Chapter 04: sequences and series of functions

Notes from prof zeglaoui course

January 19, 2026

1 Sequences of functions

1.1 Convergence of a sequence of functions

Definition 1. *Let D be a subset (non-empty) of $\mathbb{R}$, if we denote by $\mathcal{F}(D, \mathbb{R})$ the $\mathbb{R}$ vector space of real functions from $D \to \mathbb{R}$.*

A sequence of functions over D is a sequence in $\mathcal{F}(D, \mathbb{R})$ that is :

$$\begin{aligned} &\mathbb{N} \to \mathcal{F}(D, \mathbb{R}) \\ &n \mapsto f_n : \begin{cases} D \to \mathbb{R} \\ x \mapsto f_n(x) \end{cases} \quad \textit{which is often denoted by } (f_n) \textit{ or } (f_n)_{n \in \mathbb{N}} \end{aligned}$$

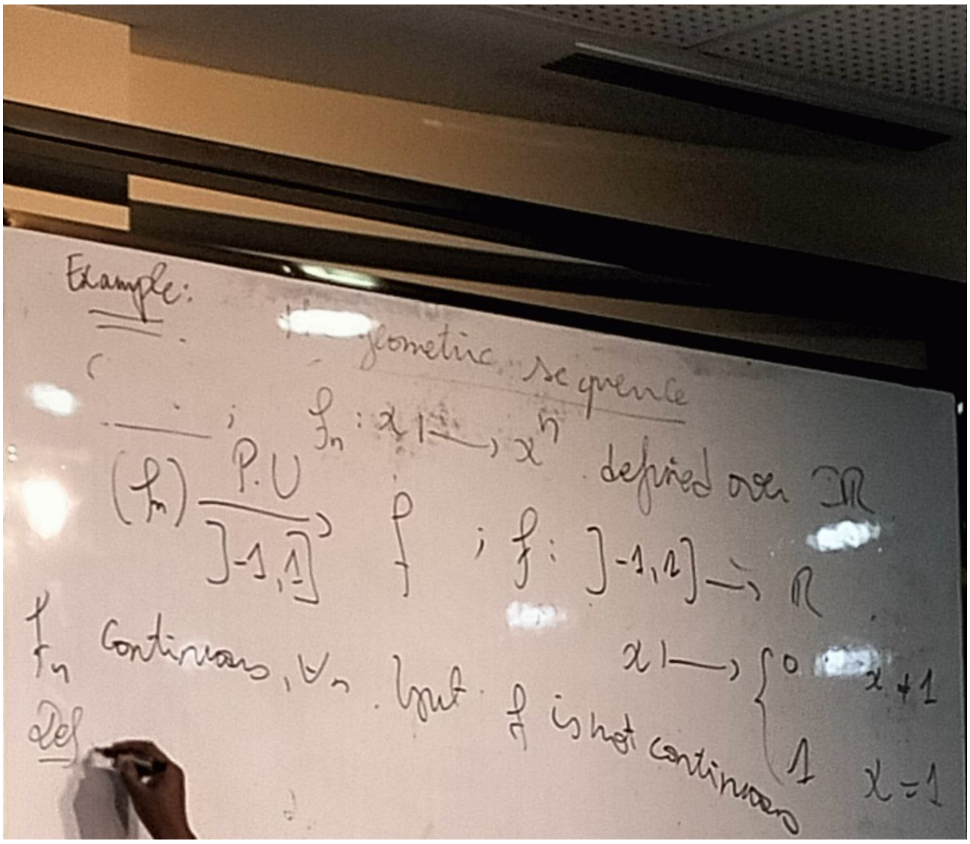
Definition 2. *Let (f_n) be a sequence of functions over D; we say that (f_n) positive convergente, or simply converges to $f \in \mathcal{F} \to (D, \mathbb{R})$, if :*

$$\forall x \in D; \lim_{n \to +\infty} f_n(x) = f(x) \textit{ or } (f_n) \xrightarrow{\textit{point-wise converge}} f$$

$$\Leftrightarrow \lim_{n \to +\infty} f_n = f$$

$$\forall x \in D, (\forall \epsilon > 0; \exists N = N(\epsilon, x) : \forall n \geq \mathbb{N} : \forall n \in \mathbb{N}, |f_n(x) - f(x)| < \epsilon)$$

Example 1. *...*



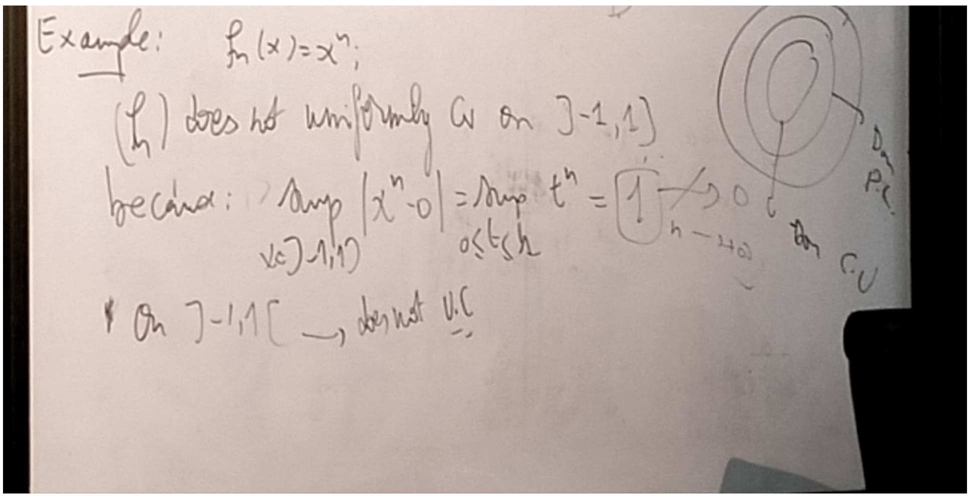
Definition 3. <u>*Uniform convergence*</u>

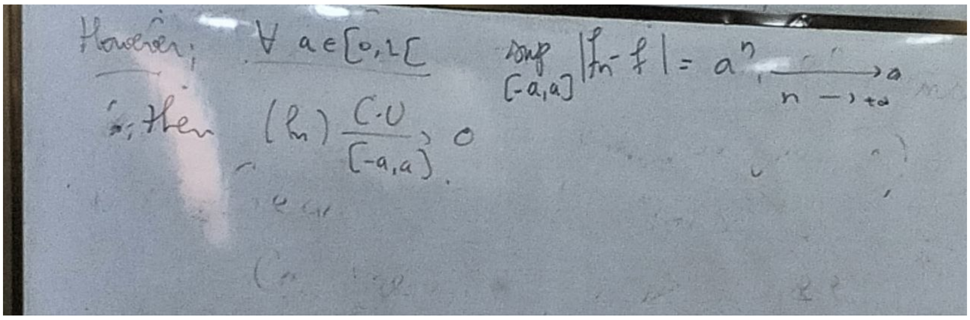
We say that $(f_n) \xrightarrow[D]{C.v} f$ *if* ; $\forall_n \in \mathbb{N}$; $(f_n - f)$ *is bounded over* D *and* $\lim_{n \to +\infty} (sup_D |f_n - f|) = 0$

$\Leftrightarrow \forall \epsilon > 0, \exists N = N(\epsilon) : \forall n \geq N, \forall x \in D, |f_n(x) - f(x) < \epsilon|$

Remark: $(f_n) \xrightarrow[D]{C.V} f \implies (f_n) \xrightarrow[D]{P.V} f$

Example 2. ...





1.2 Properties of uniformly convergent sequences of functions

Theorem 1. *(continuity)*

If

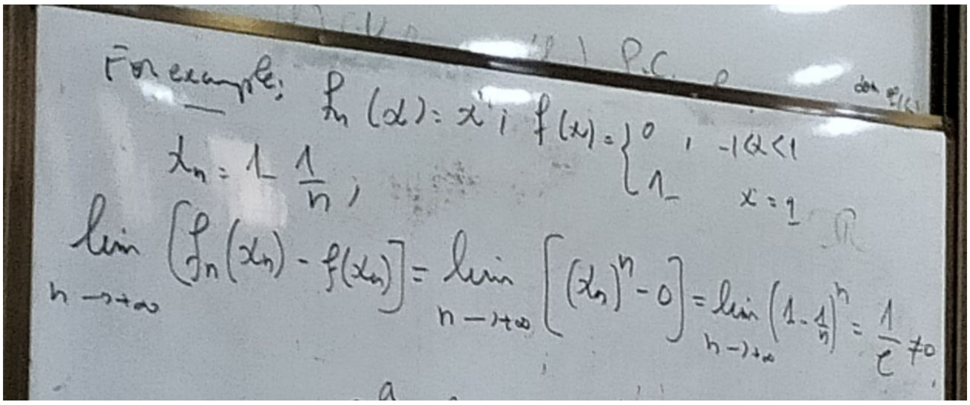
$$\begin{cases} \exists N \in \mathbb{N} \mid \forall \geq N, \ f_n \ \textit{continuous on } D \\ \textit{and } (f_n) \xrightarrow[D]{C.V} f \end{cases} \quad \textit{then } f \ \textit{is continuous}$$

$$\begin{aligned} \forall x_0 \in D; \lim_{x \to x_0} f(x) &= \lim_{x \to x_0} (\lim_{n \to +\infty} f_n(x)) \\ &= \lim_{n \to +\infty} (\lim_{x \to x_0} f(x_0)) = f(x_0) \end{aligned}$$

Remark: if $(f_n) \xrightarrow[D]{P.C} f$ and $(x_n) \subset D \mid \lim_{n \to +\infty} [f_n(x_n) - f(x_n)] \neq 0$ then $(f_n)_{n \in \mathbb{N}}$ does not unifromly convergent to f on D because :

$$(f_n) \xrightarrow[D]{C.V} f \implies \forall (x_n) \subset D : \lim_{n \to +\infty} [f_n(x_n) - f(x_n)] = 0$$

Example 3. ..



Theorem 2. *(integrability)*

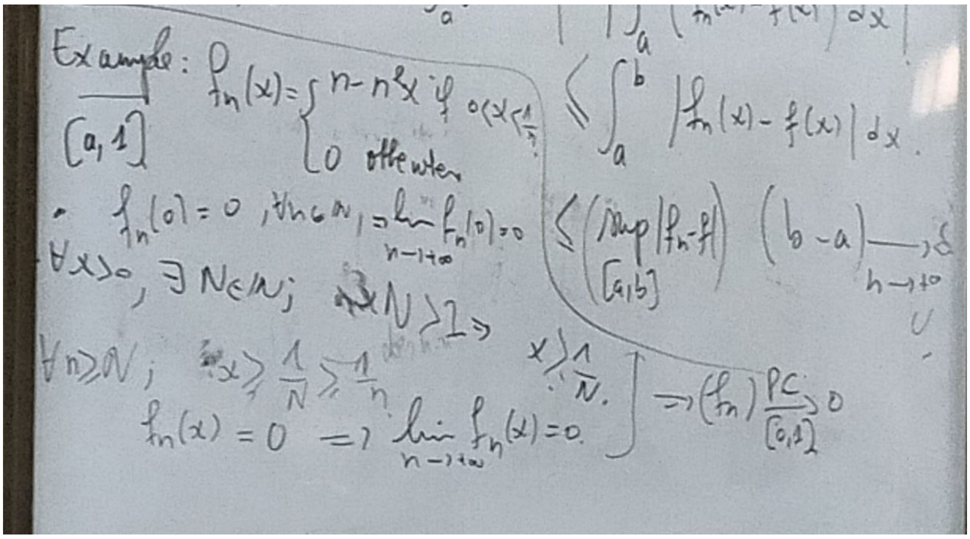
if $f_n : [a, b] \to \mathbb{R}$ *is Riemann-integrable,* $\forall n \in N$ *and* $(f_n) \xrightarrow[D]{C.V} f$*; then :*

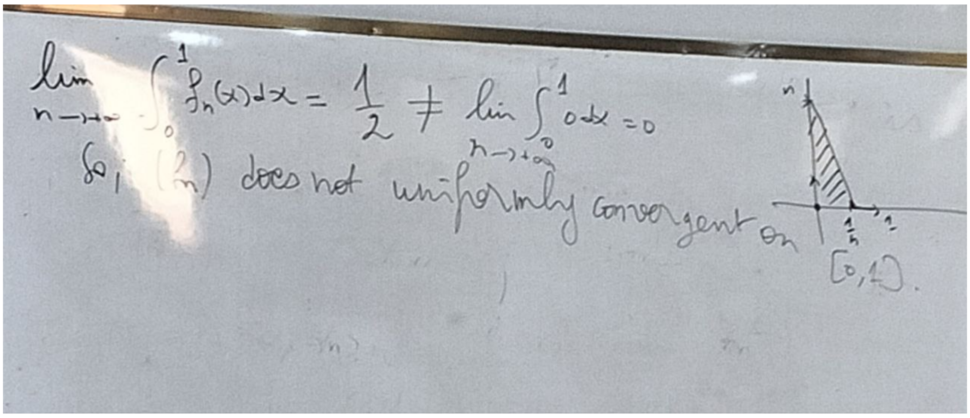
$$\lim_{n \to +\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx = \int_a^b (\lim_{n \to +\infty} f_n(x))dx$$

proof:

$$\begin{aligned}
|\int_a^b f_n(x)dx - \int_a^b f(x)dx| &= |\int_a^b (f_n(x) - f(x))dx| \\
&\leq \int_a^b |f_n(x) - f(x)|dx \\
&\leq (sup_{[a,b]}(f_n - f))(b-a) \xrightarrow[n\to+\infty]{} 0
\end{aligned}$$

Example 4. ..





Theorem 3. <u>*(differentiability)*</u>

Let I an <u>interval</u> ; $(f_n)_n$ a sequence of class C^1 functions on I such that : $(f'_n)_n \xrightarrow[I]{C.V} g$ and $\exists x_0 \in I \setminus$ the numerical sequence $f_n(x_0)$ converges to a real number l

then : $(f'_n)_n \xrightarrow[I]{C.V} f$ / where f is defined by : $\begin{cases} f' = g \\ f(x_0) = l \end{cases}$

so ;

$$\lim_{n\to+\infty} f'_n(x) = g(x) = f'(x) = \left(\lim_{n\to+\infty} f_n(x)\right)'$$

Proof:

$$f(x) = l + \int_{x_0}^{x} g(t)dt, \forall x \in I$$

Recall that : $f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt$

Using the integrability theorem we get : $(f_n) \xrightarrow[I]{P.C} f$

(P.C means point-wise continous)