Chapter 04: Sequences of Functions

Notes from Prof. Zeglaoui's Course

February 10, 2026

1 Sequences of Functions

1.1 Convergence of a Sequence of Functions

Definition 1. *Let D be a subset (non-empty) of $\mathbb{R}$. We denote by $\mathcal{F}(D, \mathbb{R})$ the $\mathbb{R}$-vector space of real functions from $D \to \mathbb{R}$.*

A ***sequence of functions*** *over D is a sequence in $\mathcal{F}(D, \mathbb{R})$, i.e.:*

$$\mathbb{N} \to \mathcal{F}(D, \mathbb{R}), \quad n \mapsto f_n : \begin{cases} D \to \mathbb{R} \\ x \mapsto f_n(x) \end{cases}$$

It is often denoted by (f_n) or $(f_n)_{n \in \mathbb{N}}$.

Definition 2. *Let (f_n) be a sequence of functions over D. We say that (f_n)* ***pointwise converges****, or simply* ***converges****, to $f \in \mathcal{F}(D, \mathbb{R})$ if:*

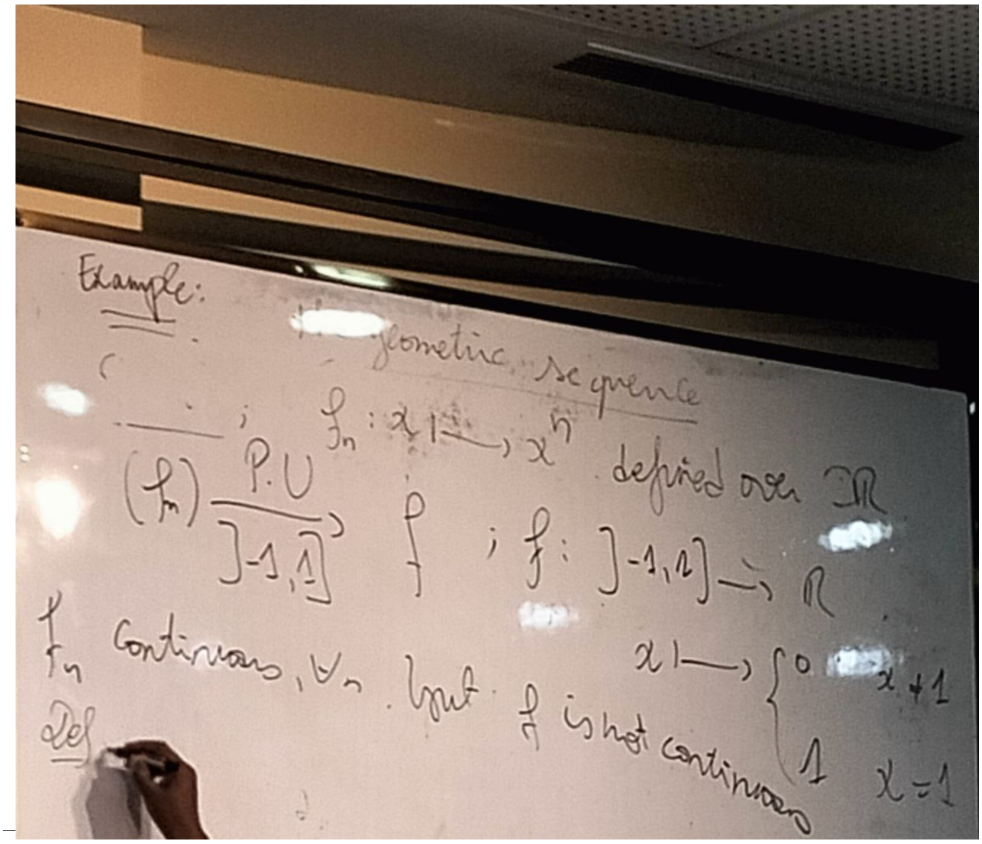
$$\forall x \in D, \ \lim_{n \to +\infty} f_n(x) = f(x)$$

or equivalently, $(f_n) \xrightarrow{\text{pointwise}} f$.

In logical form:

$$\forall x \in D, \ \forall \epsilon > 0, \ \exists N = N(\epsilon, x) : \forall n \geq N, \ |f_n(x) - f(x)| < \epsilon.$$

Example 1. *...*



Definition 3 (Uniform Convergence). *We say that* $(f_n) \xrightarrow{C_V} f$ *if for all* $n \in \mathbb{N}$, $(f_n - f)$ *is bounded over* D *and*

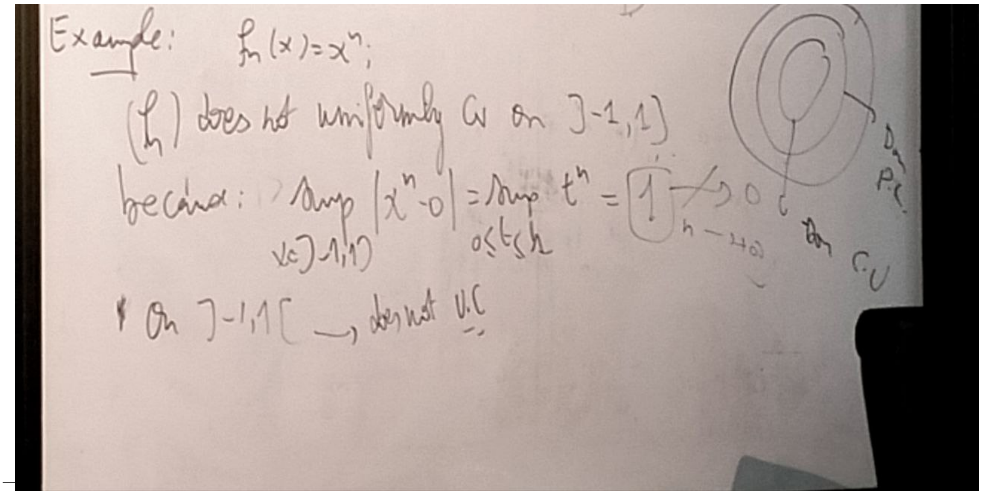
$$\lim_{n \to +\infty} \left(\sup_D |f_n - f| \right) = 0.$$

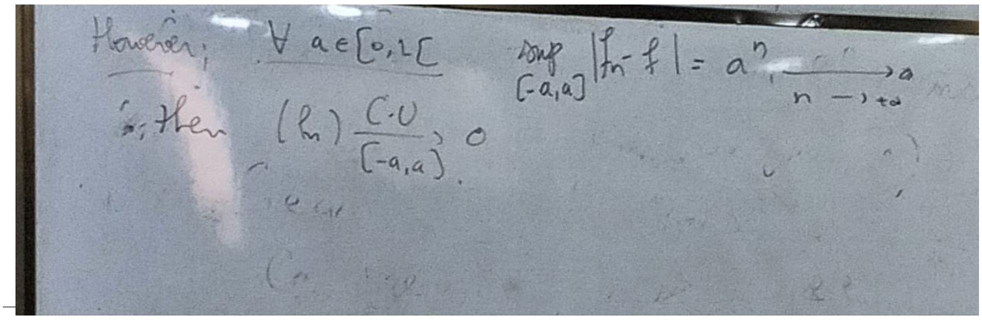
Equivalently:

$$\forall \epsilon > 0,\ \exists N = N(\epsilon) : \forall n \geq N,\ \forall x \in D,\ |f_n(x) - f(x)| < \epsilon.$$

Remark 1. *Uniform convergence implies pointwise convergence:* $(f_n) \xrightarrow{C_V} f \Rightarrow (f_n) \xrightarrow{P_V} f$.

Example 2. ...





1.2 Properties of Uniformly Convergent Sequences of Functions

Theorem 1 (Continuity)**.** *If*

$$\begin{cases} \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ f_n \text{ is continuous on } D, \\ \text{and } (f_n) \xrightarrow{C.V} f \text{ on } D, \end{cases}$$

then f is continuous on D.

In other words:

$$\forall x_0 \in D,\ \lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(\lim_{n \to +\infty} f_n(x) \right) = \lim_{n \to +\infty} \left(\lim_{x \to x_0} f_n(x_0) \right) = f(x_0).$$

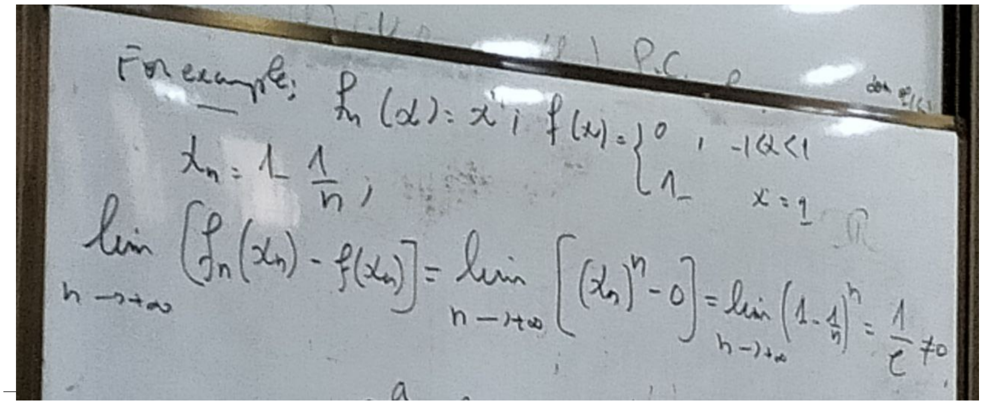
Remark 2. *If $(f_n) \xrightarrow{P.C} f$ and there exists a sequence $(x_n) \subset D$ such that*

$$\lim_{n \to +\infty} [f_n(x_n) - f(x_n)] \neq 0,$$

then (f_n) does not converge uniformly to f on D, because:

$$(f_n) \xrightarrow{C.V} f \implies \forall (x_n) \subset D,\ \lim_{n \to +\infty} [f_n(x_n) - f(x_n)] = 0.$$

Example 3. ...



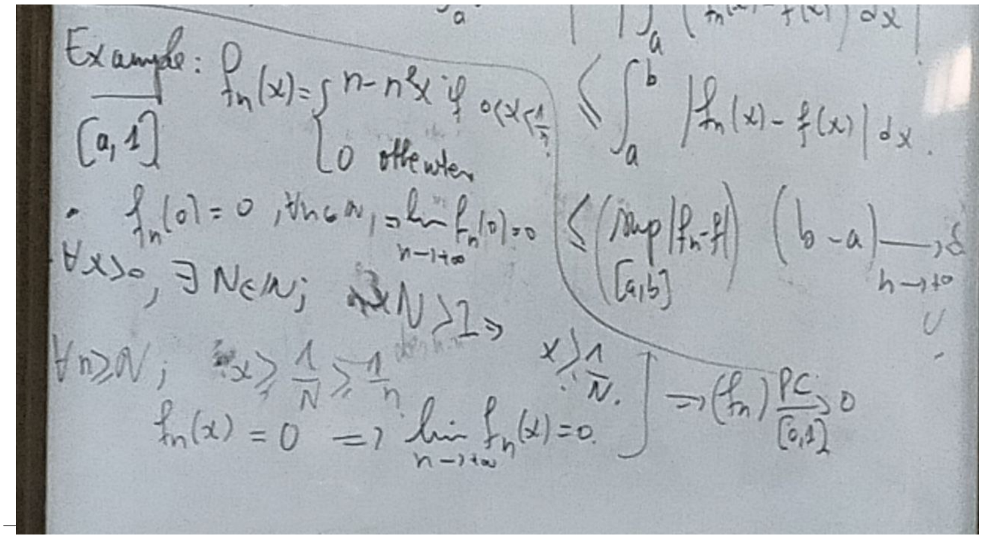
Theorem 2 (Integrability)**.** *If $f_n : [a, b] \to \mathbb{R}$ is Riemann-integrable for all $n \in \mathbb{N}$, and $(f_n) \xrightarrow{C.V} f$ on $[a, b]$, then:*

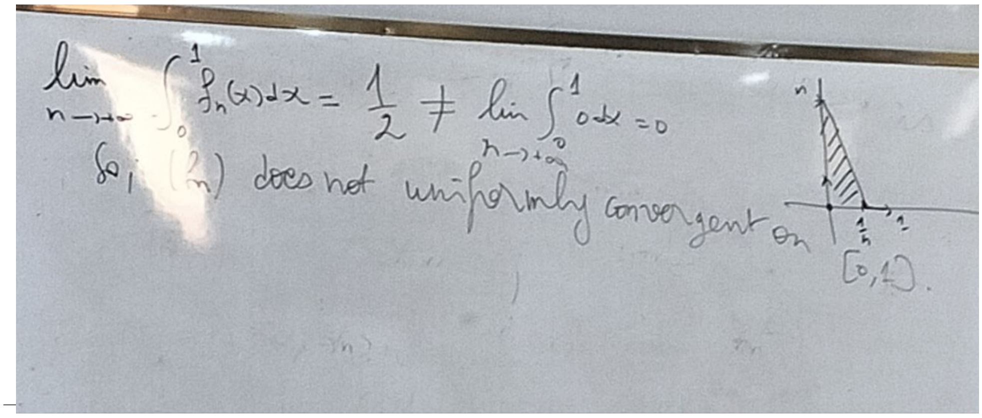
$$\lim_{n \to +\infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x = \int_a^b \left(\lim_{n \to +\infty} f_n(x) \right) \mathrm{d}x.$$

Proof:

$$\begin{aligned}
\left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| &= \left|\int_a^b (f_n(x) - f(x))\,\mathrm{d}x\right| \\
&\leq \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \\
&\leq \left(\sup_{[a,b]} |f_n - f|\right)(b-a) \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

Example 4. ...





Theorem 3 (Differentiability)**.** *Let I be an interval and $(f_n)_n$ a sequence of class C^1 functions on I such that:*

$$(f'_n)_n \xrightarrow{C.V} g \quad \textit{and} \quad \exists x_0 \in I \textit{ such that the numerical sequence } f_n(x_0) \textit{ converges to } l \in \mathbb{R}.$$

Then $(f_n)_n$ converges uniformly to a function f on I, where f is defined by:

$$\begin{cases} f' = g, \\ f(x_0) = l. \end{cases}$$

Thus:

$$\lim_{n\to+\infty} f'_n(x) = g(x) = f'(x) = \left(\lim_{n\to+\infty} f_n(x)\right)'.$$

Proof:

$$f(x) = l + \int_{x_0}^{x} g(t)\,\mathrm{d}t, \quad \forall x \in I.$$

Recall that:

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,\mathrm{d}t.$$

Using the integrability theorem, we get:

$$(f_n) \xrightarrow[I]{P.C} f,$$

where P.C means pointwise continuous.

2 Convergence of series of functions

Definition 4 (Series of functoin - convergence domain)**.** *Let $(f_n)_n$ be a sequence of functions on D, When we consider the family of numerical series $\sum_{n\in\mathbb{N}} f_n(x)$, parametrized by D, we speak about the series of functions $\sum_{n\in\mathbb{N}} f_n$*

By convergence domain of $\sum f_n$ (point-wise convergence) , we mean the set

$\{x \in D; \quad \sum_{n\in\mathbb{N}} f_n(x) \text{ converges}\}$

the sum of $\sum_{n\in\mathbb{N}} f_n$ is defined on the domain of convergence by:

$$x \mapsto \sum_{n=0}^{+\infty} = F(x) = \lim_{n\to+\infty} F_n(x)$$

where

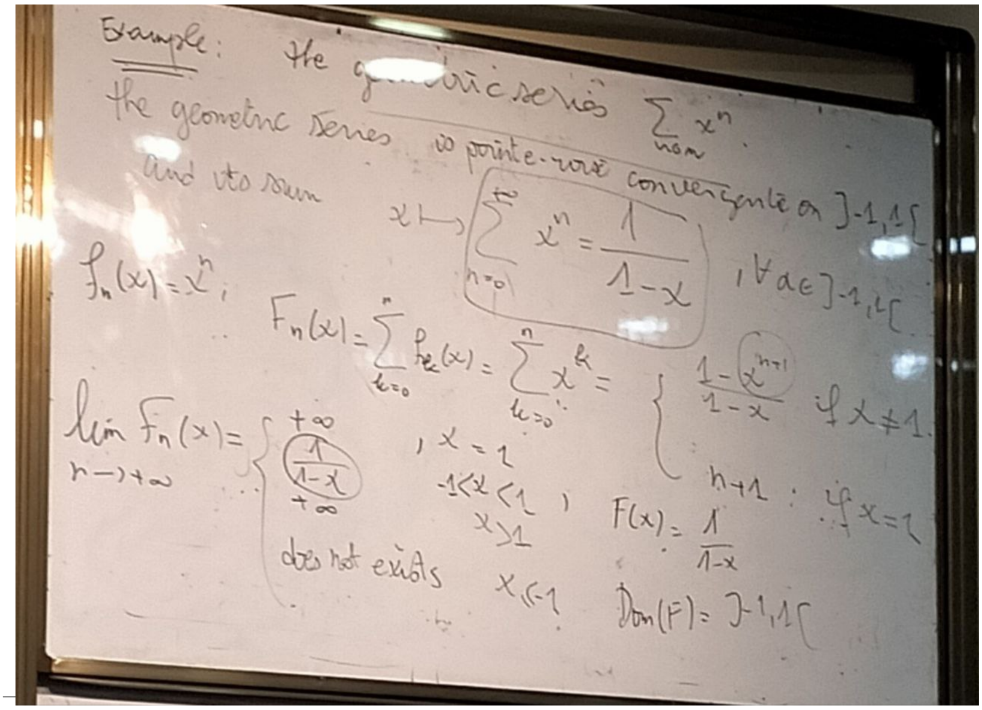
$$F_n = f_0 + f_1 + \cdots + f_n = \sum_{k=0}^{n} f_k$$

$$R_n = F - F_n = \sum_{k=n+1}^{+\infty} f_k \quad , \forall n \in \mathbb{N}$$

Example 5. *The geometric series $\sum_{n\in\mathbb{N}} x^n$*

the geometric series is pointe-wise convergente on $[-1, 1]$ and its sum :

$$x \mapsto \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x} \quad , \forall a \in [-1, 1]$$



Definition 5 (Uniform convergence and absolute convergence)**.** *1) We say that* $\sum_{n\in\mathbb{N}} f_n$ *is absolutly convergent on* D *if* $\sum_{n\in\mathbb{N}} |f_n|$ *is pointwise convergent*

2) We say that $\sum f_n$ *is uniformly convergent if,* $(R_n) \xrightarrow[D]{C.U} 0$ *(*$C.U \to$ *uniformly converges*

Remark: $\sum f_n$ C.U on $D \implies (f_n) \xrightarrow[D]{C.U} 0$ for example $\sum_{n\in\mathbb{N}} x^n$ does not uniformly CV on $[-1,1]$

Remark: If there exists a real sequence $(x_n) \subset D$ such that the numerical serie $\sum_{n\in\mathbb{N}} f_n(x_n)$ diverge then $\sum f_n$ does not uniformly CV on D for example $\sum_{n\in\mathbb{N}} ne^{-nx}$ is point-wise Cv on $]0,+\infty]$ but not uniformly convergent because $x_n = \dfrac{1}{n}$

$$f_n(x_n) = ne^{-n\frac{1}{n}} = \frac{n}{e} \xrightarrow[n\to+\infty]{} 0 \implies \sum f_n(\frac{1}{n}) \text{ div}$$

$$\implies \sum f_n \text{ does not uniformly Cv on } [0,+\infty]$$

Theorem 4 (Abel's criterion for uniform convergence)**.** *Let* $f_n = a_n b_n \quad, \forall n \in \mathbb{N}$*;* $(a_n)(b_n)$ *sequences of functions such that :*

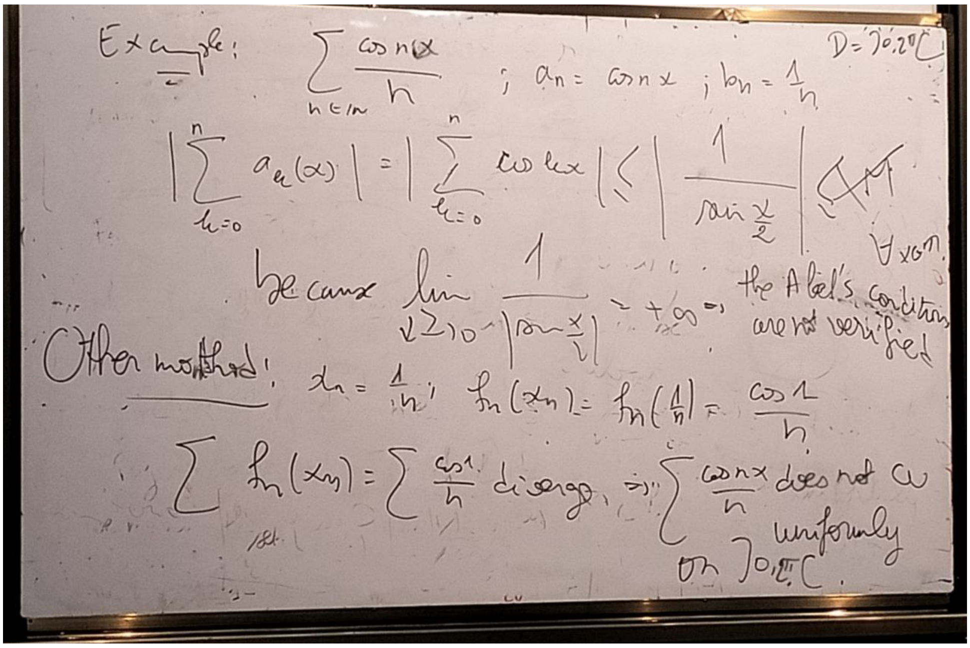
1) $\forall x \in D, \quad (b_n(x))_{n\in\mathbb{N}}$ *is positive, decreasing and* $(b_n) \xrightarrow[D]{C.U} 0$

2)

$$\exists M \geq 0, \forall x \in D; |\sum_{k=0}^{n} a_k(x)| \leq M; \quad \forall n \in \mathbb{N}$$

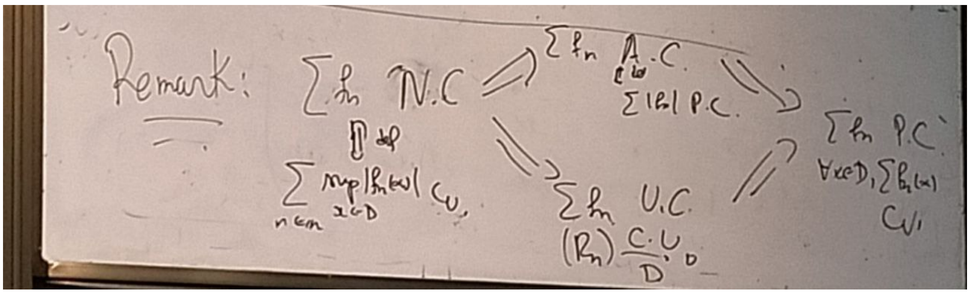
then $\sum f_n$ *Converges uniformly on* D

Example 6. ..



Definition 6 (Normal Convergence)**.** *the series $\sum F_n$ is said to be normally convergent over D if the positive numerical serie $\sum_{n\in\mathbb{N}} sup_{x\in D}|f_n(x)|$ Converge*

Remark:



Theorem 5. *Let $\sum f_n$ be a series of functions on D if there exists a real positive sequence $(V_n)_{n\in\mathbb{N}}$, such that:*

1) $\sum f_n N.C$ on D (normally convergent) 2) $\forall x \in D, \forall n \in \mathbb{N}; \quad |f_n(x)| \leq V_n$

then $\sum f_n$ $N.C$ on D

Example 7.

$$\sum \frac{cos(nx)}{n^\alpha} \text{ and } \sum \frac{sin(nx)}{n^\alpha} \text{ are } N.C \Leftrightarrow \alpha > 1$$

Because $\forall x \in \mathbb{R}; \forall n \in \mathbb{N}; \quad |\frac{cos(nx)}{n^\alpha}| \leq \frac{1}{n^\alpha}$ and $|\frac{sin(nx)}{n^\alpha}| \leq \frac{1}{n^\alpha}$

3 Properties of uniformly convergent series theorem(continuty):

Let $\sum f_n$ be a series of contiuous functions that convergent uniformly on D, its sum is also continuous on D . then :

$$\forall a \in D; \sum_{n=0}^{+\infty} f_n(a) = \sum_{n=0}^{+\infty} \left(\lim_{x \to a} f_n(x) \right) = \lim_{x \to a} \left(\sum_{n=0}^{+\infty} f_n(x) \right)$$

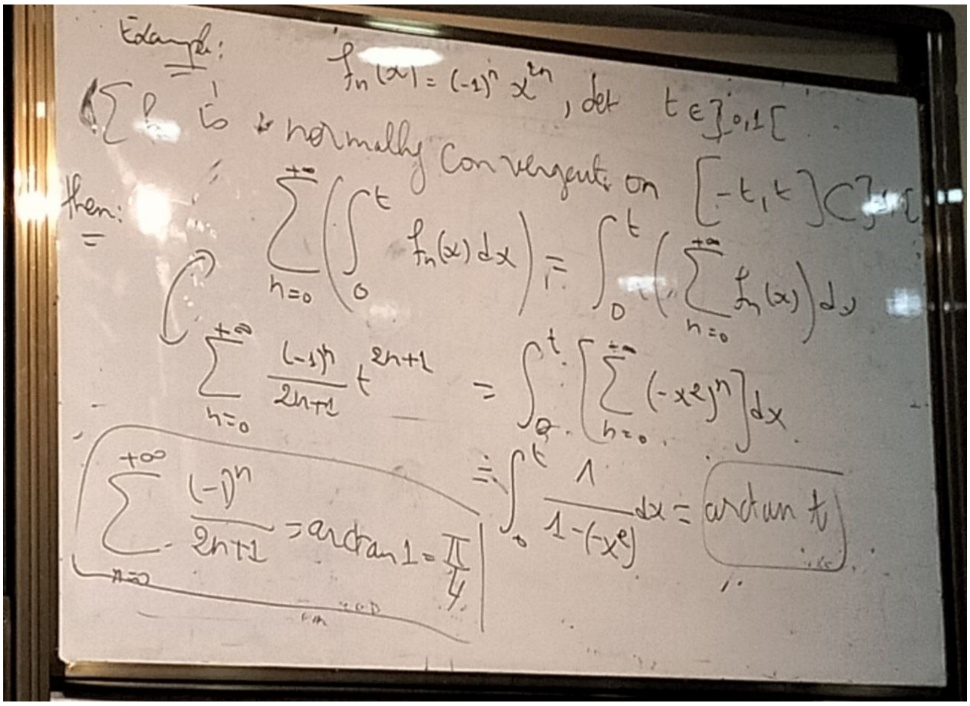
Example 8. *the function $f : x \mapsto \sum_{n=1}^{+\infty} \dfrac{1}{n^2 + x^2}$*

is continuous on $\mathbb{R}$

Theorem 6 (Integrability)**.** *Let $\sum_{n\in\mathbb{N}} f_n$ be uniformly convegent on $[a, b]$,where f_n is Riemann-integrable on $[a, b]$, then the sum is Riemann-integrable on $[a, b]$ and*

$$\int_a^b \left(\sum_{n=0}^{+\infty} f_n(x) \right) dx = \sum_{n=0}^{+\infty} \left(\int_a^b f_n(x)dx \right)$$

Example 9. ..

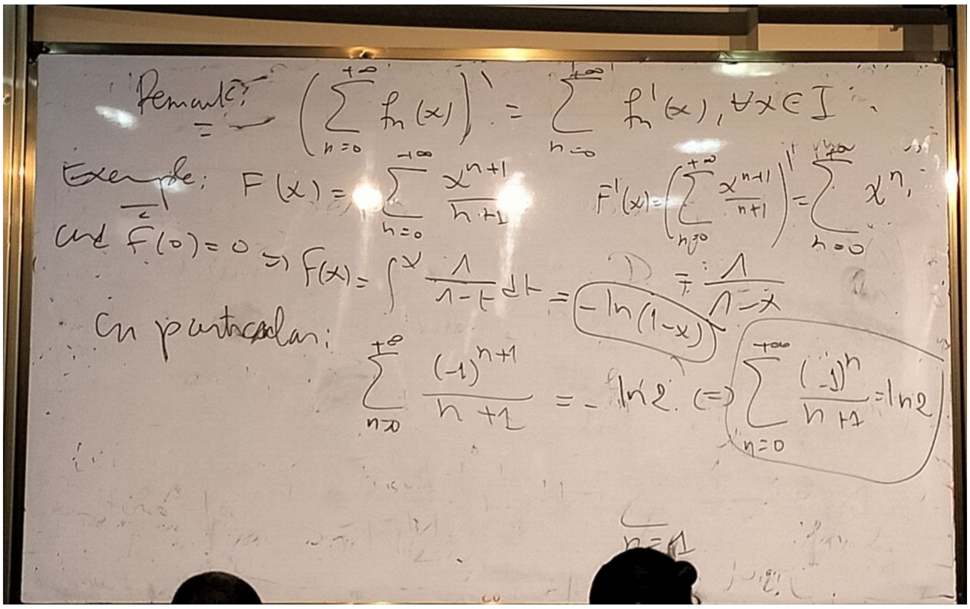


Theorem 7 (derivability)**.** *Let $\sum f_n$ defined on an interval $I \subset R$ such that : 1) f_n is of class C^1 on I; $\forall n \in \mathbb{N}$*

2) $\sum_{n\in\mathbb{N}} f_n'$ U.C on I with sum G

3) $\exists x_0 \in I|$ the numerical series $\sum_{n\in\mathbb{N}} f_n(x_0)$ convergence, then $\sum f_n$ is pointwise convergent on I, with sum F where F is defined by F' $= G$ and $F(x_0) = \sum_{n=0}^{+\infty} f_n(x_0)$

Remark:


Remarcă: $\left(\sum_{n=0}^{+\infty} f_n(x)\right)' = \sum_{n=0}^{+\infty} f_n'(x), \forall x \in I$.
Exemple: $F(x) = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$
$F'(x) = \left(\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}\right)' = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$
cu $F(0) = 0 \Rightarrow F(x) = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$
În particular: $\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n+1} = -\ln 2 \iff \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} = \ln 2$