Chapter 01: Double integrals

Notes from prof zeghlaoui course

November 6, 2025

1 Reminder:

Recall that if $f : [a, b] \to \mathbb{R}$ is bounded.

for any subdivision $\sigma = \{x_0 = a < x_1 < x_2 < ... < x_{n+1} = b\}$ of $[a, b]$

$$S(f,\sigma) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{and} \quad s(f,\sigma) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$\text{where } m_i = \inf_{x\in[x_{i-1},x_i]} f(x), \quad M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$$

called respectively, the upper and the lower Darboux sums of f on $[a, b]$ with respect to the subdivision σ;

S: the area of the region plane delimited by:

$$\begin{cases} x = a & y = 0 \\ x = b & y = f(x) \end{cases}$$

(graph showing the lower and upper of an integral)

$$s(f,\sigma) \leq S \leq S(f,\sigma) \quad \text{if } \sigma_1 \subset \sigma_2, \text{ then:}$$
$$s(f,\sigma_1) \leq s(f,\sigma_2) \leq S \leq S(f,\sigma_2) \leq S(f,\sigma_1)$$

$$S^+ = \inf(S(f,\sigma)) \quad , \quad S^- = \sup(s(f,\sigma))$$

$$S^- \leq S \leq S^+$$

Definition 1. *We say that f is Riemann integrable on $[a,b]$ if :*

$$S^+ = S^- = S = \int_a^b f(x)dx$$

Definition 2. *Riemann sums*

Let σ a subdivision of $[a,b]$ and $\zeta = \{\zeta_1, \ldots, \zeta_n\} \quad / \zeta_i \in [x_{i-1}, x_i]$

$$s(f,\sigma) \leq R(f,\sigma,\zeta) = \sum_{i=1}^{n} f(\zeta_i)(x_i - x_{i-1}) \leq S(f,\sigma)$$

If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then:

$\exists(\sigma_n)$ a sequence of subdivision of $[a,b]$

$\exists(\zeta_n)$ associated with $(\sigma_n)[a,b] \quad \zeta_n = (\zeta_n^i),\ \zeta_n^i \in [x_{i-1}^n, x_i^n]$

$\lim\limits_{n\to\infty} R(f,\sigma_n,\zeta_n) = \int_a^b f(x)dx$

<u>In particular:</u>

$$\sigma_n = \{x_i^n = a + \frac{i}{n}(b-a) : i \in 0, \ldots, n\}$$

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\zeta_i) = \int_a^b f(x)dx$$

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{i}{n}(b-a)) = \int_a^b f(x)dx \quad , x_i^n - x_{i-1}^n = \frac{b-a}{n}$$

2 Double integrals:

2.1 Definition and first properties:

Definition 3. *Darboux sums- Riemann sums:*

Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function

1) A subdivision σ of Δ is a set $\sigma = \{\Delta_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$

with $(x_i)_{i=1,\ldots,n}$ is a subdivision of $[a,b]$ and $(y_j)_{j=0,\ldots,m}$ a subdivision of $[c,d]$ we note : $A(\Delta_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$

and $\delta(\sigma) = \max_{i,j} diam(\Delta_{ij})$

2)

$$\forall i,j \quad M_{ij} = \sup_{(x,y)\in\Delta_{ij}} f(x,y) \quad and \quad m_{ij} = \inf_{(x,y)\in\Delta_{ij}} f(x,y)$$

the upper Darboux sum of f with respect to σ is :

$$S(f,\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}A(\Delta_{ij})$$

the lower Darboux sum of f with respect to σ is :

$$s(f,\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}A(\Delta_{ij})$$

3) if we give $\zeta = \{(\zeta_i^1, \zeta_j^2) \mid \zeta_i^1 \in [x_{i-1}, x_i]$ and $\zeta_j^2 \in [y_{j-1}, y_j]\}$

The Riemann sum with respect to σ and ζ is :

$$R(f,\sigma,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\zeta_i^1,\zeta_j^2)A(\Delta_{ij})$$

$$R(f,\sigma,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\zeta_i^1,\zeta_j^2)(x_i - x_{i-1})(y_j - y_{j-1})$$

2.2 Example:

$$\begin{gathered} f(x,y) = \alpha \quad ;\forall (x,y) \in \Delta \\ M_{ij} = m_{ij} = f(\zeta_i^1,\zeta_j^2) = \alpha \\ S(f,\sigma) = s(f,\sigma) = R(f,\sigma,\zeta) = \alpha(b-a)(d-c) = \alpha A(\Delta) \end{gathered}$$

Definition 4. *Riemann integrability:*

With the same notations above, we put:

$$S^{+}(f) = \inf(S(f,\sigma)) \quad \text{and} \quad S^{-}(f) = \sup(s(f,\sigma))$$

We say that f is Riemann integrable on Δ if $S^{+}(f) = S^{-}(f)$

$$\Leftrightarrow \forall \epsilon > 0; \exists \sigma \quad S(f,\sigma) - s(f,\sigma) < \epsilon$$

and we denote the common value by $\iint_{\Delta} f(x,y)dxdy$

$$\Leftrightarrow \exists (\sigma_n) \text{ of subdivisions of } \Delta / \lim S(f,\sigma_n) = \lim s(f,\sigma_n)$$

Corollary: if $f : \Delta = [a,b] \times [c,d]$ is Riemann integrable then:

$$\lim_{n\to+\infty} \frac{(b-a)(d-c)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} f(a + \frac{i}{n}(b-a), c + \frac{j}{n}(d-c))$$
$$= \iint_{\Delta} f(x,y)dxdy$$

2.3 Example:

$$f(x,y) = x^2 + y^2 \quad , \Delta = [0,1]^2 = [0,1] \times [0,1]$$

Definition 5. *Riemann integrability on a bounded domain:*

Let D be a bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ a bounded function.

We say that f is Riemann integrable on D if for any rectangle $\Delta \supset D$, the function :

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in \Delta \setminus D \end{cases}$$

is Riemann integrable, in this case :

$$\iint_D f(x,y)dxdy := \iint_\Delta \tilde{f}(x,y)dxdy$$

Remark: *Any continuous function is Riemann integrable.*

2.4 Properties:

1)<u>Linearity:</u> If $f, g : D \to \mathbb{R}$ are Riemann integrable then: $\forall \alpha, \beta \in \mathbb{R} \quad , \alpha f + \beta g$ is Riemann integrable and

$$\iint_D (\alpha f + \beta g)(x,y)dxdy = \alpha \iint_D f(x,y)dxdy + \beta \iint_D g(x,y)dxdy$$

2)

If $f \geq 0$ on $D \implies \iint_D f(x,y)dxdy \geq 0$ which is the volume of $\omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } 0 \leq z \leq f(x,y)\}$

So; if $f \geq g \implies \iint_D [f(x,y) - g(x,y)]\, dxdy$

$= vol\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } g(x,y) \leq z \leq f(x,y)\}$

3)<u>Chasles's relation:</u>

If $D = D_1 \cup D_2$ such that $D_1 \cap D_2$ is at most a curve then:

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$

Definition 6. *$Vol(D) = A(D) = \iint_D dxdy$ and the average of f on D is: $\dfrac{1}{Vol(D)} \iint_D f(x,y)dxdy = \dfrac{\iint_D f(x,y)dxdy}{\iint_D dxdy}$*

Remark:

If $f \equiv C$, the average of f is C

2.5 Example:

D: the triangle with vertices $(0,0),(1,0),(0,1)$ (graph) and $f(x,y)=xy$

$$\tilde{f}(x,y)=\begin{cases} xy & \text{if } x+y\leq 1 \\ 0 & \text{if } x+y>1 \end{cases}$$

$\forall x,y \in [0,1]\times[0,1]$

$$\iint_D xydxdy = \lim_{n\to+\infty} \frac{(1-0)(1-0)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{f}(\frac{i}{n},\frac{j}{n})$$

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \frac{i}{n}\times\frac{j}{n}\right]$$

$$U_n = \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{i}{n}\times\frac{j}{n}\right]$$

$$U_n = \frac{1}{n^4}\sum_{i=1}^{n} i\left[\sum_{j=1}^{n-i} j\right] = \frac{1}{n^4}\sum_{i=1}^{n} i\left(\frac{(n-i)(n-i+1)}{2}\right)$$

$$U_n = \frac{1}{2n^4}\sum_{i=1}^{n} i(n^2+i^2-2in+n-i)$$

$$U_n = \frac{1}{2n^4}\sum_{i=1}^{n}\left[i(n^2+n)-i^2(2n+1)+i^3\right]$$

$$U_n = \frac{1}{2n^4}\left[(n^2+n)\frac{(n+1)n}{2}-(2n+1)\frac{n(n+1)(2n+1)}{6}+\frac{n^2(n+1)^2}{4}\right]$$

$$U_n = \frac{1}{4}(1+\frac{1}{n})^2-\frac{1}{12}(1+\frac{1}{n})(2+\frac{1}{n})+\frac{1}{8}(1+\frac{1}{n})^2$$

$$\iint_D xydxdy = \lim_{n\to+\infty} U_n = \frac{1}{4}-\frac{1}{12}\cdot 2+\frac{1}{8} = \frac{1}{24}$$

Theorem 1. *Fubini's theorem:*

Let $f: D\to\mathbb{R}$ Riemann integrable

1) if

$D=\{(x,y)\in\mathbb{R}^2 : a\leq x\leq b \textit{ and } \exists g_1,g_2:[a,b]\to\mathbb{R} \textit{ continuous} \mid g_1(x)\leq y\leq g_2(x), \forall x\in[a,b]\}$

then: $\iint_D f(x,y)dxdy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx$

2) If

$$D = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \exists h_1, h_2 : [c,d] \to \mathbb{R} \text{ continuous} \mid h_1(y) \leq x \leq h_2(y), \forall y \in [c,d]\}$$

then: $\iint_D f(x,y)dxdy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy$

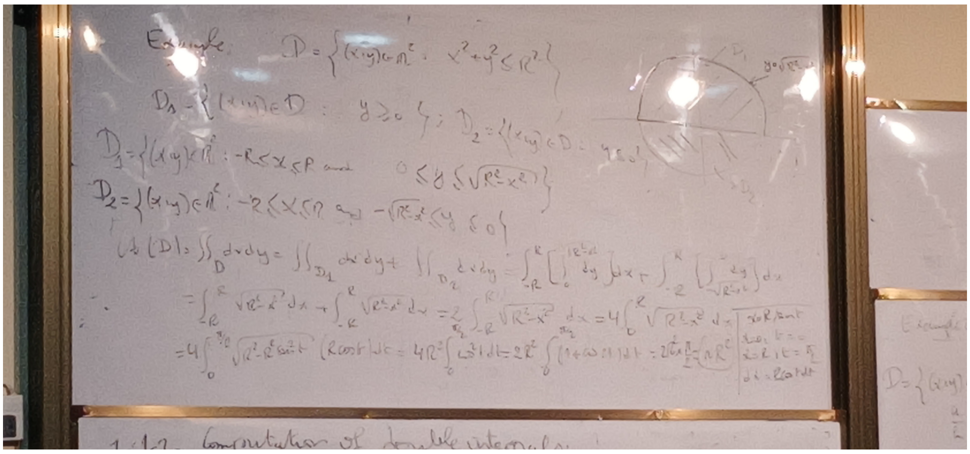
2.6 Remark:

We can use the Chasles's formula to apply this theorem.

2.7 Example 01:

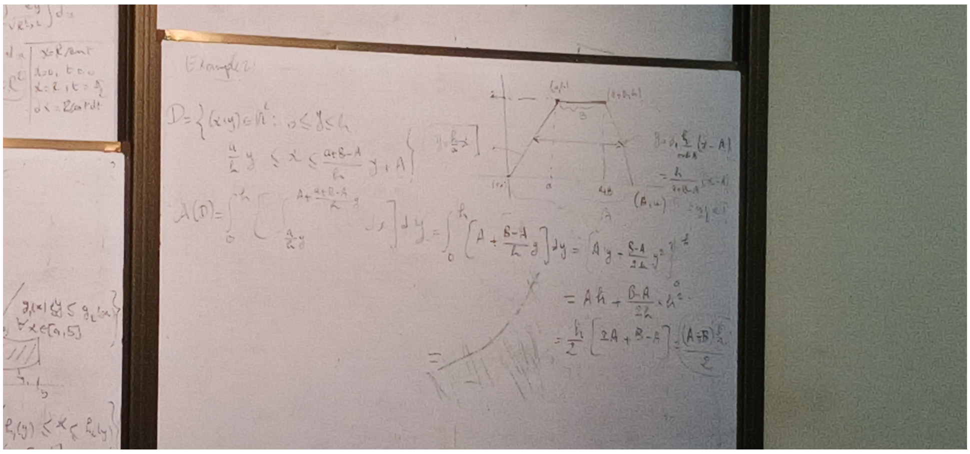
$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$$
$$D_1 = \{(x,y) \in D : y \geq 0\}; D_2 = \{(x,y) \in D : y \leq 0\}$$
$$D_1 = \{(x,y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } 0 \leq y \leq \sqrt{R^2 - x^2}\}$$
$$D_2 = \{(x,y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } -\sqrt{R^2 - x^2} \leq y \leq 0\}$$

...



2.8 Example 02:

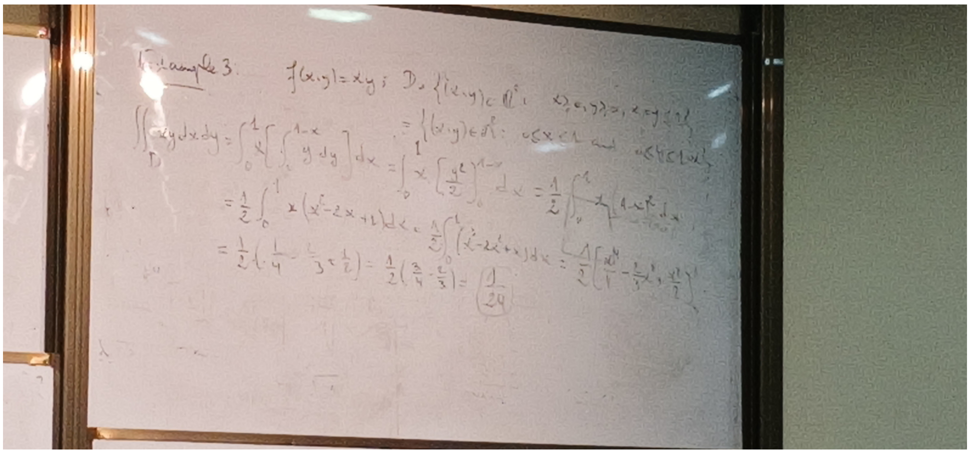
(trapezoid area proof)



2.9 Example 03:

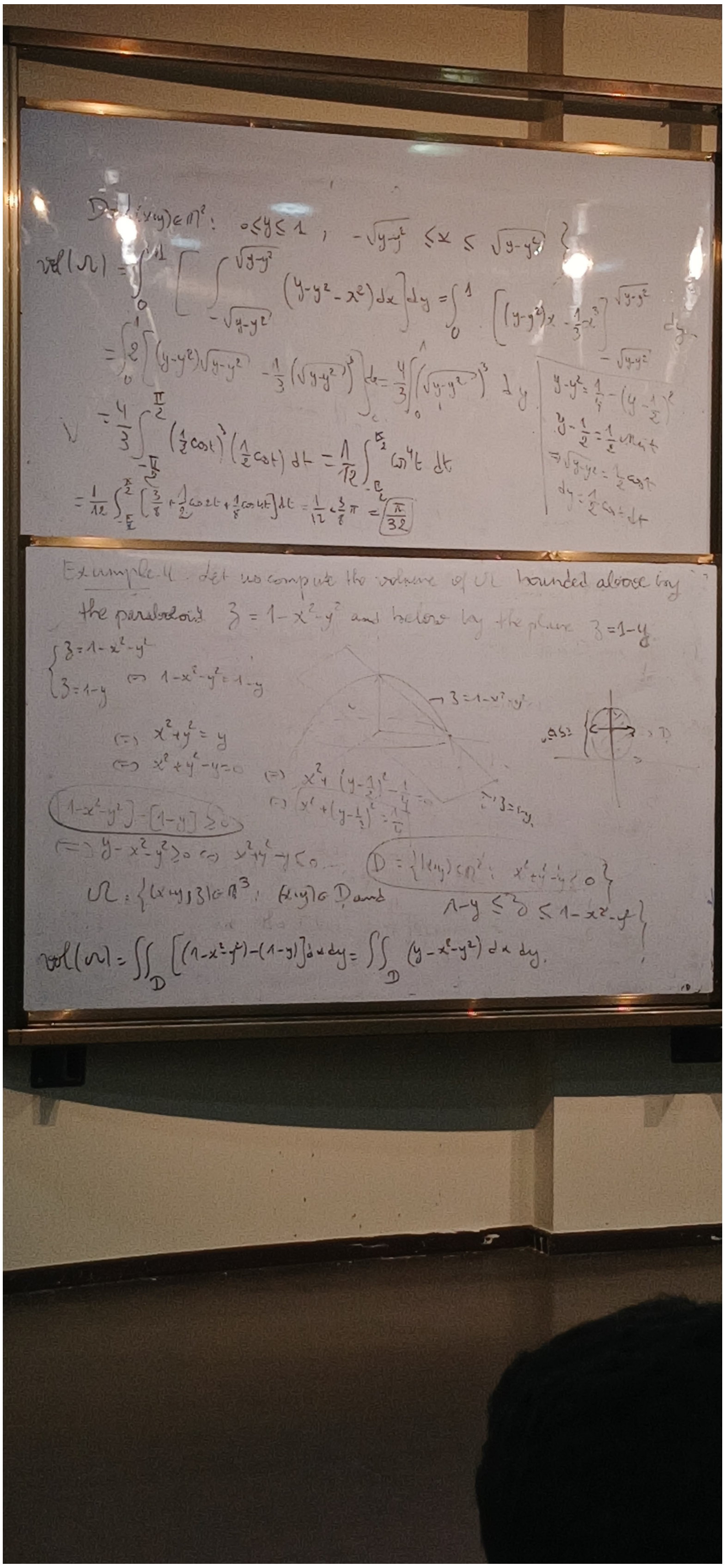
$$f(x, y) = xy$$
$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$$
$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}$$

...



2.10 Example 04:

Let us compute volume of Ω bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$



$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ -\sqrt{y-y^2} \le x \le \sqrt{y-y^2}\}$

$$vol(\Omega) = \int_0^1 \left[\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y-y^2-x^2)\,dx\right] dy = \int_0^1 \left[(y-y^2)x - \frac{1}{3}x^3\right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} dy$$

$$= \int_0^1 2\left[(y-y^2)\sqrt{y-y^2} - \frac{1}{3}\left(\sqrt{y-y^2}\right)^3\right] dy = \frac{4}{3}\int_0^1 \left(\sqrt{y-y^2}\right)^3 dy$$

$$= \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\cos t\right)^3 \left(\frac{1}{2}\cos t\right) dt = \frac{1}{12}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t\, dt$$

$$= \frac{1}{12}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t\right] dt = \frac{1}{12}\cdot\frac{3}{8}\pi = \boxed{\frac{\pi}{32}}$$

$y - y^2 = \frac{1}{4} - \left(y - \frac{1}{2}\right)^2$

$y - \frac{1}{2} = \frac{1}{2}\sin t$

$\Rightarrow \sqrt{y-y^2} = \frac{1}{2}\cos t$

$dy = \frac{1}{2}\cos t\, dt$

Example 4. Let us compute the volume of Ω bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$.

$\begin{cases} z = 1 - x^2 - y^2 \\ z = 1 - y \end{cases} \Leftrightarrow 1 - x^2 - y^2 = 1 - y$

$\Leftrightarrow x^2 + y^2 = y$

$\Leftrightarrow x^2 + y^2 - y = 0 \Leftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = 0$

$\Leftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$

$\to z = 1 - x^2 - y^2$

$\to z = 1 - y$

$[1 - x^2 - y^2] - [1 - y] \ge 0$

$\Leftrightarrow y - x^2 - y^2 \ge 0 \Leftrightarrow x^2 + y^2 - y \le 0$

$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - y \le 0\}$

$\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D$ and $1 - y \le z \le 1 - x^2 - y^2\}$

$$vol(\Omega) = \iint_D \left[(1-x^2-y^2) - (1-y)\right] dx\,dy = \iint_D (y - x^2 - y^2)\,dx\,dy.$$

3 Change of variables:

Let S, D two bounded subsets of $\mathbb{R}^2$, $\mathbb{R}^2 \to \mathbb{R}$ three functions such that :

$$1) \quad (g(u,v), h(u,v)) \in D \subset Dom(f), \forall (u,v) \in S$$

2) The functions are of class C^1 3) The function f is continuous on D; then we have :

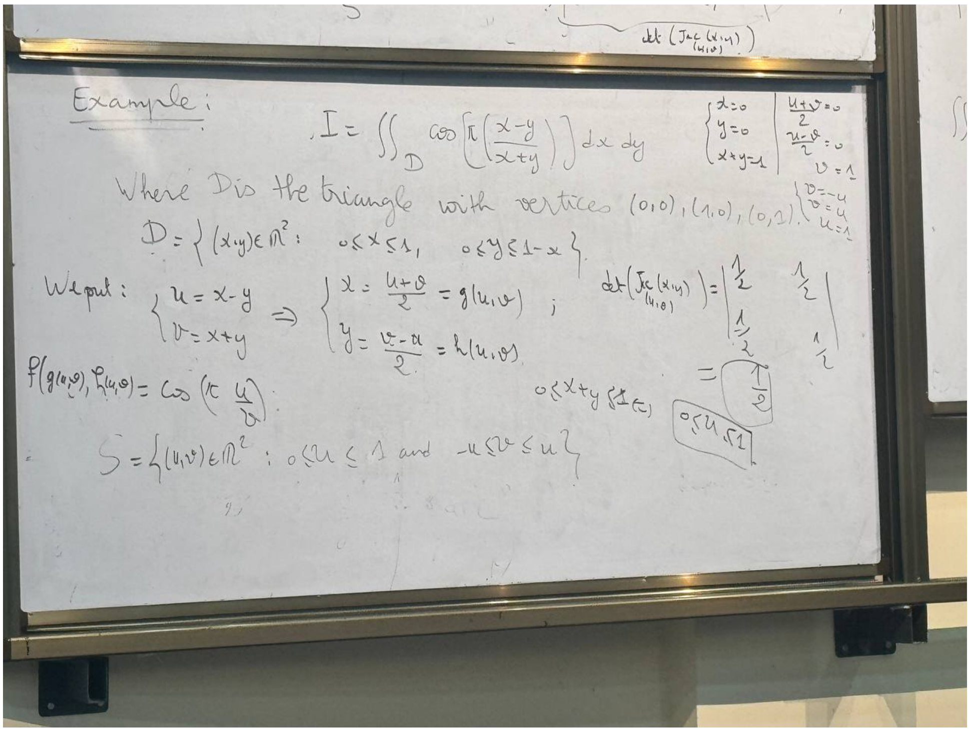
$$\iint_D f(x,y)dxdy = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} \right| dudv$$

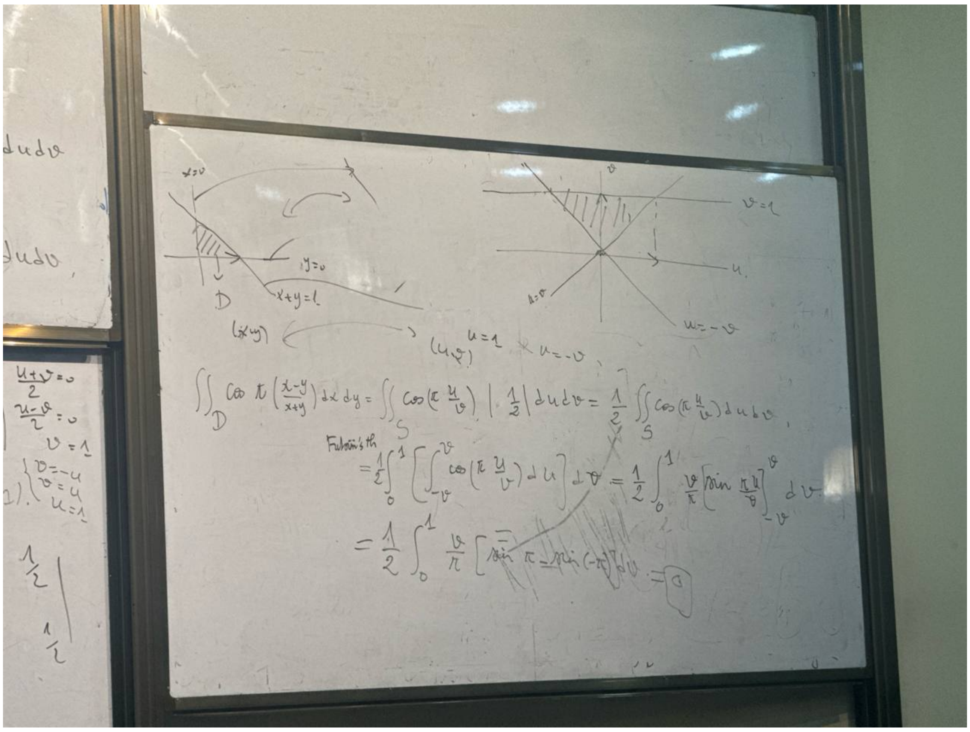
$$\iint_S f(x(u,v), y(u,v)) \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| dudv$$

$$note: \quad \det(Jac_{(u,v)}(x,y)) = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right|$$

3.1 Example 01:

$$I = \iint \cos \pi \frac{x-y}{x+y} dxdy$$





3.2 Particular case: Polar coordinates:

$$\begin{cases} x(r,\theta) = r\cos(\theta) \\ y(r,\theta) = r\sin(\theta) \end{cases}$$

$r = \sqrt{x^2+y^2}$ the angle between $\vec{i}$ and $O\vec{M}$

$$O\vec{M} = x\vec{i} + y\vec{j}$$

$$O\vec{M} = \sqrt{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}\right]$$

$$\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1$$

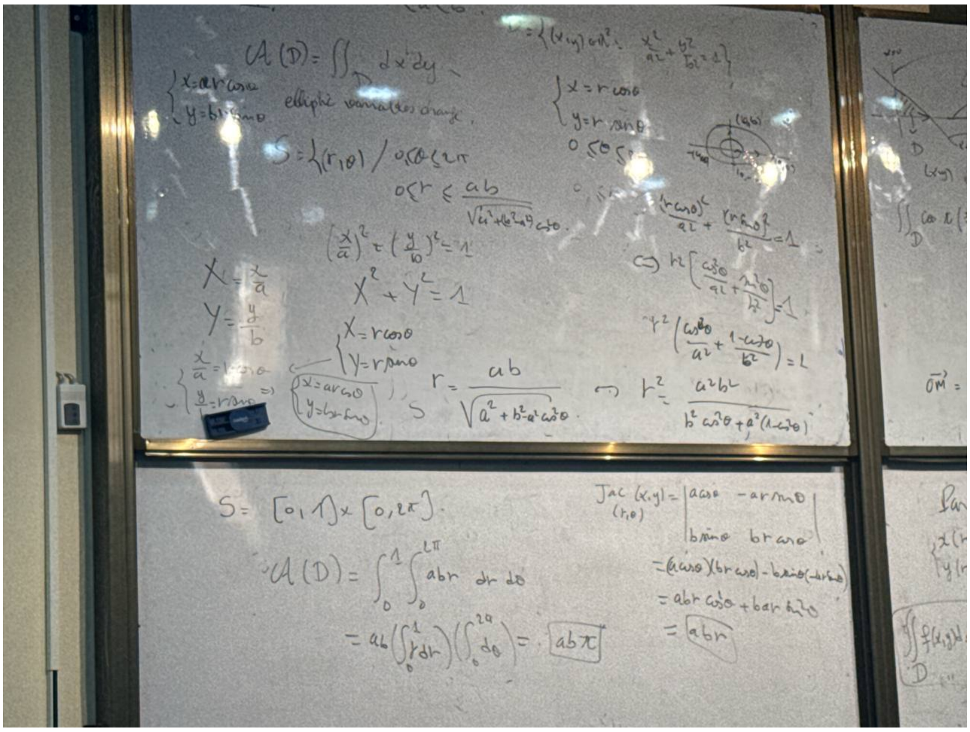
$$\exists\theta\in\mathbb{R} \quad / \quad \cos(\theta) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \quad ; \quad \sin(\theta) = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$Jac_{(r,\theta)}(x,y) = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$
$$= \cos(\theta)(r\cos(\theta)) - \sin(\theta)(-r\sin(\theta))$$
$$= r\cos^2(\theta) + r\sin^2(\theta) = r > 0$$

$$\iint_D f(x,y)dxdy = \iint_S f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

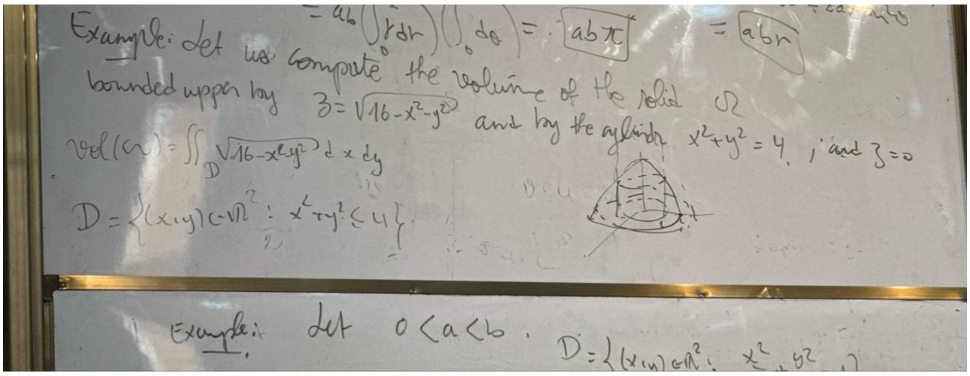
3.3 Example 02:

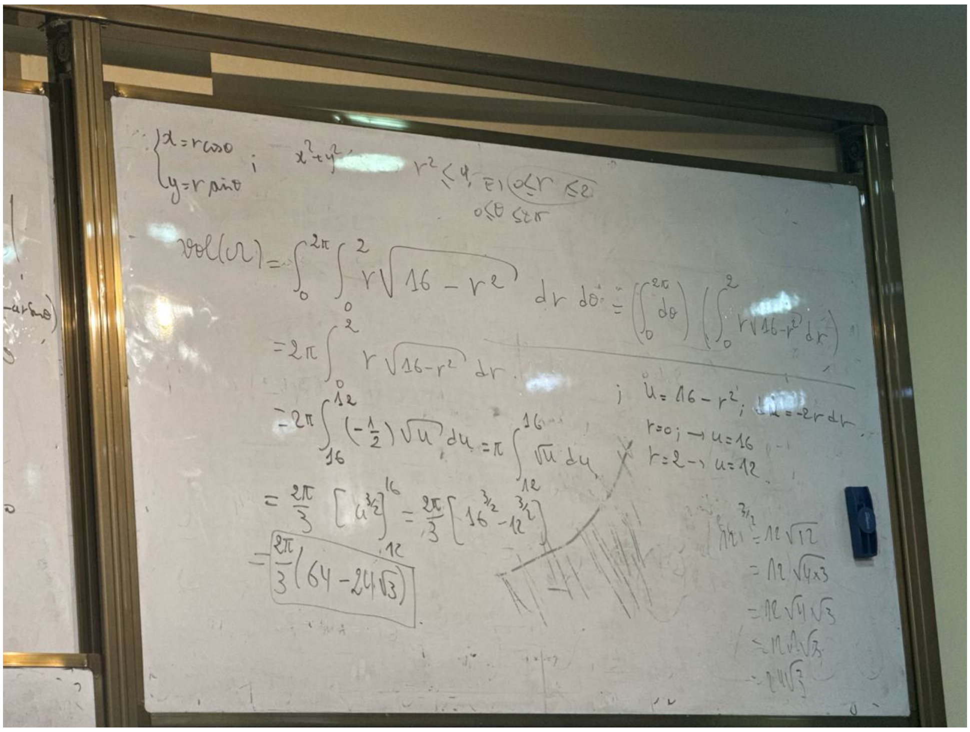
Calculate elliptic area using double integral:



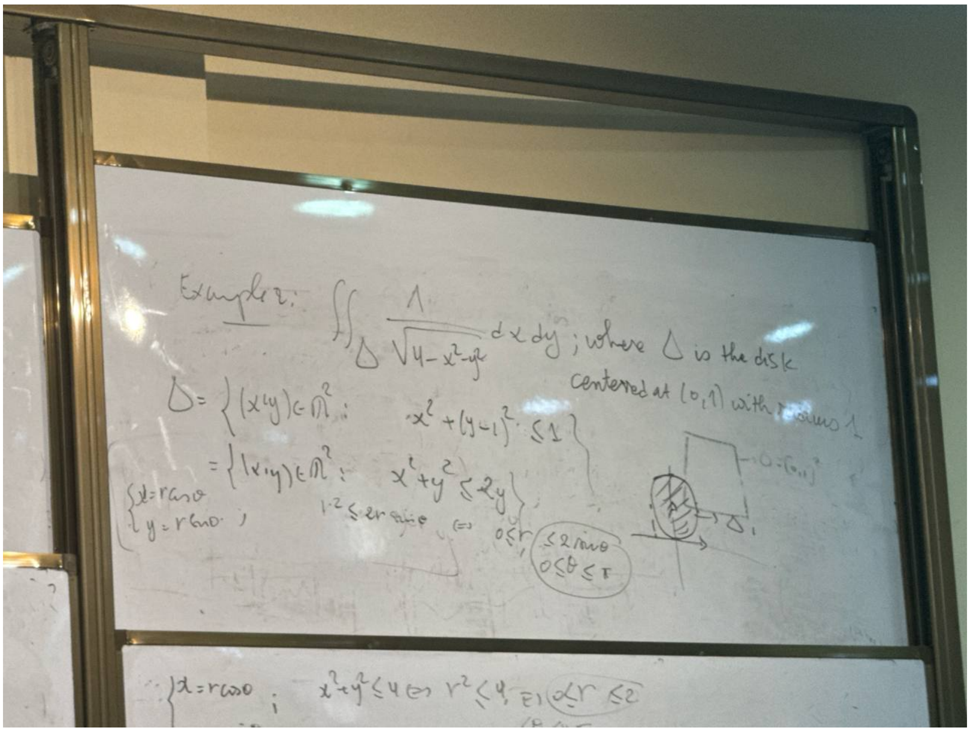
3.4 Example 03:

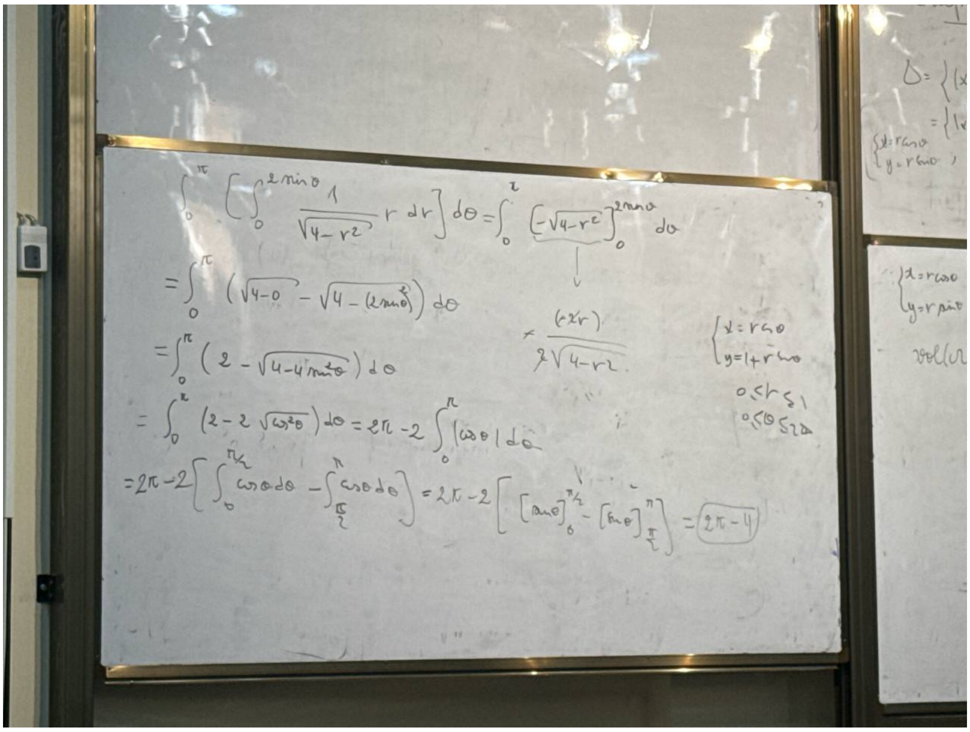
3.5 Example: Volume calculation using polar coordinates





3.6 Example 04:





4 Applications of double integrals

Definition 7. *Let $\rho : D \to \mathbb{R}$ be a continuous density function of the lamina corresponding to the plane region D, the mass $m(D)$ of the lamina is defined by:*

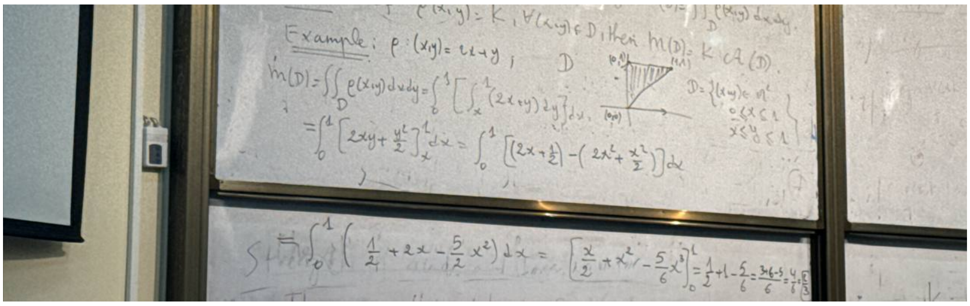
$$m(D) = \iint_D \rho(x, y)dxdy$$

<u>**Remark:**</u>

if $\rho(x, y) = K, \forall(x, y) \in D, \quad$ then $m(D) = KA(D)$

Example 1.

$$\rho : (x, y) = 2x + y$$



Definition 8. <u>*Moments and center of mass*</u>

With the same hypothesis and and notations as definition above

1) the moment of mass with respect the x and y axes respectively are:

$$M_x = \iint_D y\rho(x,y)dxdy; \quad M_y = \iint_D x\rho(x,y)dxdy$$

the center of mass of D is $(\bar{x},\bar{y}) = (\frac{M_y}{m(D)}, \frac{M_x}{m(D)})$

2) the second moments, or the moments of inertia of D about the x and y axes respectively are:

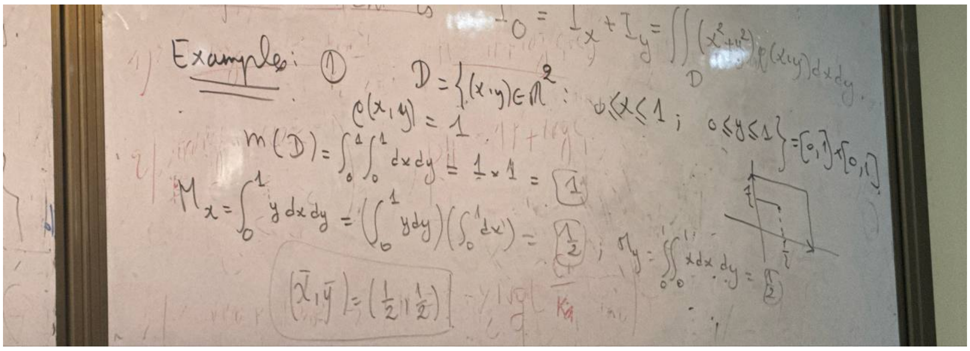
$$I_x = \iint_D y^2\rho(x,y)dxdy; \quad I_y = \iint_D x^2\rho(x,y)dxdy$$

3) the polar moment is $I_O = I_x + I_y = \iint_D (x^2+y^2)\rho(x,y)dxdy$

Example 2. *1)*

$$D = \{(x,y) \in \mathbb{R}^2; \quad 0 \le x \le 1; \quad 0 \le y \le 1\} = [0,1]\,[0,1]$$

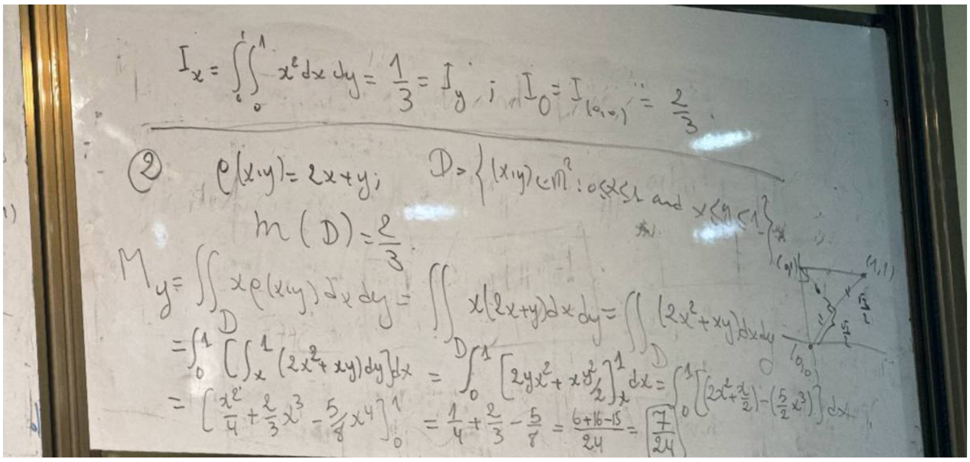
$\rho(x,y) = 1$

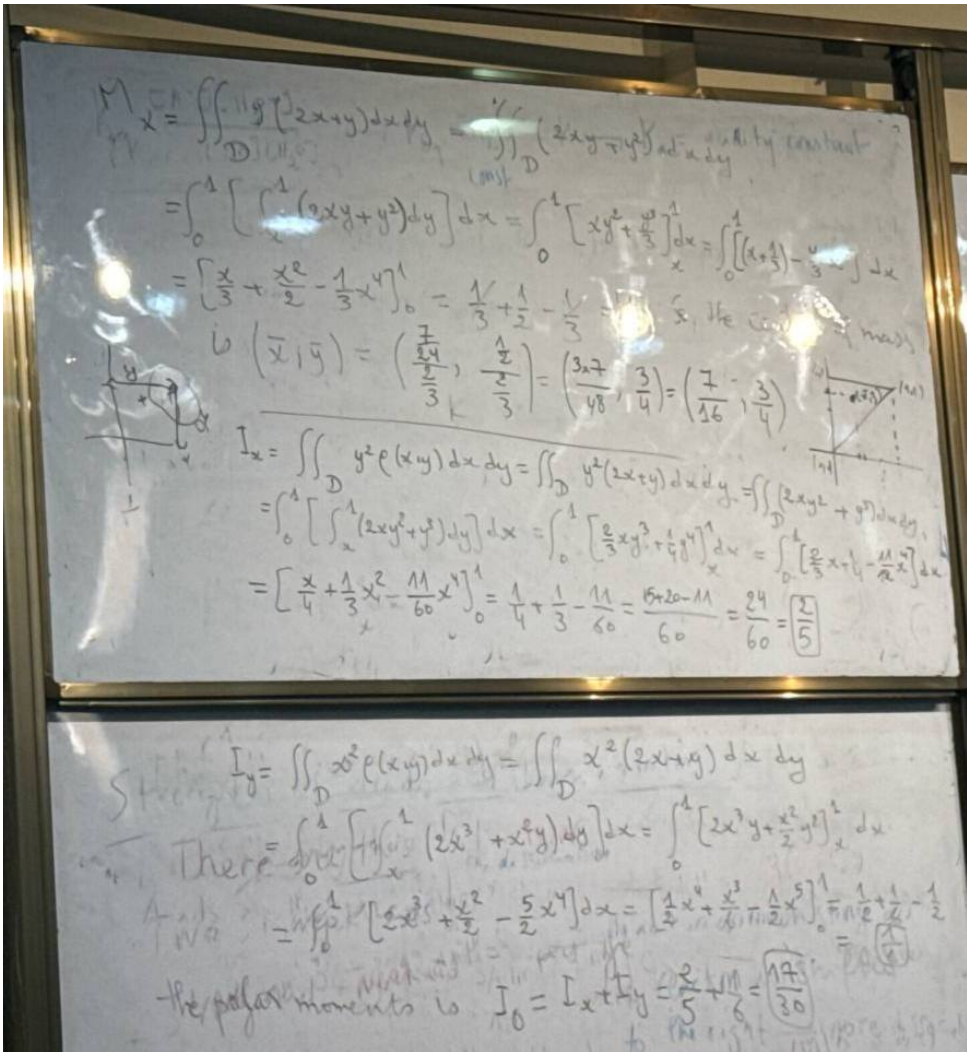


2)

$$\rho(x,y) = 2x + y$$

(the first line belongs to the first example (the I_x and I_y integrals))





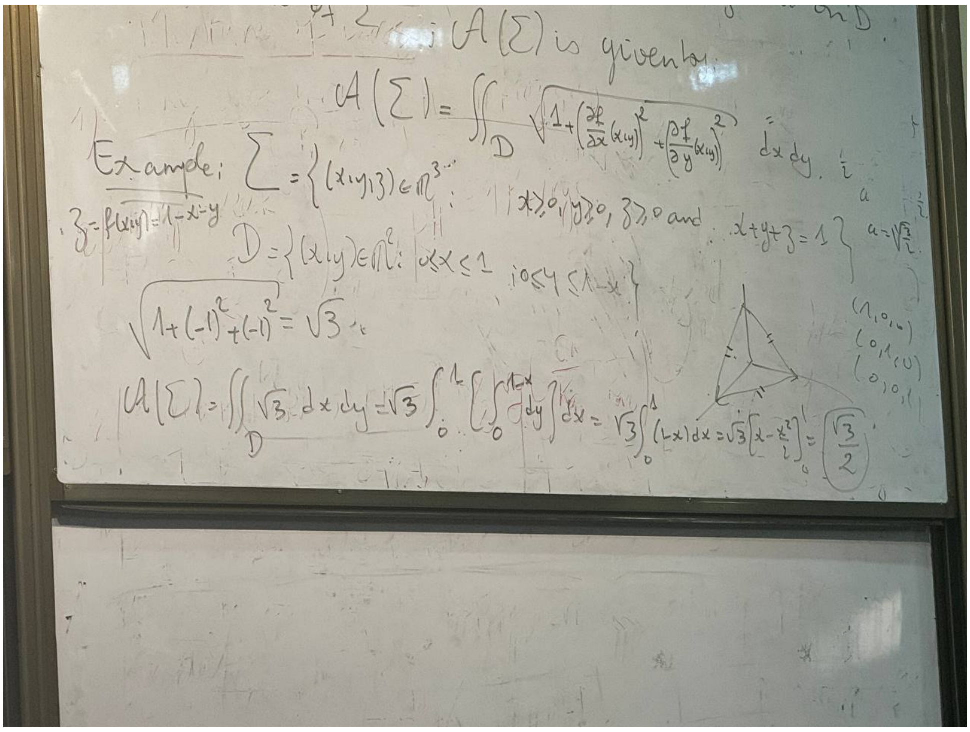
Definition 9. *Let* $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : \quad (x, y) \in D, \quad z = f(x, y)\}$ *be bounded surface, with* f *a function such that* $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ *exist on D then :*

the are of Σ*;* $A(\sigma)$ *is given by:*

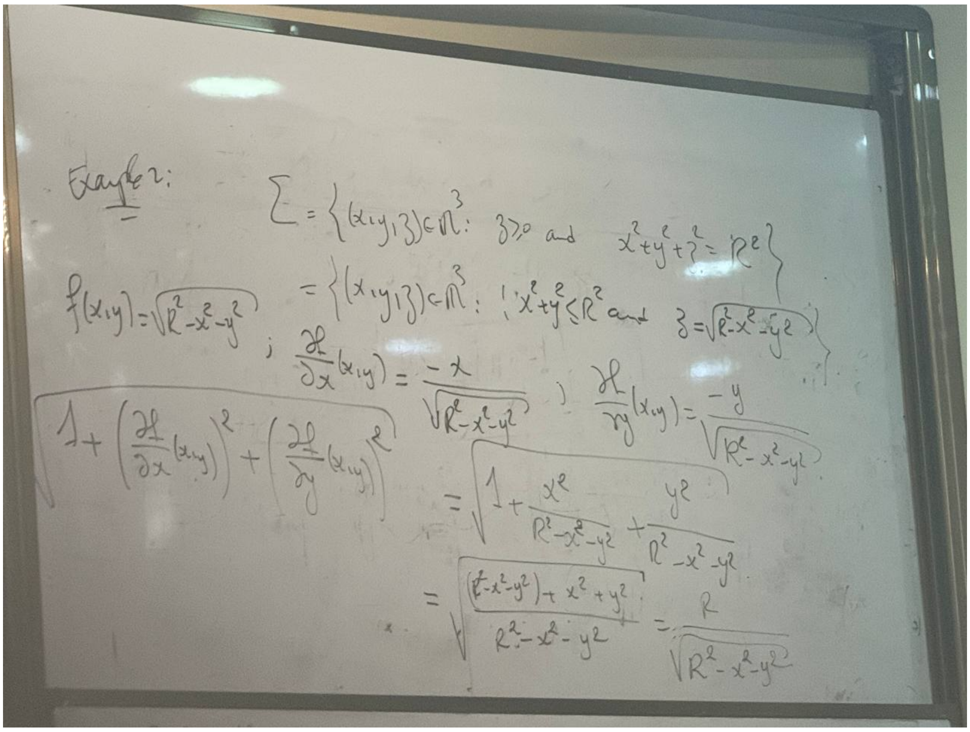
$$A(\sigma) = \iint_D \sqrt{1 + (\frac{\partial f}{\partial x}(x, y))^2 + (\frac{\partial f}{\partial y}(x, y))^2 dxdy}$$

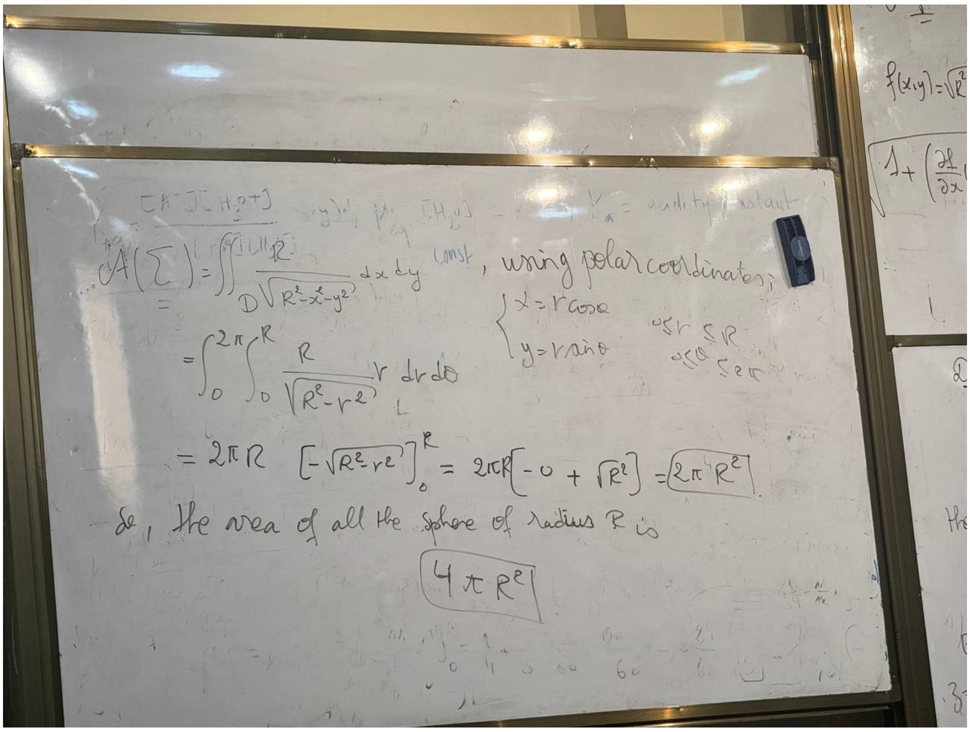
Example 3.

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : \quad x \geq 0, y \geq 0, z \geq 0 \text{ and } x + y + z = 1\}$$


A(Σ) is given by:
A(Σ) = ∬_D √(1+(∂f/∂x(x,y))²+(∂f/∂y(x,y))²) dx dy.
Example: Σ = {(x,y,z) ∈ ℝ³ : x≥0, y≥0, z≥0 and x+y+z = 1}
z = f(x,y) = 1−x−y
D = {(x,y) ∈ ℝ²: 0≤x≤1 , 0≤y≤1−x}
√(1+(−1)²+(−1)²) = √3
(1,0,0)
(0,1,0)
(0,0,1)
A(Σ) = ∬_D √3 dx dy = √3 ∫_0^1 [∫_0^{1−x} dy] dx = √3 ∫_0^1 (1−x) dx = √3 [x − x²/2]_0^1 = √3/2

Example 4. ...


Example 2:
Σ = {(x,y,z) ∈ ℝ³: z≥0 and x²+y²+z² = R²}
= {(x,y,z) ∈ ℝ³: x²+y² ≤ R² and z = √(R²−x²−y²)}
f(x,y) = √(R²−x²−y²)
∂f/∂x(x,y) = −x/√(R²−x²−y²) ; ∂f/∂y(x,y) = −y/√(R²−x²−y²)
√(1+(∂f/∂x(x,y))²+(∂f/∂y(x,y))²)
= √(1 + x²/(R²−x²−y²) + y²/(R²−x²−y²))
= √(((R²−x²−y²)+x²+y²)/(R²−x²−y²)) = R/√(R²−x²−y²)

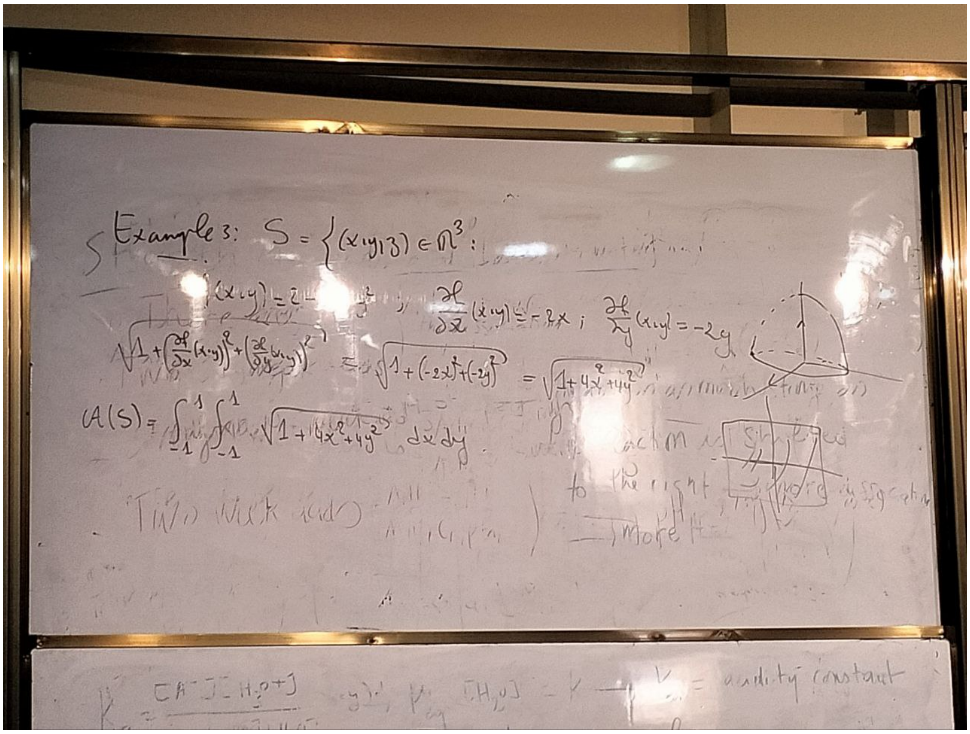


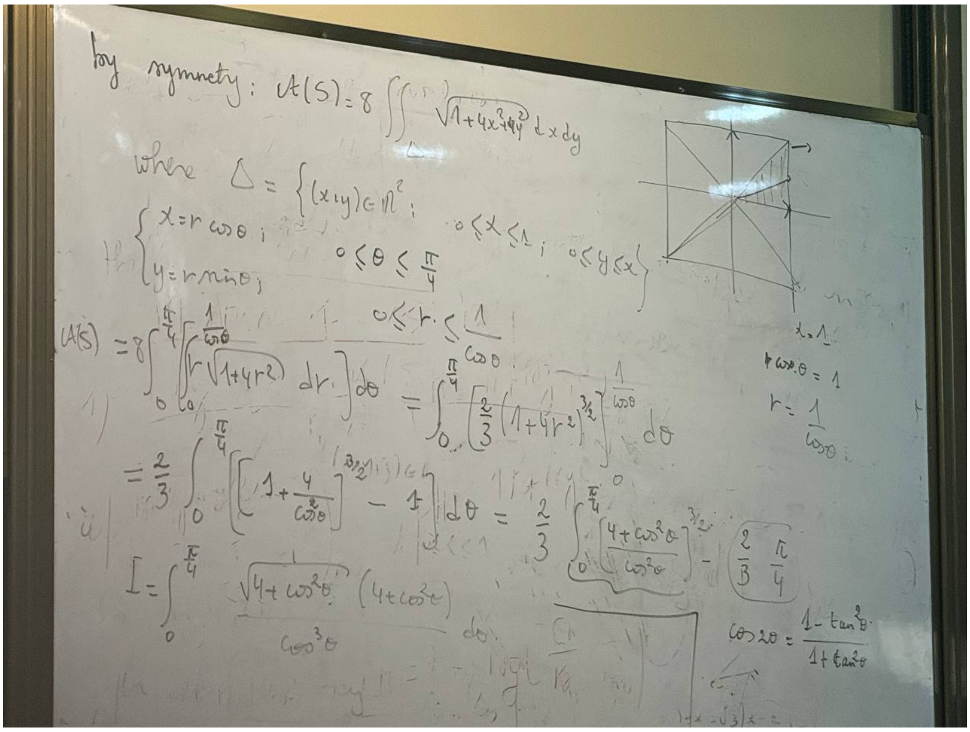
Example 5. ...

(first two lines)

$$S = \{(x, y, z) \in \mathbb{R}^3 : \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad z = 2 - x^2 - y^2\}$$

$$f(x, y) = 2 - x^2 - y^2 \quad ; \quad \frac{\partial f}{\partial x}(x, y) = -2x \quad ; \quad \frac{\partial f}{\partial y} = -2y$$





5 Triple Intergrals:

5.1 Definition and Cacluations:

the triple integral of a function $f : \mathbb{R}^3 \to \mathbb{R}$ on a bounded region $\Omega \subset \mathbb{R}^3$ is defined in the same way as double integrals with the same properties (linearity, Chasles's relation, increasing), we denot this integral by: $\iiint_\Omega f(x,y,z)dxdydz$

In the particular case : $f \equiv 1$ on Ω

$$\iiint_\Omega dxdydz = Vol(\Omega)$$

Theorem 2. *Evaluation by iterated integrals*

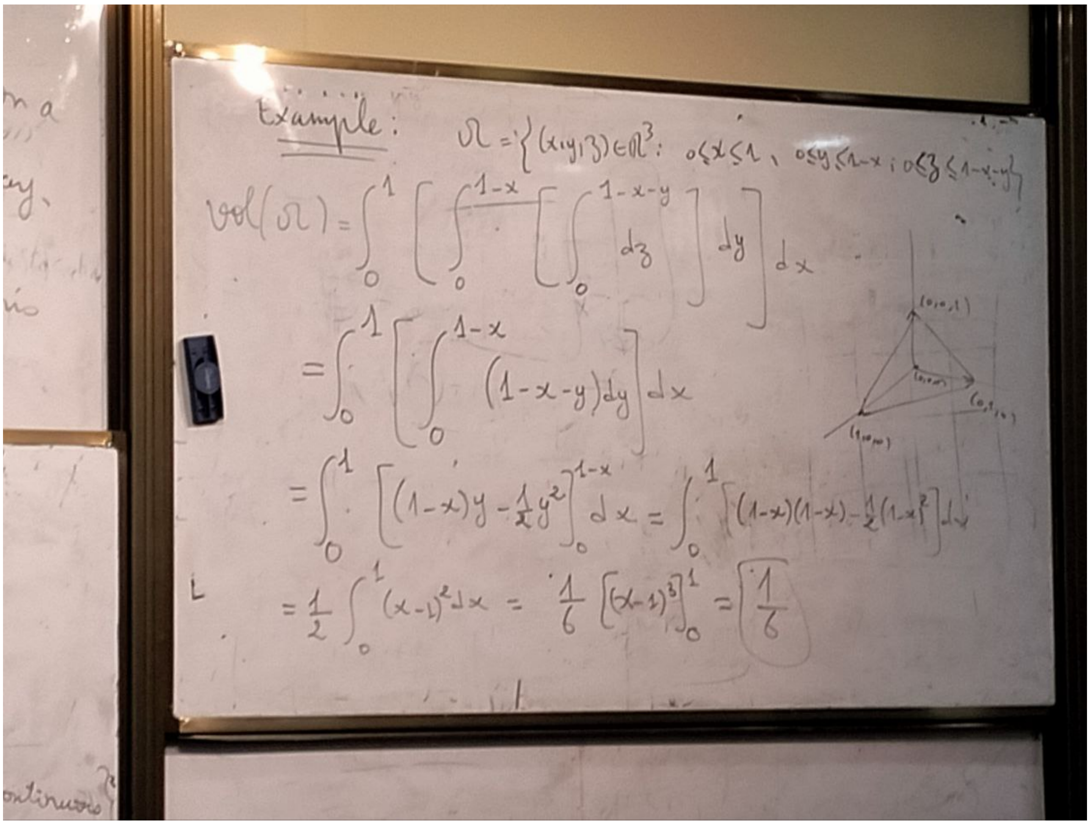
If $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid a \leq x \leq b \quad ; \quad h_1(x) \leq y \leq h_2(x) \quad ; \quad g_1(x,y) \leq z \leq g_2(x,y) \quad / h_1, h_2, g_1, g_2 \text{ are continuous}\}$

then:

$$\iiint_\Omega f(x,y,z)dxdydz = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} \left[\int_{g_1(x,y)}^{g_2 x,y} f(x,y,z)dz \right] dy \right] dx$$

Example 6.

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1 \quad ,0 \leq y \leq 1-x \quad ;0 \leq z \leq 1-x-y\}$$



5.2 Change of Variable:

Let $f : \Omega \to \mathbb{R}$ Riemann-integrable; let S a bounded region of $\mathbb{R}^3$; $g_1, g_2, g_3 \in S \to \mathbb{R}$ of class C^1 such that $\forall (y, v, w) \in S; \quad (g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)) \in \Omega$

then:

$$\iint_{\Omega} f(x, y, z)dxdydz = \iint_{S} f(g_1(u, v, w), g_2(u, v, w), g_3(u, v, w))|det(Jac_{u,v,w}(g_1, g_2, g_3))|dudvdw$$

$$det(Jac_{u,v,w}(g_1, g_2, g_3)) = det(Jac_{u,v,w}(x, y, z)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} & \frac{\partial g_3}{\partial w} \end{vmatrix}$$

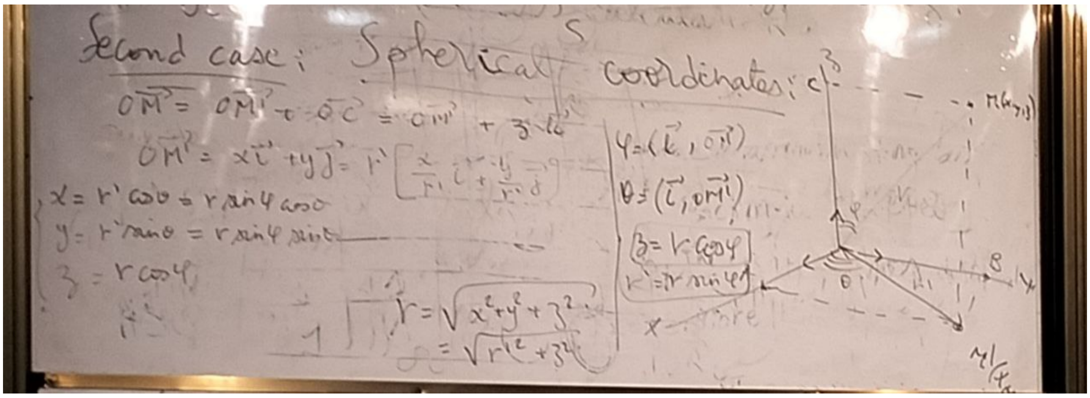
5.3 Two particular cases:

First case: Cylindrical coordinated

$$\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \\ z = z \end{cases} \quad det(Jac_{(u,v,w)}(x,y,z)) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} cos(\theta) & -rsin(\theta) & 0 \\ sin(\theta) & rcos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{S} rf(rcos(\theta), rsin(\theta), z)drd\theta dz$$

second case: spherical coordinates

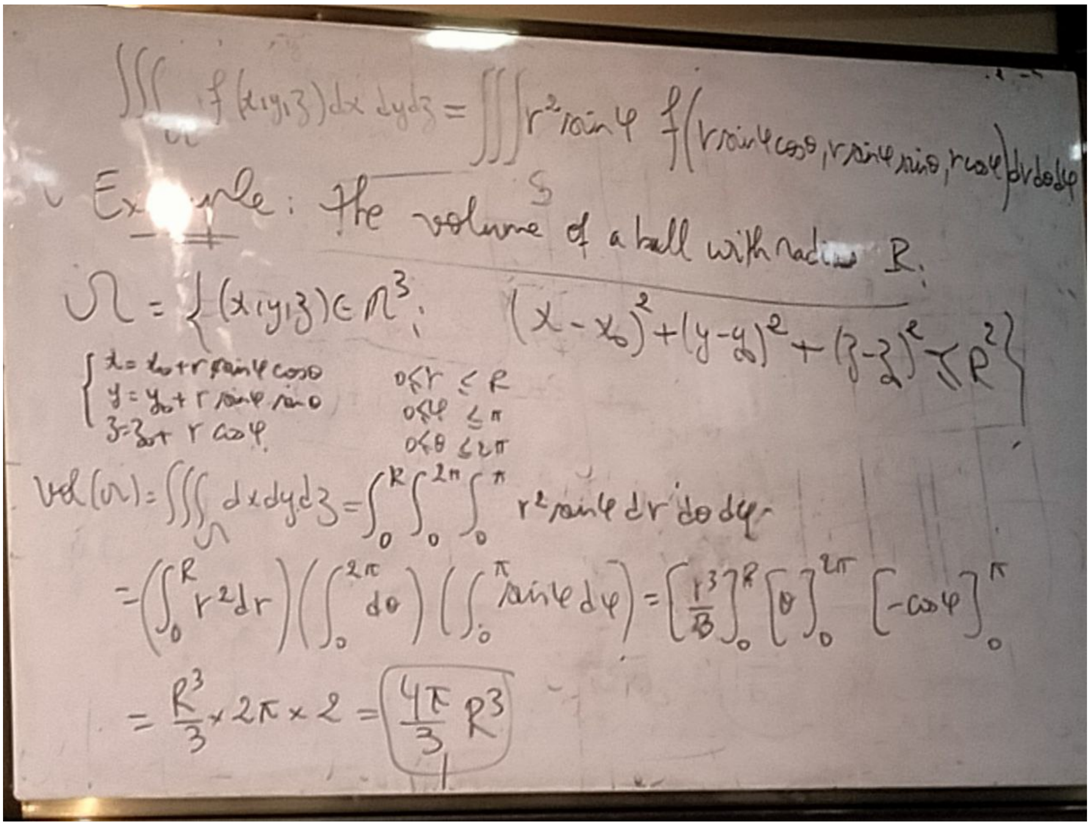


...

$$\begin{aligned} det_{(r,\theta,\varphi)}(x,y,z) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \\ &= \begin{vmatrix} sin(\varphi)cos(\theta) & -rsin(\varphi)sin(\theta) & rcos(\varphi)cos(\theta) \\ sin(\varphi)sin(\theta) & rsin(\varphi)cos(\theta) & rcos(\varphi)sin(\theta) \\ cos(\varphi) & 0 & -rsin(\varphi) \end{vmatrix} \\ &= r(rsin(\varphi)) \begin{vmatrix} sin(\varphi)cos(\theta) & -sin(\theta) & cos(\varphi)cos(\theta) \\ sin(\varphi)sin(\theta) & cos(\theta) & cos(\varphi)sin(\theta) \\ cos(\varphi) & 0 & -sin(\varphi) \end{vmatrix} \\ &= -r^2 sin(\varphi) \end{aligned}$$

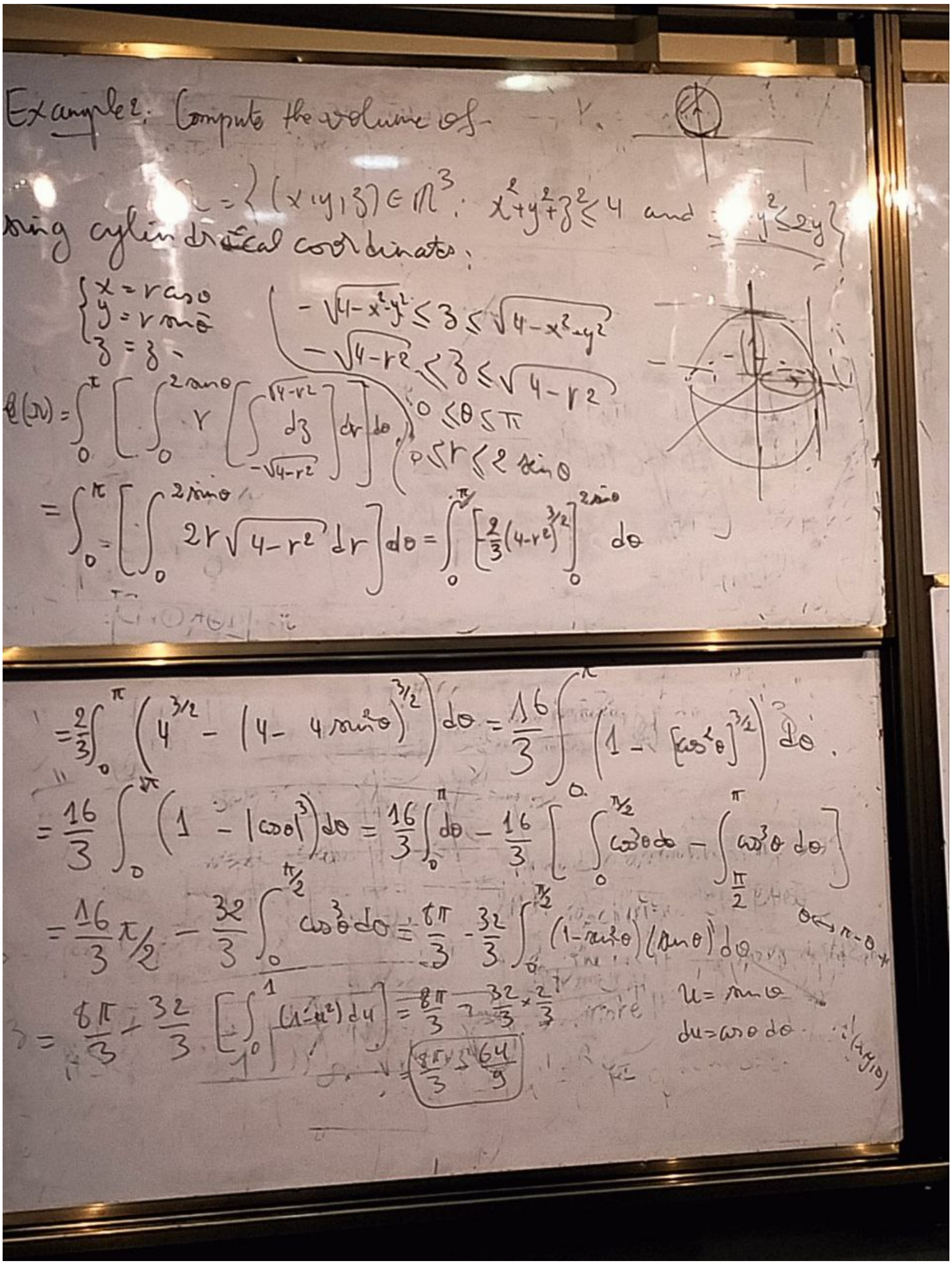
$$\iiint_{\Omega} f(x,y,z)dxdydz = \iint_{S} r^2 sin(\varphi) f(rsin(\varphi)cos(\theta), rsin(\varphi)sin(\theta), rcos(\varphi))drd\theta d\varphi$$

Example 7. <u>*the volume of a ball with radius R:*</u>

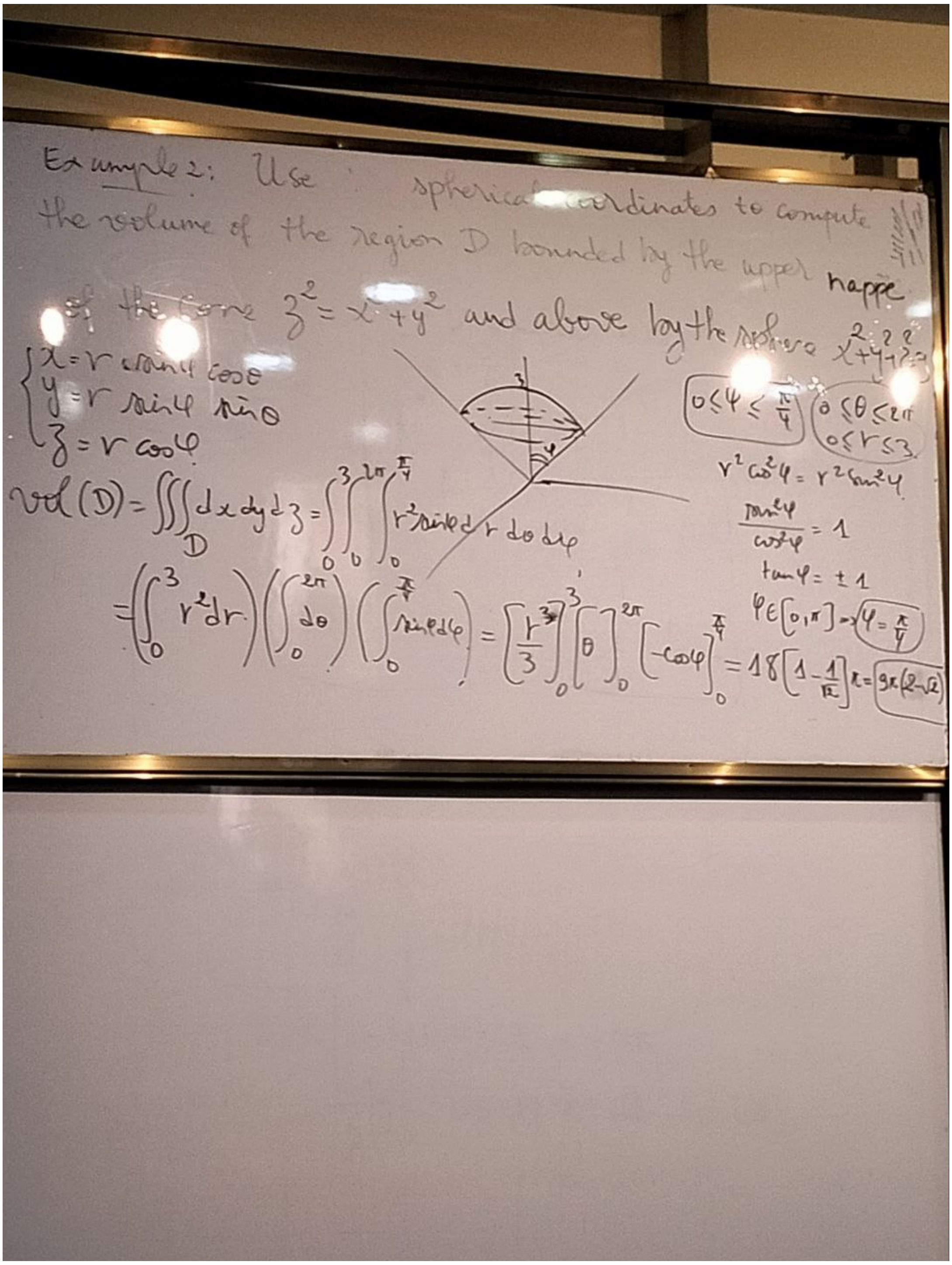


Example 8. <u>*Compute the volume of Intersection between ball and cylinder*</u> *Using cylindrical coordinates :*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4 \text{ and } x^2 + y^2 \leq 2y$$



Example 9. *Use spherical coordinates to compute the volume of the region D bounded by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$:*



5.4 Applications of triple integrals

1) If $f : D \to \mathbb{R}_+$ is a positive function on a bounded region $D \subset \mathbb{R}^2$; then:

$$\iint_D f(x,y)dxdy = \iint_D \left[\int_0^{f(x,y)} dz \right] dxdy = \iiint_\Omega dxdydz = Vol(\Omega)$$

Where $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \text{ and } 0 \le z \le f(x,y)\}$

2) Consider the solid $\Omega \subset \mathbb{R}^3$ bounded with the mass density function

$\rho : \Omega \to \mathbb{R} \quad , (x, y, z) \mapsto \rho(x, y, z)$

the mass of Ω is $m = \iiint_\Omega \rho(x, y, z)dxdydz$

In particular if $\rho \equiv K \implies m = kVol(\Omega)$

the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is:

$$\begin{cases} \bar{x} = \dfrac{\iiint_\Omega x\rho(x, y, z)dxdydz}{\iiint_\Omega \rho(x, y, z)dxdydz} \\ \bar{y} = \dfrac{\iiint_\Omega y\rho(x, y, z)dxdydz}{\iiint_\Omega \rho(x, y, z)dxdydz} \\ \bar{z} = \dfrac{\iiint_\Omega z\rho(x, y, z)dxdydz}{\iiint_\Omega \rho(x, y, z)dxdydz} \end{cases}$$

M_{yz}, M_{xz} and M_{xy} are called the first moments of Ω with respect the $(yz)-\quad, (xz)-\quad,$ and (xy)plar respectively

3)

$$I_{xy} = \iiint_\Omega z^2\rho(x, y, z)dxdydz$$

$$I_{xz} = \iiint_\Omega y^2\rho(x, y, z)dxdydz$$

$$I_{yz} = \iiint_\Omega x^2\rho(x, y, z)dxdydz$$

are the second moments (or moments of inertia) of Ω with respect the planes

$(yz)-\quad, (xz)-\quad,$ and (xy) respectively

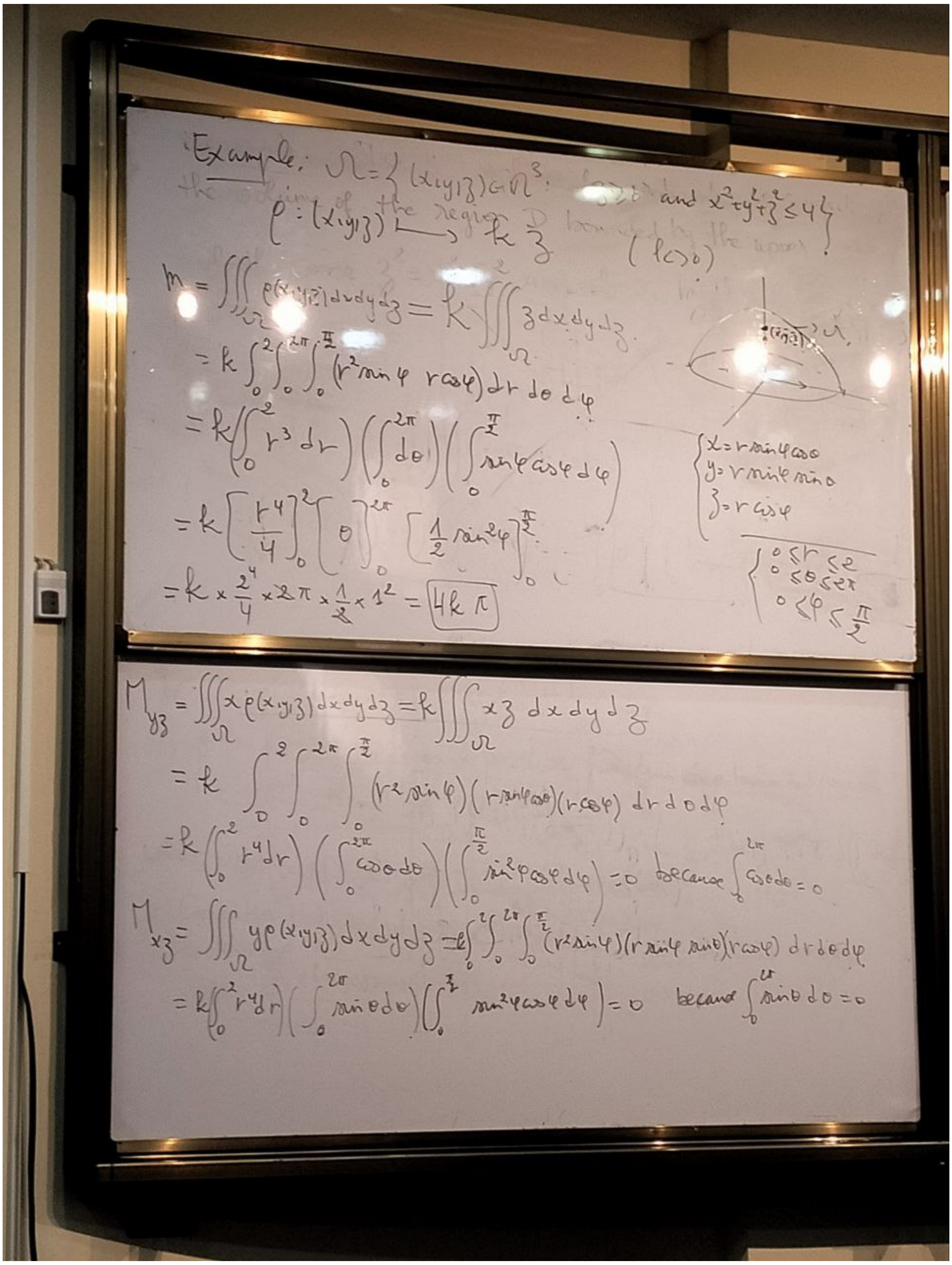
$I_x = I_{xy} + I_{xz} = \iiint_\Omega (y^2 + z^2)\rho(x, y, z)dxdydz$ the moment of inertia with respec the axe (ox)

$I_y = I_{xy} + I_{yz} = \iiint_\Omega (x^2 + z^2)\rho(x, y, z)dxdydz$ the moment of inertia with respec the axe (oy)

$I_z = I_{xz} + I_{yz} = \iiint_\Omega (x^2 + y^2)\rho(x, y, z)dxdydz$ the moment of inertia with respec the axe (oz)

Example 10.

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0 \ and \ x^2 + y^2 + z^2 \leq 4\}$$



Example: $\Omega = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \text{ and } x^2+y^2+z^2 \leq 4\}$

$\rho : (x,y,z) \longmapsto k\,z \quad (k>0)$

$$m = \iiint_\Omega \rho(x,y,z)\,dx\,dy\,dz = k\iiint_\Omega z\,dx\,dy\,dz$$

$$= k\int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (r^2\sin\varphi\ r\cos\varphi)\,dr\,d\theta\,d\varphi$$

$$= k\left(\int_0^2 r^3\,dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\,d\varphi\right)$$

$$= k\left[\frac{r^4}{4}\right]_0^2 \left[\theta\right]_0^{2\pi} \left[\frac{1}{2}\sin^2\varphi\right]_0^{\frac{\pi}{2}}$$

$$= k \times \frac{2^4}{4} \times 2\pi \times \frac{1}{2} \times 1^2 = \boxed{4k\pi}$$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \qquad \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

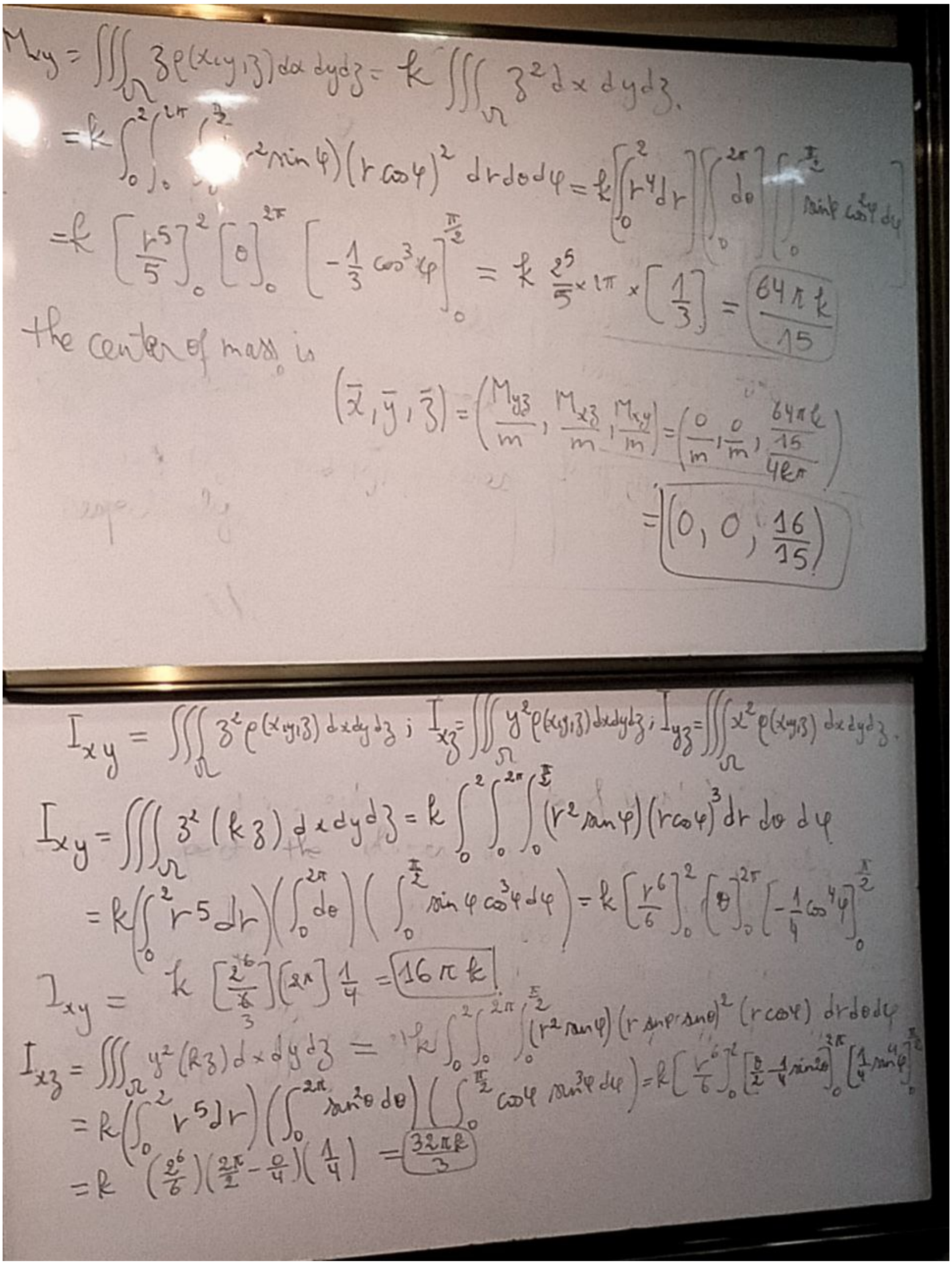
$$M_{yz} = \iiint_\Omega x\,\rho(x,y,z)\,dx\,dy\,dz = k\iiint_\Omega xz\,dx\,dy\,dz$$

$$= k\int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (r^2\sin\varphi)(r\sin\varphi\cos\theta)(r\cos\varphi)\,dr\,d\theta\,d\varphi$$

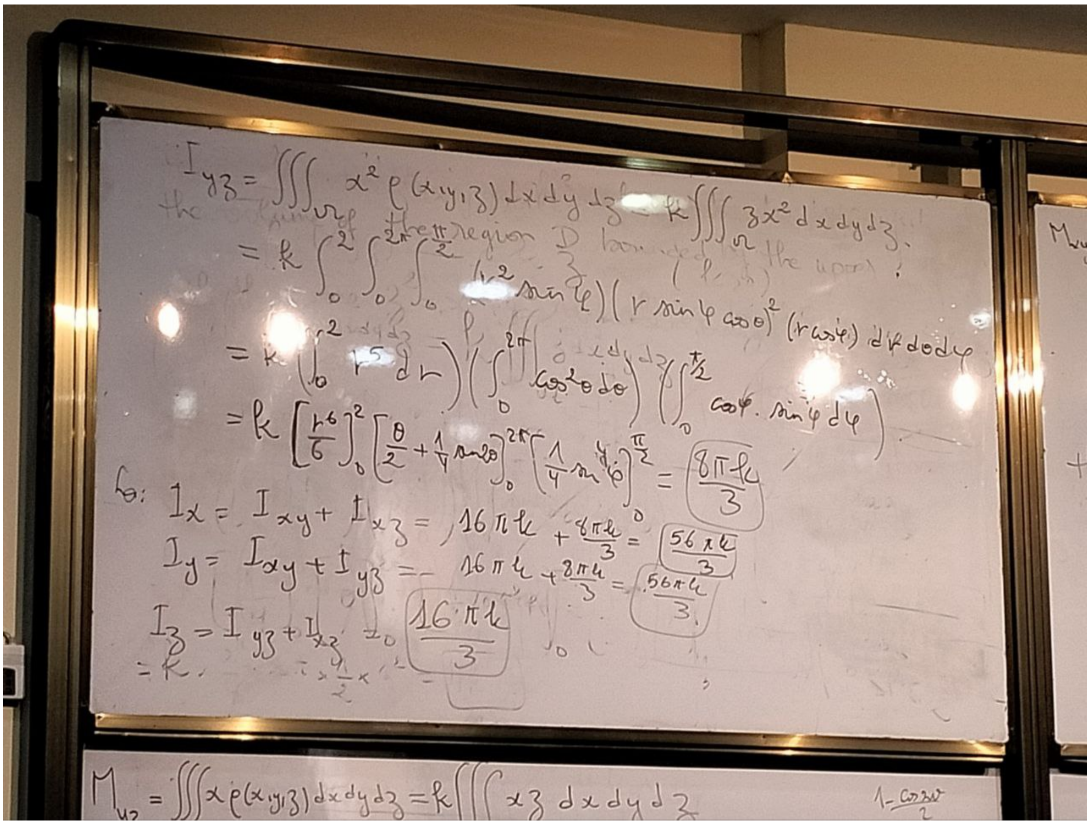
$$= k\left(\int_0^2 r^4\,dr\right)\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\varphi\,d\varphi\right) = 0 \text{ because } \int_0^{2\pi}\cos\theta\,d\theta = 0$$

$$M_{xz} = \iiint_\Omega y\,\rho(x,y,z)\,dx\,dy\,dz = k\int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (r^2\sin\varphi)(r\sin\varphi\sin\theta)(r\cos\varphi)\,dr\,d\theta\,d\varphi$$

$$= k\left(\int_0^2 r^4\,dr\right)\left(\int_0^{2\pi}\sin\theta\,d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\varphi\,d\varphi\right) = 0 \text{ because } \int_0^{2\pi}\sin\theta\,d\theta = 0$$



the last result equals $\frac{8\pi k}{3}$ *not* $\frac{32\pi k}{3}$



I_yz = ∭_Ω x² ρ(x,y,z) dx dy dz = k ∭_Ω z x² dx dy dz.
= k ∫_0^2 ∫_0^2π ∫_0^π/2 (r² sin φ)(r sin φ cos θ)² (r cos φ) dφ dθ dr
= k (∫_0^2 r^5 dr)(∫_0^2π cos²θ dθ)(∫_0^π/2 cos φ . sin³φ dφ)
= k [r^6/6]_0^2 [θ/2 + 1/4 sin 2θ]_0^2π [1/4 sin⁴φ]_0^π/2 = 8πk/3
So: I_x = I_xy + I_xz = 16πk + 8πk/3 = 56πk/3
I_y = I_xy + I_yz = 16πk + 8πk/3 = 56πk/3
I_z = I_yz + I_xz = 16πk/3
M_yz = ∭ x ρ(x,y,z) dx dy dz = k ∭ xz dx dy dz