Math 226- HW 1 Due: Sep 13 by Midnight

1. (6 points) Let A and B be two non-empty sets, $f : A \to B$ be an injective function, and a_0 is a fixed element of A. Define $g : B \to A$ such that

$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

   a) (3 points) Show that g is a left inverse of f. That is for all $a \in A$, $g \circ f(a) = a$
   b) (3 points) Explain why g might not be the right inverse of f. Under which conditions g can be a right inverse of f?
   c) optional: If $f : A \to B$ be surjective then it would have right inverse. Define the right inverse.

2. (12 points) Given two functions $f : A \to B$ and $g : B \to C$, the *composition of g with f*, denoted $g \circ f$ is the map $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$. Show that the set of bijection functions are closed under $\circ$. That is

   (a) (5 points) Show if f and g are injections, then $g \circ f$ is an injection. Hint: Recall the injective and left inverse relation.
   (b) (5 points) Show if f and g are surjections, then $g \circ f$ is a surjection. Hint: Use the definition of surjectivety first for g and then for f.
   c) (2 points) Show if f and g are bijections, then $g \circ f$ is a bijection. Hint: You don't need a hint.

3. (9 points) Let $B = \{m + n\sqrt{2} \ : \ m, n \in \mathbb{Z}\}$.

   a) (3 points) Show that B is *closed under addition*. In other words, prove "if $x \in B$ and $y \in B$, then their sum $x + y$ must be in B as well.
   b) (3 points) Show that B is *closed under multiplication*. In other words, prove "if $x \in B$ and $y \in B$, then their product xy must be in B as well.
   c) (3 points) For every integer $k \geq 1$, prove that $(-1 + \sqrt{2})^k \in B$ - Try to use an inductive argument for this problem.

4. (16 points) Define $\mathbb{Z}\{\sqrt{3}\} := \{a + b\sqrt{3}, \text{ where } a, b \in \mathbb{Z}\}$. Below we will validate that $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$, with usual addition and multiplication does not define a field.

   a) (4 points) Show that if $a^2 - 3b^2 = \pm 1$, then $a + b\sqrt{3}$ has a multiplicative inverse. Hint: Check out the definition of the multiplicative inverse, you can directly give it in each case.
   b) (10 points) Show that the above statement is iff, i.e., if $a + b\sqrt{3}$ has a a multiplicative inverse then $a^2 - 3b^2 = \pm 1$ Hint: Solve the problem first for the case when a, b are prime to each other. That will give you the understanding of the rest.
   c) (2 points) Use a) and b) to conclude that $(\mathbb{Z}\{\sqrt{3}\}, +, \cdot)$, with usual addition and multiplication does not define a field.
   - (optional) Prove that the set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \ : \ a, b \in \mathbb{Q}\}$ is indeed a field. Notice that you don't need the conditions like in b, c here.

5. (9 points) Part a) requires familiarity with the vector space $\mathcal{F}(S, \mathbb{F})$. Check out Example 3 of Section 1.2. This example will be discussed in ULA section.

a) (6 points) (Sec 1.2, Problem 12) A real valued function f defined on the real line is called an even function if $f(-t) = f(t)$ for each number of t. Prove that the set of even functions defined on the real line with the operations of addition, scalar multiplication (as in the Example 3) is a vector space.

b) (3 points) (Problem 13 in the book) Let $V := \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. If (a_1, a_2) and (b_1, b_2) are elements on V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad c(a_1, a_2) = (ca_1, a_2)$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Practice Problems : Sec 1.2 : 10 - 22