UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing

Homework 12

Professor Z.-P. Liang Due: May 14, 2021

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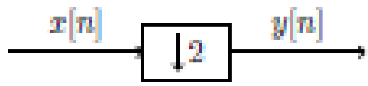
1. Consider the system shown below.



(a) Let $x[n] = \sin\left(n\frac{\pi}{2}\right)$. Determine and sketch $y[n]$.

(b) Let

$$X_d(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

show that

$$Y_d(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

Solution

(a) $x[n] = \sin\left(n\frac{\pi}{2}\right) = \{0, 1, 0, -1, 0, 1, 0, ...\}$.
After downsampling we are left with nothing:

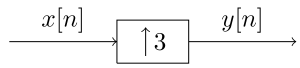
$$y[n] = \{0, 0, 0, 0, ...\}$$

Notice down-sampling is not shift invariant, $\{1, -1, 1, -1, ...\}$ is not an accepted answer.

(b) Recall downsamping in frequency domain:

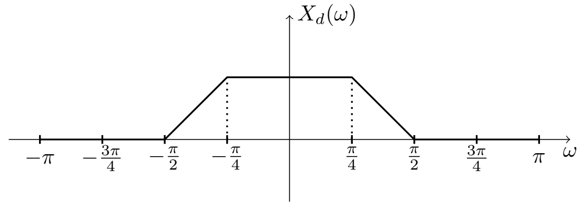
$$\begin{aligned}
Y_d(\omega) &= \frac{1}{D}\sum_{l=0}^{D-1} X_d\left(\frac{\omega-2\pi l}{D}\right) \\
&= \frac{1}{2}[X_d(\omega/2) + X_d(\omega/2-\pi)] \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{-j\omega/2}} + \frac{1}{1-\frac{1}{2}e^{-j(\omega/2-\pi)}}\right) \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{-j\omega/2}} + \frac{1}{1+\frac{1}{2}e^{-j(\omega/2)}}\right) \\
&= \frac{1}{2}\frac{1-\frac{1}{2}e^{-j\omega/2}+1+\frac{1}{2}e^{-j(\omega/2)}}{(1-\frac{1}{2}e^{-j\omega/2})(1+\frac{1}{2}e^{-j(\omega/2)})} \\
&= \frac{1}{1-\frac{1}{4}e^{-j\omega}}
\end{aligned}$$

2. Consider the system shown below:



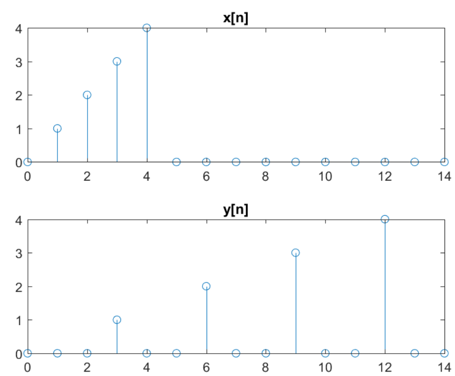
(a) Let $x[n] = n(u[n] - u[n-5])$. Determine and sketch $y[n]$.

(b) Let $X_d(\omega)$ be as shown in the figure below. Determine and sketch $Y_d(\omega)$.



Solution:

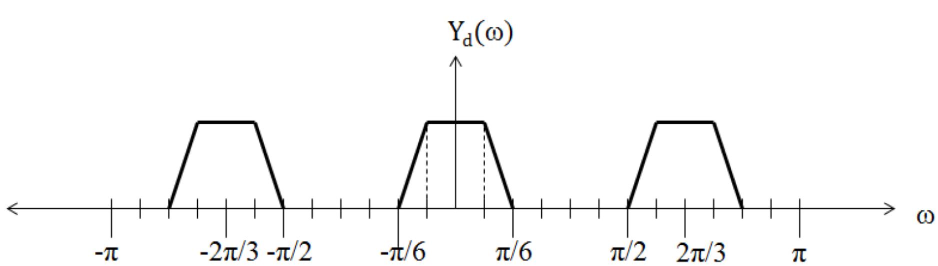
(a) See figure for $x[n]$ and $y[n]$



(b) Recall upsampling in the frequency domain:

$$Y_d(\omega) = X_d(I\omega)$$

We just have to shrink the whole axis by 3. Note that since $X_d(\omega)$ is 2π periodic, the copies at 2π's are also shrank into the $[-\pi, \pi]$ range of $Y_d(\omega)$



3. Let

$$y[n] = \begin{cases} x[n/M], & n = \ell M, \text{with integer} \ell \\ 0, & \text{otherwise} \end{cases}$$

Show that $Y_d(\omega) = X_d(M\omega)$.

Solution:

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n/m]e^{-j\omega n} = \sum_{\hat{n}=-\infty}^{\infty} x[\hat{n}]e^{-jM\hat{n}\omega} = X_d(M\omega)$$

where $\hat{n} = n/M$

4. Let $y[n] = x[Ln]$. Show that

$$Y_d(\omega) = \frac{1}{L}\sum_{\ell=0}^{L-1} X_d\left(\frac{\omega - 2\ell\pi}{L}\right)$$

Solution:

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \tilde{x}[Ln]e^{-j\omega n} = \sum_{\hat{l}=-\infty}^{\infty} \tilde{x}[\hat{l}]e^{-j\frac{\omega}{L}\hat{l}} = \tilde{X}_d(\frac{\omega}{L})$$

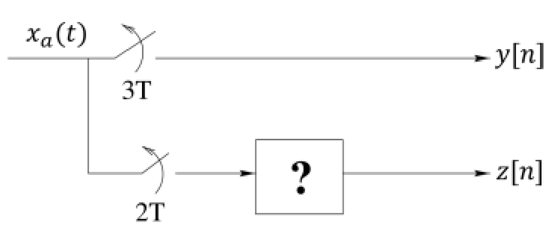
where

$$\tilde{x}[n] = x[n]p[n] = x[n]\frac{1}{L}\sum_{l=0}^{L-1} e^{j\frac{2\pi}{L}\ln}$$

$$\tilde{X}_d(\omega) = \sum_{n=-\infty}^{\infty} \tilde{x}[n]e^{-j\omega n} = \frac{1}{L}\sum_{l=0}^{L-1}\sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega - \frac{2\pi l}{L})n} = X_d(\omega - l\frac{2\pi}{L})$$

$$\Rightarrow Y_d(\omega) = \frac{1}{L}\sum_{l=0}^{L-1} X_d(\frac{\omega - 2\pi l}{L})$$

5. Consider the following system consisting of two synchronized ideal A/D convertors. Assume that the input analog signal $x_a(t)$ is bandlimited to $\Omega_0 = \pi/(3T)$. Design a digital rate conversion subsystem marked with "?" using down-sampler(s), up-sampler(s), and digital filter(s) as necessary such that $y[n] = z[n]$. Draw a block diagram and determine all the essential parameters of the subsystem.



Solution:

$$\rightarrow \boxed{\uparrow 2} \rightarrow \boxed{LPF} \rightarrow \boxed{\downarrow 3} \rightarrow$$

where

$$T_1 = 1.5T_2 \Rightarrow \alpha = \frac{L}{M} = \frac{3}{2} \Rightarrow L = 3, M = 2$$

LPF