

Chapter 01: Double integrals

Notes from prof zeglaoui course

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1 Reminder:

Recall that if $f : [a, b] \rightarrow \mathbb{R}$ is bounded.

for any subdivision $\sigma = \{x_0 = a < x_1 < x_2 < \dots < x_{n+1} = b\}$ of $[a, b]$

$$S(f, \sigma) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \quad \text{and} \quad s(f, \sigma) = \sum_{i=1}^n m_i(x_i - x_{i-1})$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$

called respectively, the upper and the lower Darboux sums of f on $[a, b]$ with respect to the subdivision σ ;

S : the area of the region plane delimited by:

$$\begin{cases} x = a & y = 0 \\ x = b & y = f(x) \end{cases}$$

(graph showing the lower and upper of an integral)

$$s(f, \sigma) \leq S \leq S(f, \sigma) \quad \text{if } \sigma_1 \subset \sigma_2, \text{ then:}$$
$$s(f, \sigma_1) \leq s(f, \sigma_2) \leq S \leq S(f, \sigma_2) \leq S(f, \sigma_1)$$

$$S^+ = \inf(S(f, \sigma)) \quad , \quad S^- = \sup(s(f, \sigma))$$

$$S^- \leq S \leq S^+$$

Definition 1. We say that f is Riemann integrable on $[a, b]$ if :

$$S^+ = S^- = S = \int_a^b f(x)dx$$

Definition 2. Riemann sums

Let σ a subdivision of $[a, b]$ and $\zeta = \{\zeta_1, \dots, \zeta_n\}$ / $\zeta_i \in [x_{i-1}, x_i]$

$$s(f, \sigma) \leq R(f, \sigma, \zeta) = \sum_{i=1}^n f(\zeta_i)(x_i - x_{i-1}) \leq S(f, \sigma)$$

If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, then:

$\exists(\sigma_n)$ a sequence of subdivision of $[a, b]$

$\exists(\zeta_n)$ associated with $(\sigma_n)[a, b]$ $\zeta_n = (\zeta_n^i)$, $\zeta_n^i \in [x_{i-1}^n, x_i^n]$

$$\lim_{n \rightarrow \infty} R(f, \sigma_n, \zeta_n) = \int_a^b f(x)dx$$

In particular:

$$\sigma_n = \{x_i^n = a + \frac{i}{n}(b-a) : i \in 0, \dots, n\}$$

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f(\zeta_i) = \int_a^b f(x)dx$$

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f(a + \frac{i}{n}(b-a)) = \int_a^b f(x)dx , x_i^n - x_{i-1}^n = \frac{b-a}{n}$$

2 Double integrals:

2.1 Definition and first properties:

Definition 3. Darboux sums- Riemann sums:

Let $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a bounded function

1) A subdivision σ of Δ is a set $\sigma = \{\Delta_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$

with $(x_i)_{i=1, \dots, n}$ is a subdivision of $[a, b]$ and $(y_j)_{j=0, \dots, m}$ a subdivision of $[c, d]$ we note : $A(\Delta_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$

and $\delta(\sigma) = \max_{i,j} \text{diam}(\Delta_{ij})$

2)

$$\forall i, j \quad M_{ij} = \sup_{(x,y) \in \Delta_{ij}} f(x, y) \quad \text{and} \quad m_{ij} = \inf_{(x,y) \in \Delta_{ij}} f(x, y)$$

the upper Darboux sum of f with respect to σ is :

$$S(f, \sigma) = \sum_{i=1}^n \sum_{j=1}^m M_{ij} A(\Delta_{ij})$$

the lower Darboux sum of f with respect to σ is :

$$s(f, \sigma) = \sum_{i=1}^n \sum_{j=1}^m m_{ij} A(\Delta_{ij})$$

3) if we give $\zeta = \{(\zeta_i^1, \zeta_j^2) \mid \zeta_i^1 \in [x_{i-1}, x_i] \text{ and } \zeta_j^2 \in [y_{j-1}, y_j]\}$

The Riemann sum with respect to σ and ζ is :

$$R(f, \sigma, \zeta) = \sum_{i=1}^n \sum_{j=1}^m f(\zeta_i^1, \zeta_j^2) A(\Delta_{ij})$$

$$R(f, \sigma, \zeta) = \sum_{i=1}^n \sum_{j=1}^m f(\zeta_i^1, \zeta_j^2) (x_i - x_{i-1})(y_j - y_{j-1})$$

2.2 Example:

$$\begin{aligned} f(x, y) &= \alpha \quad ; \forall (x, y) \in \Delta \\ M_{ij} &= m_{ij} = f(\zeta_i^1, \zeta_j^2) = \alpha \\ S(f, \sigma) &= s(f, \sigma) = R(f, \sigma, \zeta) = \alpha(b-a)(d-c) = \alpha A(\Delta) \end{aligned}$$

Definition 4. Riemann integrability:

With the same notations above, we put:

$$S^+(f) = \inf(S(f, \sigma)) \quad \text{and} \quad S^-(f) = \sup(s(f, \sigma))$$

We say that f is Riemann integrable on Δ if $S^+(f) = S^-(f)$

$$\Leftrightarrow \forall \epsilon > 0; \exists \sigma \quad S(f, \sigma) - s(f, \sigma) < \epsilon$$

and we denote the common value by $\iint_{\Delta} f(x, y) dx dy$

$$\Leftrightarrow \exists (\sigma_n) \text{ of subdivisions of } \Delta / \lim S(f, \sigma_n) = \lim s(f, \sigma_n)$$

Corollary: if $f : \Delta = [a, b] \times [c, d]$ is Riemann integrable then:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{(b-a)(d-c)}{n^2} \sum_{i=1}^n \sum_{j=1}^n f(a + \frac{i}{n}(b-a), c + \frac{j}{n}(d-c)) \\ = \iint_{\Delta} f(x, y) dx dy \end{aligned}$$

2.3 Example:

$$f(x, y) = x^2 + y^2 \quad , \Delta = [0, 1]^2 = [0, 1] \times [0, 1]$$

Definition 5. Riemann integrability on a bounded domain:

Let D be a bounded subset of \mathbb{R}^2 and $f : D \rightarrow \mathbb{R}$ a bounded function.

We say that f is Riemann integrable on D if for any rectangle $\Delta \supset D$, the function :

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in \Delta \setminus D \end{cases}$$

is Riemann integrable, in this case :

$$\iint_D f(x, y) dx dy := \iint_{\Delta} \tilde{f}(x, y) dx dy$$

Remark: Any continuous function is Riemann integrable.

2.4 Properties:

1) Linearity: If $f, g : D \rightarrow \mathbb{R}$ are Riemann integrable then: $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Riemann integrable and

$$\iint_D (\alpha f + \beta g)(x, y) dx dy = \alpha \iint_D f(x, y) dx dy + \beta \iint_D g(x, y) dx dy$$

2)

If $f \geq 0$ on $D \implies \iint_D f(x, y) dx dy \geq 0$ which is the volume of $\omega = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}$

$$\begin{aligned} \text{So; if } f \geq g \implies \iint_D [f(x, y) - g(x, y)] dx dy \\ = \text{vol}\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } g(x, y) \leq z \leq f(x, y)\} \end{aligned}$$

3) Chasles's relation:

If $D = D_1 \cup D_2$ such that $D_1 \cap D_2$ is at most a curve then:

$$\iint_D f(x, y) dx dy = \iint_{D_1} f(x, y) dx dy + \iint_{D_2} f(x, y) dx dy$$

Definition 6. $\text{Vol}(D) = A(D) = \iint_D dx dy$ and the average of f on D is: $\frac{1}{\text{Vol}(D)} \iint_D f(x, y) dx dy = \frac{\iint_D f(x, y) dx dy}{\iint_D dx dy}$

Remark:

If $f \equiv C$, the average of f is C

2.5 Example:

D : the triangle with vertices $(0, 0), (1, 0), (0, 1)$ (graph) and $f(x, y) = xy$

$$\tilde{f}(x, y) = \begin{cases} xy & \text{if } x + y \leq 1 \\ 0 & \text{if } x + y > 1 \end{cases}$$

$$\forall x, y \in [0, 1] \times [0, 1]$$

$$\iint_D xy dxdy = \lim_{n \rightarrow +\infty} \frac{(1-0)(1-0)}{n^2} \sum_{i=1}^n \sum_{j=1}^n \tilde{f}\left(\frac{i}{n}, \frac{j}{n}\right)$$

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^n \left[\sum_{j=1}^n \frac{i}{n} \times \frac{j}{n} \right]$$

$$U_n = \frac{1}{n^2} \left[\sum_{i=1}^n \sum_{j=1}^n \frac{i}{n} \times \frac{j}{n} \right]$$

$$U_n = \frac{1}{n^4} \sum_{i=1}^n i \left[\sum_{j=1}^{n-i} j \right] = \frac{1}{n^4} \sum_{i=1}^n i \left(\frac{(n-i)(n-i+1)}{2} \right)$$

$$U_n = \frac{1}{2n^4} \sum_{i=1}^n i(n^2 + i^2 - 2in + n - i)$$

$$U_n = \frac{1}{2n^4} \sum_{i=1}^n [i(n^2 + n) - i^2(2n + 1) + i^3]$$

$$U_n = \frac{1}{2n^4} \left[(n^2 + n) \frac{(n+1)n}{2} - (2n+1) \frac{n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4} \right]$$

$$U_n = \frac{1}{4}(1 + \frac{1}{n})^2 - \frac{1}{12}(1 + \frac{1}{n})(2 + \frac{1}{n}) + \frac{1}{8}(1 + \frac{1}{n})^2$$

$$\iint_D xy dxdy = \lim_{n \rightarrow +\infty} U_n = \frac{1}{4} - \frac{1}{12} \cdot 2 + \frac{1}{8} = \frac{1}{24}$$

Theorem 1. *Fubini's theorem:*

Let $f : D \rightarrow \mathbb{R}$ Riemann integrable

1) if

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \exists g_1, g_2 : [a, b] \rightarrow \mathbb{R} \text{ continuous} \mid g_1(x) \leq y \leq g_2(x), \forall x \in [a, b]\}$$

then: $\iint_D f(x, y) dx dy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$

2) If

$D = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \exists h_1, h_2 : [c, d] \rightarrow \mathbb{R} \text{ continuous} \mid h_1(y) \leq x \leq h_2(y), \forall y \in [c, d]\}$

then: $\iint_D f(x, y) dx dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$

2.6 Remark:

We can use the Chasles's formula to apply this theorem.

2.7 Example 01:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$$

$$D_1 = \{(x, y) \in D : y \geq 0\}; D_2 = \{(x, y) \in D : y \leq 0\}$$

$$D_1 = \{(x, y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } 0 \leq y \leq \sqrt{R^2 - x^2}\}$$

$$D_2 = \{(x, y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } -\sqrt{R^2 - x^2} \leq y \leq 0\}$$

...

Example: $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$

$D_1 = \{(x, y) \in D : y \geq 0\}; D_2 = \{(x, y) \in D : y \leq 0\}$

$D_1 = \{(x, y) \in D : -R \leq x \leq R \text{ and } 0 \leq y \leq \sqrt{R^2 - x^2}\}$

$D_2 = \{(x, y) \in D : -R \leq x \leq R \text{ and } -\sqrt{R^2 - x^2} \leq y \leq 0\}$

$\iint_D f(x, y) dx dy = \iint_{D_1} f(x, y) dx dy + \iint_{D_2} f(x, y) dx dy = \int_{-R}^R \left[\int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} f(x, y) dy \right] dx$

$= \int_{-R}^R \sqrt{R^2 - x^2} dx + \int_{-R}^R \sqrt{R^2 - x^2} dx = 2 \int_0^R \sqrt{R^2 - x^2} dx = 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2 - R^2 \sin^2 \theta} R \cos \theta d\theta$

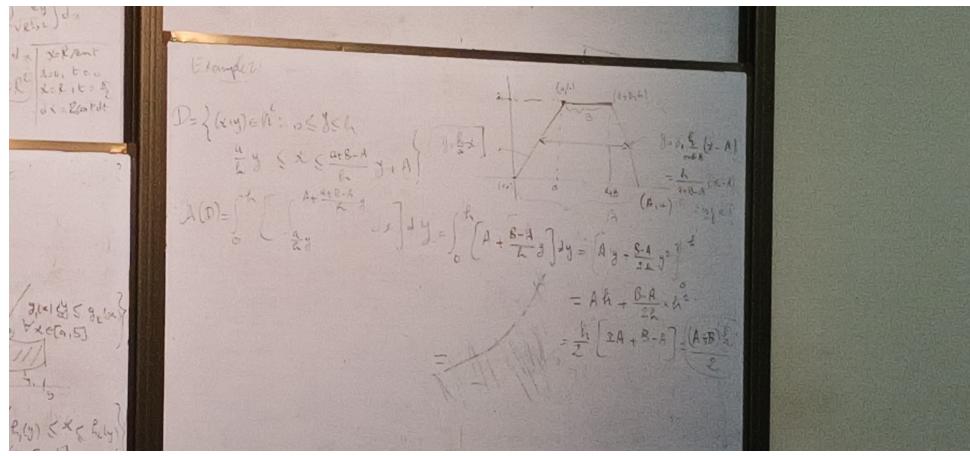
$= 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2(1 - \sin^2 \theta)} R \cos \theta d\theta = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = 2R^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2R^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = 2R^2 \left[\frac{\pi}{2} + 0 \right] = \pi R^2$

$D = \{(x, y)\}$

1.1.7 Computation of double integrals

2.8 Example 02:

(trapezoid area proof)



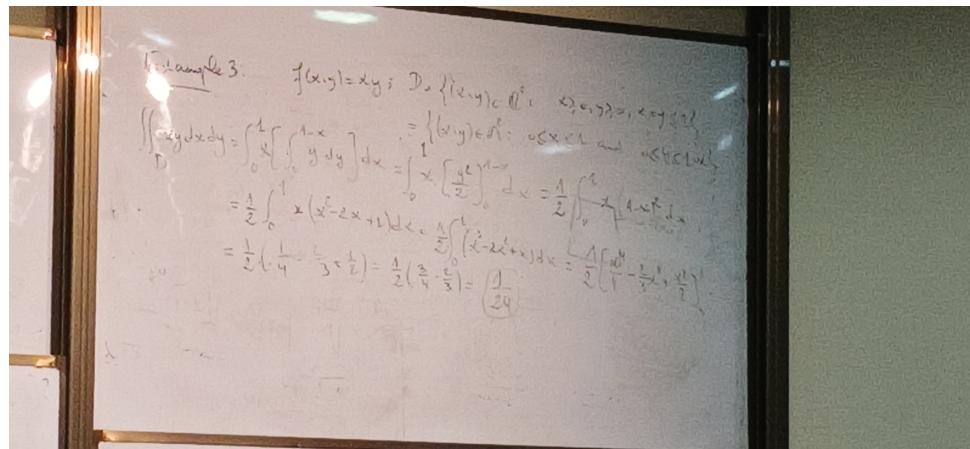
2.9 Example 03:

$$f(x,y) = xy$$

$$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$$

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1-x\}$$

...



2.10 Example 04:

Let us compute volume of Ω bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$

Defining Ω : $0 \leq y \leq 1$, $-\sqrt{y^2 - x^2} \leq x \leq \sqrt{y^2 - x^2}$

$$\text{vol}(\Omega) = \int_0^1 \left[\int_{-\sqrt{y^2 - x^2}}^{\sqrt{y^2 - x^2}} (y - y^2 - x^2) dx \right] dy = \int_0^1 \left[(y - y^2)x - \frac{1}{3}x^3 \right]_{-\sqrt{y^2 - x^2}}^{\sqrt{y^2 - x^2}} dy$$

$$= \int_0^1 \left[(y - y^2)\sqrt{y^2 - x^2} - \frac{1}{3}(\sqrt{y^2 - x^2})^3 \right] dy = \frac{4}{3} \int_0^1 (\sqrt{y^2 - x^2})^3 dy$$

$$\sqrt{y^2 - x^2} = \sqrt{y^2 - (\frac{y^2 - 1}{y})^2} = \sqrt{\frac{y^2(y^2 - 1)^2}{y^2}} = \frac{y(y^2 - 1)}{y} = y^2 - 1$$

$$= \frac{4}{3} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} \cos t \right) \left(\frac{1}{2} \cos t \right) dt = \frac{1}{12} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt$$

$$= \frac{1}{12} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{3}{8} + \frac{1}{2} \cos 2t + \frac{1}{2} \cos 4t \right] dt = \frac{1}{12} \cdot \frac{3}{8} \pi = \frac{\pi}{32}$$

Example 4. Let us compute the volume of Ω bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$.

$$\begin{cases} z = 1 - x^2 - y^2 \\ z = 1 - y \end{cases} \Rightarrow 1 - x^2 - y^2 = 1 - y \Rightarrow x^2 + y^2 = y$$

$$\Rightarrow x^2 + y^2 - y = 0 \Rightarrow x^2 + (y - \frac{1}{2})^2 - \frac{1}{4} = 0 \Rightarrow x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

$$(1 - x^2 - y^2) - (1 - y) \geq 0 \Rightarrow y^2 - y \leq 0 \Rightarrow 0 \leq y \leq 1$$

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{y^2 - y}\}$$

$$\text{vol}(\Omega) = \iint_D [(1 - x^2 - y^2) - (1 - y)] dx dy = \iint_D (y - x^2 - y^2) dx dy$$

3 Change of variables:

Let S, D two bounded subsets of \mathbb{R}^2 , $\mathbb{R}^2 \rightarrow \mathbb{R}$ three functions such that :

- 1) $(g(u, v), h(u, v)) \in D \subset \text{Dom}(f), \forall (u, v) \in S$
- 2) The functions are of class C^1 3) The function f is continuous on D ; then we have :

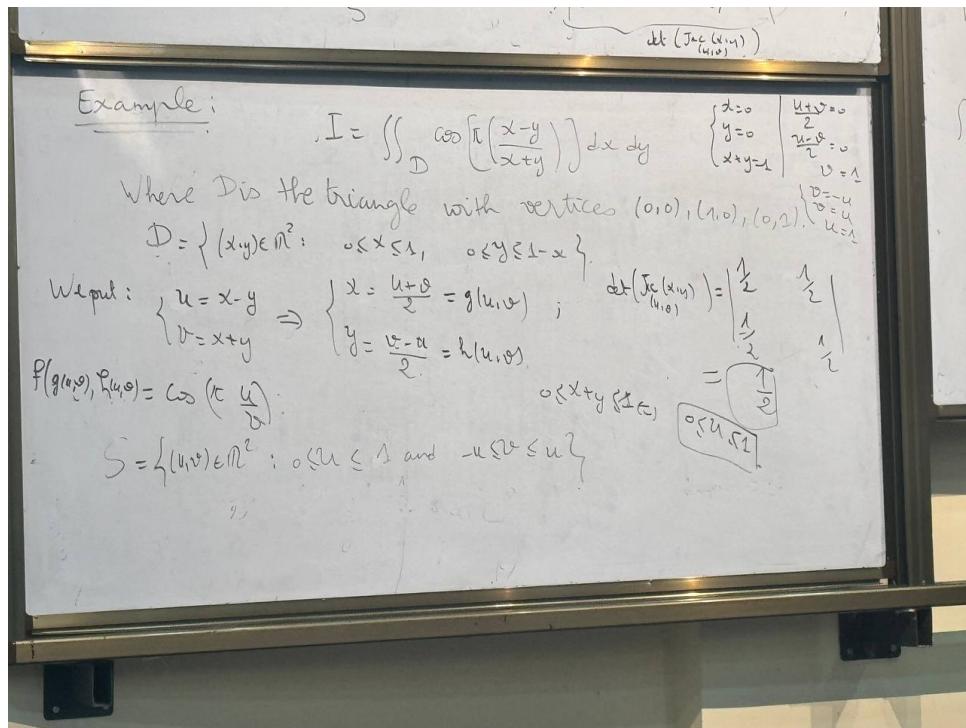
$$\iint_D f(x, y) dx dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial g}{\partial u} \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \frac{\partial h}{\partial u} \right| du dv$$

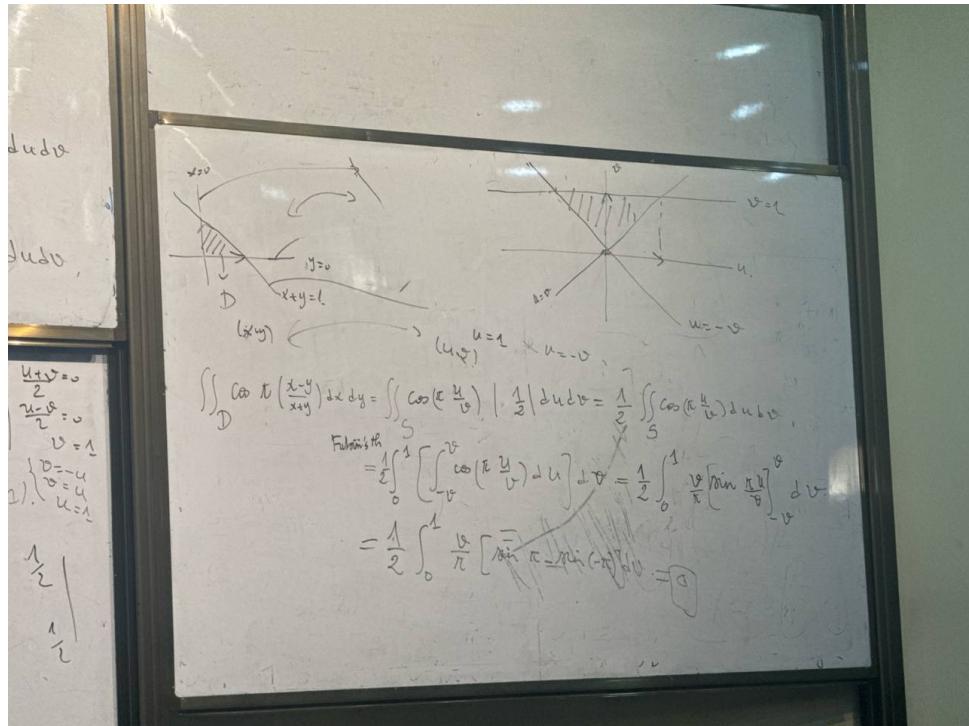
$$\iint_S f(x(u, v), y(u, v)) \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| du dv$$

$$\text{note : } \det(Jac_{(u,v)}(x, y)) = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|$$

3.1 Example 01:

$$I = \iint \cos \pi \frac{x-y}{x+y} dx dy$$





3.2 Particular case: Polar coordinates:

$$\begin{cases} x(r, \theta) = r \cos(\theta) \\ y(r, \theta) = r \sin(\theta) \end{cases}$$

$r = \sqrt{x^2 + y^2}$ the angle between \vec{i} and $O\vec{M}$

$$O\vec{M} = x\vec{i} + y\vec{j}$$

$$O\vec{M} = \sqrt{x^2 + y^2} \left[\frac{x}{\sqrt{x^2 + y^2}} \vec{i} + \frac{y}{\sqrt{x^2 + y^2}} \vec{j} \right]$$

$$\left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 = 1$$

$$\exists \theta \in \mathbb{R} \quad / \quad \cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \quad ; \quad \sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}$$

$$\begin{aligned} Jac_{(r,\theta)}(x, y) &= \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix} \\ &= \cos(\theta)(r \cos(\theta)) - \sin(\theta)(-r \sin(\theta)) \\ &= r \cos^2(\theta) + r \sin^2(\theta) = r > 0 \end{aligned}$$

$$\iint_D f(x, y) dxdy = \iint_S f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$

3.3 Example 02:

Calculate elliptic area using double integral:

Handwritten derivation of elliptic area formula using polar coordinates:

$$A(D) = \iint_D dxdy$$

Given elliptic variables change:

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$$

$$S = \{(r, \theta) | 0 \leq r \leq ab, 0 \leq \theta \leq 2\pi\}$$

$$0 \leq r \leq \sqrt{a^2 + b^2 \cos^2 \theta}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$X = \frac{x}{a}, Y = \frac{y}{b}$$

$$X^2 + Y^2 = 1$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$r^2 \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1$$

$$r^2 \left(\frac{a^2\cos^2\theta + b^2\sin^2\theta}{a^2 + b^2 \cos^2\theta}\right) = 1$$

$$r^2 = \frac{a^2b^2}{b^2 \cos^2\theta + a^2(1 - \cos^2\theta)} = \frac{a^2b^2}{b^2 \cos^2\theta + a^2 \sin^2\theta}$$

$$S = [0, 1] \times [0, 2\pi]$$

$$A(D) = \int_0^1 \int_0^{2\pi} abr dr d\theta$$

$$= ab \left(\int_0^1 r dr \right) \left(\int_0^{2\pi} d\theta \right) = ab\pi$$

Jac $(x, y) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = ab(r\cos\theta - b\sin\theta)$

$$= ab(r\cos\theta + b\sin\theta)$$

$$= abr$$

3.4 Example 03:

3.5 Example: Volume calculation using polar coordinates

Example: Let us compute the volume of the solid Ω bounded upper by $z = \sqrt{16 - x^2 - y^2}$ and by the cylinder $x^2 + y^2 = 4$, and $z = 0$.

$\text{Vol}(\Omega) = \iint_D \sqrt{16 - x^2 - y^2} dx dy$

$D = \{(x, y) | x^2 + y^2 \leq 4\}$

Volume diagram showing a cylindrical shell of radius r and height $z = \sqrt{16 - r^2}$.

Example: Let $0 < a < b$. $D = \{(x, y) | x^2 + y^2 \leq a^2\}$

$\begin{cases} d = r_{\text{min}} \\ y = r_{\text{max}} \end{cases}; \quad x^2 + y^2 \leq 4 \Rightarrow r^2 \leq 4 \Rightarrow 0 \leq r \leq 2$
 $0 \leq \theta \leq \pi$

$$\text{Vol}(R) = \int_0^{2\pi} \int_0^2 r \sqrt{16 - r^2} dr d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^2 r \sqrt{16 - r^2} dr \right)$$

$$= 2\pi \int_0^2 r \sqrt{16 - r^2} dr; \quad u = 16 - r^2, \quad du = -2r dr$$

$$= 2\pi \int_{12}^0 (-\frac{1}{2}) \sqrt{u} du = \pi \int_{16}^{12} \sqrt{u} du; \quad \begin{matrix} r=0 \rightarrow u=16 \\ r=2 \rightarrow u=12 \end{matrix}$$

$$= \frac{2\pi}{3} \left[u^{\frac{3}{2}} \right]_{12}^{16} = \frac{2\pi}{3} \left[16^{\frac{3}{2}} - 12^{\frac{3}{2}} \right]$$

$$= \frac{8\pi}{3} (64 - 24\sqrt{3})$$

$$\begin{aligned} R_1 &= 16\sqrt{3} \\ &= 16\sqrt{4 \cdot 3} \\ &= 16\sqrt{4} \cdot \sqrt{3} \\ &\approx 114.6 \end{aligned}$$

3.6 Example 04:

Example 2: $\iint_D \frac{1}{\sqrt{4-x^2-y^2}} dx dy$; where D is the disk centered at $(0,1)$ with radius 1.

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \leq 1\}$$

$$= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2y\}$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{matrix} r^2 \leq 2r \sin \theta \\ 0 \leq r \leq 2 \sin \theta \\ 0 \leq \theta \leq \pi \end{matrix}$$

$$\begin{cases} d = r_{\text{min}} \\ y = r_{\text{max}} \end{cases}; \quad x^2 + y^2 \leq 4 \Rightarrow r^2 \leq 4 \Rightarrow 0 \leq r \leq 2$$

$$\int_0^{\pi} \left[\int_0^{2\sin\theta} \frac{1}{\sqrt{4-r^2}} r dr \right] d\theta = \int_0^{\pi} \left[-\sqrt{4-r^2} \right]_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi} (\sqrt{4-0} - \sqrt{4-(2\sin\theta)^2}) d\theta \quad \times \frac{(2r)}{2\sqrt{4-r^2}}$$

$$= \int_0^{\pi} (2 - \sqrt{4-4\sin^2\theta}) d\theta = 2\pi - 2 \int_0^{\pi} \sqrt{4\sin^2\theta} d\theta$$

$$= 2\pi - 2 \left[\int_0^{\pi/2} \cos\theta d\theta - \int_{\pi/2}^{\pi} \cos\theta d\theta \right] = 2\pi - 2 \left[[\sin\theta]_0^{\pi/2} - [\sin\theta]_{\pi/2}^{\pi} \right] = 2\pi - 4$$

4 Applications of double integrals

Definition 7. Let $\rho : D \rightarrow \mathbb{R}$ be a continuous density function of the lamina corresponding to the plane region D , the mass $m(D)$ of the lamina is defined by:

$$m(D) = \iint_D \rho(x, y) dx dy$$

Remark:

if $\rho(x, y) = K, \forall (x, y) \in D$, then $m(D) = KA(D)$

Example 1.

$$\rho : (x, y) = 2x + y$$

$$\text{Example: } \rho(x, y) = 2x + y, \quad D = \begin{cases} 0 & \leq x \leq 1 \\ 0 & \leq y \leq x \end{cases}$$

$$m(D) = \iint_D \rho(x, y) dx dy = \int_0^1 \left[\int_x^1 (2x+y) dy \right] dx$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2} \right]_x^1 dx = \int_0^1 [(2x+\frac{1}{2}) - (2x+\frac{x^2}{2})] dx$$

$$= \int_0^1 \left(\frac{1}{2} + 2x - \frac{5}{2}x^2 \right) dx = \left[\frac{x}{2} + 2x^2 - \frac{5}{6}x^3 \right]_0^1 = \frac{1}{2} + 2 - \frac{5}{6} = \frac{3}{6} = \frac{1}{2}$$

Definition 8. Moments and center of mass

With the same hypothesis and notations as definition above

1) the moment of mass with respect the x and y axes respectively are:

$$M_x = \iint_D y\rho(x, y)dxdy; \quad M_y = \iint_D x\rho(x, y)dxdy$$

the center of mass of D is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m(D)}, \frac{M_x}{m(D)}\right)$

2) the second moments, or the moments of inertia of D about the x and y axes respectively are:

$$I_x = \iint_D y^2\rho(x, y)dxdy; \quad I_y = \iint_D x^2\rho(x, y)dxdy$$

3) the polar moment is $I_O = I_x + I_y = \iint_D (x^2 + y^2)\rho(x, y)dxdy$

Example 2. 1)

$$D = \{(x, y) \in \mathbb{R}^2; \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 1\} = [0, 1] [0, 1]$$

$$\rho(x, y) = 1$$

Example: ① $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$m(\mathcal{D}) = \int_0^1 \int_0^1 dx dy = 1 \times 1 = 1$

$M_x = \int_0^1 y dx dy = \left(\int_0^1 y dy \right) \left(\int_0^1 dx \right) = \left(\frac{1}{2} \right); \quad M_y = \int_0^1 x dx dy = \left(\int_0^1 x dx \right) \left(\int_0^1 dy \right) = \left(\frac{1}{2} \right)$

$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2} \right)$

2)

$$\rho(x, y) = 2x + y$$

(the first line belongs to the first example (the I_x and I_y integrals))

$I_x = \iint_D x^2 dx dy = \frac{1}{3} = \frac{1}{3}; \quad I_O = I_x + I_y = \frac{2}{3}$

② $\rho(x, y) = 2x + y; \quad \mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

$m(\mathcal{D}) = \frac{2}{3}$

$M_y = \iint_D x \rho(x, y) dx dy = \iint_D x(2x + y) dx dy = \left(\int_0^1 (2x^2 + xy) dx \right) \left(\int_0^1 y dy \right)$

$= \int_0^1 \left[\frac{2x^3}{3} + \frac{x^2 y}{2} \right]_0^1 dx = \int_0^1 \left[\frac{2x^3}{3} + \frac{x^2}{2} \right] dx = \int_0^1 \left[\left(2x^3 + \frac{x^2}{2} \right) - \left(\frac{2 \cdot 0^3}{3} + \frac{0^2}{2} \right) \right] dx = \frac{6+16-5}{24} = \frac{7}{24}$

$M_x = \iint_D xy(2x+y) dx dy$
 $= \int_0^1 \left[\frac{1}{2} (2xy + y^2) \right] dy$ (by constant
 $\int_0^1 \left[xy^2 + \frac{y^3}{3} \right] dy = \int_0^1 [(x+1) - \frac{y}{3}] dy$
 $= \left[\frac{x}{3} + \frac{x^2}{2} - \frac{1}{3}x^4 \right]_0^1 = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$
 $\therefore (\bar{x}, \bar{y}) = \left(\frac{7}{34}, \frac{1}{2} \right) = \left(\frac{35}{48}, \frac{3}{4} \right) = \left(\frac{7}{16}, \frac{3}{4} \right)$

$I_x = \iint_D y^2 \rho(x,y) dx dy = \iint_D y^2 (2x+y) dx dy = \iint_D (2xy^2 + y^3) dx dy$
 $= \int_0^1 \left[\int_x^1 (2xy^2 + y^3) dy \right] dx = \int_0^1 \left[\frac{2}{3}xy^3 + \frac{1}{4}y^4 \right]_x^1 dx = \int_0^1 \left[\frac{2}{3}x + \frac{1}{4} - \frac{11}{12}x^4 \right] dx$
 $= \left[\frac{x}{4} + \frac{1}{3}x^2 - \frac{11}{60}x^4 \right]_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{11}{60} = \frac{15+20-11}{60} = \frac{24}{60} = \boxed{\frac{2}{5}}$

$I_y = \iint_D x^2 \rho(x,y) dx dy = \iint_D x^2 (2x+y) dx dy$
 $\text{There } = \int_0^1 \left[\int_x^1 (2x^3 + x^2y) dy \right] dx = \int_0^1 \left[2x^3y + \frac{x^2y^2}{2} \right]_x^1 dx$
 $= \int_0^1 \left[2x^3 + \frac{y^2}{2} - \frac{5}{2}x^4 \right] dx = \left[\frac{1}{2}x^4 + \frac{x^3}{6} - \frac{5}{2}x^4 \right]_0^1 = \frac{1}{2} + \frac{1}{6} - \frac{1}{2}$
 $\therefore \text{The polar moment is } I_0 = I_x + I_y = \frac{2}{5} + \frac{1}{6} = \boxed{\frac{17}{30}}$

Definition 9. Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, z = f(x, y)\}$ be bounded surface, with f a function such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist on D then :

the area of Σ ; $A(\sigma)$ is given by:

$$A(\sigma) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}(x, y) \right)^2 + \left(\frac{\partial f}{\partial y}(x, y) \right)^2} dxdy$$

Example 3.

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0 \text{ and } x + y + z = 1\}$$

$A(\Sigma)$ is given by

$$A(\Sigma) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2} dx dy$$

Example: $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0, y \geq 0, x \geq 0 \text{ and } x+y+z=1\}$

$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$

$$\sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}$$

$$A(\Sigma) = \iint_D \sqrt{3} dx dy = \sqrt{3} \int_0^1 \left[\int_0^{1-x} dy \right] dx = \sqrt{3} \int_0^1 (1-x) dx = \sqrt{3} \left[x - \frac{x^2}{2} \right]_0^1 = \frac{\sqrt{3}}{2}$$

Example 4. ...

Example 2:

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \text{ and } x^2 + y^2 \leq R^2\}$$

$$f(x,y) = \sqrt{R^2 - x^2 - y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2} = \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}} = \sqrt{\frac{(R^2 - y^2) + x^2 + y^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

$\text{CAT-JE H}_2\text{O}^+$ $P_{\text{H}_2\text{O}} = K$ $V = \text{activity constant}$
 $A(\Sigma) = \iint_D \frac{1}{\sqrt{R^2 - x^2 - y^2}} dx dy$ $\text{using polar coordinates:}$
 $= \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} r dr d\theta$
 $\left\{ \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array} \right. \quad \begin{array}{l} 0 \leq r \leq R \\ 0 \leq \theta \leq 2\pi \end{array}$
 $= 2\pi R \left[-\sqrt{R^2 - r^2} \right]_0^R = 2\pi R [-0 + \sqrt{R^2}] = 2\pi R^2$
 So, the area of all the sphere of radius R is
 $4\pi R^2$

Example 5. ...

(first two lines)

$$S = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1, z = 2 - x^2 - y^2\}$$

$$f(x, y) = 2 - x^2 - y^2 \quad ; \quad \frac{\partial f}{\partial x}(x, y) = -2x \quad ; \quad \frac{\partial f}{\partial y} = -2y$$

Example 3: $S = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1, z = 2 - x^2 - y^2\}$
 $f(x, y) = 2 - x^2 - y^2$, $\frac{\partial f}{\partial x}(x, y) = -2x$, $\frac{\partial f}{\partial y}(x, y) = -2y$
 $\sqrt{1 + (\frac{\partial f}{\partial x}(x, y))^2 + (\frac{\partial f}{\partial y}(x, y))^2} = \sqrt{1 + (-2x)^2 + (-2y)^2} = \sqrt{1 + 4x^2 + 4y^2}$
 $A(S) = \int_{-1}^1 \int_{-1}^1 \sqrt{1 + 4x^2 + 4y^2} dx dy$
 TWO WORK (and) \rightarrow to the right \rightarrow more integration
 $K = \frac{1}{2} \text{CAT-JE H}_2\text{O}^+$, $P_{\text{H}_2\text{O}} = K$ $V = \text{activity constant}$

by symmetry: $A(S) = 8 \iint_{\Delta} \sqrt{1+4x^2+4y^2} dx dy$

where $\Delta = \{(x, y) \in \mathbb{R}^2 : x = r \cos \theta, 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq \theta \leq \frac{\pi}{4}\}$

$$A(S) = 8 \int_0^{\frac{\pi}{4}} \left[\int_0^{r \cos \theta} \sqrt{1+4r^2} dr \right] d\theta = 8 \int_0^{\frac{\pi}{4}} \left[\frac{2}{3} (1+4r^2)^{\frac{3}{2}} \right]_{0}^{r \cos \theta} d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{4}} \left[\left(1 + \frac{4}{\cos^2 \theta} \right)^{\frac{3}{2}} - 1 \right] d\theta = \frac{16}{3} \int_0^{\frac{\pi}{4}} \left[\frac{4 + \cos^2 \theta}{\cos^2 \theta} \right]^{\frac{3}{2}} d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\sqrt{4 + \cos^2 \theta}}{\cos^3 \theta} (4 + \cos^2 \theta) d\theta = \int_0^{\frac{\pi}{4}} \frac{4 + \cos^2 \theta}{\cos^3 \theta} d\theta$$

$$\cos^2 \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

5 Triple Integrals:

5.1 Definition and Calculations:

the triple integral of a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ on a bounded region $\Omega \subset \mathbb{R}^3$ is defined in the same way as double integrals with the same properties (linearity, Chasles's relation, increasing), we denote this integral by: $\iiint_{\Omega} f(x, y, z) dx dy dz$

In the particular case : $f \equiv 1$ on Ω

$$\iiint_{\Omega} dx dy dz = Vol(\Omega)$$

Theorem 2. Evaluation by iterated integrals

If $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b ; h_1(x) \leq y \leq h_2(x) ; g_1(x, y) \leq z \leq g_2(x, y) / h_1, h_2, g_1, g_2 \text{ are continuous}\}$

then:

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} \left[\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz \right] dy \right] dx$$

Example 6.

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$$

Example: $\Omega = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1-x; 0 \leq z \leq 1-x-y\}$

$$\begin{aligned} \text{vol}(\Omega) &= \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} dz \right] dy \right] dx \\ &= \int_0^1 \left[\int_0^{1-x} (1-x-y) dy \right] dx \\ &= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_{0}^{1-x} dx = \int_0^1 \left[(1-x)(1-x) - \frac{1}{2}(1-x)^2 \right] dx \\ &= \frac{1}{2} \int_0^1 (x-1)^2 dx = \frac{1}{6} \left[(x-1)^3 \right]_0^1 = \frac{1}{6} \end{aligned}$$

5.2 Change of Variable:

Let $f : \Omega \rightarrow \mathbb{R}$ Riemann-integrable; let S a bounded region of \mathbb{R}^3 ; $g_1, g_2, g_3 \in S \rightarrow \mathbb{R}$ of class C^1 such that $\forall (y, v, w) \in S; (g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)) \in \Omega$

then:

$$\iint_{\Omega} f(x, y, z) dxdydz = \iint_S f(g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)) |det(Jac_{u,v,w}(g_1, g_2, g_3))| du dv dw$$

$$det(Jac_{u,v,w}(g_1, g_2, g_3)) = det(Jac_{u,v,w}(x, y, z)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} & \frac{\partial g_3}{\partial w} \end{vmatrix}$$

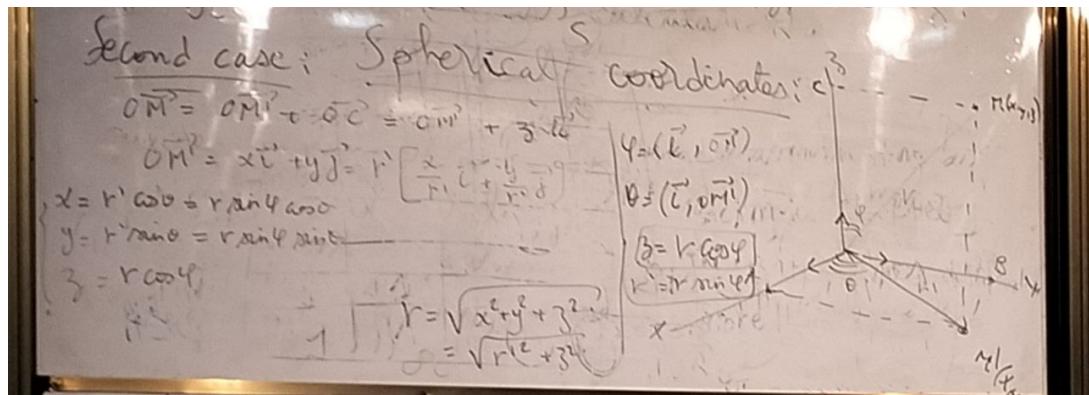
5.3 Two particular cases:

First case: Cylindrical coordinates

$$\left\{ \begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{array} \right. \quad \det(Jac_{(u,v,w)}(x, y, z)) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_S r f(r\cos(\theta), r\sin(\theta), z) dr d\theta dz$$

second case: spherical coordinates



...

$$\begin{aligned} \det_{(r,\theta,\varphi)}(x, y, z) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \\ &= \begin{vmatrix} \sin(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ \cos(\varphi) & 0 & -r\sin(\varphi) \end{vmatrix} \\ &= r(r\sin(\varphi)) \begin{vmatrix} \sin(\varphi)\cos(\theta) & -\sin(\theta) & \cos(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & \cos(\theta) & \cos(\varphi)\sin(\theta) \\ \cos(\varphi) & 0 & -\sin(\varphi) \end{vmatrix} \\ &= -r^2 \sin(\varphi) \end{aligned}$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iint_S r^2 \sin(\varphi) f(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)) dr d\theta d\varphi$$

Example 7. the volume of a ball with radius R:

The image shows a handwritten derivation on a whiteboard. At the top, there is a general formula for a triple integral in Cartesian coordinates: $\iiint_V f(x,y,z) dx dy dz = \iiint_{\Omega} r^2 \sin \varphi f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) dr d\theta d\varphi$. Below this, a note says "Ex ample: the volume of a ball with radius R". The volume is given by the set $\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq R^2\}$. This is defined by the following constraints:

$$\begin{cases} x = x_0 + r \sin \varphi \cos \theta \\ y = y_0 + r \sin \varphi \sin \theta \\ z = z_0 + r \cos \varphi \end{cases} \quad \begin{cases} 0 \leq r \leq R \\ 0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

The volume is calculated as:

$$V(\Omega) = \iiint_{\Omega} dx dy dz = \int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin \varphi dr d\theta d\varphi$$

$$= \left(\int_0^R r^2 dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\pi \sin \varphi d\varphi \right) = \left[\frac{r^3}{3} \right]_0^R \left[\theta \right]_0^{2\pi} \left[-\cos \varphi \right]_0^\pi$$

$$= \frac{R^3}{3} \times 2\pi \times 2 = \boxed{\frac{4\pi}{3} R^3}$$

Example 8. Compute the volume of Intersection between ball and cylinder Using cylindrical coordinates :

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4 \text{ and } x^2 + y^2 \leq 2y\}$$

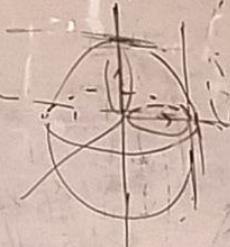
Example 2. Compute the volume of

using cylindrical coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \left\{ \begin{array}{l} (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4 \text{ and } z \leq y^2 \end{array} \right.$$

$$V = \int_0^\pi \left[\int_0^{2\pi} \left[\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \left[\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz dr d\theta \right] dr \right] d\theta \right] d\theta$$

$$= \int_0^\pi \left[\int_0^{2\pi} 2r \sqrt{4-r^2} dr \right] d\theta = \int_0^\pi \left[\frac{2}{3}(4-r^2)^{3/2} \right]_0^{2\pi} d\theta$$



$$\begin{aligned} &= \frac{2}{3} \int_0^\pi \left(4^{3/2} - (4 - 4\cos^2 \theta)^{3/2} \right) d\theta = \frac{16}{3} \int_0^\pi \left(1 - [\cos^2 \theta]^{3/2} \right) d\theta. \\ &= \frac{16}{3} \int_0^\pi \left(1 - [1 - \sin^2 \theta]^3 \right) d\theta = \frac{16}{3} \int_0^\pi d\theta - \frac{16}{3} \left[\int_0^{\pi/2} \cos^3 \theta d\theta - \int_0^{\pi/2} \sin^3 \theta d\theta \right] \\ &= \frac{16}{3} \pi - \frac{32}{3} \int_0^{\pi/2} \cos^3 \theta d\theta = \frac{16}{3} \pi - \frac{32}{3} \int_0^{\pi/2} (1 - \sin^2 \theta)(\sin \theta) d\theta \\ &= \frac{8\pi}{3} - \frac{32}{3} \left[\int_0^1 (1-u^2) du \right] = \frac{8\pi}{3} - \frac{32}{3} \times \frac{2}{3} \quad u = \sin \theta, \quad du = \cos \theta d\theta \quad (1-u^2) \\ &\quad \boxed{\frac{8\pi}{3} - \frac{64}{9}} \end{aligned}$$

Example 9. Use spherical coordinates to compute the volume of the region D bounded by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$:

Example 2: Use spherical coordinates to compute the volume of the region D bounded by the upper hemisphere of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

Spherical coordinates:

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

Volume integral setup:

$$Vol(D) = \iiint_D dx dy dz = \int_0^3 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} r^2 \sin\theta dr d\theta d\phi$$

$$= \left(\int_0^3 r^2 dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{4}} r^2 \sin\theta d\phi \right) = \left[\frac{r^3}{3} \right]_0^3 \left[\theta \right]_0^{2\pi} \left[-\cos\phi \right]_0^{\frac{\pi}{4}} = 18 \left[1 - \frac{1}{8} \right] \pi = 9\pi(2-\frac{1}{8})$$

Integration limits and results:

- $0 \leq \theta \leq \frac{\pi}{4}$
- $0 \leq \theta \leq 2\pi$
- $0 \leq r \leq 3$
- $r^2 \cos^2\theta = r^2 \sin^2\phi$
- $\frac{\tan\theta}{\cos\phi} = 1$
- $\tan\theta = \pm 1$
- $\theta \in [0, \pi] \Rightarrow \theta = \frac{\pi}{4}$

5.4 Applications of triple integrals

1) If $f : D \rightarrow \mathbb{R}_+$ is a positive function on a bounded region $D \subset \mathbb{R}^2$; then:

$$\iint_D f(x, y) dxdy = \iint_D \left[\int_0^{f(x,y)} dz \right] dxdy = \iiint_{\Omega} dxdydz = Vol(\Omega)$$

Where $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}$

2) Consider the solid $\Omega \subset \mathbb{R}^3$ bounded with the mass density function

$$\rho : \Omega \rightarrow \mathbb{R}, (x, y, z) \mapsto \rho(x, y, z)$$

the mass of Ω is $m = \iiint_{\Omega} \rho(x, y, z) dx dy dz$

In particular if $\rho \equiv K \implies m = k Vol(\Omega)$

the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is:

$$\begin{cases} \bar{x} = \frac{\iiint_{\Omega} x \rho(x, y, z) dx dy dz}{\iiint_{\Omega} \rho(x, y, z) dx dy dz} \\ \bar{y} = \frac{\iiint_{\Omega} y \rho(x, y, z) dx dy dz}{\iiint_{\Omega} \rho(x, y, z) dx dy dz} \\ \bar{z} = \frac{\iiint_{\Omega} z \rho(x, y, z) dx dy dz}{\iiint_{\Omega} \rho(x, y, z) dx dy dz} \end{cases}$$

M_{yz} , M_{xz} and M_{xy} are called the first moments of Ω with respect the $(yz)-$, $(xz)-$, and (xy) planes respectively

3)

$$\begin{aligned} I_{xy} &= \iiint_{\Omega} z^2 \rho(x, y, z) dx dy dz \\ I_{xz} &= \iiint_{\Omega} y^2 \rho(x, y, z) dx dy dz \\ I_{yz} &= \iiint_{\Omega} x^2 \rho(x, y, z) dx dy dz \end{aligned}$$

are the second moments (or moments of inertia) of Ω with respect the planes

$(yz)-$, $(xz)-$, and (xy) respectively

$$I_x = I_{xy} + I_{xz} = \iiint_{\Omega} (y^2 + z^2) \rho(x, y, z) dx dy dz \quad \text{the moment of inertia with respect to the axis (ox)}$$

$$I_y = I_{xy} + I_{yz} = \iiint_{\Omega} (x^2 + z^2) \rho(x, y, z) dx dy dz \quad \text{the moment of inertia with respect to the axis (oy)}$$

$$I_z = I_{xz} + I_{yz} = \iiint_{\Omega} (x^2 + y^2) \rho(x, y, z) dx dy dz \quad \text{the moment of inertia with respect to the axis (oz)}$$

Example 10.

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0 \text{ and } x^2 + y^2 + z^2 \leq 4\}$$

Example: $\Omega = \{(x, y, z) \in \mathbb{R}^3 : r \geq 0 \text{ and } x^2 + y^2 + z^2 \leq 4\}$
 the point (x, y, z) in the region Ω bounded by the cone $r \cos \theta > 0$

$$\begin{aligned} M &= \iiint_{\Omega} r(x^2 + y^2) dx dy dz = k \iiint_{\Omega} r^3 dx dy dz \\ &= k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r^2 \sin \varphi \cos \theta) dr d\theta d\varphi \\ &= k \left(\int_0^2 r^3 dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{2}} r^2 \sin^2 \varphi d\varphi \right) \\ &= k \left[\frac{r^4}{4} \right]_0^2 \left[\varphi \right]_0^{2\pi} \left[\frac{1}{2} \sin^2 \varphi \right]_0^{\frac{\pi}{2}} \\ &= k \times \frac{2^4}{4} \times 2\pi \times \frac{1}{2} \times 1^2 = \boxed{4k\pi} \end{aligned}$$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} 0 < r < 2 \\ 0 < \theta < \pi/2 \\ 0 < \varphi < \pi/2 \end{cases}$$

$$\begin{aligned} M_{yz} &= \iiint_{\Omega} x \rho(x, y, z) dx dy dz = k \iiint_{\Omega} x z dx dy dz \\ &= k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r^2 \sin \varphi)(r \sin \theta \cos \theta)(r \cos \varphi) dr d\theta d\varphi \\ &= k \left(\int_0^2 r^4 dr \right) \left(\int_0^{2\pi} \cos \theta d\theta \right) \left(\int_0^{\frac{\pi}{2}} r^2 \sin^2 \varphi d\varphi \right) = 0 \text{ because } \int_0^{2\pi} \cos \theta d\theta = 0 \\ M_{xz} &= \iiint_{\Omega} y \rho(x, y, z) dx dy dz = k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r^2 \sin^2 \varphi)(r \sin \theta \cos \theta)(r \cos \varphi) dr d\theta d\varphi \\ &= k \left(\int_0^2 r^4 dr \right) \left(\int_0^{2\pi} \sin^2 \theta d\theta \right) \left(\int_0^{\frac{\pi}{2}} r^2 \sin^2 \varphi \cos \varphi d\varphi \right) = 0 \text{ because } \int_0^{2\pi} \sin^2 \theta d\theta = 0 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \iiint_V z \rho(x, y, z) dx dy dz = k \iiint_V z^2 \rho(x, y, z) dx dy dz, \\
 &= k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^2 \sin \varphi (\cos \varphi)^2 dr d\theta d\varphi = k \left[\int_0^2 r^4 dr \right] \left[\int_0^{2\pi} d\theta \right] \left[\int_0^{\frac{\pi}{2}} \sin \varphi \cos^2 \varphi d\varphi \right] \\
 &= k \left[\frac{r^5}{5} \right]_0^2 \left[\theta \right]_0^{2\pi} \left[-\frac{1}{3} \cos^3 \varphi \right]_0^{\frac{\pi}{2}} = k \frac{2^5}{5} \times 2\pi \times \left[\frac{1}{3} \right] = \boxed{\frac{64\pi k}{15}}
 \end{aligned}$$

the center of mass is

$$\begin{aligned}
 (\bar{x}, \bar{y}, \bar{z}) &= \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\frac{0}{m}, \frac{0}{m}, \frac{\frac{64\pi k}{15}}{4\pi k} \right) \\
 &= \boxed{(0, 0, \frac{16}{15})}
 \end{aligned}$$

$$\begin{aligned}
 I_{xy} &= \iiint_V z^2 \rho(x, y, z) dx dy dz; I_{xz} = \iiint_V y^2 \rho(x, y, z) dx dy dz; I_{yz} = \iiint_V x^2 \rho(x, y, z) dx dy dz, \\
 I_{xy} &= \iiint_V z^2 (kz) dx dy dz = k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r^2 \sin \varphi) (r \cos \varphi)^3 dr d\theta d\varphi \\
 &= k \left(\int_0^2 r^5 dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \sin \varphi \cos^3 \varphi d\varphi \right) = k \left[\frac{r^6}{6} \right]_0^2 \left[\theta \right]_0^{2\pi} \left[-\frac{1}{4} \cos^4 \varphi \right]_0^{\frac{\pi}{2}} \\
 I_{xy} &= k \left[\frac{2^6}{6} \right] [2\pi] \frac{1}{4} = \boxed{16\pi k}, \\
 I_{xz} &= \iiint_V y^2 (kz) dx dy dz = k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (r^2 \sin \varphi) (r \sin \theta)^2 (r \cos \varphi) dr d\theta d\varphi \\
 &= k \left(\int_0^2 r^5 dr \right) \left(\int_0^{2\pi} \sin^2 \theta d\theta \right) \left(\int_0^{\frac{\pi}{2}} \cos \varphi \sin^3 \varphi d\varphi \right) = k \left[\frac{r^6}{6} \right]_0^2 \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \left[\frac{1}{4} \sin^4 \varphi \right]_0^{\frac{\pi}{2}} \\
 &= k \left(\frac{2^6}{6} \right) \left(\frac{2\pi}{2} - \frac{0}{4} \right) \left(\frac{1}{4} \right) = \boxed{\frac{32\pi k}{3}}
 \end{aligned}$$

the last result equals $\frac{8\pi k}{3}$ not $\frac{32\pi k}{3}$

$$\begin{aligned}
 I_{yz} &= \iiint_D x^2 \rho(x, y, z) dx dy dz = k \iiint_D 3x^2 dx dy dz \\
 &\text{the sum of the region } D \text{ bounded in the womb,} \\
 &= k \int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^5 (\sin \varphi)^2 (\cos \theta)^2 (r^2 \sin \varphi d\theta) dr d\varphi d\theta \\
 &= k \left(\int_0^2 r^5 dr \right) \left(\int_0^{2\pi} \cos^2 \theta d\theta \right) \left(\int_0^{\frac{\pi}{2}} \cos \theta \sin \varphi d\varphi \right) \\
 &= k \left[\frac{r^6}{6} \right]_0^2 \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \left[\frac{1}{4} \sin^2 \theta \right]_0^{\frac{\pi}{2}} = \boxed{\frac{8\pi k}{3}}
 \end{aligned}$$

∴ $I_x = I_{xy} + I_{xz} = 16\pi k + \frac{8\pi k}{3} = \boxed{\frac{56\pi k}{3}}$

$I_y = I_{xy} + I_{yz} = 16\pi k + \frac{8\pi k}{3} = \boxed{\frac{56\pi k}{3}}$

$I_z = I_{yz} + I_{xz} = \boxed{\frac{16\pi k}{3}}$

$$M_{u_2} = \iiint_D x \rho(x, y, z) dx dy dz = k \iiint_D x_3 dx dy dz$$