ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 8

Topics:

- ✓ Continued discussion of Z-transform
- ✓ Key properties of Z-transform

Educational Objectives:

- ✓ Understand linearity of Z-transform; effect on ROC
- ✓ Understand shifting property of Z-transform; effect on ROC
- ✓ Understand convolution property of Z-transform; effect on ROC

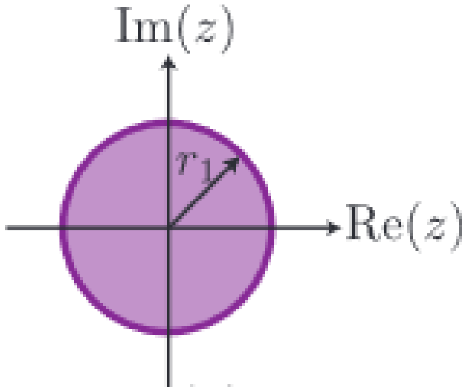
Z-Transform

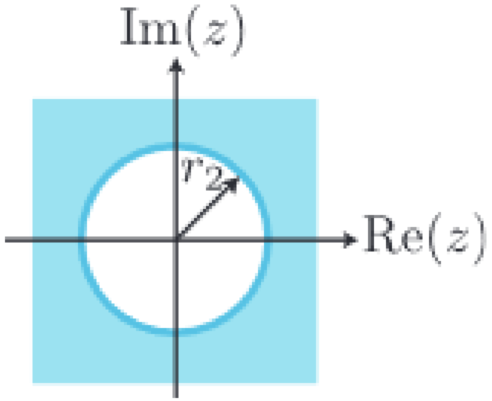
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

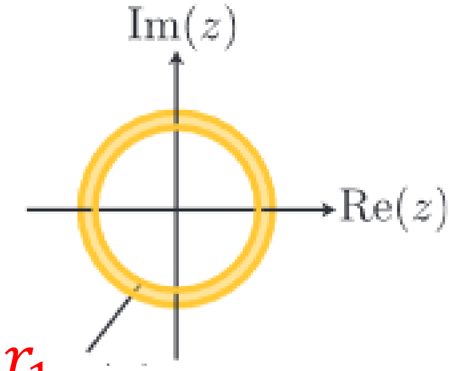
$$X(z) = \sum_{n=-\infty}^{-1} x[n] z^{-n} + \sum_{n=0}^{\infty} x[n] z^{-n}$$

Converge for |z| small enough $|z| < r_1$

Converge for |z| large enough $|z| > r_2$







Region of Convergence (ROC): $r_2 < |z| < r_1$

Linearity

$$Z\{ax[n]+by[n]\} = aX(z)+bY(z)$$

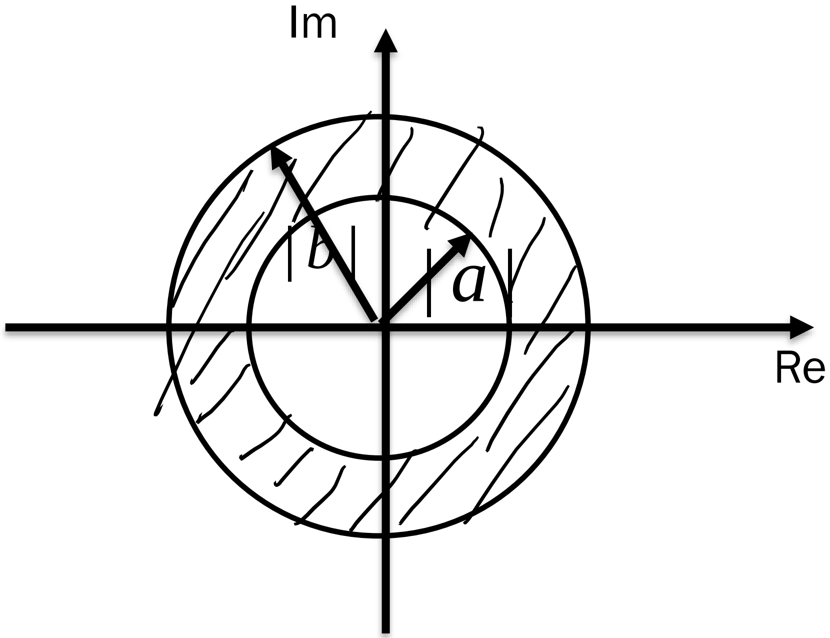
$$ROC = ROC_x \cap ROC_y$$

$$or \quad \supset ROC_x \cap ROC_y$$ (pole-zero cancellation)

Example

$$x[n] = \begin{cases} a^n, & n \geq 0 \\ b^n, & n \leq 1 \end{cases} = a^n u[n] + b^n u[-n-1]$$

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$



ROC: $|a| < |z| < |b|$

If $|b| < |a|$, $X(z)$ does not exist!

Example

$$x[n] = 3^n(u[n] - u[n-10])$$
$$= 3^n u[n] - 3^n u[n-10]$$

$$X(z) = \frac{z}{z-3} - 3^{10} z^{-10} \frac{z}{z-3}$$
$$= \frac{z(z^{10} - 3^{10})}{z^{10}(z-3)}$$

$$ROC: \quad |z| > 0$$

Shifting Property

$$Z\{x[n \pm k]\} = z^{\pm k} X(z)$$

$$ROC = ROC_x$$

$$\textit{with possible addition or deletion of } z = 0, |z| = \infty$$

$$\text{One-sided } z\text{-transform}: \quad X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$Z\{x[n-k]\} = z^{-k} X(z) + \sum_{m=1}^{k} x[-m] z^{m-k}$$

Example

$$x[n] = \delta[n] \quad \rightarrow \quad X(z) = 1, \quad ROC: entire\ z - plane$$

$$x[n] = \delta[n-3] \quad \rightarrow \quad X(z) = z^{-3}, \quad ROC: |z| > 0$$

$$x[n] = \delta[n+3] \quad \rightarrow \quad X(z) = z^{3}, \quad ROC: |z| < \infty$$

Differentiation

$$Z\{nx[n]\} = -z\frac{dX(z)}{dz}$$

Convolution

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$Y(z) = H(z)X(z)$$

$$ROC_Y = ROC_H \cap ROC_X$$

$$\supset ROC_H \cap ROC_X$$ Pole-zero cancellation

Fast way to evaluate convolution!

Example

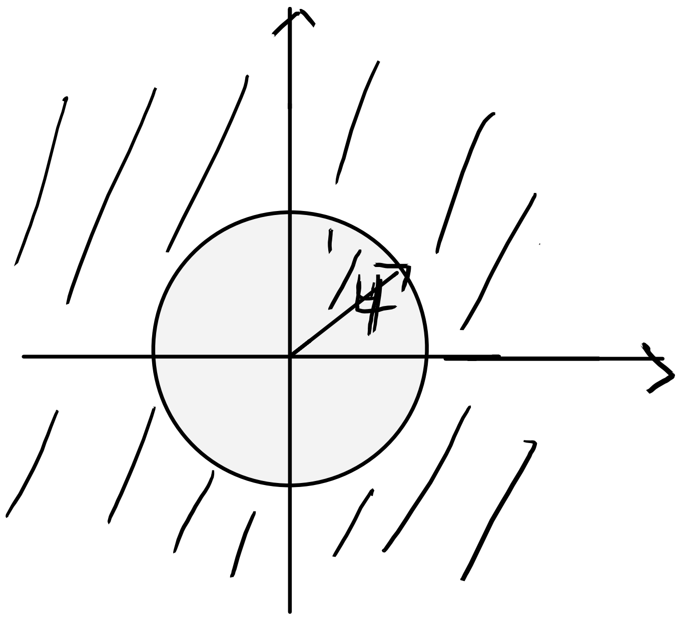
$$h[n] = (\frac{1}{2})^n u[n] \qquad x[n] = \delta[n] - (\frac{1}{4})^n u[n-1]$$

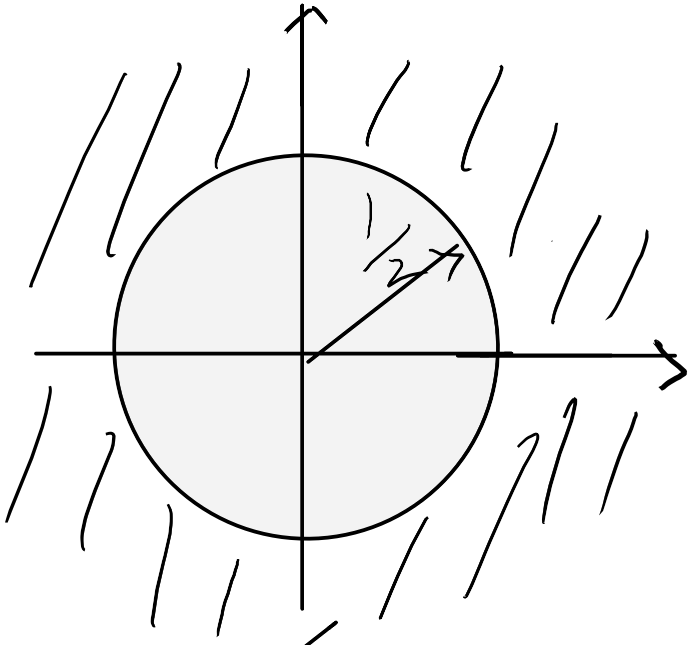
$$h[n] \quad \rightarrow \quad \frac{z}{z - \frac{1}{2}} \qquad |z| > \frac{1}{2}$$

$$x[n] \quad \rightarrow \quad 1 - \frac{1}{4}\frac{1}{z - \frac{1}{4}} = \frac{z - \frac{1}{4} - \frac{1}{4}}{z - \frac{1}{4}} = \frac{z - \frac{1}{2}}{z - \frac{1}{4}} \qquad |z| > \frac{1}{4}$$

Example

$$x[n]*h[n] \;\rightarrow\; \frac{z-\frac{1}{2}}{z-\frac{1}{4}}\,\frac{z}{z-\frac{1}{2}} = \frac{z}{z-\frac{1}{4}} \qquad |z|>\frac{1}{4}$$





But $ROC_H \cap ROC_X$: $|z|>\frac{1}{2}$