Chapter 0 : Functions of Several Variables

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A Day

1 Generalities

1.1 Definition of a functions of several variables

Let E, F be two sets.

The product $E \times F = \{(x, y) \mid x \in E \text{ and } y \in F\}$.

Definition 1. *A relation $\mathcal{R}$ from E to F is a given subset G (or G_R) of $E \times F$.*

We say that y is the image of x (and x is the pre-image of y) if $(x, y) \in G$.

The inverse relation $\mathcal{R}^{-1}$ of $\mathcal{R}$ is from F to E defined by:

$$y\mathcal{R}^{-1}x \Leftrightarrow x\mathcal{R}y$$

$$G_{\mathcal{R}^{-1}} = s(G_{\mathcal{R}}), \text{ where } s : E \times F \to F \times E, (x, y) \mapsto (y, x)$$

Definition 2. *A function $f : E \to F$ is a relation from E to F such that:*

For each $x \in E$, there exists at most one image $y \in F$.

The domain of f, denoted $Dom(f)$, is the subset:

$$Dom(f) = \{x \in E \mid \textit{there exists an image of } x\}$$

If $x \in Dom(f)$, we denote $f(x)$ its image.

When $Dom(f) = E$, f is called a map from E to F.

The graph of f is:

$$G_f = \{(x, y) \in E \times F \mid x \in Dom(f) \text{ and } y = f(x)\}$$

If $F = \mathbb{R}$: $f : E \to \mathbb{R}$ is a real function.

If $F = \mathbb{C}$: $f : E \to \mathbb{C}$ is a complex function.

Definition 3. *Let $n, m \in \mathbb{N}$. A function $\mathbb{R}^n \to \mathbb{R}^m$ is called a function of n variables:*

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

We denote by f_j the j-th component (or projection Pr_j) of f:

$$f_j : \mathbb{R}^n \to \mathbb{R}$$

$$(x_1, \dots, x_n) \mapsto f_j(x_1, \dots, x_n)$$

Another way to write it:

$$Pr_j : \mathbb{R}^m \to \mathbb{R}, \quad \forall j \in \{1, \dots, m\}$$

$$(y_1, y_2, \dots, y_j, \dots, y_m) \mapsto y_j$$

$$f = (f_1, \dots, f_m) = \sum_{j=1}^{m} f_j e_j \quad \text{where } e_j = (0, 0, \dots, 1, \dots, 0)$$

If $m = 1$, f is a real function.

If $m > 1$, f is a vector function.

The range (or image) of f is the subset of $\mathbb{R}^m$ defined by:

$$range(f) = \{y \in \mathbb{R}^m \mid \exists x \in Dom(f), y = f(x)\}$$

Example 1.

1. *$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ - the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ - is a subset of $F(\mathbb{R}^n, \mathbb{R}^m)$*

   where $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of all functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

2. *An affine function:*

   $$f : \mathbb{R}^n \to \mathbb{R}^m$$

   $$x \mapsto Ax + b$$

   where $b \in \mathbb{R}^m$ and $A \in M_{(m,n)}(\mathbb{R})$.

3. *For $\mathbb{R}^2 \to \mathbb{R}$:*

   $$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha x + \beta y$$

1.2 Operations on Functions of Several Variables

1) $\forall f, g \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m), \forall \alpha, \beta \in \mathbb{R}$

$$\alpha f + \beta g : \mathbb{R}^n \to \mathbb{R}^m$$

$$x \mapsto \alpha f(x) + \beta g(x)$$

$(\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m); +; \cdot)$ is a vector space.

2) If $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ and $g \in \mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ then:

$$g \circ f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^p)$$

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

$$x \mapsto f(x) \mapsto g(f(x))$$

3) Let $f, g \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ be two scalar functions.

The product $f \cdot g \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ defined by:

$$(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$$

If $g(x) \neq 0$, $\forall x \in \mathrm{Dom}(g)$, then $\dfrac{1}{g} : x \to \dfrac{1}{g(x)}$

$$\mathrm{Dom}\left(\frac{1}{g}\right) = \{x \in \mathrm{Dom}(g) \mid g(x) \neq 0\}$$

and $\dfrac{f}{g} = f \cdot \left(\dfrac{1}{g}\right)$.

Example 2. *1. A monomial function of degree P is:*

$$(x_1, \ldots, x_n) \mapsto \alpha x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$$

$$p_1 + p_2 + \cdots + p_n = P$$

2. A homogeneous polynomial of degree P is a finite sum of monomial functions of degree P.

3. A polynomial function is a finite sum of homogeneous polynomial functions.

4. A rational function is the quotient of two polynomial functions.

Definition 4. *The graph of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the subset of $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ defined by:*

$$G_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in Dom(f) \text{ and } y = f(x)\}$$

Remark 1. *If we consider the function:*

$$\tilde{f} : Dom(f) \to \mathbb{R}^{n+m}$$

$$x \mapsto (x, f(x))$$

then $range(\tilde{f}) = G_f$.

$\tilde{f}$ *is a parametrization of* G_f.

Example 3. 1. *Affine function* $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto ax + b$:

$$G_f = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R} \text{ and } y = ax + b\}$$

2. $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \sqrt{16 - 4x^2 - y^2}$:

$$Dom(f) = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + y^2 \leq 16\}$$

$$range(f) = \{z \in \mathbb{R} \mid \exists (x, y) \in Dom(f), z = \sqrt{16 - 4x^2 - y^2}\}$$

Since $4x^2 + y^2 \geq 0$, *we have:*

$$0 \leq 16 - 4x^2 - y^2 \leq 16 \Rightarrow 0 \leq \sqrt{16 - 4x^2 - y^2} \leq 4$$

Thus $range(f) = [0, 4]$.

The graph is:

$$G_f = \left\{ (x, y, z) \in \mathbb{R}^3 \mid (x, y) \in Dom(f) \text{ and } \begin{cases} z^2 + 4x^2 + y^2 = 16 \\ z \geq 0 \end{cases} \right\}$$

2 Limits and Continuity

2.1 Limits

Definition 5. *Let* $f : \mathbb{R}^n \to \mathbb{R}$, $l \in \mathbb{R}$, *and* $a \in \mathbb{R}^n$.

$$\lim_{x \to a} f(x) = l$$

$$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in Dom(f); 0 < d(x, a) < \delta \implies |f(x) - l| < \epsilon$$

$$\lim_{x \to a} f(x) = +\infty \ (resp. \ -\infty)$$

$$\Leftrightarrow \forall \alpha > 0, \exists \delta > 0, \forall x \in Dom(f); 0 < d(x, a) < \delta \implies f(x) > \alpha \ (resp. \ f(x) < -\alpha)$$

where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ *is one of the following distances:*

Euclidean distance:

$$d_2((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}$$

Manhattan distance:

$$d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{n}|y_i - x_i|$$

Chebyshev distance:

$$d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max_{1\le i\le n}|y_i - x_i|$$

Remark 2. *If $n=1$, then $d_1 = d_2 = d_\infty$.*

Proposition 1.

$$\lim_{x\to a} f(x) = l \Leftrightarrow \forall (x_n) \subset Dom(f),\ \lim_{n\to+\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = l$$

Remark 3. *If the limit l exists, it is unique.*

Definition 6. *Let $f = (f_1,\ldots,f_m) : \mathbb{R}^n \to \mathbb{R}^m$*

$$x \mapsto (f_1(x),\ldots,f_m(x))$$

$$\forall j \in \{1,\ldots,m\},\quad l_j = \lim_{x\to a} f_j(x)$$

If $\forall j; l_j \in \mathbb{R}$, then $\lim_{x\to a} f(x) = l = (l_1,\ldots,l_m) \in \mathbb{R}^m$.

If $\exists j; l_j = \pm\infty$, then $\lim_{x\to a} f(x)$ does not exist.

Definition 7. *$f = (f_1,\ldots,f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point $a \in Dom(f)$ if:*

$$\lim_{x\to a} f(x) = f(a)$$

Example 4. *f continuous at $x = a$*

$$\Leftrightarrow \forall (x_n) \subset Dom(f),\ \lim_{n\to+\infty} x_n = a \Rightarrow \lim_{n\to+\infty} f(x_n) = f(a)$$

Remark 4. *To prove that* $\lim_{x \to a} f(x)$ *does not exist:*

1st method:

We can give two sequences $(x_n), (y_n)$ *from* $Dom(f)$ *that converge to* a

$$\begin{cases} \lim_{n\to\infty} x_n = a \\ \lim_{n\to\infty} y_n = a \end{cases}$$

and $\lim_{n\to+\infty} f(x_n) \neq \lim_{n\to+\infty} f(y_n)$.

2nd method:

We give two paths (continuous maps $[0, \delta) \to Dom(f)$*) on* $Dom(f)$ *with* $\gamma(0) = a$

$$\begin{cases} \gamma_1 : [0, \delta_1) \to Dom(f), \gamma_1(0) = a \\ \gamma_2 : [0, \delta_2) \to Dom(f), \gamma_2(0) = a \end{cases}$$

If $\lim_{t\to0^+} f(\gamma_1(t)) \neq \lim_{t\to0^+} f(\gamma_2(t))$, *then the limit does not exist.*

Example 5. *Consider* $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

$$\begin{cases} \gamma_1 : [0, +\infty) \to \mathbb{R}^2 \\ t \mapsto (t, 0) \end{cases}$$

$$\begin{cases} \gamma_2 : [0, +\infty) \to \mathbb{R}^2 \\ t \mapsto (0, t) \end{cases}$$

$$f(\gamma_1(t)) = \frac{t^2 - 0^2}{t^2 + 0^2} = 1$$

$$f(\gamma_2(t)) = \frac{0^2 - t^2}{0^2 + t^2} = -1$$

Since $\lim_{t\to0^+} f(\gamma_1(t)) \neq \lim_{t\to0^+} f(\gamma_2(t))$, *the limit does not exist.*

Properties:

1) The Linearity:

$$\lim_{x\to a} f(x) = l_1 \in \mathbb{R}^m \text{ and } \lim_{x\to a} g(x) = l_2 \in \mathbb{R}^m$$

$$\text{then } \forall \alpha, \beta \in \mathbb{R}, \lim_{x\to a}[\alpha f(x) + \beta g(x)] = \alpha l_1 + \beta l_2$$

2) Limit of Product and quotient

$$f \times g : \mathbb{R}^n \to \mathbb{R}$$

$$\text{If } \lim_{x \to a} f(x) = l_1 \text{ and } \lim_{x \to a} g(x) = l_2$$
$$\text{Then } \lim_{x \to a} (fg)(x) = l_1 l_2$$
$$\text{And if } g(x) \neq 0, \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$$

3) Limit of Composed functions

$$\text{If } f : \mathbb{R}^n \to \mathbb{R}^m \text{ continuous at } a \text{ and } \lim_{y \to f(a)} g(y) = l$$

$$\text{then } \lim_{x \to a} (g \circ f)(x) = l$$

Definition 8. *1) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\phi \neq V \subset Dom(f)$.*

We say that f is continuous on V if:

$$\forall (x_n) \subset V, (x_n) \to x \in Dom(f) \implies \lim_{n \to +\infty} f(x_n) = f(x)$$

2) $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ continuous on $V \subset Dom(f) = \bigcap_{j=1}^{m} Dom(f_j)$

$$\Leftrightarrow \forall j \in \{1, 2, \dots, m\}, f_j : \mathbb{R}^n \to \mathbb{R} \text{ is continuous on } V$$

2.2 The Case of $f : \mathbb{R}^2 \to \mathbb{R}$

Euclidian distance

$$d_1((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|$$

$$d_2((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$
$$d_\infty((x_1, x_2), (y_1, y_2)) = \max(|y_1 - x_1|, |y_2 - x_2|)$$

limit:

$$\lim_{(x,y) \to (a,b)} f(x, y) = l \quad /l \in \mathbb{R}$$

$$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0; 0 < d((x, y), (a, b)) < \delta \implies |f(x, y) - l| < \epsilon$$

Example:

$$f(x,y) = \frac{x^3}{x^2+y^2}$$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

$$|f(x,y)| = \left|\frac{x^3}{x^2+y^2}\right| = |x|\frac{x^2}{x^2+y^2} \le |x| \le |x|+|y| = d_1((x,y),(0,0))$$

$$\Leftrightarrow \forall\epsilon>0, \exists\delta=\epsilon>0, \forall(x,y)\in\mathbb{R}^2-\{(0,0)\}, \quad 0<|x|+|y|<\delta \implies |f(x,y)|<\epsilon$$

Using polar coordinates:

$$\text{Let: } \begin{cases} x = a + r\cos\theta & ;\theta\in\mathbb{R} \\ y = b + r\sin\theta & r>0 \end{cases}$$

$$d_2((x,y),(a,b)) = \sqrt{(x-a)^2+(y-b)^2} = \sqrt{(r\cos\theta)^2+(r\sin\theta)^2} = r$$

$$l \in \overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\}$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

$$\Leftrightarrow \forall\epsilon>0, \exists\delta>0, 0<r<\delta \implies |f(a+r\cos\theta, b+r\sin\theta)-l|<\epsilon$$

$$\Leftrightarrow \forall\theta\in\mathbb{R}; \lim_{r\to0^+} f(a+r\cos\theta, b+r\sin\theta) = l$$

Example:

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad (a,b)=(0,0)$$

$$f(a+r\cos\theta, b+r\sin\theta) = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos^2\theta - \sin^2\theta$$

$\theta = 0, l = 1$

$\theta = \frac{\pi}{2}, l = -1$

$\implies$ limit does not exist.

Example 2:

$$f(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

$$f(r\cos\theta, r\sin\theta) = \frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = r(\cos^3\theta+\sin^3\theta)$$

$$0 \le |f(r\cos\theta, r\sin\theta)| = r|\cos^3\theta+\sin^3\theta| \le 2r \to 0$$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$

2.3 The Case of $f : \mathbb{R}^3 \to \mathbb{R}$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = l$$

Using cylindrical coordinates:

$$\begin{cases} x = x_0 + r\cos\theta & ; r > 0 \\ y = y_0 + r\sin\theta & \theta \in \mathbb{R} \\ z = z \end{cases}$$

$$d_2((x,y,z),(x_0,y_0,z_0)) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = \sqrt{r^2 + (z-z_0)^2}$$

We note $g_\theta = f(x_0 + r\cos\theta, y_0 + r\sin\theta, z)$

$$\forall \epsilon > 0; \exists \delta > 0; d_2((r,z),(0,z_0)) < \delta \implies |g_\theta(r,z) - l| < \epsilon$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = l$$

$$\Leftrightarrow \forall \theta \in \mathbb{R}, \lim_{(r,z)\to(0,z_0)} g_\theta(r,z) = l$$

Example:

$$f(x,y,z) = \frac{x^2+y^2-z^2}{x^2+y^2+z^2}$$

$$g_\theta(r,z) = \frac{r^2-z^2}{r^2+z^2}$$

Since $\lim g_\theta(r,z)$ does not exist, then $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2}$ does not exist.

Using spherical coordinates:

$$\begin{cases} x = x_0 + r\cos\theta\cos\varphi \\ y = y_0 + r\cos\theta\sin\varphi \\ z = z_0 + r\sin\theta \end{cases}$$

$$g_{\theta,\varphi}(r) = f(x,y,z)$$

$$d_2((x,y,z),(x_0,y_0,z_0)) = r$$

So;

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = l$$

$$\Leftrightarrow \forall \theta, \varphi, \quad \lim_{r\to 0^+} g_{\theta,\varphi}(r) = l$$

Example:

$$f(x,y,z)=\frac{x^2+y^2-z^2}{x^2+y^2+z^2}$$

Since $\lim\limits_{r\to 0^+} g_{\theta,\varphi}(r)$ depends on θ and φ, the limit does not exist.

3 differentiation

3.1 First partial derivatives

The process of "partial differentiation" consists of deriving a function of several variables with respect to one of its independent variables.

The result is referred to as the "partial derivative of f with respect to the chosen independent variable.

Definition 9. *Let* $f:\mathbb{R}^2\to\mathbb{R}$; $\quad (a,b)\in Dom(f)$

1) If the limit $\lim\limits_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}\in\mathbb{R}$ *exists, we call it the first partial derivative of* f *with respect to the first independent variable* x*, denoted by:* $\dfrac{\partial f}{\partial x}(a,b)$

2) If the limit $\lim\limits_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}\in\mathbb{R}$ *exists, we call it the first partial derivative of* f *at* (a,b) *with respect to the second independent variable, denoted by:* $\dfrac{\partial f}{\partial y}(a,b)$

3) We define two functions, $f_x=\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *by:*

$$f_x=\frac{\partial f}{\partial x}:\mathbb{R}^2\to\mathbb{R}$$

$$(x,y)\mapsto\frac{\partial f}{\partial x}(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y=\frac{\partial f}{\partial y}:\mathbb{R}^2\to\mathbb{R}$$

$$(x,y)\mapsto\frac{\partial f}{\partial y}(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

example:

1) $f(x,y)=\phi(x)\psi(y)$

$$\frac{\partial f}{\partial x}(x,y)=\phi'(x)\psi(y)$$

$$\frac{\partial f}{\partial y}(x,y) = \phi(x)\psi'(y)$$

Geometric Interpretation

The tangent line of the curve $z = f(x,b)$ at $(a, f(a,b))$ is directed by $(1, 0, \frac{\partial f}{\partial x}(a,b))$

$$\begin{cases} z = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) \\ y = b \end{cases}$$

The tangent line of the curve $z = f(a,y)$ at $(a, b, f(a,b))$ is directed by $(0, 1, \frac{\partial f}{\partial y}(a,b))$

$$\begin{cases} z = f(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ x = a \end{cases}$$

The tangent plane of the graph of f at $(a, b, f(a,b))$, denoted by $T_{(a,b,f(a,b))}G_f$ is defined by

$$T_{(a,b,f(a,b))}G_f = \{(x,y,z) \in \mathbb{R}^3 : z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)\}$$

Definition 10. *1) Let*

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$$
$$\forall i \in \{1, 2, \ldots, n\}, \quad f_{x_i} = \frac{\partial f}{\partial x_i} : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \ldots, x_n) \mapsto \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

$\frac{\partial f}{\partial x_i}$ is the i-th first derivative of f or the first partial derivative of f with respect to the i-th independent variable.

When $m > 1$: $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$

$$\forall_{i \in \{1,\ldots,n\}};\ \frac{\partial f}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_m}{\partial x_i}\right) : \mathbb{R}^n \to \mathbb{R}^m$$

The tangent plane of $G_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x)\}$ at $(a, f(a))$, denoted by $T_{(a,f(a))}G_f$ is:

$$T_{(a,f(a))}G_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(a) + \sum_{i=1}^{n}(x_i - a_i)\frac{\partial f}{\partial x_i}(a)\} \subset \mathbb{R}^{n+m}$$

Example:1

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad (x, y, z) \mapsto xy + yz^2 + xz$$

$$\frac{\partial f}{\partial x} = y + z, \quad \frac{\partial f}{\partial y} = x + z^2, \quad \frac{\partial f}{\partial z} = 2yz + x$$

3.2 Second Order Partial Derivatives

Definition 11. *1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ its first partial derivatives:*

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x, y) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x + h, y) - \frac{\partial f}{\partial x}(x, y)}{h}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x, y) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x, y + h) - \frac{\partial f}{\partial x}(x, y)}{h}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x, y) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x + h, y) - \frac{\partial f}{\partial y}(x, y)}{h}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x, y) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(x, y + h) - \frac{\partial f}{\partial y}(x, y)}{h}$$

2) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\frac{\partial f}{\partial x_i}; i \in \{1, \ldots, n\}$ its first partial derivatives.

$\forall i, j \in \{1, \ldots, n\}$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x_j}(x_1, \ldots, x_i + h, \ldots, x_n) - \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n)}{h}$$

When $i = j$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_i^2}$ is called the second partial derivative of f with respect to x_i.

When $i \neq j$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is called the mixed second partial derivative of f with respect to x_i and x_j.

There are n^2 second partial derivatives of f.

3) If $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$

$$\forall_{i,j} \in \{1, \ldots, n\}; \frac{\partial^2 f}{\partial x_i \partial x_j} = \left(\frac{\partial^2 f_1}{\partial x_i \partial x_j}, \ldots, \frac{\partial^2 f_m}{\partial x_i \partial x_j}\right)$$

Theorem 1 (Schwarz's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$.*

If $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous on an open subset $D \subset \mathbb{R}^n$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{on } D$$

Definition 12. *Higher order partial derivatives*

Let $p \in \mathbb{N}^*$; $f : \mathbb{R}^n \to \mathbb{R}^m$. The p-th partial derivatives of f are :

$$\frac{\partial^p f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_{p-1}} x_{i_p}} := \frac{\partial}{\partial x_{i_1}}\left(\frac{\partial}{\partial x_{i_2}}\left(\dots\left(\frac{\partial f}{\partial x_{i_{p-1}}}\left(\frac{\partial f}{\partial x_{i_p}}\right)\right)\right)\right), \quad \text{where } i_1, i_2, \dots, i_p \in \{1, 2, \dots, n\}$$

Definition 13. *1) We say that a function is C^1 (of class C^1) if all its first partial derivatives are continuous.*

2) We say that a function is of class C^p if all its partial derivatives of degree p are continuous.

3) We say that a function is C^∞ if all partial derivatives of degree p exist; $\forall p \in \mathbb{N}$.

Corollary

If f is of class C^p, all the p-th partial derivatives can be written as

$$\frac{\partial^p f}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}} \quad \forall p_1, \dots, p_n \in \mathbb{N} \text{ such that } p_1 + p_2 + \dots + p_n = p$$

Example:

If $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^3, the third partials derivatives are of the form: $\dfrac{\partial^3 f}{\partial x^p \partial y^q}$ where $p + q = 3$

$$\frac{\partial^3 f}{\partial x^3}, \quad \frac{\partial^3 f}{\partial y^3}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2}$$
$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial y^2}$$

3.3 Differentials of a Function

Recall that if

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$$

is derivable function and and f' is its derivative function, then the differential of f, denoted df, is the function

$$df : \mathbb{R} \to \mathcal{L}(\mathbb{R}, \mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \text{ linear}\}$$

$$x \mapsto d_x f : \mathbb{R} \to \mathbb{R}, \quad h \mapsto f'(x) \cdot h$$

Definition 14. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function such that all first partial derivatives exist.*

Then the differential of f is the function

$$df : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$x \mapsto d_x f : \mathbb{R}^n \to \mathbb{R}^m, \quad h = (h_1, \ldots, h_n) \mapsto \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x)$$

Remark: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then:

$$\forall x \in \mathbb{R}^n : \quad d_x f = f$$

In Particular: $f = \mathrm{pr}_i : (x_1, \ldots, x_n) \mapsto x_i$

$$\forall x, d_x f = f = \mathrm{pr}_i = dx_i : \mathbb{R}^n \to \mathbb{R}, \quad (h_1, \ldots, h_n) \mapsto h_i$$

If $m = 1$:

$$d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) dx_i$$

Example:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto f(x, y)$$

$$d_{(x,y)} f = \frac{\partial f}{\partial x}(x, y) dx + \frac{\partial f}{\partial y}(x, y) dy$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$f(x, y) = xy; \quad d_{(x,y)} f = y dx + x dy$

Application: $w = f(x_1, \ldots, x_n)$; $\Delta w = f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) - f(x_1, \ldots, x_n)$

Then:

$$\Delta w \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\Delta x_i$$

Definition 15. ***Differentiability***

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, and $a \in \mathbb{R}^n$

We say that f is differentiable at a if there exists a linear map

$$\mathcal{L}_a : \mathbb{R}^n \to \mathbb{R}^m, \quad h \mapsto \mathcal{L}_a(h)$$

such that:

$$\lim_{h \to 0_{\mathbb{R}^n}} \frac{1}{\|h\|}(f(a+h) - f(a) - \mathcal{L}_a(h)) = 0$$

where $\|h\| = \sqrt{h_1^2 + \cdots + h_n^2}$.

Remark: If f is differentiable at $(a_1, \ldots, a_n)$, then:

$$\forall i \in \{1, \ldots, n\}, \text{ if we consider } h = (0, \ldots, 0, t, 0, \ldots, 0) = te_i$$

$$\lim_{t \to 0} \frac{1}{|t|}(f(a + te_i) - f(a) - \mathcal{L}_a(te_i)) = 0$$

$$\Rightarrow \lim_{t \to 0} \frac{1}{t}(f(a + te_i) - f(a)) = \mathcal{L}_a(e_i)$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(a) = \mathcal{L}_a(e_i) \quad \forall i \in \{1, \ldots, n\}$$

$$\mathcal{L}_a(h) = \mathcal{L}_a\left(\sum_{i=1}^{n} h_i e_i\right) = \sum_{i=1}^{n} h_i \mathcal{L}_a(e_i) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) = d_a f(h)$$

So, $\mathcal{L}_a = d_a f$.

Thus, f is differentiable at a if and only if all its first partial derivatives exist and

$$f(a+h) = f(a) + d_a f(h) + o(\|h\|)$$

where $\lim_{h \to 0_{\mathbb{R}^n}} \frac{1}{\|h\|} o(\|h\|) = 0$.

Proposition:

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then f is continuous at a.

Proof: By the relation:

$$f(a+h) - f(a) = d_a f(h) + \|h\|\epsilon(h)$$

where $\lim_{h \to 0_{\mathbb{R}^n}} \epsilon(h) = 0$. Since $d_a f$ is a linear map, it is continuous, then:

$$\lim_{h \to 0_{\mathbb{R}^n}} [f(a+h) - f(a)] = 0 \Leftrightarrow \lim_{x \to a} f(x) = f(a)$$

Example:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f is continuous at $(0,0)$, because:

$$|f(x,y)| = |x| \left(\frac{y^2}{x^2+y^2} \right) \leq |x| \xrightarrow[(x,y)\to(0,0)]{} 0$$

So $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Now check differentiability:

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 \\
\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0 \\
&\implies d_{(0,0)}f(h_1,h_2) = 0, \quad \forall (h_1,h_2) \in \mathbb{R}^2 \\
\lim_{(h_1,h_2)\to(0,0)} & \frac{f(0+h_1,0+h_2) - f(0,0) - d_{(0,0)}f(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} \\
&= \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1h_2^2}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} \\
&= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1h_2^2}{(h_1^2+h_2^2)^{3/2}}
\end{aligned}$$

This limit does not exist because along $\gamma : t \mapsto (t,t)$:

$$\frac{t \cdot t^2}{(t^2+t^2)^{3/2}} = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2\sqrt{2}|t|^3} \to \frac{1}{2\sqrt{2}} \neq 0$$

So f is not differentiable at $(0,0)$.

Remark:

If all the first partial derivatives are continuous, then the function is differentiable.

Proposition

Linear combination, product, quotient, composition of functions of class C^p ($p \in \mathbb{N} \cup \{\infty\}$) are of Class C^p.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ differentiable at a and $g : \mathbb{R}^m \to \mathbb{R}^l$ differentiable at $f(a)$, then :*

$$g \circ f : \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^l$$
$$x \mapsto f(x) \mapsto g(f(x))$$

is differentiable at a and :

$$d_a(g \circ f) = d_{f(a)} g \circ d_a f$$
$$d_a(g \circ f)(h) = d_{f(a)} g(d_a f(h)) \quad \forall h \in \mathbb{R}^n$$

Definition 16. *Jacobian matrix - Hessian matrix*

1) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto (f_1(x), \dots, f_m(x))$ differentiable at a, the Jacobian matrix of f at a, denoted by $Jac_a(f)$ is the matrix associated with the linear map $d_a f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$:

$$Jac_a(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \dots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

$$d_a f(h) = Jac_a(f) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \left(\sum_{i=1}^n h_i \frac{\partial f_1}{\partial x_i}(a), \sum_{i=1}^n h_i \frac{\partial f_2}{\partial x_i}(a), \dots, \sum_{i=1}^n h_i \frac{\partial f_m}{\partial x_i}(a) \right)$$

2) Let $f : \mathbb{R}^n \to \mathbb{R}$ such that all second partial derivatives exist at a point a. The square matrix

$$Hess_a(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(a) & \frac{\partial^2 f}{\partial x_n \partial x_2}(a) & \dots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}$$

is called the Hessian matrix of f at a.

Remark: $n = m = 1 \quad f : \mathbb{R} \to \mathbb{R}$

$$Jac_a(f) = (f'(a)) \quad Hess_a(f) = (f''(a) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(a))$$

Corollary Chain rule:

$$Jac_a(g \circ f) = Jac_{f(a)}(g) \times Jac_a(f)$$

<u>**Example**</u>

$$z : \mathbb{R} \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$t \mapsto (x(t), y(t)) \xrightarrow{f} f(x(t), y(t)) = z(t)$$

$$Jac_{(x,y)} f = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right); \quad Jac_t(\gamma) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$\begin{aligned} Jac_t(z) = (z'(t)) &= \left(\frac{\partial f}{\partial x}(\gamma(t)), \frac{\partial f}{\partial y}(\gamma(t)) \right) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\ &= x'(t) \frac{\partial f}{\partial x}(\gamma(t)) + y'(t) \frac{\partial f}{\partial y}(\gamma(t)) \\ \text{So: } \frac{\mathrm{d}z}{\mathrm{d}t}(t) &= \frac{\mathrm{d}x}{\mathrm{d}t}(t) \frac{\partial f}{\partial x}(x(t), y(t)) + \frac{\mathrm{d}y}{\mathrm{d}t}(t) \frac{\partial f}{\partial y}(x(t), y(t)) \end{aligned}$$

In the case of 3 variables:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t} \frac{\partial f}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t} \frac{\partial f}{\partial z}$$

$$\mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$(t, s) \mapsto (x(t, s), y(t, s)) \mapsto f \circ \varphi(t, s) = w(t, s)$$

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial x}{\partial t} \dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial t} \dfrac{\partial f}{\partial y} \\ \dfrac{\partial w}{\partial s} = \dfrac{\partial x}{\partial s} \dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial s} \dfrac{\partial f}{\partial y} \end{cases}$$

$$\mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

$$(t, s) \mapsto (x(t, s), y(t, s), z(t, s)) \mapsto f \circ \varphi(t, s) = w(t, s)$$

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial x}{\partial t} \dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial t} \dfrac{\partial f}{\partial y} + \dfrac{\partial z}{\partial t} \dfrac{\partial f}{\partial z} \\ \dfrac{\partial w}{\partial s} = \dfrac{\partial x}{\partial s} \dfrac{\partial f}{\partial x} + \dfrac{\partial y}{\partial s} \dfrac{\partial f}{\partial y} + \dfrac{\partial z}{\partial s} \dfrac{\partial f}{\partial z} \end{cases}$$

<u>**Example:**</u>

$$\varphi(t) = (t^2 - 2t, t^3)$$

$$f(x, y) = x^2 + 2yx$$

$$\begin{cases} x(t) = t^2 - 2t \\ y(t) = t^3 \end{cases} \qquad \begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x + 2y = 2(x+y) \\ \dfrac{\partial f}{\partial y}(x,y) = 2x \end{cases}$$

$$\begin{aligned} z(t) = (f \circ \varphi)(t) &= (t^2 - 2t)^2 + 2t^3(t^2 - 2t) \\ &= t^4 - 4t^3 + 4t^2 + 2t^5 - 4t^4 \\ &= 2t^5 - 3t^4 - 4t^3 + 4t^2 \\ z'(t) &= 10t^4 - 12t^3 - 12t^2 + 8t \end{aligned}$$

Using chain rule:

$$\begin{aligned} z'(t) &= x'(t)\,[2(x(t) + y(t))] + y'(t) \cdot 2x(t) \\ &= (2t-2) \cdot 2(t^2 - 2t + t^3) + 3t^2 \cdot 2(t^2 - 2t) \\ &= (2t-2)(2t^3 + 2t^2 - 4t) + 6t^2(t^2 - 2t) \\ &= 4t^4 + 4t^3 - 8t^2 - 4t^3 - 4t^2 + 8t + 6t^4 - 12t^3 \\ &= 10t^4 - 12t^3 - 12t^2 + 8t \end{aligned}$$

4 Extrema of function from $\mathbb{R}^2$ to $\mathbb{R}$

4.1 Taylor expansion of $f : \mathbb{R}^2 \to \mathbb{R}$

Definition 17. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^{p+1} at (a,b). The Taylor polynomial function of f at (a,b) of degree p is denoted by $T^p_{(a,b)}f$ and is defined as the Taylor polynomial function of degree p at 0 of the function*

$$\varphi : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto f(a + t(x-a), b + t(y-b))$$

evaluated at $t = 1$:

$$\begin{aligned} T^p_{(a,b)}f(x,y) := T^p_0\varphi(1) &= \sum_{i=0}^{p} \frac{1}{i!}\varphi^{(i)}(0) \\ &= \varphi(0) + \frac{\varphi'(0)}{1!} + \frac{\varphi''(0)}{2!} + \cdots + \frac{\varphi^{(p)}(0)}{p!} \end{aligned}$$

$$\begin{aligned} T^p_{(a,b)}f(x,y) &= \sum_{i=0}^{p} \frac{1}{i!}\left[\sum_{k=0}^{i} \binom{i}{k} \frac{\partial^i f}{\partial x^k \partial y^{i-k}}(a,b)(x-a)^k(y-b)^{i-k}\right] \\ &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\,\partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right] + \ldots \end{aligned}$$

$$\begin{cases} x(t) = a + t(x-a); \quad y(t) = b + t(y-b) \\ \varphi(t) = f(x(t), y(t)) \end{cases}$$

$$\varphi(0) = f(a + 0(x-a), b + 0(y-b)) = f(a,b)$$

$$\varphi'(t) = x'(t)\frac{\partial f}{\partial x}(x(t),y(t)) + y'(t)\frac{\partial f}{\partial y}(x(t),y(t))$$

$$\varphi'(t) = (x-a)\frac{\partial f}{\partial x}(x(t),y(t)) + (y-b)\frac{\partial f}{\partial y}(x(t),y(t))$$

$$\varphi''(t) = (x-a)\left[x'(t)\frac{\partial^2 f}{\partial x\,\partial x}(x(t),y(t)) + y'(t)\frac{\partial^2 f}{\partial x\,\partial y}(x(t),y(t))\right]$$

$$+ (y-b)\left[x'(t)\frac{\partial^2 f}{\partial y\,\partial x}(x(t),y(t)) + y'(t)\frac{\partial^2 f}{\partial y\,\partial y}(x(t),y(t))\right]$$

$$\vdots$$

Definition 18. *Taylor expansion of $f : \mathbb{R}^2 \to \mathbb{R}$*

$$f(x,y) = T^p_{(a,b)} f(x,y) + \left[(x-a)^2 + (y-b)^2\right]^{p/2} \epsilon_{(a,b)}(x,y)$$

where $\lim\limits_{(x,y)\to(a,b)} \epsilon_{(a,b)}(x,y) = 0$

Example:

$$f(x,y) = e^{x+y}$$

$$\varphi(t) = f(0 + t(x-0), 0 + t(y-0)) = f(tx, ty) = e^{t(x+y)} = e^{t(x+y)}$$

Taylor expansion at $(0,0)$:

$$f(x,y) = 1 + (x+y) + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 + \dots$$

4.2 Absolute and relative extrema

Definition 19. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in Dom(f)$; $D \subset Dom(f)$*

1) We say that $f(a,b)$ is a relative maximum of f if there exists an open subset V with $(a,b) \in V \subset D$ and $f(a,b) \geq f(x,y)$ for all $(x,y) \in V$.

2) We say that $f(a,b)$ is a relative minimum of f if there exists an open subset V with $(a,b) \in V \subset D$ and $f(a,b) \leq f(x,y)$ for all $(x,y) \in V$.

3) In the two cases above, $f(a,b)$ is a relative extremum of f.

4) If $V = D$; $f(a,b)$ is an absolute extremum of f on D.

Definition 20. *Critical points*

$(a,b) \in Dom(f)$ is a critical point of f if one of the conditions is verified:

$$1)\frac{\partial f}{\partial x}(a,b)=0 \quad and \quad \frac{\partial f}{\partial y}(a,b)=0 \quad (or\ d_{(a,b)}f=0)$$

$$2)\frac{\partial f}{\partial x}(a,b)\ or\ \frac{\partial f}{\partial y}(a,b)\ doesn't\ exist$$

Proposition: First order test

If $f(a,b)$ is a relative extremum of f and f is differentiable at (a,b), then (a,b) is a critical point of f, that is:

$$\frac{\partial f}{\partial x}(a,b)=0 \text{ and } \frac{\partial f}{\partial y}(a,b)=0$$

Proof:

Suppose that $f(a,b)$ is a relative minimum of f;

$$\begin{aligned}
&\exists V \text{ such that } f(a,b)\le f(x,y) \quad \forall (x,y)\in V\\
&\Leftrightarrow f(x,y)-f(a,b)\ge 0\\
&\exists \delta>0; (a+h,b)\in V \text{ for all } |h|<\delta; -\delta<h<\delta\\
&f(a+h,b)-f(a,b)\ge 0\\
&\begin{cases} \frac{f(a+h,b)-f(a,b)}{h}\le 0, & -\delta<h<0\\ \frac{f(a+h,b)-f(a,b)}{h}\ge 0, & 0<h<\delta\end{cases}
\end{aligned}$$

Taking limits:

$$\begin{cases} \lim_{h\to 0^-}\frac{f(a+h,b)-f(a,b)}{h} = \frac{\partial f}{\partial x}(a,b)\le 0\\ \lim_{h\to 0^+}\frac{f(a+h,b)-f(a,b)}{h} = \frac{\partial f}{\partial x}(a,b)\ge 0\end{cases} \Rightarrow \frac{\partial f}{\partial x}(a,b)=0$$

With the same method, we prove that $\frac{\partial f}{\partial y}(a,b)=0$.

Remark: If (a,b) is a critical point of f; $f(a,b)$ is not necessarily a relative extremum.

For example:

$$f(x,y)=x^2-y^2$$

$\frac{\partial f}{\partial x}(0,0)=0$ and $\frac{\partial f}{\partial y}(0,0)=0$

$f(0,0)=0$ is not a relative extremum because:

$$\begin{cases} f(x,0)=x^2>0, \quad \forall x\ne 0 \Rightarrow f(0,0) \text{ is not maximum}\\ f(0,y)=-y^2<0, \quad \forall y\ne 0 \Rightarrow f(0,0) \text{ is not minimum}\end{cases}$$

Theorem 3. *Second order test*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in Dom(f)$ such that f is of class C^2 at (a,b) and (a,b) is a critical point of f.

Let $\Delta_0 = \dfrac{\partial^2 f}{\partial x^2}(a,b)\dfrac{\partial^2 f}{\partial y^2}(a,b) - \left[\dfrac{\partial^2 f}{\partial x\,\partial y}(a,b)\right]^2$

We have:

1) If $\Delta_0 > 0$ *and* $\dfrac{\partial^2 f}{\partial x^2}(a,b) > 0$, *then* $f(a,b)$ *is a relative minimum*

2) If $\Delta_0 > 0$ *and* $\dfrac{\partial^2 f}{\partial x^2}(a,b) < 0$, *then* $f(a,b)$ *is a relative maximum*

3) If $\Delta_0 < 0$ $(a,b,f(a,b))$ *is a "saddle" point of graph of* f

($f(a,b)$ is not a relative extremum of f)

4) If $\Delta_0 = 0$ *the test is not conclusive*

Proof: *Let* $A = \dfrac{\partial^2 f}{\partial x^2}(a,b)$; $B = \dfrac{\partial^2 f}{\partial x\,\partial y}(a,b)$; $C = \dfrac{\partial^2 f}{\partial y^2}(a,b)$

Using the Taylor expansion of degree 2 at (a,b):

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2}\left[A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2\right] + o((x-a)^2 + (y-b)^2) \end{aligned}$$

Since (a,b) is a critical point:

$$f(x,y) - f(a,b) = \frac{1}{2}\left[A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2\right] + o((x-a)^2 + (y-b)^2)$$

Completing the square:

$$\begin{aligned} &A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2 \\ &= A\left[(x-a)^2 + 2\frac{B}{A}(x-a)(y-b) + \frac{C}{A}(y-b)^2\right] \\ &= A\left[\left(x-a+\frac{B}{A}(y-b)\right)^2 + \left(\frac{C}{A} - \frac{B^2}{A^2}\right)(y-b)^2\right] \\ &= A\left[\left(x-a+\frac{B}{A}(y-b)\right)^2 + \frac{\Delta_0}{A^2}(y-b)^2\right] \end{aligned}$$

The sign depends on A and Δ_0.

Example:

Exercise 09: 1)

$$\begin{aligned} f(x,y) &= 8-(x-1)^2-(y+1)^2 \\ &= 8-\left[(x-1)^2+(y+1)^2\right] \end{aligned}$$

$f(1,-1)=8$ is absolute maximum of f ($8=\max_{\mathbb{R}^2} f$).

f has no minimum, because:

$$\exists (x_n,y_n)=(1,-1+n)\in\mathbb{R}^2 \text{ such that } \lim_{n\to+\infty} f(x_n,y_n)=-\infty$$

Using critical points:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y)=-2(x-1)=0 \\ \dfrac{\partial f}{\partial y}(x,y)=-2(y+1)=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=-1 \end{cases}$$

$A=-2$; $B=0$; $C=-2$; $\Delta_0=4>0$ and $A=-2<0$

So $f(1,-1)$ is a relative maximum.

Since f is C^2 on $\mathbb{R}^2$ and $(1,-1)$ is the unique critical point, then $f(1,-1)$ is absolute maximum and there is no minimum.

2) $f(x,y)=\sqrt{4x^2+y^2+4}\geq\sqrt{4}=2=f(0,0), \quad \forall (x,y)\in\mathbb{R}^2$

Then 2 is absolute minimum of f.

And $\lim_{n\to+\infty} f(0,n)=+\infty \implies f$ is not upper bounded.

Using critical point tests:

$$\frac{\partial f}{\partial x}(x,y)=\frac{4x}{\sqrt{4x^2+y^2+4}}\ ;\ \frac{\partial f}{\partial y}(x,y)=\frac{y}{\sqrt{4x^2+y^2+4}}$$

The unique critical point is $(0,0)$.

Compute second derivatives at $(0,0)$:

$$\begin{aligned} A &= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(h,0)-\frac{\partial f}{\partial x}(0,0)}{h}=\lim_{h\to 0}\frac{\frac{4h}{\sqrt{4h^2+4}}-0}{h}=2 \\ B &= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h)-\frac{\partial f}{\partial x}(0,0)}{h}=0 \\ C &= \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(0,h)-\frac{\partial f}{\partial y}(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h}{\sqrt{4+h^2}}-0}{h}=\frac{1}{2} \end{aligned}$$

$\Delta_0=2\cdot\frac{1}{2}-0^2=1>0$ and $A=2>0$

So $f(0,0)$ is a relative minimum. Since $(0,0)$ is the unique critical point and f is of class C^2 on $\mathbb{R}^2$, then $f(0,0)$ is absolute minimum and there is no maximum.

3)

$$f(x,y) = 10x + 12y - x^2 - y^2 - 64$$

$$\begin{aligned} f(x,y) &= -64 - \left[(x^2 - 10x) + (y^2 - 12y)\right] \\ &= -64 - \left[(x-5)^2 - 25 + (y-6)^2 - 36\right] \\ &= -64 + 25 + 36 - (x-5)^2 - (y-6)^2 \\ &= -3 - (x-5)^2 - (y-6)^2 \end{aligned}$$

So $f(5,6) = -3$ is the absolute maximum, and there is no minimum.