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ECE 310 Digital Signal Processing

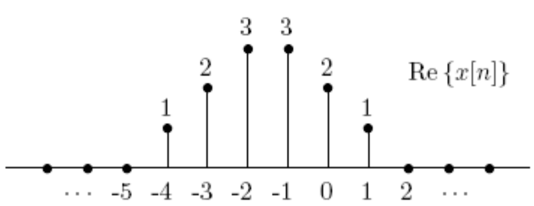
Homework 5

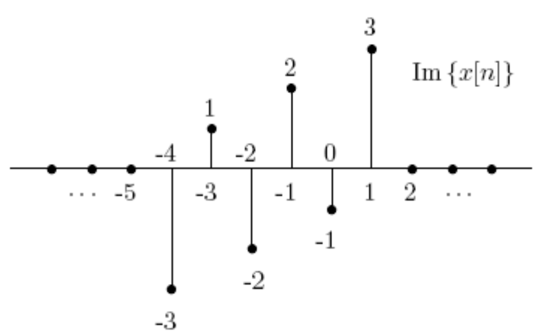
Prof. Zhi-Pei Liang **Due: March 19, 2021**

$\int_{-\infty}^{\infty} \phi(t)\,\delta^{n}(t-t_0)\,dt$

$= (-1)^{n}\phi^{n}(t_0)$

1. Evaluate the following integrals:
   (a) $\int_{-\infty}^{\infty}(t^2+5t-1)\delta(t)dt =$
   (b) $\int_{1}^{\infty}(t^2+5t-1)\delta(t)dt =$
   (c) $[e^{-t}u(t)] * \delta(5t-15) =$, where $u(t)$ is a unit step function.
2. Determine the Fourier transform of the following functions:
   (a) $\delta(2t-3)$
   (b) $\sin(\Omega_0 t + \phi_0)$, where Ω_0 and ϕ_0 are known real numbers.
   (c) $u(t) - u(t-T)$, where T is a constant.
3. Compute the discrete-time Fourier transform (DTFT) of the following sequence. $x[n] = \alpha^n \sin(\omega_0 n)u[n]$, where α and ω_0 are real constants with $|\alpha| < 1$.
4. Let $X_d(\omega)$ denote the DTFT of the complex valued signal $x[n]$, where the real and imaginary parts of $x[n]$ are given below. Perform the following calculations **without** explicitly evaluating $X_d(\omega)$.
   a) Evaluate $X_d(0)$
   b) Evaluate $X_d(\pi)$
   c) Evaluate $\displaystyle\int_{-\pi}^{\pi} X_d(\omega)d\omega$
   d) Determine and sketch the signal whose DTFT is $X_d^*(-\omega)$





5. Let $x[n]$ be an arbitrary sequence, not necessarily real-valued, with DTFT $X_d(\omega)$. Express the DTFT of the following sequences in terms of $X_d(\omega)$

   a) $x^*[n]$

   b) $x^*[-n]$

6. Consider the complex sequence $x[n] = (u[n] - u[n-N])/N$.

   a) Find closed-form expressions for $|X_d(\omega)|$ and $\angle X_d(\omega)$.

   b) For $N = 5$, plot $|X_d(\omega)|$; How will the shape of $|X_d(\omega)|$ change if N increases.

   c) For $N = 5$, plot $\angle X_d(\omega)$; How will the shape of $\angle X_d(\omega)$ change if N increases.

P1.

a) $\int_{-\infty}^{\infty} (t^2+5t-1)\,\delta(t)\,dt$

$= t^2+5t-1\big|_{t=0}$

$= -1$

b) $\int_{1}^{\infty} (t^2+5t-1)\,\delta(t)\,dt = 0$

$t=0$ is not included in the domain of t

c) $y(t) = [e^{-t}u(t)] * \delta(5t-15)$

$= \sum_{k=-\infty}^{\infty} e^{-k}u(k)\,\delta[5(t-k)-15]$

$= \sum_{k=0}^{\infty} e^{-k}\,\delta[5(t-k)-15]$

$= e^{-k}\,\delta[5(t-k)-15]\big|_{k=t-3}$

$= e^{-t+3}\,\delta[0]$

$= e^{-t+3}$

P2.

a) $\delta(2t-3)$

$\xrightarrow{FT} \int_{-\infty}^{\infty} \delta(2t-3)\,e^{-j\Omega t}\,dt$

$= \delta(2t-3)\,e^{-j\Omega t}\big|_{t=\frac{3}{2}}$

$= e^{-j\Omega\frac{3}{2}}$

$\cos\phi_0 \cos\Omega_0 t$

b) $\sin(\Omega_0 t+\phi_0) = \sin\phi_0 \sin\Omega_0 t +$

$\xrightarrow{FT} \sin\phi_0 \cdot \pi j\,(\delta(\Omega+\Omega_0) - \delta(\Omega-\Omega_0))$
$+ \cos\phi_0 \cdot \pi\,(\delta(\Omega+\Omega_0) + \delta(\Omega-\Omega_0)))$

c) $u(t) - u(t-T)$

$\xrightarrow{FT} \frac{1}{j\Omega} + \pi\delta(\Omega) - (\frac{1}{j\Omega} + \pi\delta(\Omega))\cdot e^{-j\Omega T}$

$= (\frac{1}{j\Omega} + \pi\delta(\Omega))\,(1 - e^{-j\Omega T})$

P3.

$$x[n] = \alpha^n \sin(w_0 n)\, u[n]$$

$$\xrightarrow{FT} X_d(w) = \sum_{n=-\infty}^{\infty} \alpha^n \sin(w_0 n)\, u[n]\, e^{-jwn}$$

$$\sin w_0 n \overset{F}{\longleftrightarrow} \frac{2\pi}{2j}\left(\delta(w-w_0) - \delta(w+w_0)\right)$$

$$\therefore X_d(w) = \frac{1}{2j}\left(\frac{1}{1-\alpha e^{-j(w-w_0)}} - \frac{1}{1-\alpha e^{-j(w+w_0)}}\right)$$

P4.

a) $$X_d(w) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(Re\{x[n]\} + Im\{x[n]\}\right) e^{-jwn}$$

$$X_d(0) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{0}$$

$$= \sum_{n=-\infty}^{\infty} x[n]$$

$$= 1+2+3+3+2+1 - 3j + j - 2j + 2j - j + 3j$$

$$= 12$$

b) $$X_d(\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\pi}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot (-1)$$

$$= -12$$

c) $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(w)\, e^{jwn}\, dw$$

let $n = 0$

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(w)\, e^{0}\, dw$$

$$\therefore \int_{-\pi}^{\pi} X_d(w)\, dw = x[0] \cdot 2\pi$$

$$= (2-j)2\pi$$

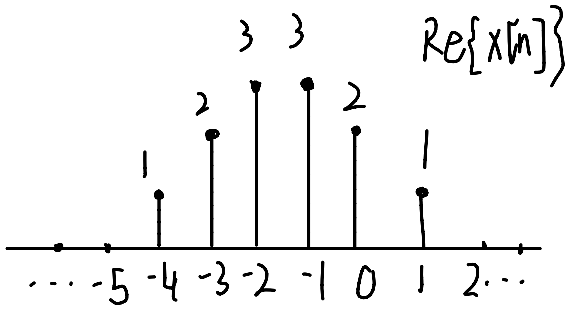
$$= 4\pi - j2\pi$$

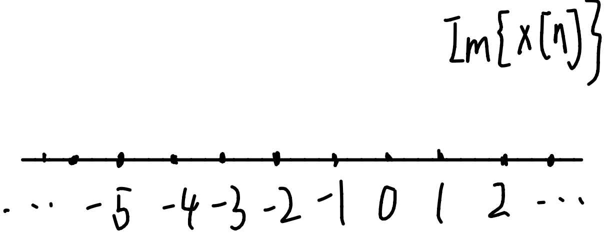
d) $$x'[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d^*(-w)\, e^{jwn}\, dw$$

when $x'[n]$ is real-valued,

$$X_d^*(-w) = X_d(w)$$

$\therefore$





P5.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} x_d(w)\, e^{jwn}\, dw$$

a) $$x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^*_d(w)\, e^{-jwn}\, dw$$

where $$\begin{cases} X_d(w) = X_{dRe}(w) + j X_{dIm}(w) \\ X^*_d(w) = X_{dRe}(w) - j X_{dIm}(w) \end{cases}$$

b) $$x^*[-n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^*_d(w)\, e^{jwn}\, dw$$

where $x^*_d(w) = X_{dRe}(w) - j X_{dIm}(w)$

P6. $$X_d(w) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

a)

$$= \sum_{n=N}^{\infty} (1 - 1/N)\, e^{-jwn} + \sum_{n=0}^{N} ((1-0)/N)\, e^{-jwn} + \sum_{n=-\infty}^{0} ((0-0)/N)\, e^{-jwn}$$

$$= \sum_{n=0}^{N} \frac{1}{N} e^{-jwn}$$

$$= \frac{1}{N} \frac{1 - e^{-jwN}}{1 - e^{-jw}}$$

$$= \frac{1}{N} \frac{e^{-j\frac{w}{2}N}\left(e^{j\frac{w}{2}N} - e^{-j\frac{w}{2}N}\right)}{e^{-j\frac{w}{2}}\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right)}$$

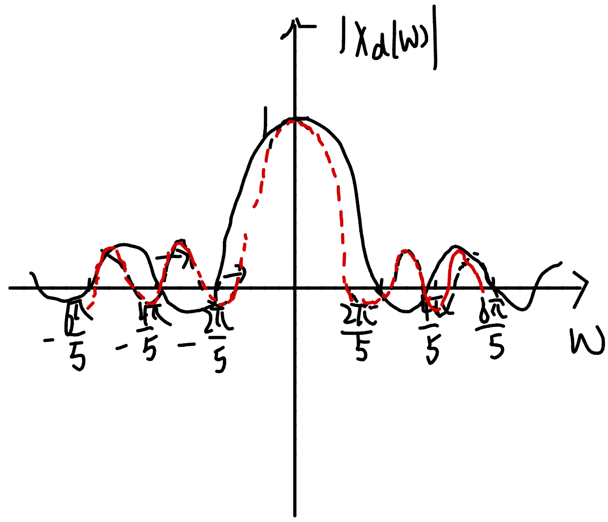
$$= e^{-j\frac{w}{2}(N-1)} \frac{1}{N} \frac{\sin(\frac{w}{2}N)}{\sin(\frac{w}{2})} \Rightarrow$$

Magnitude: $|X_d(w)| = \frac{1}{N}\left|\frac{\sin(\frac{w}{2}N)}{\sin(\frac{w}{2})}\right|$

Phase: $\angle X_d(w) = -\frac{w}{2}(N-1)$

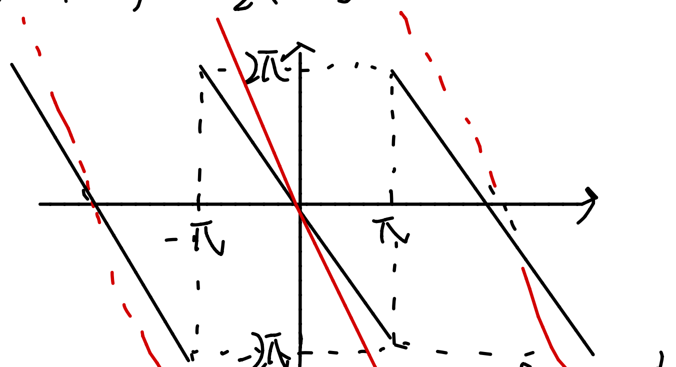
b) $N = 5$,

$$|X_d(w)| = \frac{1}{N}\left|\frac{\sin(\frac{5}{2}w)}{\sin(\frac{w}{2})}\right|$$



As N increase, the shape of $|X_d(w)|$ will be more compact to the y-axis and its intersection points with w-axis will be closer to the y-axis. (Looked as the red dotted line shows)

c) $\angle X_d(w) = -\frac{w}{2}(5-1) = -2w$



As N increase, the inclination of $\angle X_d(w)$'s shape will be larger. (As red dotted line shows)