Prop. Logic II

Proofs

A common proof type is showing that one statement is equivalent to another.

This *can* sometimes be done using a truth table.

Truth tables run out of usefulness very quickly when working with many simple statements.

Formal methods will demonstrate equivalence using substitution and well known equivalent statements.

Proof Tools - Laws Of Logic (LOL)

In many formal math classes a list of equivalent statements is understood to exist.

Definition

A **Law of Logic** is a named commonly used equivalent statement.

A Law of Logic is not typically expected to be proved in a proof.

Laws of Logic substitutions can be done as long as the correct format is met.
For our class you will be able to use any law of logic listed in this document.

Laws Of Logic - List

- Double Negation Law
- Commutative Law
- Associative Law
- Distributive Law
- DeMorgan's Law
- Absorption Law
- Idempotent Law
- Identity Law
- Inverse Law
- Domination Law
- Implication Identity

Double Negation Law

If we negate a negation, the two negations will cancel out.

$\neg\neg p \equiv p$

Can be used to remove or create negations.

Reasoning/Proof

p	$\neg p$	$\neg\neg p$
True	False	True
False	True	False

"p" has the same truth value as "$\neg\neg p$" in all assignments.

Double Negation Law - Example

Example

Prove that "$\boldsymbol{p} \vee \neg\neg\neg \boldsymbol{q}$" $\equiv$ "$\boldsymbol{p} \vee \neg \boldsymbol{q}$"

Proof.

$$\boldsymbol{p} \vee \neg\neg\neg \boldsymbol{q} \equiv \boldsymbol{p} \vee \neg \boldsymbol{q} \qquad \text{by Double Negation}$$

$\square$

The square at the end of the proof is a stand in for Q.E.D. Q.E.D. roughly translates to "we have demonstrated what was to be shown".

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Commutative Law

For “or” and “and” the order of the two parts does not matter.

$\boldsymbol{p} \vee \boldsymbol{q} \equiv \boldsymbol{q} \vee \boldsymbol{p}$
$\boldsymbol{p} \wedge \boldsymbol{q} \equiv \boldsymbol{q} \wedge \boldsymbol{p}$

Reasoning/Proof

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \vee \boldsymbol{q}$	$\boldsymbol{q} \vee \boldsymbol{p}$
True	True	True	True
True	False	True	True
False	True	True	True
False	False	False	False

“$\boldsymbol{p} \vee \boldsymbol{q}$” has the same truth value as “$\boldsymbol{q} \vee \boldsymbol{p}$” in all assignments.

Commutative Law - Example

Example

Prove that "$p \vee (\neg q \wedge r)$" $\equiv$ "$(r \wedge \neg q) \vee p$"

Proof.

$$
\begin{aligned}
p \vee (\neg q \wedge r) &\equiv p \vee (r \wedge \neg q) && \text{by Commutative Law} \\
&\equiv (r \wedge \neg q) \vee p && \text{by Commutative Law}
\end{aligned}
$$

$\square$

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Associative Law

For the "or" of 3 statements or the "and" of 3 statements the order in which the "or"s or the "and"s does not matter.

$$\boldsymbol{p} \lor (\boldsymbol{q} \lor \boldsymbol{r}) \equiv (\boldsymbol{p} \lor \boldsymbol{q}) \lor \boldsymbol{r}$$
$$\boldsymbol{p} \land (\boldsymbol{q} \land \boldsymbol{r}) \equiv (\boldsymbol{p} \land \boldsymbol{q}) \land \boldsymbol{r}$$

IMPORTANT!!!

Associative cannot be used on a mixed symbols. Distributive Law does that.
Associative cannot be used if a negation exists on the inner connective. DeMorgan's needs to be used first.

$$\boldsymbol{p} \lor \neg(\boldsymbol{q} \lor \boldsymbol{r}) \not\equiv \neg(\boldsymbol{p} \lor \boldsymbol{q}) \lor \boldsymbol{r}$$
$$\boldsymbol{p} \land (\boldsymbol{q} \lor \boldsymbol{r}) \not\equiv (\boldsymbol{p} \land \boldsymbol{q}) \lor \boldsymbol{r}$$

Associative Law - Reasoning

Reasoning/Proof

Let $S_1 \equiv p \vee (q \vee r)$ and $S_2 \equiv (p \vee q) \vee r$

p	q	r	$p \vee q$	$q \vee r$	S_1	S_2
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	True	True	True
True	False	False	True	False	True	True
False	True	True	True	True	True	True
False	True	False	True	True	True	True
False	False	True	False	True	True	True
False	False	False	False	False	False	False

S_1 has the same truth value as S_2 in all assignments.

Associative Law - Example

Example

Prove that "$\boldsymbol{p} \vee (\neg \boldsymbol{q} \vee \boldsymbol{r})$" $\equiv$ "$(\boldsymbol{p} \vee \neg \boldsymbol{q}) \vee \boldsymbol{r}$"

Proof.

$$\boldsymbol{p} \vee (\neg \boldsymbol{q} \vee \boldsymbol{r}) \equiv (\boldsymbol{p} \vee \neg \boldsymbol{q}) \vee \boldsymbol{r} \qquad \text{by Associative Law}$$

$\square$

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Practice Proof 1

Example

Prove that "$\boldsymbol{p} \vee (\neg\neg \boldsymbol{q} \vee \boldsymbol{r})$" $\equiv$ "$(\boldsymbol{q} \vee \boldsymbol{p}) \vee \boldsymbol{r}$"

Answer

Distributive Law

If we have an "and" on the outside of an "or", then the "and" can be distributed into the "or", and vice versa.

$$\boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) \equiv (\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \boldsymbol{r})$$
$$\boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \equiv (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$$

IMPORTANT!!!

Distributive cannot be used if a negation exists on the inner connective. DeMorgan's needs to be used first.

$$\boldsymbol{p} \vee \neg(\boldsymbol{q} \wedge \boldsymbol{r}) \not\equiv (\boldsymbol{p} \vee \neg \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \neg \boldsymbol{r})$$

Distributive Law - Reasoning

Reasoning/Proof

Let $S_1 \equiv p \vee (q \wedge r)$ and $S_2 \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	S_1	S_2
True	True	True	True	True	True	True	True
True	True	False	False	True	True	True	True
True	False	True	False	True	True	True	True
True	False	False	False	True	True	True	True
False	True	True	True	True	True	True	True
False	True	False	False	True	False	False	False
False	False	True	False	False	True	False	False
False	False	False	False	False	False	False	False

S_1 has the same truth value as S_2 in all assignments.

Distributive Law - Example

Example

Prove that "$p \lor (\neg q \land r)$" $\equiv$ "$(p \lor \neg q) \land (p \lor r)$"

Proof.

$$p \lor (\neg q \land r) \equiv (p \lor \neg q) \land (p \lor r) \quad \text{by Distributive Law}$$

$\square$

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DeMorgan's Law

DeMorgan's is a distribution rule for negations.

If we know that I did not "eat and swim", then either "I did not eat" or "I did not swim".

Negations, when distributed into a connective, invert the sign.

$$\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \equiv \neg \boldsymbol{p} \vee \neg \boldsymbol{q}$$
$$\neg(\boldsymbol{p} \vee \boldsymbol{q}) \equiv \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$$

IMPORTANT!!!

Do not remove the parenthesis, if combined with other connectives!

$$\boldsymbol{p} \wedge \neg(\boldsymbol{q} \wedge \boldsymbol{r}) \not\equiv \boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{r}$$

instead

$$\boldsymbol{p} \wedge \neg(\boldsymbol{q} \wedge \boldsymbol{r}) \equiv \boldsymbol{p} \wedge (\neg \boldsymbol{q} \vee \neg \boldsymbol{r})$$

DeMorgan's Law - Reasoning

Reasoning/Proof

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
True	True	True	False	False	False	False
True	False	False	False	True	True	True
False	True	False	True	False	True	True
False	False	False	True	True	True	True

Because the last two columns are identical for every row, it is the case that $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.

DeMorgan's Law - Example

Example

Prove that "$\neg(p \wedge (q \vee r))$" $\equiv$ "$\neg p \vee (\neg q \wedge \neg r)$"

Proof.

$$\begin{aligned} \neg(p \wedge (q \vee r)) &\equiv \neg p \vee \neg(q \vee r) && \text{by DeMorgan's Law} \\ &\equiv \neg p \vee (\neg q \wedge \neg r) && \text{by DeMorgan's Law} \end{aligned}$$

$\square$

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Practice Proof 2

Example

Prove that "$\neg(\neg \boldsymbol{p} \vee \neg \boldsymbol{q})$" $\equiv$ "$\boldsymbol{p} \wedge \boldsymbol{q}$"

Answer

Absorption Law

If a statement is or with itself and something else, we can ignore the something else.

$$\begin{aligned} p \vee (p \wedge q) &\equiv p \\ p \wedge (p \vee q) &\equiv p \end{aligned}$$

Reasoning/Proof

p	q	$p \wedge q$	$p \vee (p \wedge q)$
True	True	True	True
True	False	False	True
False	True	False	False
False	False	False	False

Because the last column is identical to the first column, the two statements are equivalent.

Absorption Law - Example

Example

Prove that "$\boldsymbol{p} \wedge (\boldsymbol{p} \vee (\boldsymbol{p} \wedge \boldsymbol{q}))$" $\equiv$ "$\boldsymbol{p}$"

Proof.

$$\boldsymbol{p} \wedge (\boldsymbol{p} \vee (\boldsymbol{p} \wedge \boldsymbol{q})) \equiv \boldsymbol{p} \qquad \text{by Absorption Law}$$

$\square$

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Idempotent Law

If a statement or/and with itself is just the statement.

$$\boldsymbol{p} \vee \boldsymbol{p} \equiv \boldsymbol{p}$$
$$\boldsymbol{p} \wedge \boldsymbol{p} \equiv \boldsymbol{p}$$

Reasoning/Proof

$\boldsymbol{p}$	$\boldsymbol{p} \wedge \boldsymbol{p}$
True	**True**
False	**False**

Because the two columns are identical, the two statements are equivalent.

Idempotent Law - Example

Example

Prove that "$(p \land q) \lor (p \land q)$" $\equiv$ "$p \land q$"

Proof.

$$(p \land q) \lor (p \land q) \equiv p \land q \qquad \text{by Idempotent Law}$$

$\square$

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Identity Law

The identity law is like multiplying by 1 or adding 0.

Applying the identity means that the statement does not change.

The identity for **True** and **False** is not 1 or 0, but $\mathbb{T}$ and $\mathbb{F}$.

$$\boldsymbol{p} \vee \mathbb{F} \equiv \boldsymbol{p}$$
$$\boldsymbol{p} \wedge \mathbb{T} \equiv \boldsymbol{p}$$

Reasoning/Proof

$\boldsymbol{p}$	$\mathbb{T}$	$\boldsymbol{p} \wedge \mathbb{T}$
True	**True**	**True**
False	**True**	**False**

Because the first and last columns are identical, the two statements are equivalent.

Identity Law - Example

Example

Prove that "$(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \mathbb{T}$" $\equiv$ "$\boldsymbol{p} \wedge \boldsymbol{q}$"

Proof.

$$(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \mathbb{T} \equiv \boldsymbol{p} \wedge \boldsymbol{q} \qquad \text{by Identity Law}$$

$\square$

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Inverse Law

The Inverse Laws are simple examples of Tautology or Contradiction.

$\boldsymbol{p} \vee \neg \boldsymbol{p} \;\equiv\; \mathbb{T}$
$\boldsymbol{p} \wedge \neg \boldsymbol{p} \;\equiv\; \mathbb{F}$

Reasoning/Proof

$\boldsymbol{p}$	$\mathbb{T}$	$\boldsymbol{p} \wedge \mathbb{T}$
True	**True**	**True**
False	**True**	**False**

Because the first and last columns are identical, the two statements are equivalent.

Inverse Law - Example

Example

Prove that "$\boldsymbol{p} \vee \mathbb{F}$" $\equiv$ "$\boldsymbol{p} \vee (\neg \boldsymbol{q} \wedge \neg\neg \boldsymbol{q})$"

Proof.

$$\boldsymbol{p} \vee \mathbb{F} \equiv \boldsymbol{p} \vee (\neg \boldsymbol{q} \wedge \neg\neg \boldsymbol{q}) \qquad \text{by Inverse Law}$$

$\square$

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Domination Law

Domination occurs when a statement is removed by Tautologies or Contradictions.

$\boldsymbol{p} \vee \mathbb{T} \equiv \mathbb{T}$
$\boldsymbol{p} \wedge \mathbb{F} \equiv \mathbb{F}$

Reasoning/Proof

$\boldsymbol{p}$	$\mathbb{T}$	$\boldsymbol{p} \vee \mathbb{T}$
True	**True**	**True**
False	**True**	**True**

Because the last two columns are identical, the two statements are equivalent.

Domination Law - Example

Example

Prove that "$\mathbb{T} \vee \mathbb{F}$" $\equiv$ "$\mathbb{T}$"

Proof.

$$\mathbb{T} \vee \mathbb{F} \equiv \mathbb{T} \qquad \text{by Domination Law}$$

$\square$

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Practice Proof 3

Example

Prove that "$\neg\mathbb{T}$" $\equiv$ "$\mathbb{F}$"

Answer

Implication Identity

Implication Identity is used to convert Implications into standard and/or/negation symbols.

Since implication is **True** in 3 of 4 possible outcomes, it is similar to the "or" logic.

For this reason the Identity maps implication into "or".

$\boldsymbol{p} \rightarrow \boldsymbol{q} \;\equiv\; \neg \boldsymbol{p} \vee \boldsymbol{q}$

IMPORTANT!!!

For this reason it can be shown that implication is neither commutative or associative.

Implication Identity - Reasoning

Reasoning/Proof

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
True	True	False	True	False
True	False	False	True	True
False	True	True	False	False
False	False	True	True	True

Because the last two columns are identical for every row, it is the case that the two statements are identical.

Implication Identity - Example

Example

Prove that "$p \rightarrow (q \rightarrow r)$" $\equiv$ "$\neg p \vee (\neg q \vee r)$"

Proof.

$$
\begin{aligned}
p \rightarrow (q \rightarrow r) &\equiv \neg p \vee (q \rightarrow r) && \text{by Implication Identity} \\
&\equiv \neg p \vee (\neg q \vee r) && \text{by Implication Identity}
\end{aligned}
$$

$\square$

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Practice Proof 4

Example

Prove that "$\neg(\boldsymbol{p} \rightarrow \boldsymbol{q})$" $\equiv$ "$\boldsymbol{p} \wedge \neg \boldsymbol{q}$"

Answer

Disproof

Statements might not be equivalent.

How do we show that statements are not equivalent?

To be equivalent every assignment of simple truth-values needs to have identical outcomes.

To disprove equivalence find a single truth-value assignment of the simple statement(s) that have different resulting truth-values for the non-equivalent statements.

Example

Prove that "$\boldsymbol{p} \wedge \neg(\boldsymbol{q} \wedge \boldsymbol{r})$" $\not\equiv$ "$\boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{r}$"

One possible answer

Practice Proof 1 Answer

Example

Prove that "$p \vee (\neg\neg q \vee r)$" $\equiv$ "$(q \vee p) \vee r$"

Proof.

$$
\begin{aligned}
p \vee (\neg\neg q \vee r) &\equiv p \vee (q \vee r) && \text{by Double Negation Law} \\
&\equiv (p \vee q) \vee r && \text{by Associative Law} \\
&\equiv (q \vee p) \vee r && \text{by Commutative Law}
\end{aligned}
$$

□

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Practice Proof 2 Answer

Example

Prove that "$\neg(\neg \boldsymbol{p} \vee \neg \boldsymbol{q})$" $\equiv$ "$\boldsymbol{p} \wedge \boldsymbol{q}$"

Proof.

$$\begin{aligned} \neg(\neg \boldsymbol{p} \vee \neg \boldsymbol{q}) &\equiv \neg\neg \boldsymbol{p} \wedge \neg\neg \boldsymbol{q} && \text{by DeMorgan's Law} \\ &\equiv \boldsymbol{p} \wedge \neg\neg \boldsymbol{q} && \text{by Double Negation Law} \\ &\equiv \boldsymbol{p} \wedge \boldsymbol{q} && \text{by Double Negation Law} \end{aligned}$$

$\square$

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Practice Proof 3 Answer

Example

Prove that "$\neg\mathbb{T}$" $\equiv$ "$\mathbb{F}$"

Proof.

$$
\begin{aligned}
\neg\mathbb{T} &\equiv \neg(\boldsymbol{p} \vee \neg\boldsymbol{p}) && \text{by Inverse Law} \\
&\equiv \neg\boldsymbol{p} \wedge \neg\neg\boldsymbol{p} && \text{by Demorgan's Law} \\
&\equiv \mathbb{F} && \text{by Inverse Law}
\end{aligned}
$$

$\square$

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Practice Proof 4 Answer

Example

Prove that "$\neg(p \to q)$" $\equiv$ "$p \wedge \neg q$"

Proof.

$$\begin{aligned}
\neg(p \to q) &\equiv \neg(\neg p \vee q) && \text{by Implication Identity}\\
&\equiv \neg\neg p \wedge \neg q && \text{by Demorgan's Law}\\
&\equiv p \wedge \neg q && \text{by Double Negation Law}
\end{aligned}$$

$\square$

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Disproof Answer

Example

Prove that "$p \wedge \neg(q \wedge r)$" $\not\equiv$ "$p \wedge \neg q \vee \neg r$"

Proof.

Let p be **False**, q be **True**, and r be **False**.
The left hand side is **False**, but the right hand side is **True**.
The statements cannot be equivalent, because they can have different truth values. □

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