Intersection of Convex Sets

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1 The Intersection Problem

Conventions

- **bold** lowercase letters are column vectors in $\mathbb{R}^n$
- uppercase **bold** letters are in $\mathbb{R}^{n\times n}$
- $K, K_1, K_2 \subseteq \mathbb{R}^n$ convex.
- $\|\cdot\|$ is the ℓ^2-norm.

Definition 1.1 (Oracles)**.** EO_f**:** Input $\mathbf{x}$, Output $f(\mathbf{x})$

$\mathrm{OO}_\varepsilon(\mathbf{K})$**:** Input $\mathbf{c}$, Output $\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

$\mathrm{SO}_{\varepsilon,\delta}(f)$**:** Input $\mathbf{x}$, Output: Assertion $f(\mathbf{x}) \leq \min_{\mathbf{y}} f(\mathbf{y}) + \eta$ or $\mathbf{c} \neq 0$, b with $b \leq \delta\|\mathbf{c}\|$ and $\{\mathbf{z}|\ f(\mathbf{z}) \leq f(\mathbf{x})\} \subseteq \{\mathbf{z}|\ \mathbf{c}^T\mathbf{z} \leq \mathbf{c}^T\mathbf{x} + b\}$

SGO_δ**:** Input $\mathbf{x}$, Output: $\mathbf{y} \in \{g \in \Omega|\ f(\mathbf{y}) + \delta \geq f(\mathbf{x}) + \mathbf{g}^T(y-x), \forall \mathbf{y} \in \Omega\}$.

Definition 1.2 (Strong Concavity)**.** $f\colon K \to \mathbb{R}$ is *α-strongly concave* iff $f(\mathbf{x}) + \alpha\|\mathbf{x}\|$ is concave.

Lemma 1.3. *If f is α-strongly concave with minimizer $\mathbf{x}^*$, $f(\mathbf{y}) \leq f(\mathbf{x}^*) + \varepsilon$, then $\frac{1}{2}\alpha\|\mathbf{x}^* - \mathbf{y}^*\|^2 \leq \varepsilon$, $\forall\varepsilon > 0$. Strongly concave functions have unique maximizers.*

Theorem 1.4 (Our Cutting-Plane Method)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $\alpha \in (0,1)$, $\Omega \subseteq B_\infty(\mathbf{0}, R)$ convex containing a minimizer of f. Then we can compute $\mathbf{x} \in \mathbb{R}^n$ with*

$$f(\mathbf{x}) - \min_{\mathbf{y}\in\Omega} f(\mathbf{y}) \leq \eta + \alpha(\max_{\mathbf{y}\in\Omega} f(\mathbf{y}) - \min_{\mathbf{y}\in\Omega} f(\mathbf{y}))$$

in $O(n\,\mathrm{SO}_{\eta,\delta}(f)\log(\frac{n\kappa}{\alpha}) + n^3\log^{O(1)}(\frac{n\kappa}{\alpha}))$, $\delta \in \Theta(\frac{\alpha\,\mathrm{MinWidth}(\Omega)}{n^{\frac{3}{2}}\ln(\kappa)})$, $\kappa = \frac{R}{\mathrm{MinWidth}(\Omega)}$.

Matroid Intersection Problem

Instance:	Matroids $(E, \mathcal{I}_1)$, $(E, \mathcal{I}_2)$ given via independence (or rank) oracles of complexity $\mathcal{T}_{\mathrm{ind}}$ (or $\mathcal{T}_{\mathrm{rank}}$), $\boldsymbol{w} \in \mathbb{R}^E$, $\|\boldsymbol{w}\|_\infty \leq M$.
Task:	Find $S \in \arg\min_{S\in\mathcal{I}_1\cap\mathcal{I}_2} w(S)$.

2 Solving the Intersection Problem

Assumptions

$$K_1, K_2 \subseteq B_2(0, M), M \geq 1, \quad \|\mathbf{c}\|_2 \leq M, \quad K_1 \cap K_2 \neq \emptyset \tag{A}$$

Regularize the Problem

$$\boxed{f_\lambda(\mathbf{x},\mathbf{y}) \coloneqq \frac{1}{2}\mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{c}^T\mathbf{y} - \frac{\lambda}{2}\|\mathbf{x}-\mathbf{y}\|^2 - \frac{1}{2\lambda}\|\mathbf{x}\|^2 - \frac{1}{2\lambda}\|\mathbf{y}\|^2 x}$$

Lemma 2.1. *Assuming* (A)*, f_λ has a unique maximizer $(\mathbf{x}_\lambda, \mathbf{y}_\lambda)$ on $K_1 \times K_2$ such that $\|\mathbf{x}_\lambda - \mathbf{y}_\lambda\|^2 \leq \frac{6M^2}{\lambda}$ and*

$$\max_{\mathbf{x}\in K_1\cap K_2} \mathbf{c}^T\mathbf{x} \leq f_\lambda(\mathbf{x}_\lambda, \mathbf{y}_\lambda) + \frac{M^2}{\lambda}.$$

Lemma 2.2 (Sion 1958, Corollary of 3.3)**.** *Let $f\colon X \times Y \to \mathbb{R}, (x,y) \mapsto f(x,y)$ continuous, convex in x, concave in y, X or Y compact. Then*

$$\sup_{x\in X}\inf_{y\in Y} f(\mathbf{x},\mathbf{y}) = \inf_{y\in Y}\sup_{x\in X} f(\mathbf{x},\mathbf{y}).$$

Max-Min Problem

$$\boxed{\begin{aligned}
\Omega &\coloneqq \{(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)|\ \|\boldsymbol{\theta}_1\| \leq 2M, \|\boldsymbol{\theta}_2\|, \|\boldsymbol{\theta}_3\| \leq M\}\\
g_\lambda(\mathbf{x},\mathbf{y},\boldsymbol{\theta}) &\coloneqq \left(\frac{\boldsymbol{c}}{2} + \lambda\boldsymbol{\theta}_1 + \frac{1}{\lambda}\boldsymbol{\theta}_2\right)^T \boldsymbol{x}\\
&\quad + \left(\frac{\boldsymbol{c}}{2} - \lambda\boldsymbol{\theta}_1 + \frac{1}{\lambda}\boldsymbol{\theta}_3\right)^T \boldsymbol{y}\\
&\quad + \frac{\lambda}{2}\|\boldsymbol{\theta}_1\|^2 + \frac{1}{2\lambda}(\|\boldsymbol{\theta}_2\|^2 + \|\boldsymbol{\theta}_3\|^2)\\
h_\lambda(\boldsymbol{\theta}) &\coloneqq \max_{(\boldsymbol{x},\boldsymbol{y})\in K_1\times K_2} g_\lambda(\mathbf{x},\mathbf{y},\boldsymbol{\theta}), \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)
\end{aligned}}$$

Getting an Approximately Optimal Solution

Lemma 2.3. *Assuming* (A)*, $\lambda \geq 2$,*

$$f_\lambda(\mathbf{x},\mathbf{y}) = \min_{\boldsymbol{\theta}\in\Omega} g_\lambda(\mathbf{x},\mathbf{y},\boldsymbol{\theta}).$$

If

$$h_\lambda(\boldsymbol{\theta}') \le \min_{\boldsymbol{\theta}\in\Omega} h_\lambda(\boldsymbol{\theta}) + \varepsilon,$$

then $\boldsymbol{z} = -\frac{1}{2}(\boldsymbol{\theta}_2' + \boldsymbol{\theta}_3')$ *satisfies*

$$\max_{\boldsymbol{x}\in K_1\cap K_2} \boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{c}^T\boldsymbol{z} + \frac{20M^2}{\lambda} + 20\lambda^3\varepsilon$$

and $\|\boldsymbol{z}-\boldsymbol{x}_\lambda\| + \|\boldsymbol{z}-\boldsymbol{y}_\lambda\| \le 4\sqrt{2\lambda\varepsilon} + \sqrt{\frac{6M^2}{\lambda}}$, $(\boldsymbol{x}_\lambda, \boldsymbol{y}_\lambda) \in \arg\max_{(\boldsymbol{x},\boldsymbol{y})\in K_1\times K_2} f_\lambda(\boldsymbol{x},\boldsymbol{y})$.

Getting a Separation Oracle for h_λ

Lemma 2.4. $\mathrm{SO}_{O(\sqrt{\varepsilon\lambda D}),O(\sqrt{\varepsilon\lambda D})}(h_\lambda)$ *on* $\{\boldsymbol{\theta} \mid \|\boldsymbol{\theta}\| \le D\}$ *has complexity* $O(\mathrm{OO}_\varepsilon(K_1) + \mathrm{OO}_\varepsilon(K_2))$

Solving the Intersection Problem

Theorem 2.5 (Main Theorem)**.** *Assuming* (A). *Then for any* $0 < \delta < 1$, *we can find* $\boldsymbol{z}$, $d(\boldsymbol{z}, K_1) + d(\boldsymbol{z}, K_2) \le \delta$ *such that*

$$\max_{\boldsymbol{x}\in K_1\cap K_2} \boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{c}^T\boldsymbol{z} + \delta$$

in time

$$O(n(\mathrm{OO}_\eta(K_1) + \mathrm{OO}_\eta(K_2))\log(\frac{nM}{\delta}) + n^3\log^{O(1)}(\frac{nM}{\delta})), \qquad \eta \in \Omega((\frac{\delta}{nM})^{O(1)}).$$

Lemma 2.6 (Klivans and Spielman 2001, Lemma 4)**.** *Let* C *be a set of linear forms in variables* $z_1, \dots, z_\ell$ *with coefficients in* $[K]$. *For* $z_1, \dots, z_\ell \sim \mathrm{Unif}_{[\lceil \frac{K\ell}{\varepsilon}\rceil]}$, *with probability greater than* $1-\varepsilon$, *there is a unique form of minimal value at* $z_1, \dots, z_\ell$.

- Note $\min_{x\in P} \mathbf{z}^T\mathbf{x} = \min_{x\in P} \mathbf{x}^T\mathbf{z} = \min_{x\in P} x(\mathbf{z})$, where $x \in \mathcal{L}(\mathbb{R}^n), \mathbf{z} \mapsto \mathbf{x}^T\mathbf{z}$.
- This is the reason why algorithms in the applications give guarantees only with a certain probability.

Unique Solutions and an Error Estimate

Lemma 2.7 (Uniqueness Lemma)**.** *Let* $P \coloneqq \{\mathbf{x} \mid \boldsymbol{A}\boldsymbol{x} \ge \boldsymbol{b}\} \subseteq B_M^\infty(\boldsymbol{0})$ *be integral,* $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $\boldsymbol{b} \in \mathbb{Z}^n$, $\boldsymbol{c} \in \mathbb{Z}^n$.

- *Then we can find* $\boldsymbol{z} \in \mathbb{Z}^n, \|\boldsymbol{z}\|_\infty \le 100n^2M^2\|c\|_\infty + 10nM$ *such that with probability at least* $\frac{9}{10}$, $\min_{\boldsymbol{x}\in P} \boldsymbol{z}^T\boldsymbol{x}$ *has a unique minimizer* $\boldsymbol{x}^*$ *and* $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x}\in P} \boldsymbol{c}^T\boldsymbol{x}$.
- *If* $\boldsymbol{y} \in P, \boldsymbol{z}^T\boldsymbol{y} \le \min_{\boldsymbol{x}\in P} \boldsymbol{z}^T\boldsymbol{x} + \delta$, *then* $\|\boldsymbol{y}-\boldsymbol{x}^*\| \le 2nM\delta$.

3 Applications

Weighted Matroid Intersection

Theorem 3.1. WEIGHTED MATROID INTERSECTION *with* $\boldsymbol{w} \in \mathbb{Z}^n$, $\|\mathbf{w}\|_\infty \le M$ *can be solved in*

$$O(n\,\mathrm{GO}\log(nM) + n^3\log^{O(1)}(nM))$$

with probability at least $\frac{9}{10}$, *where* GO *is the complexity of optimizing* $\min_{S\in\mathcal{I}_i} w(S)$.

For expensive GO and/or large r speedup.

Submodular Flow

SUBMODULAR FLOW PROBLEM	
Instance:	(G,c) weighted digraph, $l, u\colon V(G) \to \mathbb{R}$, $f\colon 2^{V(G)} \to \mathbb{R}$, $f(\emptyset) = f(V) = 0$
Task:	Find a flow $\varphi\colon E(G) \to \mathbb{R}$, $l(e) \le \varphi(e) \le u(e)$, $\forall e \in E(G)$, $\sum_{v\in S} \mathrm{ex}_\varphi(v) \le f(S)$.

LP formulation ($\mathbf{A}$ incidence matric of G)

$$\max_{\mathbf{x}\in P} \mathbf{c}^T\mathbf{x}, P \coloneqq \{\boldsymbol{\varphi} \mid \mathbf{l} \le \boldsymbol{\varphi} \le \mathbf{u},\ \mathbf{x} = \mathbf{A}\boldsymbol{\varphi}, \mathbf{x}(S) \le f(S), \forall S \subseteq V\}$$

Consider set P_ε of solution of value at most OPT $+\varepsilon$.

$$P_\varepsilon = \{\mathbf{x} \mid \exists\boldsymbol{\varphi}\colon \mathbf{l} \le \boldsymbol{\varphi} \le \mathbf{u},\ \mathbf{x} = \mathbf{A}\boldsymbol{\varphi}\} \cap \{\mathbf{x} \mid \mathbf{x}(S) \le f(S), \forall S \subseteq V, \mathbf{x}(V) = f(V)\}$$

Submodular Flow

Theorem 3.2. *The* SUBMODULAR FLOW PROBLEM *can be solved in*

$$O(n^2\,\mathrm{EO}_f\log(mCU)\log(n) + n^3\log^{O(1)}(mCU))$$

with probability at least $\frac{9}{10}$.

Previously best: $\tilde{O}(n^6\,\mathrm{EO}_f + n^7)$ Fleischer, Iwata, and McCormick 2002 and $O(mn^5\log(nU)\,\mathrm{EO})$ Fleischer and Iwata 2000

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