3. A Basic Projection

One of the remarkable facts that follow from $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ is that it is easy to check if two vectors are **perpendicular**. In that case $\cos\theta = 0$, i.e. $\vec{x}\cdot\vec{y} = 0$.

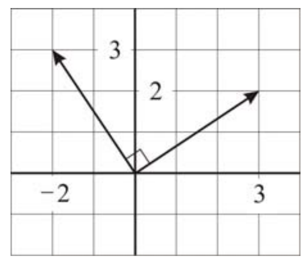
Recall that the zero vector $\vec{0}$ doesn't really have a direction. We adopt the convention that the zero vector is be perpendicular to *any* vector.

Theorem 3.1: Two vectors $\vec{x}$ and $\vec{y}$ are perpendicular if and only if $\vec{x}\cdot\vec{y} = 0$.

Note 1: This theorem works in 2D and 3D; and in fact in any higher dimensions as well.

Example 1: Since $\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}-2\\3\end{bmatrix} = 0$

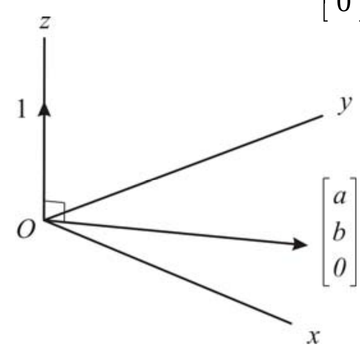
we know that $\begin{bmatrix}3\\2\end{bmatrix}\perp\begin{bmatrix}-2\\3\end{bmatrix}$.



Example 2: (a) Find a vector that is perpendicular to $\begin{bmatrix}1\\2\\3\end{bmatrix}$. There are many answers possible. For example $\begin{bmatrix}5\\-1\\-1\end{bmatrix}$ would work since $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}5\\-1\\-1\end{bmatrix} = 0$. Alternatively we can take a vector with zero first component $\begin{bmatrix}0\\?\\?\end{bmatrix}$ and choose the others such that $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}0\\?\\?\end{bmatrix} = 0$, e.g. $\begin{bmatrix}0\\3\\-2\end{bmatrix}$. Or take for example the third component to be zero: $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\0\end{bmatrix} = 0$. Etc.

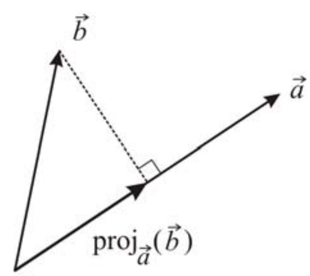
(b) A special case: *Any* vector in the $x\,y-$plane, i.e. $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, is perpendicular to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$



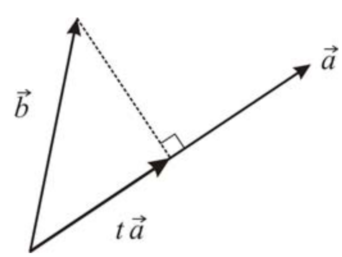
Next we'll develop a formula to compute the **orthogonal projection** of one vector onto another, non-zero vector.

Let $\vec{a} \neq \vec{0}$. With $\mathrm{proj}_{\vec{a}}(\vec{b})$ we will denote the (orthogonal) projection of $\vec{b}$ onto $\vec{a}$ (or rather the projection of $\vec{b}$ onto the line of which $\vec{a}$ is a segment.)

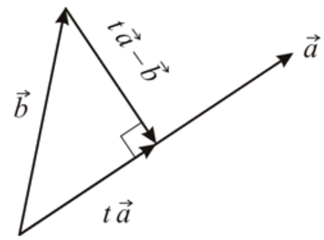


Theorem 3.2: $\mathrm{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a}$ [Of course we assume here that $\|\vec{a}\| \neq 0$]

Proof: Clearly $\mathrm{proj}_{\vec{a}}(\vec{b})$ is a multiple of $\vec{a}$ (since it is the projection of $\vec{b}$ **onto** $\vec{a}$).
Hence $\mathrm{proj}_{\vec{a}}(\vec{b}) = t\,\vec{a}$ for some real number t.



Note that the vector $t\vec{a} - \vec{b}$ is perpendicular to $\vec{a}$: $t\vec{a} - \vec{b} \perp \vec{a}$



Hence $(t\vec{a} - \vec{b})\cdot\vec{a} = 0 \quad \Rightarrow \quad t\vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} = 0$

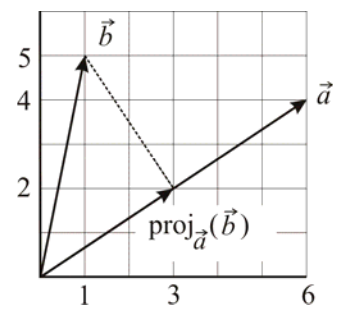
$$\Rightarrow \quad t\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{a}$$

$$\Rightarrow \quad t = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}} \qquad (\ \vec{a}\cdot\vec{a} \neq 0 \quad \text{so we can divide by it}\ )$$

So that $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a}$ $\square$

Example 3: Let $\vec{a} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

then $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{26}{52}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



Example 4: Let $A = (2,5)$, $B = (8,8)$ and $C = (3,8)$ be the vertices of triangle ABC. Find the base Q of the altitude from B using a projection vector.

Note that if

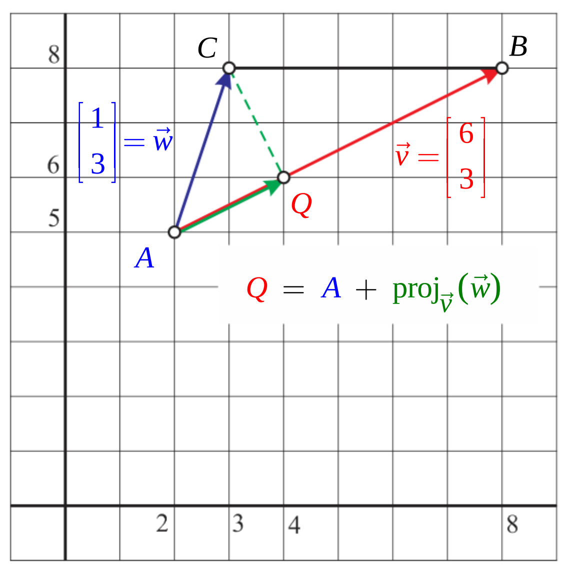
$$\vec{v} = \overrightarrow{AB} = B - A = (8,8) - (2,5) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \quad \text{and}$$

$$\vec{w} = \overrightarrow{AC} = C - A = (3,8) - (2,5) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{then}$$

$$\operatorname{proj}_{\vec{v}}(\vec{w}) = \frac{15}{45}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So that

$$Q = A + \operatorname{proj}_{\vec{v}}(\vec{w}) = (2,5) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (4,6).$$



Note that the **length** of the projection vector is

$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = \left\| \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a} \right\| = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|^2}\,\|\vec{a}\| = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|}$$

Alternatively this becomes also immediately clear by using some trigonometry:

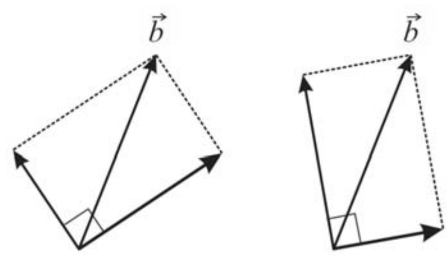
$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = \|\vec{b}\|\cdot|\cos(\theta)| = \frac{\|\vec{a}\|\cdot\|\vec{b}\|\cdot|\cos(\theta)|}{\|\vec{a}\|} = \frac{|\vec{b}\cdot\vec{a}|}{\|\vec{a}\|}$$

In particular we have that when $\vec{a}$ is a unit vector (i.e. $\|\vec{a}\|=1$) then

$$\left\| \operatorname{proj}_{\vec{a}}(\vec{b}) \right\| = |\vec{a}\cdot\vec{b}|$$

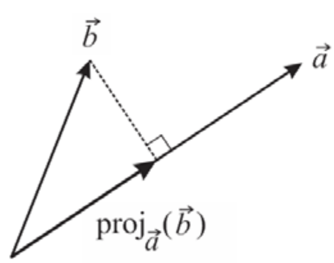
The orthogonal complement

Sometimes it is useful to decompose a vector into the sum of two orthogonal vectors.



Clearly this can be done in many ways. Usually one particular direction is given (or needed).

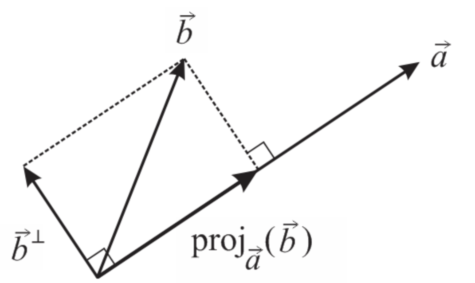
The component of the vector $\vec{b}$ in a given direction $\vec{a}$ is simply: $\operatorname{proj}_{\vec{a}}(\vec{b})$.



The other component, called the *orthogonal complement* is denoted by $\vec{b}^{\perp}$ (maybe a better notation would be $\vec{b}^{\perp\vec{a}}$, to indicate the other—given—direction as well). It can be found by

$$\vec{b}^{\perp} = \vec{b} - \operatorname{proj}_{\vec{a}}(\vec{b})$$

which is obvious because only then we would have $\vec{b}^{\perp} + \text{proj}_{\vec{a}}(\vec{b}) = \vec{b}$.

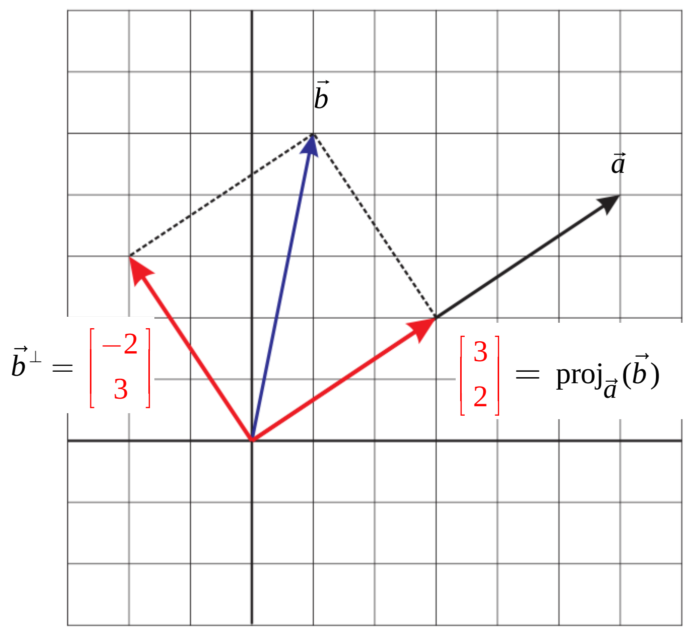


Example 5: Let $\vec{a} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ then

$$\text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and

$$\vec{b}^{\perp} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$



It is easy to check: $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -2 \\ 3 \end{bmatrix}$