UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing

Homework 11 Solution

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1. Use the windowing method to design a length-N low-pass, generalized linear phase FIR filter with cut-off frequency $\pi/3$.

   (a) Find an expression for the filter coefficients $\{h_n\}_{n=0}^{N-1}$ if the rectangular window is used for the design.

   (b) Find an expression for the filter coefficients $\{h_n\}_{n=0}^{N-1}$ if the Hamming window is used for the design.

Solution:

(a) (16 pts)
First we find the ideal filter with cutoff frequency $\omega_c = \frac{\pi}{3}$:

$$\begin{aligned} h_{d,ideal,lp}[n] &= \frac{\omega_c}{\pi} sinc(\frac{\omega_c n}{\pi}), n \in (-\infty, \infty) \\ &= \frac{1}{3} sinc(\frac{n}{3}), n \in (-\infty, \infty) \end{aligned}$$

To make the filter causal, we need to apply a time shift equal to $\frac{N}{2}$ to the ideal filter and then multiply the result with the window. In this part of the question, we are using a rectangular window, $w_r[n]$:

$$\begin{aligned} h_{d,lp}[n] &= w_R[n]\frac{1}{3} sinc(\frac{n - \frac{N-1}{2}}{3}), \text{ where } w_R[n] = u[n] - u[u-N] \\ &= \frac{1}{3}(u[n] - u[n-N]) sinc(\frac{n - \frac{N-1}{2}}{3}), or \\ &= \boxed{\frac{1}{3} sinc(\frac{n - \frac{N-1}{2}}{3}) \text{ for } n \in [0, N], 0 \text{ for } n \text{ elsewhere}} \end{aligned}$$

(b) (8 pts)
All previous part for this question is the same as the previous part, except we are using a different window $w_h[n]$:

$$\begin{aligned} h_{d,lp}[n] &= w_h[n]\frac{1}{3} sinc(\frac{n - \frac{N-1}{2}}{3}), \text{ where } w_h[n] = 0.54 - 0.46\cos(\frac{2n\pi}{N}) \text{ for } n \in [0, N], 0 \text{ for } n \text{ elsewhere} \\ &= \boxed{\frac{1}{3}(0.54 - 0.46\cos(\frac{2n\pi}{N})) sinc(\frac{n - \frac{N-1}{2}}{3}) \text{ for } n \in [0, N], 0 \text{ for } n \text{ elsewhere}} \end{aligned}$$

2. Design a length-6, anti-symmetric differentiating FIR filter ($D_d(\omega) = e^{j(\alpha - M\omega)} \cdot j\omega$) using the window design method with a simple truncation (i.e., rectangular/boxcar) window. Give the filter coefficients as your answer.

Solution: (26 pts)
The shortcut procedure of first designing filter for zero-phase, and then shift the filter to the mid-point $\frac{N-1}{2}$ (where N is the filter length) only work if EITHER:

A) The mid-point $\frac{N-1}{2}$ is integer; OR
B) The mid-point is not integer, but the filter has the sinc form (ideal Lowpass, Highpass, Bandpass). Then shifting to $\frac{N-1}{2}$ effectively we have to interpolate with another sinc function, which means multiply the filter in the frequency domain with a rectangular function, and this does not change the function form. Hence we can simply replacing $sinc(n)$ with $sinc(n - \frac{N-1}{2})$.

For differentiator and $N = 6$ even, we have to follow the full procedure:

1. Using the anti-symmetric GLP form, we have:

$$\begin{aligned} D_d(\omega) &= (j\omega)e^{-j(\frac{N-1}{2})\omega)} \\ &= \omega e^{j(\frac{\pi}{2} - (\frac{N-1}{2})\omega)} \end{aligned}$$

2. Take the inverse DTFT to get the coefficients in time (and for simplicity denote $s = n - \frac{N-1}{2}$):

$$\begin{aligned} d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} e^{j\omega n} d\omega \\ &= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega(n - \frac{N-1}{2})} d\omega \\ &(\text{assume } s = n - \frac{N-1}{2}) \\ &= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega s} d\omega \\ &(\text{using integration by parts}) \\ &= \frac{j}{2\pi}((\omega \frac{e^{\omega s}}{js})|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{j\omega s}}{js} d\omega) \\ &= \frac{\cos(\pi s)}{s} - \frac{\sin(\pi s)}{\pi s^2} \\ &= \frac{\cos(\pi(n - \frac{N-1}{2}))}{n - \frac{N-1}{2}} - \frac{\sin(\pi(n - \frac{N-1}{2}))}{\pi(n - \frac{N-1}{2})^2} \text{ for } n = 0, 1, \ldots, N-1. \\ &= \boxed{\frac{\cos(\pi(n - \frac{5}{2}))}{n - \frac{5}{2}} - \frac{\sin(\pi(n - \frac{5}{2}))}{\pi(n - \frac{5}{2})^2}} \end{aligned}$$

We know the first term is 0 as the angle falls on $\pm\frac{pi}{2}$ for the cos function on the numerator. So we can go one step further (but this is not required):

$$d[n] = -\frac{\sin(\pi(n - \frac{5}{2}))}{\pi(n - \frac{5}{2})^2}$$

One can go even further and simplify the last term:

$$d[n] = \frac{1}{\pi(n-\frac{5}{2})^2} \text{ if } n \text{ is even, and } \frac{-1}{\pi(n-\frac{5}{2})^2} \text{ if } n \text{ is odd}$$

Even length actually gives better approximation for the differentiator as $D_d(\pi) \neq 0$. Page 601 in the textbook has a discussion on this.

Note that some student miss understand the ideal frequency response to be $j\omega$ instead of ω. This will lead to an extra j term after the integration. Since we failed to clarify the details, student with this answer should receive full points.

3. Let $h[n]$ denote the unit pulse response of an ideal desired system with frequency response $H_d(\omega)$, and let $\hat{h}[n]$ and $\hat{H}_d(\omega)$ denote the unit pulse response and frequency response, respectively, of an FIR approximation to the ideal system. Assume that $\hat{h}[n] = 0$ for $n < 0$ and $n > N-1$. We wish to choose the N samples of the unit pulse response so as to minimize the mean-square error of the frequency response defined as

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_d(\omega) - \hat{H}_d(\omega)\right|^2 d\omega$$

(a) Use Parseval's relation to express the error function in term of the sequence $h[n]$ and $\hat{h}[n]$.

(b) Using the result of part (a), determine the values of $\hat{h}[n]$ for $0 \le n \le N-1$ that minimize ϵ^2.

(c) The FIR filter determined in part (b) could have been obtained by a windowing operation. That is, $\hat{h}[n]$ could have been obtained by multiplying the desired infinite-length sequence $h[n]$ by a certain finite-length sequence $w[n]$. Determine the necessary window $w[n]$ such that the optimal unit pulse response is $\hat{h}[n] = w[n]h[n]$.

Solution:

(a) (8 pts)
The core of Parseval's theorem is, for the signal power, integration in time is equal to integration in frequency domain, the discrete time version of Parseval's theorem is:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

If we treat the squared-error as a signal, then we could directly apply Parseval's theorem:

$$\begin{aligned}\epsilon^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_d(\omega) - \hat{H}_d(\omega)\right|^2 d\omega \\ &= \boxed{\sum_{n=-\infty}^{\infty} |h[n] - \hat{h}[n]|^2}\end{aligned}$$

(b) (6 pts)
Since the filter coefficient on different time delay contributes to the squared-error independent of each other, there is nothing we can do about the terms of $h[n]$ outside of the non-zero terms of $\hat{h}[n]$. Since $\hat{h}[n]$ has finite length, obviously, $\hat{h}[n] = 0$ for $n < 0$ or $n > N$. All we can do to minimize the squared-error is to set $\boxed{\hat{h}[n] = h[n] \text{ for } 0 \le n \le N}$.

(c) (6 pts)
It is not hard to see that setting $\boxed{\hat{h}[n] = h[n] \text{ for } 0 \le n \le N}$ is effectively applying a rectangular window with length N to $h[n]$. Therefore $\boxed{w[n] = w_R[n] = 1 \text{ for } 0 \le n \le N \text{ and } 0 \text{ elsewhere}}$.

4. (a) Since the filter has an even number of coefficients, the coefficients need to be antisymmetric in order to realize a high-pass filter. Therefore, the filter is type-2 GLP.

(b) One can verify that $H_d(\omega)$ has the same expression for $\frac{3\pi}{4} \le \omega \le \pi$ and $\pi \le \omega \le \frac{5\pi}{4}$ using the anti-symmetry and the 2π shift in the real part of the frequency response. Therefore,

$$H_d(\omega) = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\omega)}, & \frac{3\pi}{4} \le \omega \le \frac{5\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

In order to use the frequency sampling method, the inverse DFT of $H[m]$ is needed,

$$H[m] = \begin{cases} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)}, & 38 \le m \le 62 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
h[n] &= \frac{1}{100}\sum_{m=38}^{62} e^{j(\frac{\pi}{2} - \frac{99}{2}\frac{2\pi}{100}m)} e^{j\frac{2\pi}{100}mn} \\
&= \frac{e^{j\frac{\pi}{2}}}{100}\sum_{m=38}^{62} e^{j\frac{2\pi}{100}(n-\frac{99}{2})m} \\
&= \frac{e^{j\frac{\pi}{2}}}{100}\sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})(k+38)} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38}\sum_{k=0}^{24} e^{j\frac{2\pi}{100}(n-\frac{99}{2})k} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})25}}{1 - e^{j\frac{2\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{2\pi}{100}(n-\frac{99}{2})38} \cdot \frac{e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \cdot \frac{e^{-j\frac{\pi}{100}(n-\frac{99}{2})25} - e^{j\frac{\pi}{100}(n-\frac{99}{2})25}}{e^{-j\frac{\pi}{100}(n-\frac{99}{2})} - e^{j\frac{\pi}{100}(n-\frac{99}{2})}} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\frac{\pi}{100}(n-\frac{99}{2})(76+25-1)} \frac{\sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{\sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= \frac{e^{j\frac{\pi}{2}}}{100} e^{j\pi(n-\frac{99}{2})} \frac{\sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{\sin(\frac{\pi}{100}(n-\frac{99}{2}))} \\
&= -\frac{(-1)^n}{100} \frac{\sin(25\frac{\pi}{100}(n-\frac{99}{2}))}{\sin(\frac{\pi}{100}(n-\frac{99}{2}))}
\end{aligned}$$

5. For the following bilinear transformation

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{or} \quad z = \frac{1 + s}{1 - s}$$

four points are given in the s-plane as follows: (a) $s_1 = j$, (b) $s_2 = 2$, (c) $s_3 = -2$, and (d) $s_4 = 0$. Find the corresponding z-plane points and mark their locations in the z-plane.

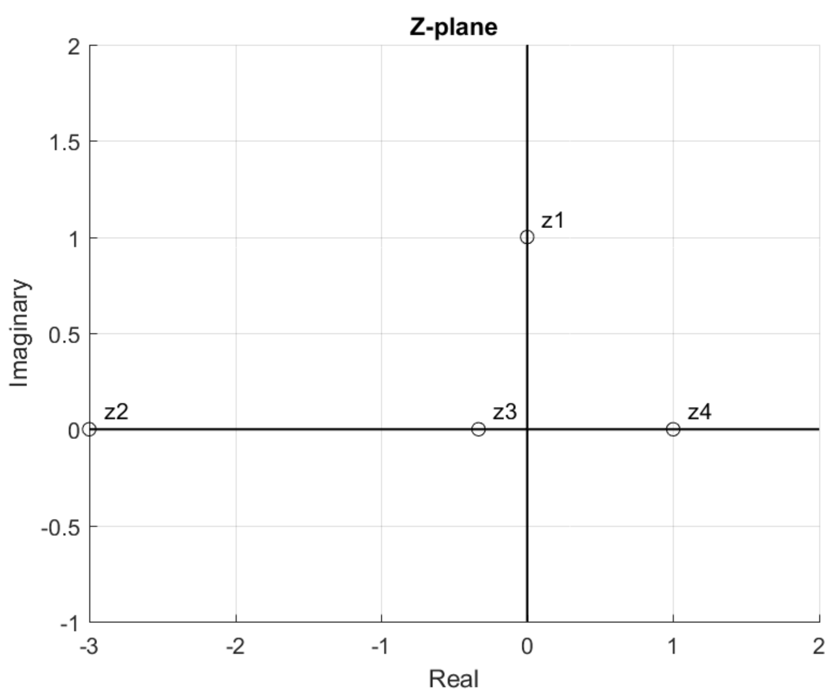
Solution:

(a) $z_1 = (1 + j)/(1 - j) = j$

(b) $z_2 = (1 + 2)/(1 - 2) = -3$

(c) $z_3 = (1 - 2)/(1 + 2) = -1/3$

(d) $z_4 = (1 + 0)/(1 - 0) = 1$



6. The bilinear transform is to be used with the analog prototype $H_L(s) = \frac{s}{s+2}$ to determine the transfer function $H(z)$ of a digital HPF with 3 dB cutoff at $\pi/3$ (i.e., $|H_d(\pi/3)|^2 = 0.5$).

   (a) Determine the 3 dB cutoff for the analog prototype Ω_c.

   (b) Find $H(z)$ in closed form.

Solution:

(a) In the continuous Fourier domain ($s = \sigma + j\Omega$, with $\sigma = 0$), we can find Ω_c using Parseval's Theorem. (Note we cannot apply the bilinear transform because we do not know the parameter used, which we will find in part b).

$$
\begin{aligned}
|H(\Omega)|^2 &= H_L(j\Omega)H_L^*(j\Omega) = (\frac{j\Omega}{j\Omega+2})(\frac{-j\Omega}{-j\Omega+2}) = \frac{\Omega^2}{\Omega^2+4} \\
|H(\Omega_c)|^2 &= \frac{\Omega_c^2}{\Omega_c^2+4} = 0.5 \\
\Omega_c &= 2 \text{ rad/s}
\end{aligned}
$$

(b) We need to first find the parameter used in the bilinear transform $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ ($\alpha = \frac{2}{T}$ using notation from lecture notes).

$$
\begin{aligned}
\Omega_c &= \frac{2}{T}\tan(\frac{\omega_c}{2}) \\
T &= \frac{2}{\Omega_c}\tan(\frac{\omega_c}{2}) = \frac{\sqrt{3}}{3}
\end{aligned}
$$

($\alpha = 2\sqrt{3}$)

$$
\begin{aligned}
H(z) &= H_L(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \frac{2(1+z^{-1})}{1+z^{-1}}} \\
&= \frac{1-z^{-1}}{1-z^{-1}+T+Tz^{-1}} \\
&= \frac{\sqrt{3}-\sqrt{3}z^{-1}}{(1+\sqrt{3})+(1-\sqrt{3})z^{-1}} \text{ , } |z| > \frac{1}{2+\sqrt{3}}
\end{aligned}
$$

7. The transformation $s = 2(1 - z^{-1})/(1 + z^{-1})$ was applied to an analog protype $H_L(s) = 1/(s^4 + 1/2)$ to design a digital filter. Calculate the (steady-state) response $y[n]$ of the digital filter for input $x[n] = 3\cos(\frac{\pi}{3}n + 45°)$.

Solution: Suppose that the ROC for the Laplace transform and the Z transforms are nice (we can then say $H_a(\Omega) = H_L(s)|_{s=j\Omega}$ and $H_d(\omega) = H(z)|_{z=e^{j\omega}}$)

$$H_a(\Omega) = H_L(s)|_{s=j\Omega} = \frac{1}{(j\Omega)^4 + 1/2} = \frac{1}{\Omega^4 + 1/2}$$

We use the transform to find: $\Omega = 2\tan(\omega/2)$

$$\begin{aligned} H_d(\omega) &= \frac{1}{(= 2\tan(\omega/2))^2 + 1/2} \\ &= \frac{1}{16 \cdot (\frac{\sin(\omega/2)}{\cos(\omega/2)})^4 + 1/2} \\ &= \frac{\cos^4(\omega/2)}{16\sin^4(\omega/2) + 1/2\cos^4(\omega/2)} \end{aligned}$$

Alternatively we could plug and chug to find $H(z)$:

$$\begin{aligned} H(z) &= H_L(s)|_{s=2(1-z^{-1})/(1+z^{-1})} \\ &= \frac{1}{2^4\frac{(1-z^{-1})^4}{(1+z^{-1})^4} + 1/2} \\ &= \frac{(1+z^{-1})^4}{2^4(1-z^{-1})^4 + 1/2(1+z^{-1})^4} \\ H_d(\omega) &= H(z)|_{z=e^{j\omega}} \\ &= \frac{(1+e^{-j\omega})^4}{2^4(1-e^{-j\omega})^4 + 1/2(1+e^{-j\omega})^4} \\ &= \frac{e^{-2j\omega}(e^{j\omega/2} + e^{-j\omega/2})^4}{e^{-2j\omega}(2^4(e^{j\omega/2} - e^{-j\omega/2})^4 + 1/2(e^{j\omega/2} + e^{-j\omega/2})^4)} \\ &= \frac{\cos^4(\omega/2)}{16\sin^4(\omega/2) + 1/2\cos^4(\omega/2)} \end{aligned}$$

Since $H_d(\omega) = H_d^*(-\omega)$, we can use the eigenfunctions:

$$|H_d(\frac{\pi}{3})| = \frac{9/16}{16 \cdot 1/16 + 1/2 \cdot 9/16} = \frac{18}{41}$$

Since $H_d(\omega)$ is real, $\angle H_d(\frac{\pi}{3}) = 0°$

$$\begin{aligned} y[n] &= 3|H_d(\frac{\pi}{3})|\cos(\frac{\pi}{3}n + 45° + \angle H_d(\frac{\pi}{3})) \\ &= 3 \cdot \frac{18}{41}\cos(\frac{\pi}{3}n + 45° + 0°) \\ &= \frac{54}{41}\cos(\frac{\pi}{3}n + 45°) \end{aligned}$$