ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 19

Topics:

- ✓ Discrete Fourier transform (DFT)

Educational Objectives:

- ✓ Understand the definition of DFT
- ✓ Understand the inverse transform
- ✓ Understand the relationship between DFT and DTFT

Overview

$$
\begin{tikzcd}
x_a(t) \arrow[rr, leftrightarrow, "\text{CTFT}"] \arrow[d] & & X(\Omega) \\
x[n] \arrow[rr, leftrightarrow, "\text{DTFT}"] \arrow[d] & & X_d(\omega) \arrow[u, dashed] \\
\{x_n\}_{n=0}^{N-1} \arrow[rr, leftrightarrow, "\text{DFT}"] & & \{X_m\}_{m=0}^{N-1} \arrow[u, dashed]
\end{tikzcd}
$$

DFT: Definition

Given $\{x_n\}_{n=0}^{N-1}$, its DFT is defined as

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn}, m = 0,1 \ldots N-1$$

Note: n → time index

m → frequency index → ω → Ω

$$\{x_n\}_{n=0}^{N-1} \xrightarrow{\text{DFT}} \{X_m\}_{m=0}^{N-1}$$

Inverse DFT

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}mn}, n = 0,1 \ldots N-1$$

Proof: Note: $\sum_{m=0}^{N-1}\alpha^m = \frac{1-\alpha^N}{1-\alpha}$

$$\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(n-l)} = \begin{cases} N, l = n \\ \frac{1-e^{j\frac{2\pi}{N}N(n-l)}}{1-e^{j\frac{2\pi}{N}(n-l)}} = 0, l \neq n \end{cases}$$

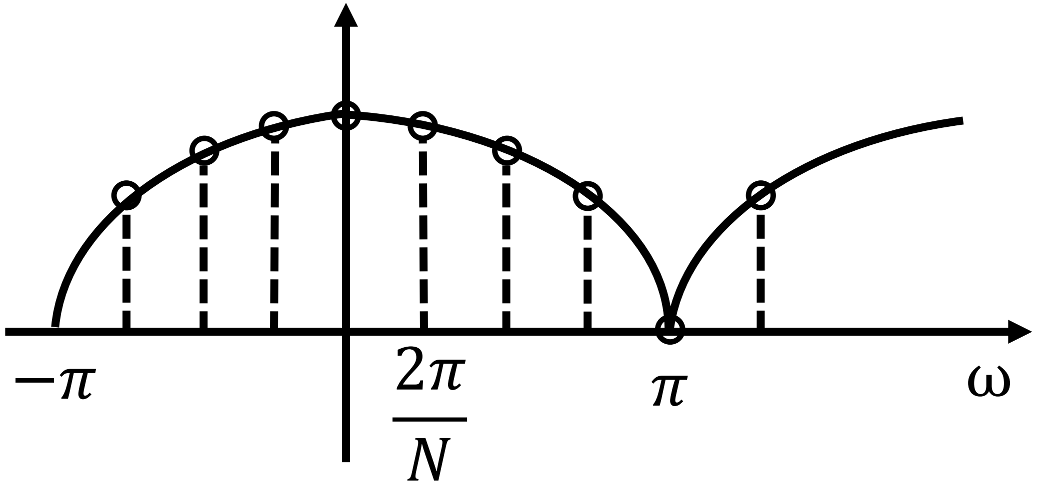
$$= N\delta[l-n]$$

Relation to DTFT

Consider $\{x_n\}_{n=0}^{N-1}$, (implying $x_n = 0$ for $n \geq N$ or $n < 0$)

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n} = \sum_{n=0}^{N-1} x_n e^{-j\omega n}$$

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn} = X_d(\frac{2\pi}{N}m)$$



Larger N → smaller the sampling interval

Zero-padded DFT (next lecture)

Periodic Extension

$$\{x_n\}_{n=0}^{N-1} \longleftrightarrow \{X_m\}_{m=0}^{N-1}$$

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn}$$

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}mn}$$

Consider

$$X_{m+lN} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}(m+lN)n}$$

$$= \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}lNn}$$

$$= X_m$$

Similarly

$$x_{n+lN} = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}m(n+lN)}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}mn} e^{j\frac{2\pi}{N}mlN}$$

$$= x_n$$