UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing

Homework 10 Solutions

Professor Z.-P. Liang **Due:** April 23, 2021

Problem 1

(15 points - 5 points each) The transfer functions of three LSI systems are given below. For each system, determine if it is an FIR or an IIR filter (justify your answer).

1. $H(z) = 1 + z^{-1} + 7z^{-6}$
2. $H(z) = \dfrac{z^2 + 3z + 2}{z + 1}$
3. $H(z) = \dfrac{z + 1}{z^2 + 3z + 2}$

Solution

There are a few ways to determine whether or not a filter is IIR or FIR. Because all IIR filters contain a feedback term, the simplest approach is to examine the poles of $H(z)$. If there are any poles not at 0 or ∞, the filter is IIR. Alternatively, we can take the inverse z-transform of $H(z)$ and examine the length of $h[n]$ directly.

(a) Obviously, the only poles of $H(z)$ will be at $z = 0$, so there is no feedback term; the filter is **FIR**. We can verify this by taking the inverse z-transform:

$$h[n] = \delta[n] + \delta[n-1] + 7\delta[n-6] = \{\boxed{1}, 1, 0, 0, 0, 0, 7\}$$

where the box indicates the time origin.

(b) Be careful - one might be tempted to say that the system is IIR, because $H(z)$ has a pole at $z = -1$. However, this is not actually a pole, as we can perform a pole-zero cancellation:

$$\frac{z^2 + 3z + 2}{z + 1} = \frac{(z+1)(z+2)}{z+1} = z + 2$$

The only pole is actually at ∞ - therefore, there is no feedback term, and the filter is **FIR**. Again, this can be verified by taking the inverse z-transform:

$$h[n] = \delta[n+1] + 2\delta[n] = \{1, \boxed{2}\}$$

where the box indicates the time origin.

(c) This is just the inverse of the transfer function given in (b), so we can immediately write

$$H(z) = \frac{1}{z+2}$$

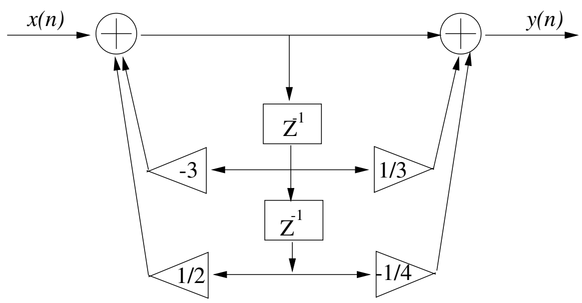
This has a pole at $z = -2$, so the filter is **IIR**. Indeed, taking the inverse z-transform gives

$$h[n] = (-2)^{(n-1)} u[n-1]$$

which is infinite-length.

Problem 2

(15 points - 10 for the transfer function and 5 for the difference equation) Derive the transfer function and the corresponding difference equation for the following block diagram



Solution

One way to proceed is to notice that the block diagram is in Direct Form II; we already know the transfer function for any system represented in this structure.

However, how would we prove this? Given any block diagram for which you're given the transfer function for, a good place to start is to label the output of each adder. Here, we denote $d[n]$ as the output of the adder on the left. This gives two equations (one for each adder):

$$d[n] = x[n] - 3d[n-1] + \frac{1}{2}d[n-2]$$
$$y[n] = d[n] + \frac{1}{3}d[n-1] - \frac{1}{4}d[n-2]$$

While these might not get us anywhere on their own (there's no obvious way to relate $x[n]$ to $y[n]$), the relationship becomes much more evident when we take the z-transform of both equations; this gives

$$D(z) = X(z) - 3z^{-1}D(z) + \frac{1}{2}z^{-2}D(z)$$
$$Y(z) = D(z) + \frac{1}{3}z^{1}D(z) - \frac{1}{4}z^{-2}D(z)$$

These give

$$X(z) = D(z)\Big(1 + 3z^{-1} - \frac{1}{2}z^{-2}\Big) \to D(z) = \frac{X(z)}{1 + 3z^{-1} - \frac{1}{2}z^{-2}}$$
$$Y(z) = D(z)\Big(1 + \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2}\Big)$$

Combining the two together gives

$$Y(z) = \frac{X(z)(1 + \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2})}{1 + 3z^{-1} - \frac{1}{2}z^{-2}}$$
$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{1 + \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2}}{1 + 3z^{-1} - \frac{1}{2}z^{-2}}}$$

as the transfer function. To find the difference equation, we simply cross-multiply and take the inverse z-transform:

$$Y(z)\Big(1 + 3z^{-1} - \frac{1}{2}z^{-2}\Big) = X(z)\Big(1 + \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2}\Big)$$
$$y[n] + 3y[n-1] - \frac{1}{2}y[n-2] = x[n] + \frac{1}{3}x[n-1] - \frac{1}{4}x[n-2]$$

We can further note that the block diagram only contains delay blocks; therefore, the system can be implemented causally, and we can write the difference equation as

$$\boxed{y[n] = x[n] + \frac{1}{3}x[n-1] - \frac{1}{4}x[n-2] - 3y[n-1] + \frac{1}{2}y[n-2]}$$

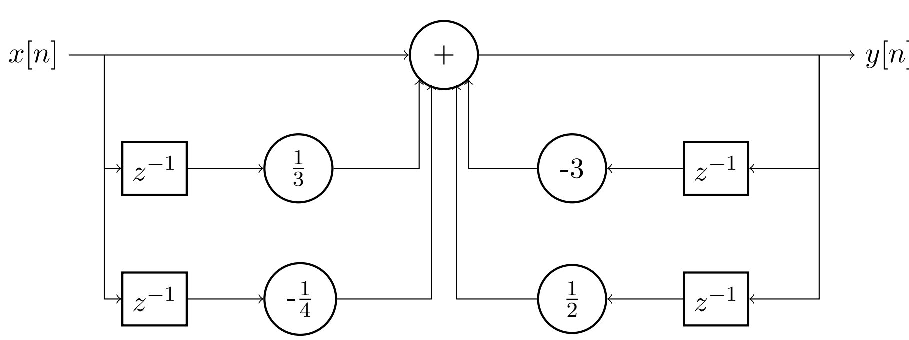
This implementation will make Problem 3 much easier, as everything is already in the correct form. *(Note: Full credit awarded for either difference equation).*

Problem 3

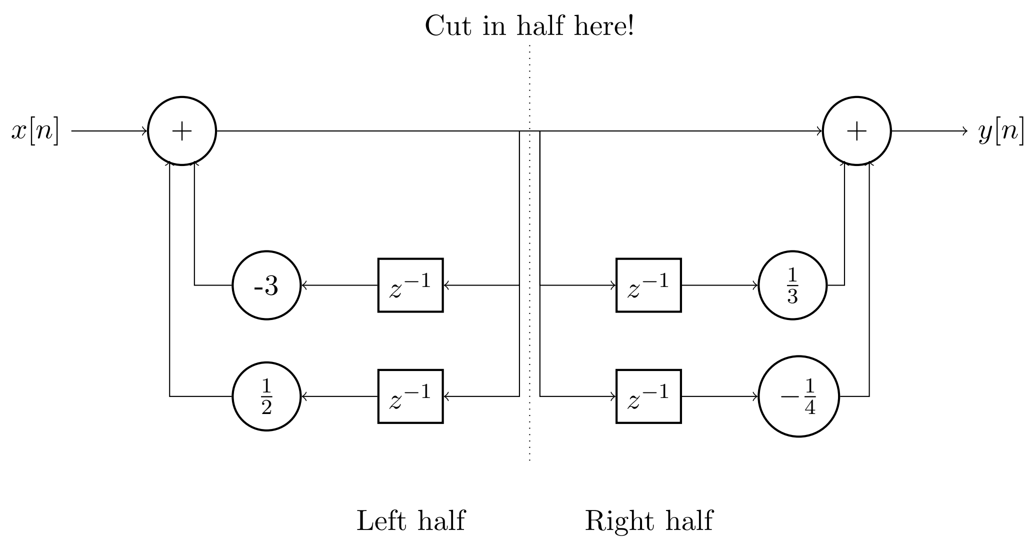
(10 points) Draw a Direct Form I block diagram for the system in Problem 2.

Solution

One approach to drawing the Direct Form I block diagram is to simply use the causal LCCDE implementation we found in Problem 2; we can treat $y[n]$ as the output of the adder, and add gain and delay terms accordingly. Doing so gives the result below.



Note that this looks very similar to the Direct Form II structure. In fact, the two are "transposes" of each other - if we draw the Direct Form II structure using four delay blocks, "cut it in half" at the specified location, and switched the orders of the two halves, we would end up with the Direct Form I structure after combining the adders together. Note that the locations of $x[n]$ and $y[n]$ remain unchanged.



Problem 4

(15 points - 7.5 for each implementation) Draw a block diagram implementation (in direct form I and II, respectively) of the system described by

$$y[n+1] - 2y[n] - 5y[n-1] = x[n] + 8x[n+1] - 2x[n-1]$$

Solution

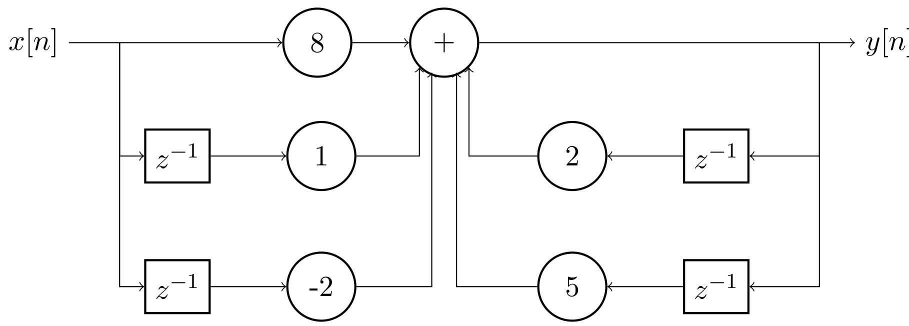
Note that we have a problem - the system requires advance elements as well as delay elements, as evidenced by the dependence on $x[n+1]$ and $y[n+1]$. However, the Direct Form I and Direct Form II structures only allow the usage of delay elements. Thankfully, because the system can be represented as an LCCDE, it is LSI, and therefore we can shift all the time indices (letting $m = n+1$, and then $n = m$). This results in the new LCCDE:

$$y[n] - 2y[n-1] - 5y[n-2] = 8x[n] + x[n-1] - 2x[n-2]$$

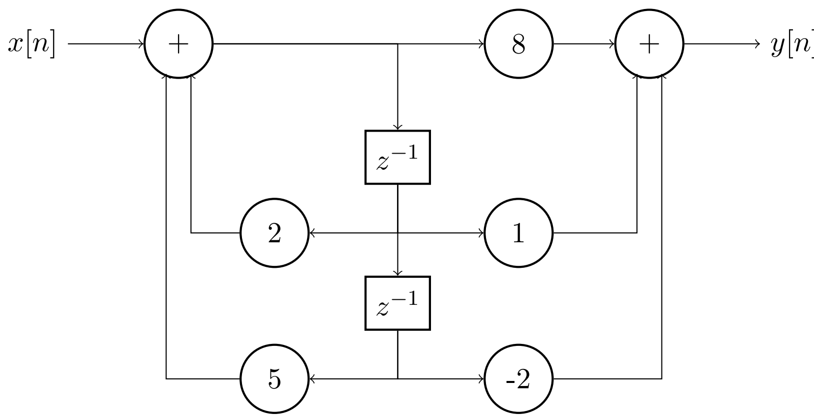
To get the Direct Form I structure, we want to place the system into causal form. Doing so gives

$$y[n] = 8x[n] + x[n-1] - 2x[n-2] + 2y[n-1] + 5y[n-2]$$

from which the structure becomes obvious; the diagram is given below.



To obtain the Direct Form II structure, we follow the method given in Problem 3. However, now we redraw the adder as two separate adders, "cut" the diagram in half between the adders, and switch the two sides. This gives



To verify that this works, it's instructive to compute the transfer function of the Direct Form II diagram. Letting $d[n]$ be the output of the first adder (as we did in Problem 2), we can again write two equations, one at the output of each adder; taking their z-transforms allows us to ascertain the transfer function.

$$d[n] = x[n] + 2d[n-1] + 5d[n-2] \to D(z) - 2z^{-1}D(z) - 5z^{-2}D(z) = X(z)$$
$$y[n] = 8d[n] + d[n-1] - 2d[n-2] \to Y(z) = 8D(z) + z^{-1}D(z) - 2z^{-2}D(z)$$

The first equation gives the relation

$$D(z) = \frac{X(z)}{1 - 2z^{-1} - 5z^{-2}}$$

which, when plugged into the second equation, gives

$$Y(z) = \frac{(8 + z^{-1} - 2z^{-2})X(z)}{1 - 2z^{-1} - 5z^{-2}} \to H(z) = \frac{Y(z)}{X(z)} = \frac{8 + z^{-1} - 2z^{-2}}{1 - 2z^{-1} - 5z^{-2}}$$

Converting back to an LCCDE by cross-multiplying and taking the inverse z-transform gives

$$y[n] = 8x[n] + x[n-1] - 2x[n-2] + 2y[n-1] + 5y[n-2]$$

which matches.

Problem 5

(18 points - 3 points each) The frequency response of a GLP filter can be expressed as $H_d(\omega) = R(\omega)e^{j(\alpha - M\omega)}$ where $R(\omega)$ is a real function. For each of the following filters, determine whether it is a GLP filter. If it is, find $R(\omega)$, M, and α, and indicate whether it is also a linear phase filter.

1. $\{h_n\}_{n=0}^{2} = \{2, 1, 2\}$
2. $\{h_n\}_{n=0}^{2} = \{1, 2, 3\}$
3. $\{h_n\}_{n=0}^{2} = \{-1, 3, 1\}$
4. $\{h_n\}_{n=0}^{4} = \{1, 1, 1, -1, -1\}$
5. $\{h_n\}_{n=0}^{2} = \{1, 0, -1\}$
6. $\{h_n\}_{n=0}^{3} = \{2, 1, 1, 2\}$

In each case, the remaining terms of the unit pulse response of the filter are zero.

Solution

Recall the different types of GLP (taken from Page 550 of the textbook):

GLP Type	Signal Symmetry	Signal Length	Restrictions on $R(\omega)$
I	Even	Odd	No Restrictions
II	Even	Even	$R(\pi) = 0$
III	Odd	Odd	$R(0) = R(\pi) = 0$
IV	Odd	Even	$R(0) = 0$

(a) We can take the DTFT and factor out an $e^{-j\omega}$:

$$H_d(\omega) = 2 + e^{-j\omega} + 2e^{-j2\omega} = e^{-j\omega}(2e^{j\omega} + 1 + 2e^{-j\omega}) = e^{-j\omega}(1 + 4\cos(\omega))$$

This is even symmetry and odd-length, so the filter has **Type I GLP**, with

$$\boxed{\alpha = 0, R(\omega) = 1 + 4\cos(\omega), M = 1}$$

The filter will have purely linear phase if $R(\omega)$ has no zero-crossings between $-\pi$ and π - that is, there are no jumps of π in the phase. Solving gives

$$R(\omega) = 0 \to 4\cos(\omega) = -1 \to \omega = \pm 1.8285$$

Since there are two zero-crossings, the filter is **not** purely linear phase.

(b) In this case, the filter exhibits no symmetry ($x[0] \neq x[2]$), so it does **not** have GLP

(c) Be careful in these cases - while the filter does have odd symmetry about the midpoint, it does **not** have GLP. This is because the middle term is not zero; observe that the DTFT is given by

$$H_d(\omega) = -1 + 3e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(-e^{j\omega} + 3 + e^{-j\omega}) = e^{-j\omega}(3 - 2j\sin(\omega))$$

The problem is that we can't pull a factor of j out because of the 3 term. If the middle term was zero, we could set $j = e^{j\frac{\pi}{2}}$ and pull it outside the parentheses to create an expression in the correct form. Therefore, for a filter odd length and odd symmetry about the midpoint, the middle term *must be zero* for the filter to exhibit GLP.

(d) By the logic given in (c), this filter will **not** have GLP because it is odd length and odd symmetry, but the middle term is not zero. To further solidify the point, taking the DTFT gives

$$\begin{aligned}
H_d(\omega) &= 1 + e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega} - e^{-j4\omega} \\
&= e^{-j2\omega}(e^{j2\omega} + e^{j\omega} + 1 - e^{-j\omega} - e^{-j2\omega}) \\
&= e^{-j2\omega}(2j\sin(2\omega) + 2j\sin(\omega) + 1)
\end{aligned}$$

We run into the same problem - if the 1 was not present, we could pull the factor of j outside and create an expression in the GLP form.

(e) This filter has odd symmetry, odd length, and the middle term is zero, so we would expect it to exhibit **Type III GLP**. We can check by taking the DTFT:

$$H_d(\omega) = 1 - e^{-j2\omega} = e^{-j\omega}(e^{j\omega} - e^{-j\omega}) = e^{-j\omega}(2j\sin(\omega)) = e^{j(\frac{\pi}{2}-\omega)}(2\sin(\omega))$$

This is indeed GLP with

$$\boxed{\alpha = \frac{\pi}{2}, R(\omega) = 2\sin(\omega), M = 1}$$

However, it will **not** exhibit linear phase, as $\sin(\omega)$ has a zero-crossing at $\omega = 0$.

(f) Since this filter has even symmetry and even length, we would expect it to exhibit **Type II GLP**. We can again check by taking the DTFT:

$$\begin{aligned}
H_d(\omega) &= 2 + e^{-j\omega} + e^{-j2\omega} + 2e^{-j3\omega} \\
&= e^{-j\omega 1.5}(2e^{j\omega 1.5} + e^{j\omega 0.5} + e^{-j\omega 0.5} + 2e^{-j\omega 1.5}) \\
&= e^{-j\omega 1.5}(4\cos(1.5\omega) + 2\cos(0.5\omega))
\end{aligned}$$

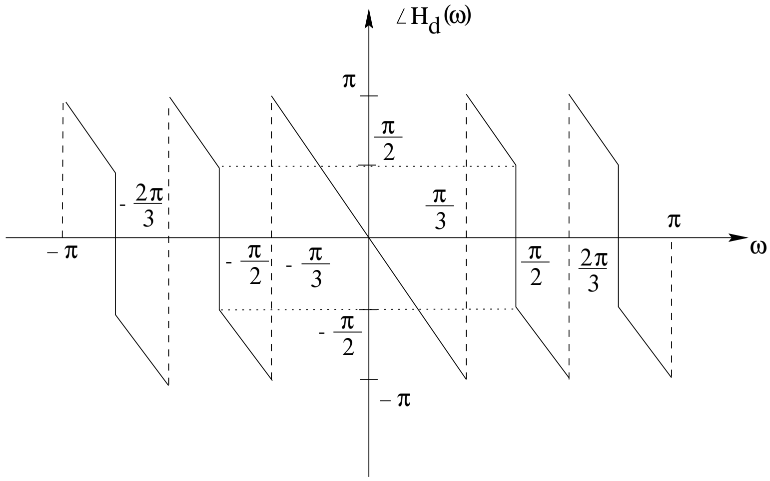
Indeed, this is GLP, with

$$\boxed{\alpha = 0, R(\omega) = 4\cos(1.5\omega) + 2\cos(0.5\omega), M = 1.5}$$

However, it will **not** exhibit linear phase, as $R(\omega)$ has zero-crossings at $\omega = \pm\frac{2\pi}{3}$.

Problem 6

(12 points - 4 points each) Given the following phase response $\angle H_d(\omega)$ of a generalized linear-phase FIR filter, answer the following questions. Explain you answers.



1. Is the filter (i) symmetric, (ii) anti-symmetric or (iii) neither symmetric nor anti-symmetric?
2. Determine the filter length from the phase plot.
3. Can you characterize the filter as (i) possibly low-pass, (ii) possibly high-pass, (iii) possibly neither high-pass nor low-pass, or (iv) the given information is insufficient to make any of the preceding statements? (Specify **all** correct answers).
4. Determine $H_d(\frac{\pi}{2})$.

Solution

(a) Since the filter is GLP, we know its DTFT can be written as $H_d(\omega) = R(\omega)e^{j(\alpha - M\omega)}$. Furthermore, we can determine the symmetry based on the value of α - if $\alpha = 0$, that means $R(\omega)$ consists of cosines, and therefore the filter is symmetric. If $\alpha = \pm\frac{\pi}{2}$, that means $R(\omega)$ consists of sines (as we needed to factor out a j to determine $R(\omega)$), and therefore the filter is antisymmetric. In this case, the phase crosses through the origin, so $\alpha = 0$, and the filter is **symmetric**. Also note that the third option makes no sense; if the filter is GLP, it must exhibit some symmetry.

(b) We know that M (the slope of the phase) and N (the length of the filter) are related through

$$M = \frac{N-1}{2}$$

This is also the factor we need to "pull out" of the DTFT to factor everything into sines and cosines. Focusing on the portion of $\angle H_d(\omega)$ given between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, we can determine that the slope is 3; therefore,

$$\boxed{N = 7}$$

(c) Since we know the filter is symmetric and odd-length, it must have **Type I GLP** - therefore, there are no restrictions on $R(\omega)$, meaning this could be *any type* of filter. So (i), (ii), and (iii) are all correct.

Note: If $R(0) = 0$, then the filter could not be low-pass. Similarly, if $R(\pi) = 0$, the filter could not be high-pass. We can observe that different types of GLP can create different filters.

(d) Since $\angle H_d(\omega)$ jumps by π at $\omega = \frac{\pi}{2}$, it must correspond to a zero-crossing of $R(\omega)$. Therefore,

$$\boxed{H_d\left(\frac{\pi}{2}\right) = 0}$$

Problem 7

(15 points for completing any one) The frequency response of a length-N symmetric or antisymmetric FIR filter with unit pulse response $h[n]$ can be expressed as

$$H_d(\omega) = R(\omega)e^{j\left(\alpha - \left(\frac{N-1}{2}\right)\omega\right)}.$$

For **ONE** of the following, show that

1. for symmetric $h[n]$ with N even,

$$R(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h[n]\cos\left(\omega\left(\frac{N-1}{2} - n\right)\right)$$

2. for symmetric $h[n]$ with N odd,

$$R(\omega) = h\left[\frac{N-1}{2}\right] + 2\sum_{n=0}^{\frac{N-3}{2}} h[n]\cos\left(\omega\left(\frac{N-1}{2} - n\right)\right)$$

3. for antisymmetric $h[n]$ with N even,

$$R(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h[n]\sin\left(\omega\left(\frac{N-1}{2} - n\right)\right)$$

4. for antisymmetric $h[n]$ with N odd,

$$R(\omega) = 2\sum_{n=0}^{\frac{N-3}{2}} h[n]\sin\left(\omega\left(\frac{N-1}{2} - n\right)\right)$$

Solution

(a) We can take the transform directly, using the fact that h is even and symmetric:

$$H_d(\omega) = h[0] + ... + h\left[\frac{N}{2} - 1\right]e^{-j(\frac{N}{2}-1)\omega} + h\left[\frac{N}{2} - 1\right]e^{-j\frac{N}{2}\omega} + ... + h[0]e^{-j\omega(N-1)}$$

We already know that we should get a sum of cosines, and we reach $\frac{N}{2} - 1$ before flipping, so we would expect $\frac{N}{2} - 1$ cosine terms. Indeed, if we factor out a complex exponential, we can rewrite this as

$$H_d(\omega) = e^{-j\omega\frac{N-1}{2}}\left(h[0]e^{j\omega\frac{N-1}{2}} + h\left[\frac{N}{2}-1\right]e^{j\omega(\frac{N-1}{2}-\frac{N}{2}-1)} + h\left[\frac{N}{2}-1\right]e^{-j\omega(\frac{N-1}{2}-\frac{N}{2}-1)} + ... + h[0]e^{-j\omega(\frac{N-1}{2})}\right)$$

Which, when simplified, becomes

$$\begin{aligned}
H_d(\omega) &= e^{-j\omega\frac{N-1}{2}}\left(h[0]e^{j\omega\frac{N-1}{2}} + h\left[\frac{N}{2}-1\right]e^{j\omega(\frac{1}{2})} + h\left[\frac{N}{2}-1\right]e^{-j\omega(\frac{1}{2})} + ... + h[0]e^{-j\omega(\frac{N-1}{2})}\right) \\
&= e^{-j\omega\frac{N-1}{2}}\left(2h[0]\cos\left(\omega\left(\frac{N-1}{2}\right)\right) + ... + 2h\left[\frac{N}{2}-1\right]\cos\left(\frac{\omega}{2}\right)\right)
\end{aligned}$$

So we can write

$$\begin{aligned}
R(\omega) &= 2h[0]\cos\left(\omega\left(\frac{N-1}{2}\right)\right) + ... + 2h\left[\frac{N}{2}-1\right]\cos\left(\frac{\omega}{2}\right) \\
&= \boxed{2\sum_{n=0}^{\frac{N}{2}-1}\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right)}
\end{aligned}$$

(b) Now, because n is odd, we can rewrite the DTFT as

$$H_d(\omega) = h[0] + ... + h\left[\frac{N-3}{2}\right]e^{-j(\frac{N-3}{2})\omega} + h\left[\frac{N-1}{2}\right]e^{-j(\frac{N-1}{2})\omega} + h\left[\frac{N-3}{2}\right]e^{-j(\frac{N-3}{2}+2)\omega} + ... + h[0]e^{-j\omega(N-1)}$$

using the fact that $\frac{N-1}{2}$ is the middle index, $\frac{N-3}{2}$ is the preceding index, and the filter exhibits even symmetry. Pulling out a factor of $e^{-j\omega\frac{N-1}{2}}$ allows us to rewrite the DTFT as

$$\begin{aligned}
H_d(\omega) &= e^{-j\omega\frac{N-1}{2}}\left(h\left[\frac{N-1}{2}\right] + h[0]e^{j\omega\frac{N-1}{2}} + h\left[\frac{N-3}{2}\right]e^{j\omega(\frac{N-1}{2}-\frac{N-3}{2})} + h\left[\frac{N-3}{2}\right]e^{-j\omega(\frac{N-1}{2}-\frac{N-3}{2})} + ... + h[0]e^{-j\omega(\frac{N-1}{2})}\right) \\
&= e^{-j\omega\frac{N-1}{2}}\left(2h[0]\cos\left(\omega\left(\frac{N-1}{2}\right)\right) + ... + 2h\left[\frac{N-3}{2}\right]\cos\left(\omega\left(\frac{N-1}{2}-\frac{N-3}{2}\right)\right) + h\left[\frac{N-1}{2}\right]\right)
\end{aligned}$$

So, we can write

$$\begin{aligned}
R(\omega) &= 2h[0]\cos\left(\omega\left(\frac{N-1}{2}\right)\right) + ... + 2h\left[\frac{N-3}{2}\right]\cos\left(\omega\left(\frac{N-1}{2}-\frac{N-3}{2}\right)\right) + h\left[\frac{N-1}{2}\right] \\
&= \boxed{h\left[\frac{N-1}{2}\right] + 2\sum_{n=0}^{\frac{N-3}{2}}\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right)}
\end{aligned}$$

(c) We can use the second approach - the only difference is that $h[N-n] = h[-n]$ instead of $h[n]$. So, if we rewrite the DTFT, we get sines instead of cosines:

$$H_d(\omega) = e^{-j\omega\frac{N-1}{2}}\Big(h[0]e^{j\omega\frac{N-1}{2}} + h[1]e^{j\omega(\frac{N-1}{2}-1)} + ...$$
$$+ h\left[\frac{N}{2}-1\right]e^{j\omega(\frac{N-1}{2}-\frac{N}{2}-1)} - h\left[\frac{N}{2}-1\right]e^{-j\omega(\frac{N-1}{2}-\frac{N}{2}-1)} - ... - h[0]e^{-j\omega(\frac{N-1}{2})}\Big)$$

Now combining the exponents gives

$$H_d(\omega) = e^{j\omega(\frac{\pi}{2}-\frac{N-1}{2})}2\sum_{n=0}^{\frac{N}{2}-1}\sin\Big(\omega\Big(\frac{N-1}{2}-n\Big)\Big)$$

and we can just read off $R(\omega)$.

(d) If N is odd, we can combine the results from (b) and (c) - we know that $h\left[\frac{N-1}{2}\right] = 0$ (to maintain symmetry, and per our results from Problem 5), and that the cosines change to sines. So we can simply write

$$H_d(\omega) = e^{-j(\frac{N-1}{2})\omega}\Big(2\sum_{n=0}^{\frac{N-3}{2}}h[n]\sin\Big(\omega\Big(\frac{N-1}{2}-n\Big)\Big)\Big)$$

giving the desired result.