ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 17

Topics:

- ✓ Frequency response of stable LSI systems
- ✓ Steady-state analysis of stable LSI systems

Educational Objectives:

- ✓ Understand what is frequency response
- ✓ Understand how frequency response is related to unit pulse response and transfer function
- ✓ Understand how to calculate system’s output for sinusoidal input

Frequency Response

$$x[n] \longrightarrow \boxed{\text{LSI}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$H_d(\omega) = DTFT\{h[n]\}$$

$$Y_d(\omega) = H_d(\omega) X_d(\omega)$$

$|H_d(\omega)|$: magnitude response

$\angle H_d(\omega)$: phase response

Steady-State Analysis of LSI Systems

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega) e^{+j\omega n} d\omega$$

$$e^{j\omega_0 n} \longrightarrow \boxed{H_d(\omega)} \longrightarrow y[n]=?$$

$$y[n] = H_d(\omega_0) e^{j\omega_0 n} = | H_d(\omega_0) | e^{j(\omega_0 n + \angle H_d(\omega_0))}$$

Proof:

Sinusoidal Function Input

$$\cos(\omega_0 n) \longrightarrow \boxed{H_d(\omega)} \longrightarrow \frac{1}{2}H_d(\omega_0)e^{j\omega_0 n} + \frac{1}{2}H_d(-\omega_0)e^{-j\omega_0 n}$$

$$\sin(\omega_0 n) \longrightarrow \boxed{H_d(\omega)} \longrightarrow \frac{1}{2j}H_d(\omega_0)e^{j\omega_0 n} - \frac{1}{2j}H_d(-\omega_0)e^{-j\omega_0 n}$$

If the system is real-valued, h[n] is real-valued

$H_d(\omega)$ has Hermitian Symmetry

$$H_d(\omega) = H_d^*(-\omega)$$

$$or\ |H_d(\omega)| = |H_d(-\omega)|$$

$$\angle H_d(\omega) = -\angle H_d(-\omega)$$

Sinusoidal Function Input

$$
\begin{aligned}
y[n] &= \frac{1}{2} H_d(\omega_0) e^{j\omega_0 n} + \frac{1}{2} H_d^*(\omega_0) e^{-j\omega_0 n} \\
&= \frac{1}{2} [H_d(\omega_0) e^{j\omega_0 n} + (H_d(\omega_0) e^{j\omega_0 n})^*] \\
&= \frac{1}{2} [|H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))} + (|H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))})^*] \\
&= \frac{1}{2} |H_d(\omega_0)| [e^{j(\omega_0 n + \angle H_d(\omega_0))} + e^{-j(\omega_0 n + \angle H_d(\omega_0))}] \\
&= H_d(\omega_0) \cos(\omega_0 n + \angle H_d(\omega_0))
\end{aligned}
$$

Sinusoidal Function Input

By the same analysis

$$\sin(\omega_0 n) \longrightarrow \boxed{H_d(\omega)} \longrightarrow |H_d(\omega_0)|\sin(\omega_0 n + \angle H_d(\omega_0))$$

Real-valued

Or

$$\sin(\omega_0 n + \phi_0) \qquad |H_d(\omega_0)|\sin(\omega_0 n + \phi_0 + \angle H_d(\omega_0))$$

Examples

$$y[n] = x[n] + 2x[n-1]$$

Find y[n] due to $x[n] = \cos(\frac{\pi}{2}(n-1)) + 1 + j^n$

a) Check “real-valued”? YES!

b) Check frequency components

$$\omega_1 = \frac{\pi}{2}, \omega_2 = 0$$

$$H_d(\frac{\pi}{2}), H_d(0)$$

Example

$$H(z) = 1 + 2z^{-1};\ ROC_d :|z| > 0$$

$$H_d(\omega) = 1 + 2e^{-j\omega}$$

$$H_d(\frac{\pi}{2}) = 1 + 2e^{-j\frac{\pi}{2}} = 1 - j2 = \sqrt{5}e^{-j63.45^o}$$

$$H_d(0) = 1 + 2 = 3e^{j0}$$

$$y[n] = \sqrt{5}\cos(\frac{\pi}{2}(n-1) - 63.45^o) + 3 + \sqrt{5}e^{j(\frac{\pi}{2}n - 63.45^o)}$$

Example

$$H_d(\omega) = \cos\omega e^{j\pi cos\omega}$$

$$x[n] = \cos(\frac{\pi}{4}n + 5^o) + e^{j\frac{\pi}{2}n}$$

Not a real-valued system, $\angle H_d(\omega)$ does not have odd symmetry

$$x[n] = \frac{1}{2}(e^{j(\frac{\pi}{4}n+5^o)} + e^{-j(\frac{\pi}{4}n+5^o)}) + e^{j\frac{\pi}{2}n}$$

$$H_d(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}e^{j\pi\frac{\sqrt{2}}{2}}$$

$$H_d(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}e^{j\pi\frac{\sqrt{2}}{2}}$$

$$H_d(\frac{\pi}{2}) = 0$$

$$y[n] = \frac{1}{2}\frac{\sqrt{2}}{2}e^{j\pi\frac{\sqrt{2}}{2}}e^{j(\frac{\pi}{4}n+5^o)} + \frac{1}{2}\frac{\sqrt{2}}{2}e^{j\pi\frac{\sqrt{2}}{2}}e^{-j(\frac{\pi}{4}n+5^o)}$$

$$= \frac{\sqrt{2}}{2}e^{j\pi\frac{\sqrt{2}}{2}}\cos(\frac{\pi}{4}n + 5^o)$$

$$\neq \frac{\sqrt{2}}{2}\cos(\frac{\pi}{4}n + 5^o + \pi\frac{\sqrt{2}}{2})$$