theta_j \rangle - t_i) = 0$ *for*
$j \in [n], i \in \{0\} \cup [n]$

Implementability *There are transfers* t *s.th.* (q, t) *is implementable*

Relation to Ratio Monotonicity

Back

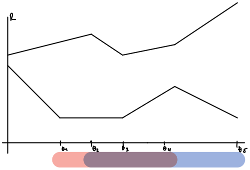
- Haghpanah and Hartline 2020 consider a concept called ratio monotonicity
- Ratio monotonicity, as a robust concept, boils down to $\frac{\sum_{k=1}^{d}\theta_i^k}{\theta_i^l} \leq \frac{\sum_{k=1}^{d}\theta_{i+1}^k}{\theta_{i+1}^l}$ for $i \in [n-1]$ and $l \in [d]$
- An equivalent formulation of monotone MRS is $\frac{\sum_{k=1}^{l}\theta_i^k}{\theta_i^l} \leq \frac{\sum_{k=1}^{l}\theta_{i+1}^k}{\theta_{i+1}^l}$ for $i \in [n-1]$ and $l \in [d]$
- (Non-trivial) calculations show that ratio-monotonicity's robust property implies that Θ is subset of a line through the origin or a $-45°$line

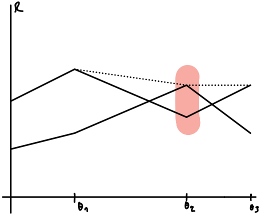
Mostly Regular

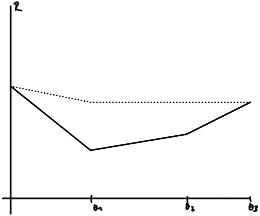
Back

Denote by $\overline{R^k}_i$ the quasi-concave closure of $i \mapsto R_i^k$.
We call a type distribution F *mostly regular* if for some $i^k \in \arg\max_{i \in [n]} R_i^k$ and any i such that $i^k < i \leq i^{k+1}$

1. If $R_i^k \neq \overline{R^k}_i$, then either $R_{i-1}^k \neq \overline{R^k}_{i-1}$ or $R_{i-1}^{l'} = \overline{R^{l'}}_{i-1}$ for $l' \in \{k-1, k+1\}$ (no overlap)
2. $R_{i^k}^l = \overline{R^l}_{i^k}$ for $l \in \{k-1, k+1\}$ (no ironing on maxima)
3. If $i^k \leq i < j \leq i^{k+1} \in [n]$ and $\overline{R^k}_r \neq R_r^k$ for any $i \leq r \leq j$, then $\theta_i^{k+1} \leq \theta_r^{k+1}$ (not too shuffled)

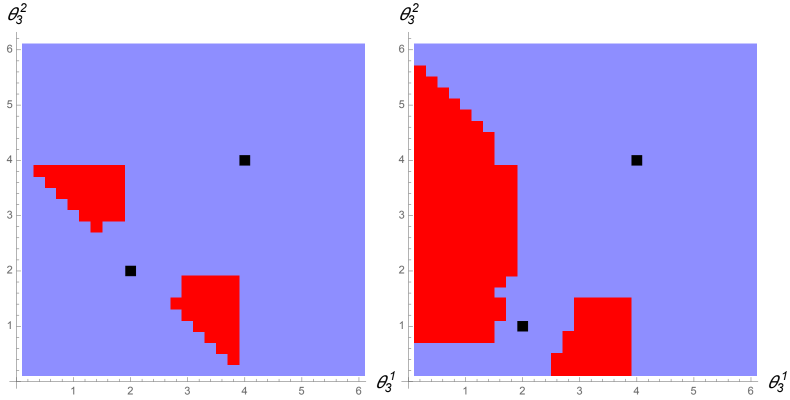






For a Fixed Distribution

◂ Back



- For fixed distribution (uniform) even more type supports support UP (in purple)
- We study thus far only robustness, but Upgrade Pricing is even more powerful for fixed distributions