ECE 310

Digital Signal Processing

Spring, 2021, ZJUI Campus

Lecture 21

Topics:

- ✓ DFT Spectral Estimation (or Spectral Analysis)

Educational Objectives:

- ✓ Understand DFT spectral analysis method
  - Sampling requirement
  - Windowing effect
  - Resolution limitation
  - Mapping from $m \to \omega \to \Omega$

DFT Spectral Analysis: Problem Formulation

- Spectral analysis: determining the frequency content of a given signal

$$x_a(t) \leftrightarrow X_a(\Omega)$$

More specifically,

$$x_a(t) = \sum_{i=1}^{M} A_i \cos(\Omega_i t)$$

Determine $\{\Omega_i, A_i\}_{i=1}^{M}$

$$X_a(\Omega) = \sum_{i=1}^{M} \pi A_i [\delta(\Omega + \Omega_i) + \delta(\Omega - \Omega_i)]$$

DFT Spectral Analysis: Procedure

$$x_a(t) \xrightarrow[T]{} \{x_n\}_{n=0}^{N-1} \longrightarrow \boxed{\text{DFT}} \longrightarrow \{X_m\}_{m=0}^{N-1}$$

$$\downarrow$$

$$X_d(\omega)$$

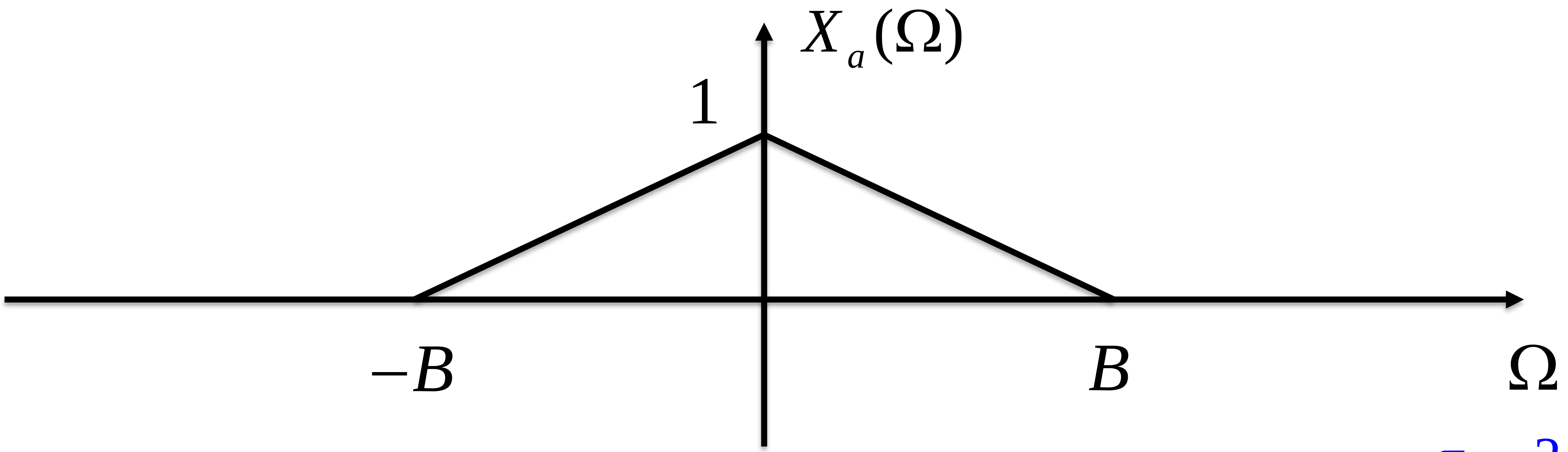
$$\downarrow$$

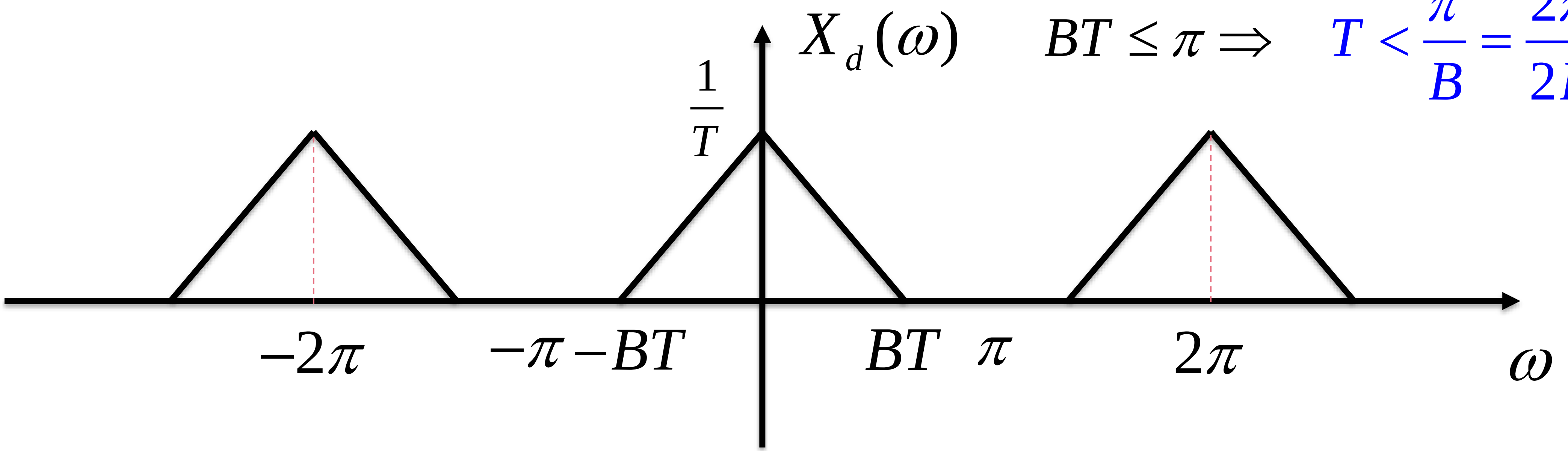
$$X_a(\Omega)$$

Effect of sampling: $x_a(t) \leftrightarrow X_a(\Omega)$

$$X_d(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega + 2k\pi}{T}\right)$$

Sampling Effect: Nyquist Criterion





$$BT \le \pi \Rightarrow \quad T < \frac{\pi}{B} = \frac{2\pi}{2B}$$

Relationship between DFT and DTFT Spectra

$$X_d(\omega) = \begin{cases} \frac{1}{T} X_a(\frac{\omega}{T}), & 0 \le \omega \le \pi \\ \frac{1}{T} X_a(\frac{\omega - 2\pi}{T}), & \pi < \omega \le 2\pi \end{cases}$$

$$X_m = X_d(\frac{2\pi}{N} m) = \begin{cases} \frac{1}{T} X_a(\frac{2\pi m}{NT}) & 0 \le m \le \frac{N-1}{2}, N\ odd \\ & 0 \le m \le \frac{N}{2}, N\ even \\ \frac{1}{T} X_a(\frac{2\pi(m-N)}{NT}) & \frac{N-1}{2} \le m \le N-1, N\ odd \\ & \frac{N}{2} < m \le N-1, N\ even \end{cases}$$

Determination of Spectral Parameters

Amplitudes : $\dfrac{A_i}{T}$

Frequencies : m_i

$$\omega_i = \frac{2\pi}{N} m_i$$

$$\Omega_i = \frac{\omega_i}{T}$$

$$\Omega_i = \frac{\omega_i}{T} - \frac{2\pi}{T}$$

Windowing Effect

$$\hat{x}[n] = x[n]w[n]$$

$$\hat{X}_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(u)W_d(\omega - u)du$$

Note:

$$A\cos(\Omega_0 nT) \rightarrow \pi A[\delta(\omega - \Omega_0 T) + \delta(\omega + \Omega_0 T)], |\omega| < \pi$$

Rectangular window:

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$$

$$W_d(\omega) = \frac{\sin(\omega \frac{N}{2})}{\sin(\frac{\omega}{2})} e^{-j\frac{\omega}{2}(N-1)}$$

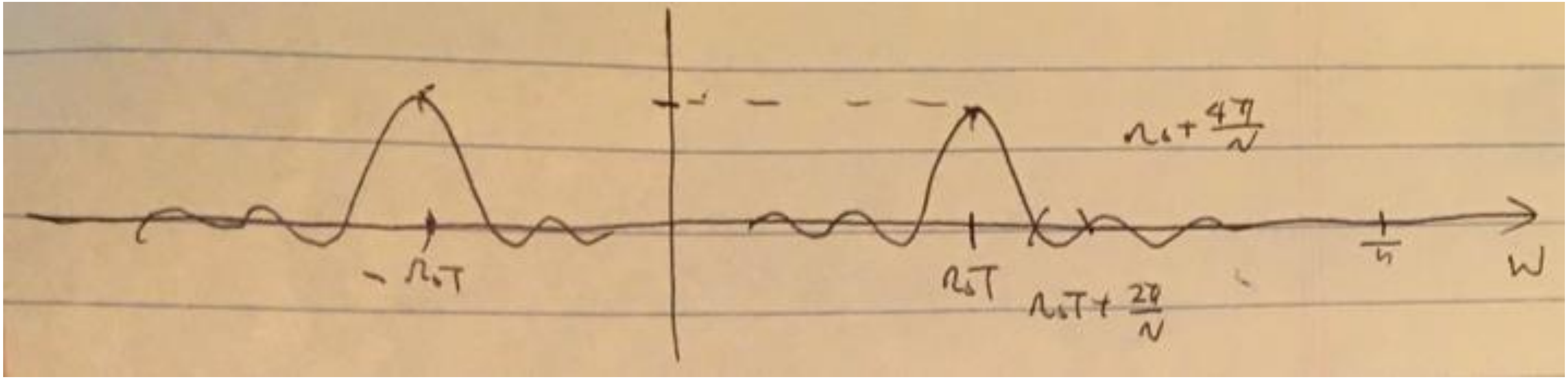
Windowing Effect

$$\hat{x}_n = A\cos(\Omega_0 nT), \qquad 0 \le n \le N-1$$

$$\downarrow$$

$$\hat{X}_d(\omega) = e^{-j(\omega-\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin[(\omega-\Omega_0 T)N/2]}{\sin(\omega-\Omega_0 T)/2} + e^{-j(\omega+\Omega_0 T)\frac{N-1}{2}} \frac{\frac{A}{2}\sin[(\omega+\Omega_0 T)N/2]}{\sin(\omega+\Omega_0 T)/2}$$

Windowing Effect



Width of the main lobe: $\frac{4\pi}{N}$

Height: $AN/2$

DFT Spectral Analysis

$$|\hat{X}_d(\omega)| = \frac{A}{2}\left|\frac{\sin((\omega-\Omega_0 T)\frac{N}{2})}{\sin(\omega-\Omega_0 T)/2}\right| + \frac{A}{2}\left|\frac{\sin((\omega+\Omega_0 T)\frac{N}{2})}{\sin(\omega+\Omega_0 T)/2}\right|$$

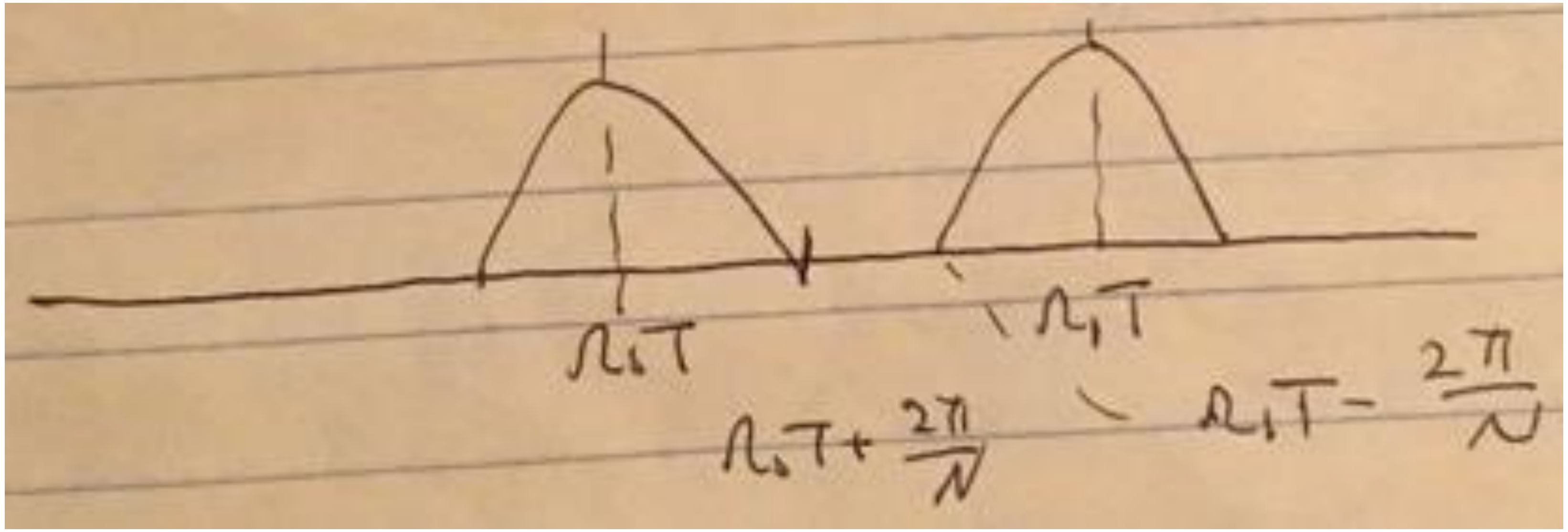
$$M : number\ of\ "peaks"\ on\ [0,\pi]$$

$$\Omega_i : location\ of\ its\ peak\ / T$$

$$A_i : height\ of\ its\ peak \times \frac{2}{N}$$

Resolution Limitation

How big N is big enough? (spectral resolution)



(Ignore sidelobes)

$$\Omega_0 + \frac{2\pi}{N} < \Omega_1 T - \frac{2\pi}{N}$$

$$(\Omega_1 - \Omega_0)T > \frac{4\pi}{N}$$

$$NT > \frac{4\pi}{\Omega_1 - \Omega_0}$$

$$NT > \frac{2\pi}{\Omega_1 - \Omega_0}$$