ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 20

Topics:

- ✓ Properties of discrete Fourier transform (DFT)

Educational Objectives:

- ✓ Understand the key properties of DFT
- ✓ Understand circular convolution
- ✓ Understand zero-padded DFT

Summary

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn}, \qquad m = 0,1\ldots N-1$$

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}mn}, \qquad n = 0,1\ldots N-1$$

$$X_d(\omega) = \sum_{n=0}^{N-1} x_n e^{-j\omega n} \longrightarrow X_m = X_d(\omega)\Big|_{\omega=\frac{2\pi}{N}m}$$

DFT Properties

a) Linearity

$$a\{x_n\}_{n=0}^{N-1} + b\{y_n\}_{n=0}^{N-1} \xleftrightarrow{DFT} a\{X_m\}_{m=0}^{N-1} + b\{Y_m\}_{m=0}^{N-1}$$

b) Periodicity (Periodic extension)

$$X_{m+lN} = X_m \qquad x_{n+lN} = x_n$$

or

$$X_m = X_{<m>_N};\ x_n = x_{<n>_N}$$

$$where <k>_N = r \quad if\ k = lN + r \quad 0 \le r \le N-1$$

Examples:

$$<7>_4 =$$

$$<4>_4 =$$

$$<-5>_4 =$$

$$<-2>_4 =$$

DFT Properties

c) Conjugate Symmetry

$$if\ \{x_n\}_{n=0}^{N-1}\ is\ real,\ then$$

$$X_m = X^*_{<N-m>_N} \qquad m = 0, 1, \ldots, N-1$$

$$|X_m| = |X_{<N-m>_N}|, \qquad \angle X_m = -\angle X_{<N-m>_N}$$

$$\mathrm{Re}\{X_m\} = \mathrm{Re}\{X_{<N-m>_N}\}; \qquad \mathrm{Im}\{X_m\} = -\mathrm{Im}\{X_{<N-m>_N}\}$$

Example:

DFT Properties

c) Time shift

$$\{x_{<n\pm k>_N}\}_{n=0}^{N-1} \xleftrightarrow{DFT} X_m e^{\pm j\frac{2\pi}{N}mk}$$

Proof:

$$\frac{1}{N}\sum_{m=0}^{N-1} X_m e^{\pm j\frac{2\pi}{N}mk} e^{j\frac{2\pi}{N}nm}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}m(n\pm k)}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}m<n\pm k>_N}$$

$$= x_{<n\pm \mathrm{k}>_N}$$

Examples

$$Let\ \{x_n\}_{n=0}^{4} = \{1,3,5,7,9\} \xleftrightarrow{DFT} \{X_0, X_1, X_2, X_3, X_4\}$$

$$\{9,1,3,5,7\} \xleftrightarrow{DFT}$$

$$\{3,5,7,9,1\} \xleftrightarrow{DFT}$$

$$\{7,9,1,3,5\} \xleftrightarrow{DFT}$$

$$\{5,7,9,1,3\} \xleftrightarrow{DFT}$$

DFT Properties

e) Duality

$$\{x_n\}_{n=0}^{N-1} \xleftrightarrow{DFT} \{X_m\}_{m=0}^{N-1}$$

$$DFT[\{X_n\}_{n=0}^{N-1}] = \{Nx_{<N-m>_N}\}_{m=0}^{N-1}$$

Applying a forward DFT twice gives $\{x_n\}$ back, but scaled by N and flipped around (except the x_0 term)

Example:

$$Let\ DFT\{2,4,6,8,10,12\} = \{X_0, X_1, X_2, ..., X_5\}$$

$$DFT\{X_0, X_1, X_2, X_3, X_4, X_5\} = 6\{12,10,8,6,4,2\}$$

DFT Properties

f) Parseval theorem

$$\sum_{m=0}^{N-1} |X_m|^2 = N \sum_{n=0}^{N-1} |x_n|^2$$

g) Cyclic Convolution (vs linear convolution)

$$\{x_n\}_{n=0}^{N-1}, \{y_n\}_{n=0}^{N-1}$$

$$\begin{aligned} z_n &= \{x_n\}_{n=0}^{N-1} \otimes \{y_n\}_{n=0}^{N-1} \\ &= \sum_{l=0}^{N-1} x_l y_{<n-l>_N} \\ &= \sum_{l=0}^{N-1} x_{<n-l>_N} y_l \end{aligned}$$

$$\{Z_m\}_{m=0}^{N-1} = \{X_m Y_m\}_{m=0}^{N-1}$$

Zero-padded DFT

$$\{x_n\}_{n=0}^{N-1} = \{x_{0,} x_{1,} \cdots, x_{N-1}\} \leftrightarrow \{X_{0,} X_{1,} \cdots, X_{N-1}\}$$

$$\{y_n\}_{n=0}^{M-1} = \{x_{0,} x_{1,} \cdots, x_{N-1}, 0, 0, \cdots, 0\} \leftrightarrow \{Y_{0,} Y_{1,} \cdots, Y_{M-1}\}$$

How are X_m related to Y_m ?