# ECE 310

# Digital Signal Processing

Spring, 2021, ZJUI Campus

# Lecture 18

## Topics:

- ✓ Analog-to-digital (A/D) converter (Shannon sampling theorem)

## Educational Objectives:

- ✓ Understand the input-output relationship of an A/D converter in time domain
- ✓ Understand the input-output relationship of an A/D converter in frequency domain
- ✓ Understand Nyquist sampling criterion
- ✓ Understand aliasing artifact

# Analog-to-Digital Convertor

$$x_a(t) \longrightarrow \boxed{\text{A/D}} \longrightarrow x[n]$$

$$x[n] = x_a(nT)$$

$$X_d(\omega) = \frac{1}{T} \sum_{l=-\infty}^{\infty} X_a\left(\frac{\omega + 2l\pi}{T}\right)$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n] \sin c\left(\frac{t - nT}{T}\right)$$

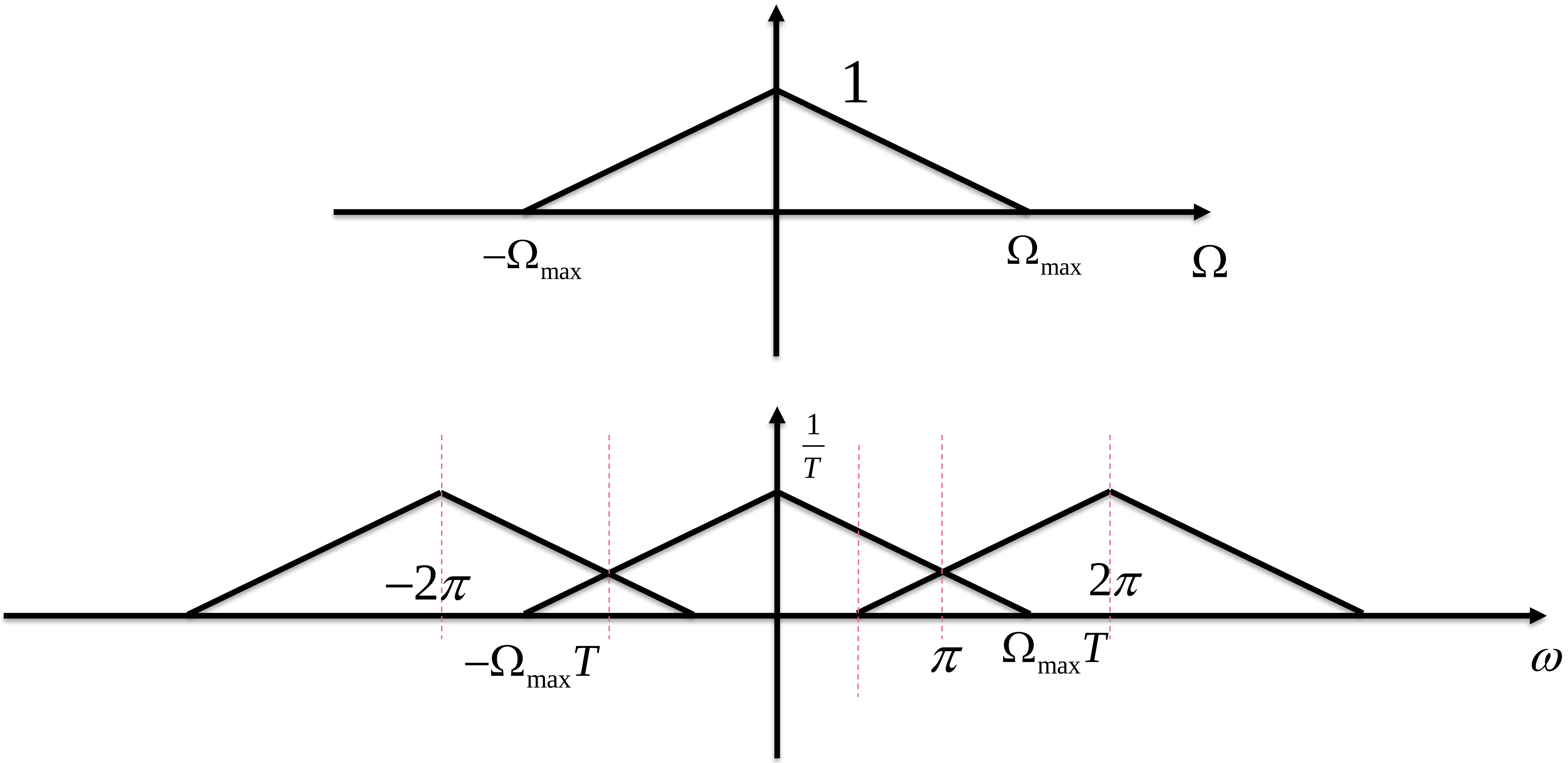
# Nyquist Sampling Criterion

$$X_d(\omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X_a\left(\frac{\omega + 2l\pi}{T}\right)$$

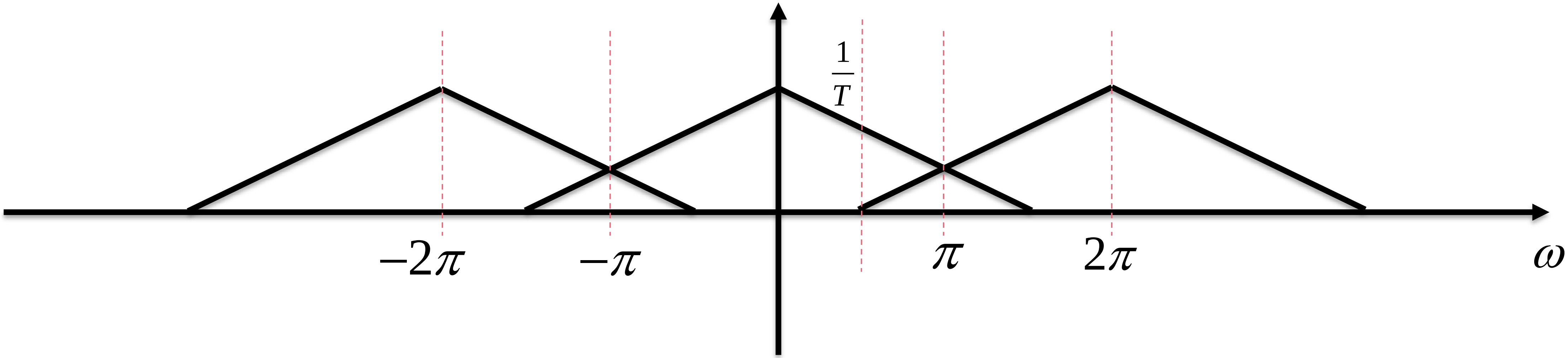
- $x_a(t)$ is bandlimited, $|X_a(\Omega)| = 0, \quad \text{for } |\Omega| > \Omega_{\max}$
- Sampling is fast enough: $\Omega_s > 2\Omega_{\max}$

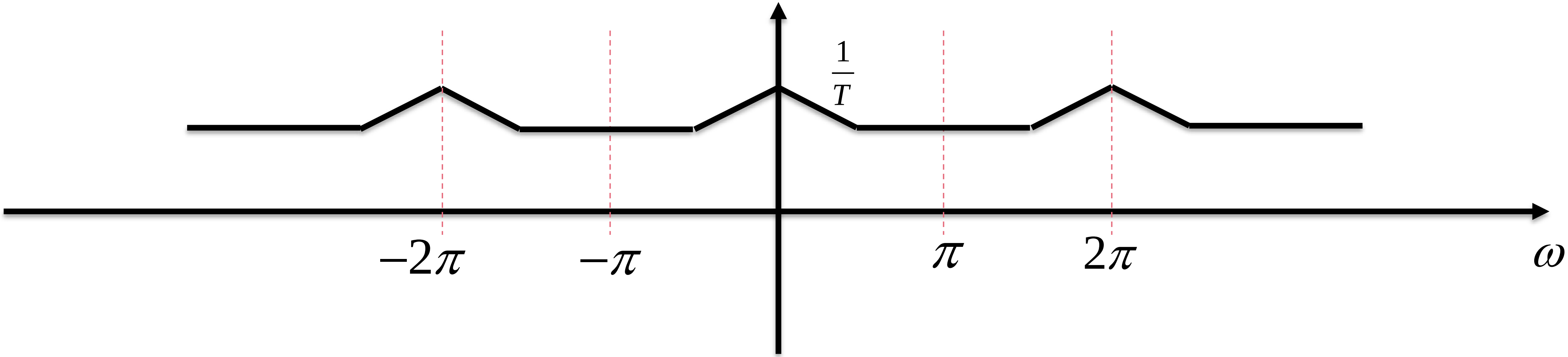
$$\Omega_s = \frac{2\pi}{T} > 2\Omega_{\max} \rightarrow T < \frac{\pi}{\Omega_{\max}} = \frac{2\pi}{BW}$$

# Aliasing Effect



# Aliasing Effect



# Example

$$x_a(t) = \cos(2\pi t)$$

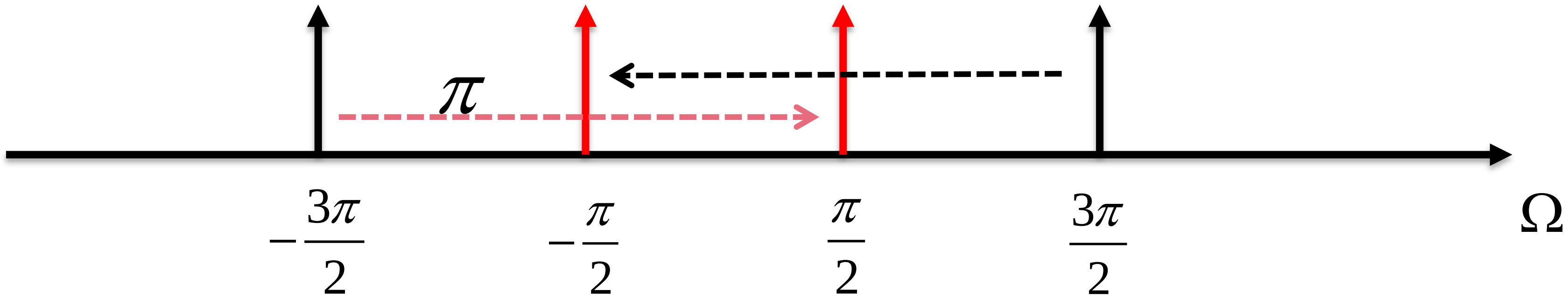

$$T < \frac{\pi}{\Omega_{\max}} = \frac{\pi}{2\pi} = \frac{1}{2}$$

# Example

Let $T = \frac{1}{4}$ $\qquad x[n] = x(nT) = \cos(2\pi\frac{n}{4}) = \cos(\frac{\pi}{2}n)$

Let $T = \frac{3}{4}$ $\qquad x[n] = x_a(nT) = \cos(2\pi\cdot\frac{3}{4}n) = \cos(\frac{3\pi}{2}n) \neq \cos(\frac{\pi}{2}n)!$

(Violate Nyquist)



# Derivation of the Frequency Relationship

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega nT}\,d\Omega\right)e^{-j\omega n}$$

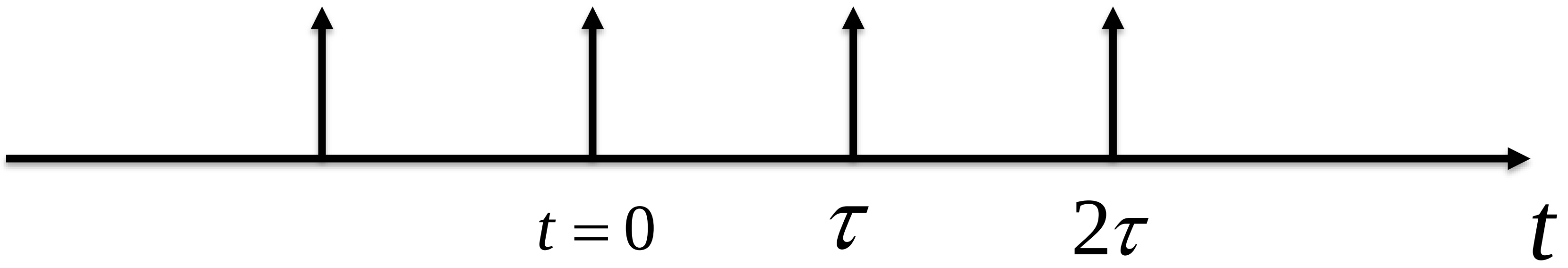
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)\sum_{n=-\infty}^{\infty} e^{jn(\Omega T-\omega)}d\Omega$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} X_a(\Omega)\sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{\omega+2k\pi}{T}\right) d\Omega$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega+2k\pi}{T}\right)$$

# Derivation of the Frequency Relationship

$$\sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{\tau}t} = \tau \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$



# Derivation of the Frequency Relationship

$$\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

$$\Omega_0 = \frac{2\pi}{\tau}$$

$$c_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \delta(t) e^{-jn\Omega_0 t} dt = \frac{1}{\tau}$$

# Derivation of the Frequency Relationship

$$\sum_{n=-\infty}^{\infty} e^{jn(\Omega T-\omega)} = \sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{\frac{2\pi}{T}}(\Omega-\frac{\omega}{T})}, \textit{ where } \frac{2\pi}{T} \to \tau$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\Omega - \frac{\omega}{T} - n\frac{2\pi}{T})$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\Omega - \frac{\omega+2n\pi}{T})$$