ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 15

Topics:

- ✓ Discrete-time Fourier transform (DTFT)

Educational Objectives:

- ✓ Understand the definition of DTFT
- ✓ Understand how to calculate inverse DTFT
- ✓ Understand key DTFT pairs
- ✓ Understand relationship between DTFT and z-transform

Discrete-time Fourier Transform (DTFT)

Given $x[n]$, its DTFT is defined as

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Note

$$X(\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt$$

So, $X_d(\omega)$ can be viewed as an approximation of $X(\Omega)$

$X_d(\omega)$ is a complex function

$$\begin{cases} |X_d(\omega)| : \text{magnitude spectrum} \\ \arg X_d(\omega) \; or \; \angle X_d(\omega) : \text{phase spectrum} \end{cases}$$

Periodicity of DTFT

Consider

$$X_d(\omega + 2k\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2k\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X_d(\omega)$$

$X_d(\omega)$ is a periodic function of 2π!

$X_d(\omega)$ is specified for $-\pi < \omega < \pi$ (base band)

How $\Omega \rightarrow \omega$, we will see in “sampling” section!

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega$$

Proof:

Examples

$$x[n] = \delta[n] + \delta[n-1]$$

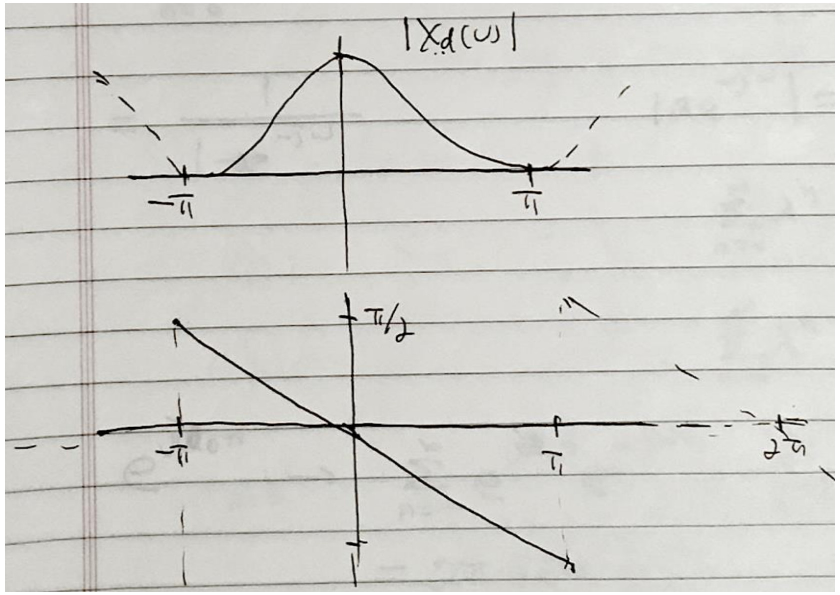
$$\mathrm{X}_d(\omega) = \sum_{n=-\infty}^{\infty} (\delta[n] + \delta[n-1])e^{-j\omega n} = 1 + e^{-j\omega}$$

$$= e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} + e^{-\frac{j\omega}{2}}\right) = e^{-\frac{j\omega}{2}}\, 2\cos(\frac{\omega}{2})$$

$$= |X_d(\omega)|e^{j\angle X_d(\omega)}$$

So: $|X_d(\omega)| = 2\cos\left(\frac{\omega}{2}\right), -\pi < \omega < \pi$

$\angle X_d(\omega) = -\frac{\omega}{2} \qquad , -\pi < \omega < \pi$



Examples

$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

$$\left|ae^{-j\omega}\right| = |a| < 1$$

For $|\lambda| < 1$

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$$

$$\sum_{n=0}^{N} \lambda^n = \frac{1-\lambda^{N+1}}{1-\lambda}$$

Examples

$$x[n] = e^{j\omega_0 n}$$

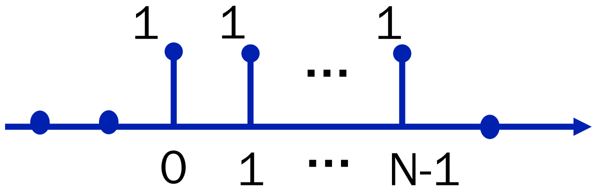
$$\mathrm{X}_d(\omega) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j(\omega-\omega_0)n}$$

$$= 2\pi\delta(\omega - \omega_0), \qquad (-\pi < \omega < \pi)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2k\pi)$$

Examples

$$x[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, \qquad\qquad\quad else \end{cases}$$

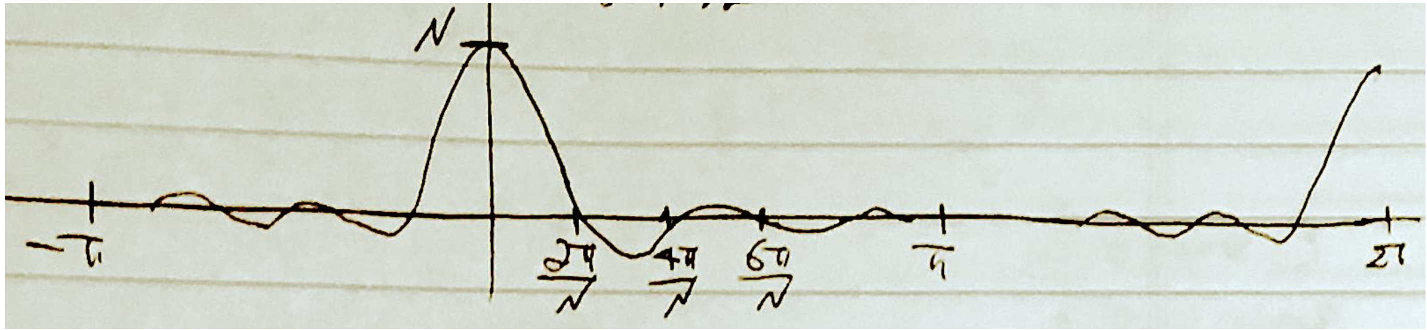


$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1 - e^{-j\omega \mathrm{N}}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega N/2}\left(e^{j\omega \mathrm{N}/2} - e^{-j\omega \mathrm{N}/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= e^{-j\frac{\omega}{2}(N-1)} \frac{2jsin(\omega N/2)}{2jsin(\omega/2)}$$

$$= \frac{sin(\omega N/2)}{sin(\omega/2)} e^{-j\frac{\omega}{2}(N-1)}$$



DTFT Pairs

$x[n]$	$\leftrightarrow$	$X_d(\omega)$	ω
1	$\leftrightarrow$	$2\pi\delta(\omega)$	$-\pi \leq \omega \leq \pi$
		$2\pi\sum\delta(\omega - 2k\pi)$	$\mathbb{R}$
$e^{j\omega_0 n}$	$\leftrightarrow$	$2\pi\delta(\omega - \omega_0)$	$-\pi \leq \omega \leq \pi$
		$2\pi\sum\delta(\omega - \omega_0 - 2k\pi)$	$\mathbb{R}$
$cos(\omega_0 n)$	$\leftrightarrow$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$	$-\pi \leq \omega \leq \pi$
		$\pi\sum(\delta(\omega - \omega_0 - 2k\pi) + \delta(\omega + \omega_0 - 2k\pi))$	$\mathbb{R}$
$sin(\omega_0 n)$	$\leftrightarrow$	$\frac{\pi}{j}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$	$-\pi \leq \omega \leq \pi$
		$\frac{\pi}{j}\sum(\delta(\omega - \omega_0 - 2k\pi) + \delta(\omega + \omega_0 - 2k\pi))$	$\mathbb{R}$

* The range of summation above is $-\infty$ to ∞

DTFT Pairs

$x[n]$	$\leftrightarrow$	$X_d(\omega)$
$\delta[n]$	$\leftrightarrow$	1
$u[n]$	$\leftrightarrow$	$\dfrac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta[\omega - 2k\pi]$
$a^n u[n]$	$\leftrightarrow$	$\dfrac{1}{1-ae^{-j\omega}}, \lvert a \rvert < 1$
$(1+n)a^n u[n]$	$\leftrightarrow$	$\dfrac{1}{(1-ae^{-j\omega})^2}, \lvert a \rvert < 1$

* Can be derived from the z-transform pairs