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Department of Physics

COURSE: Introduction to Control Systems

Semester Assignmnet: Van der Pol Oscillator and the Triode Tube

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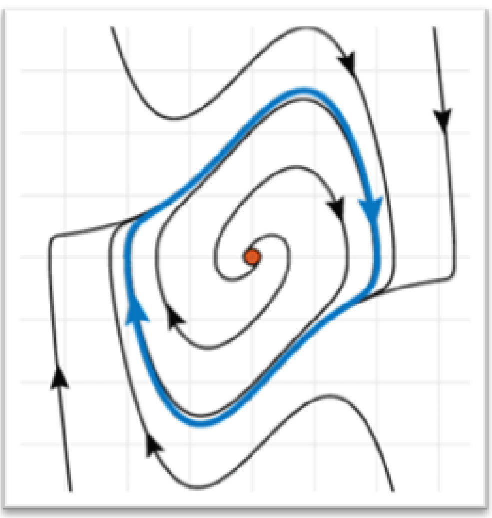

1 Balthasar Van der Pol and His Famous Equation

Figure 1: Balthasar Van der Pol

Balthasar Van der Pol (1889-1959) was Dutch, born in Utrecht, and studied at the University of Utrecht, where he received his Ph.D. in Experimental Physics in 1920. He also studied with John Ambrose Fleming and Sir J.J. Thomson in England. He joined Philips' research laboratories in 1921 and worked there until his retirement in 1949. His main scientific interests were radio wave propagation, electrical circuit theory, and mathematical physics. He was honored with the Medal of Honor from the Institute of Radio Engineers, today's IEEE (Institute of Electrical and Electronics Engineers), in 1935. His name was also given to asteroid 10443. During the first half of the 20th century, Van der Pol was a pioneer in the fields of radio and telecommunications. He was the first to introduce the appropriate equations in 1927 to describe electrical circuits using vacuum tubes. He found that these often led to stable oscillations, now called limit cycles.

Since then, thousands of papers have been published to determine better approximations to the solutions that appear in such nonlinear systems. His work in collaboration with E. Appleton in 1922 contains an early form of the equation of the oscillator with a vacuum tube triode, referred to today as the "autonomous Van der Pol equation," which is

$$\ddot{x} + \epsilon(x^2 - 1)(\dot{x} + x) = 0$$

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2 Lee de Forest and the Triode Tube Leading to This Wonderful Equation

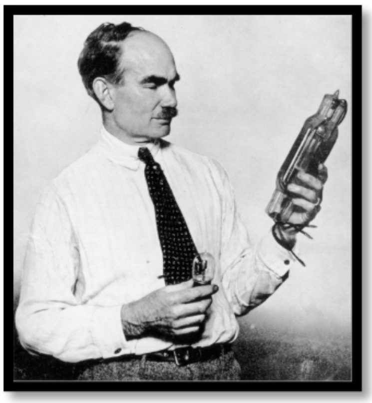


Figure 2: Lee De Forest

The triode tube used by Van der Pol was first manufactured in 1906 by American engineer Lee de Forest (1873-1961). He created a new tube with three electrodes, which he called "Audion," intended for the detection of high-frequency signals.
The third electrode that de Forest incorporated into the tube, compared to the diode, was a metallic mesh with variable potential. This mesh was permeable to some of the electrons moving from the cathode to the anode, thus controlling their flow.

Fleming, who had invented the diode tube, accused de Forest of copying his invention. However, he had not realized the significant innovation that arose from Fleming's construction, which concerned the electronic amplifier.
De Forest, who led a life full of adventures with failed ventures, lawsuits over technical inventions, and family problems, implemented the triode tube for controlling an energy flow with a weak signal. The triode tube of de Forest was used in many applications as an amplifier but mainly for audio signals and also as an electronic switch. It was later used in the construction of the first electronic computers.

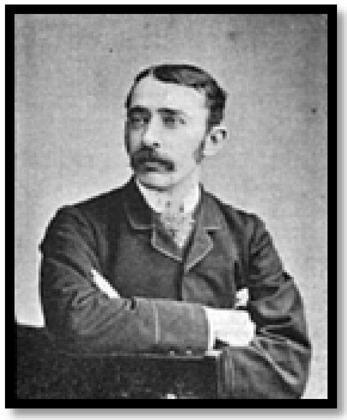


Figure 3: John Ambrose Fleming

3 Additional Information

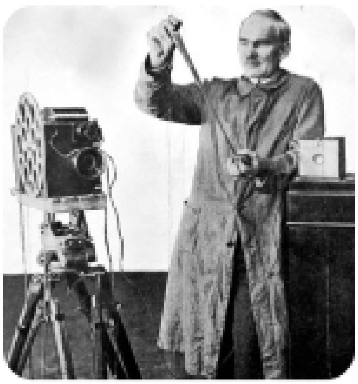


Figure 4

The development in the field of tubes was systematic, as all researchers and large companies began to study the behavior of tubes with more electrodes and special construction configurations to achieve the desired results on a case-by-case basis. As early as 1916, de Forest broadcast music with arias by Enrico Caruso and advertisements for his technical constructions. However, this technological event remained largely unknown because radio receivers were not available. De Forest was the greatest "patentee" of that era, but he was also the one who gave voice to silent cinema.

4 Structure of de Forest's Triode

Regarding the structure of de Forest's triode, as mentioned earlier, it consists of three electrodes, which are:

- Cathode (filament type or indirectly heated oxide-coated type)
- Anode (plate)
- Control grid

The grid is located between the cathode and anode and is closer to the cathode than the anode. Due to the structure of the grid, there is no direct obstacle to the flow of electrons moving towards the anode. However, it plays a significant role in the electric field created between the anode and cathode, and when there is supply to the grid, the overall flow of electrons is affected.

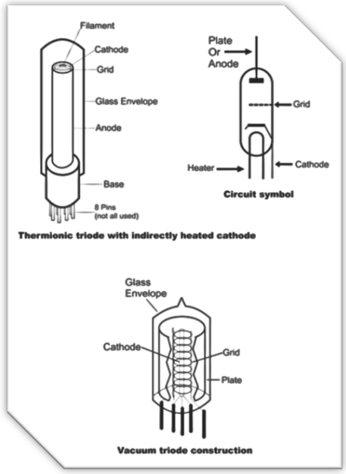


Figure 5

5 The Circuit Studied by Balthasar van der Pol in the Early 1920s

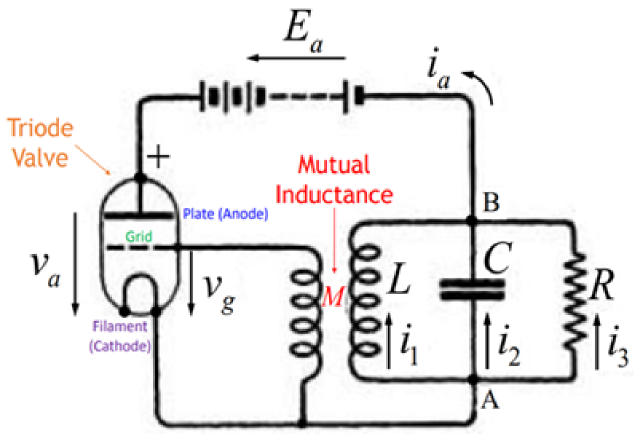


Figure 6

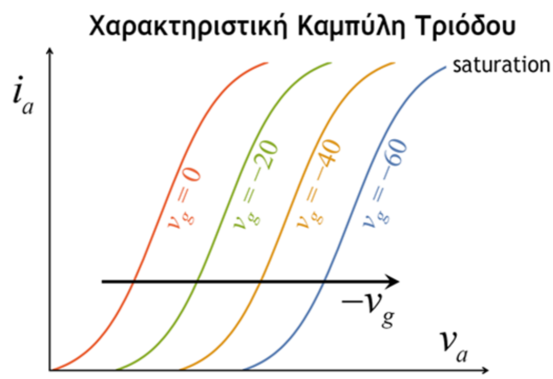


Figure 7

Based on Kirchhoff's laws for the above circuit and the RLC circuit part, we have:

$$i_a = i_1 + i_2 + i_3$$

$$V = L\frac{di_1}{dt} \Rightarrow di_1 = \frac{V}{L}dt \Rightarrow i_1 = \int V dt$$

$$M\frac{di_1}{dt} = v_g \Rightarrow v_g = \frac{M}{L}V \Rightarrow V = \frac{L}{M}v_g$$

$$i_2 = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt}$$

$$V = i_3 R$$

$$V = E_a - v_a$$

Thus, based on these relations, we have:

$$\frac{di_a}{dt} = \frac{1}{R}\frac{d}{dt} + C\frac{d^2}{dt^2} + \frac{1}{L}V \Rightarrow \frac{di_a}{dt} = \left(\frac{1}{R}\frac{d}{dt} + C\frac{d^2}{dt^2} + \frac{1}{L}\right)(E_a - v_a)$$

6 Characteristic Curve of the Triode and its Connection to the Circuit Current i_a and Voltage v_a

The differential equation:

$$\left(C\frac{d^2}{dt^2} + \frac{1}{R}\frac{d}{dt} + \frac{1}{L}\right)(v_a - E_a) + \frac{di_a}{dt} = 0$$

contains the supply voltage E_a, the plate (anode) voltage v_a, and the total anode current i_a.

We know that the characteristic curve of a triode is as follows:

$$i_a = \phi(v_a + gv_g)$$

The lower and upper parts of the curves are slightly convex, while the middle is almost linear. The maximum value of a grid voltage is always kept at zero during the design of these curves, and all curves are designed at a negative grid voltage. Thus, if $v_a = 0$ and $v_g = 0$ and I start to increase v_g, we will see a nonlinear change in both v_a and i_a. Therefore, there is ϕ which is a nonlinear function that explains the curve. The value of the plate current depends on the value of the plate voltage. The curves are parallel to each other with similar shapes. From the curves, we get that for every $\Delta v_g = 20V$, there is a constant change in the voltage v_a under constant current i_a in a linear manner, and Δv_a is not $20V$, so there is a constant g which is also called the voltage ratio of the triode. Thus, the above relation holds. Now, however, we know that $V = \frac{L}{M}v_g \Rightarrow v_g = \frac{VM}{L}$ and that $V = E_a - v_a$. Therefore, the relation transforms into:

$$i_a = \phi(v_a + gv_g) = \phi\left\{v_a + g\frac{M}{L}(E_a - v_a)\right\}$$

7 Stable and Unstable Points in the Differential Equation

The differential equation

$$(C\frac{d^2}{dt^2} + \frac{1}{R}\frac{d}{dt} + \frac{1}{L})(v_a - E_a) + \frac{di_a}{dt} = 0$$

has a stable (or unstable) point: $(v_a^*, i_a^*) = (E_a, \phi(E_a))$.

7.1 Initial Equations and Definitions

Initially, we require:

$$C\frac{d^2(v_a - E_a)}{dt^2} + \frac{1}{R}\frac{d(v_a - E_a)}{dt} + \frac{1}{L}(v_a - E_a) + \frac{di_a}{dt} = 0$$

and

$$i_a = \phi(v_a + gv_g) = \phi\left(v_a + \frac{gM}{L}(E_a - v_a)\right)$$

We define $f_1(v_a, i_a) = \frac{di_a}{dt}$ with $f_1(v_a^*, i_a^*) = 0$ and

$$f_2(v_a^*, i_a^*) = C\frac{d^2(v_a - E_a)}{dt^2} + \frac{1}{R}\frac{d(v_a - E_a)}{dt} + \frac{1}{L}(v_a - E_a) + \frac{di_a}{dt} = 0$$

From f_2, we have that $E_a = v_a^*$ and if $v_a^* = v_a$ then $i_a^* = \phi(v_a^*)$ resulting in $i_a^* = \phi(E_a)$ and thus $(v_a^*, i_a^*) = (E_a, \phi(E_a))$.

Now, for small perturbations where $V = v_a - E_a \Rightarrow v_a = V + E_a$ and

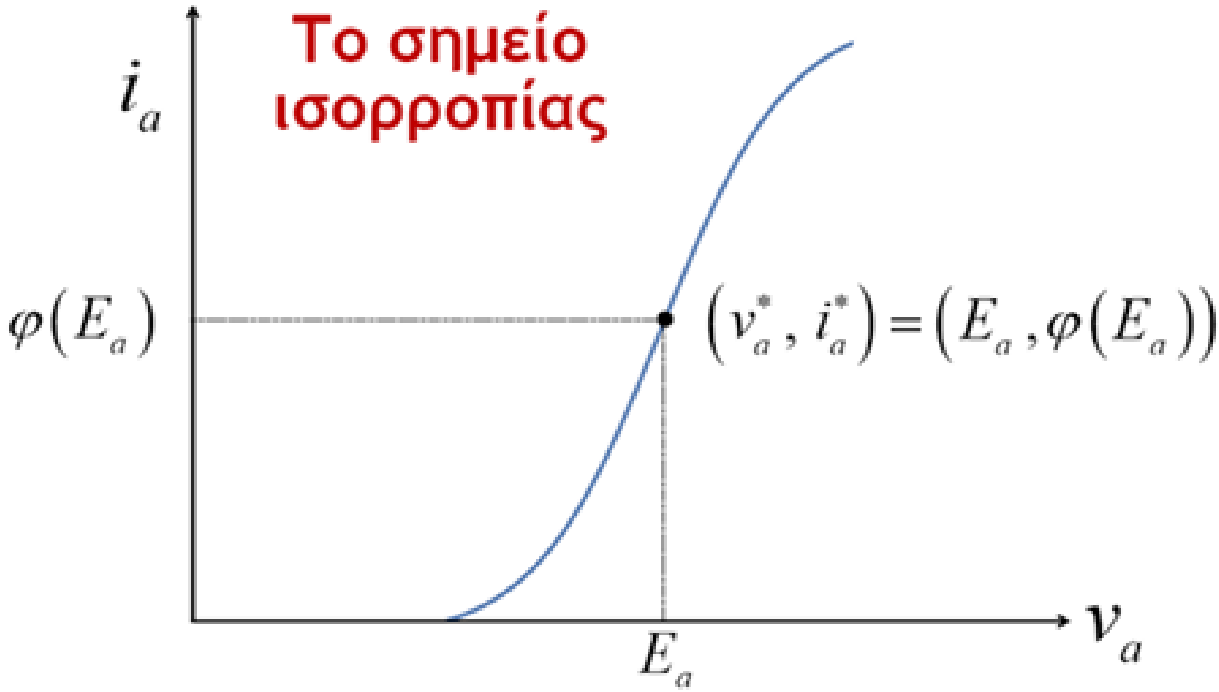
$$i_a = \phi\left(v_a + \frac{gM}{L}(E_a - v_a)\right) = \phi\left(E_a + V(1 - \frac{gM}{L})\right)$$

thus if $k = \frac{gM}{L} - 1$ then ϕ is written as $\phi(E_a - kV)$.

Now, consider a small variation in $i_a - \phi(E_a) = \phi(E_a - kV) - \phi(E_a)$ we need to show that $i = \phi(kV)$. Applying the mean value theorem (MVT) over the interval $[E_a - kV, E_a]$, let ξ be:

$$\phi'(\xi) = \frac{-\phi(E_a - kV) + \phi(E_a)}{E_a - (E_a + kV)} = -kV\phi'(\xi)$$

thus $\xi : i = \psi(kV)$



7.2 Small Perturbations

Assuming small changes in current and voltage, we have:

$$i = i_a - \phi(E_a) \implies i_a = i + \phi(E_a),$$

and for voltage,

$$V = v_a - E_a \implies v_a = V + E_a.$$

Substituting these into the differential equation we have:

$$C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V + \frac{di}{dt} = 0.$$

Given that $\frac{d\phi(E_a)}{dt} = 0$, we can simplify this to:

$$C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V + \frac{di}{dt} = 0.$$

Previously, we demonstrated that under small perturbation theory,

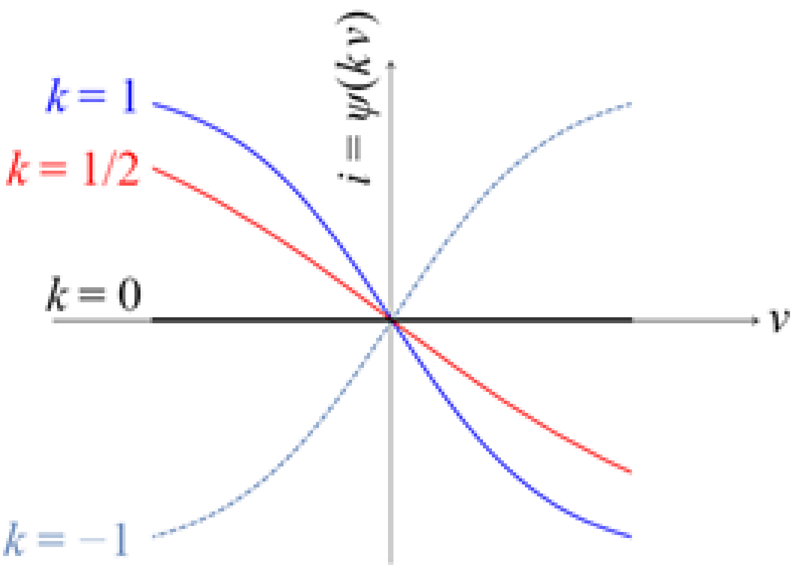
$$i = \psi(kV),$$

where ψ is a function derived from the characteristics of the system and k is a scaling factor based on the system parameters.

7.3 Behavior of the Function under Varying Conditions

In this diagram, we observe the behavior of $i = \psi(kV)$:

- For $k = 0$, $i = \psi(kV)$ is constant.
- For $k = 1$, $i = \psi(kV)$ is decreasing.
- For $k = -1$, $i = \psi(kV)$ is increasing.



Thus, we require that

$$\frac{di}{dt} < 0 \implies C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{V}{L} > 0$$

with $\psi(kV)$ being decreasing to ensure that

$$k > 0 \implies \frac{gM}{L} - 1 > 0.$$

This implies that for stability and desired behavior in the system, the parameter k, which scales the input to the function ψ, must be positive. This condition reflects the system's need for a positive feedback mechanism where the amplification factor $g\frac{M}{L}$ exceeds unity.

7.4 Taylor Series Expansion of the Function ψ for Current

Performing a Taylor series expansion of $\psi(V)$ and taking the derivative with respect to V, the terms $C_i = \frac{1}{i!}\frac{d^i\psi}{dV^i}$ contribute to variables, and thus we can write:

$$\psi(V) = aV + bV^2 + cV^3$$

However, for $k > 0$, $\psi(kV)$ is a constant which doesn't change during differentiation, therefore, we can take:

$$\psi(kV) = -\alpha V + \beta V^2 + \gamma V^3$$

with $-\alpha$ ensuring that $\frac{d\psi(kV)}{dV} < 0$ and $\alpha, \beta, \gamma > 0$.

Given that α, β, γ correspond to variables c_i through the chain rule, we have:

$$-\alpha = \left|\frac{d\psi(kV)}{dV}\right|_{(V=0)} = \frac{d\psi(kV)}{d(kV)} \cdot \frac{d(kV)}{dV} = \frac{d\psi(kV)}{d(kV)} \cdot k \implies \alpha = k\left|\frac{d\psi(kV)}{d(kV)}\right|_{(V=0)} = k\left|\frac{di}{d(kv)}\right|_{(V=0)}$$

We have:

$$i = i_a - \phi(E_a) = \phi(v_a - kV) - \phi(E_a)$$

However, with $V = 0$, $v_a = E_a$, and:

$$i_a = \phi(E_a) = \phi(v_a)$$

and thus we can express α as:

$$\alpha = k\left|\frac{di_a}{dv_a}\right|_{(v_a=E_a)}$$

Similarly for the constant β and γ:

$$\beta = \frac{k^2}{2}\left|\frac{d^2 i_a}{dv_a^2}\right|_{(v_a=E_a)}$$

$$\gamma = \frac{k^3}{6}\left|\frac{d^3 i_a}{dv_a^3}\right|_{(v_a=E_a)}$$

Now, selecting the supply voltage E_a precisely at the inflection point of the characteristic curve, we have:

$$\frac{d^2 i}{dV^2} = 0 \quad \text{and} \quad E_a = v_a$$

Thus:

$$\frac{d^2 i_a}{dv_a^2} = 2\beta/k^2 = 0 \Rightarrow \beta = \frac{k^2}{2}\left|\frac{d^2 i_a}{dv_a^2}\right|_{(v_a=E_a)} = 0$$

And hence:

$$i = \psi(kV) = -\alpha V + \gamma V^3 \implies \frac{di}{dt} = -\alpha\frac{dV}{dt} + 3\gamma V^2\frac{dV}{dt}$$

Substituting into the differential equation, we obtain:

$$C\frac{d^2V}{dt^2} + \left(\frac{1}{R} - \alpha + 3\gamma V^2\right)\frac{dV}{dt} + \frac{1}{L}V = 0$$

And:

$$i = \psi(kV) = -\alpha V + \gamma V^3$$

7.5 Derivation of the Famous Van der Pol Equation

We previously demonstrated that:

$$C\frac{d^2V}{dt^2} + \left(\frac{1}{R} - \alpha + 3\gamma V^2\right)\frac{dV}{dt} + \frac{1}{L}V = 0$$

Setting:

$$V = \left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} x, \quad t = \omega^{-1}\tau, \quad \text{with} \quad \omega = (LC)^{-1/2} \quad \text{and} \quad \epsilon \equiv \frac{1}{\omega C}\left(\alpha - \frac{1}{R}\right)$$

We derive $dV/d\tau$ and d^2V/dt^2 as follows:

$$dV/d\tau = \left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} \dot{x}\omega, \quad d^2V/dt^2 = \left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} \ddot{x}\omega^2$$

Thus, the equation becomes:

$$C\omega^2\left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} \ddot{x} + (\alpha - \frac{1}{R})(x^2 - 1)\left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} \dot{x}\omega + \frac{1}{L}x\left(\frac{\alpha - \frac{1}{R}}{3\gamma}\right)^{1/2} = 0$$

$$\implies \ddot{x} + \frac{(\alpha - \frac{1}{R})}{\omega C}(x^2 - 1)\dot{x} + \frac{1}{LC\omega^2}x = 0$$
$$\implies \ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

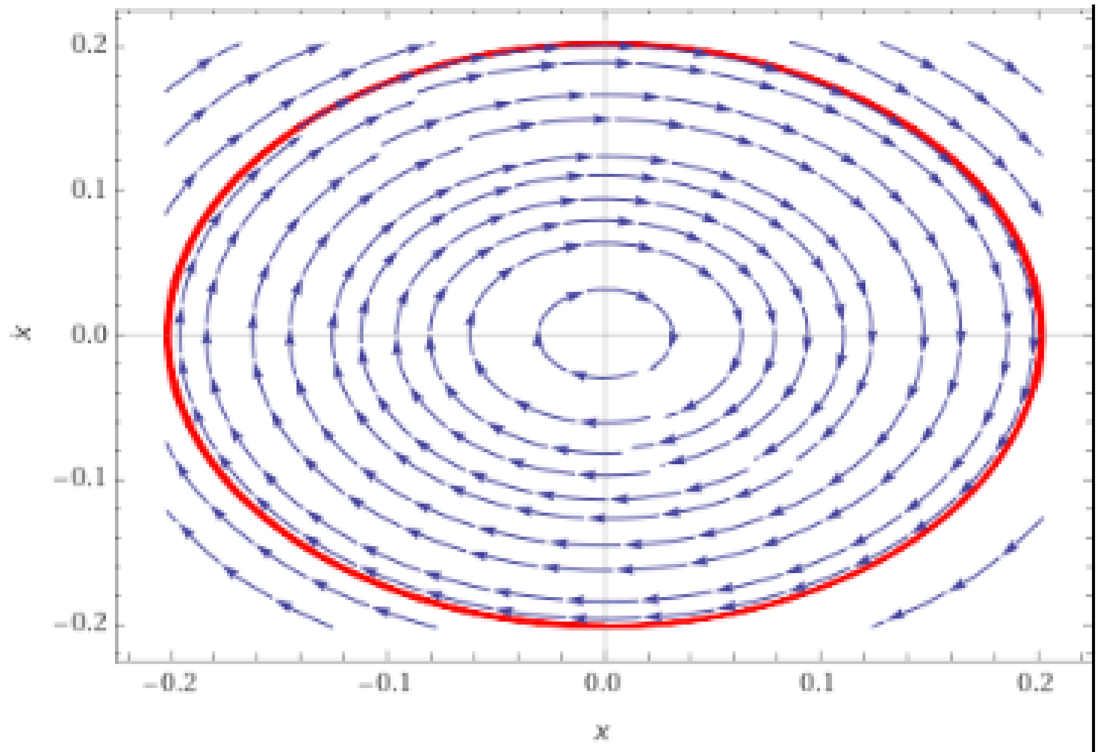
7.6 Van der Pol Equation $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$ for Small and Large ϵ

As observed, if ϵ is very close to zero, the term $(x^2 - 1)\dot{x}$ will become negligible, resulting in the equation taking the form of simple harmonic motion:

$$\ddot{x} + x = 0$$

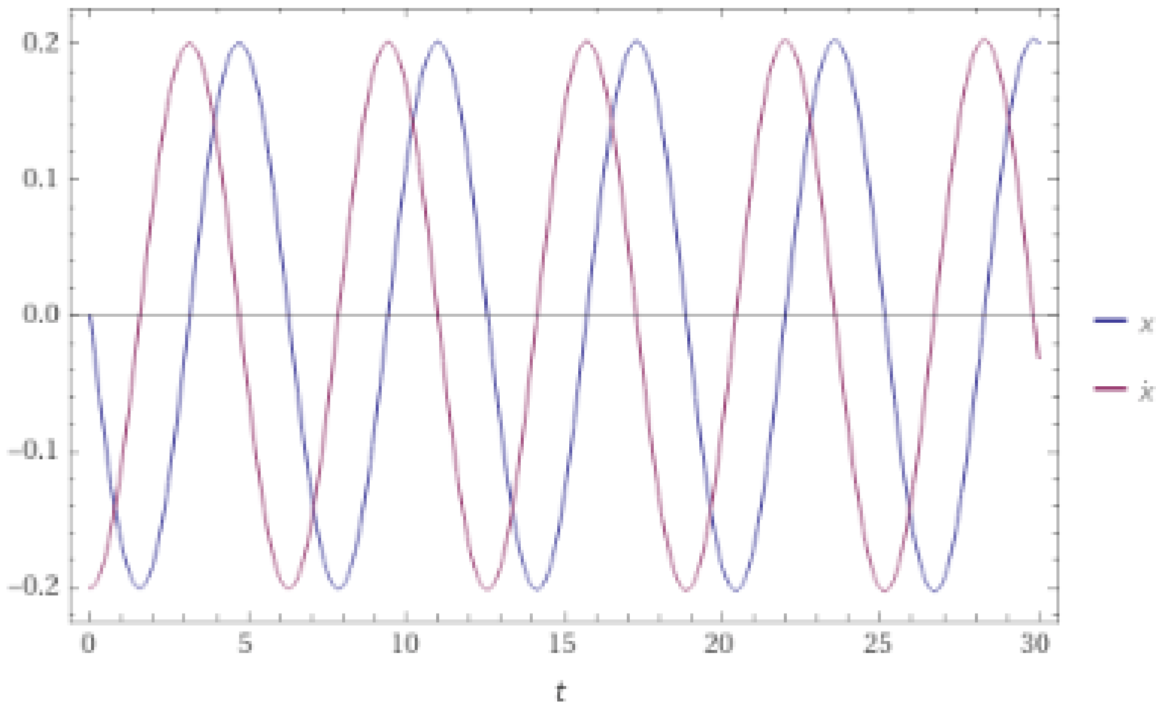
However, if $\epsilon \gg 1$, the differential equation is heavily influenced by the factor $(x^2 - 1)\dot{x}$, as small variations in x can significantly affect the dynamics. This term is the non-linear term and plays the main role in the Van der Pol equation.

For $\epsilon \ll 1$ (e.g., $\epsilon = 0.001$): - The system behaves almost like a linear oscillator.

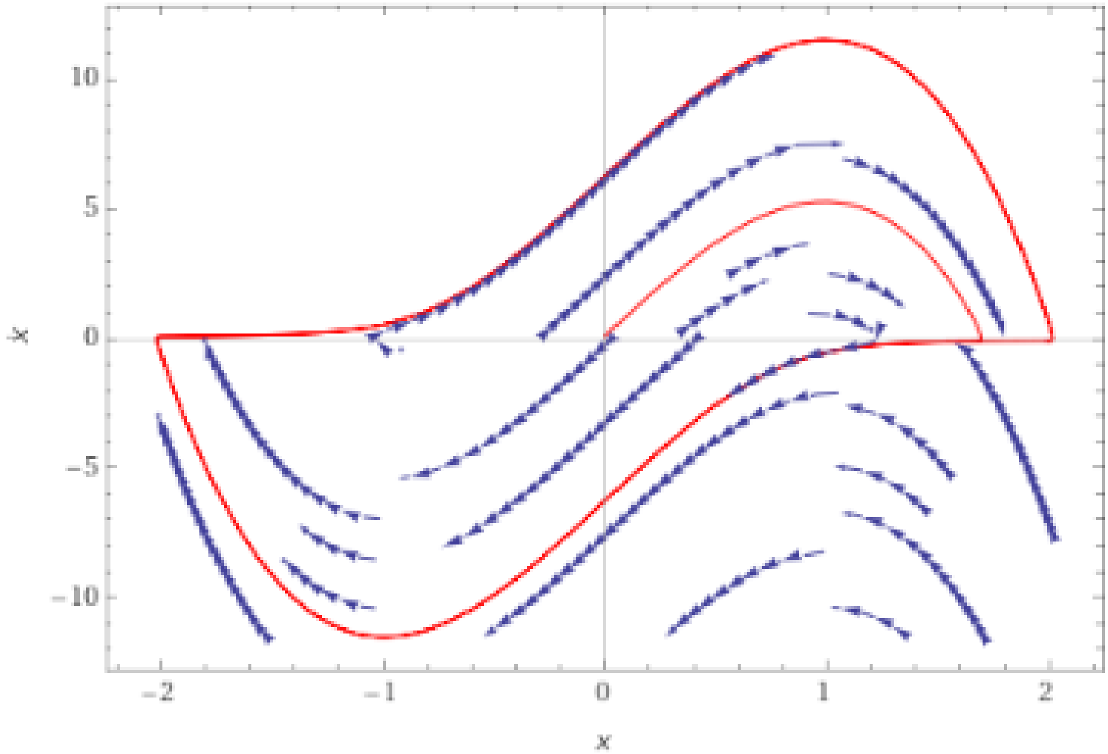


(the critical point origin is an unstable focus)

Behavior of the solution

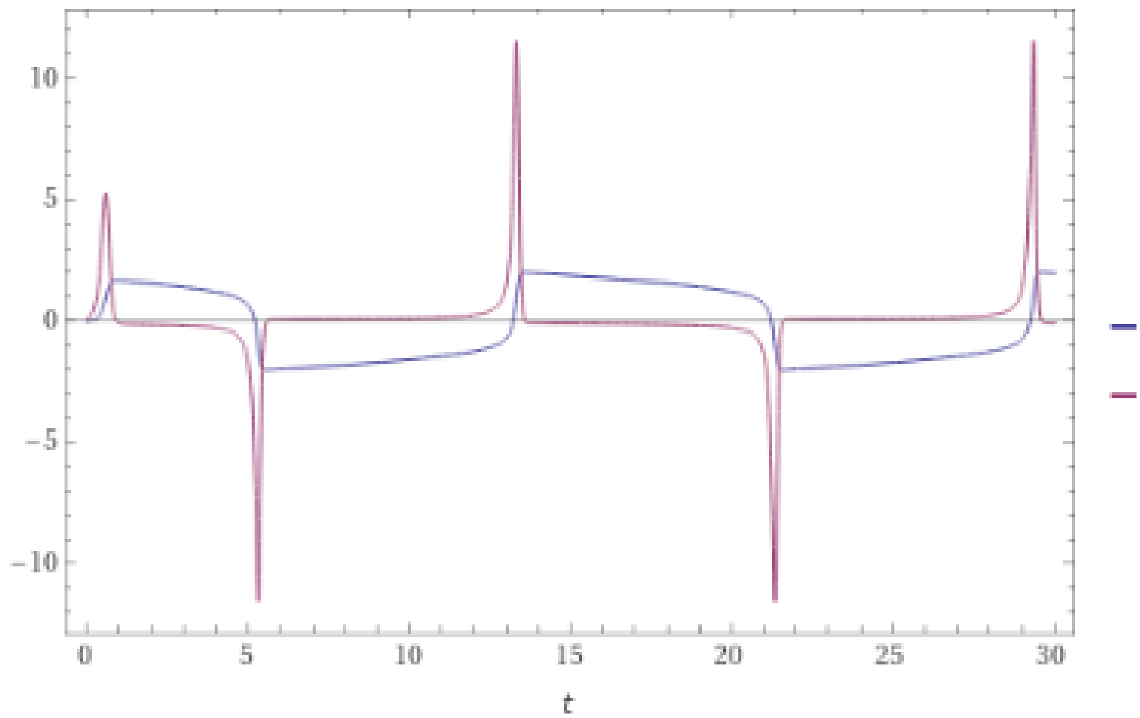


For $\epsilon > 1$ (e.g., $\epsilon = 8$): - The nonlinearity dominates the behavior, leading to typical non-linear dynamics such as limit cycles.



(the critical point origin is an unstable node)

Behavior of the solution



Clearly, the source of linearity comes from the term with the parameter $\epsilon = \frac{1}{\omega C}(\alpha - \frac{1}{R})$, where $\omega = (LC)^{-1/2}$, introduced by the characteristics of the RLC circuit.