浙江大学伊利诺伊大学厄巴纳香槟校区联合学院
Zhejiang University-University of Illinois at Urbana Champaign Institute

ECE 448: Artificial Intelligence

Lecture 17: Bayes Net Inference

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Bayes net is a **memory-efficient model** of dependencies among:

- Query *variables:* **X**
- *Evidence* (*observed*) variables and their values: **E** = **e**
- *Unobserved* variables: **Y**

Inference problem: answer questions about the query variables given the evidence variables

- This can be done using the posterior distribution P(**X** | **E** = **e**)
- The posterior can be derived from the full joint P(**X**, **E**, **Y**)
- How do we make this **computationally efficient?**

Learning problem: given some training examples, how do we learn the parameters of the model?

- Parameters = p(variable|parents), for each variable in the net

Outline：Bayesian Inference and Bayesian Learning

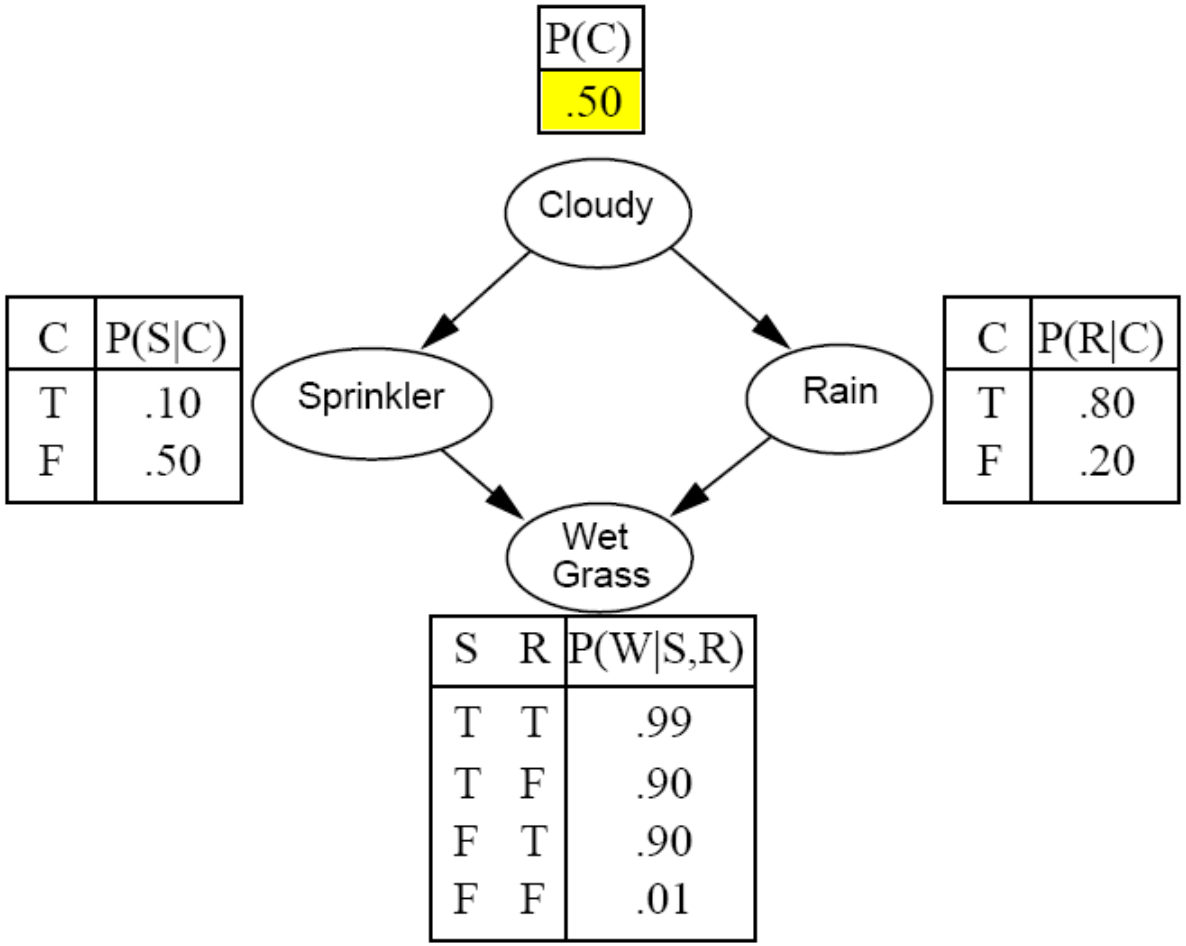
1. Inference Examples
2. Inference Algorithms
   - Trees: Sum-product algorithm
   - Poly-trees: Junction tree algorithm
   - Graphs: No polynomial-time algorithm
3. Parameter Learning

Outline：Bayesian Inference and Bayesian Learning

1. **Inference Examples**
2. **Inference Algorithms**
   - **Trees: Sum-product algorithm**
   - **Poly-trees: Junction tree algorithm**
   - **Graphs: No polynomial-time algorithm**
3. **Parameter Learning**

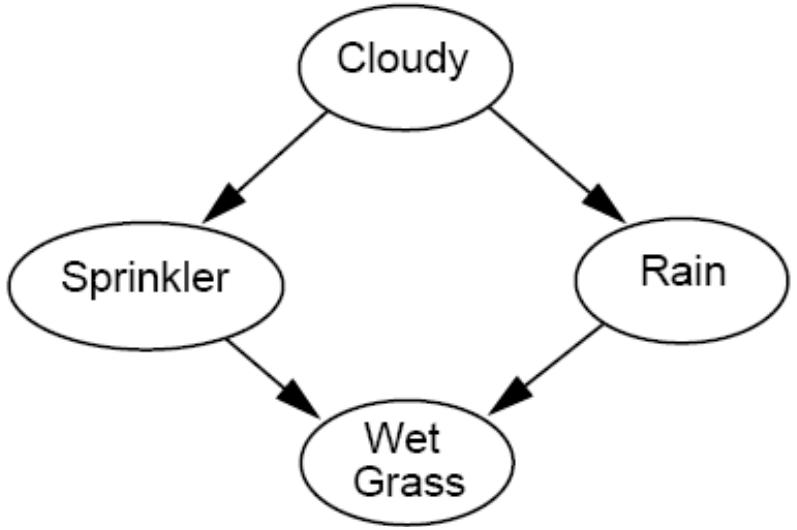
Practice example 1

- Variables: *Cloudy, Sprinkler, Rain, Wet Grass*



Practice example 1

- Given that the grass is wet, what is the probability that it has rained?



$$P(r \mid w) = \frac{P(r,w)}{P(w)} = \frac{\sum_{C=c,S=s} P(c,s,r,w)}{\sum_{C=c,S=s,R=r} P(c,s,r,w)}$$

$$= \frac{\sum_{C=c,S=s} P(c)P(s \mid c)P(r \mid c)P(w \mid r,s)}{\sum_{C=c,S=s,R=r} P(c)P(s \mid c)P(r \mid c)P(w \mid r,s)}$$

Practice example #2

- Suppose you have an observation, for example, “Jack called” (J=1)
- You want to know: was there a burglary?
- You need

$$P(B = 1|J = 1) = \frac{P(B, J = 1)}{\sum_b P(B = b, J = 1)}$$

- So you need to compute the table P(B,J) for all possible settings of (B,J)

Outline：Bayesian Inference and Bayesian Learning

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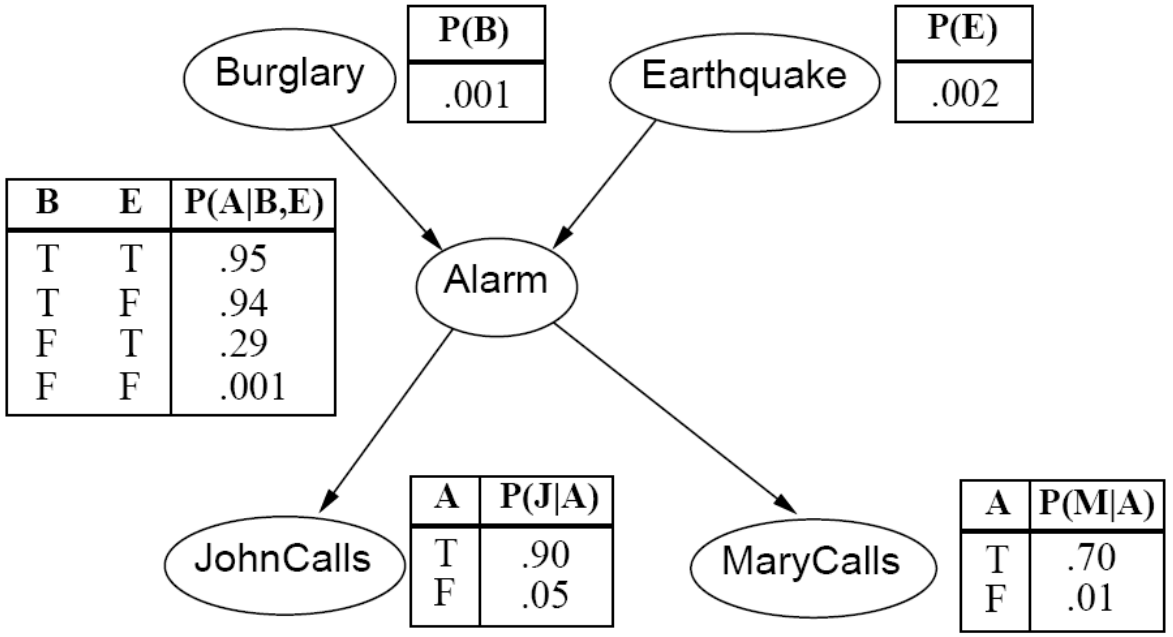
Bayes Net Inference: The Hard Way

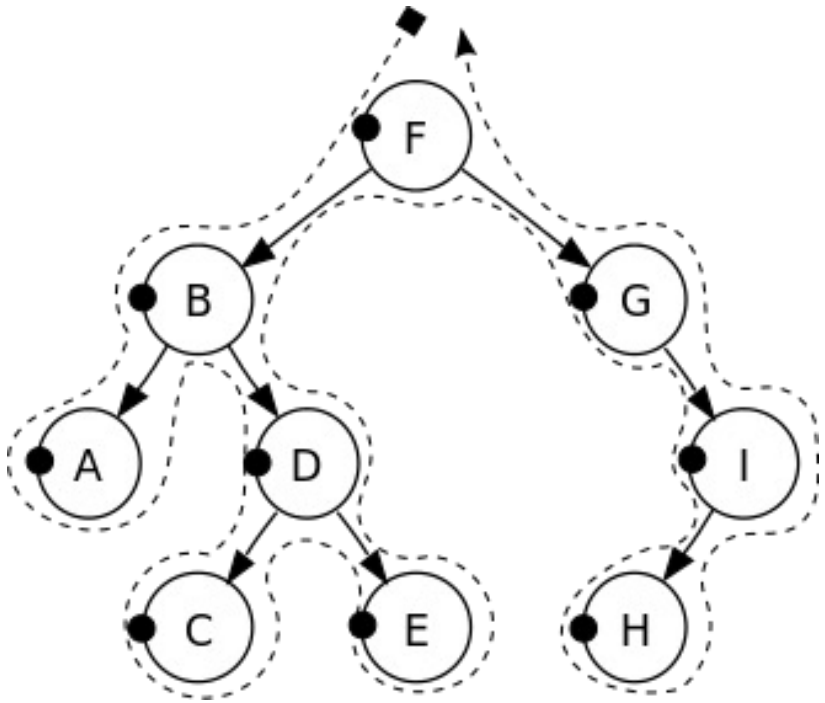


1. P(B,E,A,J,M)=P(B)P(E)P(A|B,E)P(J|A)P(M|A)
2. $P(B,J) = \sum_E \sum_A \sum_M P(B,E,A,J,M)$

Exponential complexity (#P-hard, actually): N variables, each of which has K possible values ⇒ $O\{K^N\}$ time complexity

Is there an easier way?

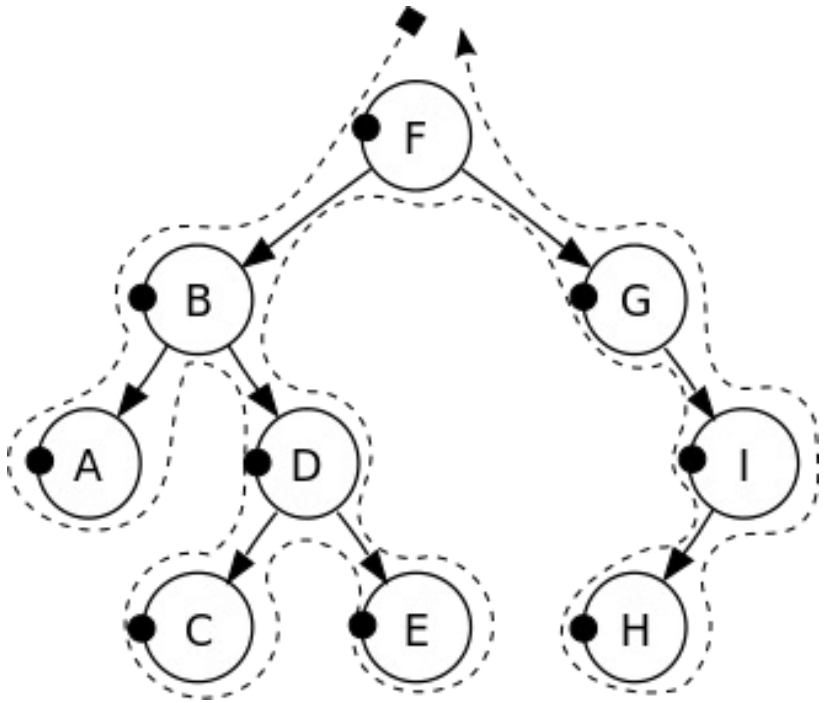
- Tree-structured Bayes nets: the sum-product algorithm
  - Quadratic complexity, $O\{NK^3\}$
- Polytrees: the junction tree algorithm
  - Pseudo-polynomial complexity, $O\{NK^M\}$, for M<N
- Arbitrary Bayes nets: #P complete, $O\{K^N\}$
  - The SAT problem is a Bayes net!
- Parameter Learning



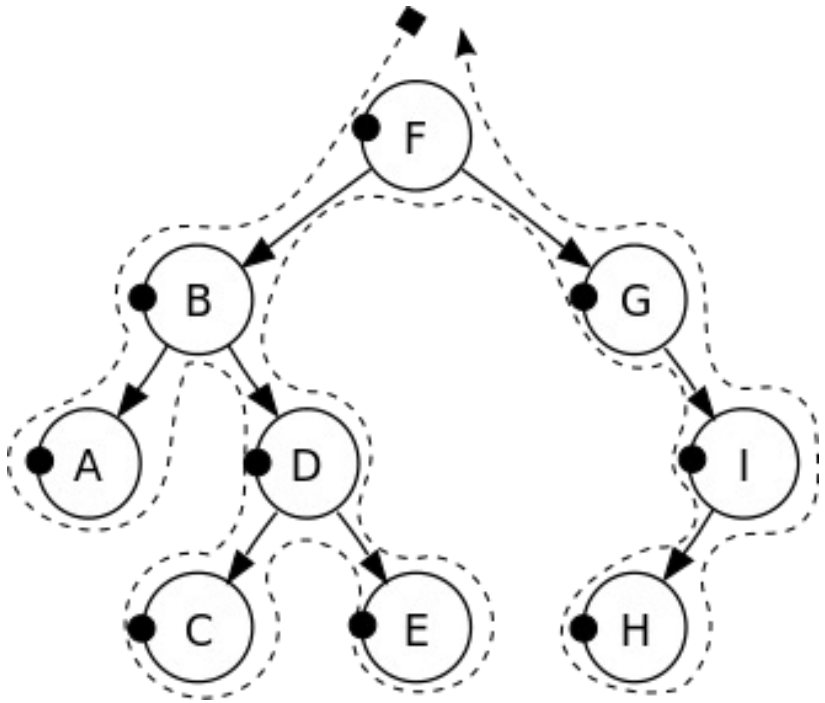
- Suppose these are all binary variables.
- We observe E=1
- We want to find P(H=1|E=1)
- Means that we need to find both P(H=0,E=1) and P(H=1,E=1) because

$$P(H = 1|E = 1) = \frac{P(H = 1, E = 1)}{\sum_h P(H = h, E = 1)}$$

The Sum-Product Algorithm (Belief Propagation)

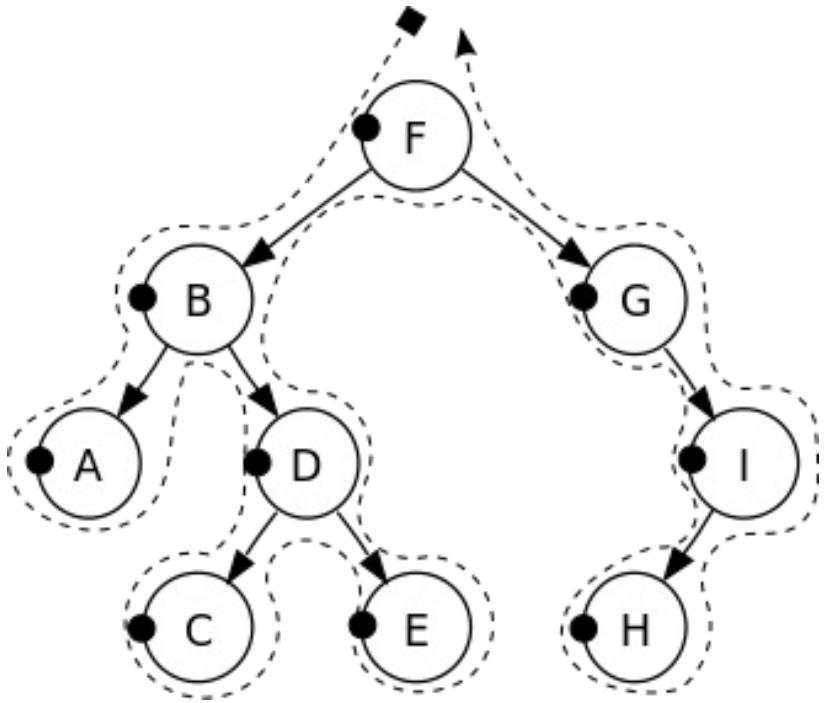


- Find the only undirected path from the evidence variable to the query variable (EDBFGIH)
- Find the directed root of this path P(F)
- Find the joint probabilities of root and evidence: P(F=0,E=1) and P(F=1,E=1)
- Find the joint probabilities of query and evidence: P(H=0,E=1) and P(H=1,E=1)
- Find the conditional probability P(H=1|E=1)



Starting with the root P(F), we find P(F,E) by alternating product steps and sum steps:

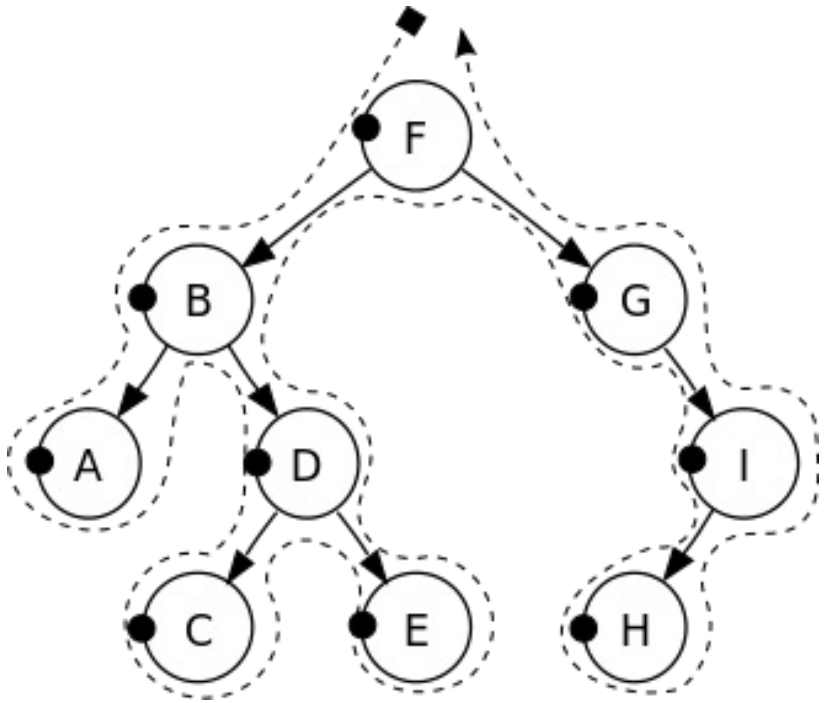
1. Product: P(B,D,F)=P(F)P(B|F)P(D|B)
2. Sum: $P(D,F) = \sum_{B=0}^{1} P(B,D,F)$
3. Product: P(D,E,F)=P(D,F)P(E|D)
4. Sum: $P(E,F) = \sum_{D=0}^{1} P(D,E,F)$



Starting with the root P(E,F), we find P(E,H) by alternating product steps and sum steps:

1. Product: P(E,F,G)=P(E,F)P(G|F)
2. Sum: $P(E,G) = \sum_{F=0}^{1} P(E,F,G)$
3. Product: P(E,G,I)=P(E,G)P(I|G)
4. Sum: $P(E,I) = \sum_{G=0}^{1} P(E,G,I)$
5. Product: P(E,H,I)=P(E,I)P(I|G)
6. Sum: $P(E,H) = \sum_{I=0}^{1} P(E,H,I)$

Time Complexity of Belief Propagation



- Each product step generates a table with 3 variables
- Each sum step reduces that to a table with 2 variables
- If each variable has K values, and if there are $O\{N\}$ variables on the path from evidence to query, then time complexity is $O\{NK^3\}$

Time Complexity of Bayes Net Inference

- Tree-structured Bayes nets: the sum-product algorithm
  - Quadratic complexity, $O\{NK^3\}$
- Polytrees: the junction tree algorithm
  - Pseudo-polynomial complexity, $O\{NK^M\}$, for M<N
- Arbitrary Bayes nets: #P complete, $O\{K^N\}$
  - The SAT problem is a Bayes net!

a. Moralize the graph (identify each variable's Markov blanket)
b. Triangulate the graph (eliminate undirected cycles)
c. Create the junction tree (form cliques)
d. Run the sum-product algorithm on the junction tree

2.a. Markov Blanket

- Suppose there is a Bayes net with variables A,B,C,D,E,F,G,H
- The “Markov blanket” of variable F is D,E,G if

$$P(F|A,B,C,D,E,G,H)$$

$$= P(F|D,E,G)$$

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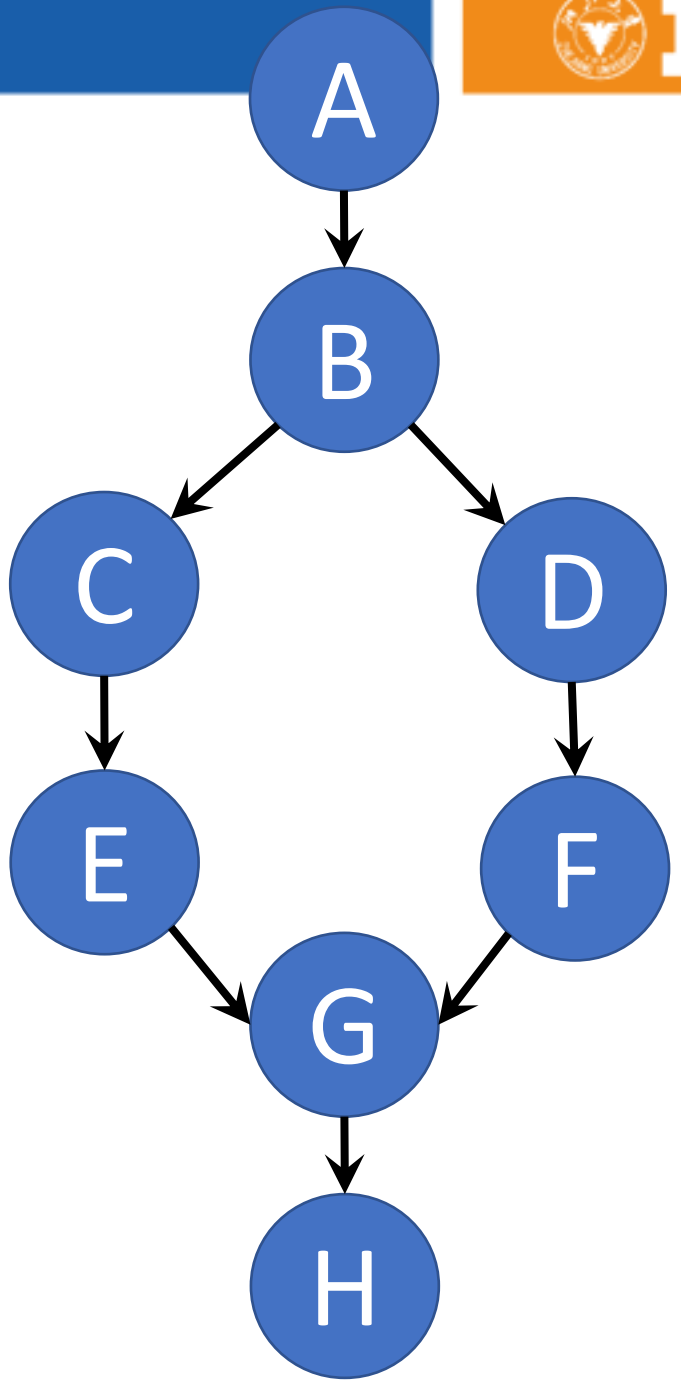


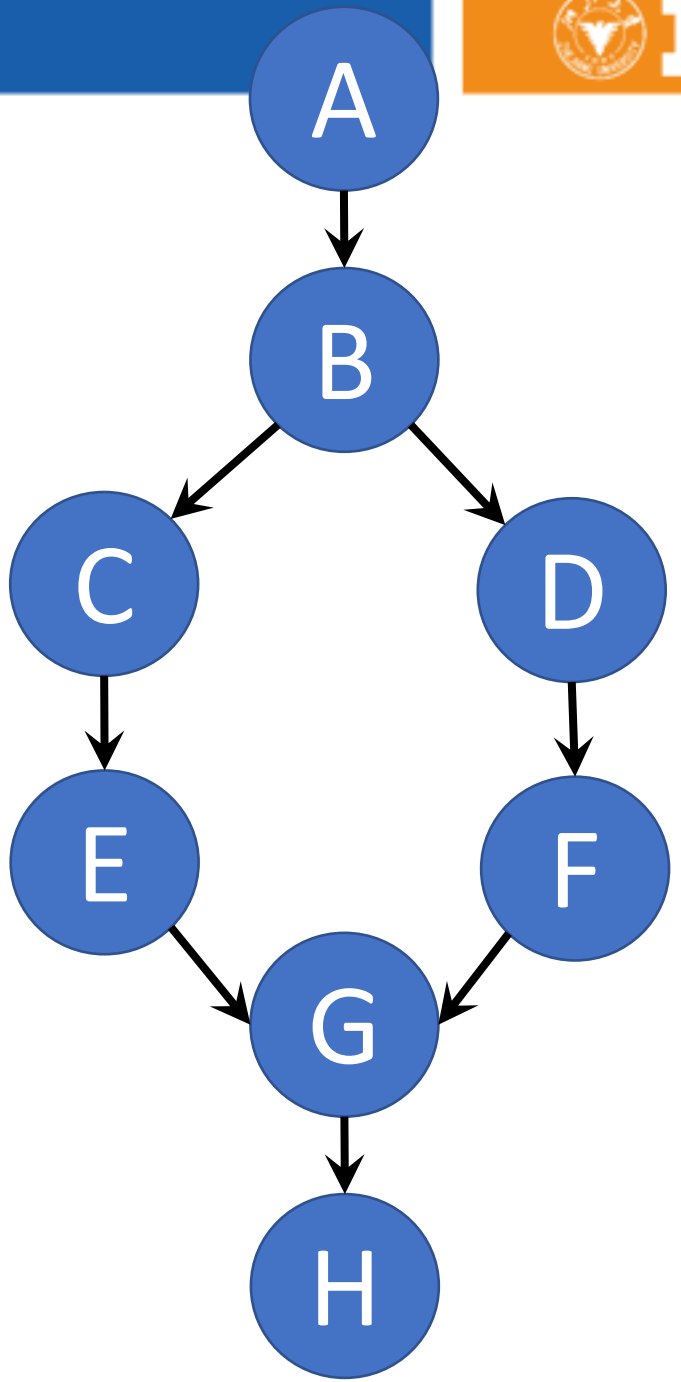
2.a. Markov Blanket

- The “Markov blanket” of variable F is D,E,G if

$$P(F|A,B,C,D,E,G,H)$$

$$= P(F|D,E,G)$$

- How can we prove that?
- P(A,…,H) = P(A)P(B|A) …
- Which of those terms include F?



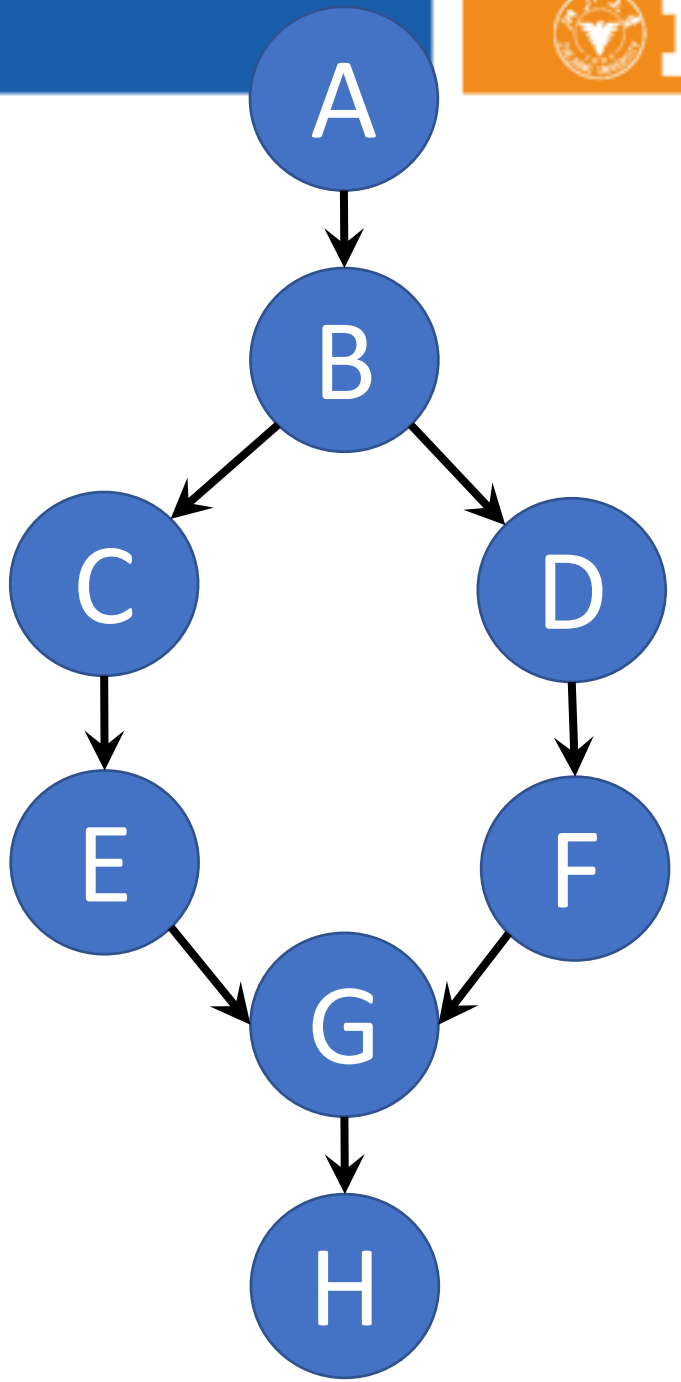
2.a. Markov Blanket

- Which of those terms include F?
- Only these two:

$$P(F|D)$$

and

$$P(G|E,F)$$



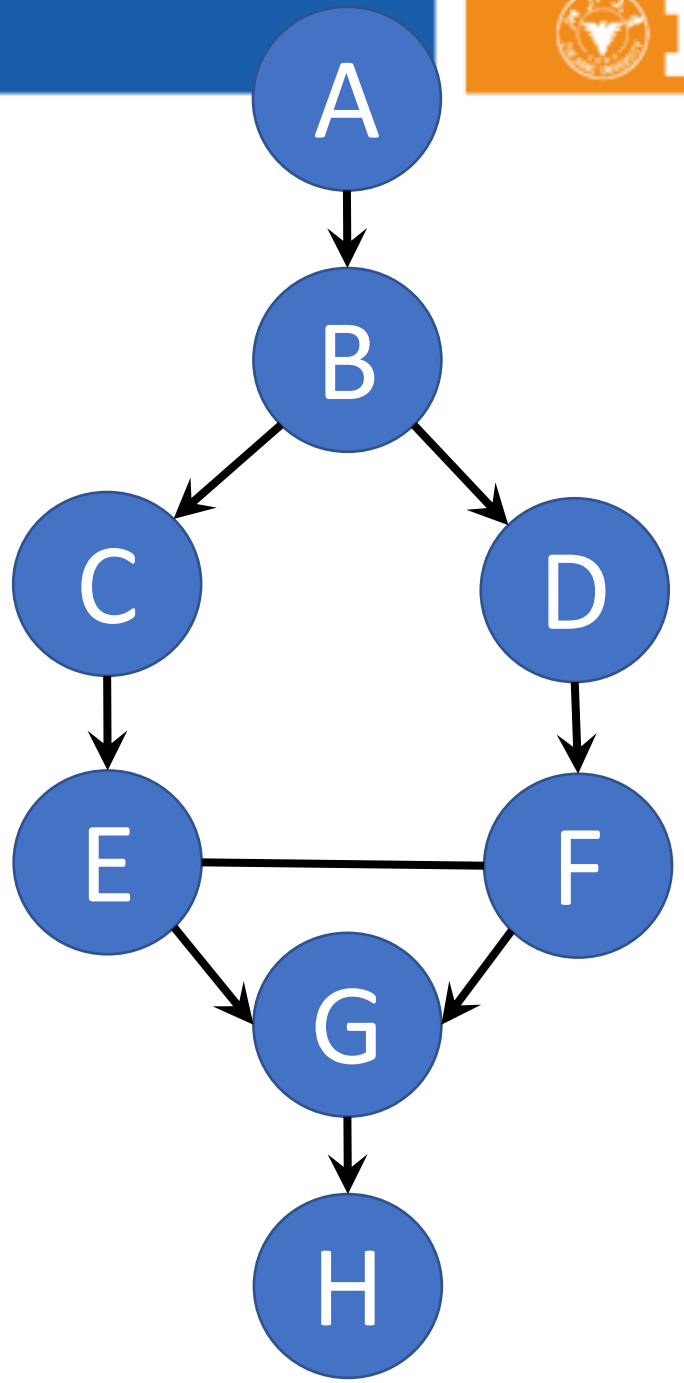
2.a. Markov Blanket

The Markov Blanket of variable F includes only its immediate family members:

- Its parent, D
- Its child, G
- The other parent of its child, E

Because $P(F|A,B,C,D,E,G,H)$

$= P(F|D,E,G)$

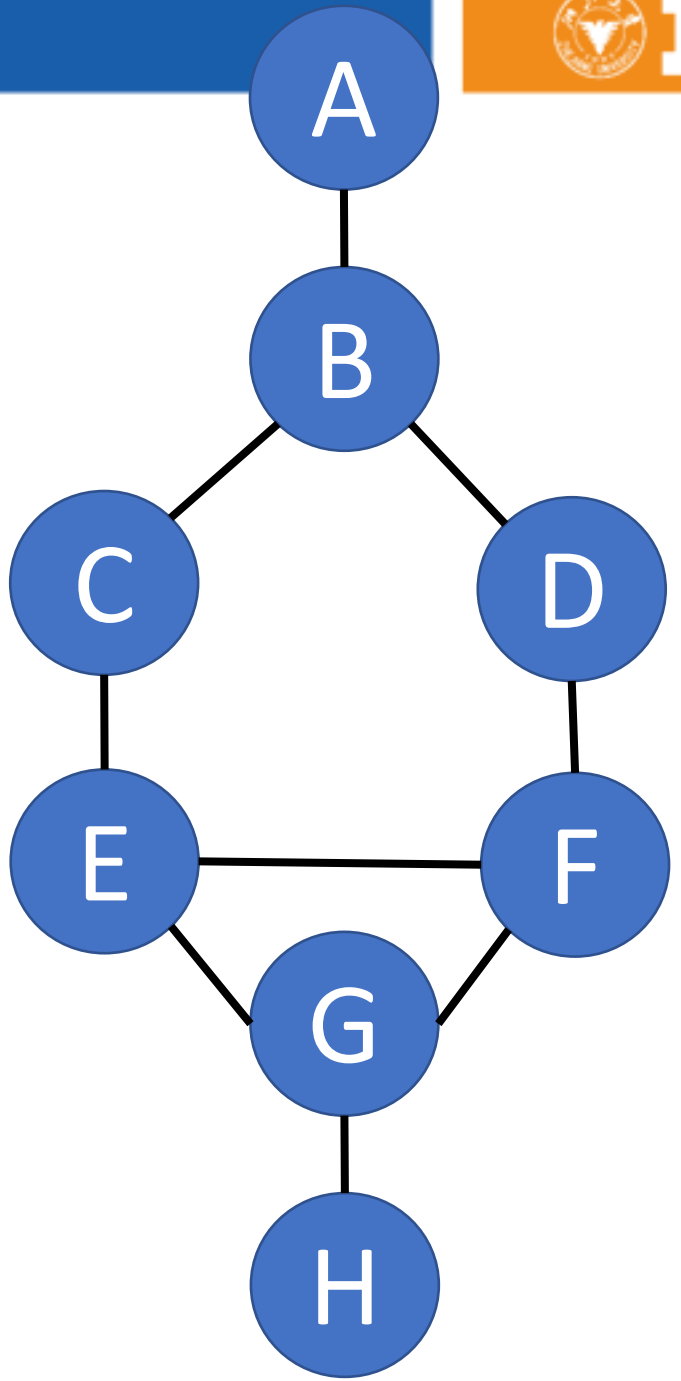


2.a. Moralization

"Moralization" =

1. If two variables have a child together, force them to get married.
2. Get rid of the arrows (not necessary any more).

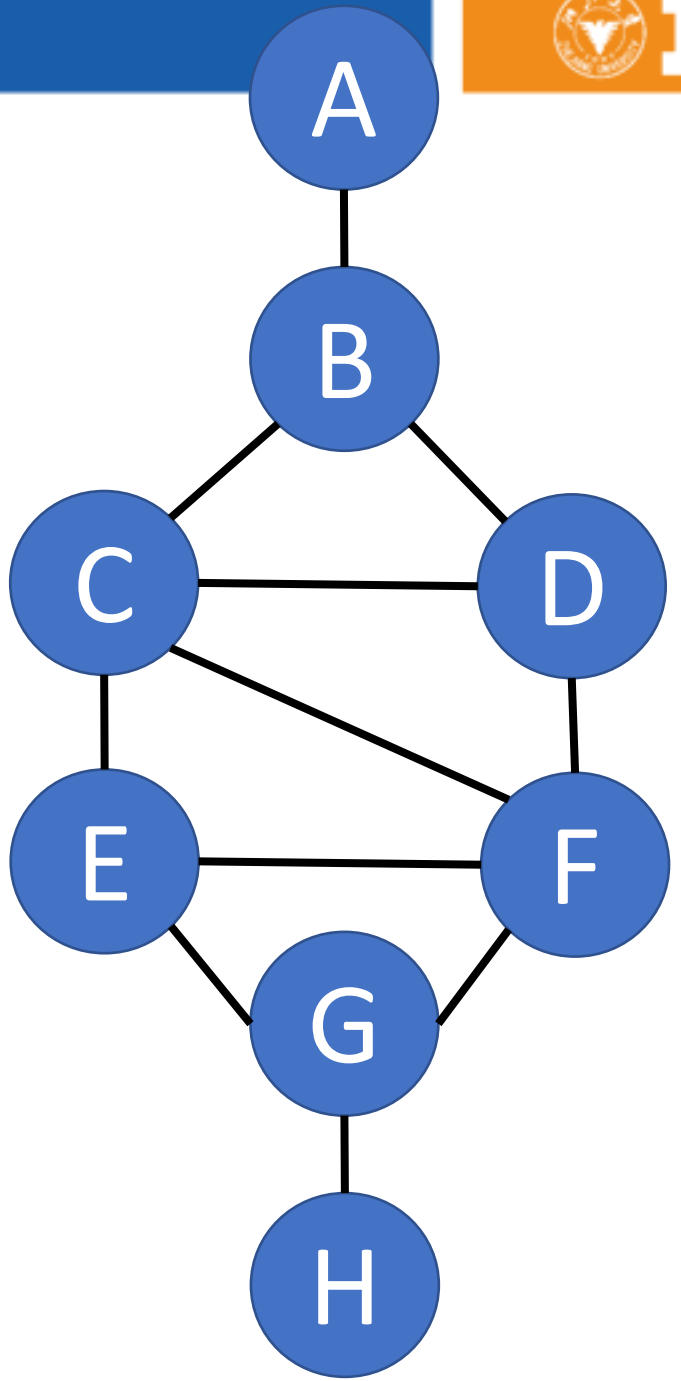
Result: Markov blanket = the set of variables to which a variable is connected.



2.b. Triangulation

Triangulation = draw edges so that there is no unbroken cycle of length > 3.

There are usually many different ways to do this. For example, here's one:

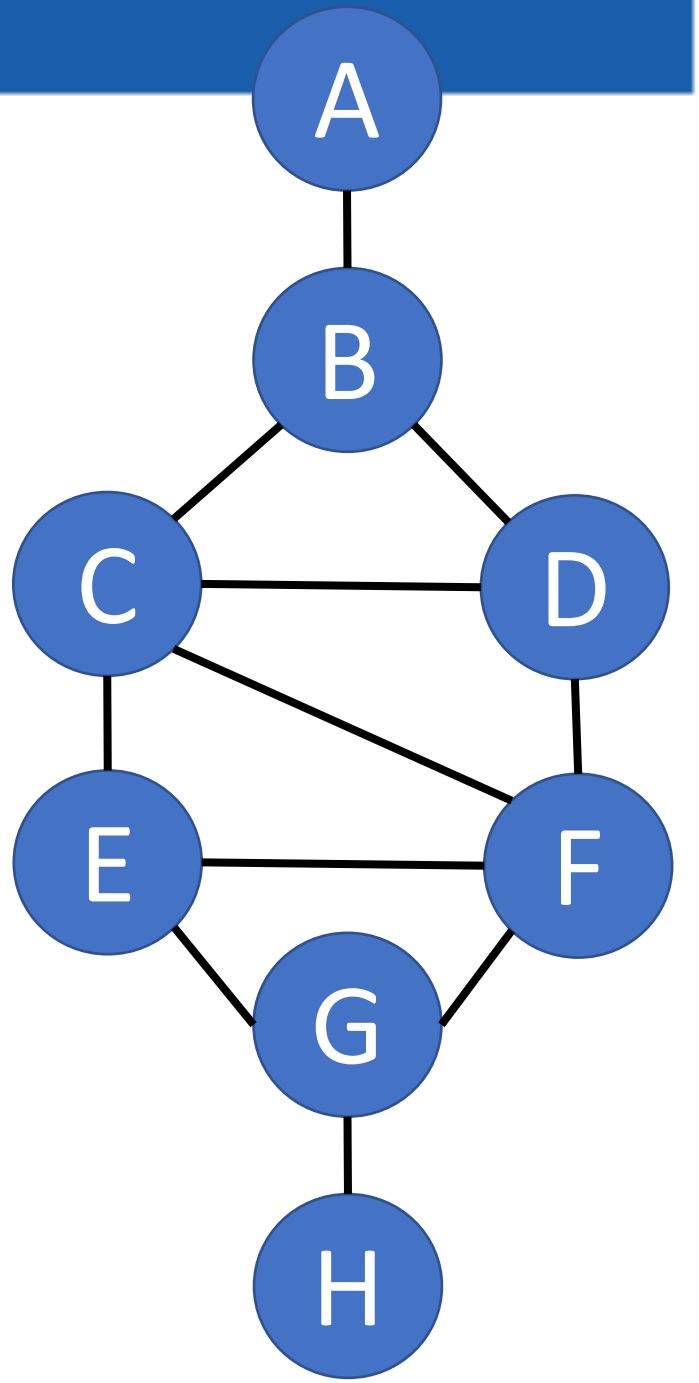


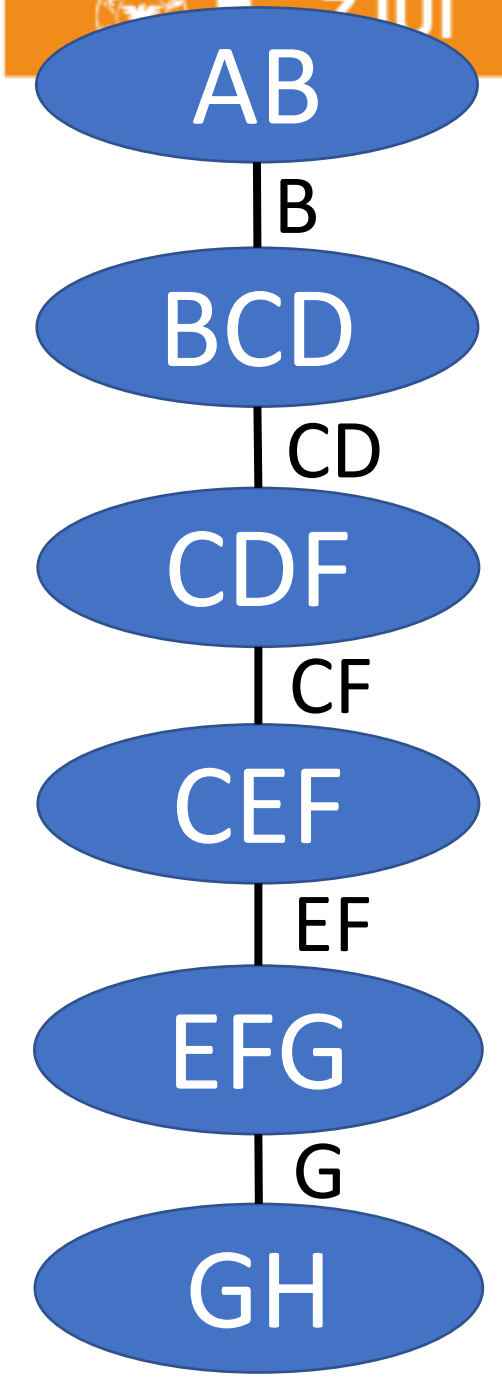
2.c. Form Cliques

Clique = a group of variables, all of whom are members of each other's immediate family.

Junction Tree = a tree in which

- Each node is a clique from the original graph,
- Each edge is an "intersection set," naming the variables that overlap between the two cliques.





Suppose we need P(B,G):

1. Product: P(B,C,D,F)=P(B)P(C|B)P(D|B)P(F|D)
2. Sum: $P(B,C,F) = \sum_D P(B,C,D,F)$
3. Product: P(B,C,E,F)=P(B,C,F)P(E|C)
4. Sum: $P(B,E,F) = \sum_C P(B,C,E,F)$
5. Product: P(B,E,F,G) = P(B,E,F)P(G|E,F)
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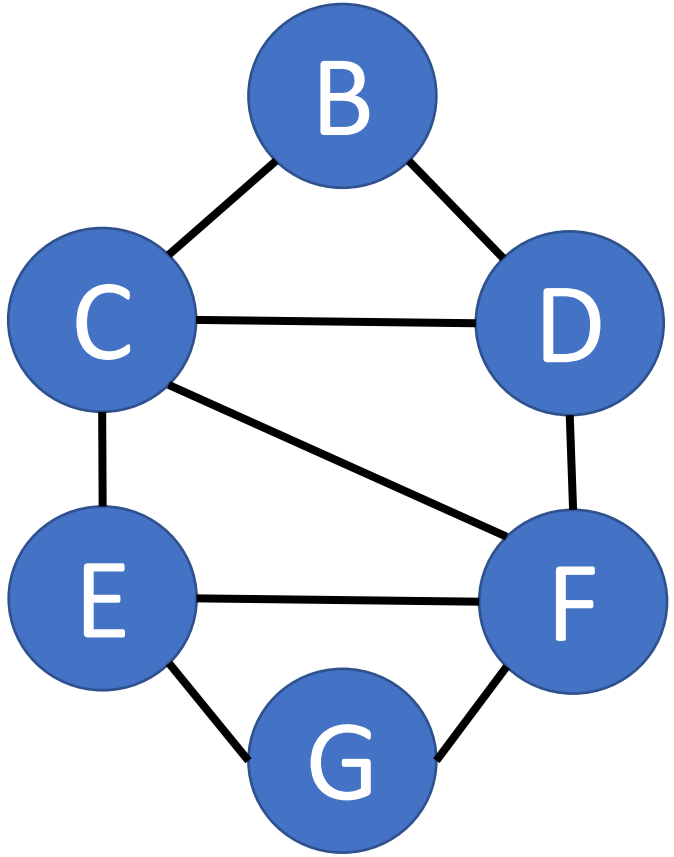
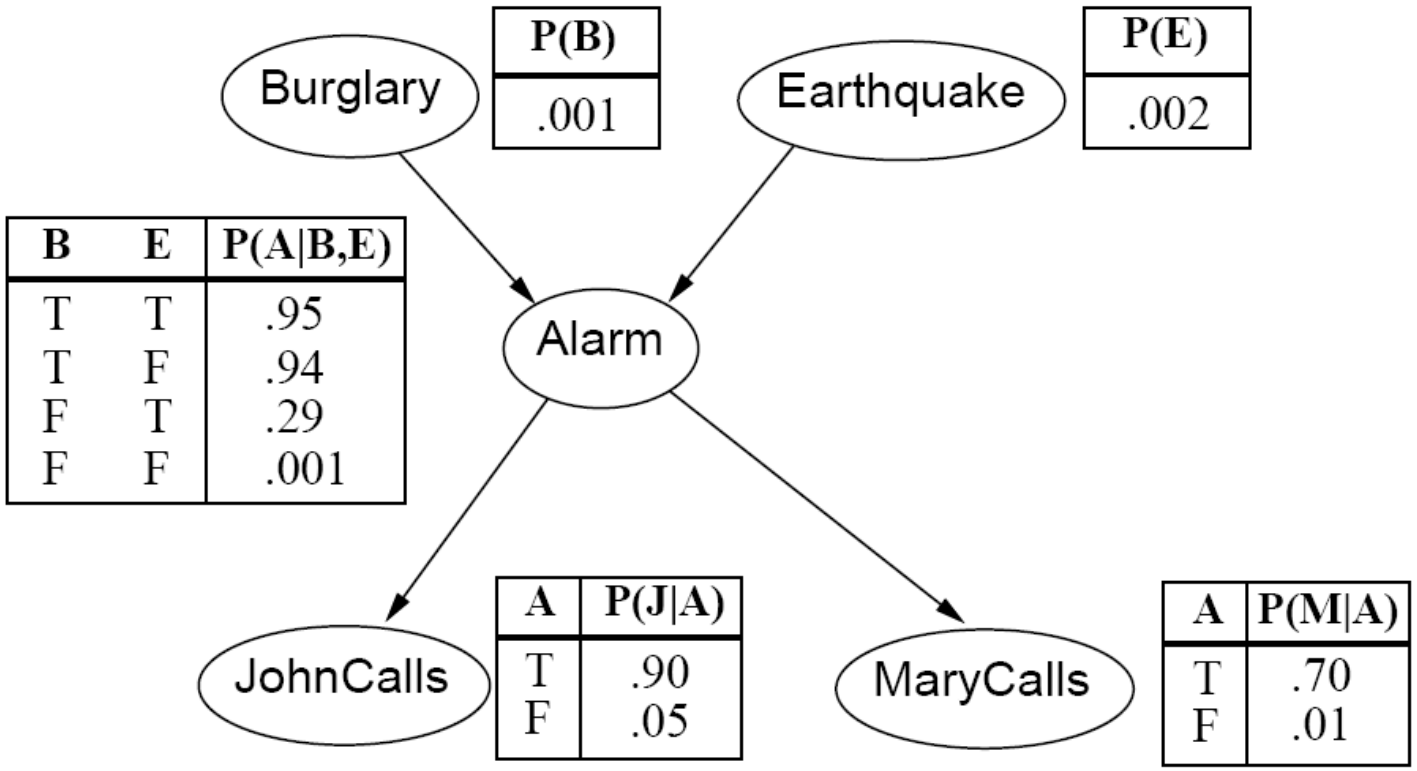


Complexity: $O\{NK^M\}$, where N=# cliques,

K = # values for each variable,

M = 1 + # variables in the largest clique

Junction Tree: Sample Test Question



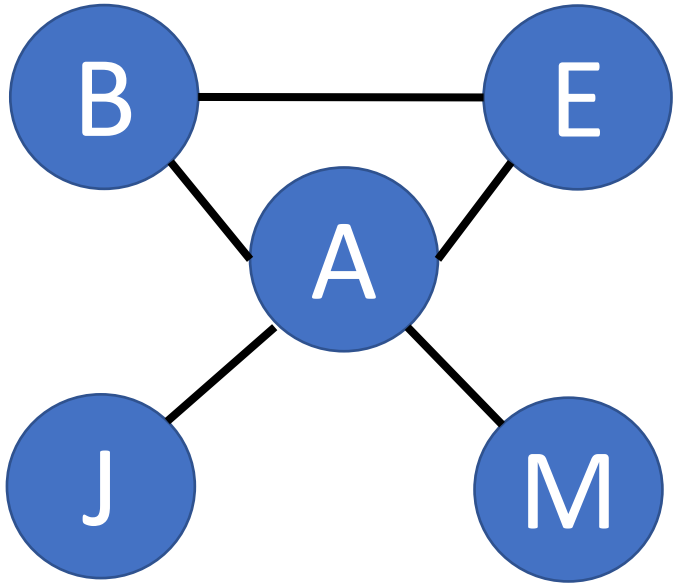
Consider the burglar alarm example.

a. Moralize this graph

b. Is it already triangulated? If not, triangulate it.

c. Draw the junction tree

Solution



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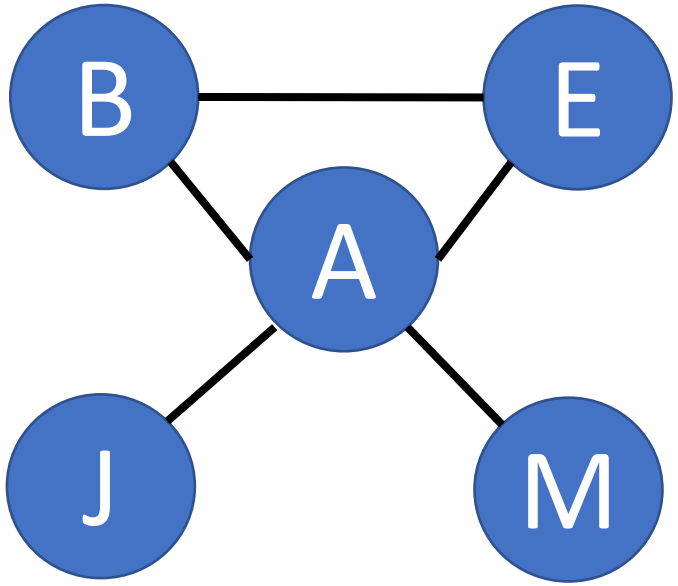
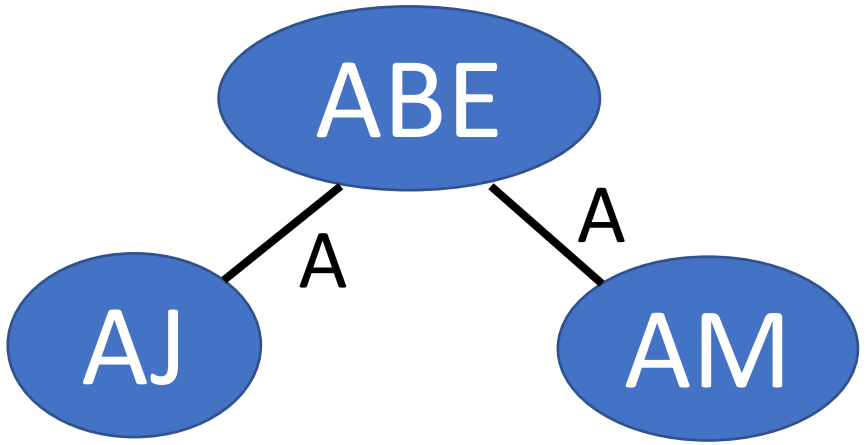


b. Is it already triangulated?

Answer: yes. There is no unbroken cycle of length > 3.

c. Draw the junction tree



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Bayesian network inference

- In full generality, NP-hard
  - More precisely, #P-hard: equivalent to counting satisfying assignments
- We can reduce satisfiability to Bayesian network inference
  - Decision problem: is P(Y) > 0?

$$Y = (U_1 \vee U_2 \vee U_3) \wedge (\neg U_1 \vee \neg U_2 \vee U_3) \wedge (U_2 \vee \neg U_3 \vee U_4)$$

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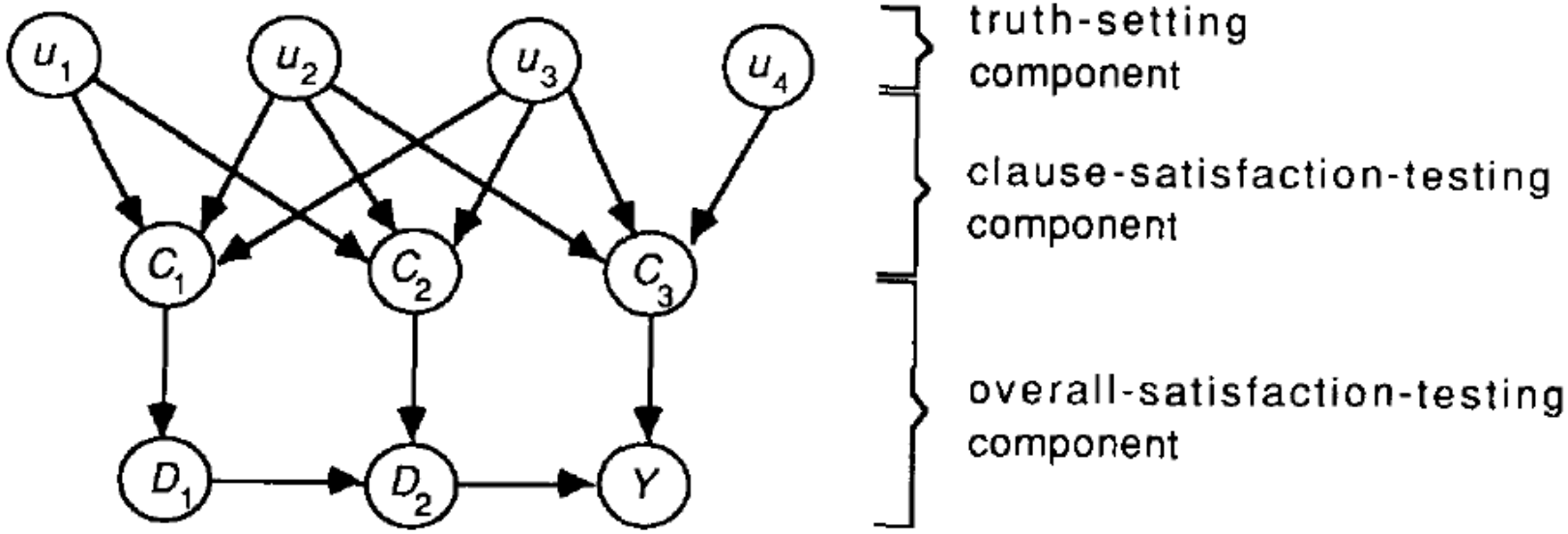
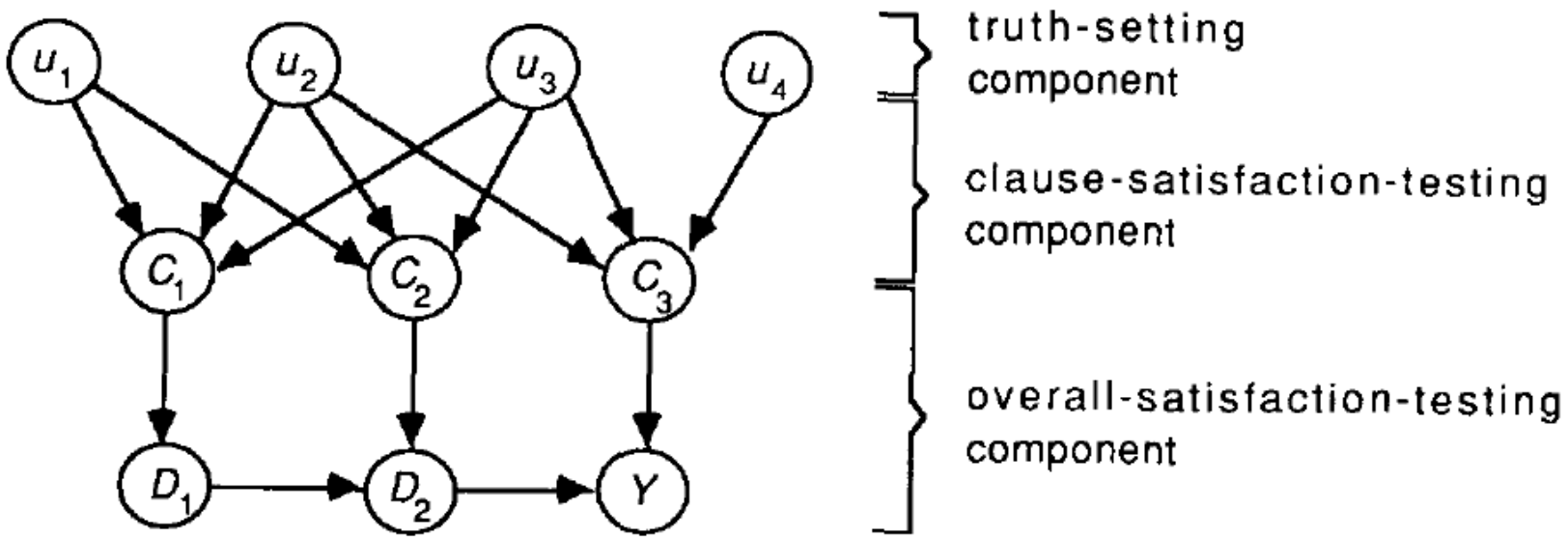


G. Cooper, 1990

Bayesian network inference

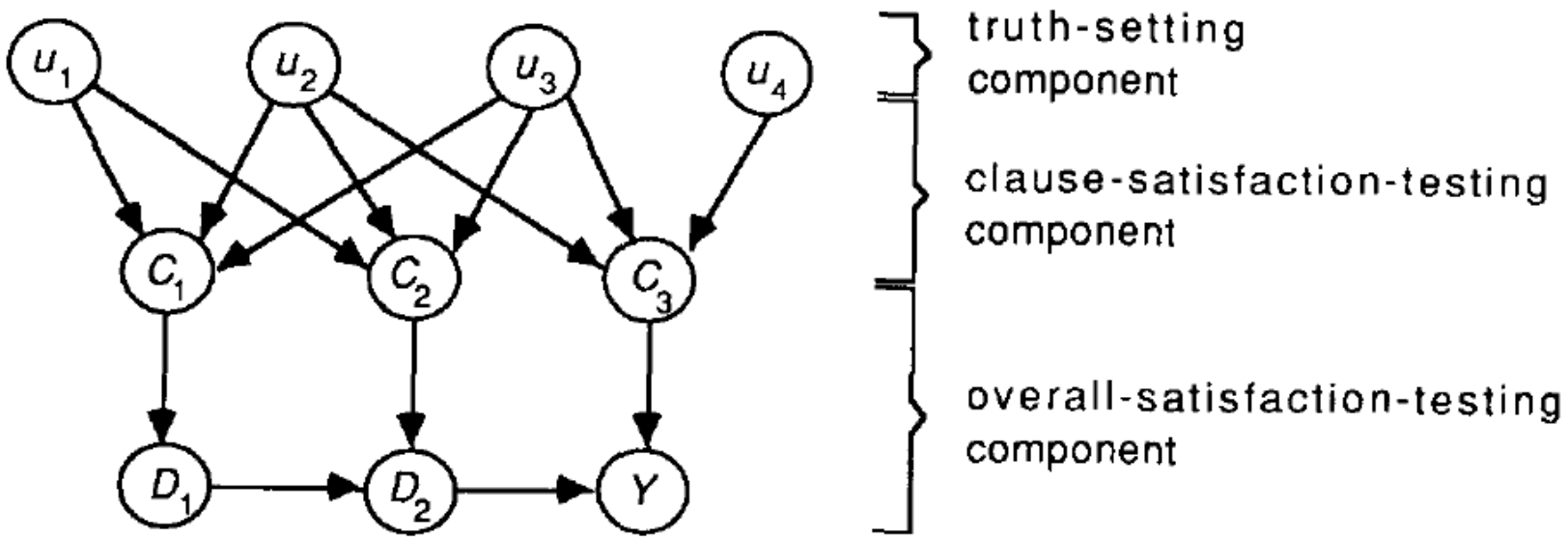


$$P(U_1,U_2,U_3,U_4,C_1,C_2,C_3,D_1,D_2,Y) =$$

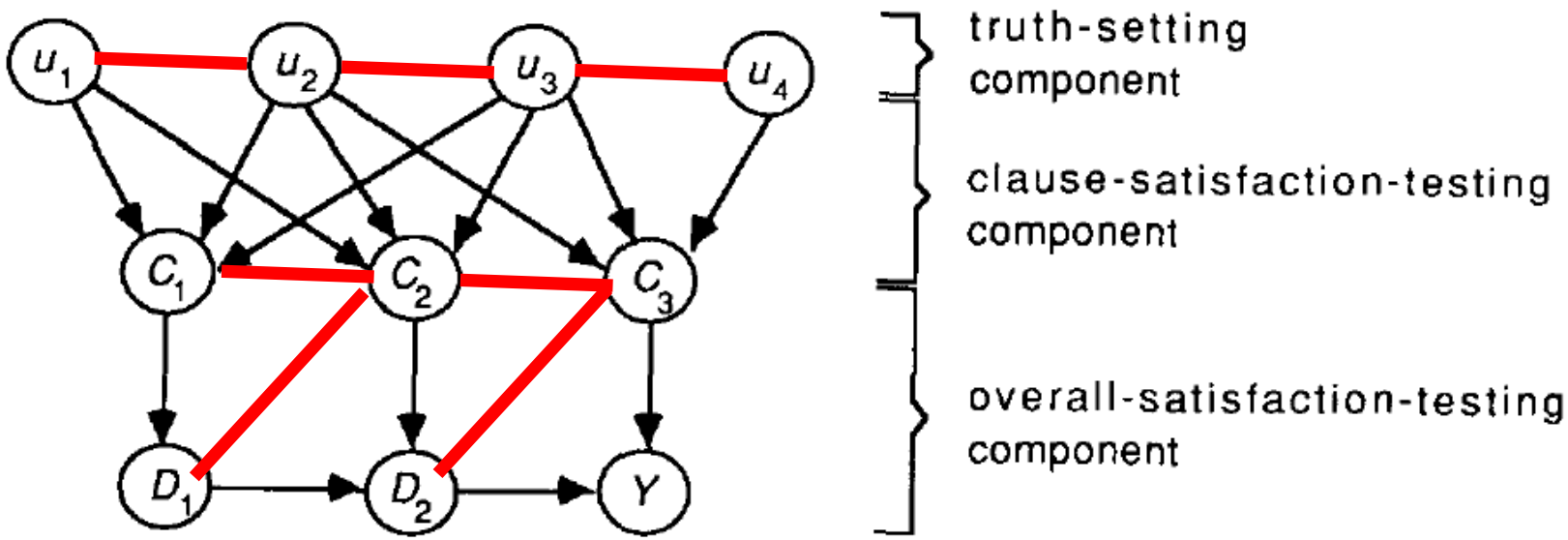
$$P(U_1)P(U_2)P(U_3)P(U_4)$$

$$P(C_1 \mid U_1,U_2,U_3)P(C_2 \mid U_1,U_2,U_3)P(C_3 \mid U_2,U_3,U_4)$$

$$P(D_1 \mid C_1)P(D_2 \mid D_1,C_2)P(Y \mid D_2,C_3)$$



Why can’t we use the junction tree algorithm to efficiently compute Pr(Y)?



Why can't we use the junction tree algorithm to efficiently compute Pr(Y)?

Answer: after we moralize and triangulate, the size of the largest clique (u2u3c1c2c3) is $M \approx N$, same order of magnitude as the original problem

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   - Trees: Sum-product algorithm
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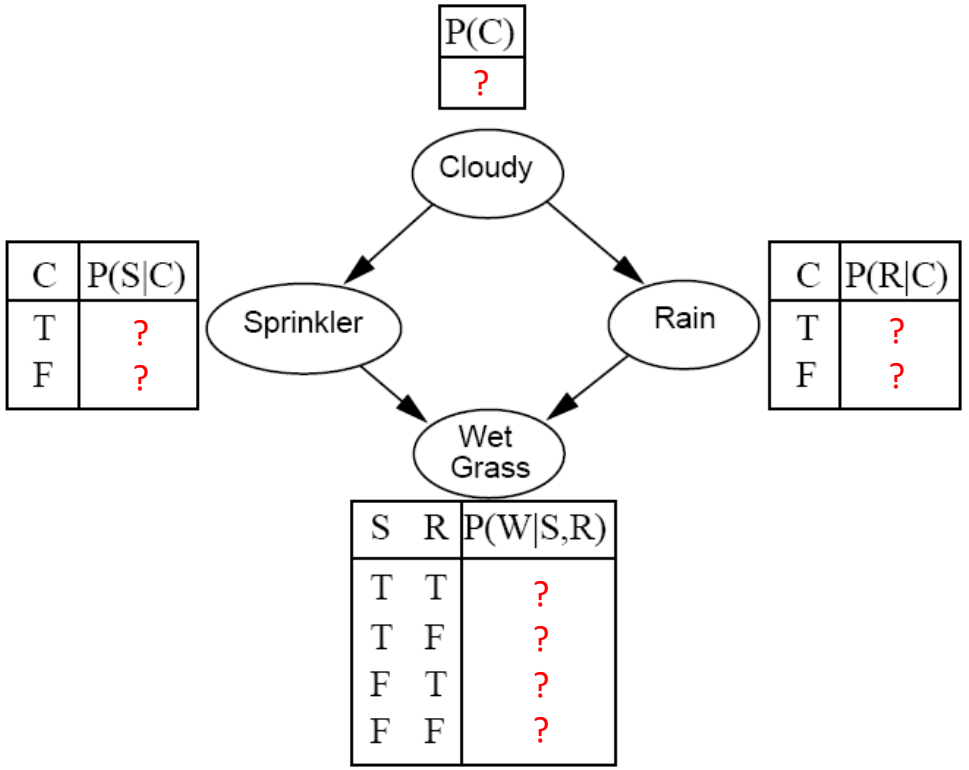
3. Parameter Learning

Parameter learning

- **Inference problem**: given values of evidence variables $\mathbf{E} = \mathbf{e}$, answer questions about query *variables* $\mathbf{X}$ using the posterior $P(\mathbf{X} \mid \mathbf{E} = \mathbf{e})$
- **Learning problem:** estimate the parameters of the probabilistic model $P(\mathbf{X} \mid \mathbf{E})$ given a *training sample* $\{(\mathbf{x}_1,\mathbf{e}_1), \ldots, (\mathbf{x}_n,\mathbf{e}_n)\}$

Parameter learning: complete data

- Suppose we know the network structure (but not the parameters), and have a training set of *complete* observations



Training set

Sample	C	S	R	W
1	T	F	T	T
2	F	T	F	T
3	T	F	F	F
4	T	T	T	T
5	F	T	F	T
6	T	F	T	F
…	…	…	….	…

Parameter learning

- Suppose we know the network structure (but not the parameters), and have a training set of *complete* observations
- Example:

$$P(S = T|C = T) = \frac{\#\text{samples with } S = T, C = T}{\#\text{ samples with } C = T} = \frac{1}{4}$$

Training set

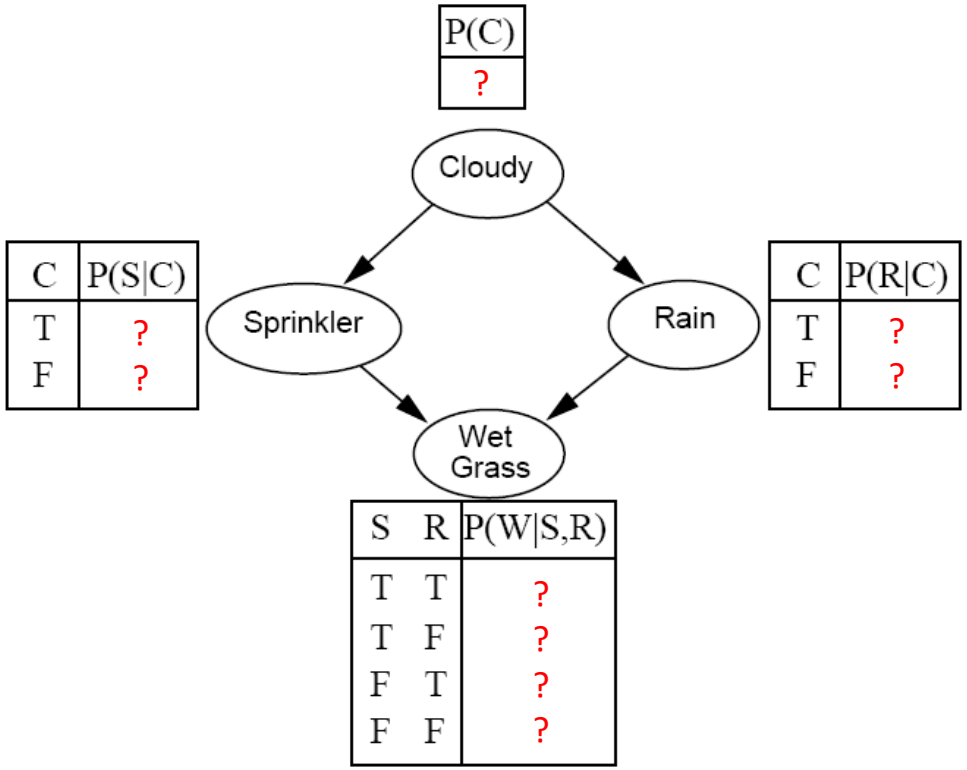
Sample	C	S	R	W
1	T	F	T	T
2	F	T	F	T
3	T	F	F	F
4	T	T	T	T
5	F	T	F	T
6	T	F	T	F
…	…	…	….	…

Parameter learning

- Suppose we know the network structure (but not the parameters), and have a training set of *complete* observations
  - P(X | Parents(X)) is given by the observed frequencies of the different values of X for each combination of parent values

Parameter learning: missing data

- Suppose we know the network structure (but not the parameters), and have a training set, but the training set is ***missing some observations***.

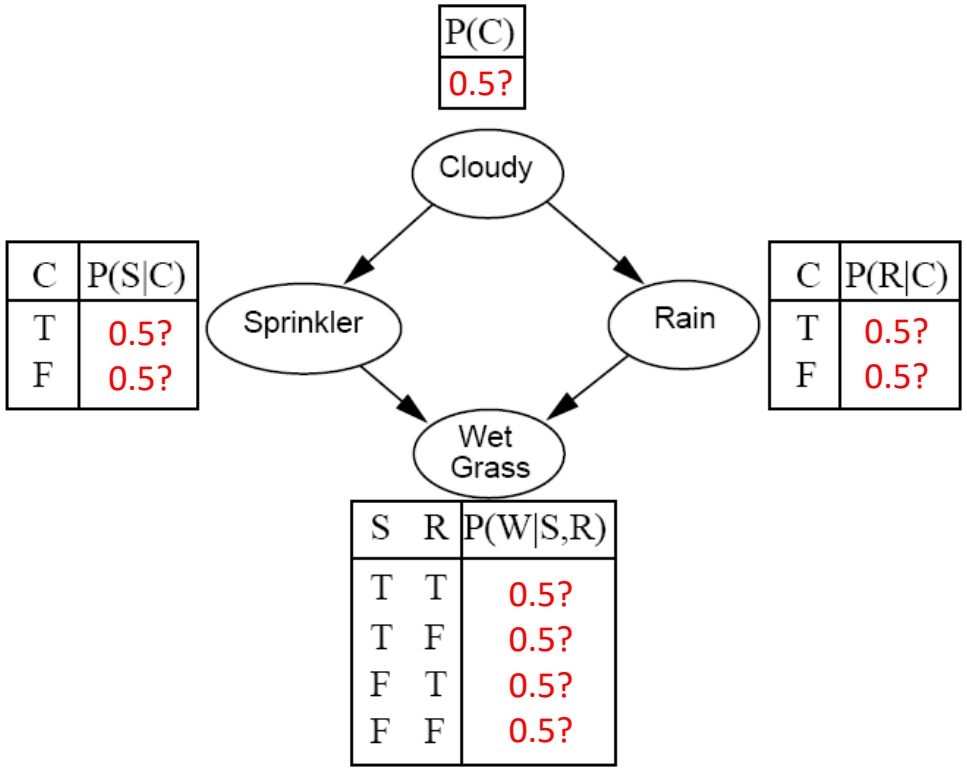


Training set

Sample	C	S	R	W
1	?	F	T	T
2	?	T	F	T
3	?	F	F	F
4	?	T	T	T
5	?	T	F	T
6	?	F	T	F
…	…	…	….	…

Missing data: the EM algorithm

- The EM algorithm starts ("Expectation Maximization") starts with an initial guess for each parameter value.
- We try to improve the initial guess, using the algorithm on the next two slides:
  - E-step
  - M-step



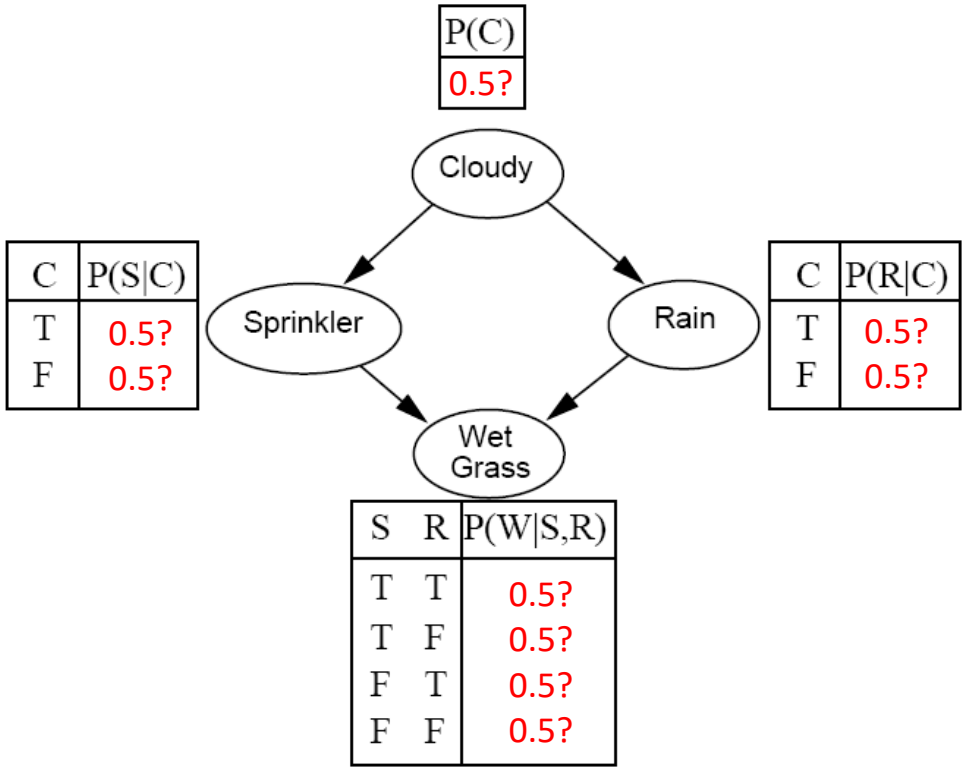
Training set

Sample	C	S	R	W
1	?	F	T	T
2	?	T	F	T
3	?	F	F	F
4	?	T	T	T
5	?	T	F	T
6	?	F	T	F
…	…	…	….	…

Missing data: the EM algorithm

- E-Step (Expectation): Given the model parameters, replace each of the missing numbers with a probability (a number between 0 and 1) using

$$P(C = 1|S, R, W) = \frac{P(C = 1, S, R, W)}{P(C = 1, S, R, W) + P(C = 0, S, R, W)}$$

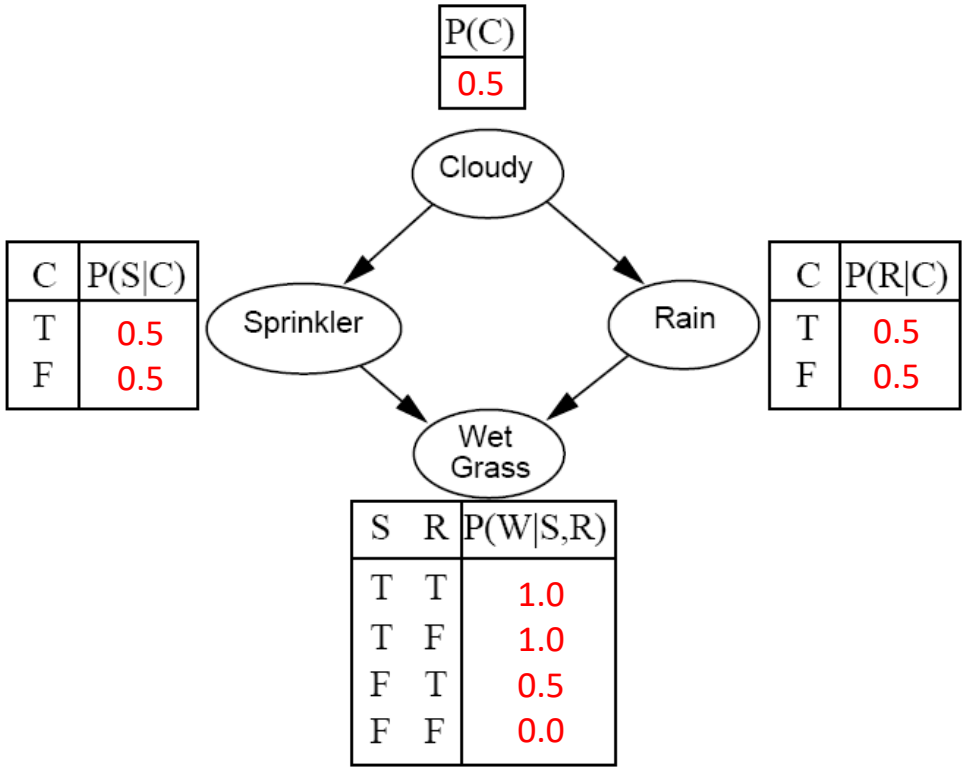


Training set

Sample	C	S	R	W
1	0.5?	F	T	T
2	0.5?	T	F	T
3	0.5?	F	F	F
4	0.5?	T	T	T
5	0.5?	T	F	T
6	0.5?	F	T	F
…	…	…	….	…

- M-Step (Maximization): Given the missing data estimates, replace each of the missing model parameters using

$$P(\text{Variable} = \text{T}|\text{Parents} = \text{value}) = \frac{E[\#\text{ times Variable} = T, \text{Parents} = \text{value}]}{E[\#\text{times Parents} = \text{value}]}$$

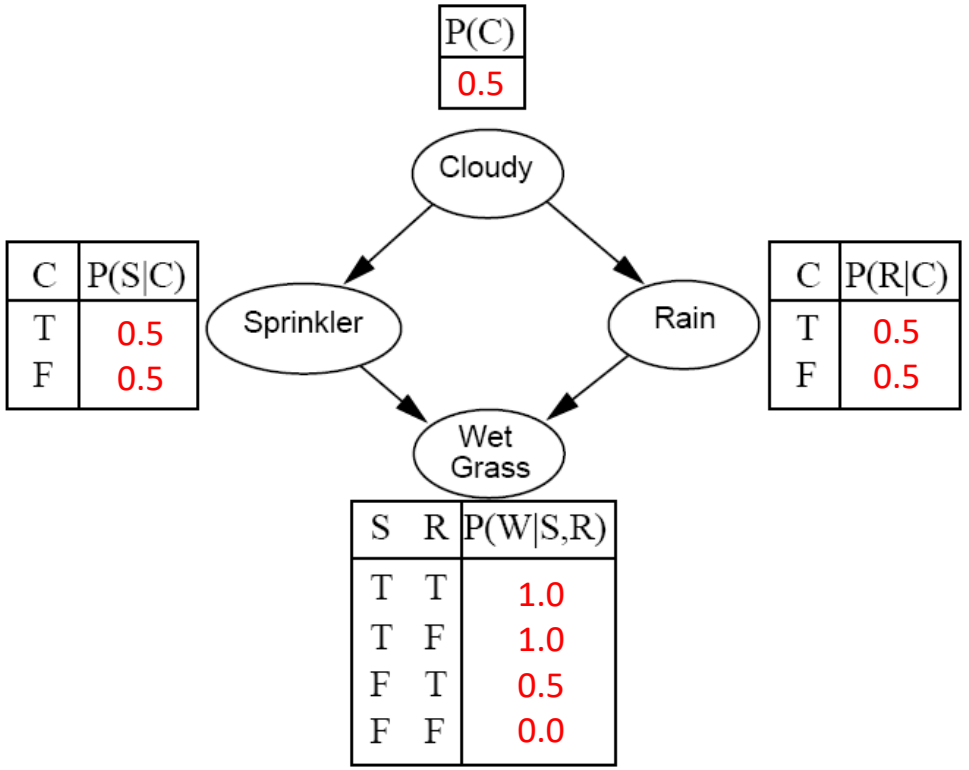


Training set

Sample	C	S	R	W
1	0.5?	F	T	T
2	0.5?	T	F	T
3	0.5?	F	F	F
4	0.5?	T	T	T
5	0.5?	T	F	T
6	0.5?	F	T	F
…	…	…	….	…

Missing data: the EM algorithm

- Iterate back and forth between E-step and M-step until the model converges.



Training set

Sample	C	S	R	W
1	0.5?	F	T	T
2	0.5?	T	F	T
3	0.5?	F	F	F
4	0.5?	T	T	T
5	0.5?	T	F	T
6	0.5?	F	T	F
…	…	…	….	…

Summary: Bayesian networks

- Structure
- Parameters
- Inference
- Learning