ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 16

Topics:

- ✓ Properties of Discrete-time Fourier transform (DTFT)

Educational Objectives:

- ✓ Understand key properties of DTFT
- ✓ Get more familiar of key DTFT pairs
- ✓ Fully understand the relationship between DTFT and z-transform

Discrete-time Fourier Transform (DTFT)

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{i\omega n} \, d\omega$$

DTFT Properties

a) Linearity

$$x[n] \leftrightarrow X_d(\omega);\ \ y[n] \leftrightarrow Y_d(\omega)$$

$$ax[n] + by[n] \leftrightarrow aX_d(\omega) + bY_d(\omega);$$

b) Periodicity

$$X_d(\omega + 2k\pi) = X_d(\omega), k \in \mathbb{Z}$$

c) For real-valued $x[n]$

$X_d(\omega) = X_d^*(-\omega)$ (complex conjugate symmetry)

$$\mathrm{Re}\{X_d(\omega)\} = \mathrm{Re}\{X_d(-\omega)\}$$

$$\mathrm{Im}\{X_d(\omega)\} = -\mathrm{Im}\{X_d(-\omega)\}$$

DTFT Properties

d) Shifting

$$x[n \pm n_0] \leftrightarrow e^{\pm j\omega n_0} X_d(\omega) = |X_d(\omega)| e^{j(\angle X_d(\omega) \pm \omega n_0)}$$

time shift ⟷ linear phase shift

e) Modulation (frequency shift)

$$x[n] e^{\pm j\omega_0 n} \leftrightarrow X_d(\omega \mp \omega_0)$$

$$x[n] \cos(\omega_0 n) \leftrightarrow \frac{1}{2}\left(X_d(\omega - \omega_0) + X_d(\omega + \omega_0)\right)$$

f) Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega$$

DTFT Properties

g) Convolution

$$y[n] = x[n] * h[n] \leftrightarrow Y_d(\omega) = X_d(\omega)\ H_d(\omega)$$

$$y[n] = x[n]h[[n] \ \leftrightarrow \ Y_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\tau)H_d(\omega-\tau)d\tau$$

e) Differentiation

$$nx[n] \leftrightarrow j\,\frac{dX_d(\omega)}{d\omega}$$

Some DTFT pairs

$x[n]$	$\leftrightarrow$	$X_d(\omega)$	ω
1	$\leftrightarrow$	$2\pi\delta(\omega)$	$-\pi \leq \omega \leq \pi$
		$2\pi \sum \delta(\omega - 2k\pi)$	$\mathbb{R}$
$e^{j\omega_0 n}$	$\leftrightarrow$	$2\pi\delta(\omega - \omega_0)$	$-\pi \leq \omega \leq \pi$
		$2\pi \sum \delta(\omega - \omega_0 - 2k\pi)$	$\mathbb{R}$
$cos(\omega_0 n)$	$\leftrightarrow$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$	$-\pi \leq \omega \leq \pi$
		$\pi \sum(\delta(\omega - \omega_0 - 2k\pi) + \delta(\omega + \omega_0 - 2k\pi))$	$\mathbb{R}$
$sin(\omega_0 n)$	$\leftrightarrow$	$\frac{\pi}{j}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$	$-\pi \leq \omega \leq \pi$
		$\frac{\pi}{j} \sum(\delta(\omega - \omega_0 - 2k\pi) + \delta(\omega + \omega_0 - 2k\pi))$	$\mathbb{R}$

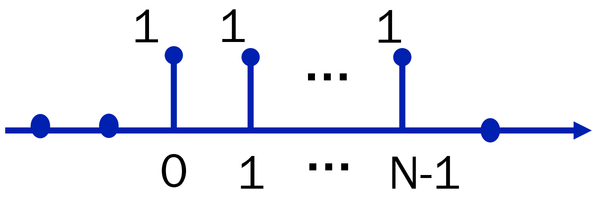
* The range of summation above is $-\infty$ to ∞

Some DTFT pairs

$x[n]$	$\leftrightarrow$	$X_d(\omega)$
$\delta[n]$	$\leftrightarrow$	1
$u[n]$	$\leftrightarrow$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta[\omega - 2k\pi]$
$a^n u[n]$	$\leftrightarrow$	$\frac{1}{1-ae^{-j\omega}}, \lvert a \rvert < 1$
$(1+n)a^n u[n]$	$\leftrightarrow$	$\frac{1}{(1-ae^{-j\omega})^2}, \lvert a \rvert < 1$

DTFT Example

$$x[n] = \begin{cases} 1, 0 \le n \le N-1 \\ 0, \quad\quad\quad\quad else \end{cases}$$



$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega \mathrm{N}}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\omega \mathrm{N}/2}(e^{j\omega \mathrm{N}/2} - e^{-j\omega \mathrm{N}/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}$$

$$= e^{-j\frac{\omega}{2}(N-1)} \frac{2jsin(\omega N/2)}{2jsin(\omega/2)}$$

$$= \frac{sin(\omega N/2)}{sin(\omega/2)} e^{-j\frac{\omega}{2}(N-1)}$$

DTFT Example

$$x[n] = u[n] - u[n-N]$$

Alternatively:

$$\mathrm{X}_d(\omega) = \left(\frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta[\omega - 2k\pi]\right) - e^{-j\omega N}\left(\frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta[\omega - 2k\pi]\right)$$

$$= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} + \left(\pi\sum_{k=-\infty}^{\infty}\delta[\omega - 2k\pi]\right)\left(1-e^{-j\omega N}\right)$$

$$= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta[\omega - 2k\pi]\left(1-\cancelto{0}{e^{-j2k\pi N}}\right)$$

$$= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$