Nonlinear dynamics and chaos

1 Introduction

1.1 Dynamical systems

Dynamical system: any system where time plays a role.
Non-autonomous equations: time-dependent.)

> **Phase space P**
>
> > **The phase space is completely filled with trajectories**, since each point can serve as an initial condition. (Strogatz)
>
> **Space of evolutionary variable E**: For us, it will always be time.
> **Evolution rule** F: Defines transition from one state to the next. In this course, we only consider deterministic F, meaning you have existence and uniqueness.

1.2 Types of evolutionary rules

- **Discrete dynamical systems (DDS)**: iterated maps
- **Continuous dynamical systems (CDS)**: 1st order systems of ODEs, with initial value problem. With this, you define the **flow map**, which has **important group properties**! ⚠

2 Fundamentals: existence, uniqueness, regularity of solutions of continuous dynamical systems

2.1 Peano's theorem: Existence

This is a fundamental theorem which guarantees the existence of solutions to certain initial value problems. ⚠It guarantees existence but not uniqueness! ⚠

2.2 Picard's theorem: Existence and Uniqueness

Two assumptions need to be fulfilled, the second one being **Lipschitz continuity**, which you can intuitively think of as placing an bound on all the slopes of all the tangents you could create.

2.3 Geometric consequences of uniqueness

For non-autonomous systems, intersection in phase space is possible. If you extend the phase space with variable t, then you see that you actually don't have an intersection.

2.4 Local Vs. global existence

2.4.1 Theorem: continuation of solutions

If a local solution cannot be continued up to time T, then one must have that: $|x(t)| \to \infty$ as $t \to \infty$

2.4.2 Dependence on initial conditions

Theorem: If $f \in C_x^r \quad (r \geq 1)$ and $f \in C_t^0$ Then, the solution $x(t; t_0, x_0)$ is also: $C_{x_0}^0$
It turns out that the inverse of the flow map is also continuously differentiable.

3 Stability of fixed points

3.1 Basic definitions

Consider: $\dot{x} = f(x,t), \quad x \in R^n, \quad f \in C^1$
Assume: $x = 0$ is a fixed point, i.e., $f(0,t) = 0, \quad \forall t \in R$ **Question**: How does the dynamical system behave near its equilibrium state?

3.1.1 Stability, due Lyapunov

$x = 0$ is stable if $\forall t_0, \forall \epsilon > 0$ (small enough), $\exists \delta = \delta(t_0, \epsilon)$, such that for $\forall x_0 \in R^n$ with $|x_0| \leq \delta$, we have:

$$|x(t; t_0, x_0)| \leq \epsilon, \quad \forall t \geq t_0$$

3.1.2 Asymptotic stability

$x = 0$ is asymptotically stable if:

- it is stable
- $\forall t_0, \quad \exists \delta_0(t_0)$ such that $\forall x_0 : \quad |x_0| \leq \delta_0$:

$$\lim_{t \to \infty} x(t; t_0, x_0) = 0$$

3.1.3 Domain of attraction (fixed point assumed to be 0)

Set of all x_0's for which: $x(t; t_0, x_0) \to 0$, as $t \to \infty$
Example 2: Damped oscillator
Check out page 32 of PDF with notes. To understand how he derives the rate of energy change, first check out how he defines x_1 and x_2 and then take the derivative of the energy E with respect to time:

$$\frac{dE}{dt} = \frac{1}{2} \cdot 2x_2 \cdot \dot{x_2} + \sin x_1 \cdot x_2 = x_2(\dot{x_2} + \sin x_1) = x_2(-cx_2 - \sin x_1 + \sin x_1) = -cx_2^2 \tag{1}$$

3.1.4 Invariant set

S ⊂ P is an invariant set for the flow map $F^t : p \to p$ if $F^t(S) = S, \forall t \in R$ (meaning, if I take an initial condition inside the set, I stay in the set).

3.1.5 Instability

$x = 0$ is unstable if it is not stable.

4 Stability based on linearization (for autonomous systems)

We are in a multivariate setting! x is a vector, and so is f and fixed point p.

$$\dot{x} = f(x), \quad x \in R^n, \quad f \in C^1 \tag{2}$$

Question: How can we systematically derive/determine the stability of the fixed point of the preceding system?
We assume that $f(p) = 0 \Rightarrow y = x - p$ ODE (1) in transformed coordinates:

$$\dot{y} = f(p+y) = f(p) + Df(p)y + O(|y|^2)$$
$$\dot{y} = Df(p)y + O(|y|^2)$$

Define the linearization of (1) at the fixed point p:

$$\dot{y} = Ay, \quad y \in R^n, \quad A := Df(p) \in R^{n \times n}$$

$$Df(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ . & & & \\ . & & & \\ . & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

To study:

- stability of y=0 in $\dot{y} = Ay$
- relevance of this for the nonlinear system defined in the beginning of this section

4.1 Review of linear dynamical systems

$$\dot{y} = A(t)y, \quad y \in R^n, \quad A \in R^{n \times n}, \quad A \in C_t^0$$

- Global existence and uniqueness guaranteed
- Superposition principle: linear combination of solutions is also a solution
- ∃ a set of n linearly independent solutions: $[\phi_1(t), \phi_2(t), ..., \phi_n(t)]$
- General soluton: $y(t) = \sum_{i=1}^n c_i \phi_i$
  $\Rightarrow y(t) = [\phi_1(t), \phi_2(t), ..., \phi_n(t)] \cdot [c_1, ...c_n] = \Psi(t)c$
  Where Ψ is the **fundamental matrix solution**
  Hence: $= \dot{\Psi}(t) = A\Psi(t)$
- IVP:
  $$y(t_0) = y_0 \Rightarrow \Psi(t_0)c = y_0 \Rightarrow y(t) = \Psi(t) \cdot [\Psi(t_0)]^{-1} y_0$$
  Where $\Psi(t) \cdot [\Psi(t_0)]^{-1} = \phi(t) = F_{t_0}^t \; \phi$ is the normalized fundamental matrix, which is the flow map.
  $$\phi(t_0) = I$$
- Practical construction of solutions
  - Autonomous case: $\dot{x} = Ax$ only!
  - Explicit solution: aqui andámos a brincar com ϕs e com McLaurin series da função exponencial.
  - Solution from eigenfunctions

4.2 Stability of fixed points in autonomous linear systems

Stability of y=0 in the linear system $\dot{y} = Ay$

1. Assume $Re(\lambda_j) < 0, \quad \forall j$
   Then $y = 0$ is asymptotically stable.
2. Assume $Re(\lambda_j) \leq 0, \quad \forall j$ and $\forall \lambda_k : Re\lambda_k = 0$, we have $a_k = g_k$
   Then $y = 0$ is stable.
3. Assume that $\exists k : Re(\lambda_k) > 0$
   Then $y = 0$ is unstable.

4.3 Stability of fixed points in non-linear systems

Consider two dynamical (autonomous) systems:

1. $\dot{x} = f(x), x \in R^n, f \in C^1 \Rightarrow F^t : x_0 \to x(t; x_0)$
2. $\dot{x} = g(x), x \in R^n, g \in C^1 \Rightarrow G^t : x_0 \to x(t; x_0)$

The dynamical systems (1) and (2) are C^k equivalent ($k \in \mathbb{N}$) on an open set $U \in \mathbb{R}^n$, if $\exists$ a C^k-diffeomorphism $h : U \to U$ that maps orbits of (1) into orbits of (2), preserving orientation but not becessarily the exact parameterization of the orbit by time. Specifically: $\forall x \in U, \forall t \in \mathbb{R} : \exists t_2 \in \mathbb{R}$ such that:

$$h(F^{t_1}(x)) = G^{t_2}(h(x))$$

For $k = 0$, C^k-equivalence is also called topological equivalence, meaning a continuous invertible map that takes orbits of one system to orbits of the othre one. Such an $h : U \to U$ is called homeomorphism (and the two sets are called homeomorphic sets).

> "Here topologically equivalent means that there is a homeomorphism (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved. **Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.**" Excerpt From: Steven H. Strogatz. "Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering". Apple Books.

$x = x_0$ is a hyperbolic fixed point of (1) if the eigenvalues of its linearization (2) satisfy $Re(\lambda_i) \neq 0, \quad \forall i = 1, \ldots, n$

Implicit function theorem
$\exists!$ nearby solution to $F(x, \epsilon) = 0$ of the form $x_\epsilon = x_0 + O(\epsilon)$, provided that $D_x F(x_0, 0)$ is non-singular and $F(x, \epsilon) \in C^0$ (x_ϵ is as smooth in ϵ as $F(x, \epsilon)$)

Claim: the linearized stability type of a hyperbolic fixed point is preserved under small perturbations to the linear system ($\dot{x} = f(x)$). Again this is more than just persisting hyperbolicity. Stability type is also preserved.
In contrast, for non-hyperbolic fixed points, the smallest perturbation may change their stability type.

Hartman-Grobman: if the fixed point x_0 of the nonlinear system (1) is hyperbolic, then (1) is topologically equivalent to its linearization (2) in a neighborhood of x_0. **Consequence:** for hyperbolic fixed points, linearization predicts the correct stability type AND orbit geometry near x_0.
Consequence 2: if you have eigenvalues with zero real part (they lie on the imaginary axis), then the fixed point is not hyperbolic and you cannot apply Hartman-Grobman.

4.3.1 A sufficient and necessary condition for $Re(\lambda_i) < 0, \forall i$

It's the Routh-Hurwitz stability criterion! Check out the introduction of the corresponding Wikipedia article if in doubt. The procedure is to define a series of determinants.

$$Re(\lambda_i) < 0 \iff D_i > 0, \quad 1 \leq i \leq n$$

4.4 Lyapunov's direct (2nd) method for stability

How to establish stability without reliance on linearization?

$$\dot{x} = f(x), \quad f \in C^1, \quad x \in \mathbb{R}^n, \quad \dot{x_0} = f(x_0) = 0$$

Assume: $\exists V : U \to \mathbb{R}, \quad V \in C^1(U), \quad U \subset \mathbb{R}^n$open, $\quad x_0 \in U$, such that:

- $V(x_0) = 0$
- $V(x) > 0, \quad x \in U - x_0$
- $\dot{V}(x) = \langle DV(x), f(x) \rangle \leq 0, \quad x \in U$

1. Then $x = x_0$ is **(Lyapunov) stable** and V is a **Lyapunov function.**
2. **Asymptotic stability** if: $\dot{V} < 0$
3. **Unstable** if:
   - $\dot{V} > 0$
   - Or $V(x)$ is indefinite and $\dot{V}$ is semidefinite (psd or nsd).

5 Bifurcation of fixed points

5.1 Local nonlinear dynamics near fixed points

Define the following invariant subspaces for the linearization:

- **Stable subspace:** $E^S = \underset{j}{\text{span}}\{Re(e_j), Im(e_j) : Re(\lambda_j) < 0\}$
- **Unstable subspace:** $E^U = \underset{j}{\text{span}}\{Re(e_j), Im(e_j) : Re(\lambda_j) > 0\}$
- **Center subspace:** $E^s = \underset{j}{\text{span}}\{Re(e_j), Im(e_j) : Re(\lambda_j) = 0\}$

Center Manifold Theorem

1. $\exists!$ stable manifold $W^S(p)$, such that:
   - $W^S(p)$ is a C^r manifold (surface), tangent to E^S at p, $dim(W^S(p)) = dim(E^S)$)
   - $W^S(p)$ is invariant for (*); for $x^S(p)$:
     $$|F^t(x)| \leq K_S \cdot exp(\max_{Re(\lambda_j)<0}(Re(\lambda_j) + \epsilon_s)t)$$
     For: $t \geq 0, 0 < \epsilon_S << 1, |x - p|$ small enough
2. $\exists!$ unstable manifold $W^U(p)$, such that:
   - $W^U(p)$ is a C^r manifold (surface), tangent to E^U at p, $dim(W^U(p)) = dim(E^U)$)
   - $W^U(p)$ is invariant for (*); for $x \in W^U(p)$:
     $$|F^t(x)| \leq K_U \cdot exp(\min_{Re(\lambda_j)>0}(Re(\lambda_j) - \epsilon_U)t)$$
     For: $t \geq 0, 0 < \epsilon_U << 1, |x - p|$ small enough
3. $\exists$ (not single!!) center manifold $W^C(p)$, such that:
   - $W^C(p)$ is a C^{r-1} manifold (surface), tangent to E^C at p, $dim(W^C(p)) = dim(E^C)$)
   - $W^C(p)$ is invariant for (*).

Overall dynamics **crucially depends on the center manifold**, especially when $E^U = \emptyset$ (i.e., stability is determined by $W^C(p)$).

5.2 The center manifold

How do you compute the center manifold in general?

How can we compute $h(x)$ for $W^C(x_0)$ in general?

(1) Linearize in fixed-point: $\dot{z} = Az + \mathcal{O}(z^2)$

(2) Block-diagonalize with $z = T\xi$ to transform into center and non-center directions:

- $T = [\underbrace{a_1, \ldots, a_c}_{\text{Basis in } E^C}, \underbrace{b_1, \ldots, b_d}_{\text{Basis in } E^U \oplus E^S}]$
- $\dot{\xi} = T^{-1}AT\xi + T^{-1}f(T\xi)$
- With $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$ obtain:

$$\begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{cases}$$

(3) Taylor expand in $x = 0$ as ansatz:
$h(x) = \underbrace{h(0)}_{=0} + \underbrace{Dh(0)}_{=0} x + D^2h(0)x^2 + \mathcal{O}(|x^3|)$

(4) Use invariance to obtain $\dot{y} = h'(x)\dot{x}$ $(\star)$

(5) Plug $\dot{y} = h(x)$ into ODE to obtain $(\star\star)$

(6) Equate $(\star)$ and $(\star\star)$ and compare coefficients

[h!]

6 Tricks of the nonlinear dynamic trade

6.1 Commonly used tricks

- Roots of quadratic polynomial:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- While doing a near identity transform: $\frac{1}{1+z} = 1 - z +)(z^2)$
- **Taylor series**:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
- Taking full derivative of a functional:
$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{i=1}^{n} \frac{\partial L}{\partial x_i}\frac{dx_i}{dt}$$
- **Some trig identities:**
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
Power reducing identities:
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta\cos^2\theta = \frac{1-\cos(4\theta)}{8}$$
$$\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$$
$$\cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}$$
$$\sin^3\theta\cos^3\theta = \frac{3\sin(2\theta) - \sin(6\theta)}{32}$$
$$\sin^4\theta = \frac{3 - 4\cos(2\theta) + \cos(4\theta)}{8}$$
$$\cos^4\theta = \frac{3 + 4\cos(2\theta) + \cos(4\theta)}{8}$$
$$\sin^4\theta\cos^4\theta = \frac{3 - 4\cos(4\theta) + \cos(8\theta)}{128}$$
Product-to-sum identities:
$$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$$
$$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$$
$$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$$
$$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$$
- Linear algebra: $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  1. **Trace**: $\mathrm{tr}(A) = \sum_{i=1}^{3} a_{ii} = a + d$
  2. **Inverse:** $B^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

7 Useful definitions from lecture

- **Algebraic multiplicity of an eigenvalue** = number of times lambda appears as a root of the characteristic polynomial
- **Geometric multiplicity** = dimension of the eigenspace for the eigenvalue lambda. The eigenspace for lambda is given by:
$$N(A - \lambda I)$$
$$\text{dim(original matrix)} = \text{rank} + \text{nullity}$$
That's how he gets that the geometric multiplicity is 1! Also note that:
$$\text{geometric multiplicity} \le \text{algebraic multiplicity}$$

7.1 ODEs and PDEs

7.1.1 Separable differentiable equations

You separate the y and dy from the x and dx. Then you can integrate both sides. **It is usually the first thing you try!**

$$\frac{dy}{dx} = \frac{-x}{ye^{x^2}} \iff y \cdot dy = -xe^{x^2}dx$$
$$\int_{-\infty}^{+\infty} y \cdot dy = \int_{-\infty}^{+\infty} -xe^{x^2}dx$$
$$\iff \frac{y^2}{2} + C_1 = \frac{1}{2}e^{-x^2} + C_2$$

This last formulation is called **the implicit form** . And then you solve for y.

7.1.2 Exact equations

Consider:
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
If:
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Then you have an **exact differential equation** and there is a Ψ such that:
$$\frac{d\Psi(x,y)}{dx} = 0$$
and
$$\Psi(x,y) = C, \quad \frac{\partial \Psi}{\partial x} = M, \quad \frac{\partial \Psi}{\partial y} = N$$

Example from Khan Academy to know how to solve this kind of problem.

7.2 Second order differential equations

7.2.1 Linear

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

7.2.2 Homogeneous 2nd order differential equations

$$Ay'' + By' + Cy = 0$$

- **Superposition applies:** If $g(x)$ and $h(x)$ are both solutions, then $a(x) + b(x)$ is also a solution.
- **Characteristic equation**: $Ar^2 + Br + C = d(x)$
- **Most general solution:** $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$
- **Solution for complex roots**: $r = \lambda \pm \mu i$
$$y(x) = C_1e^{(\lambda+\mu i)x} + C_2e^{(\lambda-\mu i)x}$$
$$y(x) = e^{\lambda x}(C_1e^{\mu x i} + C_2e^{-\mu x i})$$
Given that: $e^{ix} = cosx + isinx$
We get:
$$y = e^{\lambda x}(C_3cos(\mu x) + C_4sin(\mu x))$$
- **Repeated roots:** similar to what you had before but now you need to add an x in front!
$$y(x) = C_1 \cdot x \cdot e^{r_1x} + C_2e^{r_2x}$$

7.2.3 Non-homogeneous ODE

$$Ay'' + By' + Cy = g(x)$$

You define:

- **Homogeneous solution**
- **Particular solution**. You can use the **method of undetermined coefficients**.

And you find a
If $g(x)$ is of the form:

$$ke^{ax} \quad \rightarrow \quad Ce^{ax} \tag{3}$$
$$kx^n, n = 0,1,2\ldots \quad \rightarrow \quad \sum_{i=0}^{n} K_ix^i \tag{4}$$
$$k\cos(ax) \text{ or } k\sin(ax) \quad \rightarrow \quad K\cos(ax) + M\sin(ax) \tag{5}$$
$$ke^{ax}\cos(bx) \text{ or } ke^{ax}\sin(bx) \quad \rightarrow \quad e^{ax}(K\cos(bx) + M\sin(bx)) \tag{6}$$