

Chapter 01: Double integrals

Notes from prof zeglaoui course

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1 Reminder:

Recall that if $f : [a, b] \rightarrow \mathbb{R}$ is bounded.

for any subdivision $\sigma = \{x_0 = a < x_1 < x_2 < \dots < x_{n+1} = b\}$ of $[a, b]$

$$S(f, \sigma) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \quad \text{and} \quad s(f, \sigma) = \sum_{i=1}^n m_i(x_i - x_{i-1})$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$

called respectively, the upper and the lower Darboux sums of f on $[a, b]$ with respect to the subdivision σ ;

S : the area of the region plane delimited by:

$$\begin{cases} x = a & y = 0 \\ x = b & y = f(x) \end{cases}$$

(graph showing the lower and upper of an integral)

$$s(f, \sigma) \leq S \leq S(f, \sigma) \quad \text{if } \sigma_1 \subset \sigma_2, \text{ then:}$$
$$s(f, \sigma_1) \leq s(f, \sigma_2) \leq S \leq S(f, \sigma_2) \leq S(f, \sigma_1)$$

$$S^+ = \inf(S(f, \sigma)) \quad , \quad S^- = \sup(s(f, \sigma))$$

$$S^- \leq S \leq S^+$$

Definition 1. We say that f is Riemann integrable on $[a, b]$ if :

$$S^+ = S^- = S = \int_a^b f(x)dx$$

Definition 2. Riemann sums

Let σ a subdivision of $[a, b]$ and $\zeta = \{\zeta_1, \dots, \zeta_n\}$ / $\zeta_i \in [x_{i-1}, x_i]$

$$s(f, \sigma) \leq R(f, \sigma, \zeta) = \sum_{i=1}^n f(\zeta_i)(x_i - x_{i-1}) \leq S(f, \sigma)$$

If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, then:

$\exists(\sigma_n)$ a sequence of subdivision of $[a, b]$

$\exists(\zeta_n)$ associated with $(\sigma_n)[a, b]$ $\zeta_n = (\zeta_n^i)$, $\zeta_n^i \in [x_{i-1}^n, x_i^n]$

$$\lim_{n \rightarrow \infty} R(f, \sigma_n, \zeta_n) = \int_a^b f(x)dx$$

In particular:

$$\sigma_n = \{x_i^n = a + \frac{i}{n}(b-a) : i \in 0, \dots, n\}$$

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f(\zeta_i) = \int_a^b f(x)dx$$

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f(a + \frac{i}{n}(b-a)) = \int_a^b f(x)dx , x_i^n - x_{i-1}^n = \frac{b-a}{n}$$

2 Double integrals:

2.1 Definition and first properties:

Definition 3. Darboux sums- Riemann sums:

Let $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a bounded function

1) A subdivision σ of Δ is a set $\sigma = \{\Delta_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$

with $(x_i)_{i=1, \dots, n}$ is a subdivision of $[a, b]$ and $(y_j)_{j=0, \dots, m}$ a subdivision of $[c, d]$ we note : $A(\Delta_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$

and $\delta(\sigma) = \max_{i,j} \text{diam}(\Delta_{ij})$

2)

$$\forall i, j \quad M_{ij} = \sup_{(x,y) \in \Delta_{ij}} f(x, y) \quad \text{and} \quad m_{ij} = \inf_{(x,y) \in \Delta_{ij}} f(x, y)$$

the upper Darboux sum of f with respect to σ is :

$$S(f, \sigma) = \sum_{i=1}^n \sum_{j=1}^m M_{ij} A(\Delta_{ij})$$

the lower Darboux sum of f with respect to σ is :

$$s(f, \sigma) = \sum_{i=1}^n \sum_{j=1}^m m_{ij} A(\Delta_{ij})$$

3) if we give $\zeta = \{(\zeta_i^1, \zeta_j^2) \mid \zeta_i^1 \in [x_{i-1}, x_i] \text{ and } \zeta_j^2 \in [y_{j-1}, y_j]\}$

The Riemann sum with respect to σ and ζ is :

$$R(f, \sigma, \zeta) = \sum_{i=1}^n \sum_{j=1}^m f(\zeta_i^1, \zeta_j^2) A(\Delta_{ij})$$

$$R(f, \sigma, \zeta) = \sum_{i=1}^n \sum_{j=1}^m f(\zeta_i^1, \zeta_j^2) (x_i - x_{i-1})(y_j - y_{j-1})$$

2.2 Example:

$$\begin{aligned} f(x, y) &= \alpha \quad ; \forall (x, y) \in \Delta \\ M_{ij} &= m_{ij} = f(\zeta_i^1, \zeta_j^2) = \alpha \\ S(f, \sigma) &= s(f, \sigma) = R(f, \sigma, \zeta) = \alpha(b-a)(d-c) = \alpha A(\Delta) \end{aligned}$$

Definition 4. Riemann integrability:

With the same notations above, we put:

$$S^+(f) = \inf(S(f, \sigma)) \quad \text{and} \quad S^-(f) = \sup(s(f, \sigma))$$

We say that f is Riemann integrable on Δ if $S^+(f) = S^-(f)$

$$\Leftrightarrow \forall \epsilon > 0; \exists \sigma \quad S(f, \sigma) - s(f, \sigma) < \epsilon$$

and we denote the common value by $\iint_{\Delta} f(x, y) dx dy$

$$\Leftrightarrow \exists (\sigma_n) \text{ of subdivisions of } \Delta / \lim S(f, \sigma_n) = \lim s(f, \sigma_n)$$

Corollary: if $f : \Delta = [a, b] \times [c, d]$ is Riemann integrable then:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{(b-a)(d-c)}{n^2} \sum_{i=1}^n \sum_{j=1}^n f(a + \frac{i}{n}(b-a), c + \frac{j}{n}(d-c)) \\ = \iint_{\Delta} f(x, y) dx dy \end{aligned}$$

2.3 Example:

$$f(x, y) = x^2 + y^2 \quad , \Delta = [0, 1]^2 = [0, 1] \times [0, 1]$$

Definition 5. Riemann integrability on a bounded domain:

Let D be a bounded subset of \mathbb{R}^2 and $f : D \rightarrow \mathbb{R}$ a bounded function.

We say that f is Riemann integrable on D if for any rectangle $\Delta \supset D$, the function :

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in \Delta \setminus D \end{cases}$$

is Riemann integrable, in this case :

$$\iint_D f(x, y) dx dy := \iint_{\Delta} \tilde{f}(x, y) dx dy$$

Remark: Any continuous function is Riemann integrable.

2.4 Properties:

1) Linearity: If $f, g : D \rightarrow \mathbb{R}$ are Riemann integrable then: $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Riemann integrable and

$$\iint_D (\alpha f + \beta g)(x, y) dx dy = \alpha \iint_D f(x, y) dx dy + \beta \iint_D g(x, y) dx dy$$

2)

If $f \geq 0$ on $D \implies \iint_D f(x, y) dx dy \geq 0$ which is the volume of $\omega = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}$

$$\begin{aligned} \text{So; if } f \geq g \implies \iint_D [f(x, y) - g(x, y)] dx dy \\ = \text{vol}\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } g(x, y) \leq z \leq f(x, y)\} \end{aligned}$$

3) Chasles's relation:

If $D = D_1 \cup D_2$ such that $D_1 \cap D_2$ is at most a curve then:

$$\iint_D f(x, y) dx dy = \iint_{D_1} f(x, y) dx dy + \iint_{D_2} f(x, y) dx dy$$

Definition 6. $\text{Vol}(D) = A(D) = \iint_D dx dy$ and the average of f on D is: $\frac{1}{\text{Vol}(D)} \iint_D f(x, y) dx dy = \frac{\iint_D f(x, y) dx dy}{\iint_D dx dy}$

Remark:

If $f \equiv C$, the average of f is C

2.5 Example:

D : the triangle with vertices $(0, 0), (1, 0), (0, 1)$ (graph) and $f(x, y) = xy$

$$\tilde{f}(x, y) = \begin{cases} xy & \text{if } x + y \leq 1 \\ 0 & \text{if } x + y > 1 \end{cases}$$

$$\forall x, y \in [0, 1] \times [0, 1]$$

$$\iint_D xy dxdy = \lim_{n \rightarrow +\infty} \frac{(1-0)(1-0)}{n^2} \sum_{i=1}^n \sum_{j=1}^n \tilde{f}\left(\frac{i}{n}, \frac{j}{n}\right)$$

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^n \left[\sum_{j=1}^n \frac{i}{n} \times \frac{j}{n} \right]$$

$$U_n = \frac{1}{n^2} \left[\sum_{i=1}^n \sum_{j=1}^n \frac{i}{n} \times \frac{j}{n} \right]$$

$$U_n = \frac{1}{n^4} \sum_{i=1}^n i \left[\sum_{j=1}^{n-i} j \right] = \frac{1}{n^4} \sum_{i=1}^n i \left(\frac{(n-i)(n-i+1)}{2} \right)$$

$$U_n = \frac{1}{2n^4} \sum_{i=1}^n i(n^2 + i^2 - 2in + n - i)$$

$$U_n = \frac{1}{2n^4} \sum_{i=1}^n [i(n^2 + n) - i^2(2n + 1) + i^3]$$

$$U_n = \frac{1}{2n^4} \left[(n^2 + n) \frac{(n+1)n}{2} - (2n+1) \frac{n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4} \right]$$

$$U_n = \frac{1}{4}(1 + \frac{1}{n})^2 - \frac{1}{12}(1 + \frac{1}{n})(2 + \frac{1}{n}) + \frac{1}{8}(1 + \frac{1}{n})^2$$

$$\iint_D xy dxdy = \lim_{n \rightarrow +\infty} U_n = \frac{1}{4} - \frac{1}{12} \cdot 2 + \frac{1}{8} = \frac{1}{24}$$

Theorem 1. *Fubini's theorem:*

Let $f : D \rightarrow \mathbb{R}$ Riemann integrable

1) if

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \exists g_1, g_2 : [a, b] \rightarrow \mathbb{R} \text{ continuous} \mid g_1(x) \leq y \leq g_2(x), \forall x \in [a, b]\}$$

then: $\iint_D f(x, y) dx dy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$

2) If

$D = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \exists h_1, h_2 : [c, d] \rightarrow \mathbb{R} \text{ continuous} \mid h_1(y) \leq x \leq h_2(y), \forall y \in [c, d]\}$

then: $\iint_D f(x, y) dx dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$

2.6 Remark:

We can use the Chasles's formula to apply this theorem.

2.7 Example 01:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$$

$$D_1 = \{(x, y) \in D : y \geq 0\}; D_2 = \{(x, y) \in D : y \leq 0\}$$

$$D_1 = \{(x, y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } 0 \leq y \leq \sqrt{R^2 - x^2}\}$$

$$D_2 = \{(x, y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } -\sqrt{R^2 - x^2} \leq y \leq 0\}$$

...

Example: $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$

$D_1 = \{(x, y) \in D : y \geq 0\}; D_2 = \{(x, y) \in D : y \leq 0\}$

$D_1 = \{(x, y) \in D : -R \leq x \leq R \text{ and } 0 \leq y \leq \sqrt{R^2 - x^2}\}$

$D_2 = \{(x, y) \in D : -R \leq x \leq R \text{ and } -\sqrt{R^2 - x^2} \leq y \leq 0\}$

$\iint_D f(x, y) dx dy = \iint_{D_1} f(x, y) dx dy + \iint_{D_2} f(x, y) dx dy = \int_{-R}^R \left[\int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} f(x, y) dy \right] dx$

$= \int_{-R}^R \sqrt{R^2 - x^2} dx + \int_{-R}^R \sqrt{R^2 - x^2} dx = 2 \int_0^R \sqrt{R^2 - x^2} dx = 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2 - R^2 \sin^2 \theta} R \cos \theta d\theta$

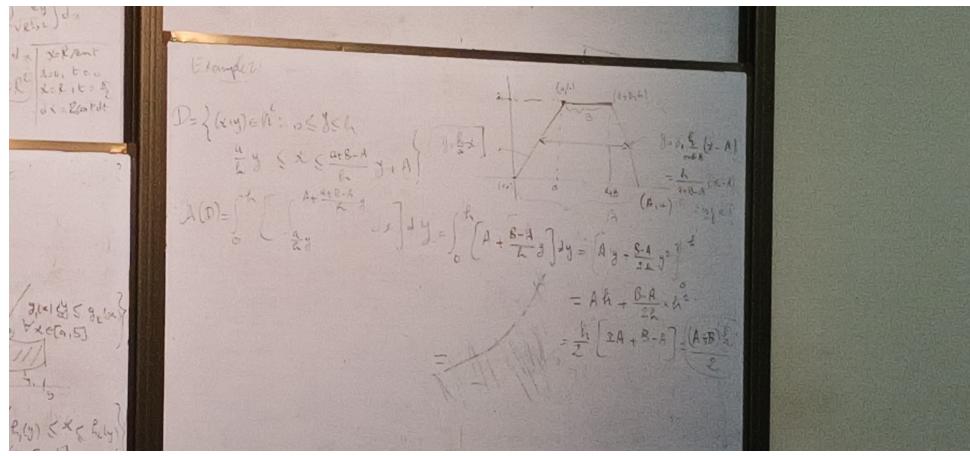
$= 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2(1 - \sin^2 \theta)} R \cos \theta d\theta = 4R^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = 2R^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2R^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = 2R^2 \left[\frac{\pi}{2} + 0 \right] = \pi R^2$

$D = \{(x, y)\}$

1.1.7 Computation of double integrals

2.8 Example 02:

(trapezoid area proof)



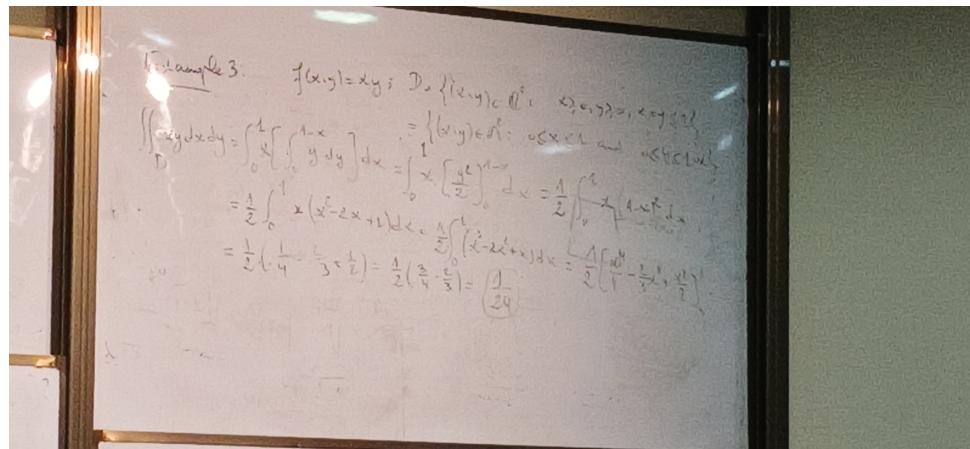
2.9 Example 03:

$$f(x,y) = xy$$

$$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$$

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1-x\}$$

...



2.10 Example 04:

Let us compute volume of Ω bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$

Defining region: $0 \leq y \leq 1$, $-\sqrt{y^2} \leq x \leq \sqrt{y^2 - y^2}$

$$V_U = \int_0^1 \left[\int_{-\sqrt{y^2 - y^2}}^{\sqrt{y^2 - y^2}} (y - y^2 - x^2) dx \right] dy = \int_0^1 \left[\left(\frac{1}{2}x^2 - \frac{1}{3}y^2x^3 \right) \Big|_{-\sqrt{y^2 - y^2}}^{\sqrt{y^2 - y^2}} \right] dy$$

$$= \int_0^1 \left[\frac{4}{3}(y - y^2)\sqrt{y^2 - y^2} - \frac{1}{3}(y - y^2)^3 \right] dy = \frac{4}{3} \int_0^1 (y - y^2)^2 dy = \frac{4}{3} \int_0^1 (y^2 - 2y^3 + y^4) dy$$

$$= \frac{4}{3} \left[\frac{1}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{5}y^5 \right]_0^1 = \frac{4}{3} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{4}{3} \cdot \frac{1}{30} = \frac{2}{45}$$

Example 4: Let us compute the volume of the bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 3 - y$.

$$\begin{aligned} & \int_D (3 - y) - (1 - x^2 - y^2) dA \\ & \Rightarrow \int_D (2 - x^2 - y^2 - y) dA \\ & \Rightarrow \int_D (2 - x^2 - y^2 - y - 2) dA = \int_D (-x^2 - y^2 - y) dA \\ & \text{Region } D: \{(x, y) | 0 \leq x^2 + y^2 \leq 1\} \end{aligned}$$

$$V_U = \iint_D (-x^2 - y^2 - y) dA$$