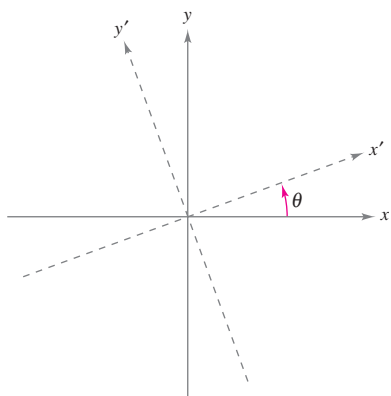


# D Rotation and the General Second-Degree Equation

- Rotate the coordinate axes to eliminate the  $xy$ -term in equations of conics.
- Use the discriminant to classify conics.

## Rotation of Axes



After rotation of the  $x$ - and  $y$ -axes counterclockwise through an angle  $\theta$ , the rotated axes are denoted as the  $x'$ -axis and  $y'$ -axis.

**Figure D.1**

Equations of conics with axes parallel to one of the coordinate axes can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \quad \text{Horizontal or vertical axes}$$

In this appendix, you will study the equations of conics whose axes are rotated so that they are *not* parallel to either the  $x$ -axis or the  $y$ -axis. The general equation for such conics contains an  $xy$ -term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \text{Equation in } xy\text{-plane}$$

To eliminate this  $xy$ -term, you can use a procedure called **rotation of axes**. The objective is to rotate the  $x$ - and  $y$ -axes until they are parallel to the axes of the conic. The rotated axes are denoted as the  $x'$ -axis and the  $y'$ -axis, as shown in Figure D.1. After the rotation, the equation of the conic in the new  $x'y'$ -plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \quad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no  $x'y'$ -term, you can obtain a standard form by completing the square.

The next theorem identifies how much to rotate the axes to eliminate the  $xy$ -term and also the equations for determining the new coefficients  $A'$ ,  $C'$ ,  $D'$ ,  $E'$ , and  $F'$ .

### THEOREM D.1 Rotation of Axes

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $B \neq 0$ , can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

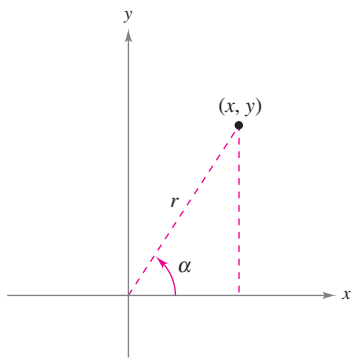
by rotating the coordinate axes through an angle  $\theta$ , where

$$\cot 2\theta = \frac{A - C}{B}.$$

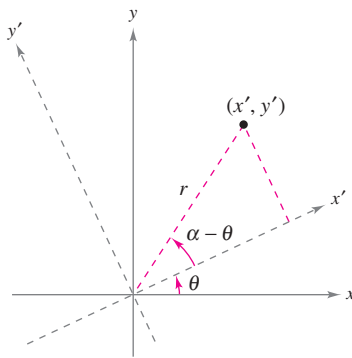
The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta.$$



Original:  $x = r \cos \alpha$   
 $y = r \sin \alpha$



Rotated:  $x' = r \cos(\alpha - \theta)$   
 $y' = r \sin(\alpha - \theta)$

Figure D.2

**Proof** To discover how the coordinates in the  $xy$ -system are related to the coordinates in the  $x'y'$ -system, choose a point  $(x, y)$  in the original system and attempt to find its coordinates  $(x', y')$  in the rotated system. In either system, the distance  $r$  between the point and the origin is the same, and so the equations for  $x, y, x',$  and  $y'$  are those given in Figure D.2. Using the formulas for the sine and cosine of the difference of two angles, you obtain

$$\begin{aligned} x' &= r \cos(\alpha - \theta) \\ &= r(\cos \alpha \cos \theta + \sin \alpha \sin \theta) \\ &= r \cos \alpha \cos \theta + r \sin \alpha \sin \theta \\ &= x \cos \theta + y \sin \theta \end{aligned}$$

and

$$\begin{aligned} y' &= r \sin(\alpha - \theta) \\ &= r(\sin \alpha \cos \theta - \cos \alpha \sin \theta) \\ &= r \sin \alpha \cos \theta - r \cos \alpha \sin \theta \\ &= y \cos \theta - x \sin \theta. \end{aligned}$$

Solving this system for  $x$  and  $y$  yields

$$x = x' \cos \theta - y' \sin \theta \quad \text{and} \quad y = x' \sin \theta + y' \cos \theta.$$

Finally, by substituting these values for  $x$  and  $y$  into the original equation and collecting terms, you obtain the following.

$$\begin{aligned} A' &= A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta \\ C' &= A \sin^2 \theta - B \cos \theta \sin \theta + C \cos^2 \theta \\ D' &= D \cos \theta + E \sin \theta \\ E' &= -D \sin \theta + E \cos \theta \\ F' &= F \end{aligned}$$

Now, in order to eliminate the  $x'y'$ -term, you must select  $\theta$  such that  $B' = 0$ , as follows.

$$\begin{aligned} B' &= 2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) \\ &= (C - A) \sin 2\theta + B \cos 2\theta \\ &= B(\sin 2\theta) \left( \frac{C - A}{B} + \cot 2\theta \right) \\ &= 0, \quad \sin 2\theta \neq 0 \end{aligned}$$

When  $B = 0$ , no rotation is necessary, because the  $xy$ -term is not present in the original equation. When  $B \neq 0$ , the only way to make  $B' = 0$  is to let

$$\cot 2\theta = \frac{A - C}{B}, \quad B \neq 0.$$

So, you have established the desired results.



**EXAMPLE 1** Rotation of Axes for a Hyperbola

Write the equation  $xy - 1 = 0$  in standard form.

**Solution** Because  $A = 0$ ,  $B = 1$ , and  $C = 0$ , you have (for  $0 < \theta < \pi/2$ )

$$\cot 2\theta = \frac{A - C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}.$$

The equation in the  $x'y'$ -system is obtained by making the following substitutions.

$$x = x' \cos \frac{\pi}{4} - y' \sin \frac{\pi}{4} = x' \left( \frac{\sqrt{2}}{2} \right) - y' \left( \frac{\sqrt{2}}{2} \right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x' \sin \frac{\pi}{4} + y' \cos \frac{\pi}{4} = x' \left( \frac{\sqrt{2}}{2} \right) + y' \left( \frac{\sqrt{2}}{2} \right) = \frac{x' + y'}{\sqrt{2}}$$

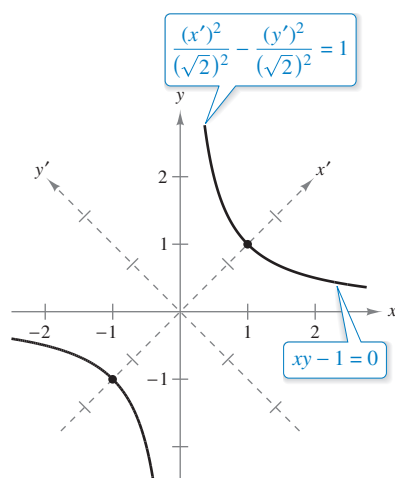
Substituting these expressions into the equation  $xy - 1 = 0$  produces

$$\left( \frac{x' - y'}{\sqrt{2}} \right) \left( \frac{x' + y'}{\sqrt{2}} \right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1. \quad \text{Write in standard form.}$$

This is the equation of a hyperbola centered at the origin with vertices at  $(\pm\sqrt{2}, 0)$  in the  $x'y'$ -system, as shown in Figure D.3.



Vertices:  
 $(\sqrt{2}, 0), (-\sqrt{2}, 0)$  in  $x'y'$ -system  
 $(1, 1), (-1, -1)$  in  $xy$ -system

**Figure D.3**

**EXAMPLE 2** Rotation of Axes for an Ellipse

Sketch the graph of  $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$ .

**Solution** Because  $A = 7$ ,  $B = -6\sqrt{3}$ , and  $C = 13$ , you have (for  $0 < \theta < \pi/2$ )

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}.$$

The equation in the  $x'y'$ -system is derived by making the following substitutions.

$$x = x' \cos \frac{\pi}{6} - y' \sin \frac{\pi}{6} = x' \left( \frac{\sqrt{3}}{2} \right) - y' \left( \frac{1}{2} \right) = \frac{\sqrt{3}x' - y'}{2}$$

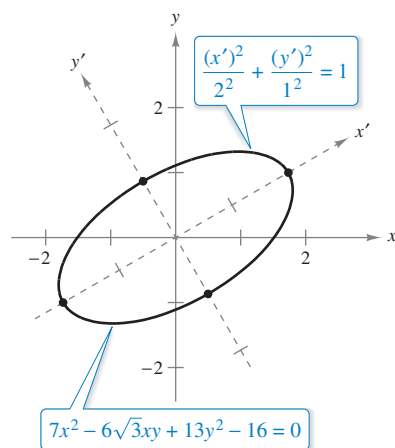
$$y = x' \sin \frac{\pi}{6} + y' \cos \frac{\pi}{6} = x' \left( \frac{1}{2} \right) + y' \left( \frac{\sqrt{3}}{2} \right) = \frac{x' + \sqrt{3}y'}{2}$$

Substituting these expressions into the original equation eventually simplifies (after considerable algebra) to

$$4(x')^2 + 16(y')^2 = 16$$

$$\frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} = 1. \quad \text{Write in standard form.}$$

This is the equation of an ellipse centered at the origin with vertices at  $(\pm 2, 0)$  and  $(0, \pm 1)$  in the  $x'y'$ -system, as shown in Figure D.4.



Vertices:  
 $(\pm 2, 0), (0, \pm 1)$  in  $x'y'$ -system  
 $(\pm\sqrt{3}, \pm 1), \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$  in  $xy$ -system

**Figure D.4**

In Examples 1 and 2, the values of  $\theta$  were the common angles  $45^\circ$  and  $30^\circ$ , respectively. Of course, many second-degree equations do not yield such common solutions to the equation

$$\cot 2\theta = \frac{A - C}{B}.$$

Example 3 illustrates such a case.

### EXAMPLE 3 Rotation of Axes for a Parabola

Sketch the graph of  $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$ .

**Solution** Because  $A = 1$ ,  $B = -4$ , and  $C = 4$ , you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

The trigonometric identity  $\cot 2\theta = (\cot^2 \theta - 1)/(2 \cot \theta)$  produces

$$\cot 2\theta = \frac{3}{4} = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

from which you obtain the equation

$$\begin{aligned} 6 \cot \theta &= 4 \cot^2 \theta - 4 \\ 0 &= 4 \cot^2 \theta - 6 \cot \theta - 4 \\ 0 &= (2 \cot \theta - 4)(2 \cot \theta + 1). \end{aligned}$$

Considering  $0 < \theta < \pi/2$ , it follows that  $2 \cot \theta = 4$ . So,

$$\cot \theta = 2 \Rightarrow \theta \approx 26.6^\circ.$$

From the triangle in Figure D.5, you obtain  $\sin \theta = 1/\sqrt{5}$  and  $\cos \theta = 2/\sqrt{5}$ . Consequently, you can write the following.

$$x = x' \cos \theta - y' \sin \theta = x' \left( \frac{2}{\sqrt{5}} \right) - y' \left( \frac{1}{\sqrt{5}} \right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta = x' \left( \frac{1}{\sqrt{5}} \right) + y' \left( \frac{2}{\sqrt{5}} \right) = \frac{x' + 2y'}{\sqrt{5}}$$

Substituting these expressions into the original equation produces

$$\left( \frac{2x' - y'}{\sqrt{5}} \right)^2 - 4 \left( \frac{2x' - y'}{\sqrt{5}} \right) \left( \frac{x' + 2y'}{\sqrt{5}} \right) + 4 \left( \frac{x' + 2y'}{\sqrt{5}} \right)^2 + 5\sqrt{5} \left( \frac{x' + 2y'}{\sqrt{5}} \right) + 1 = 0$$

which simplifies to

$$5(y')^2 + 5x' + 10y' + 1 = 0.$$

By completing the square, you obtain the standard form

$$\begin{aligned} 5(y' + 1)^2 &= -5x' + 4 \\ (y' + 1)^2 &= (-1) \left( x' - \frac{4}{5} \right). \end{aligned} \quad \text{Write in standard form.}$$

The graph of the equation is a parabola with its vertex at  $(\frac{4}{5}, -1)$  and its axis parallel to the  $x'$ -axis in the  $x'y'$ -system, as shown in Figure D.6.

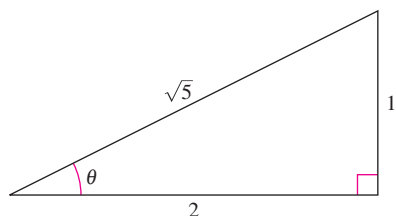
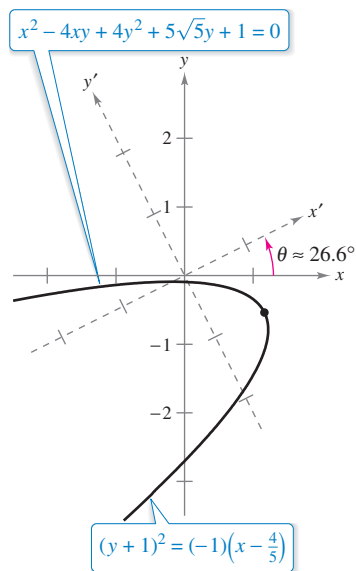


Figure D.5



Vertex:

$$\left( \frac{4}{5}, -1 \right) \text{ in } x'y'\text{-system}$$

$$\left( \frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}} \right) \text{ in } xy\text{-system}$$

Figure D.6

## Invariants Under Rotation

In Theorem D.1, note that the constant term is the same in both equations—that is,  $F' = F$ . Because of this,  $F$  is said to be **invariant under rotation**. Theorem D.2 lists some other rotation invariants. The proof of this theorem is left as an exercise (see Exercise 34).

### THEOREM D.2 Rotation Invariants

The rotation of coordinate axes through an angle  $\theta$  that transforms the equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1.  $F = F'$
2.  $A + C = A' + C'$
3.  $B^2 - 4AC = (B')^2 - 4A'C'$

You can use this theorem to classify the graph of a second-degree equation *with* an  $xy$ -term in much the same way you do for a second-degree equation *without* an  $xy$ -term. Note that because  $B' = 0$ , the invariant  $B^2 - 4AC$  reduces to

$$B^2 - 4AC = -4A'C'$$

Discriminant

which is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Because the sign of  $A'C'$  determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

the sign of  $B^2 - 4AC$  must determine the type of graph for the original equation. This result is stated in Theorem D.3.

### THEOREM D.3 Classification of Conics by the Discriminant

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is, except in degenerate cases, determined by its discriminant as follows.

1. *Ellipse or circle:*  $B^2 - 4AC < 0$
2. *Parabola:*  $B^2 - 4AC = 0$
3. *Hyperbola:*  $B^2 - 4AC > 0$

**EXAMPLE 4** Using the Discriminant

Classify the graph of each equation.

a.  $4xy - 9 = 0$

b.  $2x^2 - 3xy + 2y^2 - 2x = 0$

c.  $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

d.  $3x^2 + 8xy + 4y^2 - 7 = 0$

**Solution**

a. The graph is a hyperbola because

$$B^2 - 4AC = 16 - 0 > 0.$$

b. The graph is a circle or an ellipse because

$$B^2 - 4AC = 9 - 16 < 0.$$

c. The graph is a parabola because

$$B^2 - 4AC = 36 - 36 = 0.$$

d. The graph is a hyperbola because

$$B^2 - 4AC = 64 - 48 > 0.$$

**D Exercises**

**Rotation of Axes** In Exercises 1–12, rotate the axes to eliminate the  $xy$ -term in the equation. Write the resulting equation in standard form and sketch its graph showing both sets of axes.

- $xy + 1 = 0$
- $xy - 4 = 0$
- $x^2 - 10xy + y^2 + 1 = 0$
- $xy + x - 2y + 3 = 0$
- $xy - 2y - 4x = 0$
- $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
- $5x^2 - 2xy + 5y^2 - 12 = 0$
- $2x^2 - 3xy - 2y^2 + 10 = 0$
- $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
- $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
- $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
- $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$



**Graphing a Conic** In Exercises 13–18, use a graphing utility to graph the conic. Determine the angle  $\theta$  through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.

- $x^2 + xy + y^2 = 10$
- $x^2 - 4xy + 2y^2 = 6$
- $17x^2 + 32xy - 7y^2 = 75$
- $40x^2 + 36xy + 25y^2 = 52$
- $32x^2 + 50xy + 7y^2 = 52$
- $4x^2 - 12xy + 9y^2 + (4\sqrt{13} + 12)x - (6\sqrt{13} + 8)y = 91$

**Using the Discriminant** In Exercises 19–26, use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola.

- $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
- $x^2 - 4xy - 2y^2 - 6 = 0$
- $13x^2 - 8xy + 7y^2 - 45 = 0$
- $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
- $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
- $36x^2 - 60xy + 25y^2 + 9y = 0$
- $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
- $x^2 + xy + 4y^2 + x + y - 4 = 0$

**Degenerate Conic** In Exercises 27–32, sketch the graph (if possible) of the degenerate conic.

- $y^2 - 4x^2 = 0$
- $x^2 + y^2 - 2x + 6y + 10 = 0$
- $x^2 + 2xy + y^2 - 1 = 0$
- $x^2 - 10xy + y^2 = 0$
- $(x - 2y + 1)(x + 2y - 3) = 0$
- $(2x + y - 3)^2 = 0$

**33. Invariant Under Rotation** Show that the equation  $x^2 + y^2 = r^2$  is invariant under rotation of axes.

**34. Proof** Prove Theorem D.2.