ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 24

Topics:

- ✓ Fast convolution using FFT

Educational Objectives:

- ✓ Understand the difference between circular convolution and linear convolution
- ✓ Under the procedure of using FFT for fast convolution
- ✓ Understand why it works

Fast Linear Convolution Using FFT

- Linear Convolution

$$\{x_n\}_{n=0}^{N-1} * \{h_n\}_{n=0}^{M-1} = \{y_n\}_{n=0}^{L-1}, \qquad L = N + M - 1$$

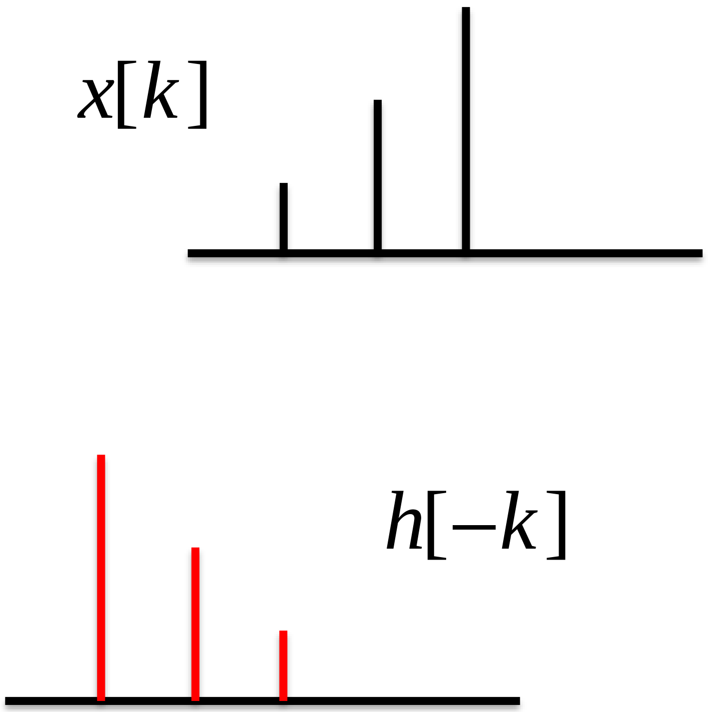
- Circular Convolution

$$\{x_n\}_{n=0}^{N-1} \circledast \{h_n\}_{n=0}^{N-1} = \{y_n\}_{n=0}^{N-1} = \sum_{k=0}^{N-1} x_k h_{<n-k>_N}$$

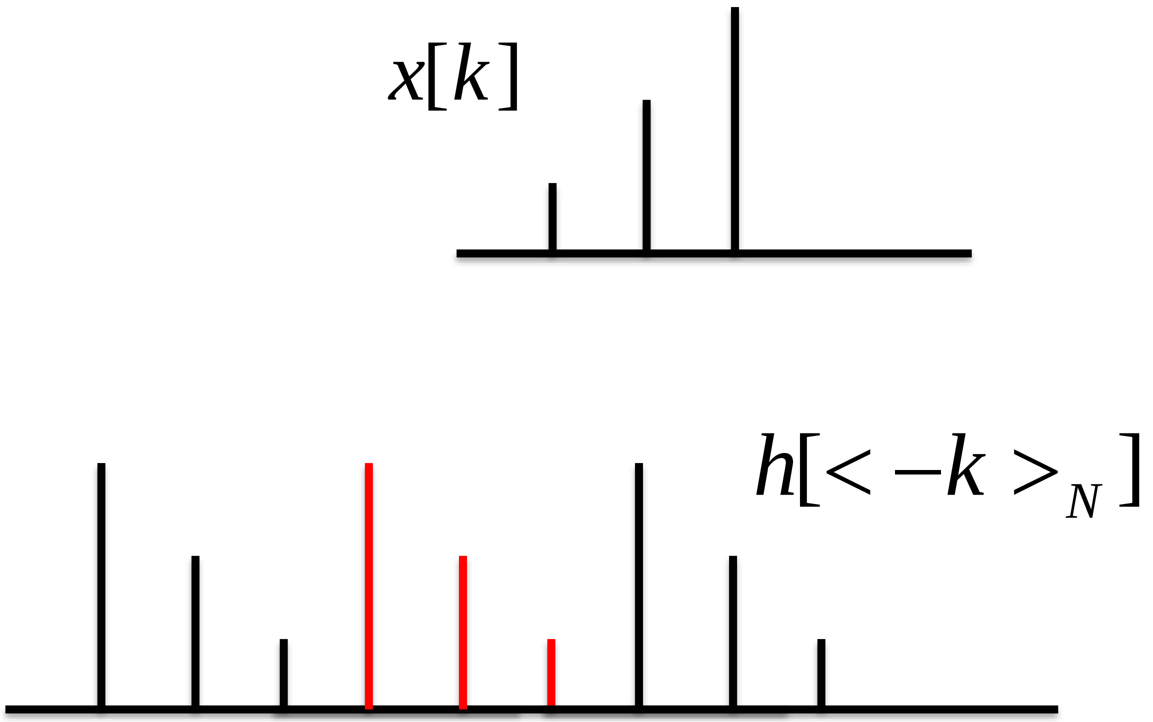
Difference between Linear and Circular Convolution

$x[n] = \{1,\ 2,\ 3\}$ $\qquad$ $h[n] = \{1,\ 2,\ 3\}$

Linear Convolution



Circular Convolution



Difference between Linear and Circular Convolution

Linear Convolution

$$y[n]=0,\ n<0$$

$$y[0]=1\times 1=1$$

$$y[1]=1\times 2+2\times 1=4$$

$$y[2]=1\times 3+2\times 2+3\times 1=10$$

$$y[3]=2\times 3+3\times 2=12$$

$$y[4]=3\times 3=9$$

$$y[n]=0,\ n\geq 5$$

$$y[n]=\{1,4,10,12,9\}$$

Circular Convolution

$$y[0]=1\times 1+2\times 3+3\times 2=13$$

$$y[1]=1\times 2+2\times 1+3\times 3=13$$

$$y[2]=1\times 3+2\times 2+3\times 1=10$$

$$y[3]=1\times 1+2\times 3+3\times 2=13=y[0]$$

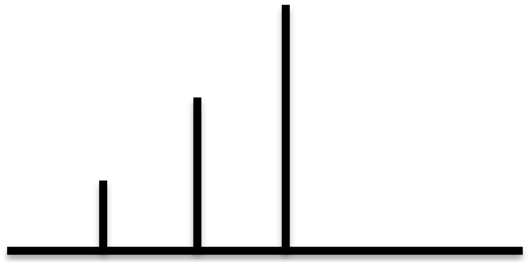
$$y[n]=\{13,13,10\}$$

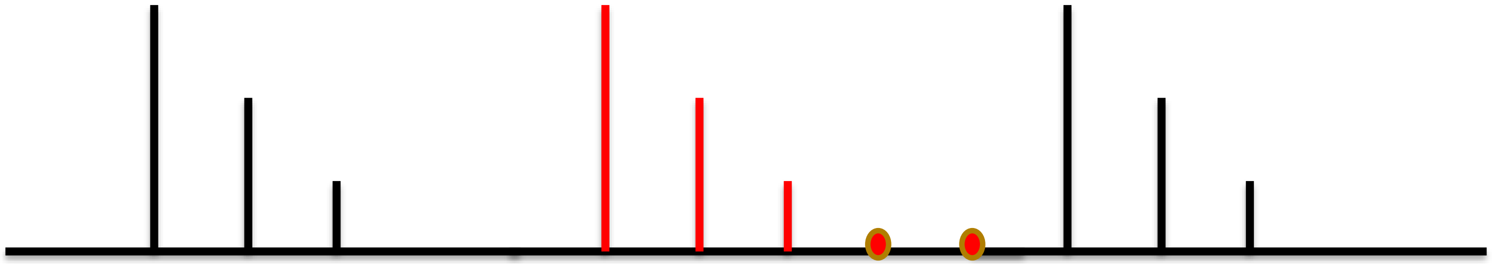
Difference between Linear and Circular Convolution

- Now consider $x[n] = h[n] = \{1, 2, 3, 0, 0\}$

  $x[n] \circledast h[n] = ?$

  $x[n] * h[n]$ stays the same





Difference between Linear and Circular Convolution

$$y[0] = 1\times1 = 1$$

$$y[1] = 1\times2 + 2\times1 = 4$$

$$y[2] = 1\times3 + 2\times2 + 3\times1 = 10$$

$$y[3] = 2\times3 + 3\times2 = 12$$

$$y[4] = 3\times3 = 9$$

$$y[n] = \{1, 4, 10, 12, 9\}$$

Therefor, in this case, we have (why?):

$$x[n] \circledast h[n] = x[n] * y[n],\ n = 0, \ldots, N-1$$

Fast Linear Convolution Using FFT

- Implication

$$y[n] = x[n] * h[n]$$

$$\underbrace{y[n]}_{N+M-1} = \underbrace{x[n]}_{N} * \underbrace{h[n]}_{M}$$

- Step 1: zero-padding to $L \geq M + N - 1$

$$x[n] = \begin{cases} x[n], 0 \leq n \leq N-1 \\ 0, N \leq n \leq L-1 \end{cases}$$

add L-N zeros

$$h[n] = \begin{cases} h[n], 0 \leq n \leq M-1 \\ 0, M \leq n \leq L-1 \end{cases}$$

add L-M zeros

Fast Linear Convolution Using FFT

- Step 2:

$$x[n] \xrightarrow{FFT} X[m]$$

$$h[n] \xrightarrow{FFT} H[m]$$

- Step 3:

$$Y[m] = X[m]H[m]$$

- Step 4:

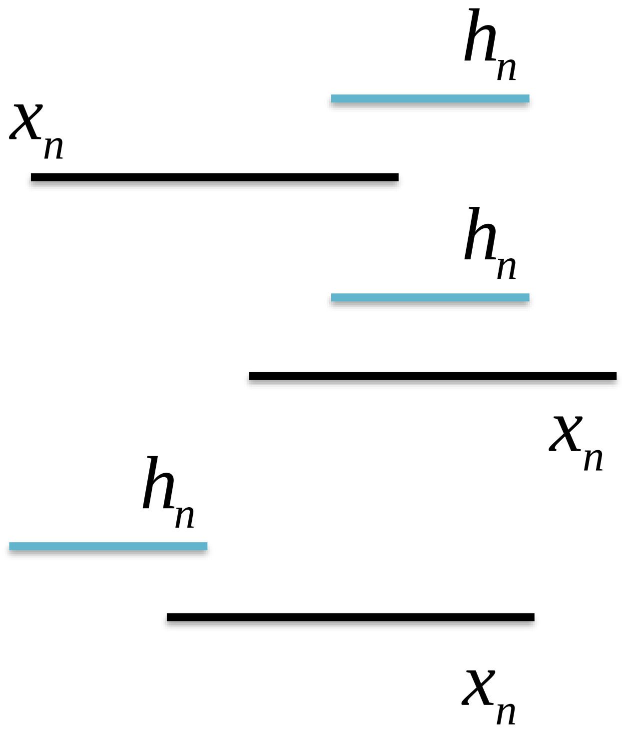
$$y[n] = FFT^{-1}\{Y[m]\}$$

$$\downarrow$$

$$y[n]$$

Example

$$\{x_n\}_{n=0}^{7000} * \{h_n\}_{n=0}^{1100}$$



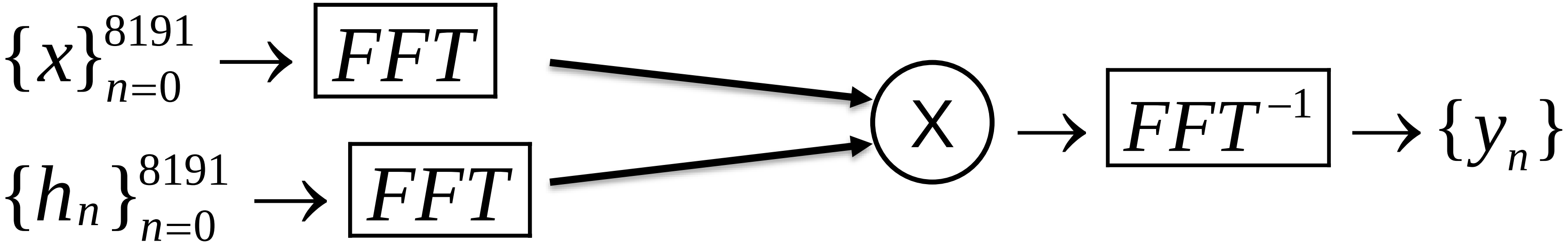
$$\#\, of\;\; MA = 1100 + 1099 + \ldots + 1 = \frac{1100 \times 1101}{2}$$

$$\#\, of\;\; MA = 1101 \times (7001 - 1100)$$

$$\#\, of\;\; MA = \frac{1100 \times 1101}{2}$$

$$total = 7,708,101$$

Example



complex MAs $= 3(N\log_2 N) + N = 3\times(8192\times 13) + 8192 = 327,680$

real MAs $= 4\times 327,680 = 1,310,720$