Recitation 8 Sol

Q1

a) What is the DFT of $x_1[n] + 3x_2[n]$?

Let

$$x_3[n] = x_1[n] + 3x_2[n]$$

$$X_3[k] = X_1[k] + 3X_2[k] = \{4, 6+2j, 2, -2\}$$

b) What is the DFT of $e^{j\pi n}x_1[n]$?

$$e^{j\pi n} = e^{-j\frac{2\pi}{4}(-2n)} = W_N^{-2n}$$

$$\begin{aligned} DFT\{e^{j\pi n}x_1[n]\} &= DFT\{W_N^{-2n}x_1[n]\} \\ &= X_1[< k-2 >_4] = \{-1, 1, 1, 2j\} \end{aligned}$$

c) What is $\sum_{n=0}^{3} |x_2[n]|^2$?

$$\sum_{n=0}^{3} |x_2[n]|^2 = \frac{1}{4}\sum_{k=0}^{3} |X_2[k]|^2 = \frac{1}{4}(1+4+1+1) = \frac{7}{4}$$

d) What is $x_2[0]$?

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-nk}$$

$$x_2[0] = \frac{1}{4}\sum_{k=0}^{3} X_2[k] = \frac{1}{4}(1+2+1-1) = \frac{3}{4}$$

e) What is $x_1[1]$?

$$x_1[1] = \frac{1}{4}\sum_{k=0}^{3} X_1[k] W_4^{-k}$$

$$W_4^{-1} = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$$

$$x_1[1] = \frac{1}{4}[1 + 2j(j) - (j)^2 + (j)^3] = \frac{-j}{4}$$

Q2

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{9} x[n] e^{-j\frac{2\pi}{10}mn}$$

$$Y[m] = X[m] e^{-j\frac{2\pi}{5}mn_0} = X[m] e^{-j\frac{2\pi}{5}m\cdot 3}$$

Recall **time shift property**

$$\{x_{<n\pm k>_N}\}_{n=0}^{N-1} < -DFT- > X[m] e^{\pm j\frac{2\pi}{N}mk}$$

In this case, N=10. Therefore, we want to make

$$\frac{2\pi}{10}mk = \frac{2\pi}{5}m \cdot 3$$

We have $k = 6$.

Thus,

$$y[n] = \{x_{<n-6>_N}\}_{n=0}^{9} = \{-3, 4, 0, 0, 0, 0, 1, -1, 2, 3\}$$

Q3

$$Y[m] = \sum_{n=0}^{255} y[n] e^{-j\frac{2\pi}{256}mn}$$

$$Y[32] = \sum_{n=0}^{255} y[n] e^{-j\frac{2\pi}{256}32n} = \sum_{n=0}^{255} y[n] e^{-j\frac{2\pi}{8}n}$$

We want $Y[32] = X[m_0]$ where

$$X[m_0] = \sum_{n=0}^{239} x[n] e^{-j\frac{2\pi}{240}nm_0}$$

Since $y[n]$ is obtained by zero-padding $x[n]$, we have $y[n] = x[n]$ for $n \in [0, 239]$. And $y[n] = 0$ for $n \in [240, 255]$.

$$Y[32] = \sum_{n=0}^{239} x[n] e^{-j\frac{2\pi}{8}n} = X[30]$$

Thus, $m_0 = 30$.

Q4 Cyclic Convolution

$$z_n = \{x_n\}_{n=0}^{N-1} \otimes \{y_n\}_{n=0}^{N-1} = \sum_{l=0}^{N-1} x_l y_{<n-l>_N} = \sum_{l=0}^{N-1} x_{<n-l>_N} y_l$$

$$\{Z_m\}_{m=0}^{N-1} = \{X_m Y_m\}_{m=0}^{N-1}$$

$$z_n = IDFT\{Z_m\} = \frac{1}{N} \sum_{m=0}^{N-1} X_m Y_m W_N^{-mn}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} [\sum_{k=0}^{N-1} x[k] W_N^{km}][\sum_{l=0}^{N-1} y[l] W_N^{lm}] W_N^{-mn}$$

$$= \sum_{k=0}^{N-1} x[k] \sum_{l=0}^{N-1} y[l] [\frac{1}{N} \sum_{m=0}^{N-1} W_N^{m(k+l-n)}]$$

Using the orthogonality condition:

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{m(k+l-n)} = \begin{cases} 1, & k+l-n = rN \\ 0, & O.W. \end{cases}$$

Therefore, the last summation in the above equation is nonzero as long as $k = n - l + rN =< n - l >_N$. Equivalently, $l =< n - k >_N$

Plug the relationship into above equation, we have.

$$z[n] = \sum_{k=0}^{N-1} x[k]y[< n - l >_N] = \sum_{l=0}^{N-1} y[l]x[< n - k >_N]$$