ECE 310 Recitation 4 Solution
Thursday Mar 11, 2021

**Concept check**

- √ Inverse z-transform: a general approach
  - Partial fraction expansion: determine the fractions and coefficients
  - Determine ROC (based on causality)
  - Table look-up
- √ Causality of LSI system
  - Time-domain: $h[n] = 0, n < 0$ (right-hand sided)
  - Frequency-domain: $ROC_H: |z| > R$
- √ BIBO stability of LSI system
  - Time-domain: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (absolute summability)
  - Frequency-domain: $ROC_H$ includes the unit circle
- √ Finding a counter example for BIBO
  - Boundedness of a signal vs. BIBO stability of a system
    - Unit circle? Single pole? Double pole?
  - "match the single poles on UC"

**Exercise**

1. Find the inverse z-transform of $\frac{2z^3-7z^2}{(z+1)(z-2)^2}$

$$X(z) = \frac{2z^3 - 7z^2}{(z+1)(z-2)^2} = \frac{2 - 7z^{-1}}{(1+z^{-1})(1-2z^{-1})^2} = \frac{A}{1+z^{-1}} + \frac{B}{1-2z^{-1}} + \frac{C2z^{-1}}{(1-2z^{-1})^2}$$

$$\Rightarrow A(1-2z^{-1})^2 + B(1+z^{-1})(1-2z^{-1}) + C2z^{-1}(1+z^{-1})$$

$$= (A+B) + (-4A - B + 2C)z^{-1} + (4A - 2B + 2C)z^{-2}$$
$$= 2 - 7z^{-1}$$

$$\begin{cases} A + B = 1 \\ -4A - B + 2C = -7 \\ 4A - 2B + 2C = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \\ C = -1 \end{cases}$$

$$\Rightarrow X(z) = \frac{1}{1+z^{-1}} + \frac{1}{1-2z^{-1}} - \frac{2z^{-1}}{(1-2z^{-1})^2}$$

Poles: {-1, 2}

i. ROC: $|z| > 2$
$x[n] = (-1)^n u[n] + 2^n u[n] - 2^n n u[n]$

ii. ROC: $1 < |z| < 2$
$x[n] = (-1)^n u[n] - 2^n u[-n-1] + 2^n n u[-n-1]$

iii. ROC: $0 \leq |z| < 1$
$x[n] = -(-1)^n u[-n-1] - 2^n u[-n-1] + 2^n n u[-n-1]$

2. Discuss whether each of the following transfer functions represent a BIBO stable system. For each case in which the system is determined to be unstable, find a bounded input that will produce an unbounded output.

a. $H(z) = \frac{z}{z-a}, |a| > 1$

i. If causal: $ROC_H: |z| > |a| > 1$, does not include unit circle, NOT BIBO stable. Input example: $x[n] = \delta[n]$, then $y[n] = h[n] = a^n u[n]$ is unbounded.

ii. If anti-causal: $ROC_H: |z| < |a|$, includes unit circle, BIBO stable.

b. $H(z) = \frac{z}{z-a}, |a| < 1$

i. If causal: $ROC_H: |z| > |a|$, includes unit circle, BIBO stable.

ii. If anti-causal: $ROC_H: |z| < |a| < 1$, does not include unit circle, NOT BIBO stable. Input example: $x[n] = \delta[n]$, then $y[n] = h[n] = -a^n u[-n-1]$ is unbounded.

c. $H(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z+j)(z-j)}$

i. If causal: $ROC_H: |z| > 1$, does not include unit circle, NOT BIBO stable.

ii. If anti-causal: $ROC_H: |z| < 1$, does not include unit circle, NOT BIBO stable.

Input example: $X(z) = \frac{z^2}{(z+j)(z-j)}$ ($x[n] = \cos\left(\frac{\pi}{2}n\right) u[n]$),
then $Y(z) = \frac{z^4}{(z+j)^2(z-j)^2}$ has double poles on UC, so $y[n]$ is unbounded

*d.* $H(z) = \frac{z}{(z-1)^2(z+1)}$

i. If causal: $ROC_H: |z| > 1$, does not include unit circle, NOT BIBO stable.

ii. If anti-causal: $ROC_H: |z| < 1$, does not include unit circle, NOT BIBO stable.

Input example: $x[n] = \delta[n]$, $Y(z) = H(z) = \frac{z}{(z-1)^2(z+1)}$ has double poles on UC, so $y[n]$ is unbounded.

$$Y(z) = \frac{z}{(z-1)^2(z+1)} = \frac{1}{2}\frac{z}{(z-1)^2} - \frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\frac{z}{z+1}$$

$ROC_H: |z| > 1 \implies y[n] = \frac{1}{2}nu[n] - \frac{1}{4}u[n] + \frac{1}{4}(-1)^n u[n]$,
the term nu[n] is unbounded

$ROC_H: |z| < 1 \implies y[n] = -\frac{1}{2}nu[-n-1] + \frac{1}{4}u[-n-1] - \frac{1}{4}(-1)^n u[-n-1]$,
the term nu[-n-1] is unbounded