# HKN ECE 310 Exam 2 Review

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# DTFT Exercise 1: Part (a)

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

## Compute the DTFT of $h[n]$:

$$
\begin{aligned}
\mathcal{F}(h[n]) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \\
&= 1 + 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(2 + e^{j\omega} + e^{-j\omega}) \\
&= e^{-j\omega}(2 + 2\cos(\omega)).
\end{aligned}
$$

# DTFT Exercise 1: Part (b)

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

## Plot the magnitude response:

$$H_d(\omega) = e^{-j\omega}(2 + 2\cos(\omega))$$

$$|H_d(\omega)| = \sqrt{H_d(\omega)H_d^*(\omega)} = |2 + 2\cos(\omega)|.$$

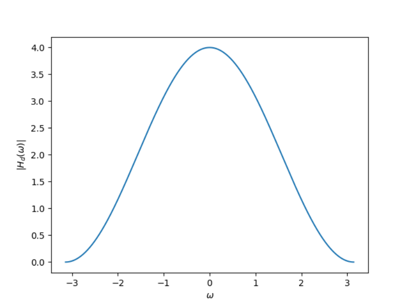


Figure 1: $|H_d(\omega)|$

# DTFT Exercise 1: Part (c)

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

## Plot the phase response:

Recall that we may decompose frequency responses as follows:

$$H_d(\omega) = \mathrm{Re}\{H_d(\omega)\} + j\mathrm{Im}\{H_d(\omega)\} = |H_d(\omega)|e^{j\angle H_d(\omega)}$$

Thus, we have

$$\angle H_d(\omega) = -\omega \text{ } (\pm\pi \text{ jumps where } H_d(\omega) \text{ changes signs.})$$

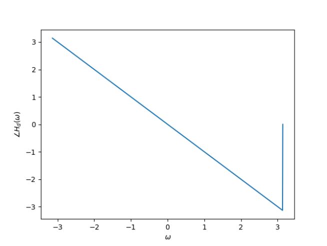


Figure 2: $\angle H_d(\omega)$

# DTFT Exercise 2: Part (a)

Given $h[n]$ real-valued and

$$|H_d(\omega)| = \begin{cases} 1, & \pi \leq \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \leq \omega < 2\pi \end{cases}, \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \leq \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \leq \omega < 2\pi \end{cases}$$

## Plot $|H_d(\omega)| \in [-\pi, \pi]$:

We are given $h[n]$ is real-valued, thus Hermitian symmetry is our best friend here.

$$H_d(\omega) = H_d^*(-\omega), |H_d(\omega)| = |H_d(-\omega)|, \ \angle H_d(\omega) = -\angle H_d(-\omega)$$

Also, utilizing the $2\pi$ periodicity of the DTFT, we have:

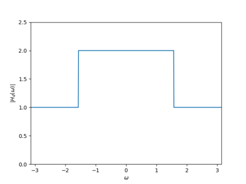


Figure 3: $|H_d(\omega)|$

# DTFT Exercise 2: Part (b)

Given $h[n]$ real-valued and

$$|H_d(\omega)| = \begin{cases} 1, & \pi \leq \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \leq \omega < 2\pi \end{cases}, \quad \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \leq \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \leq \omega < 2\pi \end{cases}$$

## Plot $\angle H_d(\omega) \in [-\pi, \pi]$:

Similarly as in Part (a),

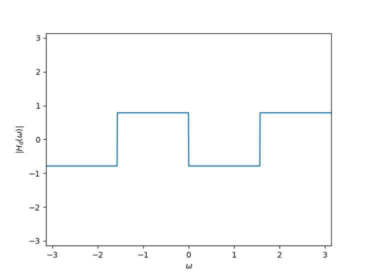


Figure 4: $\angle H_d(\omega)$

# Sinusoidal Response Exercise 1: Part (a)

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

**Find transfer function $H(z)$:**

$$\begin{aligned} y[n] &= x[n] - 2x[n-1] + x[n-2] \\ \overset{\mathcal{Z}\{\cdot\}}{\leftrightarrow} Y(z) &= X(z)(1 - 2z^{-1} + z^{-2}) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 - 2z^{-1} + z^{-2}. \end{aligned}$$

# Sinusoidal Response Exercise 1: Part (c)

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

**Find frequency response $H_d(\omega)$:**

The solution in part (a) is clearly BIBO stable since we have no poles. Thus,

$$\begin{aligned}
H_d(\omega) &= H(z)\Big|_{z=e^{j\omega}} \\
&= 1 - 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(e^{j\omega} + e^{-j\omega} - 2) \\
&= e^{-j\omega}(2\cos(\omega) - 2) \\
&= e^{j(-\omega+\pi)}(2 - 2\cos(\omega))
\end{aligned}$$

# Sinusoidal Response Exercise 1: Part (c)

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

**Find the output $y[n]$ for signals:**

From our solution in (b), we have:

$$|H_d(\omega)| = |2 - 2\cos(\omega)|,\ \angle H_d(\omega) = e^{j(-\omega+\pi)}.$$

$x_1[n] = 2 + \cos(\pi n)$ :

$$|H_d(0)| = 0,\ \angle H_d(0) = \pi,\ |H_d(\pi)| = 4,\ \angle H_d(\pi) = 0$$

$$\begin{aligned} y_1[n] &= 2(0)\cos(0n + \pi) + (4)\cos(\pi n + 0) \\ &= 4\cos(\pi n) \end{aligned}$$

# Sinusoidal Response Exercise 1: Part (c)

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

**Find the output $y[n]$ for signals:**

From our solution in (b), we have:

$$|H_d(\omega)| = |2 - 2\cos(\omega)|, \ \angle H_d(\omega) = e^{j(-\omega+\pi)}.$$

$x_2[n] = e^{j\frac{\pi}{4}n} + \sin\left(-\frac{\pi}{2}n\right)$ :

$$H_d\left(\frac{\pi}{4}\right) = (\sqrt{2}-2)e^{-j\frac{\pi}{4}}, \ \left|H_d\left(-\frac{\pi}{2}\right)\right| = 2, \ \angle H_d\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}.$$

$$\begin{aligned} y_2[n] &= \left((\sqrt{2}-2)e^{-j\frac{\pi}{4}}\right)e^{j\frac{\pi}{4}n} + (2)\sin\left(-\frac{\pi}{2}n - \frac{\pi}{2}\right) \\ &= (\sqrt{2}-2)e^{j\left(\frac{\pi}{4}n-\frac{\pi}{4}\right)} + 2\sin\left(-\frac{\pi}{2}n - \frac{\pi}{2}\right) \end{aligned}$$

# Sampling Exercise 1

For analog signal $x_a(t) = \cos(\Omega_0 t)$ sampled at $T = \frac{1}{1000}$s to obtain $x[n] = \cos\left(\frac{\pi}{4}n\right)$

## What are possible values for $\Omega_0$?

Always remember $\omega_d = \Omega_a T$! Moreover, by $2\pi$ periodicity of the DTFT,

$$x[n] = \cos\left(\frac{\pi}{4}n\right) \equiv \cos\left(\left(\frac{\pi}{4} + 2\pi k\right) n\right), \ k \in \mathbb{Z}.$$

Thus,

$$\begin{aligned} \Omega_0 T &= \left(\frac{\pi}{4} + 2\pi k\right) \\ \Omega_0 &= 250\pi + 2000\pi k, \ k \in \mathbb{Z}. \end{aligned}$$

Possible values for $\Omega_0$ are then $250\pi$ ( choice a), -$1750\pi$ (choice c) and $4250\pi$ (choice d).

# Sampling Exercise 2: Part (a)

The maximum frequency of $X_a(\Omega)$ is $4000\pi$.

## Sketch DTFT spectrum when $T = \frac{1}{8000}$

In each part, we should focus on where the maximum frequency maps. This will help us identify aliasing. If there is no aliasing, the shape of the spectrum will not change. By $\omega_d = \Omega_a T$,

$$\omega_d = \frac{4000\pi}{8000} = \frac{\pi}{2}.$$

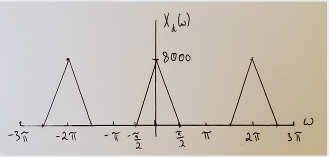


Figure 5: $T_1 = \frac{1}{8000}s$

# Sampling Exercise 2: Part (b)

The maximum frequency of $X_a(\Omega)$ is $4000\pi$.

## Sketch DTFT spectrum when $T = \frac{1}{4000}$

By $\omega_d = \Omega_a T$,

$$\begin{aligned}\omega_d &= \frac{4000\pi}{4000} \\ &= \pi.\end{aligned}$$

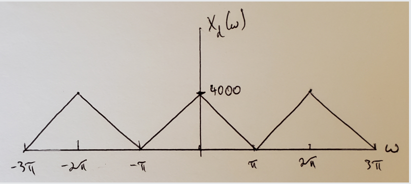


Figure 6: $T_1 = \frac{1}{4000}s$

# Sampling Exercise 2: Part (c)

The maximum frequency of $X_a(\Omega)$ is $4000\pi$.

## Sketch DTFT spectrum when $T = \frac{1}{2000}$

By $\omega_d = \Omega_a T$,

$$\omega_d = \frac{4000\pi}{2000}$$
$$= 2\pi.$$

Aliasing!

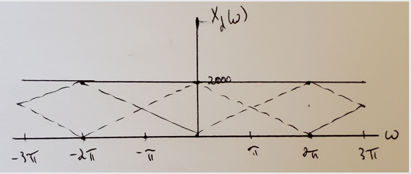


Figure 7: $T_1 = \frac{1}{2000} s$

# DFT Exercise 1: Part (a)

Given $x[n] = \cos\left(\frac{\pi}{3}n\right),\ 0 \leq n < 18,$

## For which value(s) of $k$ is $X[k]$ largest?

$$\omega_k = \frac{2\pi k}{N}$$
$$\frac{\pi}{3} = \frac{\pi}{9}k$$
$$k_1 = 3.$$

We also have energy at $\omega = -\frac{\pi}{3}$:

$$-\frac{\pi}{3} = \frac{\pi}{9}k$$
$$k_2 = -3$$

Instead by $N$ periodicity of the DFT, we should say

$$k_2 \equiv k_2 + N = 15$$

# DFT Exercise 1: Part (a)

Given $x[n] = \cos\left(\frac{\pi}{3}n\right),\ 0 \leq n < 18$,

**For which $k$ is $X[k]$ largest if $x[n]$ is padded with 72 zeros?**

New length is $N = 90$:

$$\omega_k = \frac{2\pi k}{N}$$
$$\frac{\pi}{3} = \frac{\pi}{45}k$$
$$k_1 = 15.$$

$$-\frac{\pi}{3} = \frac{\pi}{45}k$$
$$k_2 = -15$$
$$\implies k_2 = 75.$$

Thanks everyone!
Good luck studying!