Lyapunov Function

Lyapunov functions are scalar functions that may be used to prove the Stability of an Equilibrium of an ODE.

For some classes of ODEs, the existence of a Lyapunov function is a necessary and sufficient condition for stability. Whereas there is no general technique for constructing Lyapunov functions for ODEs, in many specific cases the construction of Lyapunov functions is known.

Definition

A Lyapunov function for a autonomous dynamical system

$$\begin{cases} g : \mathbb{R}^n \to \mathbb{R}^n \\ \dot{y} = g(y) \end{cases}$$

with an equilibrium point at $y = 0$ is a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ that is

- Continuous
- Has continuous first derivatives
- Is strictly positive for $y \neq 0$
- $\dot{V} = \nabla V \cdot g$ is non-positive or written as $\nabla V \cdot g$ is locally negative definite

Basic Theorem

Let $x^* = 0$ be an EP of the autonomous system

$$\dot{x} = f(x)$$

and use the notation $\dot{V}(x)$ to denote the time derivative of the Lyapunov-candidate-function V:

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \nabla V \cdot \dot{x} = \nabla V \cdot f(x)$$

Notes

- See Lyapunov's Stability Theory for some applications

- Is used for control purposed through Control-Lyapunov Function