Linear Quadratic Regulation

Using pole placement to calculate the full-state feedback gain K is always a trade-off between speed of regulation response and control input effort.

A direct way of trading off various performance requirements is to introduce a cost function and minimize it, i.e. find an optimal design. This is where LQR can be used

A typical optimal control problem is formulated as follows:
Find a control input $u(t)$ that minimises

$$J = \int_0^{t_f} h(\underline{x}, \underline{u})\, dt \qquad \text{a positive cost function}$$

subject to

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}) \qquad \text{a constraint (the dynamic plant)}$$

Linear Quadratic Regulation (LQR)

Given the **linear*** plant state equation

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}; \quad \underline{x}(0) = \underline{x}_0$$

find a **regulating** control input, $\underline{u}(t)$ that minimises

$$J = \int_0^{\infty} (\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u})\, dt$$

Where

- J is the scalar cost function in quadratic form
- $Q_{n \cdot n}$ is the state weighting matrix
- $Rm \cdot m$ is the input weighting matrix

Notes:

- Can add a third term to LQR minimise function to be

- $$J = \int_0^\infty (\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u} + \underline{x}^T N \underline{u})\, dt$$
- We are defining a state regulation problem because J_{min} is finite only when both $\underline{x}$ and $\underline{u}$ end up at zero.
- When Q is 'large in value' and R is 'small in value' then more weighting is being placed on state regulation compared to control effort.
- If Q & R are diagonal then $J = \int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + \cdots + q_n x_n^2 + r_1 u_1^2 + r_2 u_2^2 + \cdots + r_m u_m^2)\, dt$

Linear Algebra Definitions - Matrices

Def: Matrix A is *positive-semidefinite* (written $A \geq 0$) means A is square, symmetric, and all eigenvalues* of A are ≥ 0.
*Recall that all eigenvalues of a symmetric matrix are purely real

Def: Matrix A is *positive-definite* (written $A > 0$) means A is square, symmetric, and all eigenvalues of A is > 0

Solution to the LQR problem (minimising J) by Kalman:

If:

1. Q is positive-semidefinite
2. R is positive-definite
3. (A, B)Controllable
   then

- $u = -K\underline{x}$ is the optimal regulation controller
  Where $K = R^{-1}B^T P$ the optimal control gain and $P_{n \cdot n}$ is the unique positive-definite solution to the algebraic Riccati equation (ARE)
- $$A^T P + PA + Q = PBR^{-1}B^T P$$
- the closed-loop system $(A - BK)$ is always stable
- the minimum cost is $J_{min} = \underline{x}_0^T P \underline{x}_0$

Steps to finding a LQR controllers:

1. Choose Q & R to trade off the refulation performance of particular state with the effort of particular control inputs. This is based on the control objectives. Typically, it is sufficient to choose Q and R as diagonal matrices.
   - E.g. Bryson's rule - Initially, choose diagonal elements of Q and R by

$$q_i = \frac{1}{\text{max. acceptable value of } x_i^2} \quad \text{and} \quad r_i = \frac{1}{\text{max. acceptable value of } u_i^2}$$

2. Check that the 3 conditions are met for finding an optimal controller.
3. Solve the Riccati equation for P (positive-definite), then calculate K using $K = R^{-1}B^TP$. This process is tedious by hand but computers are efficient at it, See. MATLAB Functions for `lqr`