ECE 310 Recitation 7 Solution
Thursday Apr 1, 2021

Concept check

√ Frequency response
  - $\mathcal{H}\{e^{j\omega_0 n}\} = H_d(\omega_0)e^{j\omega_0 n} = |H_d(\omega_0)|e^{j(\omega_0 n + \angle H_d(\omega_0))}$
  - For real system: $h[n]$ real or $H_d(\omega) = H_d^*(-\omega)$
    - $\mathcal{H}\{\cos(\omega_o n + \phi_0)\} = |H_d(\omega_0)|\cos(\omega_0 n + \phi_0 + \angle H_d(\omega_0))$

√ Sampling: A/D converter
  - Time domain: $x[n] = x_a(nT)$
  - Frequency domain: $X_d(\omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X_a(\frac{\omega+2l\pi}{T})$
  - Nyquist criterion: $f_S > 2f_{max}$
  - Aliasing effect

Exercise

1. The frequency responses of two LSI systems are respectively $H_{d1}(\omega) = cos\omega e^{jsin\omega}$ and $H_{d2}(\omega) = sin\omega e^{jcos\omega}$.
   The input is $x[n] = 5 + 10\cos\left(\frac{\pi}{4}n + 45°\right) + j^n$
   Determine the corresponding system output $y_1[n]$ and $y_2[n]$.

   (a) $H_{d1}(\omega) = cos\omega e^{jsin\omega}$

   First check if the system is real:
   $$H_{d1}^*(-\omega) = \cos(-\omega)\, e^{-jsin(-\omega)} = \cos\omega\, e^{jsin\omega} = H_{d1}(\omega)$$

   The system is real, so we can use $\mathcal{H}\{\cos(\omega_o n + \phi_0)\} = |H_d(\omega_0)|\cos(\omega_0 n + \phi_0 + \angle H_d(\omega_0))$.

   Rewrite $x[n]$ as:
   $$x[n] = 5 + 10\cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right) + e^{j\frac{\pi}{2}n}$$

   Calculate the DTFT values:
   $$H_{d1}(0) = 1, \qquad H_{d1}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}, \qquad H_{d1}\left(\frac{\pi}{2}\right) = 0$$
   Output $y[n]$ is:
   $$y[n] = 5H_{d1}(0) + 10\left|H_{d1}\left(\frac{\pi}{4}\right)\right|\cos\left(\frac{\pi}{4}n + 45° + \angle H_{d1}\left(\frac{\pi}{4}\right)\right) + e^{j\frac{\pi}{2}n}H_{d1}\left(\frac{\pi}{2}\right)$$
   $$= 5 + 5\sqrt{2}\cos\left(\frac{\pi}{4}n + \frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)$$

(b) $H_{d2}(\omega) = sin\omega e^{jcos\omega}$

First check if the system is real:

$$H_{d2}^*(-\omega) = \sin(-\omega)\, e^{-jcos(-\omega)} = -\sin\omega\, e^{-jcos\omega} \neq H_{d2}(\omega)$$

The system is not real, we have to break the sinusoidal terms into complex exponentials through Euler formula.

Rewrite $x[n]$ as:

$$x[n] = 5 + 5e^{j\frac{\pi}{4}n}e^{j\frac{\pi}{4}} + 5e^{-j\frac{\pi}{4}n}e^{-j\frac{\pi}{4}} + e^{j\frac{\pi}{2}n}$$

Calculate the DTFT values:

$$H_{d2}(0) = 0, \qquad H_{d2}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}, \qquad H_{d2}\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}, \qquad H_{d2}\left(\frac{\pi}{2}\right) = 1$$

Output $y[n]$ is:

$$\begin{aligned}
y[n] &= 5H_{d2}(0) + 5H_{d2}\left(\frac{\pi}{4}\right)e^{j\frac{\pi}{4}n}e^{j\frac{\pi}{4}} + 5H_{d2}\left(-\frac{\pi}{4}\right)e^{-j\frac{\pi}{4}n}e^{-j\frac{\pi}{4}} + e^{j\frac{\pi}{2}n}H_{d1}\left(\frac{\pi}{2}\right) \\
&= 0 + 5\cdot\frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}e^{j\frac{\pi}{4}n}e^{j\frac{\pi}{4}} - 5\cdot\frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}e^{-j\frac{\pi}{4}n}e^{-j\frac{\pi}{4}} + e^{j\frac{\pi}{2}n} \\
&= 5\cdot\frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}\left(e^{j\frac{\pi}{4}n}e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}n}e^{-j\frac{\pi}{4}}\right) + e^{j\frac{\pi}{2}n} \\
&= 5\cdot\frac{\sqrt{2}}{2}e^{j\frac{\sqrt{2}}{2}}\cdot 2jsin\left(\frac{\pi}{4}(n+1)\right) + e^{j\frac{\pi}{2}n} \\
&= e^{j\frac{\sqrt{2}+\pi}{2}}\cdot 5\sqrt{2}sin\left(\frac{\pi}{4}(n+1)\right) + e^{j\frac{\pi}{2}n}
\end{aligned}$$

2. A continuous-time signal $x_a(t) = \sin(at)$ is sampled with a sampling period T to obtain a discrete-time signal $x[n] = \sin(bn)$, where a, b are constants. Determine a set of choices of T consistent with the information given.

   Intuitively,

$$x[n] = x_a(nT) = \sin(aTn) = \sin(bn) \implies aT = b \implies T = \frac{b}{a}$$

   But this is not unique. Consider the periodicity of sinusoidal.

$$\begin{aligned}
x[n] = x_a(nT) = \sin(aTn) = \sin\big(n(aT + 2k\pi)\big) = \sin(bn) \\
\implies aT + 2k\pi = b \implies T = \frac{b - 2k\pi}{a}, k \in \mathbb{Z}
\end{aligned}$$

   Any choice of k such that $T > 0$ is consistent with the information given.

3. Suppose $x[n] = x_a(nT)$, prove that $X_d(\omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X_a(\frac{\omega+2l\pi}{T})$.

Hint 1: $X_d(\omega) = \sum_{l=-\infty}^{\infty} x[n]e^{-j\omega l}$

Hint 2: $x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t}d\Omega$

Hint 3: $\sum_{k=-\infty}^{\infty} e^{j2\pi kt} = \sum_{k=-\infty}^{\infty} \delta(t-k)$

Hint 3 comes from the Fourier series of an infinite-sum of delta functions.

Recall that any continuous function $f(t)$ with period T can be represented as Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\frac{2\pi}{T}t}\ , where\ c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\frac{2\pi}{T}t}\mathrm{dt}$$

Let $f(t)$ be a train of impulse with period T:

$$f(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

Then

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\frac{2\pi}{T}t}\mathrm{dt} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\frac{2\pi}{T}t}\mathrm{dt} = \frac{1}{T}$$

Hence

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

Let $\mathrm{t}' = \mathrm{t}/\mathrm{T}$

$$\sum_{k=-\infty}^{\infty} e^{jk2\pi t'} = \sum_{k=-\infty}^{\infty} T\delta(t'T-kT) = \sum_{k=-\infty}^{\infty} \delta(t'-k)$$