Chapter 03: Numerical series

Notes from prof zeglaoui course

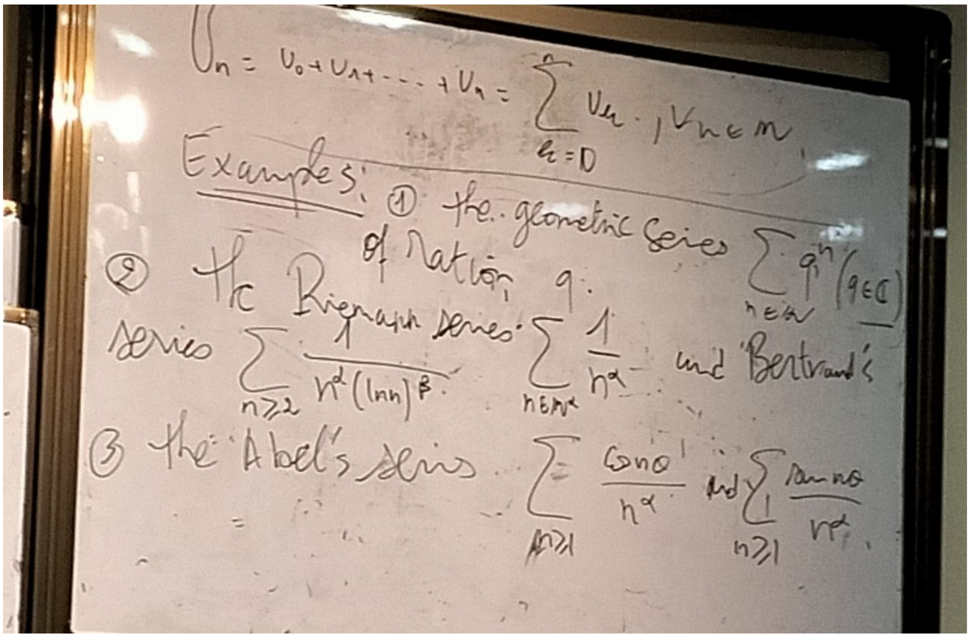
February 10, 2026

1 Generalities

Definition 1. *Let (U_n) be a sequence of real(or complex) numbers. By numerical series of the general term U_n , that we denote by $\sum_{n\in\mathbb{N}} U_n$, we mean the couple$((U_n), (\mathcal{U}_n))$ of real (or complex) sequences, where $(\mathcal{U}_n)$ is the sequence of partial sums of $\sum U_n$, defined by ;*

$$\mathcal{U}_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} U_k \quad ; \forall_k \in \mathbb{N},$$

Example 1. ..



Definition 2. *Let $\sum_{n\in\mathbb{N}} U_n$ numerical series and $(\mathcal{U}_n)$. its sequence of partial sums*

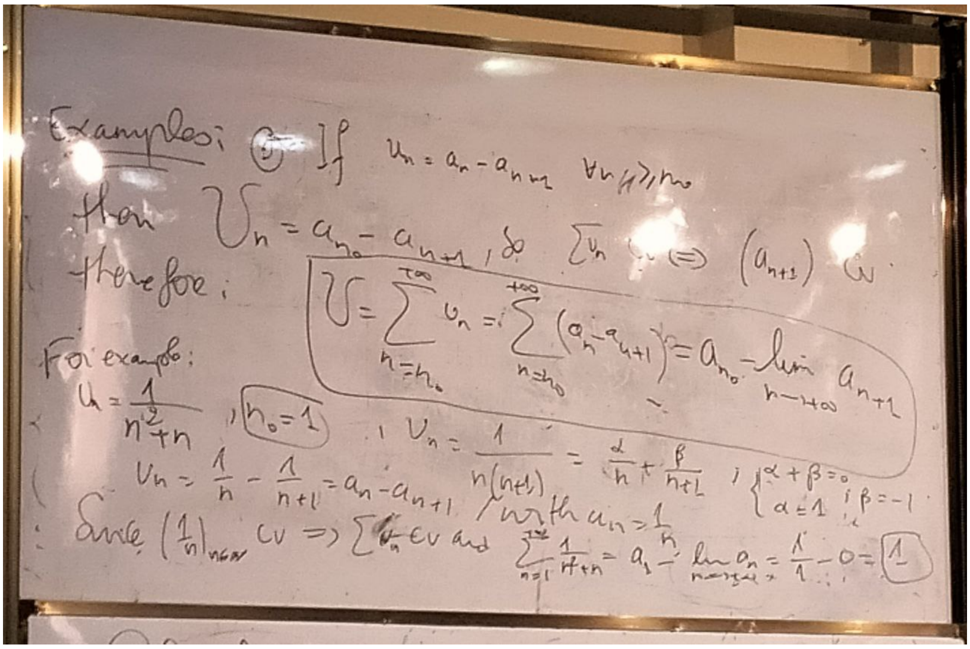
1) the series $\sum_{n\in\mathbb{N}} U_n$ is said to be convergent(Cv)if

($\mathcal{U}_n$ *converges; the limit* $\mathcal{U} = \lim\limits_{n\to+\infty} \mathcal{U}_n$ *is called the sum of the series* $\sum_{n\in\mathbb{N}}$*, often denoted by :* $\mathcal{U} = \sum_{n=0}^{\infty} U_n$

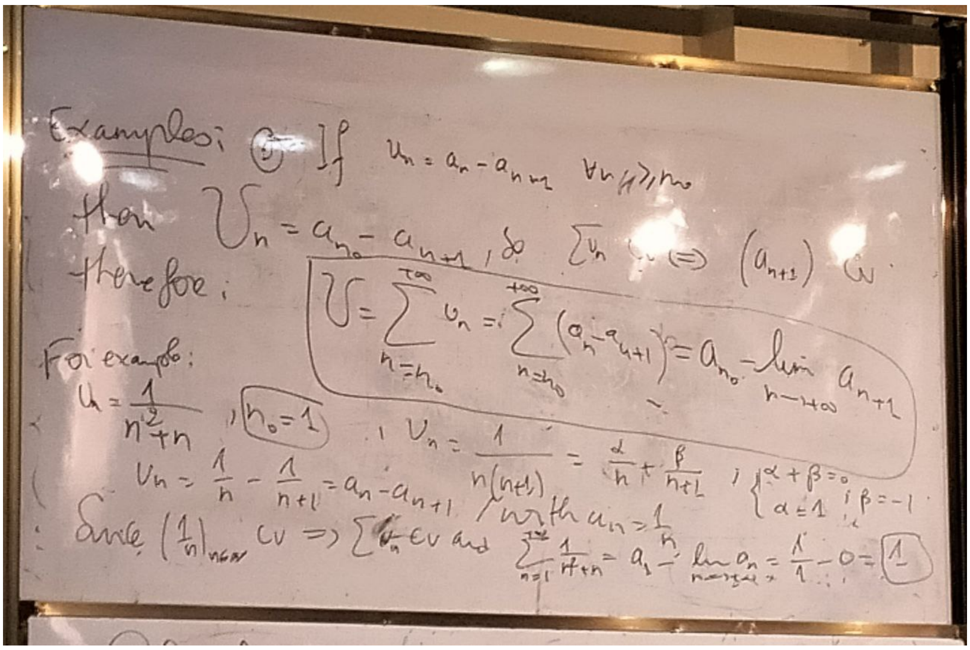
2) the series $\sum U_n$ *is said to be divergent if* $(\mathcal{U}_n)$ *diverges*

3) the nature of $\sum_{n\in\mathbb{N}} U_n$ *is the fact that it is convergent or divergent*

Example 2. *1)*



2)



Theorem 1.

$$\sum U_n \quad converge \implies \lim_{n \to +\infty} (U_n) = 0$$

Proof

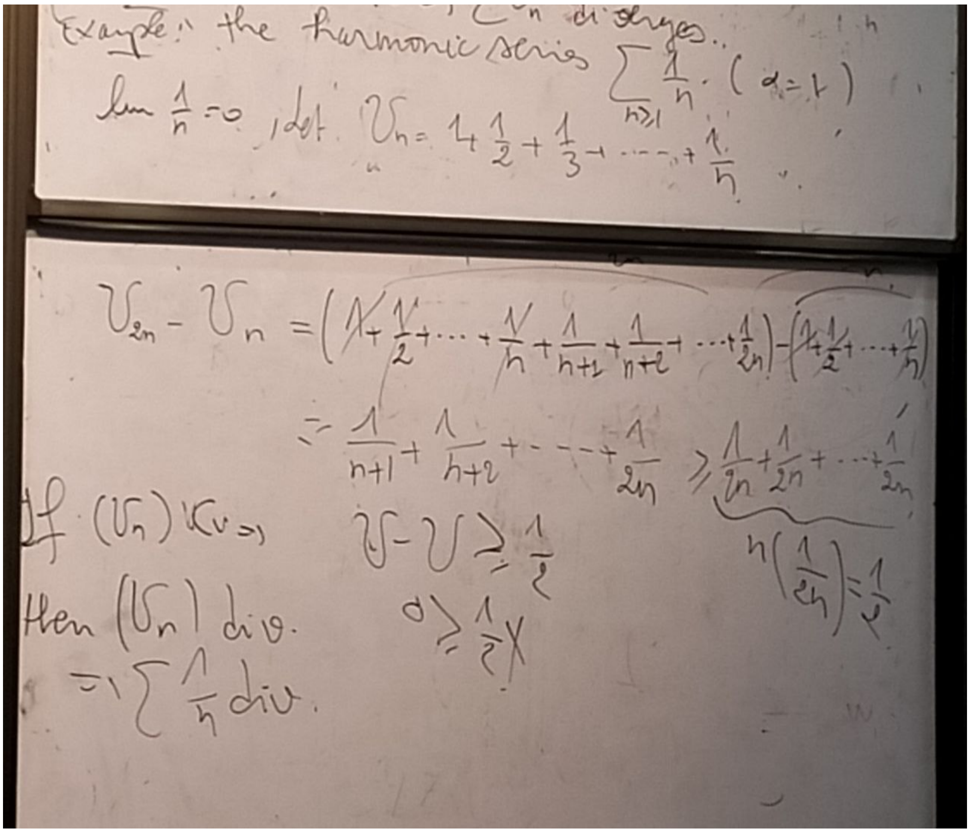
Because $U_n = \mathcal{U}_n - \mathcal{U}_{n-1} \quad , \forall n$

$$U_n \quad converges \Leftrightarrow (\mathcal{U}_n) converges \implies \lim U_n = \mathcal{U} - \mathcal{U} = 0$$

Remark

$$\lim U_n \neq 0 \implies \sum U_n \text{ diverges}$$

Example 3. *the harmonic series*



2 Positive series:

Definition 3. *the real series* $\sum_{n \in \mathbb{N}} U_n$ *is said to be positive if there exists* $N \in \mathbb{N}$ *st:* $\forall n \geq N \quad U_n \geq 0$

Remark: if $\sum_{n \in \mathbb{N}} U_n$ is positive then the partial sums sequence $\mathcal{U}_n$ is an increasing sequence

Recall that if $(\alpha_n)_{n\in\mathbb{N}}$ is an increasing real sequence either (α_n) converges to its supremum(i.e $\lim \alpha_n = sup_{n\in\mathbb{N}}(\alpha_n)$ when it's upper bounded) or $\lim \alpha_n = +\infty$ if not

proposition: comparision criterion

Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two positive series st:

$$\exists N \in \mathbb{N} \quad, \forall n \geq N \quad U_n \leq V_n$$

so, $\sum_{n\in\mathbb{N}} V_n$ converges $\implies$ $\sum_{n\in\mathbb{N}} U_n$ converges

then $\sum_{n\in\mathbb{N}} U_n$ diverges $\implies$ $\sum_{n\in\mathbb{N}} V_n$ diverges

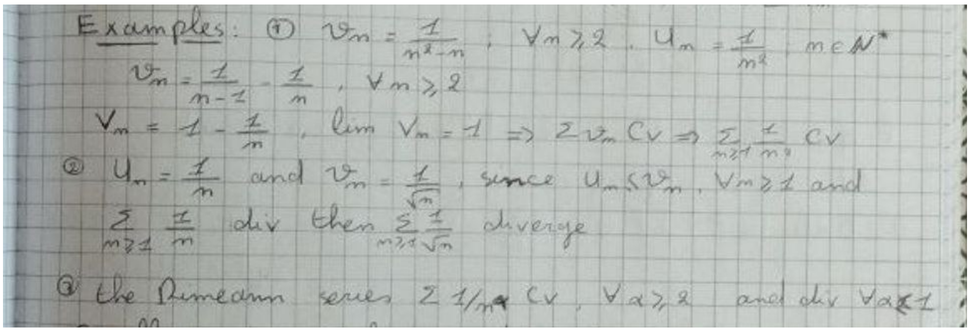
Proof:

if (U_n) is the sequence of partial sums of $\sum_{n\in\mathbb{N}} U_n$

and (V_n) is the sequence of partial sums of $\sum_{n\in\mathbb{N}} V_n$

$U_n \leq V_n$ hence: $\sum_{n\in\mathbb{N}} V_n$ converges(converges) $\Leftrightarrow (V_n)_n$converges $\implies$ $(V_n)_n$ is upper bounded $\implies$ $(U_n)_n$ is upper bounded too $\implies$ (U_n) converges $\implies \sum_{n\in\mathbb{N}} U_n$ converges

Example 4. ..



Examples: ① $v_n = \frac{1}{n^2-n}$; $\forall n \geq 2$, $U_n = \frac{1}{n^2}$ $n \in \mathbb{N}^*$

$v_n = \frac{1}{n-1} - \frac{1}{n}$, $\forall n \geq 2$

$V_n = 1 - \frac{1}{n}$, $\lim V_n = 1 \Rightarrow \sum v_n$ Cv $\Rightarrow \sum_{n\geq 1} \frac{1}{n^2}$ Cv

② $U_n = \frac{1}{n}$ and $v_n = \frac{1}{\sqrt{n}}$, since $U_n \leq v_n$, $\forall n \geq 1$ and $\sum_{n\geq 1} \frac{1}{n}$ div then $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverge

③ the Rimeann series $\sum 1/n^\alpha$ Cv, $\forall \alpha \geq 2$ and div $\forall \alpha \leq 1$

Corollary: equivalance criterion

Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be a positive series

1) if $\exists a > b > 0, \quad \exists N \in \mathbb{N}; \quad \forall n > N$

$0 \leq a \leq \dfrac{U_n}{V_n} \leq b$

$U_n = O(V_n)$

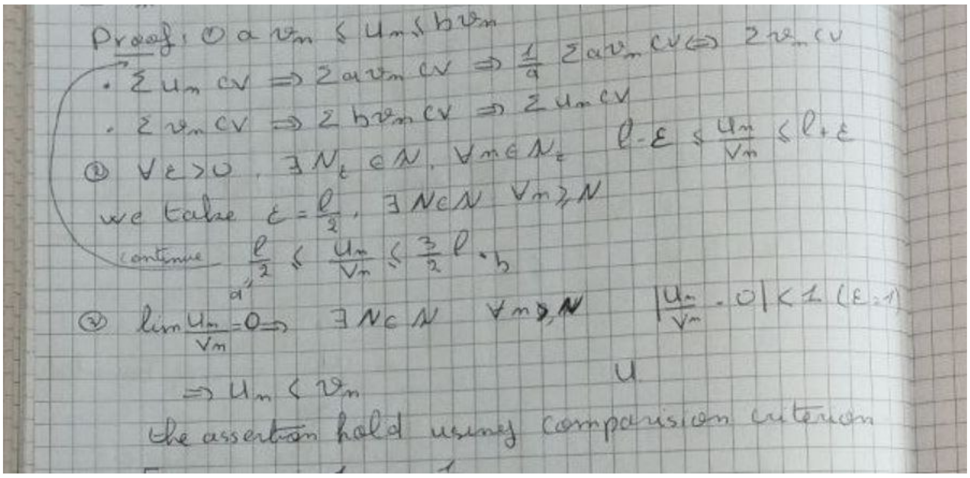
then $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ have the same nature

2) if $\lim \dfrac{U_n}{V_n} = l > 0$ then $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} U_n$ have the same nature also($U_n \sim V_n$)

3) if $U_n = O(V_n)$, that is $\lim \dfrac{U_n}{V_n} = 0$ then :

$\sum V_n$ converges $\implies$ $\sum U_n$ converges (also ,$\sum U_n$ div $\implies$ $\sum V_n$ div)

proof:



Proof: ① $a v_n \le u_n \le b v_n$

• $\sum u_n$ cv $\Rightarrow \sum a v_n$ cv $\Rightarrow \frac{1}{a} \sum a v_n$ cv $\Leftrightarrow \sum v_n$ cv

• $\sum v_n$ cv $\Rightarrow \sum b v_n$ cv $\Rightarrow \sum u_n$ cv

② $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$, $\forall n \in N_\varepsilon$ $l - \varepsilon \le \frac{u_n}{v_n} \le l + \varepsilon$

we take $\varepsilon = \frac{l}{2}$, $\exists N \in \mathbb{N}$ $\forall n \ge N$

continue $\underset{a}{\frac{l}{2}} \le \frac{u_n}{v_n} \le \frac{3}{2} l$. b

③ $\lim \frac{u_n}{v_n} = 0 \Rightarrow$ $\exists N \in \mathbb{N}$ $\forall n \ge N$ $\left|\frac{u_n}{v_n} - 0\right| < 1$ ($\varepsilon = 1$)

$\Rightarrow u_n < v_n$ U

the assertion hold using comparision criterion

Landau's symbols:

$$U_n = o(V_n) \text{ if } \lim \frac{U_n}{V_n} = 0$$

$$U_n \sim (V_n) \text{ if } \lim \frac{U_n}{V_n} = 1$$

$$U_n = O(V_n) \text{ if } (\frac{U_n}{V_n}) \text{ is bounded}$$

Example 5. *for* $\alpha > 0$

$$sin(\frac{1}{n^\alpha}) \sim \frac{1}{n^\alpha}$$

$$ln(1 + \frac{1}{n^\alpha}) \sim \frac{1}{n^\alpha}$$

Proposition D'Alenbert's and Cauchy's criterion

1)(D'Alenbert) If: If $\lim \frac{U_{n+1}}{U_n} = l \geq 0$, then $\sum U_n$ converges if $0 \leq l < 1$

$\sum U_n$ div if $l > 1$

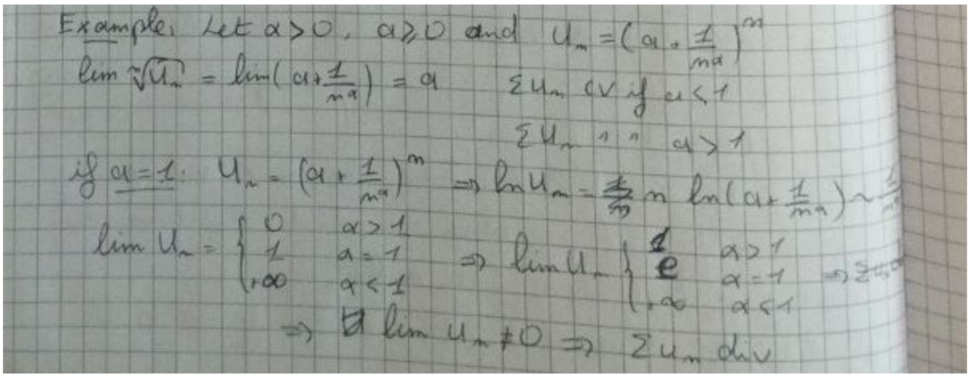
2)(Cauchy) If:

$\lim \sqrt[n]{U_n} = l \geq 0$, then $\sum U_n$ converges if $0 \leq l < 1$

$\sum U_n$ div if $l > 1$

3) if $l = 1$, we may use another criterion

Example 6. ..


Example: Let $\alpha > 0$, $a \geq 0$ and $U_n = \left(a + \frac{1}{n^\alpha}\right)^n$

$\lim \sqrt[n]{U_n} = \lim\left(a + \frac{1}{n^\alpha}\right) = a$ $\quad \sum U_n$ CV if $a < 1$

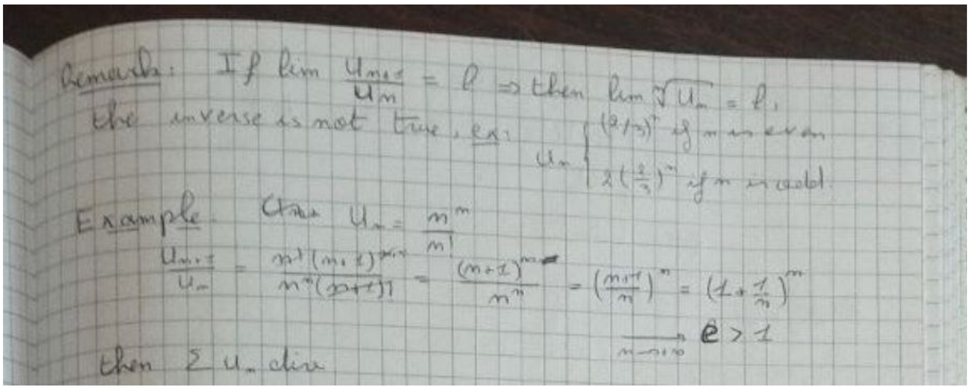
$\sum U_n$ " " $a > 1$

if $a = 1$: $U_n = \left(a + \frac{1}{n^\alpha}\right)^n \Rightarrow \ln U_n = n \ln\left(a + \frac{1}{n^\alpha}\right) \sim$

$\lim U_n = \begin{cases} 0 & \alpha > 1 \\ 1 & \alpha = 1 \\ +\infty & \alpha < 1 \end{cases} \Rightarrow \lim U_n \begin{cases} 1 & \alpha > 1 \\ e & \alpha = 1 \\ +\infty & \alpha < 1 \end{cases}$

$\Rightarrow \lim U_n \neq 0 \Rightarrow \sum U_n$ div

Remark + example


Remark: If $\lim \frac{U_{n+1}}{U_n} = \ell \Rightarrow$ then $\lim \sqrt[n]{U_n} = \ell$.

the inverse is not true, ex: $U_n \begin{cases} (2/3)^n & \text{if } n \text{ is even} \\ 2\left(\frac{2}{3}\right)^n & \text{if } n \text{ is odd.} \end{cases}$

Example: $U_n = \frac{n^n}{n!}$

$\frac{U_{n+1}}{U_n} = \frac{n!\,(n+1)^{n+1}}{n^n\,(n+1)!} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to +\infty]{} e > 1$

then $\sum U_n$ div

Theorem 2. *Comparision with an upper umpropre integral Let $f : [0, +\infty[\to [0, +\infty[$ be a decreasing continuous function and $U_n = f(n)$*

the series $\sum_{n\in\mathbb{N}} U_n = \sum_{n\in\mathbb{N}} f(n)$ have the same nature as the sequence$(\int_0^n f(x)dx)_{n\in\mathbb{N}}$ Furthermore:

$$\forall p \in \mathbb{N} \quad \int_{p+1}^{\infty} f(x)dx \leq R_p = U - U_p \leq \int_p^{\infty} f(x)dx$$

Proof Since f is decreasing on $[h, h+1]$, then:

$$f(h+1) \leq f(x) \leq f(h), \quad \forall x \in [h, h+1]$$

$$U_{h+1} \leq f(x) \leq U_n$$

$$U_{h+1} \int_h^{h+1} dx \leq \int_h^{h+1} f(x)dx \leq U_n \int_h^{h+1} dx$$

$$U_{h+1} \leq \int_h^{h+1} f(x)dx \leq U_n$$

$$\Longrightarrow \sum_{h=0}^{n-1} U_{h+1} \leq \sum_{h=0}^{n-1} \int_h^{h+1} f(x)dx \leq \sum_{h=0}^{n-1} U_n$$

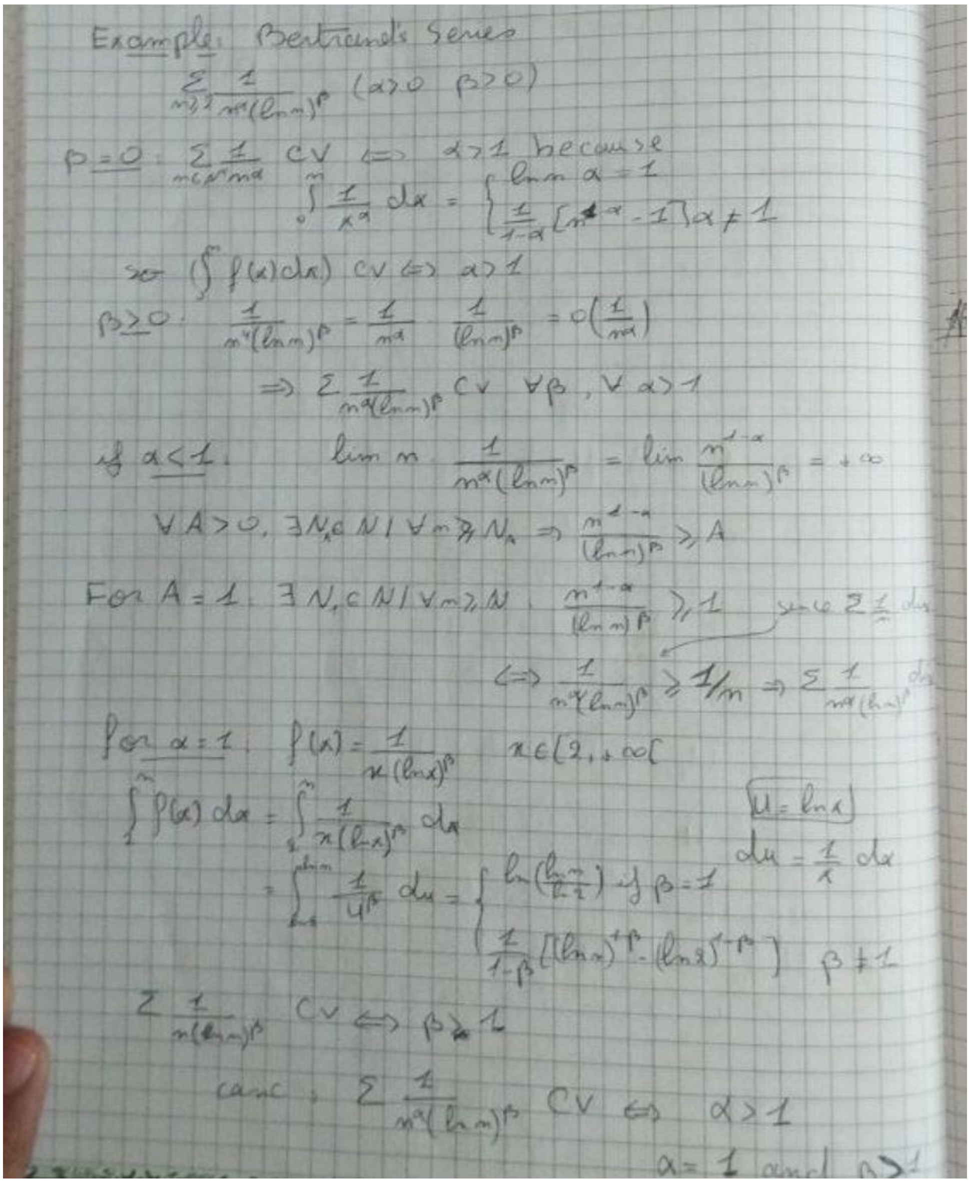
$$U_n - U_0 \leq \int_0^n f(x)dx \leq U_{n-1}$$

(U_n) and $(\int_0^n f(x)dx)$ are increasing so:

$$\begin{aligned}\sum U_n \text{ converges } &\Leftrightarrow (U_n) \text{ converges } \Leftrightarrow (U_{n-1})\text{converges}\\ &\Leftrightarrow (U_n) \text{ is upperbounded}\\ &\Longrightarrow \left(\int_0^n f(x)dx\right) \text{ is upperbounded}\\ &\Longrightarrow \left(\int_0^n f(x)dx\right) \text{ converges}\end{aligned}$$

$$\begin{aligned}\text{conversly: } \int_0^n f(x)dx \text{ converges}&\\ &\Leftrightarrow \int_0^n f(x)dx \text{ is upperbouned}\\ &\Longrightarrow (U_n) \text{ is upperbounded}\\ &\Longrightarrow (U_n) \text{ converges}\end{aligned}$$

Example 7. ..



Example: Bertrand's Series

$\sum_{n\geq 2} \frac{1}{n^\alpha(\ln n)^\beta}$ $(\alpha>0 \quad \beta>0)$

$\beta=0$: $\sum_{n\in\mathbb{N}^*} \frac{1}{n^\alpha}$ CV $\Leftrightarrow$ $\alpha>1$ because

$\int_0^n \frac{1}{x^\alpha}dx = \begin{cases} \ln n & \alpha=1 \\ \frac{1}{1-\alpha}[n^{1-\alpha}-1] & \alpha\neq 1\end{cases}$

so $(\int_0^n f(x)dx)$ CV $\Leftrightarrow$ $\alpha>1$

$\beta>0$: $\frac{1}{n^\alpha(\ln n)^\beta} = \frac{1}{n^\alpha}\frac{1}{(\ln n)^\beta} = o\left(\frac{1}{n^\alpha}\right)$

$\Rightarrow \sum \frac{1}{n^\alpha(\ln n)^\beta}$ CV $\forall\beta$, $\forall\alpha>1$

if $\alpha<1$: $\lim n\frac{1}{n^\alpha(\ln n)^\beta} = \lim \frac{n^{1-\alpha}}{(\ln n)^\beta} = +\infty$

$\forall A>0, \exists N_A\in\mathbb{N} / \forall n\geq N_A \Rightarrow \frac{n^{1-\alpha}}{(\ln n)^\beta}\geq A$

For $A=1$ $\exists N\in\mathbb{N} / \forall n\geq N$, $\frac{n^{1-\alpha}}{(\ln n)^\beta}\geq 1$ since $\sum\frac{1}{n}$ dv

$\Leftrightarrow \frac{1}{n^\alpha(\ln n)^\beta}\geq 1/n \Rightarrow \sum\frac{1}{n^\alpha(\ln n)^\beta}$ dv

For $\alpha=1$: $f(x)=\frac{1}{x(\ln x)^\beta}$ $x\in[2,+\infty[$

$\int_2^n f(x)dx = \int_2^n \frac{1}{x(\ln x)^\beta}dx$ $\quad\boxed{u=\ln x}$ $\quad du=\frac{1}{x}dx$

$= \int_{\ln 2}^{\ln n}\frac{1}{u^\beta}du = \begin{cases}\ln\left(\frac{\ln n}{\ln 2}\right) & \text{if } \beta=1\\ \frac{1}{1-\beta}[(\ln n)^{1-\beta}-(\ln 2)^{1-\beta}] & \beta\neq 1\end{cases}$

$\sum\frac{1}{n(\ln n)^\beta}$ CV $\Leftrightarrow$ $\beta>1$

conc: $\sum\frac{1}{n^\alpha(\ln n)^\beta}$ CV $\Leftrightarrow$ $\alpha>1$

$\alpha=1$ and $\beta>1$

3 Alternating series

Definition 4. *A numerical series* $\sum U_n$ *is said to be alternating if its general term* U_n *change sign infinitly many times*

Remark

$\sum U_n$ is alternating if there exist two subsequences

$(a_n) = (U_{\varphi_n})$ and $(b_n) = (U_{\psi_n})$st:

$(a_n) > 0$ and $(b_n) < 0$

Example 8. Examples: ① The Abel's series $\sum_{n\geq 1} \frac{\cos n\theta}{n^\alpha}$ and $\sum_{n\geq 1} \frac{\sin n\theta}{n^\alpha}$ are alternating series. in particular $\sum_{n\geq 1} \frac{(-1)^n}{n}$ is an alternating series.
② If $\sum a_n$ is a positive series then $\sum (-1)^n a_n$ is an alternating series, called Leibniz's series

Definition 5. *Absolute convergence and semi convergence*

A numerical series $\sum U_n$ *is said to be absolutely convergent if the positive series* $\sum_{n\in\mathbb{N}} |U_n|$ *converges*

Theorem 3. *if* $\sum U_n$ *is absolutely convergent then* $\sum U_n$ *converges*

Proof

Let $V_n = U_n + |U_n| > 0 \quad, \forall n \in \mathbb{N}$ and $0 \leq V_n \leq 2|U_n|$, by comparision criterion

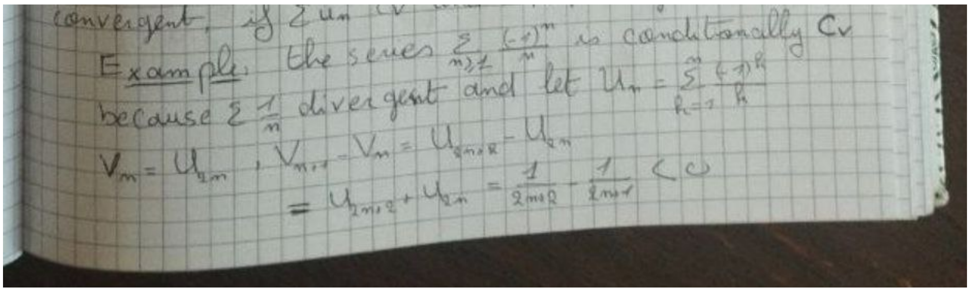
if $\sum U_n$ converges then $\sum V_n$ converges $\implies \sum U_n = \sum V_n - \sum |U_n|$ converges

Example 9. *The Abel's series* $\sum \dfrac{cos(n\theta)}{n^\alpha}$ *and* $\sum \dfrac{sin(n\theta)}{n^\alpha}$ *convergent absolutely*

$$\textit{if } \alpha > 1 \textit{ because } \left|\frac{cos(n\theta)}{n^\alpha}\right| \leq \frac{1}{n^\alpha} \textit{ and } \left|\frac{sin(n\theta)}{n^\alpha}\right| \leq \frac{1}{n^\alpha}$$

Definition 6. *We say that* $\sum U_n$ *conditionally (or semi) convergent, if* $\sum U_n$ *converges and* $\sum |U_n|$ *divergent*

Example 10. *..*



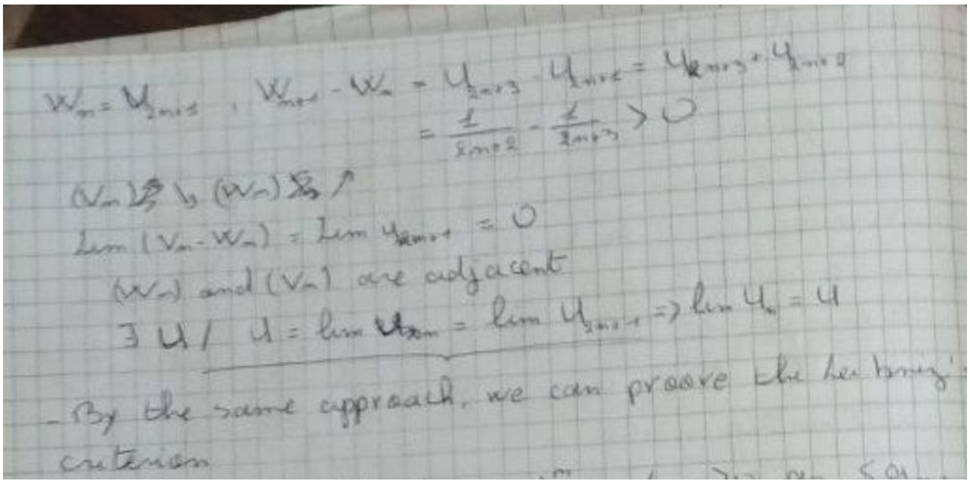
convergent, if $\sum u_n$ ...

Example: the series $\sum_{n\geq 1} \frac{(-1)^n}{n}$ is conditionally Cv

because $\sum \frac{1}{n}$ divergent and let $U_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$

$V_n = U_{2n}$, $V_{n+1} - V_n = U_{2n+2} - U_{2n}$

$= U_{2n+2} + U_{2n} = \frac{1}{2n+2} - \frac{1}{2n+1} < 0$



$W_n = U_{2n+1}$, $W_{n+1} - W_n = U_{2n+3} - U_{2n+1} = U_{2n+3} + U_{2n+2}$

$= \frac{1}{2n+2} - \frac{1}{2n+3} > 0$

(V_n) is $\searrow$ (W_n) is $\nearrow$

$\lim (V_n - W_n) = \lim u_{2n+1} = 0$

(W_n) and (V_n) are adjacent

$\exists\, U \,/\, U = \lim U_{2n} = \lim U_{2n+1} \Rightarrow \lim U_n = U$

- By the same approach, we can proove the Leibniz' criterion

Theorem 4. <u>*Leibniz*</u>

Let $U_n = (-1)^n a_n \mid a_n > 0 \quad, a_{n+1} \leq a_n \quad, \forall n \in \mathbb{N}$ *and* $\lim a_n = 0$ *,then:*

$\sum (-1)^n a_n$ *Converges*

Example 11. $\sum \dfrac{(-1)^n}{n^\alpha}$ *converges* $\Leftrightarrow \alpha > 0$

$$S = \sum_{n=3}^{+\infty} \frac{1}{n^2+3n+2} = \sum_{n=3}^{+\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=3}^{+\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 1/4$$

Theorem 5. *(Abel's criterion) If:* $U_n = a_n b_n$ *st:*

1) $a_n \geq 0 \quad ; a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$ *and* $\lim a_n = 0$

2)$\exists M \geq 0 \quad, \forall n \in \mathbb{N} \quad |\sum_{h=0}^{n} b_h| \leq M$ *then* $\sum U_n$ *converges*

Example 12. .

$\forall m \in \mathbb{N} \quad \left|\sum_{k=0} b_k\right| \leq M$ then $\sum u_n$ cv

Example: $\sum \frac{\cos n\theta}{n^\alpha}$ and $\sum \frac{\sin n\theta}{n^\alpha}$ cv

$\forall \theta \neq k\pi/2, \ k \in \mathbb{Z}, \ \forall \alpha > 0$

Let $a_n = \frac{1}{n^\alpha}$ and $b_n = \cos n\theta, \quad c_n = \sin n\theta$

$$A_n = b_n + i c_n = \cos n\theta + i \sin n\theta = (e^{i\theta})^n$$

$$\sum_{k=0}^{n} A_k = \left(\sum_{k=0}^{n} b_k\right) + i \sum_{k=0}^{n} c_k = \sum_{k=0}^{n} (e^{i\theta})^k = \frac{(e^{i\theta})^n - 1}{e^{i\theta} - 1}$$

$\forall \theta \neq 2\pi k$

$$= \cancel{e^{i\frac{\theta}{2}}} \frac{e^{i\frac{\theta}{2}(n+1)} \left[e^{i(\theta/2)(n+1)} - e^{-i\frac{\theta}{2}(n+1)}\right]}{e^{i\frac{\theta}{2}} \cdot \left[e^{i\theta/2} - e^{-i\theta/2}\right]}$$

$$= \frac{\cancel{2i} \sin\left(\frac{\theta}{2}(n+1)\right)}{\cancel{2i} \sin\frac{\theta}{2}} \cancel{e^{i\frac{\theta}{2}(n+1)}} \times e^{i\frac{\theta}{2}n}$$

$$\left|\sum_{k=0}^{n} b_k\right| = \left|\operatorname{Re}\left(\sum_{k=0}^{n} A_k\right)\right| = \left|\frac{\sin\frac{\theta}{2}(n+1) \cdot \cos\frac{\theta}{2}n}{\sin\theta/2}\right|$$

$$\leq \frac{1}{\left|\sin\frac{\theta}{2}\right|} = M$$

$$\left|\sum_{k=0}^{n} c_k\right| = \left|\operatorname{Im}\left(\sum_{k=0}^{n} A_k\right)\right| = \frac{\sin\frac{\theta}{2}(n+1) \cdot \sin\frac{\theta}{2}n}{\sin\theta/2} \leq \frac{1}{\left|\sin\frac{\theta}{2}\right|} = M$$

- Using Abel's criterion the series $\sum \frac{\cos n\theta}{n^\alpha}$ and $\sum \frac{\sin n\theta}{n^\alpha}$ cv.