# Chapter 01: Double integrals

Notes from prof zeghlaoui course

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## 1 Reminder:

Recall that if $f : [a, b] \to \mathbb{R}$ is bounded.

for any subdivision $\sigma = \{x_0 = a < x_1 < x_2 < ... < x_{n+1} = b\}$ of $[a, b]$

$$S(f, \sigma) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad and \quad s(f, \sigma))$$

$$S(f, \sigma) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \text{where } m_i = inf(f(x)) \quad x \in [a, b] \quad M_i sup(f(x)), x \in [x_{i-1}, x_i]$$

called respectively, the upper and the lower Darbous sumf of $f$ on $[a, b]$ with respect to the subdivision G;

S: the area of the region plane delimited by:

$$\begin{cases} x = 1 & y = 0 \\ x = b & y = f(x) \end{cases}$$

(graph showing the lower and upper of an integral)

$$s(f, \sigma) \leq S \leq S(f, \sigma) \quad if \sigma_1 \subset \sigma_2, \text{ then:}$$
$$s(f, \sigma_1) \leq s(f, \sigma_2) \leq S \leq S(f, \sigma_2) \leq S(f, \sigma_1)$$

$$S^+ = inf(S(f, \sigma)) \quad , \quad S^- = sup((s(f, \sigma)))$$

$$S^- \leq S \leq S^+$$

**Definition 1.** *We say that $f$ is Riemann integral on $[a,b]$ if :*

$$S^+ = S^- = S = \int_a^b f(x)dx$$

**Definition 2.** *Riemann sums*

*Let $G$ a subdivision of $[a,b]$ and $\zeta = \{\zeta_1, \ldots, \zeta_n\} \quad /\zeta \in [x_i - 1, x_i]$*

$$s(f,\sigma) \leq R(f,\sigma,\zeta) = \sum_{i=1}^{n} f(\zeta_i)(x_i - x_i - 1) \leq S(f,\sigma)$$

*If $f : [a,b] \to \mathbb{R}$ is Riemann integral, then:*

*$\exists(\sigma_n)$ a sequence of subdivision of $[a,b]$*

*$\exists(\zeta_n)$ associated with $(\sigma_n)[a,b] \quad \zeta_n = (\zeta_n^i),\ \zeta_n^i \in [x_{i-1}^n, x_i^n]$*

$\lim\limits_{n\to\infty} R(f,\sigma_n,\zeta_n) = \int_a^b f(x)dx$

*Inparticular:*

$$\sigma_n = \{x_i^n = a + \frac{i}{n}(b-a) : i \in 0, \ldots, n\}$$

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\zeta_i) = \int_a^b f(x)dx$$

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{i}{n}(b-a)) = \int_a^b f(x)dx \quad , x_i^n - x_{i-1}^n = \frac{b-a}{n}$$

# 2 Double integrals:

## 2.1 Definition and first properties:

**Definition 3.** *Darboux sums- Riemann sums:*

*Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a bounded funtion*

*1) A subdivision $\sigma$ of $\Delta$ is a set $\sigma = \{\Delta_y = [x_{i-1}, x_i] \times [y_{i-1}, y_i]\}$*

*with $(x_i)_{i=1,\ldots,n}$ is a subdivision of $[a,b]$ and $(y_j)_{j=0,\ldots,m}$ a subdivision of $[c,d]$ we note : $A(\Delta_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$*

*and $\delta(\sigma) = Max_{i,j} A(\Delta_{ybar})$*

*2)*

$$\forall i,j \quad /M_y = sup_{(x,y)\in(\Delta_y)} f(x,y) \quad and \quad m_i = inf_{(x,y)\in(\Delta_y)} f(x,y)$$

*the upper Darbous sum of $f$ with respect $G$ is :*

$$S(f,\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}A(\Delta_y)$$

*the lower Darbous sum of $f$ with respect $G$ is :*

$$S(f,\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}A(\Delta_y)$$

*3) if we give $\zeta = \{(\zeta_i^1, \zeta_j^2)\ mid\zeta_i^1 \in [x_{i-1}, x_i]\ and\ \zeta_i^1 \in [y_{i-1}, y_i]\}$*

*The Riemann sum with respet to $G$ and $\zeta$ is :*

$$R(f,\sigma,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\zeta_i^1, \zeta_j^2)A(\Delta_y)$$

$$R(f,\sigma,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\zeta_i^1, \zeta_j^2)(x_i - x_{i-1})(y_i - y_{i-1})$$

## 2.2 example:

$$f : (x,y) = \alpha \quad ; \forall (x,y) \in \Delta$$
$$M_{ij} = m_{ij} = f(\zeta_i^1, \zeta_j^2) = \alpha$$
$$S(f,\sigma) = s(f,\sigma) = R(f,\sigma,\zeta) = \alpha(b-a)(c-d) = \alpha A(\Delta)$$

**Definition 4.** *Riemann integrability:*

*with the same notations above , we put:*

$$S^+(f) = inf(S(f,\sigma))\ and\ S^- = sup(s(f,\sigma))$$

*We say that $f$ is : Riemann integrable on $\Delta$ if $S^+(f) = S^-(f)$*

$$\Leftrightarrow \forall_{\epsilon>0}; \exists(\sigma^2)/ \quad S(f,\sigma^2) - s(f,\sigma^2) < \epsilon = \int\int_\Delta f(x,y)dxdy$$

$$\Leftrightarrow \exists(\sigma_n) of\ subdivsion\ of\ \Delta / \lim S(f,\sigma_n) = \lim s(l,\sigma_n)$$

*corollay: if $f : \Delta\, [a,b] \times [c,d]$ then :*

$$\lim_{n\to+\infty} \frac{(b-a)(d-c)}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} f(a + \frac{i}{n}(b-a), c + \frac{d}{n}(d-c))$$
$$= \int\int_\Delta f(x,y)dxdy$$

## 2.3 Example:

$$f(x,y) = x^2 + y^2 \quad , \Delta = [0,1]^2 = [0,1] \times [1,0]$$

**Definition 5.** *Riemann integrability on a bounded domaine:*

*Let d be a bounded subsed on* $\mathbb{R}^2$ *and* $f : D \to \mathbb{R}$ *a bounded function*

*we say that* $f$ *is Riemann integrable on* $D$ *if for any rectangle* $D \subset \Delta$*, the function :*

$$f^{tilde}(x,y) = \begin{cases} f(x,y) & if(x,y) \in D \\ 0 & if(x,y) \in \Delta - D \end{cases}$$

*is Riemann integrable, in this case :*

$$\int\int f(x,y)dxdy := \in \int f^{tilde}(x,y)dxdy$$

***Remark:*** *Any continuous function is Riemann integrable*

## 2.4 properties:

*1)Linearity: if* $f, g : D \to \mathbb{R}$ *are Riemann integrables then:* $\forall \alpha, \beta \in \mathbb{R} \quad , \alpha f + \beta y$ *is Riemann inegrable and*

$$\int\int_D (\alpha f + \beta y)(x,y)dxdy = \alpha \int\int_D f(x,y)(x,y)dxdy + \beta \int\int_D g(x,y)(x,y)dxdy$$

*2)*

*If* $f \geq 0 on D \implies \int\int_D f(x,y)dxdy \geq$ *with is the volume of* $\omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D and 0 \leq z \leq f(x,y)\}$

*So ; if* $f \geq g \implies \int\int_D [f(x,y) - g(x,y)]\, dxdy$

$= vol\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } g(x,y) \leq z \leq f(x,y)\}$

*3)Chasles's relation:*

*If* $D = D_1 \cup D_2$ *such that* $D_1 \cap D_2$ *is at most a curve then :*

$$\int\int_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$

**Definition 6.** $Vol(D) = A(D) = \int\int_D dxdy$ *and the average of* $f$ *on* $D$ *is :* $\dfrac{1}{Vol(D)} \int\int_D f(x,y)dxdy = \dfrac{\int\int_D f(x,y)dxdy}{\int\int dxdy}$

**Remark:**

If $f \equiv C \quad$ , the average of $f$ is $C$

## 2.5 Example:

$D$: the triangle with vertices $(0,0),(1,0),(0,1)$ (graph) and $f(x,y)=xy$

$$\tilde{f}(x,y)=\begin{cases} xy \quad if \quad x+y\leq 1 \\ 0 \quad if \quad x+y>1 \end{cases}$$

$\forall_{x,y}\in[0,1]\,[0,1]$

$$\int\int_D xydxdy=\lim_{n\to+\infty}\frac{(1-0)(1-0)}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{f}(\frac{i}{n},\frac{d}{n})$$

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\frac{i}{n}\times\frac{d}{n}\right]$$

$$U_n=\frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i}{n}\times\frac{d}{n}\right]$$

$$U_n=\frac{1}{n^4}\sum_{i=1}^{n}i\left[\sum_{j=1}^{n-i}j\right]==\frac{1}{n^4}\sum_{i=1}^{n}i\left(\frac{(n-i)(n-i+1)}{2}\right)$$

$$U_n=\frac{1}{2n^4}\sum_{i=1}^{n}i(n^2+i^2-2in+n-i)$$

$$U_n=\frac{1}{2n^4}\sum_{i=1}^{n}\left[i(n^2+n)-i^2(2n+1)+i^3\right]$$

$$U_n=\frac{1}{2n^4}\left[(n^2+n)\frac{(n+1)n}{2}-(2n+2)\frac{n(n+1)(2n+1)}{6}+\frac{(n^2)(n+1)^2}{4}\right]$$

$$U_n=\frac{1}{4}(1+\frac{1}{n})^2-\frac{1}{12}(1+\frac{1}{n})(2+\frac{1}{n})^2+\frac{1}{8}(1+\frac{1}{n})^2$$

$$\iint_D xydxdy=\lim_{n\to+\infty}U_n=\frac{1}{4}-\frac{1}{12}*4+\frac{1}{8}=\frac{1}{24}$$

**Theorem 1.** *Fubini's theorem;;*

*Let $f:D\to\mathbb{R}$ Riemann integrable*

*1) if*

$$D=\{(x,y)\in\mathbb{R}^2:\quad a\leq x\leq b\ and\ \exists g_1,g_2:[a,b]\to\mathbb{R}\ continuous\mid\begin{cases} g_1(x)\leq y\leq g_2(x) \\ \forall x\in[a,b]\end{cases}\quad\}$$

*then :* $\iint_D f(x,y)dxdy=\int_a^b\left[\int_{g_1(x)}^{g_2(x)}f(x,y)dy\right]dx$

*2) If*

$$D = \{(x,y) \in \mathbb{R}^2 : \quad c \leq y \leq d \ \textit{and} \ \exists h_1, h_2 : [c,d] \to \mathbb{R} \ \textit{continuous} \begin{cases} h_1(y) \leq\leq h_2(y) \\ \forall y \in [c,d] \end{cases} \quad \}$$

*then :* $\iint_D f(x,y)dxdy = \int_a^b \left[ \int_{h_1(x)}^{h_2(x)} f(x,y)dx \right] dy$

## 2.6 Remark:

We can use the Chasles's formula to apply this theorem

## 2.7 Example 01:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$$
$$D_1 = \{x,y\} \in D : \quad y \geq 0; D_2 = \{(x,y) \in D : \quad y \leq 0\}$$
$$D_1 = \{(x,y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } \quad 0 \leq y \leq \sqrt{R^2 - x^2}\}$$
$$D_2 = \{(x,y) \in \mathbb{R}^2 : -R \leq x \leq R \text{ and } \quad -\sqrt{R^2 - x^2} \leq y \leq 0\}$$

...

## 2.8 Example 02:

(trpezoid area proof )

## 2.9 Example 03:

$$f(x,y) = xy$$
$$D = \{(x,y) \in \mathbb{R}^2 : \quad x \geq 0, y \geq 0, x + y \leq 1\}$$
$$D = \{(x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$

## 2.10 Example 04:

Let us compute volume of $\Omega$ bounded above by the parboloid $z = 1 - x^2 - y^2$ and beloo by the plane $z = 1 - y$

$$S^+(f) = \inf_\sigma S(f,\sigma) \quad , \quad S^-(f) = \sup_\sigma s(f,\sigma)$$