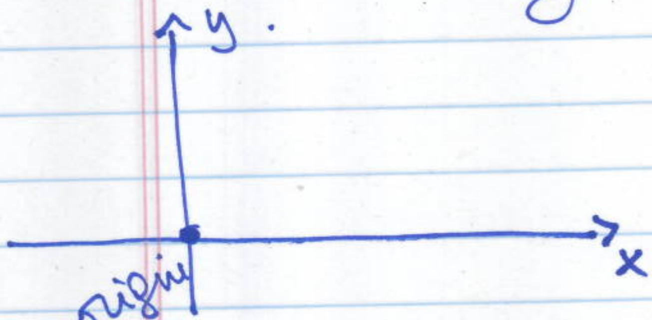
8.3 Polar form of complex numbers.

complex numbers.

$$z = a + bi$$

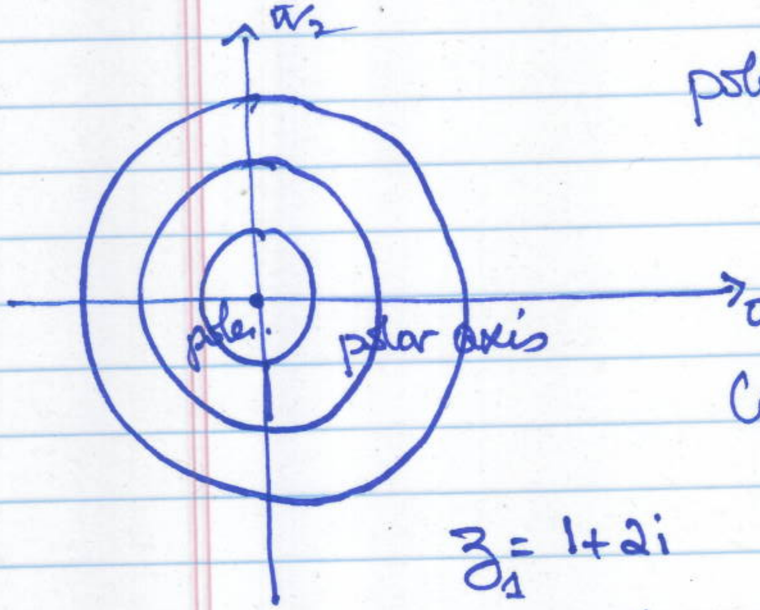
a, b are real numbers

$i = \sqrt{-1}$ or $i^2 = -1$



xy-plane $y = f(x)$

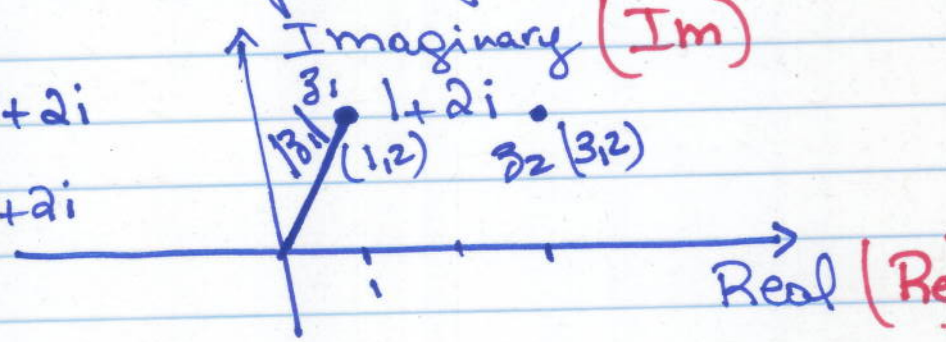
polar coordinate.
grid



Complex plane.

$z_1 = 1+2i$

$z_2 = 3+2i$



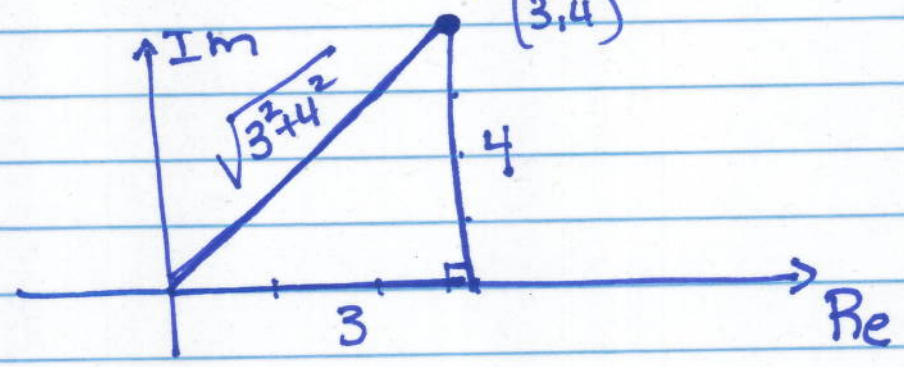
Find the modulus of $8-5i$ and $3+$

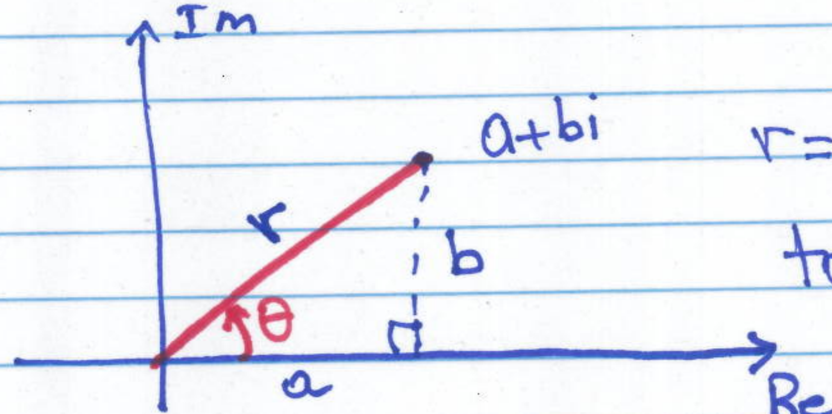
$z = a+bi$ modulus $|z| = \sqrt{a^2+b^2}$

$z = 8 - 5i \quad \therefore \quad |z| = \sqrt{8^2 + 5^2}$

$= \sqrt{64 + 25} = \sqrt{89}$

$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \boxed{5}$





$r = |a+bi|$

$\tan\theta = \dfrac{b}{a}$

$\cos\theta = \dfrac{a}{r}$

$\sin\theta = \dfrac{b}{r}$

$a = r\cos\theta$

$b = r\sin\theta$

$r = \sqrt{a^2+b^2}$

$z = a + bi$

$z = r\cos\theta + r\sin\theta\, i$

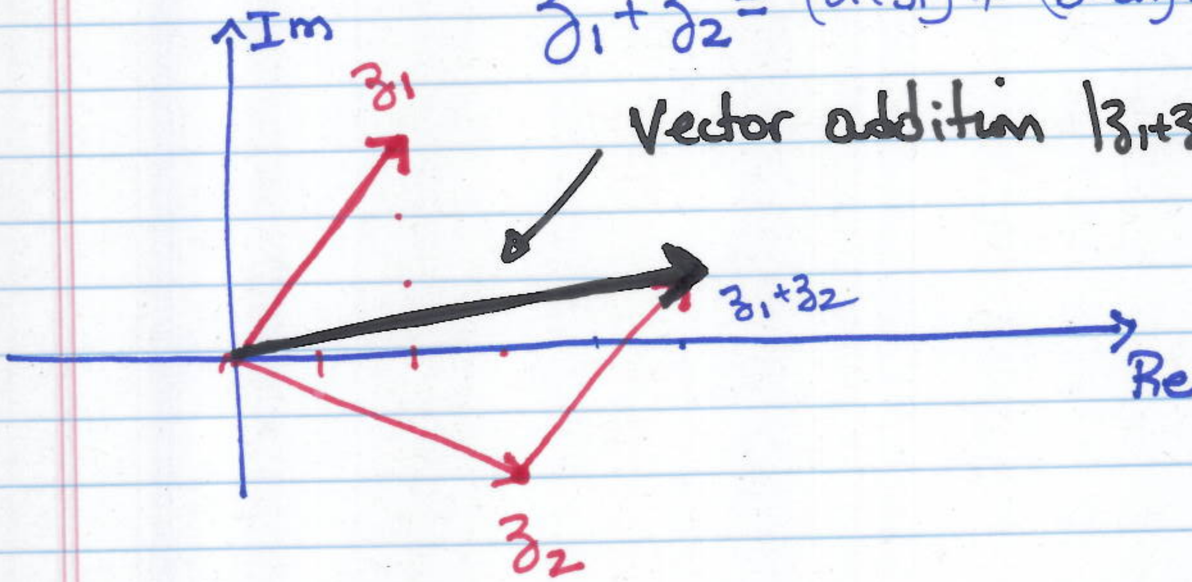
$z = r(\cos\theta + i\sin\theta)$. where $\boxed{\tan\theta = b/a \quad r = \sqrt{a^2+b^2}}$

Graph $z_1 = 2+3i \quad \therefore |z_1| = \sqrt{2^2+3^2} = \sqrt{13}$

$z_2 = 3-2i \quad \therefore |z_2| = \sqrt{3^2+2^2} = \sqrt{13}$

$z_1+z_2 = (2+3i) + (3-2i) = 5+i$

Vector addition $|z_1+z_2| = \sqrt{5^2+1}$
$= \sqrt{26}$



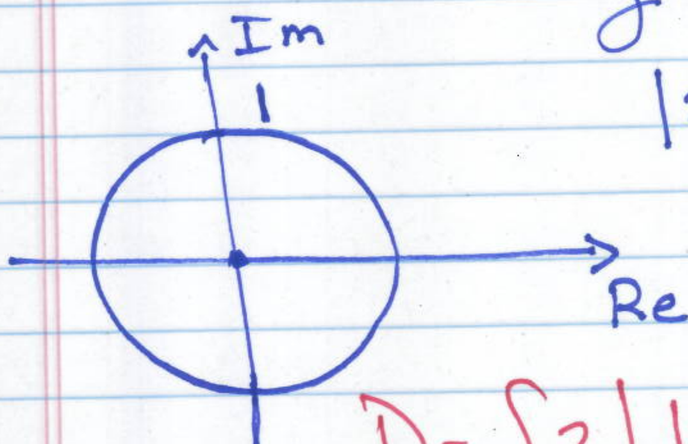
graph each set of complex number.

$$C = \{ z \mid |z| = 1 \}.$$

$$z = x+yi$$

$$|z| = \underbrace{\sqrt{x^2+y^2}} = 1$$

$$x^2+y^2 = 1$$



Circle with center $(0,0)$ and radius 1

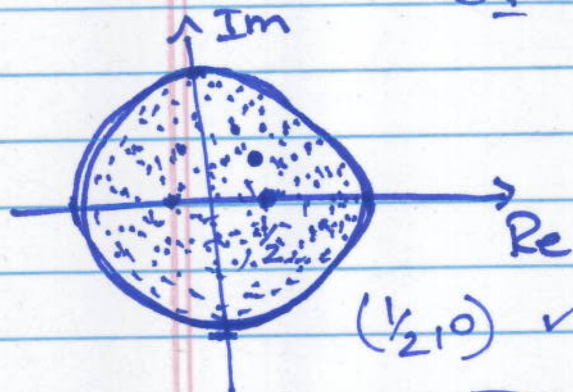
$$D = \{ z \mid |z| \leq 1 \}$$

$z = x + yi$

$|z| = \sqrt{x^2+y^2} \leq 1$

$0 \leq \sqrt{x^2+y^2} \leq 1$

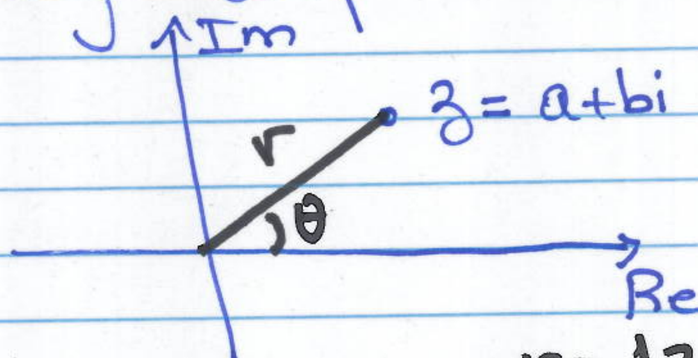
$x^2+y^2 \leq 1$



$(1/2, 0)$ ✓ $(\frac{1}{2})^2 + 0 \leq 1$

Disk

Polar form of Complex numbers.



$r = |z| = \sqrt{a^2+b^2}$

$\tan\theta = \frac{b}{a}$

$\cos\theta = \frac{a}{r} \therefore a = r\cos\theta$

$\sin\theta = \frac{b}{r} \therefore b = r\sin\theta$

$z = a + bi = r\cos\theta + r\sin\theta\, i$

$= r(\cos\theta + i\sin\theta) \qquad r = \sqrt{a^2+b^2}$

Examples: write complex numbers in polar form.

QI

$1+i$ $\quad r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1$

$(\sqrt{2}, \pi/4)$. $\quad \boxed{\theta = \pi/4}$ or $\theta = 5\pi/4$

$1+i = \sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)$ ✓

Polar form

$z = r(\cos\theta + i\sin\theta)$

QI

$z = 3+4i$

$r = \sqrt{9+16} = 5$

$\tan\theta = 4/3 \therefore \theta = \tan^{-1}(4/3)$

Final Answer $3+4i = 5\left(\cos\left(\tan^{-1}(4/3)\right) + i\sin\left(\tan^{-1}\right.\right.$

Multiply and division of complex numbers.

$$z_1 = a + bi \qquad z_2 = c + di$$

$$z_1 z_2 = (a+bi)(c+di).$$
$$= (ac - bd) + i(ad + bc)$$

But $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$.

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

$$z_1 z_2 = r_1 r_2 \left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \right)$$

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \right).$$

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3).$$

$$z_1 z_2 z_3 = r_1 r_2 r_3\left(\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\right)$$

Multiply.

$$z_1 = 2\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right).$$

$$z_2 = 5\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right)$$

$$z_1 z_2 = 10\left(\cos\left(\tfrac{\pi}{4}+\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{4}+\tfrac{\pi}{3}\right)\right).$$

$$= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

Division:. $$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$z_1 = 2\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right);\ z_2 = 5\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right).$$

$$\frac{z_1}{z_2} = \frac{2}{5}\left[\cos\left(\tfrac{\pi}{4}-\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{4}-\tfrac{\pi}{3}\right)\right]$$

$$= \frac{2}{5}\left[\cos\left(-\pi/12\right) + i\,\sin\left(-\pi/12\right)\right].$$

$$\boxed{\frac{z_1}{z_2} = \frac{2}{5}\left[\cos \pi/12 - i\,\sin \pi/12\right]}$$

De Moivre's theorem

If $z = r(\cos\theta + i\sin\theta)$.

$$z^n = r^n\left(\cos(n\theta) + i\,\sin(n\theta)\right)$$

for any integer n

$\boxed{n=2}$ $z^2 = z \cdot z = r(\cos\theta + i\sin\theta)\, r(\cos\theta + i\sin\theta)$

$$= r^2\left(\cos 2\theta + i\,\sin(2\theta)\right).$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\sin(a-b) = \sin a \cos b - \sin b \cos a$

n^{th} Roots of complex numbers.

$\sqrt{i} = a + bi$ find a, b.

$i = (a+bi)^2 = a^2 - b^2 + 2abi$

$0 + 1i = a^2 - b^2 + 2abi$

$$\begin{cases} a^2 - b^2 = 0 \\ 2ab = 1 \end{cases} \longrightarrow \begin{cases} a^2 = b^2 \\ ab = \frac{1}{2} \end{cases}$$

either $a = b$ or $\cancel{a = -b}$

$\downarrow$

$aa = \frac{1}{2} \therefore a^2 = \frac{1}{2} \therefore a = \pm\sqrt{\frac{1}{2}}$

$\boxed{a = b = \pm \frac{\sqrt{2}}{2}}$

answer

$\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

or $\sqrt{i} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$