ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 7

Topics:

- ✓ Z-transform: definition and properties

Educational Objectives:

- ✓ Understand the definition of Z-transform
- ✓ Understand the region of convergence (ROC) of Z-transform
- ✓ Understand the characteristics of ROCs
- ✓ Understand key properties of Z-transform

Z-Transform: Definition

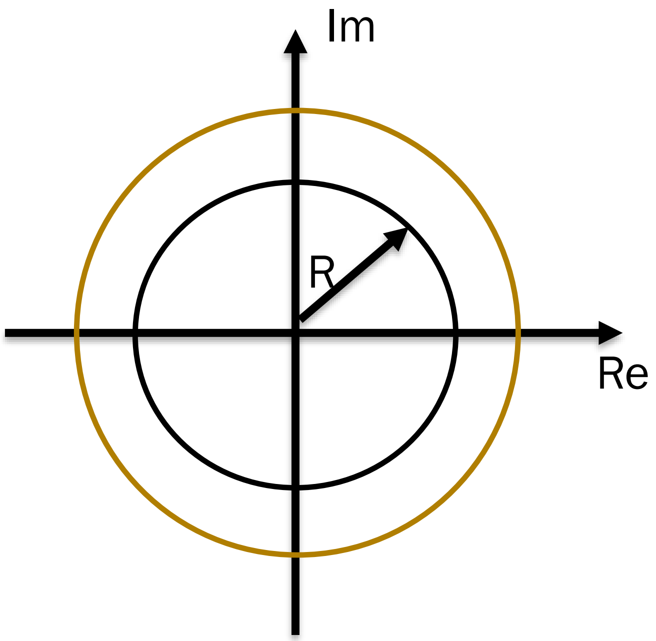
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = \sigma + j\omega$$

$$X(z) = \sum_{n=-\infty}^{-1} x[n] z^{-n} + \sum_{n=0}^{\infty} x[n] z^{-n}$$

Converge for |z| small enough $|z| < R_1$

Converge for |z| large enough $|z| > R_1$



Region of Convergence (ROC): $R_1 < |z| < R_2$

Key Formulas

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}, \qquad a \neq 1$$

$$\sum_{n=N}^{\infty} a^n = \frac{a^N}{1-a}, \qquad |a|<1$$

$$\sum_{n=-\infty}^{-N} a^n = \frac{a^{N+1}}{a-1}, \qquad |a|<1$$

Special cases:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \qquad |a|<1$$

$$\sum_{n=-\infty}^{-1} -a^n = \frac{1}{1-a}, \qquad |a|<1$$

Example

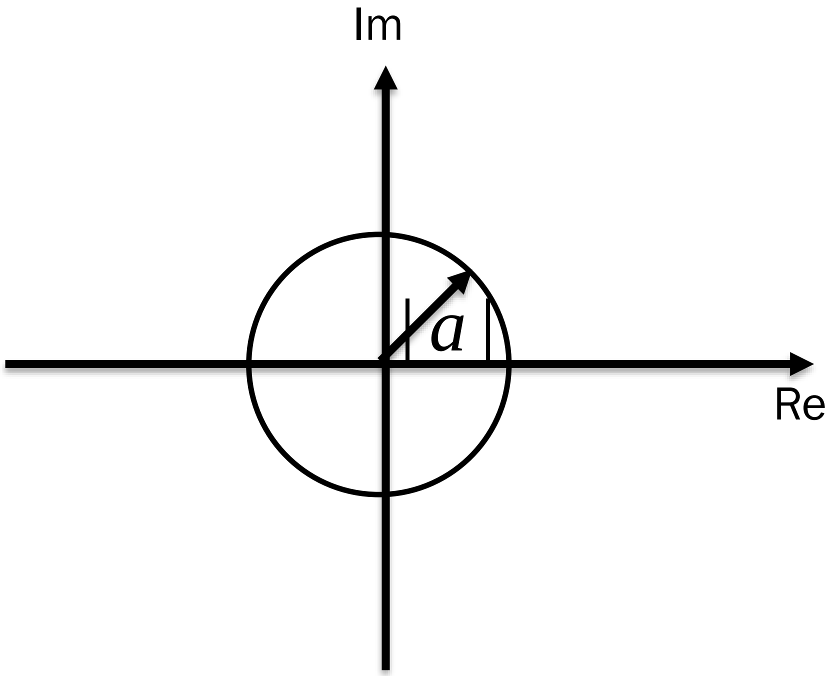
$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (\frac{a}{z})^n$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}, \qquad |\frac{a}{z}| < 1$$

$$|z| > |a|$$



Example

$$x[n] = -a^n u[-n-1]$$

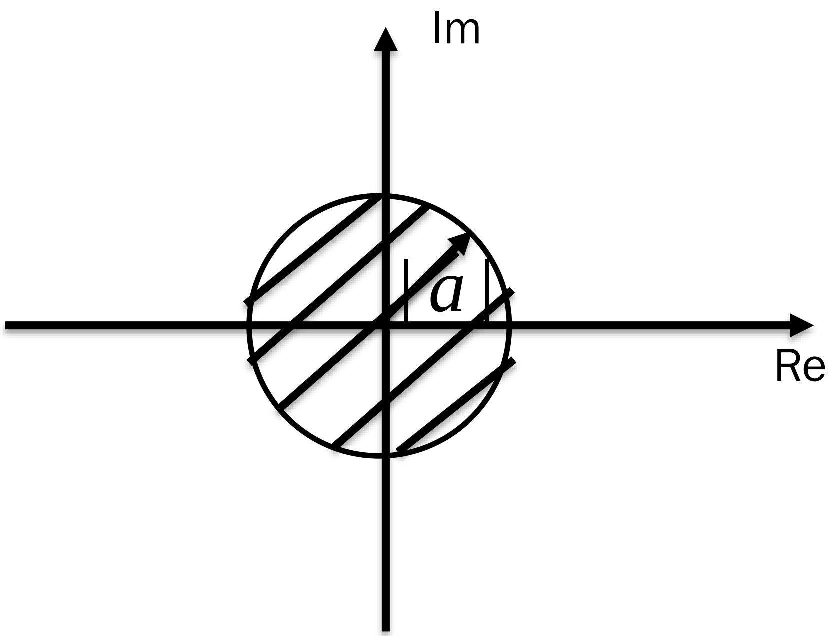
$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n = 1 - \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= 1 - \frac{1}{1-\frac{z}{a}} = 1 - \frac{a}{a-z}$$

$$= \frac{z}{z-a}$$



$$ROC: \quad \left|\frac{z}{a}\right| < 1 \quad \rightarrow \quad |z| < |a|$$

Importance of ROC

$$a^n u[n] \quad \leftrightarrow \quad \frac{z}{z-a} \quad ROC: \quad |z|>|a|$$

$$Let \quad a = 2,$$

$$Note \quad X(1) = -1$$

$$But \quad \sum_{n=0}^{\infty} 2^n 1^n = \infty$$

$$X(1) \neq Z\{x[n]\}(1) \quad !$$

Example

$$x[n] = \{0, 0, 8, \underset{\uparrow}{3}, -2, 0, 1\}$$

$$X(z) = 8z + 3 - 2z^{-1} + z^{-3}$$

$$ROC: \quad z \neq 0, \quad z \neq \infty$$

$$0 < |z| < \infty$$

Example

$$x[n] = \delta[n-k]$$

$$X(z) = z^{-k}$$

$ROC:$ $\quad k = 0,\quad$ entire z-plane

$$k > 0, \qquad z \neq 0 \quad or \quad |z| > 0$$

$$k < 0, \qquad |z| < \infty$$

Key Z-transform Pairs

$$a^n u[n] \leftrightarrow \frac{z}{z-a}, \qquad ROC: |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{z}{z-a}, \qquad ROC: |z| < |a|$$

$$na^n u[n] \leftrightarrow \frac{az}{(z-a)^2}, \qquad ROC: |z| > |a|$$

$$-na^n u[-n-1] \leftrightarrow \frac{az}{(z-a)^2}, \qquad ROC: |z| < |a|$$

$$\cos \omega_0 n u[n] \leftrightarrow \frac{z^2 - z\cos\omega_0}{z^2 - 2z\cos\omega_0 + 1}, \qquad |z| > 1$$

Properties of Z-transform

a) Linearity

b) Shifting

c) Convolution

Linearity

$$Z\{ax[n]+by[n]\}=aX(z)+bY(z)$$

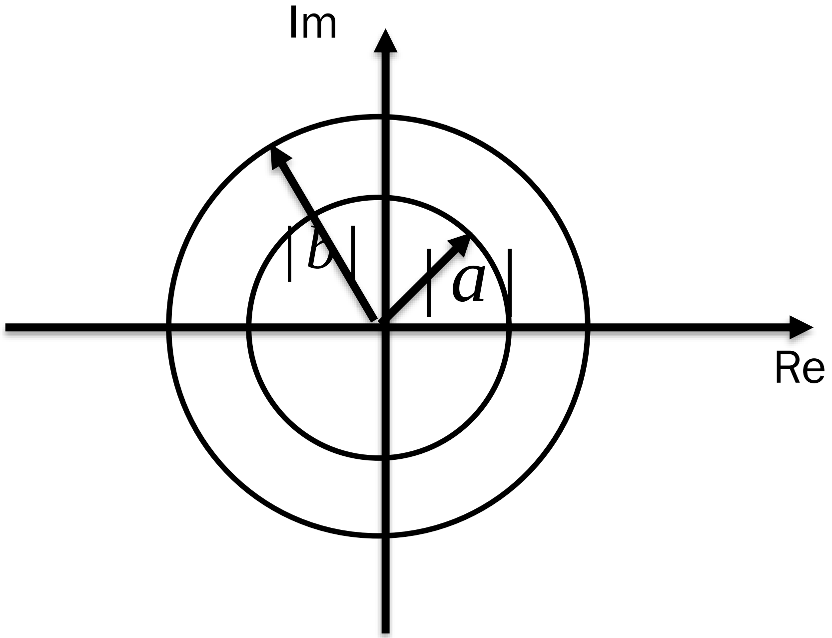
$$ROC=ROC_x\cap ROC_y$$

$$or \quad \supset ROC_x\cap ROC_y$$ (pole-zero cancellation)

Example

$$x[n] = \begin{cases} a^n, & n \geq 0 \\ b^n, & n \leq 1 \end{cases} = a^n u[n] + b^n u[-n-1]$$

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$



ROC: $\quad |a| < |z| < |b|$

If $|b| < |a|$, $X(z)$ does not exist!

Example

$$x[n] = 3^n (u[n] - u[n-10])$$

$$= 3^n u[n] - 3^n u[n-10]$$

$$X(z) = \frac{z}{z-3} - 3^{10} z^{-10} \frac{z}{z-3}$$

$$= \frac{z(z^{10} - 3^{10})}{z^{10}(z-3)}$$

$$ROC: \quad |z| > 0$$