Implementing and Almost Ideal Demand System of mobility expenditure in Stan

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Expenditure Systems

are used in microeconomics to ...

- investigate welfare effects of price or income changes, or of taxes or subsidies
- predict across households or over time how demand changes in response to price/income changes

Expenditure Systems

need to account for...

- complementarity and substitution between goods
- cross-equation error correlation $\rightarrow$ Seemingly unrelated regression (Zellner, 1962)
- censoring
  - Tobit model
  - Double Hurdle Model
  - Infrequency of Purchase Model (Blundell & Meghir, 1987)

Expenditure System Estimation

- commonly done using frequentist methods
- Bayesian approach by Tiffin and Arnoult (2010)
  - Gibbs Sampling (Casella and George, 1992)

Almost Ideal Demand System (Deaton and Muellbauer, 1980)

expenditure share of good i

$$s_{ig} = \alpha_i + \sum_{j=1}^{M} \gamma_{ij} \ln(p_{jg}) + \beta_i \ln\left(\frac{m_g}{P_g^*}\right) \tag{1}$$

where

- s_{ig} expenditure share of transport mode i of household group g
- p_{jg} is the price of transport mode j of household group g
- m_g is total household group expenditure for transport by group g
- $\ln(P_g^*)$ is the translog price index, which can be approximated by Stones price index $\ln(P_g^*) = \sum_k s_{ig} \ln(p_{mj})$ where s_{ig} are replaced by $\bar{s}_i$

Almost Ideal Demand System (cont.)

A system of demand functions derived from the maximisation of a utility function under a linear budget constraint, automatically satisfies some general restrictions.

- Adding-up $\sum_{i=1}^{M} \alpha_i = 1 \sum_{i=1}^{M} \gamma_{ij} = 0 \forall j \; \sum_{i=1}^{M} \beta_i = 0$ (all budget is spent)
- Homogeneity of degree zero $\sum_j \gamma_{ij} = 0$ (a proportional rise in prices & expenditure has no effect on demand)
- Symmetry $\gamma_{ij} = \gamma_{ji}$ (substitution matrix is symmetric)
- Negativity (own price elasticities are negative)

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- ~~Negativity (own price elasticities are negative)~~

$\rightarrow$ restrictions need to be imposed $\rightarrow$ reduces the number of parameters to be estimated

Model specification

Adding-up implies that one equation is dropped

Rewrite model in (1) as

$$s = XB + \varepsilon \tag{2}$$

where
s is a $G(M-1)$ vector
X is a (M-1) block diagonal matrix
B is $(M-1)k$

Parameter restrictions

- Homogeneity and symmetry are imposed on the set of M-1 equations using q parameter restrictions

$$\mathbf{R}B^{rest} = 0 \tag{3}$$

where

- $\mathbf{R}$ is an orthonormal matrix defining the restrictions
- B^{rest} is the restricted version of B which can be expressed as $B^{rest} = \mathbf{R}_{\perp}B^{free}$

Parameter restrictions (cont.)

How to make $\mathbf{R}_\perp$?

If M=4 equations, B looks like this

$$\begin{array}{cccccc} \alpha_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \beta_1 \\ \alpha_2 & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \beta_2 \\ \alpha_3 & \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \beta_3 \end{array}$$

- make zero matrix **Z** with 18 columns and q=6 rows
- in a given row where **Z** is set to 1 the equivalent parameters will sum to zero

Using simulated data, in R:

- compute $\mathbf{R}_{\perp}$

```
rperp = Null(Z)
q = ncol(rperp)
```

- generate $vec(B^{free})$

```
bfree.true = matrix(rnorm(k.free),k.free,1)
```

- compute restricted B^{rest}

```
bvec.true = rperp%*%bfree.true
```

- compute dependent variable

```
Ystar.t <- X%*%bvec.true+Evec.t
```

```
> bvec.true
[1,] -0.34945072
[2,]  1.03534811
[3,] -0.25708502
[4,] -0.46490645
[5,] -0.31335664
[6,] -0.01590289
[7,]  0.41809309
[8,] -0.25708502
[9,]  0.29166730
[10,]  0.09911658
[11,] -0.13369886
[12,] -0.69465199
[13,] -0.78064472
[14,] -0.46490645
[15,]  0.09911658
[16,] -0.62194387
[17,]  0.98773374
[18,] -0.87921086
```

For estimation in Stan need to compute

```
Xr <- X%*%rperp
```

where Xr is $G(M-1) \times (m-1)k$

In stan:

```
data {
int<lower=1> K; // total number of free parameters
int<lower=1> Kr; // number of restricted parameters per equation
int<lower=1> D; // number of equations
int<lower=0> N; // number of observations
int<lower=0,upper=1> y_ind[N,D]; // indicator if censored at zero
vector[D] y[N];
matrix[N*D,K] Xr; // restricted matrix
int<lower=0> index[N,D];
matrix[Kr*D,K] rperp;
}
```

```
parameters {
vector[K] beta;
cholesky_factor_corr[D] L_Omega;
vector<lower=0>[D] L_sigma;
vector<upper=0>[N_neg] z_neg;
}
```

```
model {
...
mu_vec = Xr*beta;
// reshape vector to array
for (j in 1:D){
for (i in 1:N){
mu[i, j] = mu_vec[index[i,j]];
}}
L_Sigma = diag_pre_multiply(L_sigma, L_Omega);

// priors
to_vector(beta) ~ normal(0, 1);
L_Omega ~ lkj_corr_cholesky(3);
L_sigma ~ normal(0, 1);

z ~ multi_normal_cholesky(mu, L_Sigma);
}
```

Simulated results: free parameters

```
                          beta.post.free
beta[1]   -1.8149578      -1.8563195    L_sigma[1] L_sigma[2] L_sigma[3]
beta[2]   -0.6274798      -0.7048428    1.548713   2.416292   2.786170
beta[3]   -2.0716180      -2.1260447
beta[4]   -0.4943079      -0.5482699    1.5        2.5        3.0
beta[5]   -0.1943162      -0.1096420
beta[6]    0.3787821       0.2448027
beta[7]   -0.5280202      -0.4398354
beta[8]    0.5477599       0.6080177
beta[9]    0.2072907       0.1712506
beta[10]   0.3162866       0.2353760
beta[11]  -0.7562456      -0.7908830
beta[12]   1.3238252       1.3674655
```

Simulated results: restricted parameters

```
> beta.post.mat
[,1]              [,2]         [,3]          [,4]         [,5]         [,6]
[1,] -0.2229613  1.3582597645 -0.1053980 0.0008544288 -1.2537162 -0.103
[2,] -1.8563195 -0.1053979817 -1.1339773 0.3569497885  0.8824255  0.244
[3,] -0.4398354  0.0008544288  0.3569498 0.3342273951 -0.6920316  1.367
> bmat.true
[,1]            [,2]          [,3]          [,4]         [,5]         [,6]
[1,] -0.1651147  1.33403919 -0.08059262 -0.06276309 -1.1906835 -0.13165
[2,] -1.8149578 -0.08059262 -1.08795582  0.37920244  0.7893460  0.3787
[3,] -0.5280202 -0.06276309  0.37920244  0.37804645 -0.6944858  1.3238
```

Aim of this application

To investigate substitute and complementary relationships between different transport modes so as to better understand the impact of recent changes in costs on transport demand.

Data on mobility expenditure

Mobility data usually do not contain information on prices $\rightarrow$ combine two data sets

1. 'Income and expenditure survey 2018' (EVS, 2018) contains data on quarterly household expenditure in **Euros** for car, train, and bus and taxi usage, which is scaled to daily expenditure
2. 'Mobility in Germany 2017' (MiD, 2017) contains data on how many **kilometers** a household travelled by car, train, bus and taxi on a given day

Household groups

Expenditure and travel data are aggregated over groups of households, which have attributes relevant to transport poverty:

- equivalised net household income (Quintiles 1-5)
- economic activity (0 persons, 1+ persons)
- region (agglomeration, urbanised, rural)
- car ownership (0 cars, 1+ cars)

This results in 60 possible combinations, of which data are available for all but one combination.

Variables

- $G = 59$ sample size
- M=3 transport modes car, train, bus/taxi (dropped)
- "price" = unit value = $\left(\frac{\text{median daily Euros}}{\text{median daily kilometres}}\right)$ for each travel mode $i = 1, \ldots, M$ for each household group $g = 1, \ldots, G$
- censoring 46% car and 0% train

Estimating mobility expenditure system

4 chains, 1000 iterations

```
Divergences:
0 of 2000 iterations ended with a divergence.
Tree depth:
0 of 2000 iterations saturated the maximum tree depth of 20.
Energy:
E-BFMI indicated no pathological behavior.
```

Computing elasticities

- Use restrictions to recover parameters the dropped equation.

$$\text{uncompensated price elasticity } \epsilon_{ij} = -\delta_{ij} + \frac{\gamma_{ij}}{\bar{s}_i} - \beta_i \frac{\bar{s}_j}{\bar{s}_i} \tag{4}$$

$$\text{expenditure elasticity } \epsilon_i = 1 + \frac{\beta_i}{\bar{s}_i} \tag{5}$$

Results

Uncompensated price elasticities

	car	train	bus/taxi
car	-1.42	-0.23	-0.30
train	0.31	-0.79	0.22
bus/taxi	0.45	0.19	-0.68

Expenditure elasticities

car	1.95
train	0.26
bus/taxi	0.04

Conclusions

- accounting for censoring is easy
- imposing restrictions done via a pre-computation of restricted data matrix
- free parameters have no straight forward interpretation $\rightarrow$ difficult to include prior knowledge
- the system is highly constraint $\rightarrow$ not responsive to changes in the priors
- next steps: imposing negativity

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