**Theorem.** Let  $\Omega$  be a sample space and  $A \subseteq \Omega$  be an event. If P(A) = 0 or P(B) = 1 then A is independent of every other event.

*Proof.* Let  $B \neq A$  be an event of  $\Omega$ . In the case P(A) = 0, we have  $P(A)P(B) = 0 \cdot P(B) = 0$ . We also know that  $AB \subseteq A$ , so  $P(AB) \leq P(A)$ . That is,  $P(AB) \leq 0$ . Seeing that  $AB \subseteq A \subseteq \Omega$ , then AB is also an event of  $\Omega$ . By the first axiom of probability,  $P(AB) \geq 0$ . So, P(AB) = 0. Overall,

$$P(A)P(B) = P(AB).$$

That is to say, A and B are independent events. Given  $B \neq A$  was chosen arbitrarily, then A is independent of every other event of  $\Omega$ .

In the case P(A)=1, we have  $P(B)=1\cdot P(B)=P(A)P(B)$ . Additionally,  $AB\subseteq B$ , so  $P(AB)\le P(B)=P(A)P(B)$ . Suppose that P(AB)< P(B). Then  $P(AB)=P(A)+P(B)-P(A\cup B)< P(B)$ . That is  $1-P(A\cup B)<0$ , or  $1< P(A\cup B)$ , a contradiction to the first axiom of probability. So, P(AB)=P(B)=P(A)P(B), meaning A and B are independent events