ECE 310 Recitation 9 Solution
Thursday Apr 15, 2021

Concept check

√ DFT spectral analysis

- Determine the frequency content of a given signal: $x_a(t) = \sum_{i=1}^{M} A_i \cos(\Omega_i t)$, determine $\{\Omega_i, A_i\}_{i=1}^{M}$
- Ideally
  - $X_a(\Omega) = \sum_{i=1}^{M} \pi A_i[\delta(\Omega + \Omega_i) + \delta(\Omega - \Omega_i)]$
  - $X_d(\omega) = \begin{cases} \frac{1}{T} X_a(\frac{\omega}{T}), & 0 \le \omega \le \pi \\ \frac{1}{T} X_a(\frac{\omega - 2\pi}{T}), & \pi < \omega \le 2\pi \end{cases}$
  - $X_m = X_d(\frac{2\pi}{N} m)$
- Spectral parameters
  - Amplitudes: $\frac{A_i}{T}$
  - Frequencies: $m_i \rightarrow \omega = \frac{2\pi}{N} m_i \rightarrow \Omega_i = \frac{\omega_i}{T}$ or $\Omega_i = \frac{\omega_i}{T} - \frac{2\pi}{T}$
- Windowing effect:
  - $\hat{x}[n] = x[n]w[n], \widehat{X_d}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(u) W_d(\omega - u) du$
  - For sinusoidal input $x[n] = A\cos(\Omega_0 nT)$ and rectangular window $w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & else \end{cases}$
  - $\widehat{X_d}(\omega) = e^{-j(\omega - \Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2} sin[(\omega - \Omega_0 T)\frac{N}{2}]}{\sin[(\omega - \Omega_0 T)\frac{1}{2}]} + e^{-j(\omega + \Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2} sin[(\omega + \Omega_0 T)\frac{N}{2}]}{\sin[(\omega + T)\frac{1}{2}]}$
  - Main lobe height: $\frac{AN}{2}$, main lobe width $\frac{4\pi}{N}$, peak location $\omega = \pm\Omega_0 T$

√ FFT

- $\begin{cases} X_m = Y_m + W_N^m Z_m \\ X_{m+N/2} = Y_m - W_N^m Z_m \end{cases}, \begin{cases} Y_m = DFT\{x[2l]\}_{l=0}^{n/2-1} \ (even) \\ Z_m = DFT\{x[2l+1]\}_{l=0}^{n/2-1} \ (odd) \end{cases}, W_N^m = e^{-j\frac{2\pi m}{N}}$
- Butterfly diagram, bit-reverse indexing

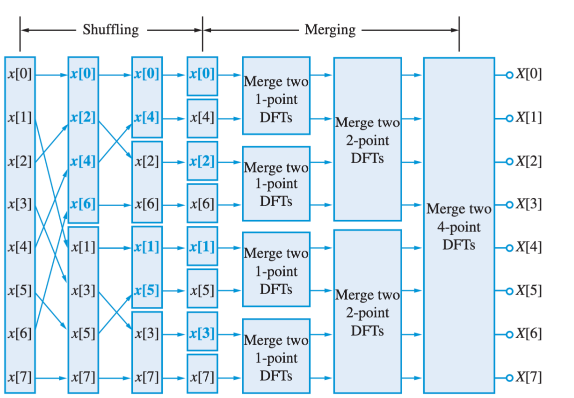


Figure 8.5 The shuffling and merging operations required for recursive computation of the 8-point DFT using the decimation-in-time FFT algorithm.

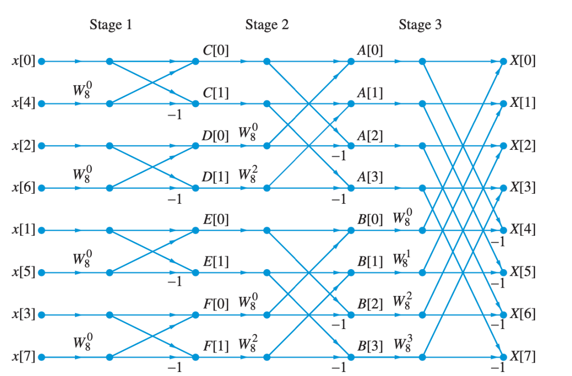


Figure 8.6 Flow graph of 8-point decimation-in-time FFT algorithm using the butterfly computation shown in Figure 8.4. The trivial twiddle factor $W_8^0 = 1$ is shown for the sake of generality.

Exercise

1. A continuous-time signal $x_c(t) = \cos(\frac{\pi}{3}t)$ is sampled at a rate of 30Hz for 12s to produce a discrete-time signal $x[n]$ with length $L = 360$.

   (a) Let $X[k]$ be the length-L DFT of $x[n]$. At what value(s) of k will $X[k]$ have the greatest magnitude?

   $x[n] = \{x_c(nT)\}_{n=0}^{359} = \left\{\cos(\frac{\pi}{90}n)\right\}_{n=0}^{359}$

   If $x[n]$ was not truncated, the DTFT of a cosine would be a pair of shifted delta functions for every period. Since there is a truncation, the DTFT now has the deltas replaced with sinc-like functions.
   Given the resemblance between the DTFT of the cosine and truncated cosine, the greatest magnitudes for both DTFTs are at the same values of ω.

   $$\frac{2\pi k}{360} = \frac{\pi}{90} \to k = 2; \quad 2\pi - \frac{2\pi k}{360} = \frac{\pi}{90} \to k = 358$$

   (b) Suppose that $x[n]$ is zero-padded to a total length of $L = 512$. At what value(s) of k will $X[k]$ have the greatest magnitude?

   Similarly,

   $$\frac{2\pi k}{512} = \frac{256}{90} \to k = 2.84 \approx 3; \quad 2\pi - \frac{2\pi k}{512} = \frac{\pi}{90} \to k = 509.15 \approx 509$$

   Round to the nearest integer gives the largest $X[k]$.

   For the exact derivation of $X[k]$, we can use the above formula:

$$X[k] = X_d\left(\frac{2\pi}{L}k\right)$$
$$= \left(e^{-j(\omega-\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin\left[(\omega-\Omega_0 T)\frac{N}{2}\right]}{\sin\left[(\omega-\Omega_0 T)\frac{1}{2}\right]} + e^{-j(\omega+\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin\left[(\omega+\Omega_0 T)\frac{N}{2}\right]}{\sin\left[(\omega+T)\frac{1}{2}\right]}\right)\Bigg|_{\omega=\frac{2\pi}{L}k}$$
$$= e^{-j\left(\frac{\pi k}{256}-\frac{\pi}{90}\right)\frac{360-1}{2}} \frac{\sin\left[\left(\frac{\pi k}{256}-\frac{\pi}{90}\right)\frac{360}{2}\right]}{2\sin\left[\left(\frac{\pi k}{256}-\frac{\pi}{90}\right)\frac{1}{2}\right]} + e^{-j\left(\frac{\pi k}{256}+\frac{\pi}{90}\right)\frac{360-1}{2}} \frac{\sin\left[\left(\left(\frac{\pi k}{256}+\frac{\pi}{90}\right)\right)\frac{360}{2}\right]}{2\sin\left[\left(\frac{\pi k}{256}+\frac{\pi}{90}\right)\frac{1}{2}\right]}$$

   Then we can check the values near peak to determine the exact largest $X[k]$.

   (c) Suppose that $x(t)$ is only sampled for 2s, so the length of $X[k]$ is $L = 60$. At what value(s) of k will $X[k]$ have the greatest magnitude?

$$x[n] = \{x_c(nT)\}_{n=0}^{59} = \left\{\cos(\frac{\pi}{90}n)\right\}_{n=0}^{59}$$

$$\frac{2\pi k}{60} = \frac{\pi}{90} \to k = \frac{1}{3} \approx 0; \quad 2\pi - \frac{2\pi k}{60} = \frac{\pi}{90} \to k = \frac{179}{3} \approx 60\ (in\ next\ period)$$

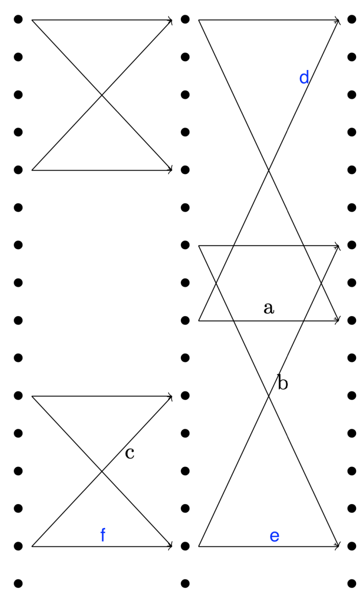
What happens in time domain?
Period of cosine $T_0 = \frac{2\pi}{\pi/3} = 6s$, sampling time $T_s = 2s < T_0$. Not even sampled an entire period! (Information loss!)

(d) Suppose that the $x[n]$ from (c) is zero-padded to a total length of $L = 64$. At what value(s) of k will $X[k]$ have the greatest magnitude?

$$\frac{2\pi k}{64} = \frac{\pi}{90} \to k = \frac{32}{90} \approx 0; 2\pi - \frac{2\pi k}{64} = \frac{\pi}{90} \to k = \frac{179 \times 32}{90} \approx 64\ (in\ next\ period)$$

Zero padding cannot save an "incomplete" sample!

2. The diagram below represents a part of the computation in a 16-point decimation-in-time radix-2 FFT. Indicate the values of the three branch weights, d, e, and f.



- Branches from the top half are always 1 (coefficient of Y_m is 1).
- Branches going up from the bottom half are W_N^m (coefficient of Z_m).
- Branches going straight across in the bottom half are $-W_N^m$ (coefficient of Z_m).
- N is the current DFT size, m is index of current DFT

$$d = W_{16}^0 = 1$$
$$e = -W_{16}^6 = -e^{-j\frac{2\pi 6}{16}} = -e^{-j\frac{3\pi}{4}} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$
$$f = -W_8^2 = -e^{-j\frac{2\pi 2}{8}} = -e^{-j\frac{\pi}{2}} = j$$