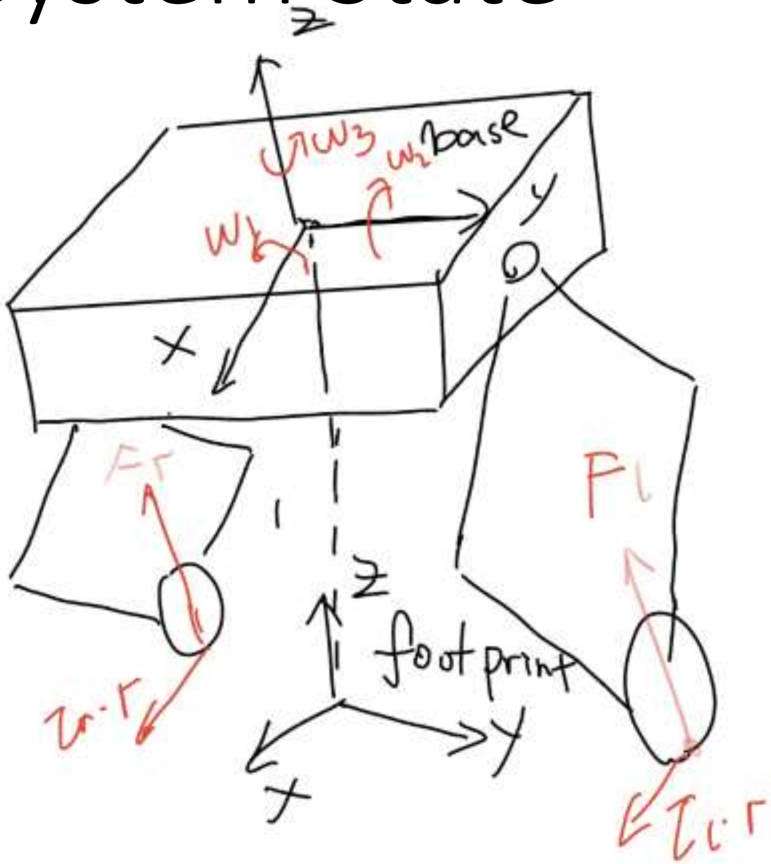
System State



$$state: \begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_x^{base} \\ \omega_y^{base} \\ \omega_z^{base} \\ \dot{x} \\ z \\ \dot{z} \end{bmatrix}$$

$$input: \begin{bmatrix} F_r \\ F_l \\ \tau_r \\ \tau_l \end{bmatrix}$$

- Use Z-Y-X (YPR) Euler Angles
- Footprint frame is the projection of base frame
- All states can be calculated under base frame: state estimation is not needed.

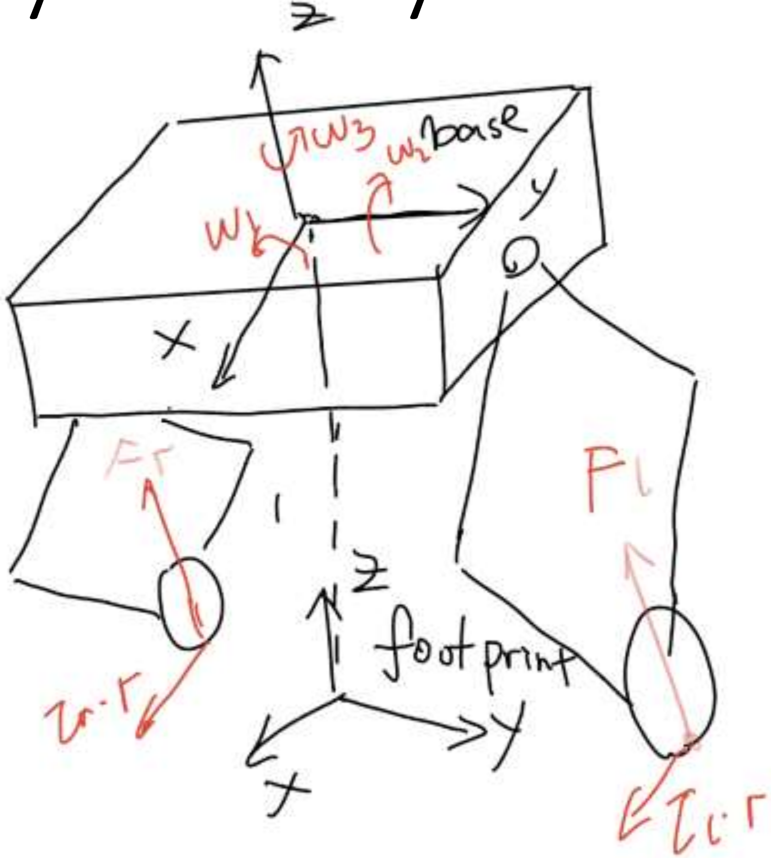
MPC Formulation

$$\begin{cases} \min\limits_{\mathbf{u}(.)} \; \phi(\mathbf{x}(t_I)) + \int_{t_0}^{t_I} l(\mathbf{x}(t), \mathbf{u}(t), t)\, dt & \\ \text{s.t.} \;\; \mathbf{x}(t_0) = \mathbf{x}_0 & \text{initial state} \\ \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) & \text{system flow map} \end{cases}$$

Cost formulation:

$$J = \frac{1}{2} x(t_f)^T E x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

System Dynamics



Step 1: Calculate Euler rate from base angular velocity

$$\begin{bmatrix} 1 & 0 & -sin(\theta) \\ 0 & \cos(\psi) & cos(\theta)sin(\phi) \\ 0 & -\sin(\psi) & cos(\theta)cos(\phi) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \omega^{base}$$

$$\begin{bmatrix} 1 & 0 & -sin(\theta) \\ 0 & \cos(\psi) & cos(\theta)sin(\phi) \\ 0 & -\sin(\psi) & cos(\theta)cos(\phi) \end{bmatrix}^{-1} \omega^{base} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

System Dynamics

Step 2: Calculate angular acceleration of base
The rigid body dynamics is governed by:

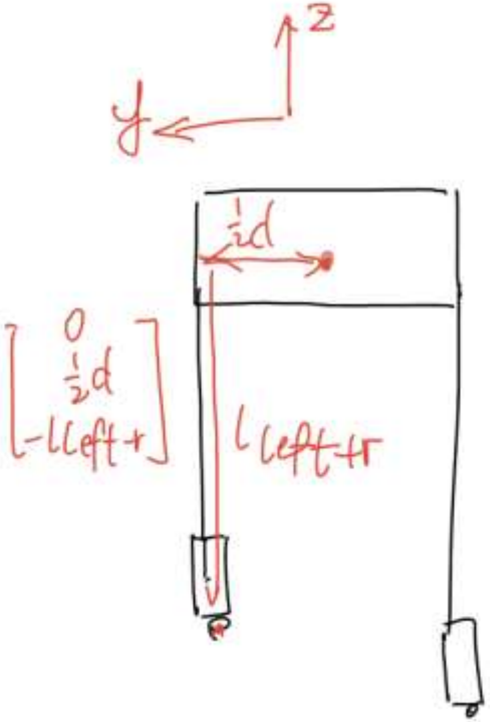
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega}^{base}) = \sum_{i=1}^{n} \mathbf{r}_i^{base} \times \mathbf{f}_i^{base}$$

$$\mathbf{I}\dot{\boldsymbol{\omega}}^{base} + \boldsymbol{\omega}^{base} \times (\mathbf{I}\boldsymbol{\omega}^{base}) = \sum_{i=1}^{n} \mathbf{r}_i^{base} \times \mathbf{f}_i^{base}$$

$$\dot{\boldsymbol{\omega}}^{base} = \mathbf{I}^{-1}(\sum_{i=1}^{n} \mathbf{r}_i^{base} \times \mathbf{f}_i^{base} - \boxed{\boldsymbol{\omega}^{base} \times (\mathbf{I}\boldsymbol{\omega}^{base}))}$$

System Dynamics

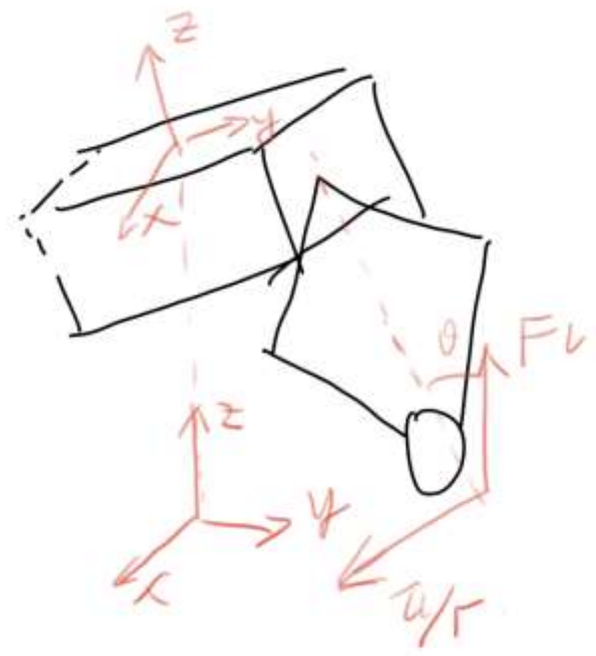
Take left leg as example.
The point of contact is easy to find under base frame:



$$\mathbf{r}_{left}^{base} = \begin{bmatrix} 0 \\ d/2 \\ -l_{left} - r_{wheel} \end{bmatrix}$$

System Dynamics

The contact force is easy to calculate under footprint frame: the support force always points upward and the friction always points forward.

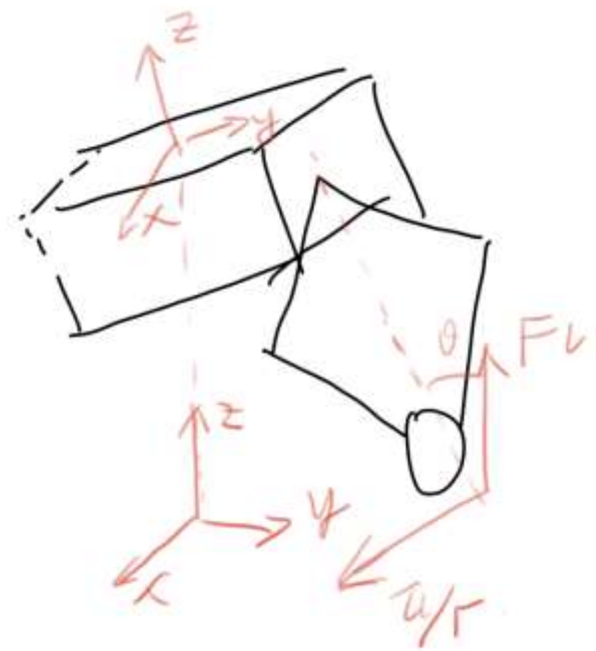


$$\mathbf{f}_{left}^{footprint} = \begin{bmatrix} \tau_{left}/r \\ 0 \\ F_{left} \end{bmatrix}$$

Transform to base frame:

$$\mathbf{f}_{left}^{base} = R_{footprint}^{base}\mathbf{f}_{left}^{footprint}$$
$$\mathbf{f}_{left}^{base} = (R_y(\theta)R_x(\phi))^T\mathbf{f}_{left}^{footprint}$$

System Dynamics



Linear acceleration terms:

$$\ddot{x} = (\tau_{left}/r + \tau_{right}/r)/\sum_{i=1}^{n} m_i$$

$$\ddot{z} = \sum F_z^{footprint}/\sum_{i=1}^{n} m_i$$

System Observation

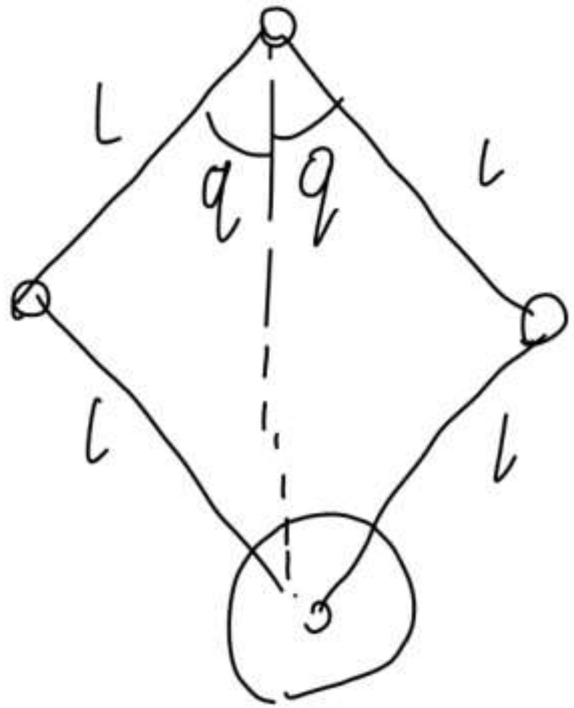
$$state: \begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_x^{base} \\ \omega_y^{base} \\ \omega_z^{base} \\ \dot{x} \\ z \\ \dot{z} \end{bmatrix}$$

These terms can be acquired from IMU readings.

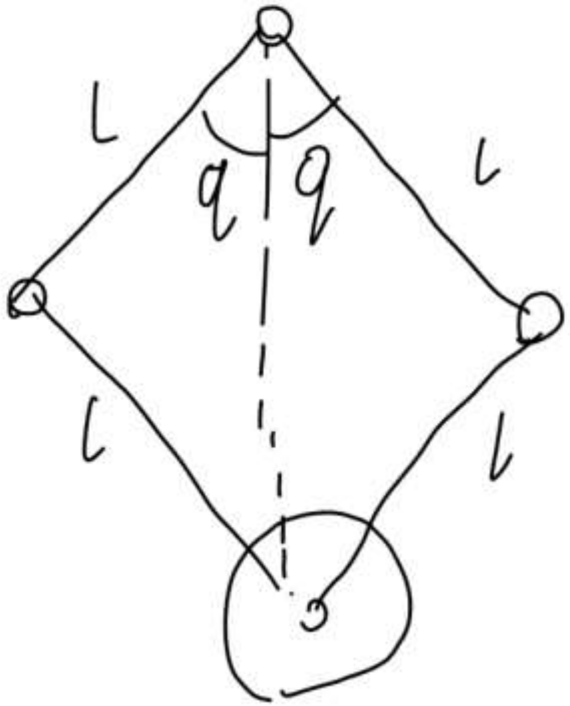
$$\dot{x} = (\omega_{left} + \omega_{right})r/2$$

$$z = 2l(cos(q_{left}) + cos(q_{right}))$$

$$\dot{z} = 2l(sin(q_{left})\dot{q}_{left} + sin(q_{left})\dot{q}_{right})$$



Joint-level torque control:



$\tau = J^T F$

* Use VMC
* TBD