Control-Lyapunov Function

A control-Lyapunov function (CLF) is an extension of the idea of Lyapunov Function $V(x)$ to systems with control inputs. A control-Lyapunov function is used to test whether a system is asymptotically stabilizable, that is whether for any state x there exists a control $u(x,t)$ such that the system can be brought to the zero state asymptotically by applying the control u

Consider an autonomous dynamical system with inputs:

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^n$ is the control vector. Suppose out goal is to drive the system to an equilibrium $x^* \in \mathbb{R}^n$ from every initial state in some domain $D \subset \mathbb{R}^n$. Suppose the equilibrium is at $x^* = 0$

Def: A control-Lyapunov function is a function $V : D \to \mathbb{R}$ that is continuously differentiable, positive-definite, and such that for all $x \in \mathbb{R}^n (x \neq 0)$, there exist $u \in \mathbb{R}^m$ such that

$$\dot{V}(x, u) := \langle \nabla V(x), f(x, u) \rangle < 0,$$

where $\langle u, v \rangle$ denotes the Inner Product of $u, v \in \mathbb{R}^n$

The last condition is for each x we can find a control u that will reduce the "energy" V. Intuitively, if in each state we can always find a way to reduce the energy, we should eventually be able to bring the energy asymptotically to zero, that is to bring the system to a stop.

Some results apply only to Control-Affine Systems i.e. control systems in the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}^n$ for all $i = 1, \ldots, m$

Constructing the Stabilizing Input

It is often difficult to find a control-Lyapunov function for a given system, but if one is found, then the feedback stabilization problem simplifies considerably.

For the control-affine system [Sontag's formula] (http://www.sontaglab.org/FTPDIR/sontag_mathematical_control_theory_springer98.pdf | 5.5.6) gives the feedback law $k : \mathbb{R}^n \to \mathbb{R}^n$ directly in terms of the derivatives of the CLF. In the special case of a single input system $m = 1$ the formula can be written as:

$$k(x) = \begin{cases} -\frac{L_f V(x) + \sqrt{[L_f V(x)]^2 + [L_g V(x)]^4}}{L_g V(x)} & \text{if } L_g V(x) \neq 0 \\ 0 & \text{if } L_g V(x) = 0 \end{cases}$$

where $L_f V(x) := \langle \nabla V(x), f(x) \rangle$ and $L_g V(x) := \langle \nabla V(x), g(x) \rangle$ are the Lie Derivatives of V along $f$and g respectively.

For a general nonlinear system, the input u can be found by solving a static nonlinear programming problem:

$$u^* = \arg\min \nabla V(x) \cdot f(x, u)$$

for each state x

Source: https://en.wikipedia.org/wiki/Control-Lyapunov_function