Lyapunov's Stability Theory

See Types of Stability and Lyapunov Function

Lyapunov's Theorem

- If the linearized system ($\underline{\dot{x}} = \underline{Ax}$) is strictly stable (i.e., if all eigenvalues of A are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).
- If the linearized system is unstable (i.e., if at least one eigenvalue of A is strictly in the right-half complex plane), then the equilibrium point is unstable (for the actual nonlinear system) .
- If the linearized system is marginally stable (i.e., all eigenvalues of A are in the left-half complex plane, but at least one of them is on the imaginary axis), then one cannot conclude anything from the linear approximation for the EP of the actual nonlinear system

Given $\dot{x} = f(x)$ and $V(x)$ defined on Ω

1. $V(x) = 0$ when $x = x^*$
2. $V(x) > 0$, for all x in Ω except $x = x^*$
3. $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ for all x in Ω
   $\implies x^*$ is stable
   If also
4. $\dot{V}(x) < 0$ for all x in Ω (except x^*)
   $\implies x^*$ is locally asymptotically stable (L.A.S)
   If also
5. $\Omega = \mathbb{R}^n$
6. $V(x) \to \infty$ as $\sqrt{x_1^2 + \cdots + x_n^2} \to \infty$
   $\implies x^*$ is globally asymptotically stable (G.A.S)

Source

This can be solved for using two methods:

1. Lyapunov's Indirect (Linearization) Method
2. Lyapunov's Direct Method

Useful References

https://underactuated.mit.edu/lyapunov.html