ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 9

Topics:

- ✓ Inverse Z-transform

Educational Objectives:

- ✓ Understand how to use the long division method
- ✓ Understand how to use the partial fraction expansion method

General Inversion Formula

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- ✓ Not practically useful in engineering applications
- ✓ Assume X(z) =B(z)/A(z)
- ✓ Long division method
- ✓ Partial fraction expansion method

Long Division Method

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example:

$$X(z) = \frac{z}{z-0.5}, \qquad |z| > 0.5$$

$$X(z) = \frac{z}{z-0.5}, \qquad |z| < 0.5$$

Partial Fraction Expansion Method

- ✓ Partial Fraction Expansion
- ✓ Table look-up

Key transform pairs			ROC
$\delta[n]$	$\leftrightarrow$	1	$all\ z$
$a^n u[n]$	$\leftrightarrow$	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\leftrightarrow$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\leftrightarrow$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\leftrightarrow$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$cos(\omega_0 n)u[n]$	$\leftrightarrow$	$\dfrac{z^2 - z\ cos\omega_0}{z^2 - z\ 2cos\omega_0 + 1}$	$\lvert z\rvert > 1$
$sin(\omega_0 n)u[n]$	$\leftrightarrow$	$\dfrac{z\ sin\omega_0}{z^2 - z\ 2cos\omega_0 + 1}$	$\lvert z\rvert > 1$
$r^n cos(\omega_0 n)u[n]$	$\leftrightarrow$	$\dfrac{z^2 - z\ rcos\omega_0}{z^2 - z\ 2rcos\omega_0 + r^2}$	$\lvert z\rvert > \lvert r\rvert$
$r^n sin(\omega_0 n)u[n]$	$\leftrightarrow$	$\dfrac{z\ rsin\omega_0}{z^2 - z\ 2rcos\omega_0 + r^2}$	$\lvert z\rvert > \lvert r\rvert$

Partial Fraction Expansion

a) $\hat{X}(z) = \frac{X(z)}{z}$

b) Check if $\hat{X}(z)$ is proper:

$$\hat{X}(z) = \frac{B(z)}{A(z)}, \qquad order\big(B(z)\big) < order\big(A(z)\big)$$

c) Factoring of $A(z)$

$$\hat{X}(z) = \frac{B(z)}{(z-p_1)(z-p_2)\ldots(z-p_n)}$$

d) Single poles: $\frac{c_i}{z-p_i}$

e) Multiple poles: $\frac{1}{(z-p_i)^2} \rightarrow \frac{A}{(z-p_i)^2} + \frac{B}{z-p_i}$

f) Determine coefficients:

Example 1

$$\mathrm{X}(z) = \frac{z}{(z-2)(z-3)(z-4)}, \qquad |z| > 4$$

$$\hat{X}(z) = \frac{X(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$A = \hat{X}(z)(z-2)\Big|_{z=2} = \frac{1}{(z-3)(z-4)}\Big|_{z=2} = \frac{1}{2}$$

$$B = \hat{X}(z)(z-3)\Big|_{z=3} = \frac{1}{(z-2)(z-4)}\Big|_{z=3} = -1$$

$$C = \hat{X}(z)(z-4)\Big|_{z=4} = \frac{1}{(z-2)(z-3)}\Big|_{z=4} = \frac{1}{2}$$

$$X(z) = \frac{\frac{1}{2}z}{z-2} - \frac{z}{z-3} - \frac{\frac{1}{2}z}{z-4} \quad \Rightarrow \quad x[n] = \frac{1}{2}2^n u[n] - 3^n u[n] + \frac{1}{2}4^n u[n]$$

Example 2

$$\mathrm{X}(z) = \frac{z^2}{(z+1)(z-1)^2}, |z| > 1$$

$$\hat{X}(z) = \frac{X(z)}{z} = \frac{z}{(z+1)(z-1)^2} = \frac{C_0}{z+1} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}$$

$$C_0 = \hat{X}(z)(z+1)\Big|_{z=-1} = \frac{z}{(z-1)^2}\Big|_{z=-1} = -\frac{1}{4}$$

$$C_2 = \hat{X}(z)(z-1)^2\Big|_{z=1} = \frac{z}{(z+1)}\Big|_{z=1} = \frac{1}{2}$$

Example 2

$$X(z) = \frac{z^2}{(z+1)(z-1)^2}, |z| > 1 \qquad \hat{X}(z) = \frac{C_0}{z+1} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}$$

Two ways to determine C_1

$$C_0 = -\frac{1}{4} \qquad C_2 = \frac{1}{2}$$

a)

$$C_1 = \frac{d}{dz}\left[\hat{X}(z)(z-1)^2\right]\Big|_{z=1} = \frac{d}{dz}\left[\frac{z}{z+1}\right]\Big|_{z=1} = \frac{1}{(z+1)^2}\Big|_{z=1} = \frac{1}{4}$$

b)

$$\frac{z}{(z+1)(z-1)^2} = \frac{-\frac{1}{4}}{z+1} + \frac{C_1}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$= \frac{-\frac{1}{4}(z-1)^2 + C_1(z+1)(z-1) + \frac{1}{2}(z+1)}{(z+1)(z-1)^2}$$

$$= \frac{\left(C_1 - \frac{1}{4}\right)z^2 + z + \frac{1}{4} - C_1}{(z+1)(z-1)^2} \quad \Rightarrow \quad C_1 = \frac{1}{4}$$

Example 2

$$\mathrm{X}(z) = \frac{z^2}{(z+1)(z-1)^2}, \qquad |z| > 1$$

$$\hat{X}(z) = \frac{C_0}{z+1} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}$$

$$C_0 = -\frac{1}{4} \qquad C_2 = \frac{1}{2} \qquad C_1 = \frac{1}{4}$$

$$X(z) = \frac{-\frac{1}{4}z}{z+1} + \frac{\frac{1}{4}z}{z-1} + \frac{\frac{1}{2}z}{(z-1)^2}$$

$$\Rightarrow \quad x[n] = -\frac{1}{4}(-1)^n u[n] + \frac{1}{4}u[n] + \frac{1}{2}nu[n]$$

$$= \left(-\frac{1}{4}(-1)^n + \frac{1}{2}n + \frac{1}{4}\right)u[n]$$

Example 3

$$X(z) = \frac{z^3}{z^2 + 3z + 2}, \qquad |z| > 2$$