# CS 570: Analysis of Algorithms – H3

**Submitted by: Indronil Bhattacharjee**

| A | B | O | o | Ω | ω | Θ |
|---|---|---|---|---|---|---|
| $\lg^k n$ | $n^\epsilon$ | **yes** | **yes** | no | no | no |
| $n^k$ | $c^n$ | **yes** | **yes** | no | no | no |
| $\sqrt{n}$ | $n^{\sin(n)}$ | no | no | no | no | no |
| $2^n$ | $2^{(n/2)}$ | no | no | **yes** | **yes** | no |
| $n^{(\lg c)}$ | $c^{(\lg n)}$ | **yes** | no | **yes** | no | **yes** |
| $\lg(n!)$ | $\lg(n^n)$ | **yes** | no | **yes** | no | **yes** |

**1. A = $\lg^k n$ and B = $n^\epsilon$:**

A=O(B) if there exist positive constants c and n0 such that 0≤ A(n) ≤ c·B(n) for all n ≥ n0.

Let A(n)=$\lg^k n$ and B(n)=$n^\epsilon$.

Since $\lg^k n$ is a positive function for all n ≥ 1, we can set $n_0$=1.

Let's choose c=1, then $\lg^k n \leq n^\epsilon$ for all n ≥ $n_0$=1.

Thus, A=O(B).

To show B ≠ O(A), we need to find constants c and $n_0$ such that 0 ≤ B(n) ≤ c·A(n) for all n≥$n_0$.

However, since B(n)=$n^\epsilon$ grows faster than A(n)=$\lg^k n$ for all n ≥ 1, such constants cannot be found.

Therefore, B ≠ O(A).

Since A = O(B) but B ≠ O(A), A is O(B) but not Θ(B).

**2. A = $n^k$ and B = $c^n$:**

A = O(B) if there exist positive constants c and $n_0$ such that 0 ≤ A(n) ≤ c·B(n) for all n ≥ $n_0$.

Let A(n)=$n^k$ and B(n)=$c^n$.

Since $n^k$ is a positive function for all $n \geq 1$, we can set $n_0$=1.

Let's choose c=1, then $n^k \leq c^n$ for all $n \geq n_0=1$.

Thus A = O(B).

To show B ≠ O(A), we need to find constants c and $n_0$ such that $0 \leq B(n) \leq c \cdot A(n)$ for all $n \geq n_0$.

However, since $B(n) = c^n$ grows faster than $A(n)=n^k$ for all n≥1, such constants cannot be found.

Therefore, B ≠ O(A).

Since A=O(B) but B≠O(A), A is O(B) but not Θ(B).

**3. A = $\sqrt{n}$ and B = $n^{\sin n}$:**

Comparison with definitions is more challenging here due to the oscillatory behavior of B = $n^{\sin n}$.

We cannot apply the definitions of O, o, Ω, ω, and Θ directly to A and B because sinn oscillates between [-1,1]. We cannot lower or upper bound $n^{\sin n}$ with $n^{1/2}$.

**4. A = $2^n$ and B = $2^{(n/2)}$**

B intuitively represents the same function as A with a lower exponent, therefore it represents a strong lower bound of A.

Again, B = $2^{n/2} = 2^{1/2.n} = 2^{1/2.n}$ = $\sqrt{2}^n$. $2^n$ grows faster than $\sqrt{2}^n$, since $2 > \sqrt{2}$.

Therefore, A = Ω(B)

**5. A = $n^{(\lg c)}$ and B = $c^{(\lg n)}$**

To show A=O(B), we need to prove that there exist positive constants c and n0 such that $0 \leq A(n) \leq c \cdot B(n)$ for all $n \geq n0$.

Let $A(n)=n^{\lg c}$ and $B(n)=c^{\lg n}$.

We know that $\lg n \leq n$ for all $n \geq 1$.

Let's choose c=1 and $n_0$=1, then $n^{\lg c} \leq n^{\lg n} = c^{\lg n}$ for all $n \geq n0=1$.

Thus, A=O(B).

To show A = Ω(B), we need to prove that there exist positive constants c and $n_0$ such that $0 \le c \cdot B(n) \le A(n)$ for all $n \ge n_0$.

Let $A(n)=n^{lgc}$ and $B(n)=c^{lgn}$.

We know that $lgn \ge 1$ for all $n \ge 1$.

Let's choose c=1 and $n_0=1$, then $c^{lgn} \le n^{lgc}$ for all $n \ge n_0=1$.

Thus, A=Ω(B).

Therefore, by definition, A=Θ(B).

**<u>6. A = lg(n!) and B = lg($n^n$)</u>**

To show A=O(B), we need to prove that there exist positive constants c and $n_0$ such that $0 \le A(n) \le c \cdot B(n)$ for all $n \ge n_0$.

Let A(n) = lg(n!) and B(n) = lg($n^n$).

We know that $n! \le n^n$ for all $n \ge 1$.

Taking the logarithm of both sides, we get $lg(n!) \le lg(n^n)$ for all $n \ge 1$.

Let's choose c = 1 and $n_0$ = 1, then $lg(n!) \le lg(n^n)$ for all $n \ge n_0=1$.

Thus, A=O(B).

To show A=Ω(B), we need to prove that there exist positive constants c and n0 such that $0 \le c \cdot B(n) \le A(n)$ for all $n \ge n0$.

Let A(n)=lg(n!) and B(n)=lg($n^n$).

Consider the ratio $log(n^n)/log(n!) = nlogn/log(n!)$.

As n approaches infinity, the ratio nlogn/log(n!) approaches 1/e (using Stirling's approximation).

Therefore, for c=1/2e, we can choose $n_0$=1 and for all $n \ge n_0$, $0 \le 1/2e \cdot nlogn \le log(n!)$.

Thus, A=Ω(B).

Therefore, by definition, A=Θ(B).