ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 11

Topics:

- ✓ Further Discussion of BIBO Stability of LSI systems using z-transform

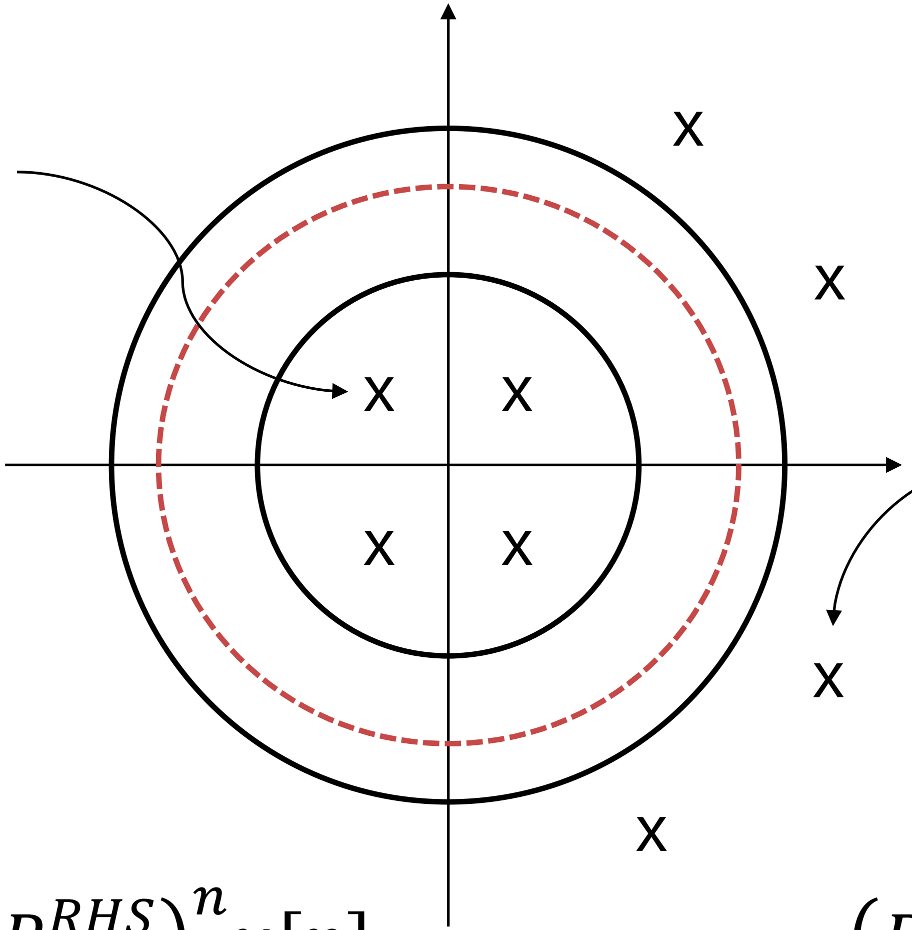
Educational Objectives:

- ✓ Know how to determine the BIBO stability of an LSI system based on $H(z)$
- ✓ Know how the boundedness of a signal is related to its poles
- ✓ Know how to drive an unstable system to give an unbounded output with a bounded input

ROC of H(z) vs Absolute Summability of *h[n]*

P_i^{RHS}

$|P_i^{RHS}| < 1$



P_i^{LHS}

$|P_i^{LHS}| > 1$

simple poles $\left(P_i^{RHS}\right)^n u[n]$ $\left(P_i^{LHS}\right)^n u[-n-1]$

multiple poles $n^l\left(P_i^{RHS}\right)^n u[n]$ $n^l\left(P_i^{LHS}\right)^n u[-n-1]$

Example 1

$$H(Z) = \frac{3 - 4Z^{-1}}{1 - 3.5Z^{-1} + 1.5Z^{-2}}$$

Specify the ROC of $H(Z)$ for the following conditions:

a) The system is stable

b) The system is causal

c) The system is anti-causal

Example 1

$$H(Z) = \frac{3Z^2 - 4Z}{(Z - \frac{1}{2})(Z - 3)}$$

$a)\quad ROC_H: \frac{1}{2} < |Z| < 3$

System is noncausal (not anti-causal)
System is stable

$b)\quad ROC_H: |Z| > 3$

System is causal
System is BIBO unstable

$c)\quad ROC_H: |Z| < 0.5$

System is noncausal (and anti-causal)
System is unstable

Example 2

$$H(Z) = \frac{Z - 2}{Z^2 + \frac{1}{16}}$$

Determine all possible ROCs with associated system properties

$$Z_{1,2} = \sqrt{-\frac{1}{16}} = \pm j\sqrt{\frac{1}{16}} = \pm j\frac{1}{4}$$

Example 2

$$H(Z) = \frac{Z-2}{(Z - j\frac{1}{4})(Z + j\frac{1}{4})}$$

a) $ROC_H: |Z| > \frac{1}{4}$

System: causal & stable

b) $ROC_H: |Z| < \frac{1}{4}$

System anti-causal & unstable

Example 3

$$H(Z) = \frac{Z}{Z^2 + j}$$

$$Z_{1,2} = \sqrt{-j} = \sqrt{e^{-j\frac{\pi}{2}}}$$

$$Z_1 = e^{-j\frac{\pi}{4}}, \qquad Z_2 = e^{+j\frac{3\pi}{4}} = -e^{-j\frac{\pi}{4}}$$

Case 1: $|Z| > 1$: causal and unstable

Case 2: $|Z| < 1$: anti-causal and unstable

Boundedness vs Pole Distributions

Causal Signals
(RHS signal) $\longrightarrow$

- ✓ Poles inside UC
- ✓ Single poles on UC

Anti-causal Signals
(LHS signal) $\longrightarrow$

- ✓ Poles outside UC
- ✓ Single poles on UC

Bounded Input → Unbounded Output

	Source instability of $H(Z)$	Bounded input
Causal:	Pole(s) outside UC	$\delta[n]$
	Double pole(s) on UC	$g[n]$
	Single pole(s) on UC	Matching single poles on UC
Anti-causal:	Pole(s) inside UC	$g[n]$
	Double pole(s) on UC	$\delta[n]$
	Single pole(s) on UC	Matching single poles on UC

Bounded Input → Unbounded Output

	Source instability of $H(Z)$	**Bound input**
Non-causal:	RHS pole(s) outside UC	$g[n]$
	LHS pole(s) inside UC	$\delta[n]$
	Double pole(s) on UC	$\delta[n]$
	Single pole(s) on UC	Matching poles on UC

Example

$$H(Z) = \frac{Z}{Z^2 + j} = \frac{Z}{(Z - e^{-j\frac{\pi}{4}})(Z + e^{-j\frac{\pi}{4}})}$$

$ROC_H\colon |Z| < 1$ unstable

$ROC_H\colon |Z| > 1$ unstable

Reason: single poles on UC

System's BIBO Stability

Case 1: $$X(Z) = -\frac{Z}{Z - e^{-j\frac{\pi}{4}}} \qquad |Z| < 1$$

$$x[n] = e^{-jn\frac{\pi}{4}} u[-n-1]$$

Case 2: $$X(Z) = \frac{Z}{Z - e^{-j\frac{\pi}{4}}} \qquad |Z| > 1$$

$$x[n] = e^{-jn\frac{\pi}{4}} u[n]$$

Example

$$H(Z) = \frac{Z}{Z-2}$$

a) $ROC_H: |Z| > 2$, unstable

$\delta[n] \longrightarrow h[n] = 2^n u[n]$, unbounded

b) $ROC_H: |Z| < 2$, stable

No unbounded input would give an unbounded output

Example

$$H(Z) = \frac{Z^2 - Z\cos(\alpha)}{(Z - e^{j\alpha})(Z - e^{-j\alpha})}, \qquad |Z| > 1$$

$$h[n] = \cos(\alpha n)u[n]$$

$$\mathrm{X}(Z) = \frac{AZ}{Z - e^{j\alpha}}$$

$$\mathrm{X}(Z) = \frac{AZ}{Z - e^{+j\alpha}}$$

$$\mathrm{X}(Z) = H(Z)$$