# The Boris Method


Ryan Cox

PHYS 415: Electromagnetism


April 8, 2022

## Contents



## I. Theory

The motion of a charged particle subject subject only to forces from electric and magnetic fields is governed by the Newton-Lorentz force law

$$m\mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

Given an initial position $\mathbf{r}(t_0)$ and velocity $\mathbf{v}(t_0)$ we can increment the time in small steps $\Delta t$. The most basic process of going from $t$ to $t + \Delta t$ is

1. Update position $\mathbf{r}$ as $\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \mathbf{v}(t)\Delta t$
2. Update velocity $\mathbf{v}$ as $\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \mathbf{a}(t)\Delta t$ where $\mathbf{a}(t)$ is derived from Equation 1

Repeating this many time approximates the particle motion over a given period. However, this method has some significant drawbacks that cause its numerical solution to rapidly diverge from the analytical solution.

Firstly, it assumes $\mathbf{E}$ and $\mathbf{B}$ are constant across the distance traversed each iteration, giving constant acceleration over each step. We can approximate this scenario by choosing a small timestep.

Secondly, it calculates particle displacement from $t$ to $t + \Delta t$ by using the velocity at time $t$. It is easy to see that this leads to an inaccurate result when any significant acceleration is present. This can be corrected by using the average velocity over that step. As we assume constant acceleration each step this average is given by

$$\frac{\mathbf{v}(t + \Delta t) + \mathbf{v}(t - \Delta t)}{2} = \mathbf{v}(t + \frac{\Delta t}{2}) \tag{2}$$

This method can be easily implemented by stepping the initial velocity $\mathbf{v}(t_0)$ back in time by $\Delta t/2$. This causes all future velocity steps to be correctly offset. We call this the leapfrog method.[1]

While this works well for the electric force, it is inaccurate for the magnetic force as it depends on velocity. We want to calculate the velocity as

$$\mathbf{v}(t + \frac{\Delta t}{2}) = \mathbf{v}(t - \frac{\Delta t}{2}) + \frac{q}{m}(\mathbf{E} + \mathbf{v_a}(t_i) \times \mathbf{B})) \tag{3}$$

where

$$\mathbf{v_a}(t) = \frac{\mathbf{v}(t + \frac{\Delta t}{2}) + \mathbf{v}(t - \frac{\Delta t}{2})}{2} \tag{4}$$

The Boris method is a computationally efficient solution to this equation. First we define new vectors such that

$$\mathbf{v}(t - \frac{\Delta t}{2}) = \mathbf{v}^{-} - \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} \tag{5}$$

$$\mathbf{v}(t + \frac{\Delta t}{2}) = \mathbf{v}^{+} + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} \tag{6}$$

Using this, Equation 4 and Equation 3 we find

$$\frac{\mathbf{v}^{+} - \mathbf{v}^{-}}{\Delta t} = \frac{q}{2m}\left(\mathbf{v}^{+} + \mathbf{v}^{-}\right) \times \mathbf{B} \tag{7}$$

By geometric arguments[2] we can define additional vectors such that

$$\mathbf{v\prime} = \mathbf{v}^{-} + \mathbf{v}^{-} \times \mathbf{t} \tag{8}$$

$$\mathbf{v}^{+} = \mathbf{v}^{-} + \mathbf{v\prime} \times \mathbf{s} \tag{9}$$

where

$$\mathbf{t} \equiv \frac{q\mathbf{B}}{m}\frac{\Delta t}{2} \tag{10}$$

$$\mathbf{s} \equiv \frac{2\mathbf{t}}{1 + t^2} \tag{11}$$

Thus we can proceed along the chain

$$\mathbf{v}(t - \frac{\Delta t}{2}) \rightarrow \mathbf{v}^{-} \rightarrow \mathbf{v\prime} \rightarrow \mathbf{v}^{+} \rightarrow \mathbf{v}(t + \frac{\Delta t}{2}) \tag{12}$$

In effect, we apply half the electric force, rotate the velocity vector due to the magnetic field and then apply the rest of the electric force.[3]

Using this method to update velocity in our leapfrog method gives us the Boris method. It is efficient to compute and remains accurate for long periods.

## II. Results

From the plots it can be seen that the numerical and analytical methods agree within a reasonable margin over 1000 iterations with timestep $3 \times 10^{-11}$ s.

## References

[1] *Leapfrog particle push (velocity integration)*, Particle In Cell Consulting, (July 10, 2011) https://www.particleincell.com/2011/velocity-integration/ (visited on 04/02/2022).

[2] C. K. Birdsall and A. B. Langdon, *Plasma physics via computer simulation* (Adam Hilger, New York, 1991), pp. 77–82.

[3] *Particle push in magnetic field (boris method)*, Particle In Cell Consulting, (July 10, 2011) https://www.particleincell.com/2011/vxb-rotation/ (visited on 04/02/2022).

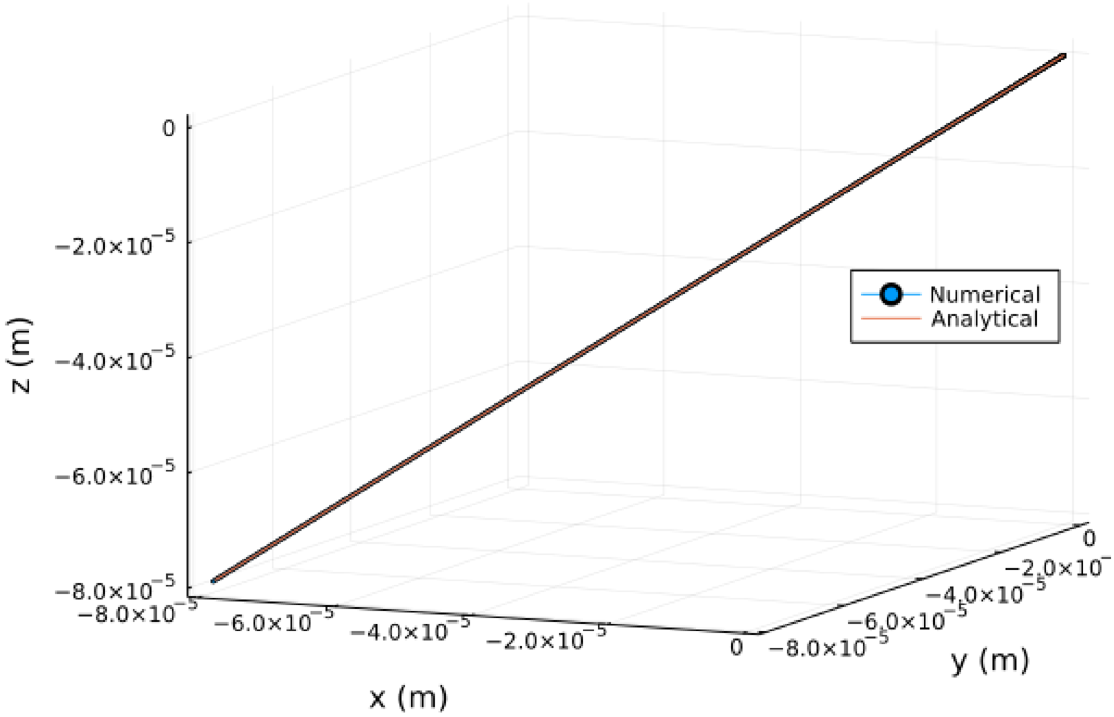


**Figure 1:** $\mathbf{E} = (1, 1, 1)\,\mathrm{V\,m^{-1}}$, *zero magnetic field and initial velocity, initial position at origin.*

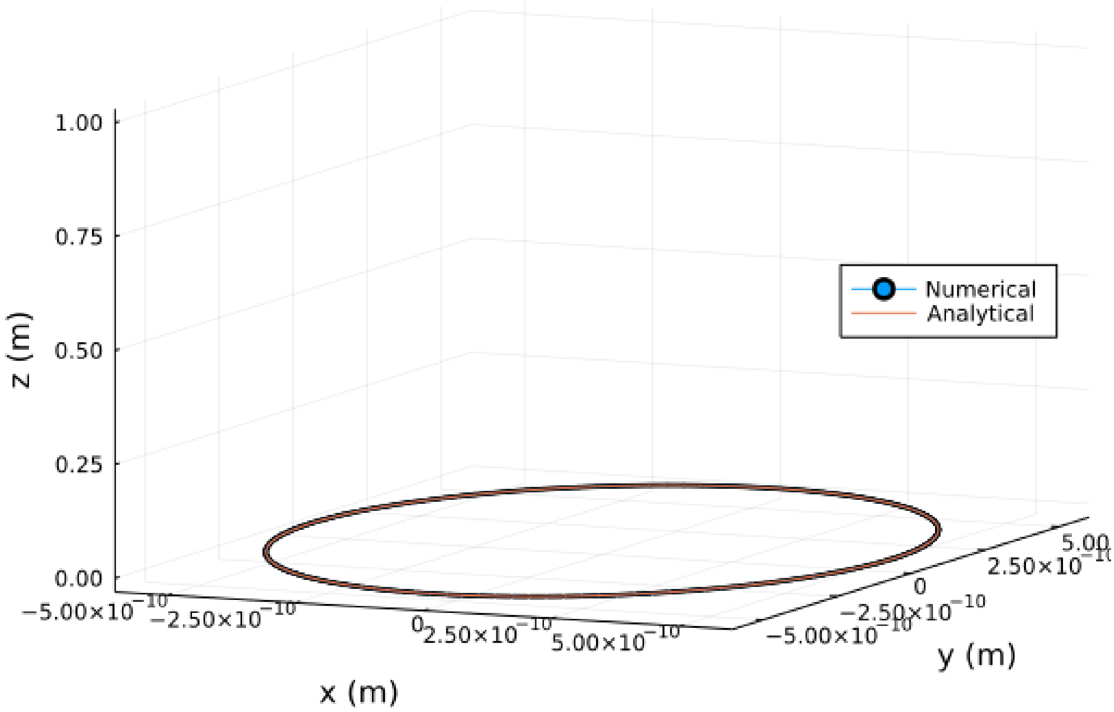


**Figure 2:** $\mathbf{B} = (0, 0, 0.01)\mathrm{T}$, *zero electric field, initial velocity* $(0, 1, 0)\mathrm{m}$, *initial position* $(5.686017478152309 \times 10^{-10}, 0, 0)\mathrm{m}$.

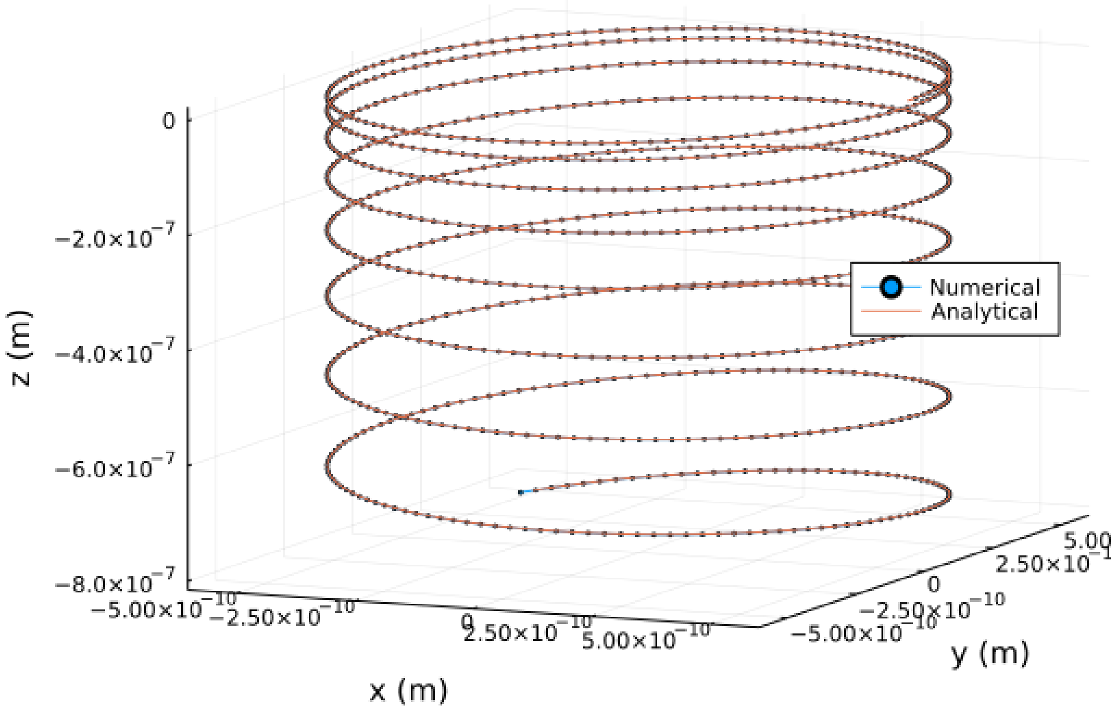


**Figure 3:** $\mathbf{E} = (0, 0, 0.01)\,\mathrm{V\,m^{-1}}$, $\mathbf{B} = (0, 0, 0.01)\,\mathrm{T}$, *initial velocity* $(0, 1, 0)\mathrm{m}$, *initial position* $(5.686017478152309 \times 10^{-10}, 0, 0)\mathrm{m}$.

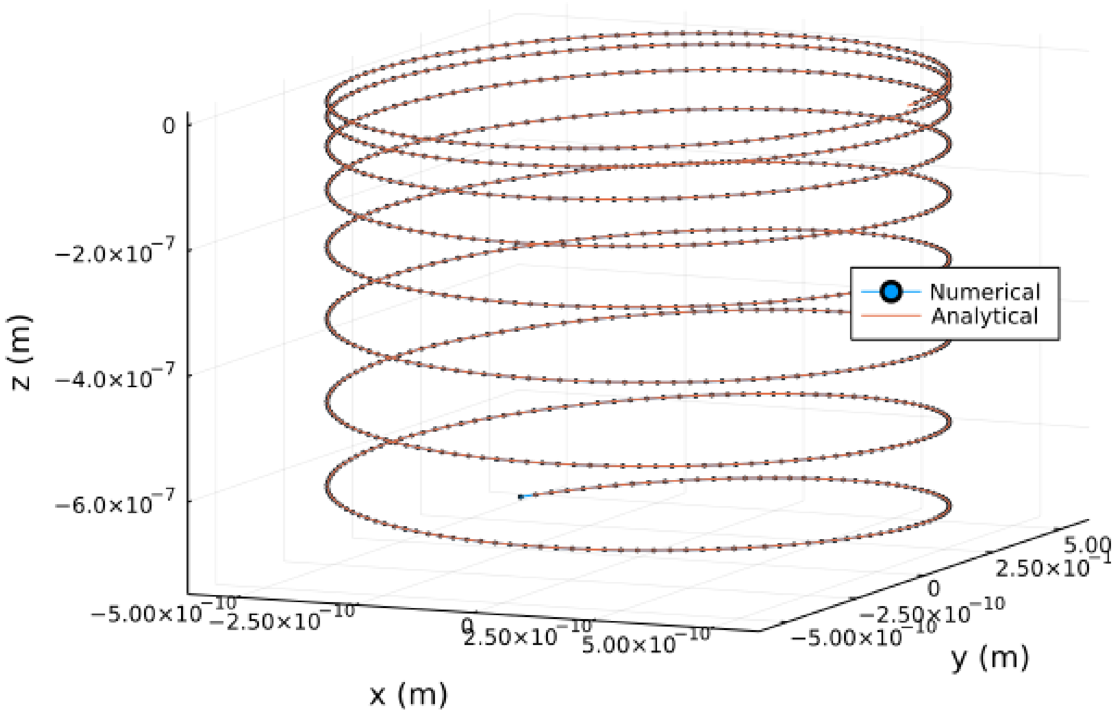


**Figure 4:** $\mathbf{E} = (0, 0, 0.01)\,\mathrm{V\,m^{-1}}$ *when* $z > -4 \times 10^{-7}\,\mathrm{m}$, $\mathbf{B} = (0, 0, 0.01)\,\mathrm{T}$, *initial velocity* $(0, 1, 0)\mathrm{m}$, *initial position* $(5.686017478152309 \times 10^{-10}, 0, 0)\mathrm{m}$.