ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 2

Topics:

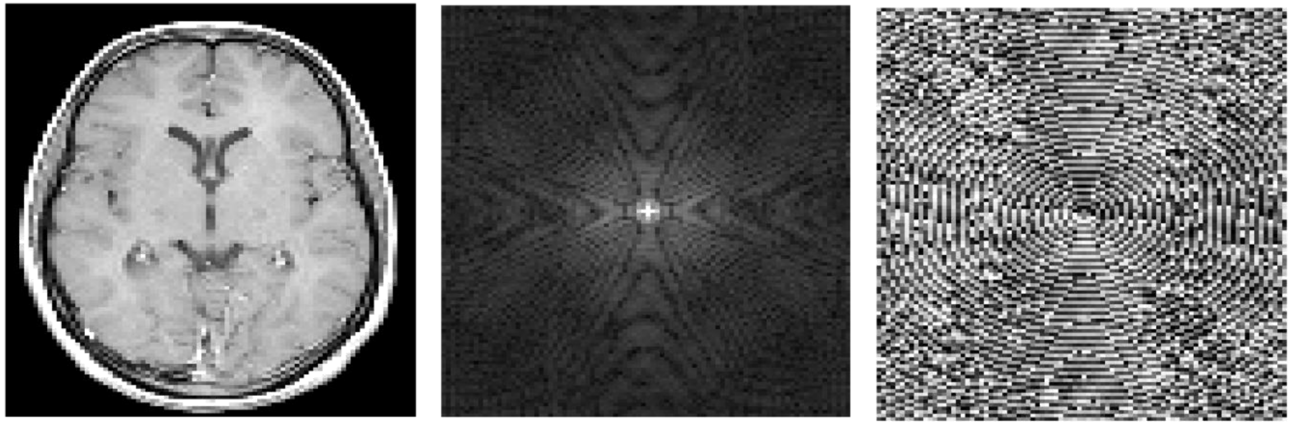
- ✓ Complex variables and functions

Objectives:

- ✓ Understand the need to use complex variables to describe signal representations and processing schemes
- ✓ Understand the two representations (Cartesian & polar forms) of complex variables
- ✓ Understand operations on complex variables

Need to Use Complex Variables

- ✓ **Fourier representations are complex-valued, even for real-valued signals**
- ✓ **Magnitude and phase provide complementary information about a signal**
- ✓ **Processing algorithms are designed to have different effects on magnitude and phase**



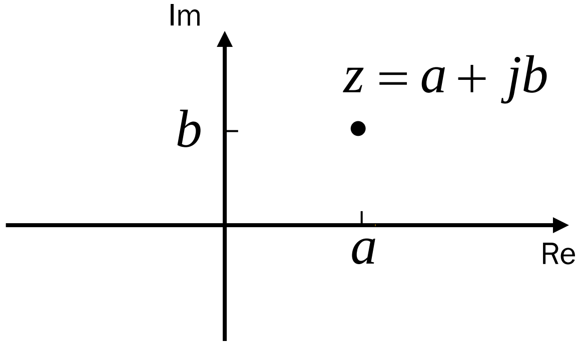
Representations

a) Cartesian form:

$$z = a + jb, \qquad j = \sqrt{-1}$$

$$\operatorname{Re} z = a$$

$$\operatorname{Im} z = b$$

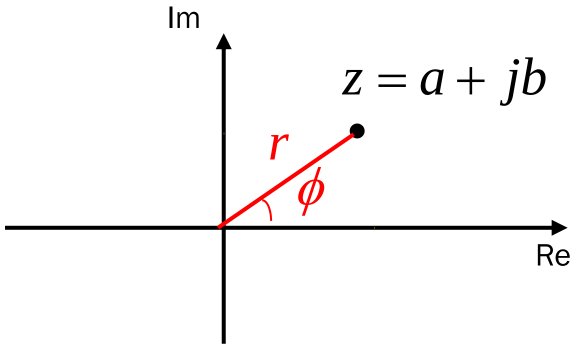


b) Exponential / Polar form

$$re^{j\phi}, \quad r\angle\phi$$

$$r = \sqrt{a^2 + b^2} \longrightarrow r = |z|$$

$$\phi = \tan^{-1}\frac{b}{a} \longrightarrow \phi = \arg(z)$$



Representations

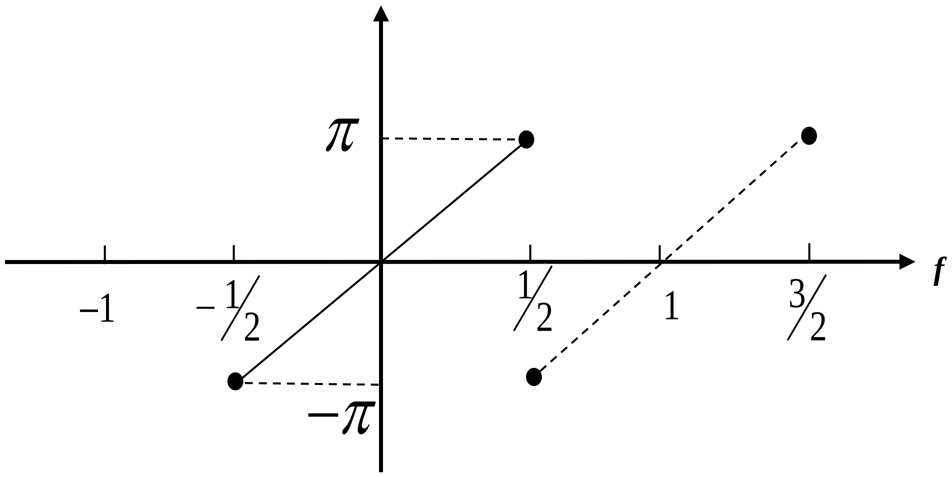
Important point:

ϕ is uniquely defined only in the principal value range $(-\pi, \pi)$

Example: $z(f) = e^{+j2\pi f}$

$|z(f)| = 1$

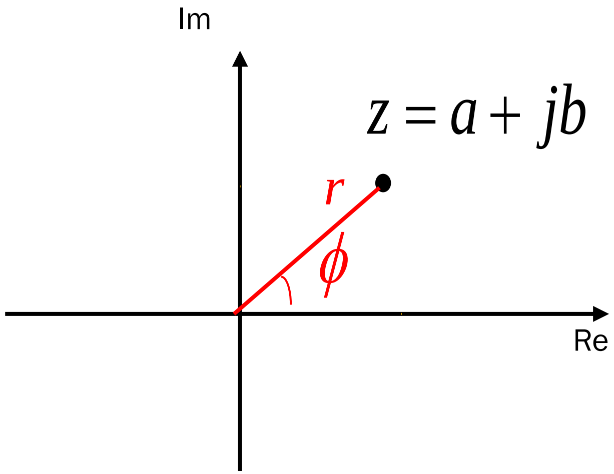
$\arg(z) = \phi(f) = 2\pi f, \qquad (-\pi, \pi)$



Conversion between Cartesian and polar forms

- ✓ **Euler formula:**

$$z = re^{j\phi} = r(\cos\phi + j\sin\phi)$$



- ✓ **De Moivre's Formula:**

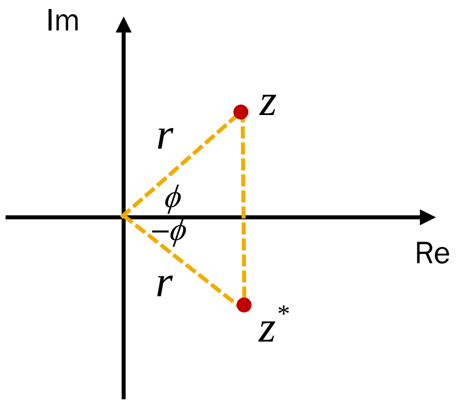
$$e^{jn\phi} = (\cos\phi + j\sin\phi)^n = \cos n\phi + j\sin n\phi$$

Complex number manipulations

- Complex conjugation

$$z = a + jb \qquad \longrightarrow \qquad z^* = a - jb$$

$$re^{j\phi} \qquad \qquad z^* = re^{-j\phi}$$



- Addition/subtraction

$$(a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2)$$

Complex number manipulations

- Multiplication/division

$$(a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 + ja_1b_2 + ja_2b_1 - b_1b_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)$$

Special case

$$(a + jb)(a - jb) = a^2 + b^2 = |z|^2$$

$$\frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{a_2^2 + b_2^2} = \frac{(a_1a_2 + b_1b_2) + j(-a_1b_2 + a_2b_1)}{a_2^2 + b_2^2}$$

In exponential form

$$r_1e^{j\phi_1} \cdot r_2e^{j\phi_2} = (r_1r_2)e^{j(\phi_1+\phi_2)} \qquad \frac{r_1e^{j\phi_1}}{r_2e^{j\phi_2}} = (\frac{r_1}{r_2})e^{j(\phi_1-\phi_2)}$$

Example 1

$$z = \frac{1+j}{3-2j}, \quad \text{find } Re\{z\}, \ Im\{z\}$$

$$z = \frac{1+j}{3-2j} \cdot \frac{3+2j}{3+2j} = \frac{3+2j+3j-2}{3^2+2^2}$$

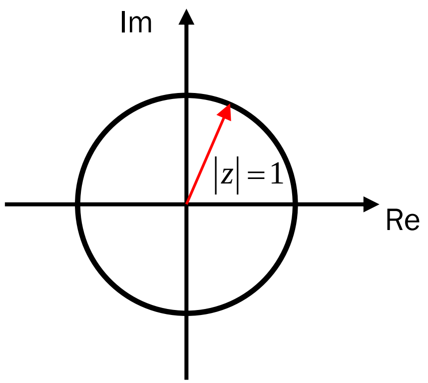
$$= \frac{1}{13} + j\frac{5}{13}$$

Example 2

- **Unit circle**

$$|z| = 1$$

$$e^{j\omega},\ 0 \le \omega \le 2\pi$$



Example 3

$$2z^3 + 1 = 0$$

3rd–order → 3 roots in the complex plane

$$z = \sqrt[3]{-\frac{1}{2}} = \sqrt[3]{\frac{1}{2}}\sqrt[3]{-1} \quad = \sqrt[3]{\frac{1}{2}}\sqrt[3]{e^{j\pi}}$$

$$= \sqrt[3]{\frac{1}{2}}\sqrt[3]{e^{j(\pi+2n\pi)}}$$

$$z_1 = \frac{1}{\sqrt[3]{2}} e^{j\frac{\pi}{3}} \quad (n=0) \qquad z_2 = \frac{1}{\sqrt[3]{2}} e^{j\pi} \quad (n=1) \qquad z_3 = \frac{1}{\sqrt[3]{2}} e^{j\frac{5\pi}{3}} = \frac{1}{\sqrt[3]{2}} e^{-j\frac{\pi}{3}} \quad (n=2)$$

What happen for $n > 2$?

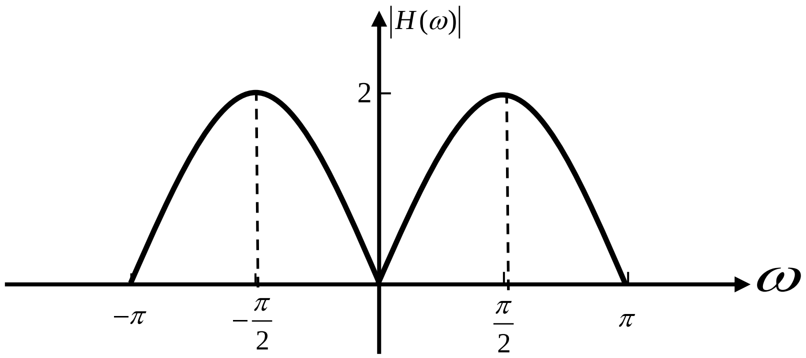
Example 4

$$H(\omega) = 1 - e^{-j2\omega}$$

$$H(\omega) = e^{-j\omega}(e^{+j\omega} - e^{-j\omega})$$

$$= e^{-j\omega}(2j\sin\omega)$$

$$= e^{j(\frac{\pi}{2}-\omega)} 2\sin\omega$$



$$|H(\omega)| = 2|\sin\omega|$$

$$\angle H(\omega) = \begin{cases} \frac{\pi}{2} - \omega & 0 \le \omega < \pi \\ -\frac{\pi}{2} - \omega & -\pi \le \omega < 0 \end{cases}$$