20. Reflection matrices in $\mathbb{R}^2$

In this sections we will discuss the matrices of 2D line reflections. Here is the condensed version of the main theorems

Theorem 20.1: The matrix of a **reflection** in the line $\vec{x} = t\vec{v}$: $\quad M = \boxed{\dfrac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

Theorem 20.2: The matrix of a **reflection** in $\vec{n}\cdot\vec{x} = 0$: $\quad M = \boxed{I - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}}$

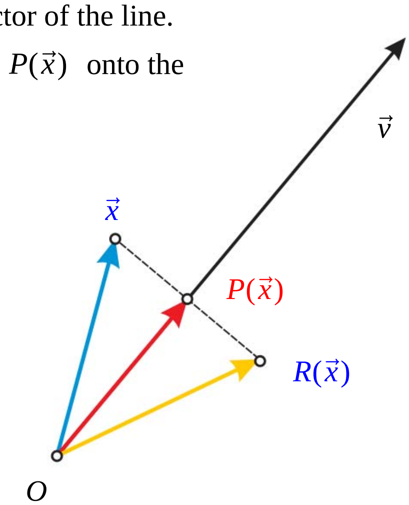
Theorem 20.3: The matrix of a **skew-reflection** in $\vec{n}\cdot\vec{x} = 0$ in the direction $\vec{v}$: $\quad M = \boxed{I - \dfrac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$

Reflections

With only minor modifications we can produce the reflection matrices using the methods we used for projections in the previous section.

(a) The line is given in vector form (The method presented here also works in higher dimensional space)

Let l: $\vec{x} = t\vec{v}$ where $\vec{v}$ is a direction vector of the line.
Notice the relationship between the projection $P(\vec{x})$ onto the line and the reflection $R(\vec{x})$ in the line



$$\frac{\vec{x} + R(\vec{x})}{2} = P(\vec{x})$$

Hence $\quad R(\vec{x}) = 2P(\vec{x}) - \vec{x}$

So that

$$R(\vec{x}) = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} - \vec{x}$$

Hence the reflection matrix is $\dfrac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I$

or if $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then $M = \dfrac{1}{A^2 + B^2}\begin{bmatrix} A^2 - B^2 & 2AB \\ 2AB & B^2 - A^2 \end{bmatrix}$

Example 1: **2D**: Let $l: \quad \vec{x} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and the reflection in this line

is $$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \left(\frac{2}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} - I\right)\vec{x} = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}\vec{x}$$

This approach also works in higher dimensions:

Example 2: **3D**: Let $l: \quad \vec{x} = t\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ then $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and the reflection in this line

is $$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \left(\frac{2}{5}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} - I\right)\vec{x} = \frac{1}{5}\begin{bmatrix} -3 & -4 & 2 \\ -4 & 3 & -4 \\ 2 & -4 & 9 \end{bmatrix}\vec{x}$$

Theorem 20.1: Let the line l be given in vector form $\vec{x} = t\vec{v}$, then the reflection in this line is given by

$$R(\vec{x}) = \left(\frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} \quad \text{or} \quad R(\vec{x}) = \left(\frac{2\,\vec{v}\,\vec{v}^{\mathsf{T}}}{\|\vec{v}\|^2} - I\right)\vec{x}$$

R is a linear transformation with matrix $\boxed{M = \frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I}$

If $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then $M = \dfrac{1}{A^2+B^2}\begin{bmatrix} A^2-B^2 & 2AB \\ 2AB & B^2-A^2 \end{bmatrix}$

or

if $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ then $M = \dfrac{1}{A^2+B^2+C^2}\begin{bmatrix} A^2-B^2-C^2 & 2AB & 2AC \\ 2AB & -A^2+B^2-C^2 & BC \\ 2AC & 2BC & -A^2-B^2+C^2 \end{bmatrix}$

Furthermore

- $M = M^{\mathsf{T}}$
- $M^2 = I$ i.e. $M^{-1} = M$
- $\text{Trace}(M_2) = 0$ in the 2D case
- $\text{Trace}(M_3) = -1$ in the 3D case [In $\mathbb{R}^n$: $\text{Trace}(M_n) = 2-n$]
- $\det(M_2) = -1$ in the 2D case
- $\det(M_3) = +1$ in the 3D case [In $\mathbb{R}^n$: $\det(M_n) = (-1)^{n-1}$]
- The only fixed points of this transformation are the points on the line.

Proof: Only the ‘furthermore’ points remain to be proven.

- $M^{\mathsf{T}} = \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)^{\mathsf{T}} = \frac{2}{\|\vec{v}\|^2}\cdot(\vec{v}^{\mathsf{T}})^{\mathsf{T}}\,\vec{v}^{\mathsf{T}} - I^{\mathsf{T}} = \frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I = M$

- $$\begin{aligned} M^2 &= \left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\cdot\left(\frac{2}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} - I\right) \\ &= \frac{4}{\|\vec{v}\|^4}\cdot\vec{v}\,\vec{v}^{\mathsf{T}}\,\vec{v}\,\vec{v}^{\mathsf{T}} - \frac{4}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} + I^2 \\ &= \frac{4}{\|\vec{v}\|^4}\cdot\left(\left[\,\vec{v}\cdot\vec{v}\,\right]\cdot\vec{v}\right)\vec{v}^{\mathsf{T}} - \frac{4}{\|\vec{v}\|^2}\cdot\vec{v}\,\vec{v}^{\mathsf{T}} + I = I \end{aligned}$$

  and $M^2 = I$ which implies $M^{-1} = M$

- $\text{Trace}(M) = 0,\ -1$ (Immediate from the matrices. The general case: an exercise!)

- $\det(M_n) = (-1)^{n-1}$ (A direct calculation suffices for the 2D and 3D matrices. The general case requires more work.)

- Fixed points: $\left(\frac{2}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}} - I\right)\vec{x} = \vec{x} \Rightarrow \frac{1}{\|\vec{v}\|^2}\vec{v}\,\vec{v}^{\mathsf{T}}\vec{x} = \vec{x} \Rightarrow P(\vec{x}) = \vec{x}$

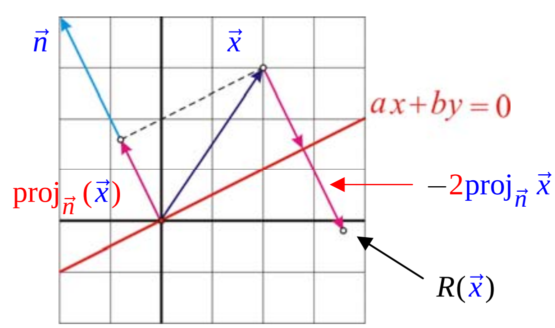
  i.e. all those points on the line (as we proved for the projection).

(b) The line is given in normal form (2D only!)

Let $l:\ ax+by = 0$ be a line through the origin, with normal $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$.

The reflection $R:\ \mathbb{R}^2 \to \mathbb{R}^2$ in this line can be computed as follows

$$\begin{aligned} R(\vec{x}) &= \vec{x} - 2\,\text{proj}_{\vec{n}}(\vec{x}) \\ &= \vec{x} - 2\frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\,\vec{n} \\ &= \vec{x} - \frac{2}{\|\vec{n}\|^2}(\vec{n}\cdot\vec{x})\,\vec{n} \\ &= \vec{x} - \frac{2}{\|\vec{n}\|^2}\,\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} \\ &= \left(I_2 - \frac{2}{\|\vec{n}\|^2}\,\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} \end{aligned}$$



Hence the reflection matrix is $I_2 - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}$. An explicit form of this matrix would be

$M = \dfrac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$ which can be easily derived as follows

$$
\begin{aligned}
I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} &= \frac{1}{a^2+b^2}\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \frac{1}{a^2+b^2}\begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix} \\
&= \frac{1}{a^2+b^2}\left(\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} - \begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix}\right) \\
&= \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}
\end{aligned}
$$

Theorem 20.2: Let $l:\ \vec{n}\cdot\vec{x} = 0$ be a line in $\mathbb{R}^2$, i.e. $l:\ ax+by = 0$ when $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$,

then the matrix of the reflection in the line l is given by

$$M = \boxed{I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}} \qquad \text{i.e.} \qquad M = \frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$$

Furthermore:

- $M = M^{\mathsf{T}}$
- $M^2 = I_2$ hence $M^{-1} = M$ (i.e. M is invertible.)
- $\text{Trace}(M) = 0$
- $\det(M) = -1$
- The only fixed points of this transformation are the points on the line.

Proof: Since $M = \dfrac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$

- $M = M^{\mathsf{T}}$ and $\text{Trace}(M) = 0$ are pretty obvious
- $M^2 = M$ follows by direct computation. [This also implies that $M^{-1} = M$.]
- $\det(M) = -1$ follows by direct computation.
- Fixed points: $M\vec{x} = \vec{x}$ $\Rightarrow \left(I_2 - \dfrac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\right)\vec{x} = \vec{x}$

$$
\begin{aligned}
&\Rightarrow \vec{x} - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{x} \\
&\Rightarrow \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0} \\
&\Rightarrow \vec{n}\,\vec{n}^{\mathsf{T}}\vec{x} = \vec{0}
\end{aligned}
$$

$$\Rightarrow \quad (\vec{n}\cdot\vec{x})\,\vec{n} \;=\; \vec{0}$$

$$\Rightarrow \quad \vec{n}\cdot\vec{x} \;=\; 0 \qquad\qquad \Rightarrow \quad \text{all points on the line } l$$

Example 3: Let $l:\ x-2y = 0$ be a line in $\mathbb{R}^2$, hence $\vec{n} = \begin{bmatrix}1\\-2\end{bmatrix}$, then the matrix of the reflection R in the line l is

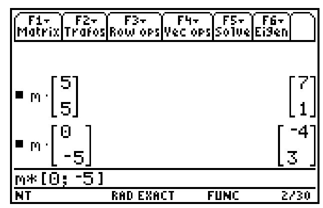
$$I_2 - \frac{2}{\|\vec{n}\|^2}\vec{n}\,\vec{n}^{\mathsf{T}} = \begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1\\-2\end{bmatrix}\begin{bmatrix}1&-2\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1&-2\\-2&4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix}$$

or

$$\frac{1}{a^2+b^2}\begin{bmatrix}-a^2+b^2 & -2ab\\ -2ab & a^2-b^2\end{bmatrix} = \frac{1}{1^2+(-2)^2}\begin{bmatrix}-1^2+(-2)^2 & -2\cdot1\cdot(-2)\\ -2\cdot1\cdot(-2) & 1^2-(-2)^2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix}$$

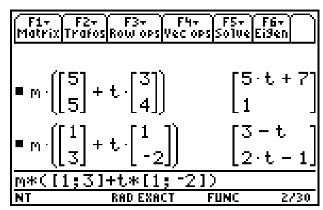
- The image of $\begin{bmatrix}5\\5\end{bmatrix}$ and $\begin{bmatrix}0\\-5\end{bmatrix}$ are $\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}5\\5\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$ and $\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}0\\-5\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$

- The image of the line $m:\ \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} + t\begin{bmatrix}3\\4\end{bmatrix}$ is



$$R\begin{bmatrix}x\\y\end{bmatrix} = R\left(\begin{bmatrix}5\\5\end{bmatrix} + t\begin{bmatrix}3\\4\end{bmatrix}\right) = R\begin{bmatrix}5\\5\end{bmatrix} + tR\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix} + t\begin{bmatrix}5\\0\end{bmatrix}$$

We get $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix} + t\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}7+5t\\1\end{bmatrix}$ i.e. the images



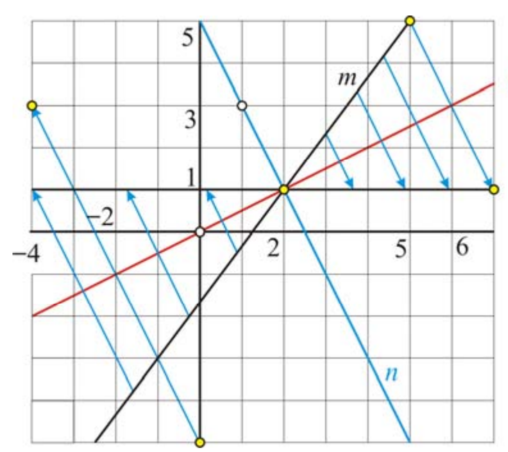
$(x',\ y')$ are on the horizontal line $y=1$.

- The image of the line $n:\ \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}$ (i.e. $2x+y=5$) is

$$R\begin{bmatrix}x\\y\end{bmatrix} = R\left(\begin{bmatrix}1\\3\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix} + t\begin{bmatrix}-1\\2\end{bmatrix}$$

Closer inspection reveals this is the same line $2x+y=5$. This makes sense once we realize that n is perpendicular to the line l in which we reflect.

(See picture)

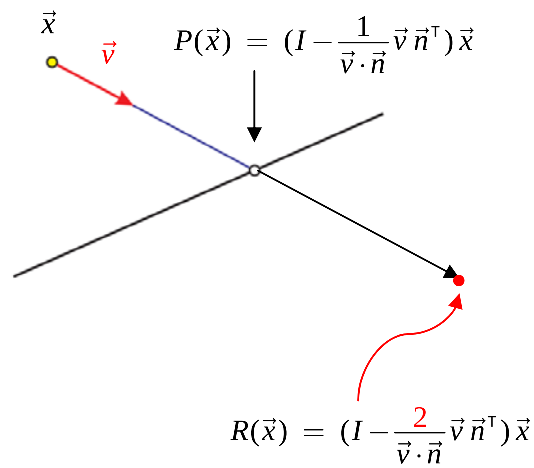


Skew Reflections

We can also create a skew-reflection, which we can derive from a skew-projection, in pretty much the same way as we derived a reflection from a projection:

Theorem 20.3: The matrix of a **skew reflection** onto $\vec{n}\cdot\vec{x} = 0$ in the direction of the vector $\vec{v}$ is

$$M = \boxed{I - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}}$$



If $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ then

$$M = \frac{1}{aA+bB}\begin{bmatrix} -aA+bB & -2bA \\ -2aB & aA-bB \end{bmatrix}$$

Furthermore:

- $M \neq M^{\mathsf{T}}$
- $M^2 = I$ so that $M^{-1} = M$
- $\text{Trace}(M) = 0$
- $\det(M) = -1.$
- The only fixed points of this transformation are the points on the line.

Proof:

Again $\dfrac{\vec{x} + R(\vec{x})}{2} = P(\vec{x})$ so that

$$R(\vec{x}) = 2P(\vec{x}) - \vec{x} = 2\left(I - \frac{1}{\vec{v}\cdot\vec{n}}\vec{v}^{\mathsf{T}}\vec{n}\right)\vec{x} - \vec{x} = \left(I - \frac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}\right)\vec{x}$$

Computing $M = I - \dfrac{2}{\vec{v}\cdot\vec{n}}\vec{v}\,\vec{n}^{\mathsf{T}}$ with $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ we get, after some algebra,

$$M = \frac{1}{aA+bB}\begin{bmatrix} -aA+bB & -2bA \\ -2aB & aA-bB \end{bmatrix}$$

It is then clear that, in general, $M \neq M^{\mathsf{T}}$, $\text{Trace}(M) = 0$ and after simple computations that $M^2 = I$ and $\det(M) = -1$. That the only fixed points are the points on the line we leave as an exercise (or compare with the skew-reflection in a plane in chapter 25).