UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing

Homework 13 Solutions

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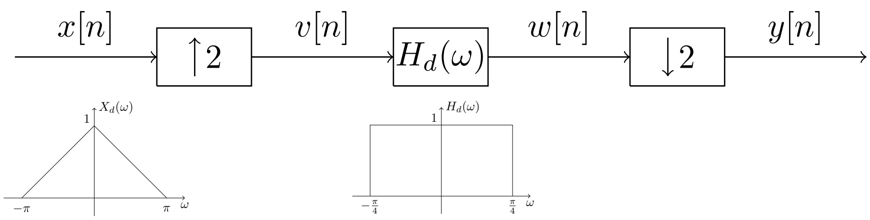
1. Consider the system shown below:



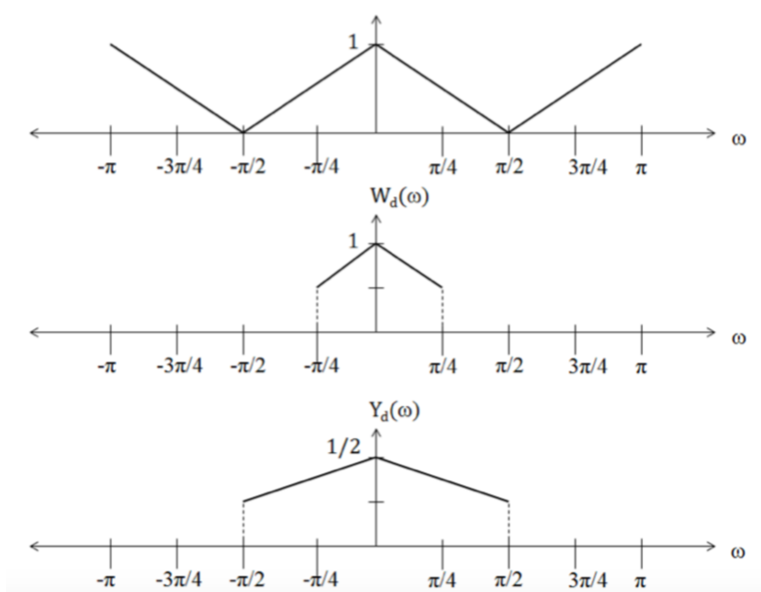
(a) If $X_d(\omega)$ is as shown, find and sketch $V_d(\omega)$, $W_d(\omega)$, and $Y_d(\omega)$. Make sure you label your axes and indicate the values of all important points.

(b) Give a close-form expression for the frequency response $H_d(\omega)$ of the entire system, from $x[n]$ to $y[n]$.

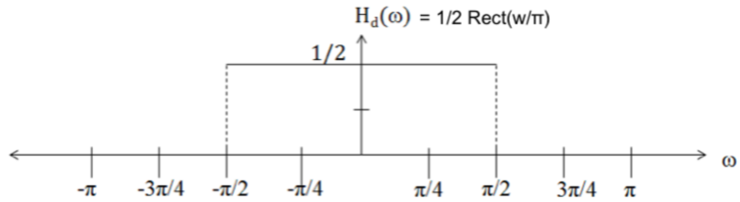
(c) Is the system LSI? why or why not?

Solution

(a) The plots can be seen below. Upsampling compresses the spectrum by a factor of 2, *including* the copies every 2π. Applying the filter is just a multiplication, and downsampling expands the spectrum by a factor of 2 (adding copies to maintain 2π-periodicity), while scaling the amplitude by a factor of $\frac{1}{2}$.



(b) In this case, by inspection, we can view the system as a single filter $H_d(\omega)$ such that $Y_d(\omega) = X_d(\omega)H_d(\omega)$. Such a filter has a frequency response given in the image below.



(c) The system is LSI. Note that it is not possible for aliasing to occur, as we upsample by 2, but immediately filter the output with an ideal LPF with a cutoff frequency of $\frac{\pi}{4}$. This means that the maximum digital frequency of the signal input to the downsampler is $\frac{\pi}{2}$, and downsampling expands the spectrum by a factor of 2, so the maximum digital frequency of $y[n]$ is π.

Furthermore, we were able to find a closed-form expression for the system's frequency response when the input contained all possible frequencies. Therefore, yes, the system is LSI.

2. A speech signal $x_a(t)$ is assumed to be bandlimited to 12kHz. It is desired to filter this signal with a bandpass filter that will pass the frequencies between 300Hz and 6kHz by using a digital filter $H_d(\omega)$ sandwiched between an A/D and an ideal D/A.
   (a) Determine the Nyquist sampling rate for the input signal.
   (b) Sketch the frequency response $H_{d,1}(\omega)$ for the necessary discrete-time filter, when sampling at the Nyquist rate.
   (c) Find the largest sampling period T for which the A/D, digital filter response ($H_{d,2}(\omega)$), and D/A can perform the desired filtering function. (Hint: some amount of aliasing may be permissible during A/D conversion for this part.)
   (d) For the system using T from part (c), sketch the necessary $H_d(\omega)$.

Solution

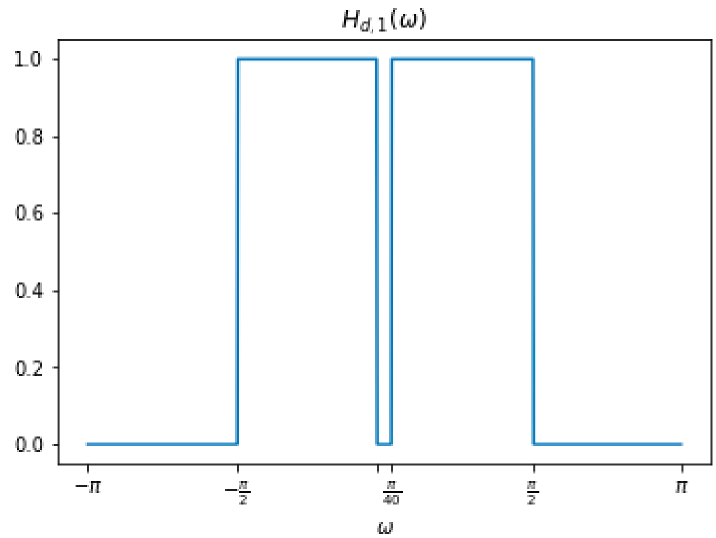
(a) The Nyquist rate is just twice the maximum frequency present in the signal. Since we're told that $x_a(t)$ is bandlimited to 12 kHz, this tells us that

$$\boxed{f_{nyquist} = 24kHz \rightarrow T_{max} = \frac{1}{24000} \text{ s}}$$

(b) Using $\omega = \Omega T$, and the Nyquist rate found in (a), we can convert the required analog frequencies to digital ones:

$$f_{min} = 300 \text{ Hz} \rightarrow \Omega_{min} = 600\pi \text{ rad/sec} \rightarrow \omega_{min} = \frac{600\pi}{24000} = \frac{\pi}{40}$$
$$f_{max} = 6000 \text{ Hz} \rightarrow \Omega_{max} = 12000\pi \text{ rad/sec} \rightarrow \omega_{max} = \frac{12000\pi}{24000} = \frac{\pi}{2}$$

This gives the filter shown below.



(c) Note that sub-Nyquist sampling might be possible here, as we don't need to pass the entire signal through the filter; only the components up to 6 kHz, or half the maximum frequency of the signal. This means that we don't care what the response is beyond the "halfway mark," as long as no aliasing occurs below that frequency. This allows us to find an even lower sampling rate that will still result in the desired filter.

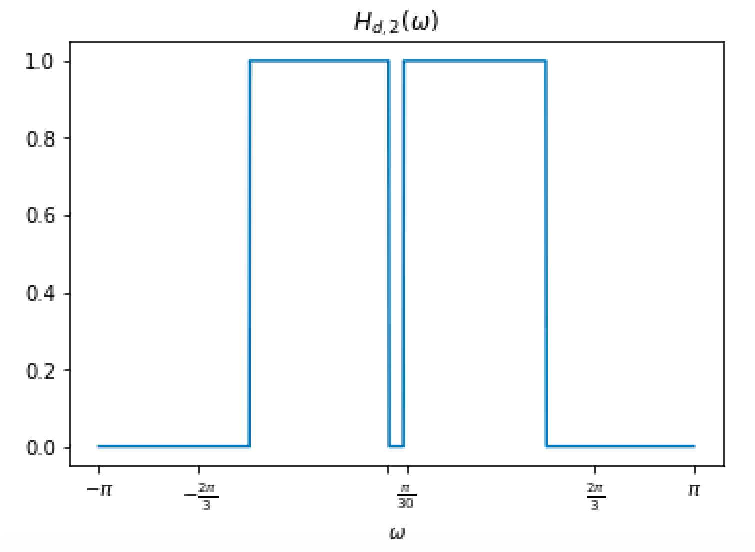
Since we only care about frequencies up to 12000π in the analog domain, that corresponds to $12000\pi T$ in the digital domain. However, the entire signal will extend up to $24000\pi T$, meaning the first aliased component will occur at $2\pi - 24000\pi T$. Therefore, for proper filter operation, we require that the first aliased component occurs after the maximum frequency we need to pass through; we need

$$12000\pi T \leq 2\pi - 24000\pi T \rightarrow \boxed{T \leq \frac{1}{18000} \text{ s}}$$

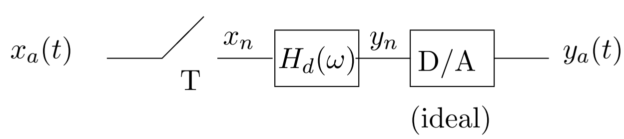
(d) Under this new rate, we can again convert the required analog frequencies to digital:

$$f_{min} = 300 \text{ Hz} \rightarrow \Omega_{min} = 600\pi \text{ rad/sec} \rightarrow \omega_{min} = \frac{600\pi}{18000} = \frac{\pi}{30}$$
$$f_{max} = 6000 \text{ Hz} \rightarrow \Omega_{max} = 12000\pi \text{ rad/sec} \rightarrow \omega_{max} = \frac{12000\pi}{18000} = \frac{2\pi}{3}$$

The new filter response is shown below.



3. Consider the following system with uniform sampling



The discrete-time system $H_d(\omega)$ is an ideal low-pass filter with cutoff frequency $\frac{\pi}{8}$.

(a) If $x_a(t)$ is bandlimited to 5 kHz, what is the maximum value of T that will avoid aliasing in the A/D converter?

(b) If $\frac{1}{T} = 10$ kHz and $x_a(t)$ is sufficiently bandlimited such that the overall system from $x_a(t)$ to $y_a(t)$ behaves as an LTI system, what will the cutoff frequency of the effective continuous-time filter be?

(c) Repeat part (b) for $\frac{1}{T} = 20$ kHz.

Solution

(a) The maximum value of T that avoids aliasing in the A/D converter is just the Nyquist rate, or twice the maximum frequency of the signal. This is

$$\boxed{f_{nyquist} = 10 \text{ kHz} \rightarrow T_{min} = \frac{1}{10000} s}$$

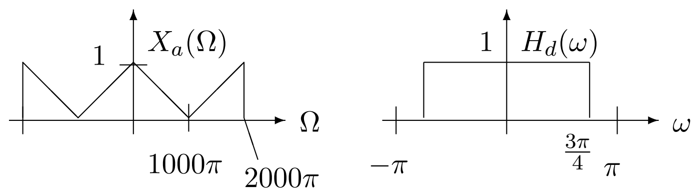
(b) If $x_a(t)$ is sufficiently bandlimited such that the overall system behaves as an LTI sytem, this means that no aliasing occurred when we sampled; from part (a), we know that this means the maximum frequency present in $x_a(t)$ is 5 kHz, corresponding to a maximum digital frequency of π. When this passes through the filter, then $Y_d(\omega)$ is bandlimited to $\frac{\pi}{8}$. Since $y[n]$ goes through an ideal D/A converter, we can use $\omega = \Omega T$ directly to find that $y_a(t)$ will be bandlimited to $\frac{\pi}{8}(10000) = \frac{10000\pi}{8}$ rad/sec. That is, the effective cutoff frequency of the equivalent continuous-time filter is

$$\boxed{\Omega_c = \frac{10000\pi}{8} \text{ rad/sec} = 625 \text{ Hz}}$$

(c) Following the same logic in (b), now the maximum frequency present in $x_a(t)$ can be 10 kHz, which again corresponds to a maximum digital frequency of π. Again, $Y_d(\omega)$ will be bandlimited to $\frac{\pi}{8}$, and using $\omega = \Omega T$ gives the new effective cutoff frequency as

$$\boxed{\Omega_c = \frac{20000\pi}{8} \text{ rad/sec} = 1250 \text{ Hz}}$$

4. For the digital system in problem 2, assume $T = 0.5$ msec, with



(a) Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$.

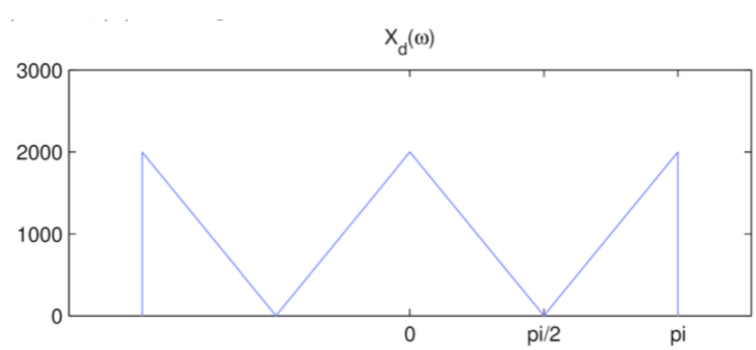
(b) Suppose the ideal D/A is now replaced by a zero-order hold, using the pulse

$$g_a(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{else.} \end{cases}$$

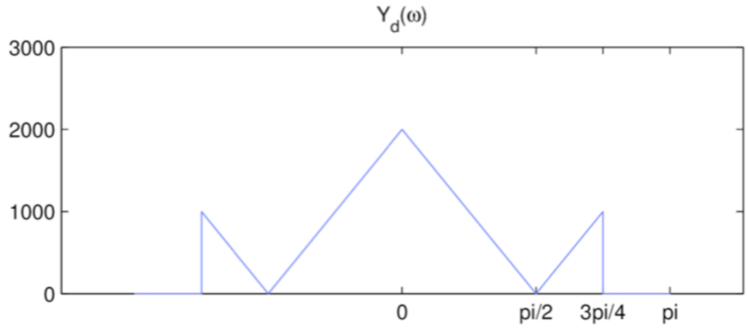
Sketch $Y_a(\Omega)$ for $|\Omega| \leq 8000\pi$. Find the amplitude of the largest unwanted (out of the band $|\Omega| \leq \frac{\pi}{T}$) component of $Y_a(\Omega)$, due to the nonideal D/A.

Solution

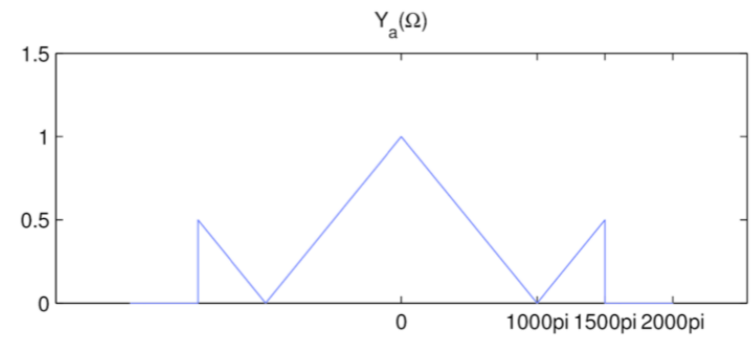
(a) Because $T = \frac{1}{2000}$, and because the original signal is bandlimited to 2000π rad/sec, no aliasing will occur. when the signal passes through the A/D converter. This means that all we have to do is compress the spectrum by a factor of $\frac{1}{T}$, and scale the amplitude by a factor of $\frac{1}{T}$. The resulting $X_d(\omega)$ can be seen below.



When this passes through the low-pass filter, everything beyond $|\omega| > \frac{3\pi}{4}$ is completely attenuated. The resulting output, $Y_d(\omega)$ is sketched below.



Finally, because we're assuming the D/A converter is ideal, to convert back, we just scale the amplitude by T and apply $\Omega = \frac{\omega}{T}$. Note that the copies that would be present from the DTFT are completely removed in the ideal case. The resulting $Y_a(\Omega)$ can be seen below.



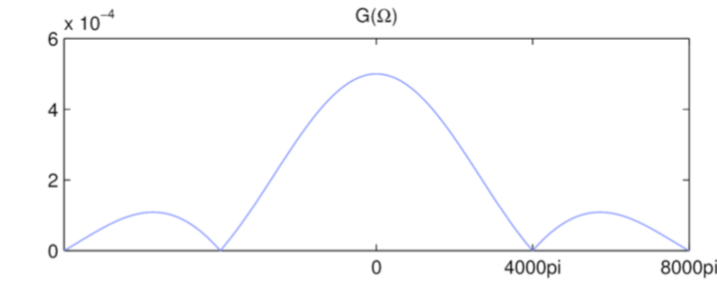
(b) Things get more complicated when the ideal D/A converter is replaced with the ZOH. We can determine its response by taking the CTFT:

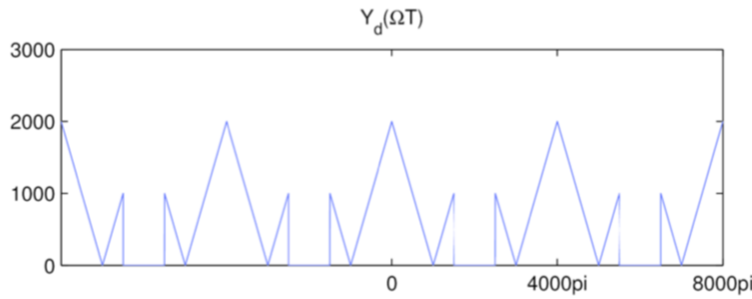
$$\begin{aligned} G_a(\Omega) &= \int_0^T e^{-j\Omega t} dt = \frac{1}{-j\Omega} e^{-j\Omega t}\Big|_0^T \\ &= \frac{1 - e^{-j\Omega T}}{j\Omega} = \frac{e^{-j\Omega \frac{T}{2}}(e^{j\Omega \frac{T}{2}} - e^{-j\Omega \frac{T}{2}})}{j\Omega} \\ &= Te^{-j\Omega \frac{T}{2}} \operatorname{sinc}\left(\frac{\Omega T}{2}\right) \end{aligned}$$

This makes sense - in the time domain, the ZOH response is just a shifted rectangle. We now get the resulting analog output by using the formula

$$Y_a(\Omega) = Y_d(\Omega T) G_a(\Omega)$$

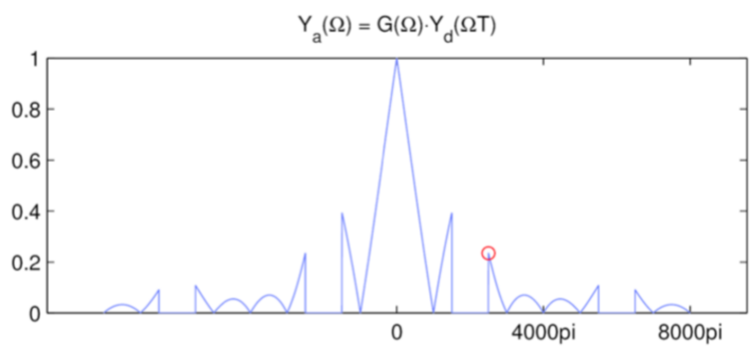
This involves multiplying the two functions given below together.



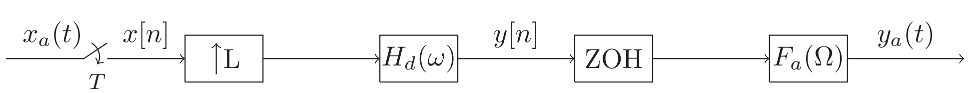


Note that, while the ZOH zeros out the peaks in the copies, it does not completely attenuate them. This means that we leave "artifacts" at higher frequencies - nonideal behavior. The resulting $Y_a(\Omega)$ is plotted below, and the largest spurious component (the amplitude of the largest out-of-band component) is circled. Its value is given as

$$1000T\text{sinc}\left(\frac{2500\pi T}{2}\right) = \boxed{0.2352}$$



5. Consider the following system, where $T = 0.01$ sec, and $H_d(\omega)$ is an ideal LPF with cutoff π/L.



$F_a(\Omega)$ is an analog compensation filter, picked such that the system from $y[n]$ to $y_a(t)$ functions as an ideal D/A, with any signal $x_a(t)$ bandlimited to 50π rad/sec. $F_a(\Omega)$ can have a transition band, in which its response can be arbitrary.

(a) Assuming $L = 1$, find the beginning and end of the transition band of $F_a(\Omega)$.

(b) Repeat (a) assuming $L = 5$.

Solution

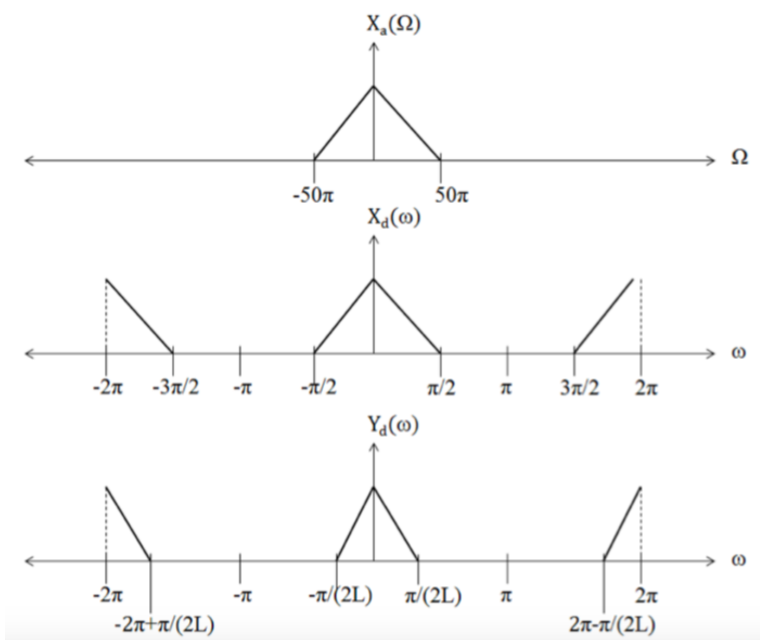
If we sample at $T = \frac{1}{100}$, and $x_a(t)$ is bandlimited to 50π rad/sec, then $X_d(\omega)$ will be bandlimited to $\frac{\pi}{2}$. Upsampling by L compresses the entire spectrum by a factor of L; we now get a base copy bandlimited to $\frac{\pi}{2L}$, and all the extra copies that would have been present between $-\pi$ and π are eliminated by the ideal LPF. Therefore, $Y_d(\omega)$ is bandlimited to $\frac{\pi}{2L}$.

For the combination of the ZOH and $F_a(\Omega)$ to act as an ideal D/A converter, the transition band needs to be narrow enough such that $F_a(\Omega)$ completely attenuates any of the high-frequency artifacts from the ZOH output; these come from the 2π-periodicity of the DTFT. So, we have two restrictions that $F_a(\Omega)$ must satisfy:

(a) It must pass through the original copy. This means that beginning of the transition band should be 50π, the bandlimit of the original signal.

(b) It must completely attenuate the copies centered around $2\pi n, n \in \mathbb{Z}$. Since the closest spurious component will occur at $2\pi - \frac{\pi}{L}$ in the digital domain, we force the filter to be zero at and above this frequency, denoted Ω_s. This gives the end of the transition band.

Graphically, $X_a(\Omega)$, $X_d(\omega)$, and $Y_d(\omega)$ are seen below.



(a) In the first case, we don't upsample at all. So, in the digital domain, the first spurious frequency will be at $\frac{3\pi}{2}$, which corresponds to 150π in the analog domain. So,

$$\boxed{\Omega_p = 50\pi, \Omega_s = 150\pi}$$

(b) However, if we upsample by 5, and apply the ideal LPF with cutoff frequency $\frac{\pi}{5}$, now $Y_d(\omega)$ is bandlimited to $\frac{\pi}{10}$, and the first spurious frequency will be at $2\pi - \frac{\pi}{10} = \frac{19\pi}{10}$, which corresponds to 950π in the analog domain. We now have

$$\boxed{\Omega_p = 50\pi, \Omega_s = 950\pi}$$

We see that, if we upsample by 5, the restriction on the filter decreases ninefold, without changing the specifications on the system.