# Appendix F: Mathematical Induction

## Introduction

In this appendix, you will study a form of mathematical proof called **mathematical induction.** To see the logical need for mathematical induction, take another look at the problem discussed in Section 8.2, Example 5.

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

$$S_4 = 1 + 3 + 5 + 7 = 4^2$$

$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first $n$ odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Although this particular formula is valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of $n$ is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of $n$ but then fails at some point. One of the most famous cases of this is the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, . . .$$

are prime. For $n = 0, 1, 2, 3,$ and 4, the conjecture is true.

$$F_0 = 3$$

$$F_1 = 5$$

$$F_2 = 17$$

$$F_3 = 257$$

$$F_4 = 65{,}537$$

The size of the next *Fermat number* $(F_5 = 4{,}294{,}967{,}297)$ is so great that it was difficult for Fermat to determine whether or not it was prime. However, another well-known mathematician, Leonhard Euler (1707–1783), later found a factorization

$$\begin{aligned} F_5 &= 4{,}294{,}967{,}297 \\ &= 641(6{,}700{,}417) \end{aligned}$$

which proved that $F_5$ is not prime and therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of $n$, you cannot simply decide that it is valid for *all* values of $n$ without going through a *legitimate proof*. Mathematical induction is one method of proof.

### What you should learn

- Use mathematical induction to prove statements involving a positive integer $n$.
- Find the sums of powers of integers.
- Find finite differences of sequences.

### Why you should learn it

Finite differences can be used to determine what type of model can be used to represent a sequence. For instance, in Exercise 59 on page F8, you will use finite differences to find a model that represents the number of sides of the $n$th Koch snowflake.

### The Principle of Mathematical Induction

Let $P_n$ be a statement involving the positive integer $n$. If

1. $P_1$ is true, and
2. the truth of $P_k$ implies the truth of $P_{k+1}$ for every positive integer $k$,

then $P_n$ must be true for all positive integers $n$.

To apply the Principle of Mathematical Induction, you need to be able to determine the statement $P_{k+1}$ for a given statement $P_k$. To determine $P_{k+1}$, substitute $k + 1$ for $k$ in the statement $P_k$.

### Study Tip

It is important to recognize that in order to prove a statement by induction, *both* parts of the Principle of Mathematical Induction are necessary.

## Example 1 A Preliminary Example

Find $P_{k+1}$ for each $P_k$.

**a.** $P_k : S_k = \dfrac{k^2(k+1)^2}{4}$

**b.** $P_k : S_k = 1 + 5 + 9 + \cdot \cdot \cdot + [4(k-1) - 3] + (4k - 3)$

**c.** $P_k : k + 3 < 5k^2$

**d.** $P_k : 3^k \geq 2k + 1$

### Solution

**a.** $$P_{k+1} : S_{k+1} = \frac{(k+1)^2(k+1+1)^2}{4} \qquad \text{Replace } k \text{ by } k+1.$$
$$= \frac{(k+1)^2(k+2)^2}{4} \qquad \text{Simplify.}$$

**b.** $$P_{k+1} : S_{k+1} = 1 + 5 + 9 + \cdot \cdot \cdot + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$$
$$= 1 + 5 + 9 + \cdot \cdot \cdot + (4k - 3) + (4k + 1)$$

**c.** $$P_{k+1} : (k+1) + 3 < 5(k+1)^2$$
$$k + 4 < 5(k^2 + 2k + 1)$$

**d.** $$P_{k+1} : 3^{k+1} \geq 2(k+1) + 1$$
$$3^{k+1} \geq 2k + 3$$

CHECKPOINT Now try Exercise 9.

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes represented by Figure F.1. If the line actually contains infinitely many dominoes, it is clear that you could not knock down the entire line by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of $P_k$ implies the truth of $P_{k+1}$ and if $P_1$ is true, the chain reaction proceeds as follows: $P_1$ implies $P_2$, $P_2$ implies $P_3$, $P_3$ implies $P_4$, and so on.

Figure F.1

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of $S_{k+1}$ as

$$S_{k+1} = S_k + a_{k+1}$$

where $a_{k+1}$ is the $(k + 1)$th term of the original sum.

## Example 2 Using Mathematical Induction

Use mathematical induction to prove the following formula.

$$S_n = 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2$$

### Solution

Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

**1.** When $n = 1$, the formula is valid because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer $k$. The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

**2.** Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all positive integer values of $n$.

CHECKPOINT Now try Exercise 11.

It occasionally happens that a statement involving natural numbers is *not* true for the first $k - 1$ positive integers but *is* true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify $P_k$ rather than $P_1$. This variation is called the *Extended Principle of Mathematical Induction*. To see the validity of this principle, note from Figure F.1 that all but the first $k - 1$ dominoes can be knocked down by knocking over the $k$th domino. This suggests that you can prove a statement $P_n$ to be true for $n \geq k$ by showing that $P_k$ is true and that $P_k$ implies $P_{k+1}$. In Exercises 29–34 in this appendix, you are asked to apply this extension of mathematical induction.

## Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

### Solution

**1.** When $n = 1$, the formula is valid because

$$S_1 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6}.$$

**2.** Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write the following.

$$\begin{aligned}
S_{k+1} &= S_k + a_{k+1} \\
&= (1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2) + (k+1)^2 && \text{Substitute for } S_k. \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By assumption} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Combine fractions.} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor.} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{Simplify.} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && S_k \text{ implies } S_{k+1}.
\end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

CHECKPOINT Now try Exercise 17.

When proving a formula by mathematical induction, the only statement that you *need* to verify is $P_1$. As a check, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying $P_2$ and $P_3$.

### Example 4 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all integers $n \geq 1$.

#### Solution

**1.** For $n = 1$ and $n = 2$, the formula is true because

$$1 < 2^1 \quad \text{and} \quad 2 < 2^2.$$

**2.** Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. Multiply each side of $k < 2^k$ by 2.

$$2(k) < 2(2^k) = 2^{k+1}$$

Because $k + 1 < k + k = 2k$ for all $k > 1$, it follows that

$$k + 1 < 2k < 2^{k+1}$$

or

$$k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all integers $n \geq 1$.

CHECKPOINT Now try Exercise 29.

## Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first $n$ positive integers are summarized below.

**Sums of Powers of Integers**

**1.** $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdot \cdot \cdot + n = \frac{n(n+1)}{2}$

**2.** $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2 = \frac{n(n+1)(2n+1)}{6}$

**3.** $\displaystyle\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdot \cdot \cdot + n^3 = \frac{n^2(n+1)^2}{4}$

**4.** $\displaystyle\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \cdot \cdot \cdot + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

**5.** $\displaystyle\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \cdot \cdot \cdot + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Each of these formulas for sums can be proven by mathematical induction. (See Exercises 17–20 in this appendix.)

## Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$: | 1 | | 2 | | 3 | | 4 | | 5 | | 6 |
| $a_n$: | 3 | | 5 | | 8 | | 12 | | 17 | | 23 |
| First differences: | | 2 | | 3 | | 4 | | 5 | | 6 | |
| Second differences: | | | 1 | | 1 | | 1 | | 1 | | |

For this sequence, the second differences are all the same. When this happens, and the second differences are nonzero, the sequence has a perfect *quadratic* model. If the first differences are all the same nonzero number, the sequence has a *linear* model—that is, it is arithmetic.

### Study Tip

For a linear model, the *first* differences are the same nonzero number. For a quadratic model, the *second* differences are the same nonzero number.

### Example 5 Finding a Quadratic Model

Find the quadratic model for the sequence 3, 5, 8, 12, 17, 23, . . . .

#### Solution

You know from the second differences shown above that the model is quadratic and has the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for $n$, you can obtain a system of three linear equations in three variables.

$a_1 = a(1)^2 + b(1) + c = 3$ Substitute 1 for $n$.

$a_2 = a(2)^2 + b(2) + c = 5$ Substitute 2 for $n$.

$a_3 = a(3)^2 + b(3) + c = 8$ Substitute 3 for $n$.

You now have a system of three equations in $a$, $b$, and $c$.

$$\begin{cases} a + b + c = 3 & \text{Equation 1} \\ 4a + 2b + c = 5 & \text{Equation 2} \\ 9a + 3b + c = 8 & \text{Equation 3} \end{cases}$$

Solving this system of equations using the techniques discussed in Chapter 7, you can find the solution to be $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 2$. So, the quadratic model is

$$a_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n + 2.$$

Check the values of $a_1$, $a_2$, and $a_3$ as follows.

#### Check

$a_1 = \frac{1}{2}(1)^2 + \frac{1}{2}(1) + 2 = 3$ Solution checks. ✓

$a_2 = \frac{1}{2}(2)^2 + \frac{1}{2}(2) + 2 = 5$ Solution checks. ✓

$a_3 = \frac{1}{2}(3)^2 + \frac{1}{2}(3) + 2 = 8$ Solution checks. ✓

CHECKPOINT Now try Exercise 55.

## F Exercises

For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1–4, fill in the blank.**

1. The first step in proving a formula by _______ is to show that the formula is true when $n = 1$.
2. The _______ differences of a sequence are found by subtracting consecutive terms.
3. A sequence is an _______ sequence when the first differences are all the same nonzero number.
4. When the _______ differences of a sequence are all the same nonzero number, then the sequence has a perfect quadratic model.

### Procedures and Problem Solving

**Finding $P_{k+1}$** **In Exercises 5–10, find $P_{k+1}$ for the given $P_k$.**

5. $P_k = \dfrac{5}{k(k+1)}$
6. $P_k = \dfrac{4}{(k+2)(k+3)}$
7. $P_k = \dfrac{2^k}{(k+1)!}$
8. $P_k = \dfrac{2^{k-1}}{k!}$

✓ 9. $P_k = 1 + 6 + 11 + \cdots + [5(k-1) - 4] + (5k - 4)$

10. $P_k = 7 + 13 + 19 + \cdots + [6(k-1) + 1] + (6k + 1)$

**Using Mathematical Induction** **In Exercises 11–24, use mathematical induction to prove the formula for all positive integers $n$.**

✓ 11. $2 + 4 + 6 + 8 + \cdots + 2n = n(n+1)$

12. $3 + 11 + 19 + 27 + \cdots + (8n - 5) = n(4n - 1)$
13. $3 + 8 + 13 + 18 + \cdots + (5n - 2) = \dfrac{n}{2}(5n + 1)$
14. $1 + 4 + 7 + 10 + \cdots + (3n - 2) = \dfrac{n}{2}(3n - 1)$
15. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$
16. $2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}) = 3^n - 1$

✓ 17. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

18. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
19. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
20. $\displaystyle\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$
21. $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$
22. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$
23. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
24. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

**Finding Sums of Powers of Integers** **In Exercises 25–28, find the sum using the formulas for the sums of powers of integers.**

25. $\displaystyle\sum_{n=1}^{50} n^3$
26. $\displaystyle\sum_{n=1}^{10} n^4$
27. $\displaystyle\sum_{n=1}^{12} (n^2 - n)$
28. $\displaystyle\sum_{n=1}^{40} (n^3 - n)$

**Proving an Inequality by Mathematical Induction** **In Exercises 29–34, prove the inequality for the indicated integer values of $n$.**

✓ 29. $n! > 2^n, \quad n \ge 4$

30. $\left(\dfrac{4}{3}\right)^n > n, \quad n \ge 7$
31. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \ge 2$
32. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, \quad n \ge 1$ and $0 < x < y$
33. $(1 + a)^n \ge na, \quad n \ge 1$ and $a > 1$
34. $3^n > n\,2^n, \quad n \ge 1$

**Using Mathematical Induction** **In Exercises 35–46, use mathematical induction to prove the property for all positive integers $n$.**

35. $(ab)^n = a^n b^n$
36. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

**37.** If $x_1 \neq 0,\ x_2 \neq 0, \ldots, x_n \neq 0$, then
$(x_1 x_2 x_3 \cdot \cdot \cdot x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdot \cdot \cdot x_n^{-1}$.

**38.** If $x_1 > 0,\ x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 \cdot \cdot \cdot x_n) = \ln x_1 + \ln x_2 + \cdot \cdot \cdot + \ln x_n$.

**39.** Generalized Distributive Law:
$x(y_1 + y_2 + \cdot \cdot \cdot + y_n) = xy_1 + xy_2 + \cdot \cdot \cdot + xy_n$

**40.** $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

**41.** A factor of $(n^3 + 3n^2 + 2n)$ is 3.

**42.** A factor of $(n^3 + 5n + 6)$ is 3.

**43.** A factor of $(n^3 - n + 3)$ is 3.

**44.** A factor of $(n^4 - n + 4)$ is 2.

**45.** A factor of $(2^{2n+1} + 1)$ is 3.

**46.** A factor of $(2^{4n-2} + 1)$ is 5.

Using Finite Differences to Classify a Sequence **In Exercises 47–54, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. Does the sequence have a linear model, a quadratic model, or neither?**

**47.** $a_1 = 0$
$a_n = a_{n-1} + 3$

**48.** $a_1 = 2$
$a_n = n - a_{n-1}$

**49.** $a_1 = 3$
$a_n = a_{n-1} - n$

**50.** $a_2 = -3$
$a_n = -2a_{n-1}$

**51.** $a_0 = 0$
$a_n = a_{n-1} + n$

**52.** $a_0 = 2$
$a_n = (a_{n-1})^2$

**53.** $a_1 = 2$
$a_n = a_{n-1} + 2$

**54.** $a_1 = 0$
$a_n = a_{n-1} + 2n$

Finding a Quadratic Model **In Exercises 55–58, find a quadratic model for the sequence with the indicated terms.**

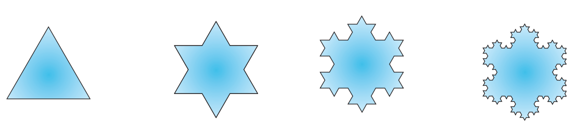
✓ **55.** 3, 3, 5, 9, 15, 23, . . .

**56.** 7, 6, 7, 10, 15, 22, . . .

**57.** $a_0 = -3,\ a_2 = 1,\ a_4 = 9$

**58.** $a_0 = 3,\ a_2 = 0,\ a_6 = 36$

**59.** *Why you should learn it* *(p. F1)* A *Koch snowflake* is created by starting with an equilateral triangle with sides one unit in length. Then, on each side of the triangle, a new equilateral triangle is created on the middle third of that side. This process is repeated continuously, as shown in the figure.

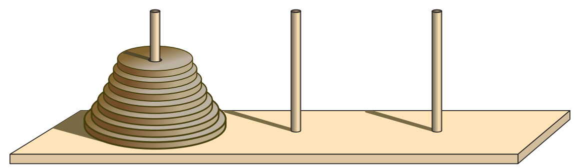


(a) Determine a formula for the number of sides of the $n$th Koch snowflake. Use mathematical induction to prove your answer.

(b) Determine a formula for the area of the $n$th Koch snowflake. Recall that the area $A$ of an equilateral triangle with side $s$ is $A = (\sqrt{3}/4)s^2$.

(c) Determine a formula for the perimeter of the $n$th Koch snowflake.

**60.** Using Mathematical Induction The *Tower of Hanoi* puzzle is a game in which three pegs are attached to a board and one of the pegs has $n$ disks sitting on it, as shown in the figure. Each disk on that peg must sit on a larger disk. The strategy of the game is to move the entire pile of disks, one at a time, to another peg. At no time may a disk sit on a smaller disk.



(a) Find the number of moves when there are three disks.

(b) Find the number of moves when there are four disks.

(c) Use your results from parts (a) and (b) to find a formula for the number of moves when there are $n$ disks.

(d) Use mathematical induction to prove the formula you found in part (c).

## Conclusions

True or False? **In Exercises 61–63, determine whether the statement is true or false. Justify your answer.**

**61.** If the statement $P_k$ is true and $P_k$ implies $P_{k+1}$, then $P_1$ is also true.

**62.** If a sequence is arithmetic, then the first differences of the sequence are all zero.

**63.** A sequence with $n$ terms has $n - 1$ second differences.

**64.** Think About It What conclusion can be drawn from the information given about each sequence $P_1, P_2, P_3, \ldots, P_n$?

(a) $P_3$ is true and $P_k$ implies $P_{k+1}$.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) $P_1, P_2$, and $P_3$ are all true, but the truth of $P_k$ does not imply that $P_{k+1}$ is true.

(d) $P_2$ is true and $P_{2k}$ implies $P_{2k+2}$.