

# Chapter 02: Vector Analysis

Notes from prof zeglaoui course

December 21, 2025

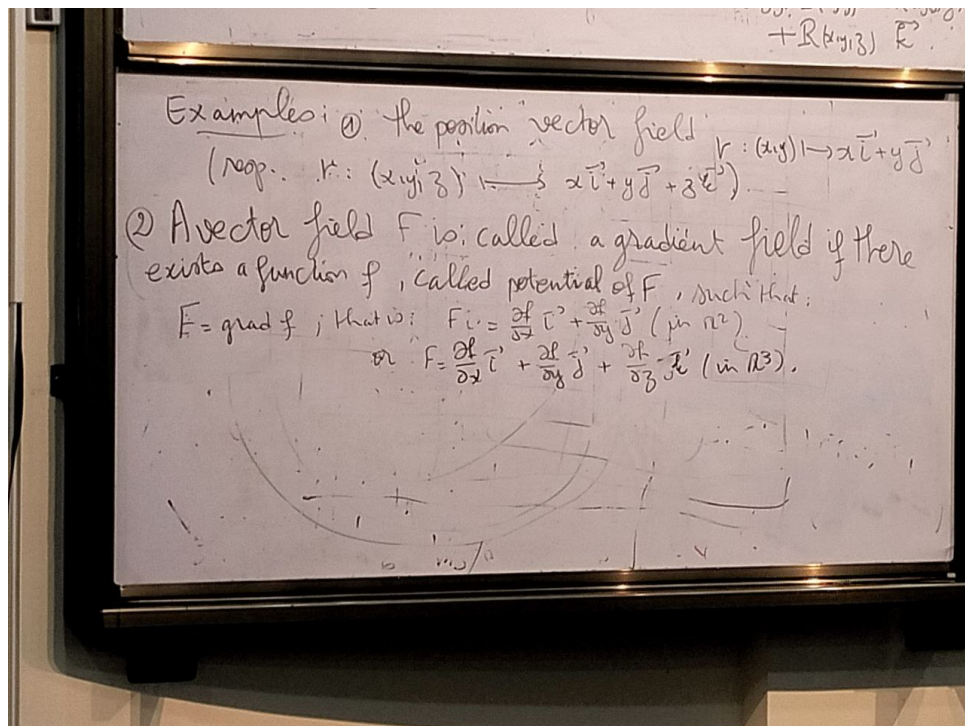
## 1 vector fields and scalar fields

**Definition 1.** 1) We call a scalar field defined on a region  $D \subset \mathbb{R}^2$  (respectively on  $\Omega \subset \mathbb{R}^3$ ), a real function defined on  $D$  (respectively  $\Omega$ )

2) a vector field  $F$  over  $D \subset \mathbb{R}^2$  (resp  $\Omega$ ) is a function  $F$  that assigns to any  $(x, y) \in D$  a vector  $F(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$  where  $M, N \in \mathcal{F}(D, \mathbb{R})$

3) a vector field  $F$  over  $\Omega \subset \mathbb{R}^3$  is a function  $F$  that assigns to any  $(x, y, z) \in \Omega$  a vector  $F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$

**Example 1.** 1) the position vector field



**Definition 2.** A vector field  $F$  is conservative if there exists a

potential function  $f$ ,  $F = \text{grad}(f)$

**Example 2.** ...

Example:  $F(x, y, z) = \frac{k}{(x^2 + y^2 + z^2)^{3/2}} (-x, -y, -z)$

$F = \text{grad } f$  where  $f(x, y, z) = \frac{-k}{\sqrt{x^2 + y^2 + z^2}}$

because:

$$\frac{\partial f}{\partial x}(x, y, z) = (-k)(2x) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} = \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial f}{\partial y}(x, y, z) = (-k)(2y) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} = \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial f}{\partial z}(x, y, z) = (-k)(2z) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-3/2} = \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} = \text{grad } f$$

**Theorem 1.** A vector field  $F = M\vec{i} + N\vec{j}$  over a convex region  $D \subset \mathbb{R}^2$  is conservative if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

*Proof.* if  $F$  is conservative

$$F = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \quad \text{where } f \text{ is of class } C^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0$$

Schwarz' theorem

$\Leftrightarrow$

$$f(x, y) = \int_0^1 [(x - x_0)M(x_0 + t(x - x_0), y_0 + t(y - y_0)) + (y - y_0)N(x_0 + t(x - x_0), y_0 + t(y - y_0))] dt$$

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \int_0^1 M(x(t), y(t)) dt \\
&\quad + \int_0^1 \left[ t(x-x_0) \frac{\partial M}{\partial x}(x(t), y(t)) + t(y-y_0) \frac{\partial N}{\partial x}(x(t), y(t)) \right] dt \\
&= \int_0^1 M(x(t), y(t)) dt + \int_0^1 \left[ t(x-x_0) \frac{\partial M}{\partial x}(x(t), y(t)) + t(y-y_0) \frac{\partial M}{\partial y}(x(t), y(t)) \right] dt \\
&= \int_0^1 \varphi'(t) dt, \text{ where } \varphi(t) = t M(x_0 + t(x-x_0), y_0 + t(y-y_0)) \\
&= \varphi(1) - \varphi(0) = M(x_0 + (x-x_0), y_0 + (y-y_0)) - 0 \\
&= M(x, y)
\end{aligned}$$

In the same way, we prove that  $\left(\frac{\partial f}{\partial y} = N\right)$

Schwarz's theorem:

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0$$

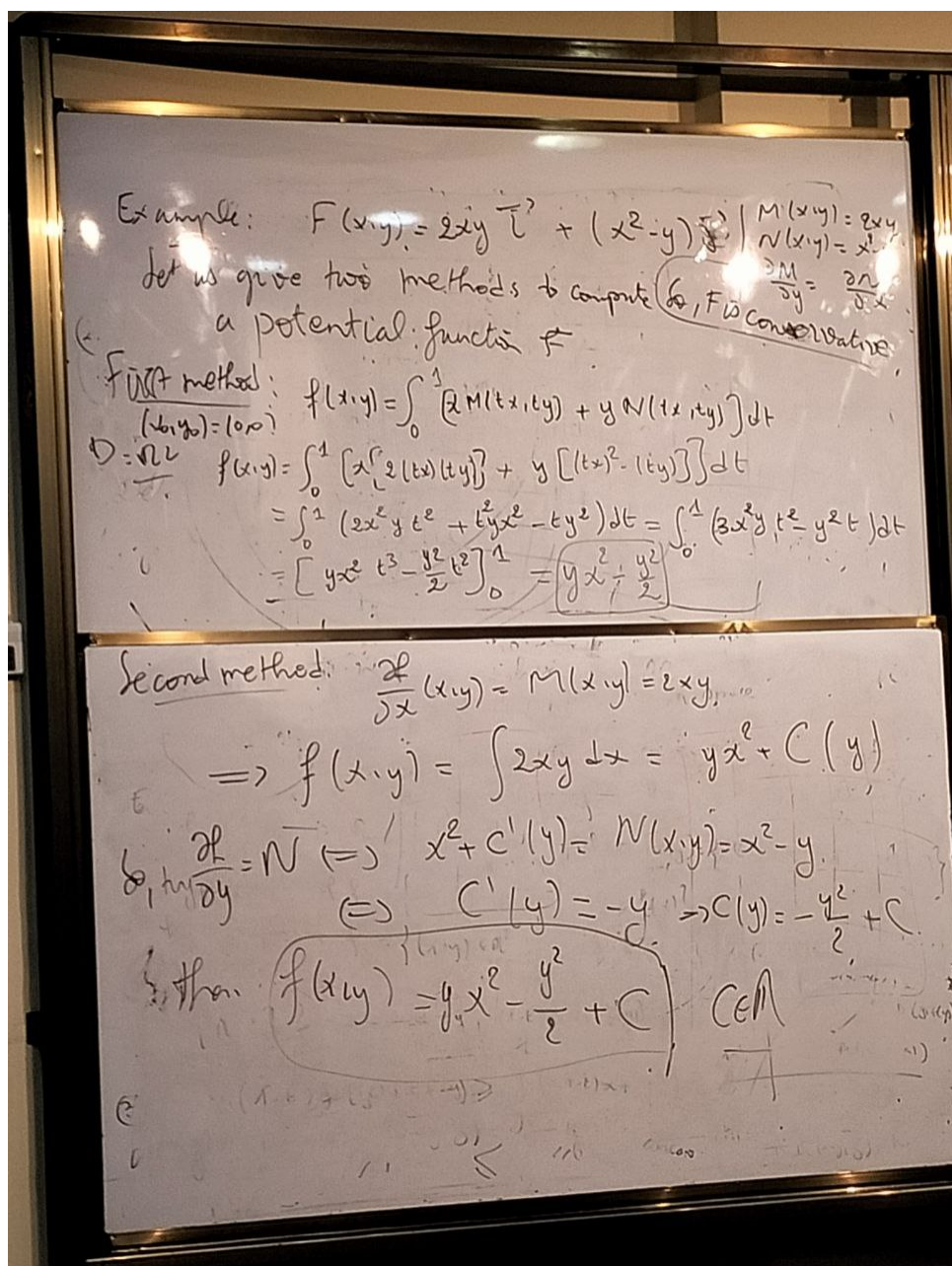
( $\Leftarrow$ )  $f(x,y) = \int_0^1 \left[ (x-x_0) M(x_0 + t(x-x_0), y_0 + t(y-y_0)) + (y-y_0) N(x_0 + t(x-x_0), y_0 + t(y-y_0)) \right] dt$

for  $(x_0, y_0) \in D$ ,  $\forall (x,y) \in D$

□

Example 3. .





**Definition 3.** Let  $F : (x, y, z) \rightarrow P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$  be a vector field over  $\Omega \subset \mathbb{R}^3$ . Such that the first partial derivatives of  $F$  exists

the curl of  $F$ , denoted by  $\text{Curl}(F)$  (Or  $\nabla \times F$ ) is the vector fields over  $\Omega$  defined by :

$$\text{Curl}(F) = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

We say that  $F$  is irrotationnal vector field if  $\text{Curl}(F) = 0$

**Example 4.** .

Example:  $F: (x, y, z) \mapsto 2xy \vec{i} + (x^2 + z^2) \vec{j} + 2yz \vec{k}$   
 $P(x, y, z) = 2xy$ ;  $Q(x, y, z) = x^2 + z^2$ ;  $R(x, y, z) = 2yz$   
 $\text{Curl}(F) = 0$  because:  $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$ ,  $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ ,  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$   
 So,  $F$  is irrotational  
 theorem

**Theorem 2.** Let  $F = P\vec{i} + Q\vec{j} + R\vec{k}$  be vector field over a convex domain  $\Omega \subset \mathbb{R}^3$  of class  $C^1$  then  $F$  is conservative  $\Leftrightarrow \text{Curl}(F) = 0$

**proof**

Proof ( $\Rightarrow$ ) If  $F = \vec{0}$  conservative, there exists

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$  of class  $C^2$  over  $\mathcal{U}$  such that

$$P = \frac{\partial f}{\partial x}; \quad Q = \frac{\partial f}{\partial y}; \quad R = \frac{\partial f}{\partial z} \quad (F = \text{grad } f)$$

$$\text{Curl}(F) = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \vec{i} + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \vec{j} + \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \vec{k}$$

Schwarz's

$$= 0 \vec{i} + 0 \vec{j} + 0 \vec{k}$$

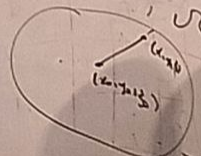
$$= \vec{0}$$

( $\Leftarrow$ ) Since  $\mathcal{U}$  is convex, let  $(x_0, y_0, z_0) \in \mathcal{U}$ ,

$\forall (x, y, z) \in \mathcal{U}, \forall t \in [0, 1]; (x(t), y(t), z(t)) \in \mathcal{U}$

where

$$\begin{cases} x(t) = x_0 + t(x - x_0) \\ y(t) = y_0 + t(y - y_0) \\ z(t) = z_0 + t(z - z_0) \end{cases} \quad t \in [0, 1]$$



$$df(x, y, z) = \int_0^1 \left[ (x - x_0) P(x(t), y(t), z(t)) + (y - y_0) Q(x(t), y(t), z(t)) + (z - z_0) R(x(t), y(t), z(t)) \right] dt$$



$\text{curl}(P) = 0$

$$\frac{\partial f}{\partial x}(x, y, z) = \int_0^1 \varphi'(t) dt; \text{ where } \varphi(t) = t P(x(t), y(t), z(t)).$$

$$\varphi'(t) = P(x(t), y(t), z(t)) + t \left[ (x-x_0) \frac{\partial P}{\partial x} + (y-y_0) \frac{\partial P}{\partial y} + (z-z_0) \frac{\partial P}{\partial z} \right]_{(x(t), y(t), z(t))}$$

$$= \varphi(1) - \varphi(0) = P(x, y, z) - 0$$

$$= P(x, y, z).$$

In the same way, we prove that  $\frac{\partial f}{\partial y} = Q$  and  $\frac{\partial f}{\partial z} = R$ .

**Example 5.**

Example:  $F(x, y, z) \mapsto 2xy \vec{i} + (x^2 + z^2) \vec{j} + 2yz \vec{k}$ ,  $\text{curl} F = 0$

$$f(x, y, z) = \int_0^1 [xP(tx, ty, tz) + yQ(tx, ty, tz) + zR(tx, ty, tz)] dt$$

$$= \int_0^1 [x(2(tx)(ty)) + y((tx)^2 + (tz)^2) + z(2(ty)(tz))] dt$$

$$f(x, y, z) = \int_0^1 [2x^2y t^2 + y(x^2 + z^2) t^2 + 2yz^2 t^2] dt$$

$$= \int_0^1 3y(x^2 + z^2) t^2 dt = \left[ y(x^2 + z^2) t^3 \right]_0^1$$

$$f(x, y, z) = y(x^2 + z^2)$$

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad \frac{\partial f}{\partial z} = R.$$

**Definition 4.** Divergence of vector field-Laplacian of scalar field

$$1) \operatorname{div}(F) = \nabla \cdot F \rightarrow \begin{cases} F = M\vec{i} + N\vec{j} \implies \operatorname{div}(F) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \\ F = P\vec{i} + Q\vec{j} + R\vec{k} \implies \operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \end{cases}$$

2) If  $F = \operatorname{grad}(f) \implies \operatorname{div}(F) = \operatorname{div}(\operatorname{grad}(f)) = \Delta(f)$  the Laplacian of  $f$

3)  $F$  is called divergence free if  $\operatorname{div}(F) = 0$

$$\text{if } f \text{ is called harmonic function if } \Delta(f) = 0 \rightarrow \begin{cases} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \end{cases}$$

**Remark:** If  $F$  is of class  $C^2$  vector field, then  $\operatorname{div}(\operatorname{curl}(F)) = 0$ , using Schwarz's theorem

## 2 Line integrals

### 2.1 Line integral of scalar field

Let  $f : D \rightarrow \mathbb{R}$  a continuous function over  $D \subset \mathbb{R}^2$  (resp :  $D \subset \mathbb{R}^3$ ), let

$$\Gamma : \begin{cases} [a, b] \rightarrow D \\ t \mapsto \gamma(t) \end{cases}$$

of  $C^1$  curve on  $D$

the line integral of  $f$  over  $\Gamma$  denoted by  $\int_{\Gamma} f dl$  is a real number

$$\int_{\Gamma} f dl = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt \quad \text{where } \|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

in particular if  $f \equiv 1$  ;

$$\int_{\Gamma} dl = l(\Gamma) \quad \text{length of } \Gamma = \int_a^b \|\gamma'(t)\| dt$$

$\int_{\Gamma} f dl$  is independant to parametrization  $([a, b], \gamma)$  of  $\Gamma$

2) If  $\Gamma$  is piece-wise  $C^1$  ; i.e  $\exists k \geq 2$

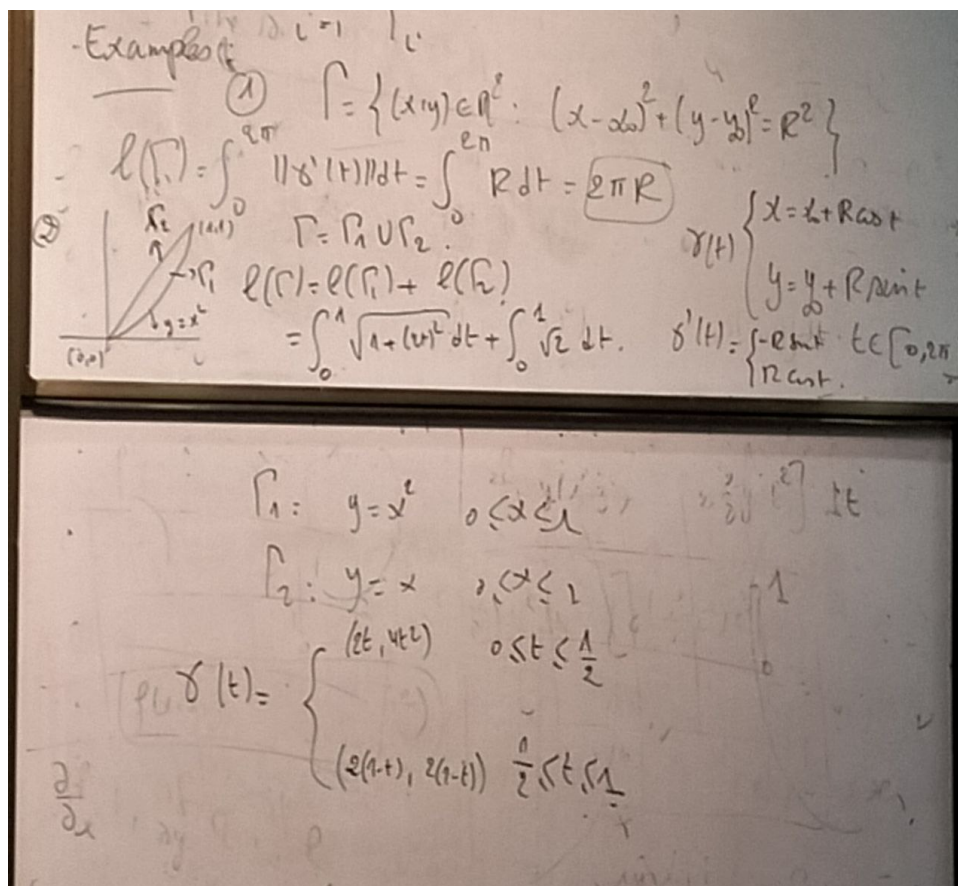
$$\forall i \in \{1, \dots, k\}; \Gamma_i \text{ is } C^1 \quad ; \Gamma_i \cap \Gamma_{i+1} = \{pt\} \quad \text{and } \Gamma_i \cap \Gamma_j = \emptyset \quad \text{if } |i - j| \geq 2$$

then:

$$\int_{\Gamma} f dl = \sum_{i=1}^k \int_{\Gamma_i} f dl$$



### Example 6.



## 2.2 Line integral of a vector field

Let  $\Gamma$  a curve parametrized by  $[a, b] \xrightarrow{\gamma} \gamma(t)$  and  $F$  a continuous vector field

the line integral of  $F$  over  $\Gamma$  ( or the circulation of  $F$  to  $\Gamma$ ) is :

$$\int_{\Gamma} F \cdot dr = \int_a^b [F(\gamma(t)) \cdot \gamma'(t)] dt$$

which is independant of the choice of parametrization

in  $\mathbb{R}^2$  if  $\gamma(t) = (x(t), y(t))$  and  $F = M\vec{i} + N\vec{j}$

$$\int_{\Gamma} F \cdot dr = \int_a^b [x'(t)M(x(t), y(t)) + y'(t)N(x(t), y(t))] dt$$

In  $\mathbb{R}^3$  if  $\gamma(t) = (x(t), y(t), z(t))$  and  $F = P\vec{i} + Q\vec{j} + R\vec{k}$

then:

$$\int_{\Gamma} F \cdot dr = \int_a^b [x'(t)P(x(t), y(t), z(t)) + y'(t)Q(x(t), y(t), z(t)) + z'(t)R(x(t), y(t), z(t))] dt$$

**Remark:**

We denoted by  $\int_{\Gamma} Mdx + Ndy = \int_{\Gamma} F \cdot dr$  in  $\mathbb{R}^2$  and by

$$\int_{\Gamma} Pdx + Qdy + Rdz = \int_{\Gamma} F \cdot dr \text{ in } \mathbb{R}^3$$

this is the differential representation of the integral of F along to  $\Gamma$

**Remark:** If  $F = \text{grad}(f)$  is conservative vector field then:

$$\begin{cases} df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = F \cdot dr & | dr = dx\vec{i} + dy\vec{j} \\ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = F \cdot dr & | dr = dx\vec{i} + dy\vec{j} + dz\vec{k} \end{cases}$$

**Example 7. ...**

Handwritten solution for Example 7:

$$\begin{aligned}
 &\underline{\text{Example:}} \quad F: (x, y, z) \longrightarrow -\frac{1}{2}x\vec{i} - \frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k} \\
 &\quad \gamma[0, 2\pi] \longrightarrow \mathbb{R}^3 \\
 &\quad t \longrightarrow \gamma(t) = (\cos t, \sin t, t) \\
 &\quad \gamma'(t) = -\sin t \vec{i} + \cos t \vec{j} + \vec{k} = \cos t \vec{i} - \sin t \vec{j} + t \vec{k} \\
 &\quad F(\gamma(t)) = -\frac{1}{2}\cos t \vec{i} - \frac{1}{2}\sin t \vec{j} + \frac{1}{4}\vec{k} \\
 &\Rightarrow F(\gamma(t)) \cdot \gamma'(t) = \left(-\frac{1}{2}\cos t\right)(-\sin t) + \left(-\frac{1}{2}\sin t\right)(\cos t) + \left(\frac{1}{4}\right)(1) \\
 &\quad = \frac{1}{4} \\
 &\int_{\Gamma} F \cdot dr = \int_0^{2\pi} \frac{1}{4} dt = \boxed{\frac{\pi}{2}}
 \end{aligned}$$

**Example 8. ...**

Example 2:

$$\vec{r}(t) = 3 \cos(t) \vec{i} + 3 \sin(t) \vec{j}$$

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$$

$$\int_{\Gamma} (y^3 dx + (x^3 + 2xy^2) dy) = \int_{\Gamma} M dx + N dy \quad \begin{cases} M(x, y) = y^3 \\ N(x, y) = x^3 + 2xy^2 \end{cases}$$

$$\begin{cases} x(t) = 3 \cos t \Rightarrow x'(t) = -3 \sin t \\ y(t) = 3 \sin t \Rightarrow y'(t) = 3 \cos t \end{cases}$$

$$= \int_0^{2\pi} [x'(t) M(x(t), y(t)) + y'(t) N(x(t), y(t))] dt$$

$$= \int_0^{2\pi} [-3 \sin t (3 \sin t)^3 + 3 \cos t ((3 \cos t)^3 + 2(3 \cos t)(3 \sin t)^2)] dt$$

$$= \int_0^{2\pi} [-81 \sin^4 t + 81 \cos^4 t + 62 \cos^2 t \sin^2 t] dt$$

$$= 81 \int_0^{2\pi} (\cos^4 t - \sin^4 t + \frac{2}{3} \cos^2 t \sin^2 t) dt$$

$$= 81 \int_0^{2\pi} \left[ \cos(2t) + \frac{1}{2} (\sin(2t))^2 \right] dt$$

$$= \frac{81}{4} \int_0^{2\pi} [4 \cos(2t) + 1 - \cos(4t)] dt = \frac{81}{2} \pi$$

the parametrization of a curve  
• each point  $(x, f(x))$

**Theorem 3.** Let  $F = \text{grad}(f)$  be a conservative vector field over  $D : \Gamma \subset D$ , parametrized by  $([a, b], \gamma)$ , then:

$$\int_{\Gamma} F \cdot d\vec{r} = f(\gamma(b)) - f(\gamma(a))$$

in particular, if  $\Gamma$  is a closed path ( $\gamma(a) = \gamma(b)$ ) then :  $\oint F \cdot d\vec{r} = 0$

**Proof**

$$F = \text{grad}(f) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}; \quad \begin{cases} \gamma(t) = x(t) \vec{i} + y(t) \vec{j} \\ \gamma'(t) = x'(t) \vec{i} + y'(t) \vec{j} \end{cases}$$

$$F(\gamma(t)) \cdot \gamma'(t) = x'(t) \frac{\partial f}{\partial x}(x(t), y(t)) + y'(t) \frac{\partial f}{\partial y}(x(t), y(t)) = (f \circ \gamma)'(t) \quad | \quad (f \circ \gamma)(t) = f(x(t), y(t))$$

$$\begin{aligned}\int_{\Gamma} F \cdot dr &= \int_{\Gamma} \text{grad}(f) \cdot dr = \int_a^b (f \circ \gamma)'(t) dt = [f(\gamma(t))]_a^b \\ &= f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

**Theorem 4.** (Green's theorem)

Let  $D$  be a simply connected and connected region of  $\mathbb{R}^2$  with piece-wise  $C^1$  boundary simple curve  $\Gamma = \partial D$

and  $F = M\vec{i} + N\vec{j}$  be a  $C^1$  vector field over  $D$ , Then :

$$\oint_{\Gamma} F \cdot dr = \oint_{\Gamma} M dx + N dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) (x, y) dx dy$$

**Example 9.** ..

Example:

$$M(x, y) = y^2, \quad N(x, y) = x^2 + 2x$$

$$x(t) = 3 \cos t$$

$$y(t) = 3 \sin t$$

$$\begin{aligned}\oint_{\Gamma} M dx + N dy &= \iint_D (3x^2 + 2y^2 - 3y^2) dx dy \\ &= \iint_D (3x^2 - y^2) dx dy \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \\ &= \int_0^{2\pi} \int_0^3 r (3r^2 \cos^2 \theta - r^2 \sin^2 \theta) dr d\theta \\ &\quad \begin{matrix} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{matrix} \\ &= \left( \int_0^3 r^3 dr \right) \left( \int_0^{2\pi} (3 \cos^2 \theta - 1 + \cos^2 \theta) d\theta \right) \\ &= \left[ \frac{r^4}{4} \right]_0^3 \int_0^{2\pi} (4 \cos^2 \theta - 1) d\theta \\ &= \frac{81}{4} \int_0^{2\pi} (1 + \cos^2 \theta) d\theta \\ &= \frac{81}{4} \cdot 2\pi = \frac{81}{2} \pi\end{aligned}$$



**Theorem 5.** Corollary Line Integral area

if  $D$  is a plane region bounded by a piece-wise simple closed curve  $\Gamma = \partial D$  oriented counterclockwise, then  $A(D) = \frac{1}{2} \oint_{\Gamma=\partial D} (x dy - y dx)$

**Proof**

$$M(x, y) = -\frac{1}{2}y; \quad N(x, y) = \frac{1}{2}x$$

by Green's theorem;

$$\oint_{\partial D} M dx + N dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) (x, y) dx dy$$

**Example 10.** .

Example,  $D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$ ;  $\Gamma = \partial D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$   
 $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$   
 $\gamma(t) = (x(t), y(t)) = (a \cos t, b \sin t)$   
 $A(D) = \frac{1}{2} \int_0^{2\pi} [(a \cos t)(b \cos t) - (b \sin t)(-a \sin t)] dt = \frac{ab}{2} \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt = \pi ab$

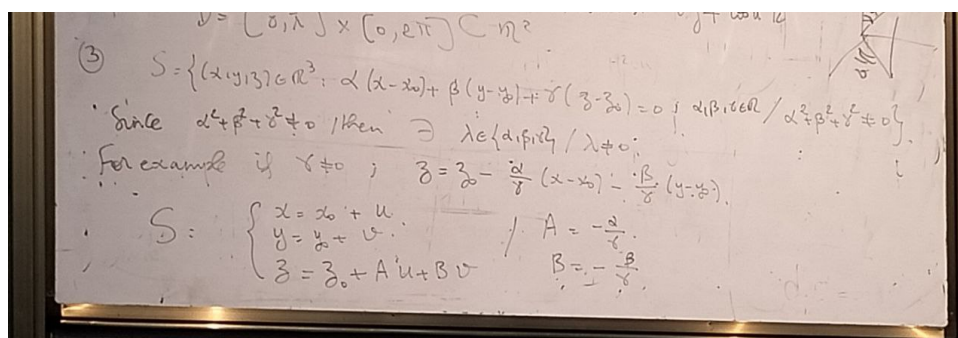
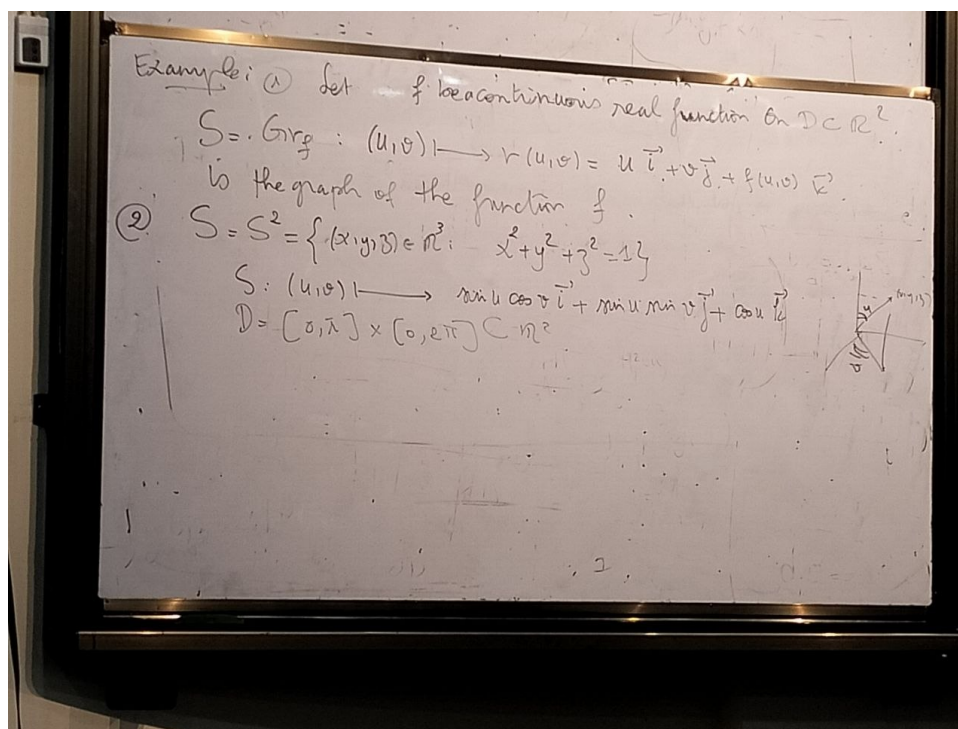
### 3 Surface integral

#### 3.1 Parametric surface

**Definition 5.** Let  $x, y, z$  be a three functions of  $u$  and  $v$ , that are continuous on  $D \subset \mathbb{R}^2$ . The set  $S$  of points  $(x, y, z)$  given by  $r(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$  be a parametrized surface. The following equations:

$$S : \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \text{ are called the parametric equation of } S$$

**Example 11.** ..



**Definition 6.** Let  $(D, r)$  be a parametrization of the surface  $S$ .

We say that  $S$  is smooth (or  $C^1$ ) if  $x, y, z$  are  $C^1$  or the vector field  $V$  is  $C^1$  on  $D$

we say that  $S$  is piece-wise smooth. if  $\exists k \in \mathbb{N}^*$ ;  $S = S_1 \cup S_2 \cup \dots \cup S_k$  |  $S_i \cup S_j$  is at most a curve and  $S_i$  is smooth

Let  $S$  be smooth surface. The tangent plane of  $S$  at a point  $(x_0, y_0, z_0) \in S$  is the affine space

$$T_{(x_0, y_0, z_0)} S : \begin{cases} x = x_0 + (u - u_0) \frac{\partial x}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial x}{\partial v}(u_0, v_0) \\ y = y_0 + (u - u_0) \frac{\partial y}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial y}{\partial v}(u_0, v_0) \\ z = z_0 + (u - u_0) \frac{\partial z}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial z}{\partial v}(u_0, v_0) \end{cases}$$

$$: r : (u, v) \mapsto r(u_0, v_0) + (u - u_0) \frac{\partial r}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial r}{\partial v}(u_0, v_0)$$

$$r(u_0, v_0) = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k} \quad ; \quad \frac{\partial r}{\partial u} = r_u \quad ; \quad \frac{\partial r}{\partial v} = r_v$$

the normal vector field of  $S$  ; denoted by  $N$  is :

$$N = r_u \times r_v = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$$

$$N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\text{So; } T_{(x_0, y_0, z_0)} S \perp N(x_0, y_0, z_0), \quad \|N(x_0, y_0, z_0)\| = \left\| \frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0) \right\|$$

**Example 12. ...**

Examples: ①  $S = \pi G_{m3}$

$$r(u, v) = u \vec{i} + v \vec{j} + f(u, v) \vec{k}$$

$$r_u(u, v) = \frac{\partial r}{\partial u}(u, v) = \vec{i} + \frac{\partial f}{\partial u}(u, v) \vec{k}, \quad r_v(u, v) = \frac{\partial r}{\partial v}(u, v) = \vec{j} + \frac{\partial f}{\partial v}(u, v) \vec{k}$$

$$N(x, y, z) = N(u, v) = r_u(u, v) \times r_v(u, v) = \left( \vec{i} + \frac{\partial f}{\partial u}(u, v) \vec{k} \right) \times \left( \vec{j} + \frac{\partial f}{\partial v}(u, v) \vec{k} \right)$$

$$= \vec{i} \times \vec{j} + \frac{\partial f}{\partial v}(u, v) \vec{i} \times \vec{k} + \frac{\partial f}{\partial u}(u, v) \vec{k} \times \vec{j}$$

$$= \vec{k} - \frac{\partial f}{\partial v}(u, v) \vec{j} - \frac{\partial f}{\partial u}(u, v) \vec{i} = -\frac{\partial f}{\partial u}(u, v) \vec{i} - \frac{\partial f}{\partial v}(u, v) \vec{j} + \vec{k}$$

$$\|N(x, y, z)\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v)\right)^2 + \left(\frac{\partial f}{\partial v}(u, v)\right)^2}$$

②  $S = S^2$

$$N(x_0, y_0, z_0) = r(x_0, y_0, z_0) = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$$

**Definition 7.** the area of a smooth surface

Let  $S : r : (u, v) \mapsto r(u, v)$  over  $D \subset \mathbb{R}^2$  be a smooth parametrized surface the area of  $S$  is

$$A(S) = \iint_D \|r_u(u, v) \times r_v(u, v)\| = \iint_D d\sigma$$

$$= \iint_D \|N(u, v)\| du dv$$

**Example 13. ...**

Example:  $S = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + (z-3)^2 = R^2\}$ .  $R > 0$

$r(u, v) = (1 + R \cos u \cos v) \vec{i} + (2 + R \sin u \cos v) \vec{j} + (3 + R \sin u) \vec{k}$

$D = [0, \pi] \times [0, 2\pi]$

$N(x, y, z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R \cos u \cos v & R \sin u \cos v & -R \sin u \\ -R \sin u \cos v & R \cos u \cos v & 0 \end{vmatrix} = R^2 \sin u \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \sin u \cos v & -\sin u \\ -\sin v & \cos v & 0 \end{vmatrix}$

$dS = (R^2 \sin u) du dv$

$A(S) = \iint_D dS = R^2 \left( \int_0^\pi \sin u du \right) \left( \int_0^{2\pi} dv \right) = R^2 \left[ -\cos u \right]_0^\pi \left[ v \right]_0^{2\pi} = 4\pi R^2$

**Remark:**

$$S = Gr_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } z = f(x, y)\}$$

$$S : r(u, v) = u\vec{i} + v\vec{j} + f(u, v)\vec{k}$$

$$d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v)\right)^2 + \left(\frac{\partial f}{\partial v}(u, v)\right)^2}$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v)\right)^2 + \left(\frac{\partial f}{\partial v}(u, v)\right)^2} du dv$$

## 3.2 Surface integral

**Definition 8.** Let  $S : r(u, v) \mapsto x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$  be a parametrized smooth surface and  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  a function continuous over  $S$ , denoted by  $\int_S f d\sigma$  is the real number

$$\iint f d\sigma = \iint_D f(r(u, v)) \|r_u(u, v) \times r_v(u, v)\| du dv$$

$$\iint f d\sigma = \iint_D f(x(u, v) + y(u, v) + z(u, v)) \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right\| du dv$$

**Remark:**



1) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : z = \varphi(x, y)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(x, y, \varphi(x, y)) \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x, y)\right)^2 + \left(\frac{\partial \varphi}{\partial y}(x, y)\right)^2} dx dy \end{aligned}$$

2) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : y = \varphi(x, z)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(x, \varphi(x, z), z) \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x, z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(x, z)\right)^2} dx dz \end{aligned}$$

3) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : x = \varphi(y, z)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(\varphi(y, z), y, z) \sqrt{1 + \left(\frac{\partial \varphi}{\partial y}(y, z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(y, z)\right)^2} dy dz \end{aligned}$$

4)

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = c, (x, y) \in D\}; A(S) = A(D) = \iint_D dx dy$$

**Example 14.** 1)...

Examples: ①  $\iint_S (y^2 + 2yz) \, dS$ , where  $S$  is the first octant of the plane  $2x + y + 2z = 2$ .

$D = \{(x,y) \in \mathbb{R}^2 : y + 2z \leq 2, x \geq 0, y \geq 0, z \geq 0\}$

$x = 1 - \frac{y}{2} - z$

$\iint_S f \, dS = \iint_D f(x(y,z), y, z) \sqrt{1 + \left(\frac{-1}{2}\right)^2 + (-1)^2} \, dy \, dz$

$= \iint_D (y^2 + 2yz) \sqrt{\frac{9}{4}} \, dy \, dz = \frac{3}{2} \int_0^2 \left[ \int_0^{2-y} (y^2 + 2yz) \, dz \right] dy$

$= \frac{3}{2} \int_0^2 \left[ y^2 \left(1 - \frac{y}{2}\right) + y \left(1 - \frac{y}{2}\right)^2 \right] dy$

$= \frac{3}{2} \int_0^2 \left(1 - \frac{y}{2}\right) \left(y^2 + y \left(1 - \frac{y}{2}\right)\right) dy = \frac{3}{2} \int_0^2 \left(1 - \frac{y}{2}\right) \left(y + \frac{y^2}{2}\right) dy$

$= \frac{3}{2} \int_0^2 y \left(1 - \frac{y}{2}\right) \left(1 + \frac{y}{2}\right) dy = \frac{3}{2} \int_0^2 y \left(1 - \frac{y^2}{4}\right) dy = \frac{3}{2} \left[ \frac{y^2}{2} - \frac{y^4}{16} \right]_0^2$

$= \frac{3}{2} \left( \frac{4}{2} - \frac{16}{16} \right) = \frac{3}{2}$

2) ..

② A cone-shaped surface lamina is given by  $z = 4 - 2\sqrt{x^2 + y^2}$ ,  $0 \leq z \leq 4$ . At each point of  $S$ , the mass density is proportional to the distance between this point and the  $z$ -axis.

$\rho: (x, y, z) \mapsto k\sqrt{x^2 + y^2}$

The mass of  $S$ :  $m = \iint_S \rho \, dS = \iint_D k\sqrt{x^2 + y^2} \sqrt{1 + \left(\frac{-2x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{-2y}{\sqrt{x^2 + y^2}}\right)^2} \, dx \, dy$

$0 \leq z \leq 4$

$0 \leq 4 - 2\sqrt{x^2 + y^2} \leq 4$

$-4 \leq -2\sqrt{x^2 + y^2} \leq 0$

$0 \leq \sqrt{x^2 + y^2} \leq 2$

$= k \iint_D \sqrt{x^2 + y^2} \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} \, dx \, dy$

$= k \sqrt{5} \iint_D \sqrt{x^2 + y^2} \, dx \, dy = k \sqrt{5} \int_0^{2\pi} \int_0^2 r^2 \, dr \, d\theta = \frac{25k\sqrt{5}}{3} \left[ \frac{r^3}{3} \right]_0^2$

$= \frac{16\pi k\sqrt{5}}{3}$

### 3.3 Flux of vector field over surface

Definition 9. Unit normal vector fields - Oriented surfaces

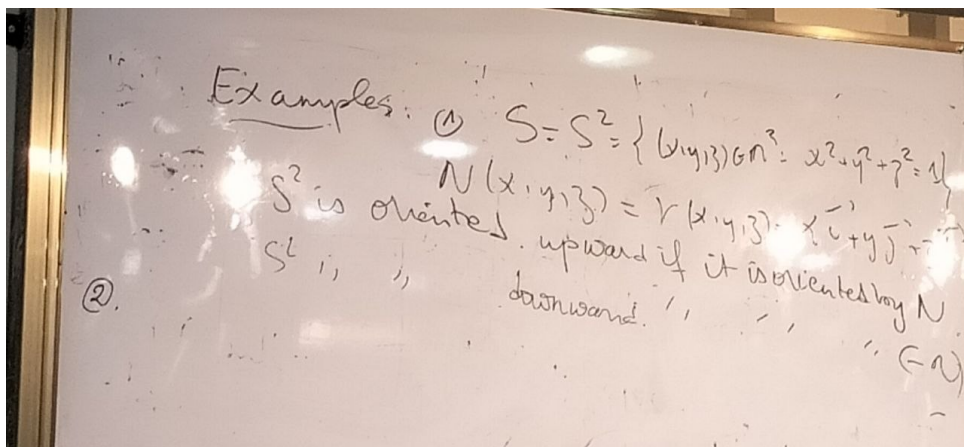
Let  $S$  be a smooth parametrized surface

$S$  is regular if  $\|r_u \times r_v\|(u, v) \neq 0, \forall (u, v) \in D \mid D$  connected

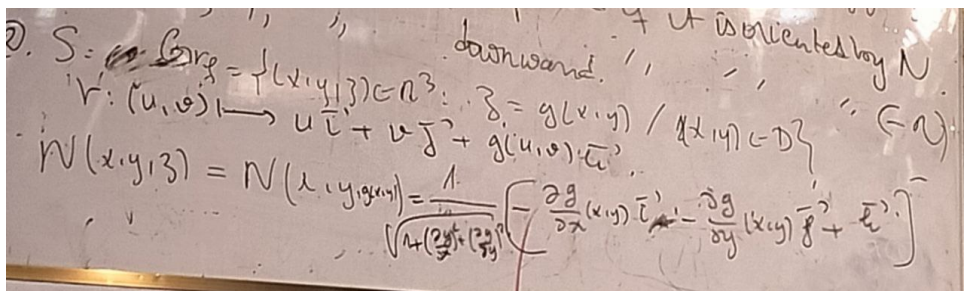
So,  $S$  can be oriented by  $N(x, y, z) = \frac{1}{\|r_u \times r_v\|(u, v)} ((r_u \times r_v)(u, v)) \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$

or,  $S$  can be also oriented by  $(-N)(x, y, z) = \frac{1}{\|r_v \times r_u\|(v, u)} ((r_u \times r_v)(u, v)) \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$

**Example 15. 1)**



2)



**Definition 10. Flux of vector field over regular parametrized**

Let  $F = P\vec{i} + Q\vec{j} + R\vec{k}$  be a continuous vector field over an open set  $\Delta \subset S$

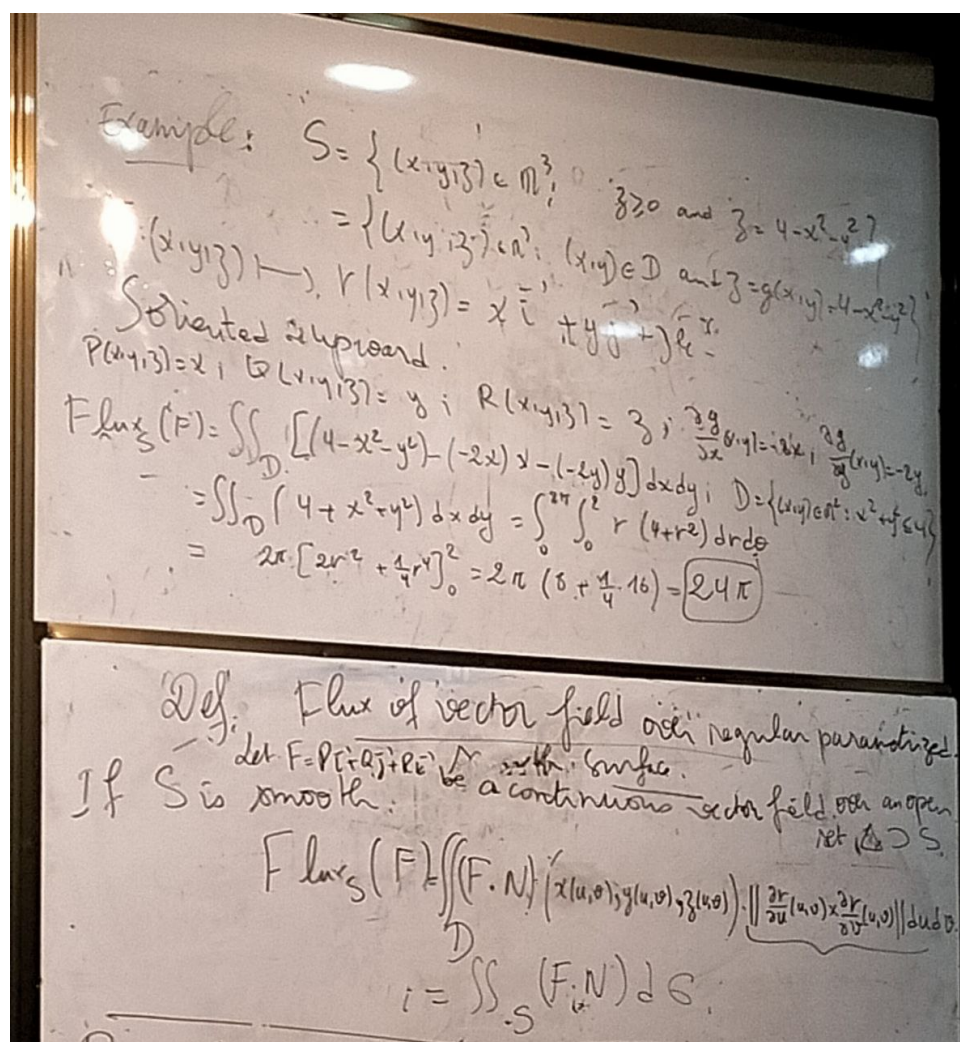
If  $S$  is smooth

$$\begin{aligned} \text{Flux}_S(F) &= \iint_D (F \cdot N)(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right\| du dv \\ &= \iint_S (F \cdot N) d\sigma \end{aligned}$$

**Remark**  $F = P\vec{i} + Q\vec{j} + R\vec{k}$   $S = Gr_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D ; z = g(x, y)\}$

$$Flux_S(f) = \begin{cases} \iint_D \left[ R - \frac{\partial g}{\partial x}(x, y)P - \frac{\partial g}{\partial y}(x, y)Q \right] (x, y, g(x, y)) dx dy & \text{if } S \text{ is oriented upward} \\ \iint_D \left[ \frac{\partial g}{\partial x}(x, y)P((x, y, g(x, y))) + \frac{\partial g}{\partial y}(x, y)Q(x, y, g(x, y)) - R(x, y, g(x, y)) \right] dx dy & \text{if } S \text{ is oriented downward} \end{cases}$$

**Example 16. 1)**



**Remark**

$$\iint_S (F \cdot N) d\sigma = \iint_D F(x(u, v), y(u, v), z(u, v)) \cdot \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right)(u, v) du dv$$



$$\iint_D \begin{vmatrix} P(x(u,v), y(u,v), z(u,v)) & Q(x(u,v), y(u,v), z(u,v)) & R(x(u,v), y(u,v), z(u,v)) \\ \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) & \frac{\partial z}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) & \frac{\partial z}{\partial v}(u,v) \end{vmatrix} du dv$$

**Example 17.** 1 ..

Example:  $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}$ ;  $F = \frac{kq}{\|r\|^3} r$

$N = \frac{1}{R} r$

$F \cdot N = \frac{kq}{\|r\|^3} r \cdot \left(\frac{1}{R} r\right) = \frac{kq}{R^2}$

$F \cdot N|_S = \frac{kq}{R^3} \cdot \frac{1}{R} (\|r\|^2) = \frac{kq}{R^4} R^2 = \frac{kq}{R^2}$

$\text{Flux}_S(F) = \iint_S \frac{kq}{R^2} dS = \frac{kq}{R^2} \iint_S dS = \frac{kq}{R^2} \mathcal{A}(S) = \frac{kq}{R^2} (4\pi R^2) = 4\pi kq$

There are two methods to compute the flux of vector field over a surface  $S$ .

1) if  $S$  is boundary of a solid domain  $\Omega \subset \mathbb{R}^3$ , ( $i - eS = \partial D$ ),

( $S$  is closed. without boundary  $\partial S = \phi$ ) we can use the divergence theorem

2) If  $S$  is not closed,  $C = \partial S$  is piece-wise curve we can use Stokes's theorem

**Theorem 6. Divergence theorem (Ostrogradsky's theorem)**

Let  $\Omega$  be a closed solid bounded region in  $\mathbb{R}^3$  by

closed smooth piece-wise regular surface  $S = \partial\Omega$  oriented outward from  $\Omega$

If  $F = P\vec{i} + Q\vec{j} + R\vec{k}$  is of class  $C^1$  over  $\Omega$  then;

$$\text{Flux}_S(F) = \iiint_{\Omega} \text{div}(F) dv = \iiint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) (x, y, z) dx dy dz$$

**Theorem 7. Stokes's theorem** Let  $S$  be oriented by  $N$ ;  $C = \partial S$  is a piece-wise smooth simple closed curve with positive orientation. If  $F = P\vec{i} + Q\vec{j} + R\vec{k}$  is  $C^1$  vector field over a open region  $\Delta \subset S$ , then:

$$\oint_{C=\partial S} F \cdot dr = \iint_S (\text{Curl}(F) \cdot N) d\sigma = \text{Flux}_S(\text{Curl}(F))$$

**Example 18.** 1)

Example:  $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, 2x+2y+z \leq 6\}$

$\text{Flux}_S(F) = \iiint_{\Omega} \text{div}(F) dV = \iiint_{\Omega} (1+2y+1) dxdydz$   $F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

$$= \int_0^3 \left[ \int_0^{3-x} \left[ \int_0^{6-2x-2y} (2+2y) dz \right] dy \right] dx$$

$$= 2 \int_0^3 \left[ \int_0^{3-x} (1+y)(6-2x-2y) dy \right] dx$$

$$= 4 \int_0^3 \left[ \int_0^{3-x} (3-x + (2-x)y - y^2) dy \right] dx$$

$$= 4 \int_0^3 \left[ (3-x)y + (2-x) \frac{y^2}{2} - \frac{1}{3} y^3 \right]_0^{3-x} dx$$

$$= 4 \int_0^3 (3-x)^2 \left[ 1 + \frac{1}{2}(2-x) - \frac{1}{3}(3-x) \right] dx$$

$$= 4 \int_0^3 (3-x)^2 \left( \frac{1}{2} + \frac{1}{6}(3-x) \right) dx$$

$$= \frac{4}{6} \int_0^3 \left[ 3(3-x)^2 + (3-x)^3 \right] dx = \frac{2}{3} \int_0^3 (3u^2 + u^3) du$$

$u = 3-x$   
 $du = -dx$   
 $x=0 \rightarrow u=3$   
 $x=3 \rightarrow u=0$

$$= \frac{2}{3} \left[ u^3 + \frac{1}{4} u^4 \right]_0^3 = 2 \left[ 9 + \frac{27}{4} \right] = 18 + \frac{27}{2} = \frac{63}{2}$$

$S = S_1 \cup S_2 \cup S_3 \cup S_4$

$S_1: z=0, P_1 = \{x \geq 0, y \geq 0, x+y \leq 3\} \quad N_1 = -\vec{k} \quad F \cdot N_1|_{S_1} = 0$

$S_2: y=0, P_2 = \{x \geq 0, z \geq 0, 2x+z \leq 6\} \quad N_2 = -\vec{j} \quad F \cdot N_2|_{S_2} = 0$

$S_3: x=0, P_3 = \{y \geq 0, z \geq 0, 2y+z \leq 6\} \quad N_3 = -\vec{i} \quad F \cdot N_3|_{S_3} = 0$

$S_4: 2x+2y+z=6, P_4 = \{x \geq 0, y \geq 0, x+y \leq 3\} \rightarrow N_4 = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$

Example 19. stokes example

Example: the Stokes th.  $\oint_{\partial G} F \cdot dr = \iint_S (\text{curl}(F)) \cdot N \, dG$

$$S = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0, 2x + 2y + z = 6\}$$

$$F : (x, y, z) \mapsto -y^2 \vec{i} + z \vec{j} + x \vec{k}$$

$$\text{curl}(F)(x, y, z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = -\vec{i} - \vec{j} + 2y \vec{k}$$

$$N(x, y, z) = \frac{2}{3} \vec{i} + \frac{2}{3} \vec{j} + \frac{1}{3} \vec{k}$$

$$[\text{curl}(F) \cdot N](x, y, z) = -\frac{2}{3} - \frac{2}{3} + \frac{2y}{3} = \frac{2}{3}(y-2)$$

$$dG = 3 \, dx \, dy, \quad D = \{P_r(S) \mid z=0\} = \{(x, y) \in \mathbb{R}^2 : x+y \leq 3, x>0, y>0\}$$

$$\begin{aligned} \oint_{\partial G} F \cdot dr &= \iint_S [\text{curl}(F) \cdot N] \, dG = \int_0^3 \left[ \int_0^{3-x} \frac{2}{3}(y-2) \, dy \right] 3 \, dx \\ &= 2 \int_0^3 \left[ \frac{y^2}{2} - 2y \right]_0^{3-x} dx = \int_0^3 [(3-x)^2 - 4(3-x)] dx \\ &= \int_0^3 (u^2 - 4u) \, du = \left[ \frac{u^3}{3} - 2u^2 \right]_0^3 = (9 - 18) = -9 \end{aligned}$$



$C = \partial S = C_1 \cup C_2 \cup C_3$

$C_1: (3, 0, 0) \rightarrow (0, 3, 0); r_1(t) = (3-t, t, 0); 0 \leq t \leq 3$

$C_2: (0, 3, 0) \rightarrow (0, 0, 6); r_2(t) = (0, 3-t, 2t); 0 \leq t \leq 3$

$C_3: (0, 0, 6) \rightarrow (3, 0, 0); r_3(t) = (t, 0, 6-2t); 0 \leq t \leq 3$

$F(r_1(t)) = F(3-t, t, 0) = -t^2 \vec{i} + (3-t) \vec{j}$

$F(r_2(t)) = F(0, 3-t, 2t) = -(3-t)^2 \vec{i} + 2t \vec{j}$

$F(r_3(t)) = F(t, 0, 6-2t) = (6-2t) \vec{j} + t \vec{k}$

$r_1'(t) = -\vec{i} + \vec{j}$

$r_2'(t) = -\vec{j} + 2\vec{k}$

$r_3'(t) = \vec{i} - 2\vec{k}$

$\oint_C F \cdot dr = \int_0^3 (t^2 - 2t - 2t) dt = \int_0^3 (t^2 - 4t) dt = \left[ \frac{t^3}{3} - 2t^2 \right]_0^3 = -9$