Chapter 02: Vector Analysis

Notes from prof zeglaoui course

December 21, 2025

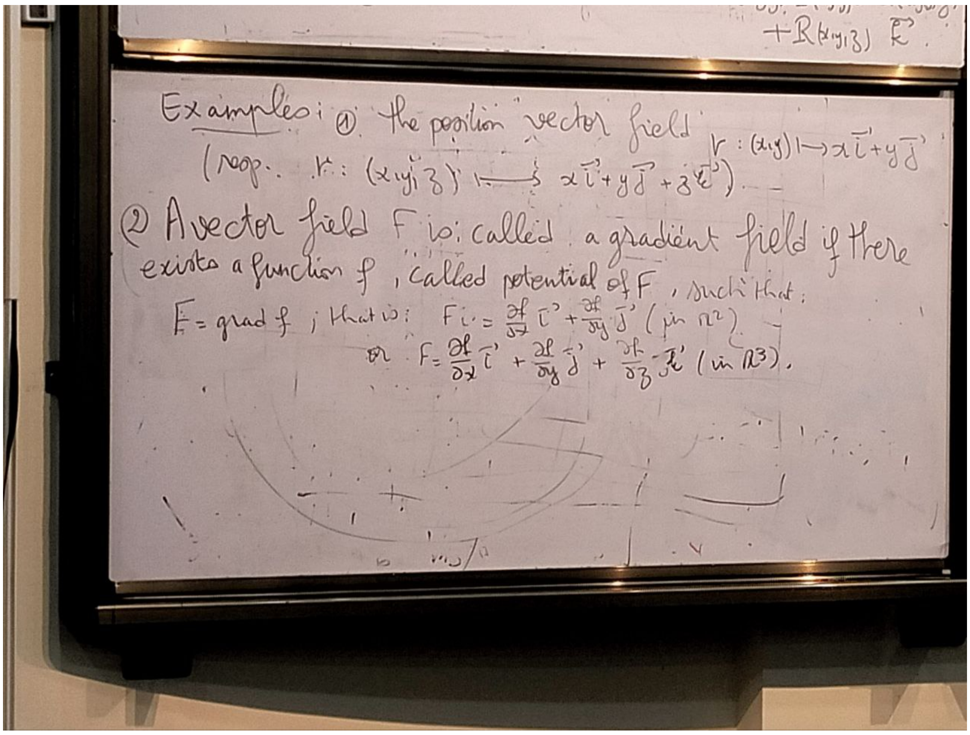
1 vector fields and scalar fields

Definition 1. *1) We call a scalar field defined on a region $D \subset \mathbb{R}^2$ (respectively on $\Omega \subset \mathbb{R}^3$), a real function defined on D (respectively Ω)*

2) a vector field F over $D \subset \mathbb{R}^2$ (resp Ω) is a function F that assigns to any $(x, y) \in D$ a vector $F(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$ where $M, N \in \mathcal{F}(D, \mathbb{R})$

3) a vector field F over $\Omega \subset \mathbb{R}^3$ is a function F that assignes to any $(x, y, z) \in \Omega$ a vector $F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$

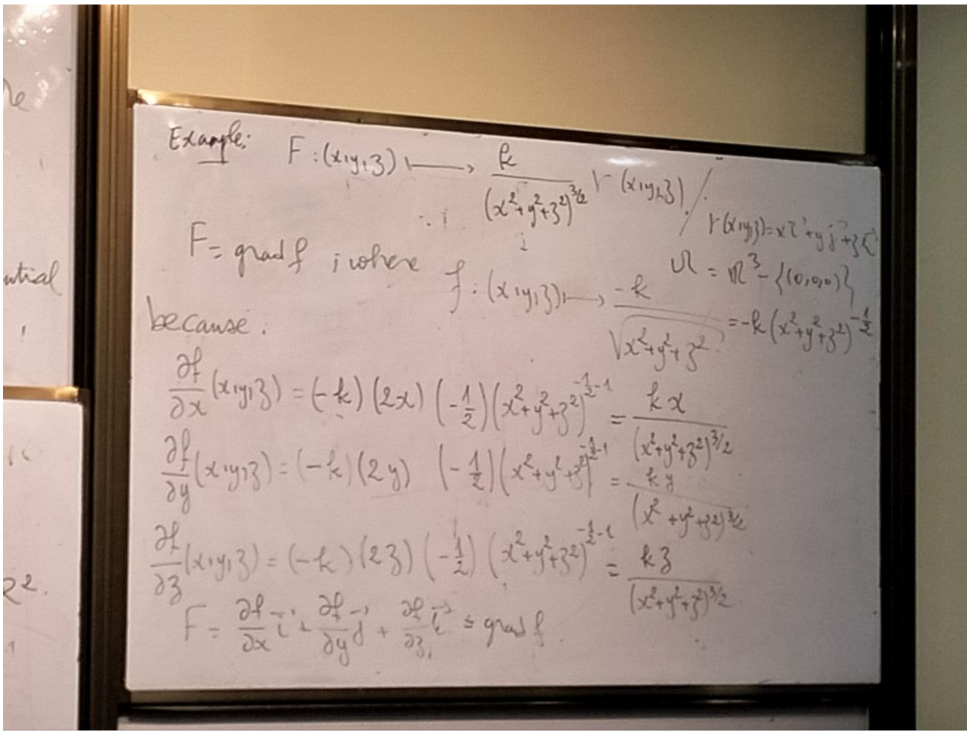
Example 1. *1) the position vector field*



Definition 2. *A vector field F is conservative if there exists a*

potential function f, $\quad F = grad(f)$

Example 2. ...



Theorem 1. *A vector field $F = M\vec{i} + N\vec{j}$ over a convex region $D \subset \mathbb{R}^2$ is conservative if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof. if F is conservative

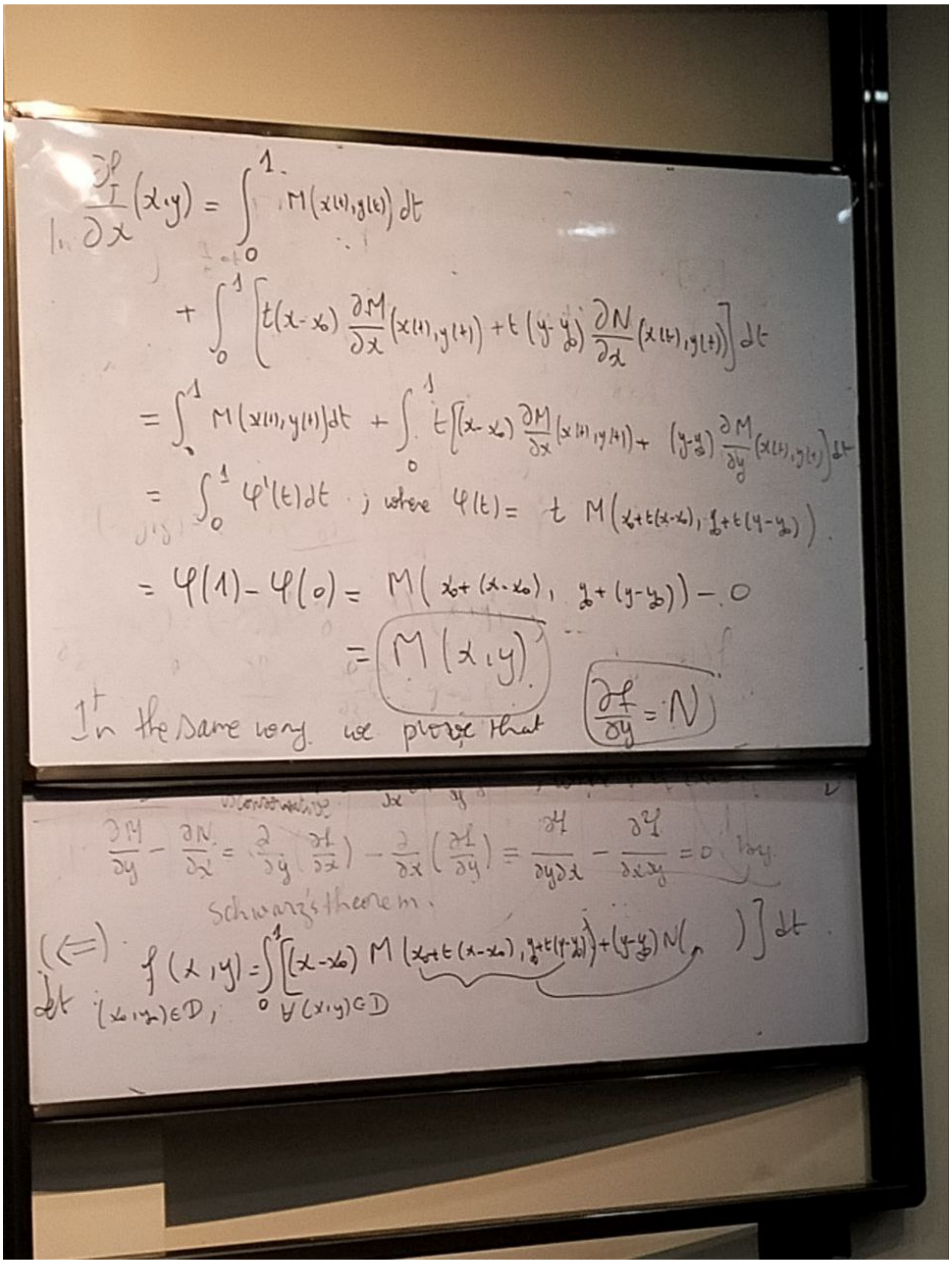
$$F = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \quad \text{where } f \text{ is of class } C^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\, \partial x} - \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

Shwarz' theorem

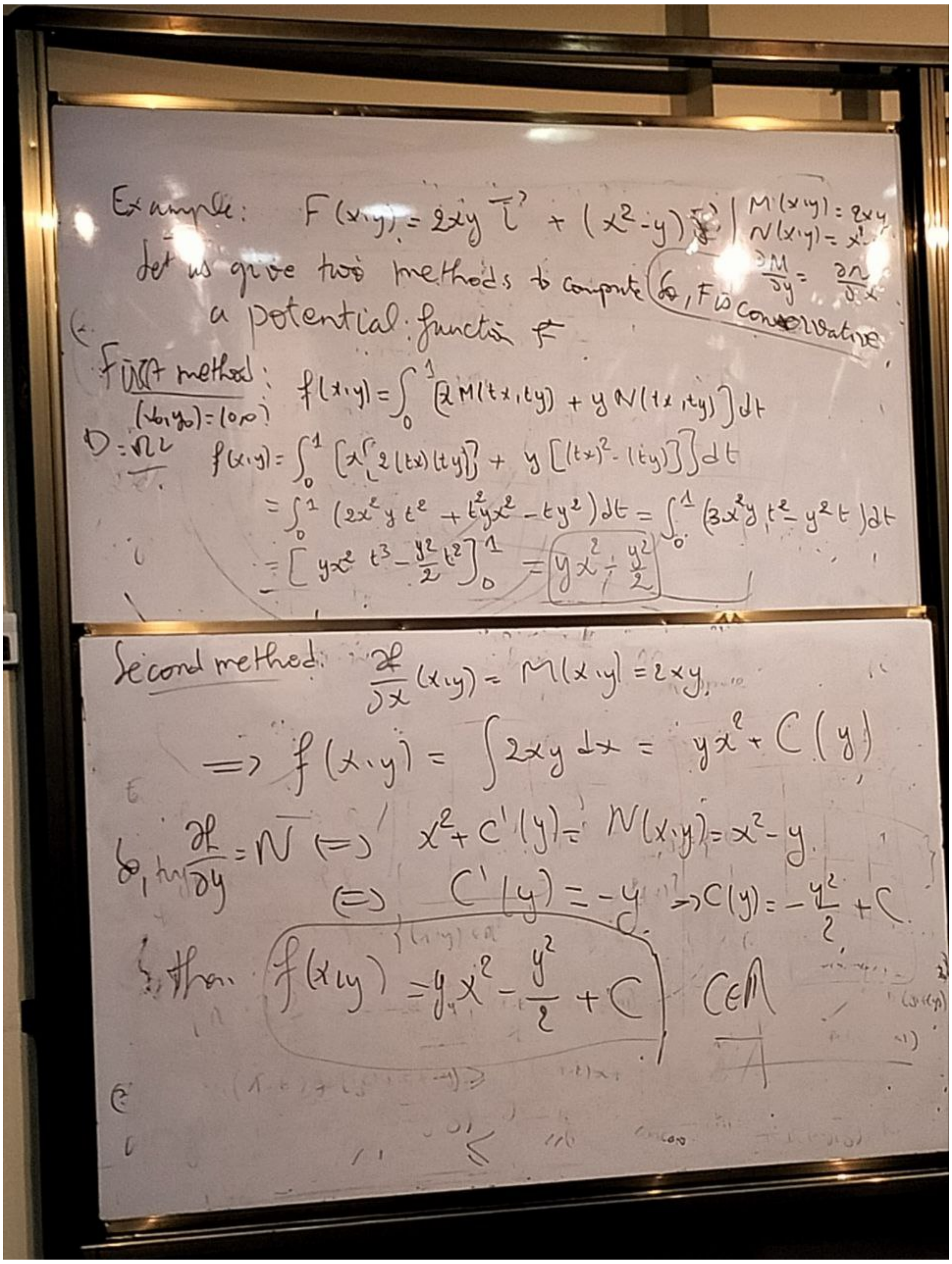
$\Leftrightarrow$

$$f(x,y) = \int_0^1 [(x-x_0)M(x_0 + t(x-x_0), y_0 + t(y-y_0)) + (y-y_0)N(x_0 + t(x-x_0), y_0 + t(y-y_0))]\, dt$$



$\square$

Example 3. .



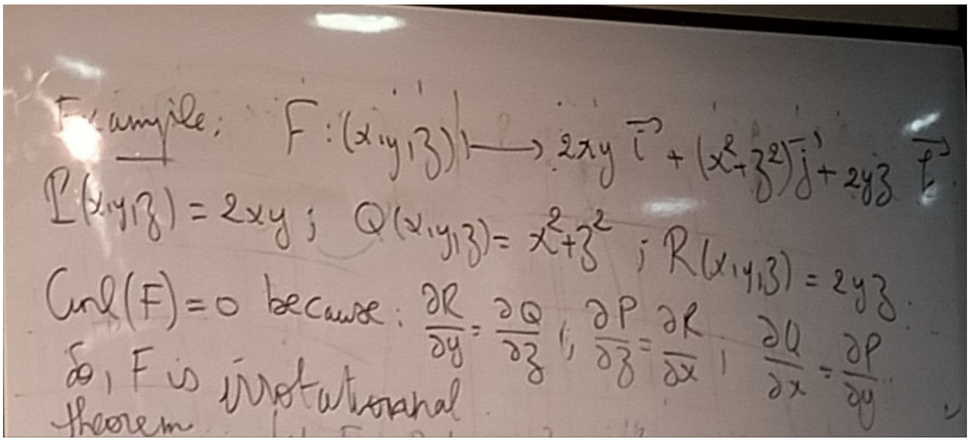
Definition 3. *Let $F : (x, y, z) \to P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{z}$ be a vector field over $\Omega \subset \mathbb{R}^3$. Such that the first partial deriatives of F exists*

the curl of F, denoted by $Curl(F)$ (Or $\nabla \times F$) is the vector fiels over Ω defind by :

$$Curl(F) = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

We say that F is irrotationnal vector field if $Curl(F) = 0$

Example 4. .



Theorem 2. *Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be vector field over a convex domain $\Omega \subset \mathbb{R}^3$ of class C^1 then F is conservative $\Leftrightarrow Curl(F) = 0$*

proof

Proof $(\Rightarrow)$ If F is conservative, there exists $f : \mathbb{R}^3 \to \mathbb{R}$ of class C^2 on Ω such that:

$$P = \frac{\partial f}{\partial x} \; ; \; Q = \frac{\partial f}{\partial y} \; ; \; R = \frac{\partial f}{\partial z} \qquad (F = \operatorname{grad} f),$$

$$\operatorname{Curl}(F) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

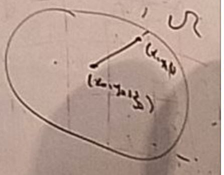
$$\overset{\text{Schwarz's th}}{=} 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k},$$

$$= 0.$$

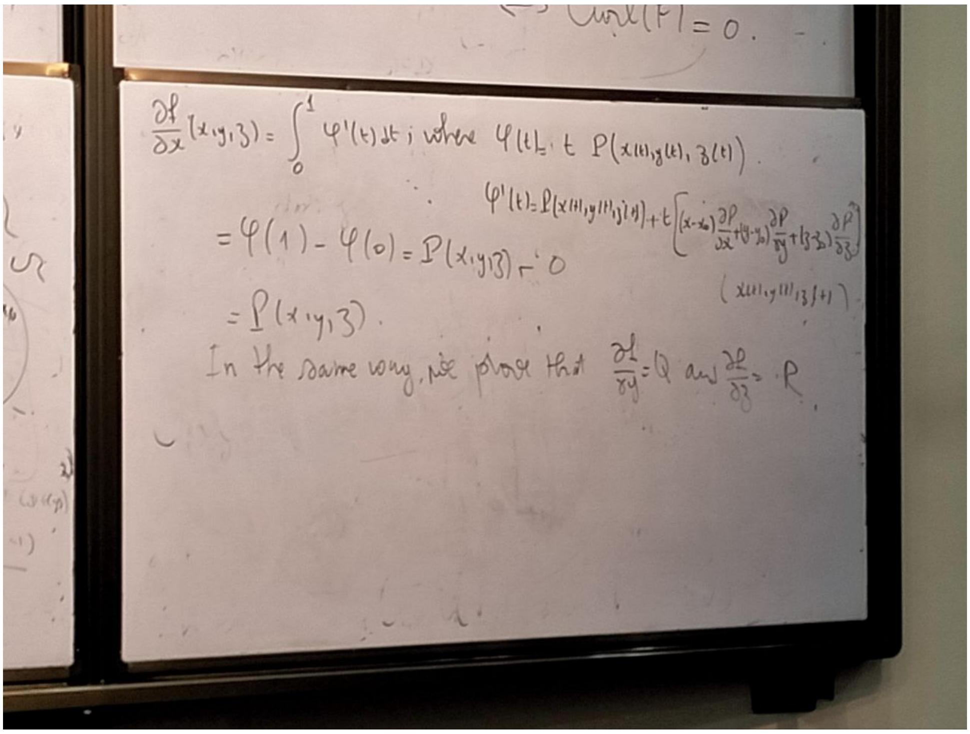
$(\Leftarrow)$ Since Ω is convex; let $(x_0, y_0, z_0) \in \Omega$,

$\forall (x,y,z) \in \Omega \; ; \; \forall t \in [0,1] \; ; \; (x(t), y(t), z(t)) \in \Omega$

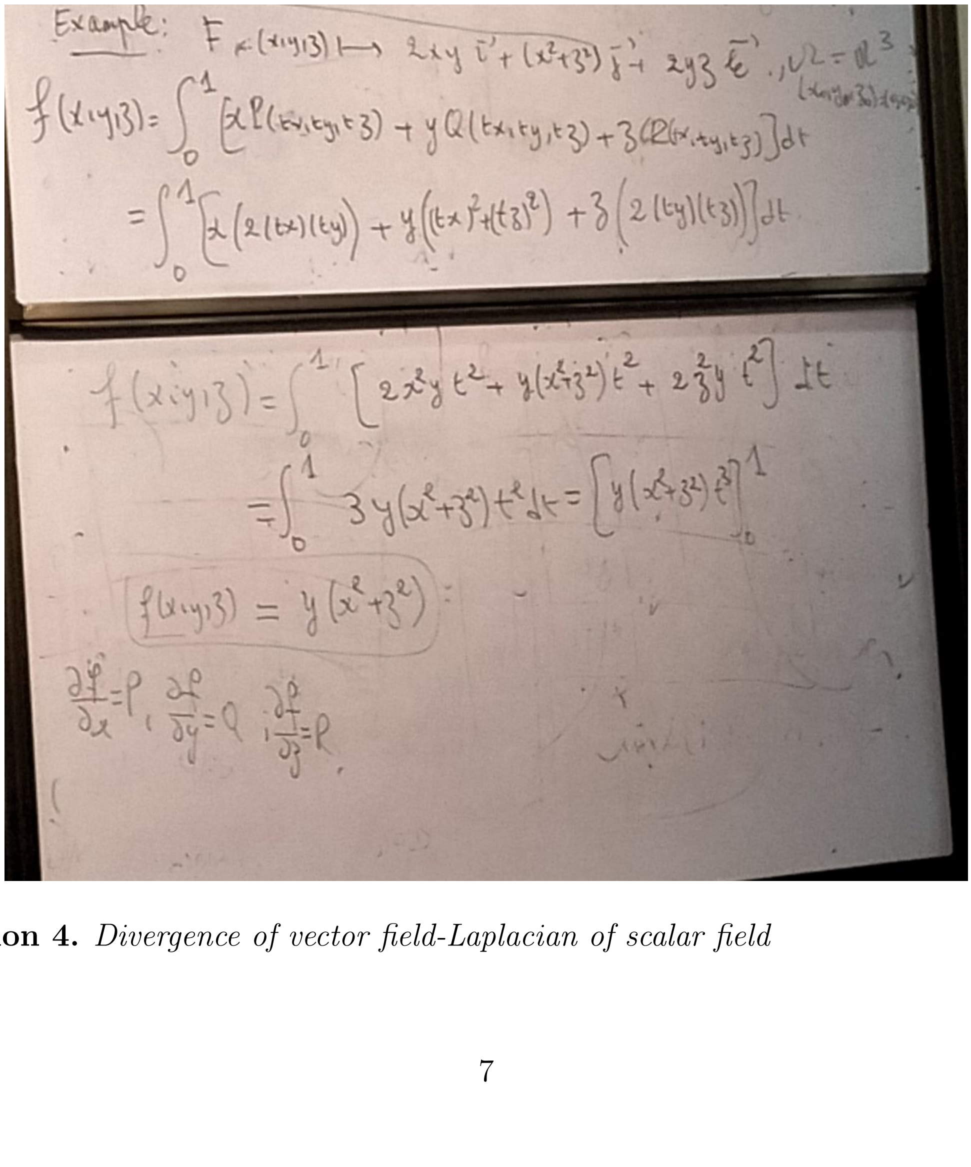
where
$$\begin{cases} x(t) = x_0 + t(x - x_0) \\ y(t) = y_0 + t(y - y_0) \; ; \quad t \in [0,1] \\ z(t) = z_0 + t(z - z_0) \end{cases}$$



let
$$f(x,y,z) = \int_0^1 \Big[(x - x_0) P(x(t), y(t), z(t)) + (y - y_0) Q(x(t), y(t), z(t)) + (z - z_0) R(x(t), y(t), z(t))\Big]\, dt$$



Example 5. .



Definition 4. *Divergence of vector field-Laplacian of scalar field*

$$1)div(F) = \nabla.F \rightarrow \begin{cases} F = M\vec{i} + N\vec{j} \implies div(F) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} \\ F = P\vec{i} + Q\vec{j} + R\vec{k} \implies div(F) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} \end{cases}$$

2) If $F = grad(f) \implies div(F) = div(grad(f)) = \Delta(f)$ *the Laplacian of f*

3) F is called divergence free if $div(F) = 0$

if is called harmonic function if $\Delta(f) = 0 \rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0 \\ \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0 \end{cases}$

Remark: If F is of class C^2 vector field, then $div(curl(F)) = 0$, using Schwarz's theorem

2 Line integrals

2.1 Line integral of scalar field

Let $f : D \rightarrow \mathbb{R}$ a continuous function over $D \subset \mathbb{R}^2 (resp : D \subset \mathbb{R}^3)$, let

$$\Gamma : \begin{cases} [a, b] \rightarrow D \\ t \mapsto \gamma(t) \end{cases}$$

of C^1 curve on D

the line integral of f over Γ denoted by $\int_\Gamma f dl$ is a real number

$$\int_\Gamma f dl = \int_a^b f(\gamma(t))||\gamma'(t)||dt \quad \text{where}||\gamma'(t)|| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

in particular if $f \equiv 1$;

$\int_\Gamma dl = l(\Gamma)$ length of $\Gamma = \int_a^b ||\gamma'(t)||dt$

$\int_\Gamma f dl$ is independant to parametrization $([a, b], \gamma)$ of Γ

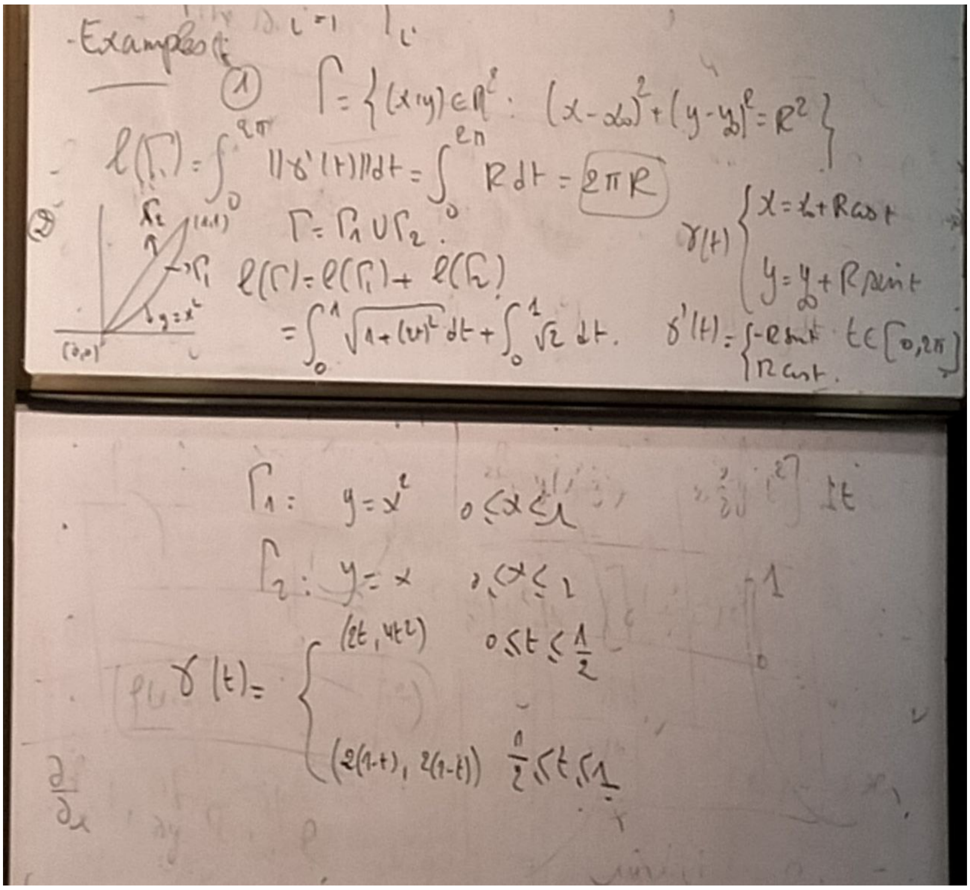
2) If Γ is piece-wise C^1 ; i.e $\exists k \geq 2$

$\forall i \in \{1, \ldots, k\}; \Gamma_i$ is C^1 ; $\Gamma_i \cap \Gamma_{i+1} = \{pt\}$ and $\Gamma_i \cap \Gamma_j = \phi \quad if \quad |i - j| \geq 2$

then:

$$\int_\Gamma f dl = \sum_{i=1}^{k} \int_{\Gamma_i} f dl$$

Example 6. .



2.2 Line integral of a vector field

Let Γ a curve parametrized by $[a,b] \xrightarrow{\gamma} \gamma(t)$ and F a continuous vector field

the line integral of F over Γ (or the circulation of F to Γ) is :

$$\int_\Gamma F.dr = \int_a^b [F(\gamma(t)) \cdot \gamma'(t)]\, dt$$

which is independant of the choice of parametrization

in $\mathbb{R}^2$ if $\gamma(t) = (x(t), y(t))$ and $F = M\vec{i} + N\vec{j}$

$$\int_\Gamma F.dr = \int_a^b [x'(t)M(x(t), y(t)) + y'(t)N(x(t), y(t))]\, dt$$

In $\mathbb{R}^3$ if $\gamma(t) = (x(t), y(t), z(t))$ and $F = P\vec{i} + Q\vec{j} + R\vec{k}$

then:

$$\int_\Gamma F.dr = \int_a^b \left[x'(t)P(x(t),y(t),z(t)) + y'(t)Q(x(t),y(t),z(t)) + z'(t)R(x(t),y(t),z(t))\right] dt$$

Remark:

We denoted by $\int_\Gamma Mdx + Ndy = \int_\Gamma F.dr$ int $\mathbb{R}^2$ and by

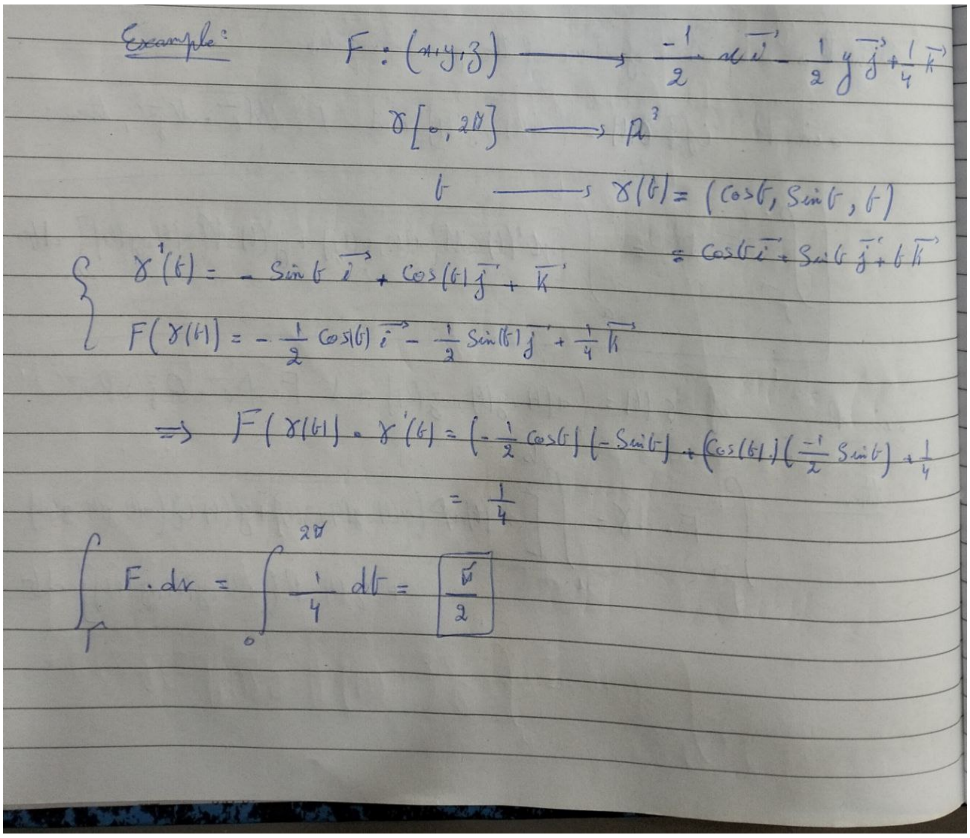
$\int_\Gamma Pdx + Qdy + Rdz = \int_\Gamma F.dr$ in $\mathbb{R}^3$

this is the differential representation of the integral of F along to Γ

Remark: If $F = grad(f)$ is conservative vector field then:

$$\begin{cases} df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = F.dr & \mid dr = dx\vec{i} + dy\vec{j} \\ df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z} = F.dr & \mid dr = dx\vec{i} + dy\vec{j} + dz\vec{k} \end{cases}$$

Example 7. ..



Example:

$F : (x,y,z) \longrightarrow -\frac{1}{2} x\vec{i} - \frac{1}{2} y\vec{j} + \frac{1}{4}\vec{k}$

$\gamma[0,2\pi] \longrightarrow \mathbb{R}^3$

$t \longrightarrow \gamma(t) = (\cos t, \sin t, t)$

$= \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

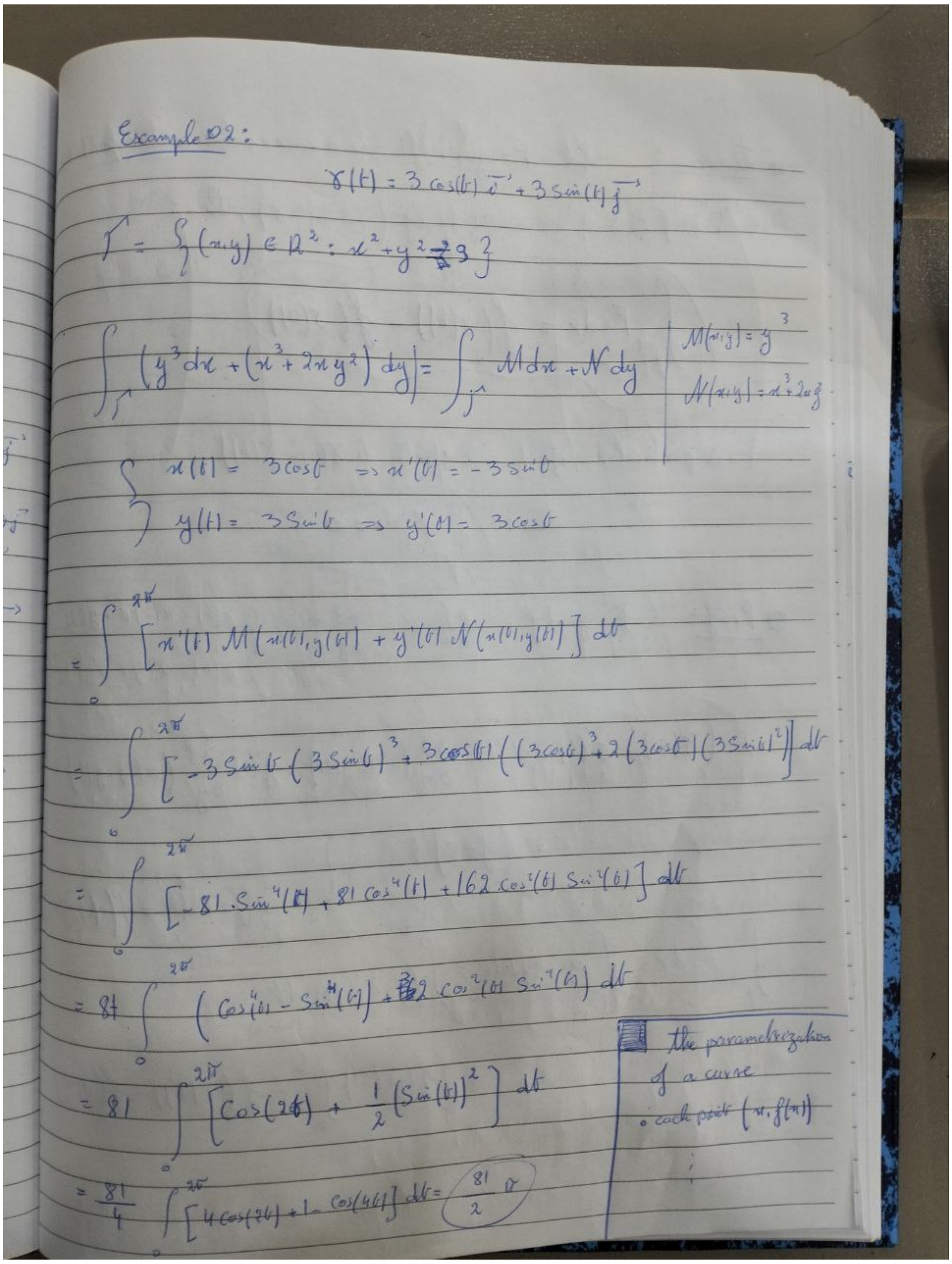
$\begin{cases} \gamma'(t) = -\sin t\,\vec{i} + \cos(t)\vec{j} + \vec{k} \\ F(\gamma(t)) = -\frac{1}{2}\cos(t)\vec{i} - \frac{1}{2}\sin(t)\vec{j} + \frac{1}{4}\vec{k} \end{cases}$

$\Rightarrow F(\gamma(t)).\gamma'(t) = (-\frac{1}{2}\cos t)(-\sin t) + (\cos(t))(\frac{-1}{2}\sin t) + \frac{1}{4}$

$= \frac{1}{4}$

$\int_\Gamma F.dr = \int_0^{2\pi} \frac{1}{4} dt = \boxed{\frac{\pi}{2}}$

Example 8. ..



Example 02:

$$\gamma(t) = 3\cos(t)\,\vec{i} + 3\sin(t)\,\vec{j}$$

$$\Gamma = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$$

$$\int_\Gamma \left(y^3 dx + (x^3 + 2xy^2)\, dy\right) = \int_\Gamma M\,dx + N\,dy$$

$M(x,y) = y^3$

$N(x,y) = x^3 + 2xy^2$

$$\begin{cases} x(t) = 3\cos t \Rightarrow x'(t) = -3\sin t \\ y(t) = 3\sin t \Rightarrow y'(t) = 3\cos t \end{cases}$$

$$= \int_0^{2\pi} \left[x'(t)\, M(x(t), y(t)) + y'(t)\, N(x(t), y(t))\right] dt$$

$$= \int_0^{2\pi} \left[-3\sin t\,(3\sin t)^3 + 3\cos(t)\left((3\cos t)^3 + 2(3\cos t)(3\sin t)^2\right)\right] dt$$

$$= \int_0^{2\pi} \left[-81\sin^4(t) + 81\cos^4(t) + 162\cos^2(t)\sin^2(t)\right] dt$$

$$= 81\int_0^{2\pi} \left(\cos^4(t) - \sin^4(t)\right) + 2\cos^2(t)\sin^2(t)\, dt$$

$$= 81\int_0^{2\pi} \left[\cos(2t) + \frac{1}{2}(\sin(t))^2\right] dt$$

$$= \frac{81}{4}\int_0^{2\pi} \left[4\cos(2t) + 1 - \cos(4t)\right] dt = \frac{81}{2}\pi$$

the parametrization of a curve

- each point $(x, f(x))$

Theorem 3. *Let $F = grad(f)$ be a conservative vector field over $D : \Gamma \subset D$, parametrized by ($[a,b]$,γ), then:*

$$\int_\Gamma F.dr = f(\gamma(b)) - f(\gamma(a))$$

in particular, if Γ is a closed path ($\gamma(a) = \gamma(b)$) then : $\oint F.dr = 0$

Proof

$$F = grad(f) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}; \begin{cases} \gamma(t) = x(t)\vec{i} + y(t)\vec{j} \\ \gamma'(t) = x'(t)\vec{i} + y'(t)\vec{j} \end{cases}$$

$$F(\gamma(t)).\gamma'(t) = x'(t)\frac{\partial f}{\partial x}(x(t), y(t)) + y'(t)\frac{\partial f}{\partial y}(x(t), y(t)) = (f \circ \gamma)'t \quad | \; (f \circ \gamma)(t) = f(x(t), y(t))$$

$$\int_\Gamma F.dr = \int_\Gamma grad(f).dr = \int_a^b (f \circ \gamma)'(t)dt = [f(\gamma(t))]_a^b$$
$$= f(\gamma(b)) - f(\gamma(a))$$

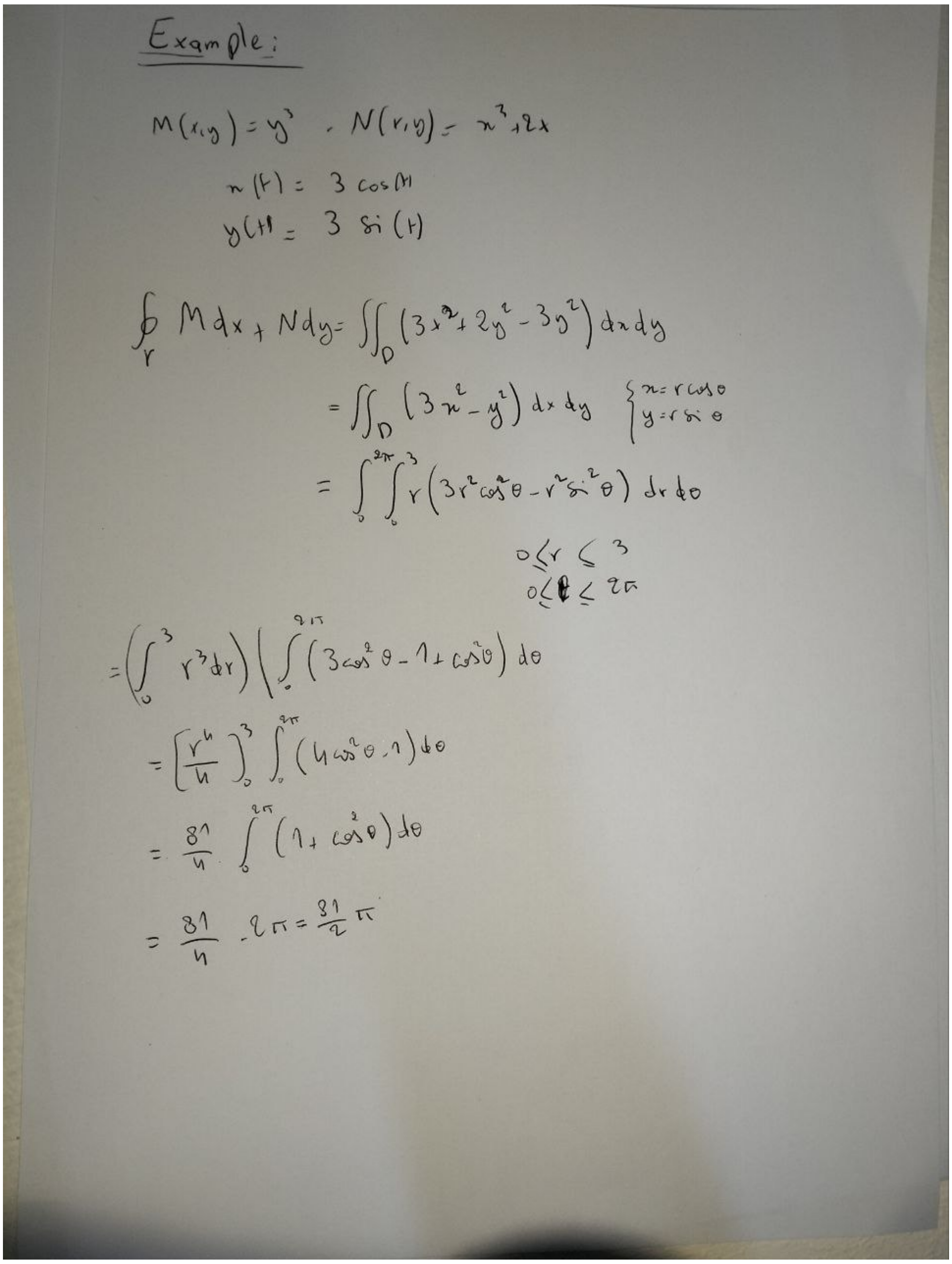
Theorem 4. *(Green's theorem)*

Let D be a simply connected and connected region of $\mathbb{R}^2$ with piece-wise C^1 boundary simple curve $\Gamma = \partial D$

and $F = M\vec{i} + N\vec{j}$ be a C^1vector field over D, Then :

$$\oint_\Gamma F.dr = \oint_\Gamma Mdx + Ndy = \iint_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})(x,y)dxdy$$

Example 9. ..



Theorem 5. *Corollary Line Integral area*

if D is a plane region bounded by a piece-wise simple closed curve $\Gamma = \partial D$ oriented counterclockwise, then $A(D) = \frac{1}{2} \oint_{\Omega=\partial D}(xdy - ydx)$

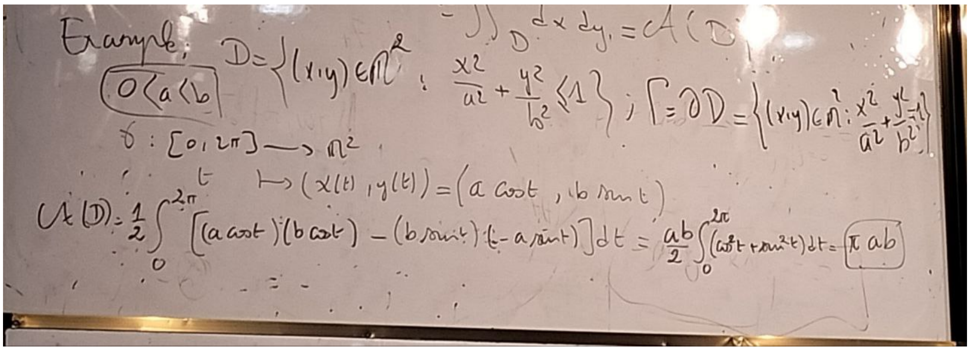
Proof

$$M(x,y) = -\frac{1}{2}y; \quad N(x,y) = \frac{1}{2}x$$

by Green's theorem;

$$\oint_{\partial D} Mdx + Ndy = \iint_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})(x,y)dxdy$$

Example 10. .



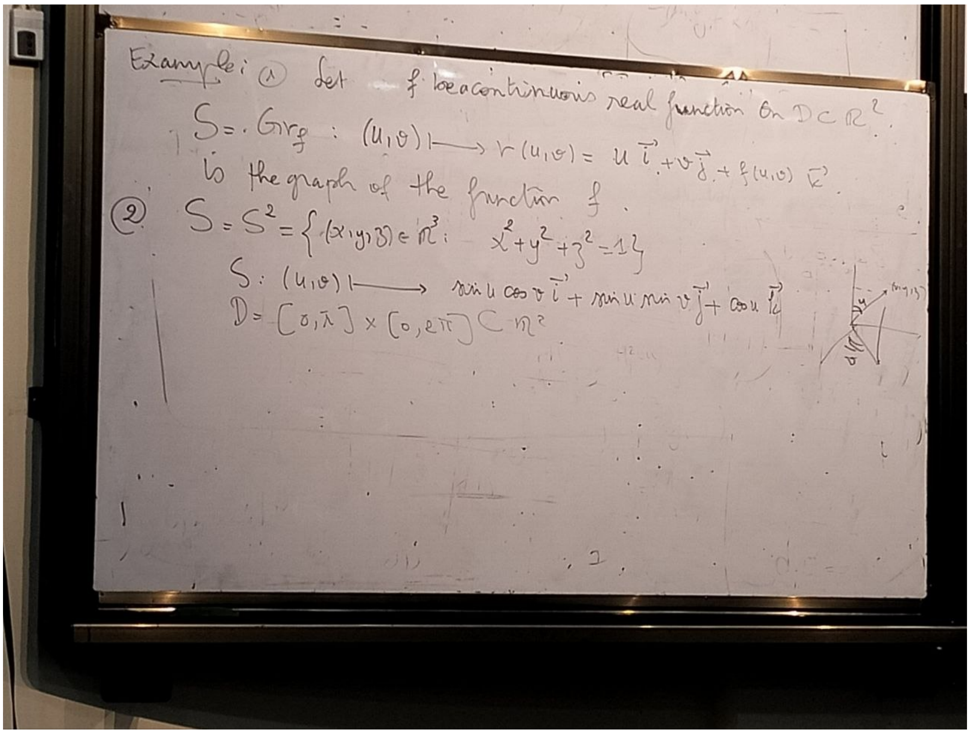
3 Surface integral

3.1 Parametric surface

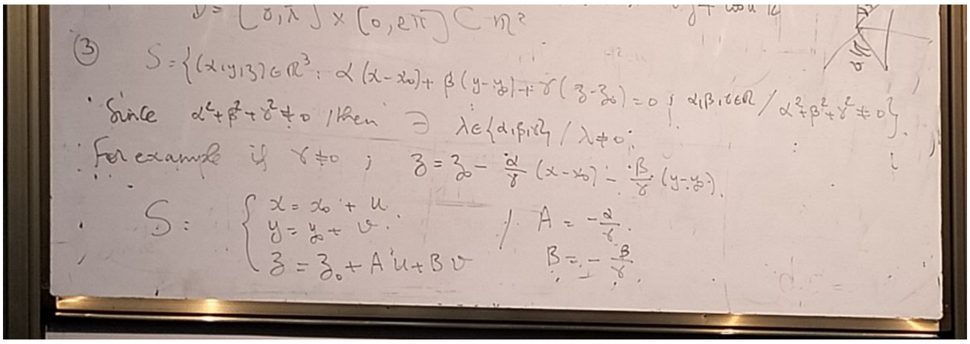
Definition 5. *Let x,y,z be a three functions of u and v, that are continuous on $D \subset \mathbb{R}^2$. The set S of points (x,y,z) given by : $r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ be a parametrized surface. The following equations:*

$$S : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \text{, are called the parametric equation of } S$$

Example 11. ..



..



Definition 6. *Let (D, r) be a parametrization of the surface S.*

We say that S is smooth (or C^1) if x,y,z are C^1 or the vector field V is C^1 on D

we say that S is piece-wise smooth. if $\exists k \in \mathbb{N}^*; S = S_1 \cup S_2 \cup ... \cup S_k \mid S_i \cup S_j$ is at most a curve and S_i is smooth

Let S be smooth surface. The tangent plane of S at a point $(x_0, y_0, z_0) \in S$ is the affine space

$$T_{(x_0,y_0,z_0)}S : \begin{cases} x = x_0 + (u-u_0)\dfrac{\partial x}{\partial u}(u_0,v_0) + (v-v_0)\dfrac{\partial x}{\partial v}(u_0,v_0) \\ y = y_0 + (u-u_0)\dfrac{\partial y}{\partial u}(u_0,v_0) + (v-v_0)\dfrac{\partial y}{\partial v}(u_0,v_0) \\ z = x_0 + (u-u_0)\dfrac{\partial z}{\partial u}(u_0,v_0) + (v-v_0)\dfrac{\partial z}{\partial v}(u_0,v_0) \end{cases}$$

$$: r : (u,v) \mapsto r(u_0,v_0) + (u-u_0)\frac{\partial r}{\partial u}(u_0,v_0) + (v-v_0)\frac{\partial r}{\partial v}(u_0,v_0)$$

$$r(u_0, v_0) = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} \quad ; \quad \frac{\partial r}{\partial u} = r_u \quad ; \quad \frac{\partial r}{\partial v} = r_v$$

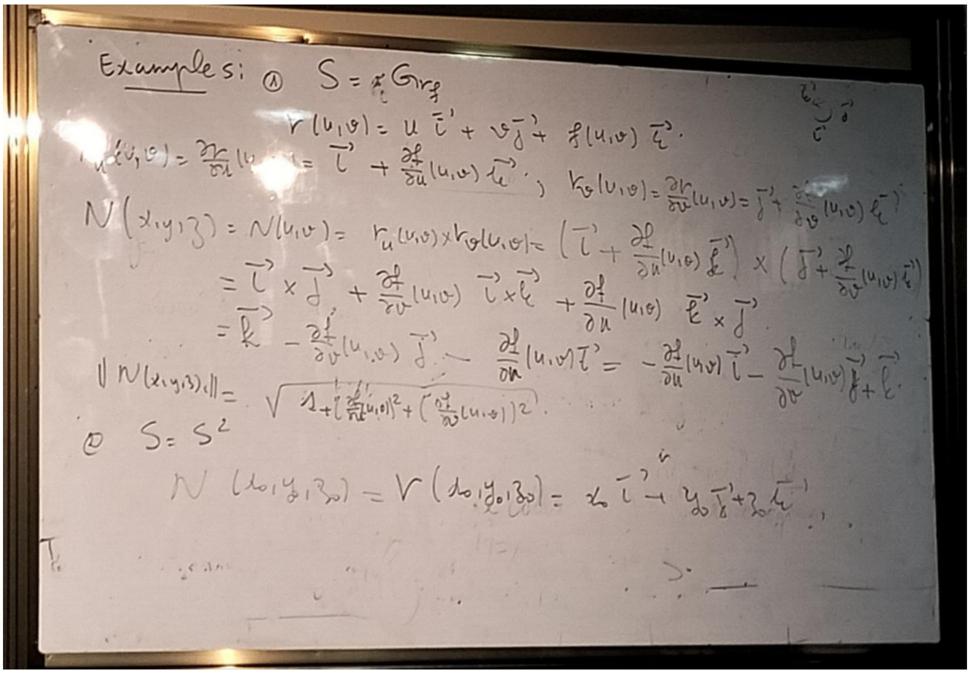
the normal vector field of S ; denoted by N is :

$$N = r_u \times r_v = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$$

$$N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

So; $T_{(x_0,y_0,z_0)}S \perp N(x_0, y_0, z_0), \quad ||N(x_0, y_0, z_0)|| = ||\frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0)||$

Example 12. ..

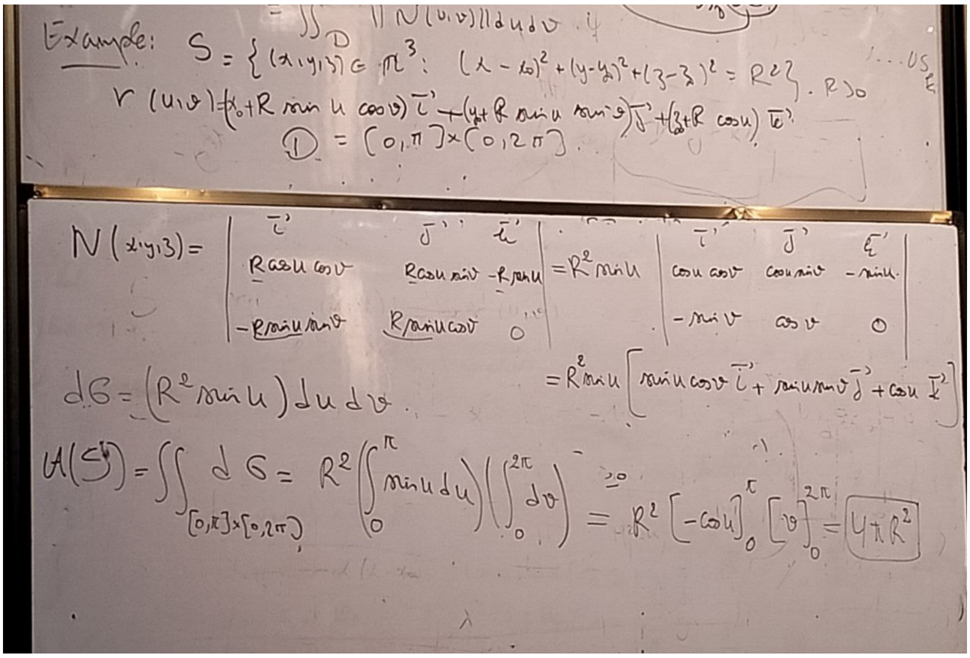


Definition 7. *<u>the area of a smooth surface</u>*

Let $S : r : (u, v) \mapsto r(u, v)$ *over* $D \subset R$ *be a smooth parametrized surface the area of* S *b*

$$A(S) = \iint_D ||r_u(u, v) \times r_v(u, v)|| = \iint_\sigma d\sigma$$
$$= ||N(u, v)||dudv$$

Example 13. ...



Remark:

$$S = Gr_f = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \subset D \text{ and } z = f(x,y)\}$$

$$S : r(u,v) = u\vec{i} + v\vec{j} + f(u,v)\vec{k}$$

$$d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial u}(u,v)\right)^2 + \left(\frac{\partial f}{\partial v}(u,v)\right)^2}$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial u}(u,v)\right)^2 + \left(\frac{\partial f}{\partial v}(u,v)\right)^2}\, dudv$$

3.2 Surface integral

Definition 8. *Let $S : r(u,v) \mapsto x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ be a parametrized smooth surface and $f : \mathbb{R}^3 \to \mathbb{R}$ a function continuous over S, denoted by $\int_S f d\sigma$ is the real number*

$\iint f d\sigma = \iint_D f(r(u,v))||r_u(u,v) \times r_v(u,v)||dudv$

$\iint f d\sigma = \iint_D f(x(u,v) + y(u,v) \ + z(u,v))||\dfrac{\partial r}{\partial u}(u,v) \times \dfrac{\partial r}{\partial v}(u,v)||dudv$

Remark:

1) If

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = \varphi(x,y)\} \to \iint_S f d\sigma$$
$$= \iint_D f(x,y,\varphi(x,y))\sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x,y)\right)^2 + \left(\frac{\partial \varphi}{\partial y}(x,y)\right)^2} dxdy$$

2) If

$$S = \{(x,y,z) \in \mathbb{R}^3 : y = \varphi(x,z)\} \to \iint_S f d\sigma$$
$$= \iint_D f(x,\varphi(x,y),z)\sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x,z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(x,z)\right)^2} dxdz$$

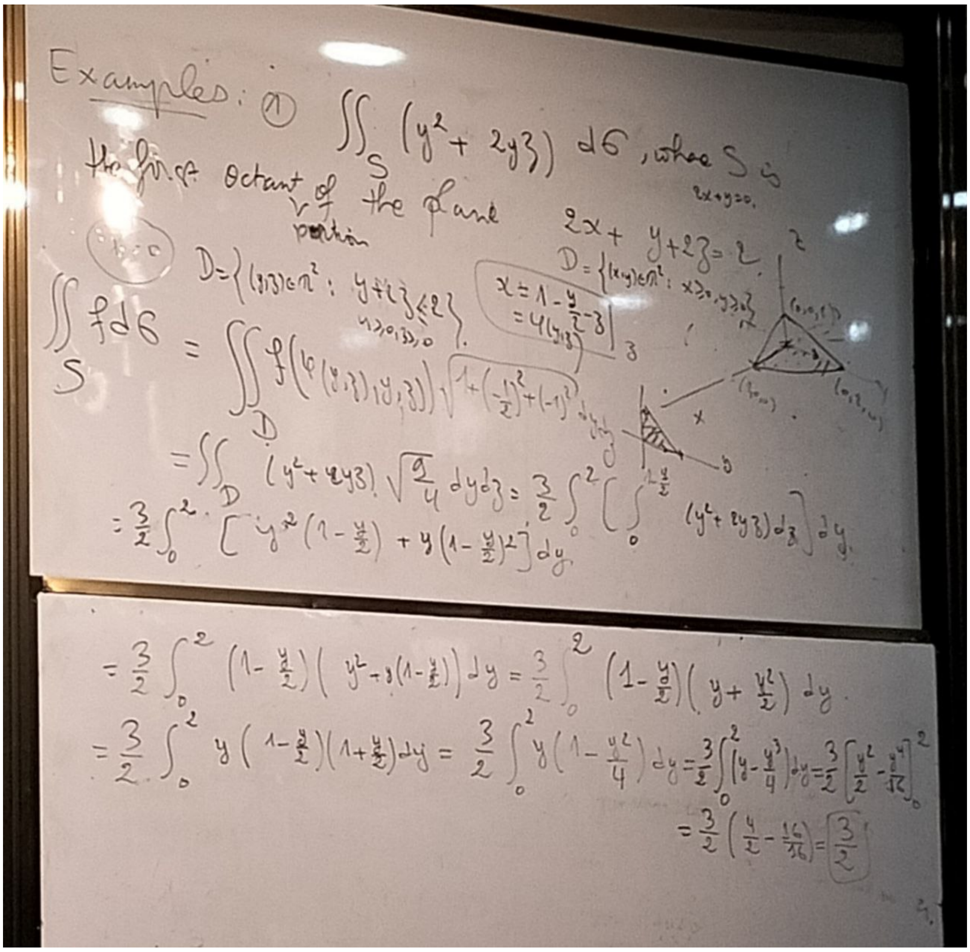
3) If

$$S = \{(x,y,z) \in \mathbb{R}^3 : x = \varphi(y,z)\} \to \iint_S f d\sigma$$
$$= \iint_D f(\varphi(y,z),y,z)\sqrt{1 + \left(\frac{\partial \varphi}{\partial y}(y,z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(y,z)\right)^2} dydz$$

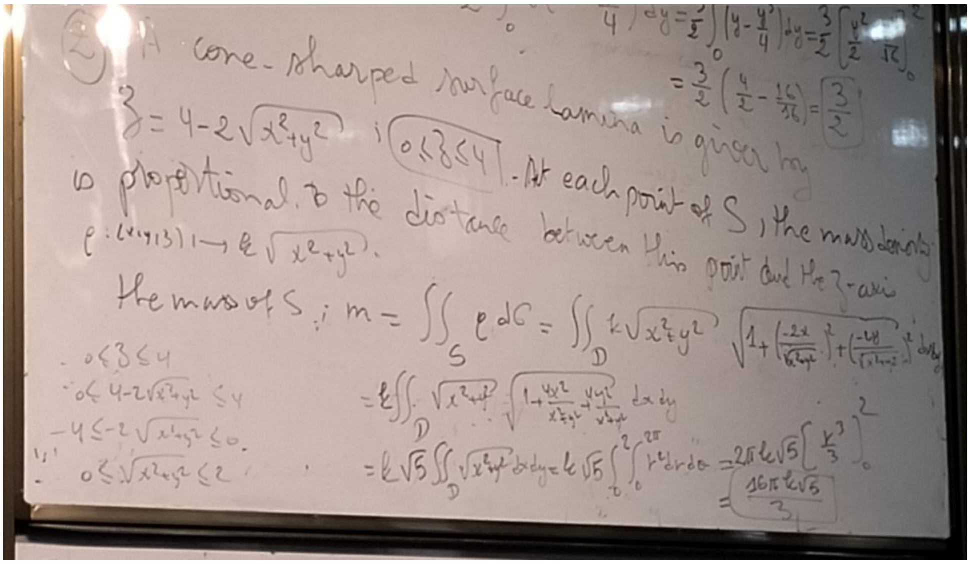
4)

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = c, (x,y) \in D\}; A(S) = A(D) = \iint_D dxdy$$

Example 14. *1)...*



2)..



3.3 Flux of vector field over surface

Definition 9. *Unit norma vector fields - Oriened surfaces*

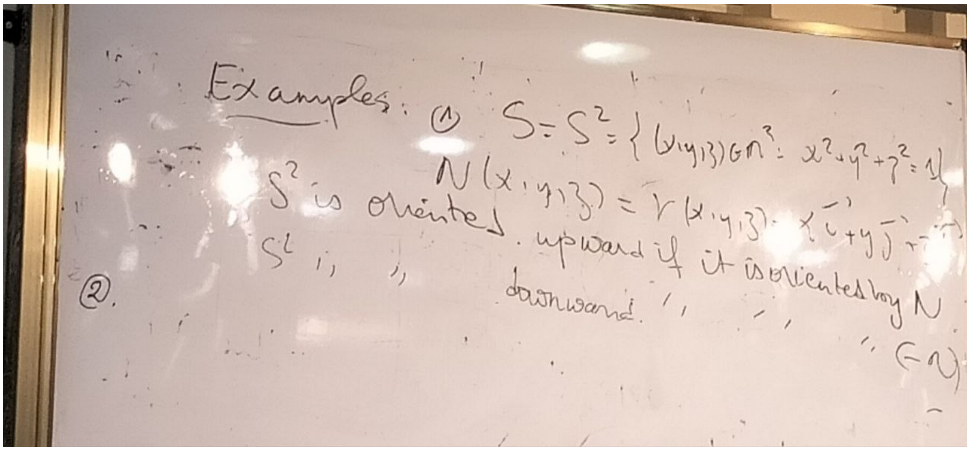
Let S be a smooth parametrized surface

S is regular if $||r_u \times r_\theta||(u,v) \neq 0, \forall (u,v) \in D \mid D$ connected

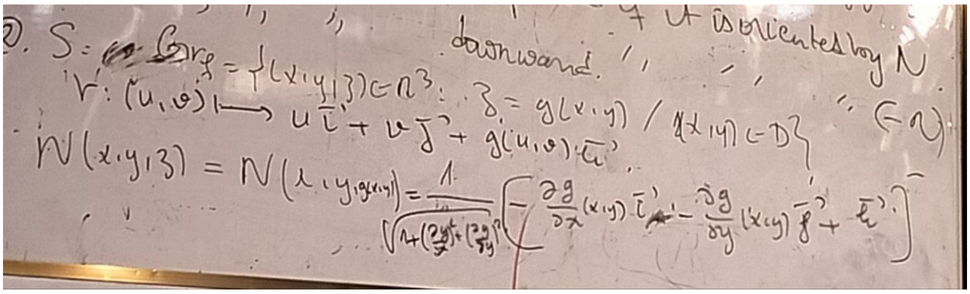
$$So,\ S\ can\ be\ oriented\ by\ N(x,y,z) = \frac{1}{||r_u \times r_v||(u,v)}((r_u \times r_v)(u,v)) \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$$or,\ S\ can\ be\ also\ oriented\ by\ (-N)(x,y,z) = \frac{1}{||r_v \times r_u||(v,u)}((r_u \times r_v)(u,v)) \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

Example 15. *1)*



2)



Definition 10. ***<u>Flux of vector field over regular parametrized</u>***

Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuous vector field over an open set $\Delta \subset S$

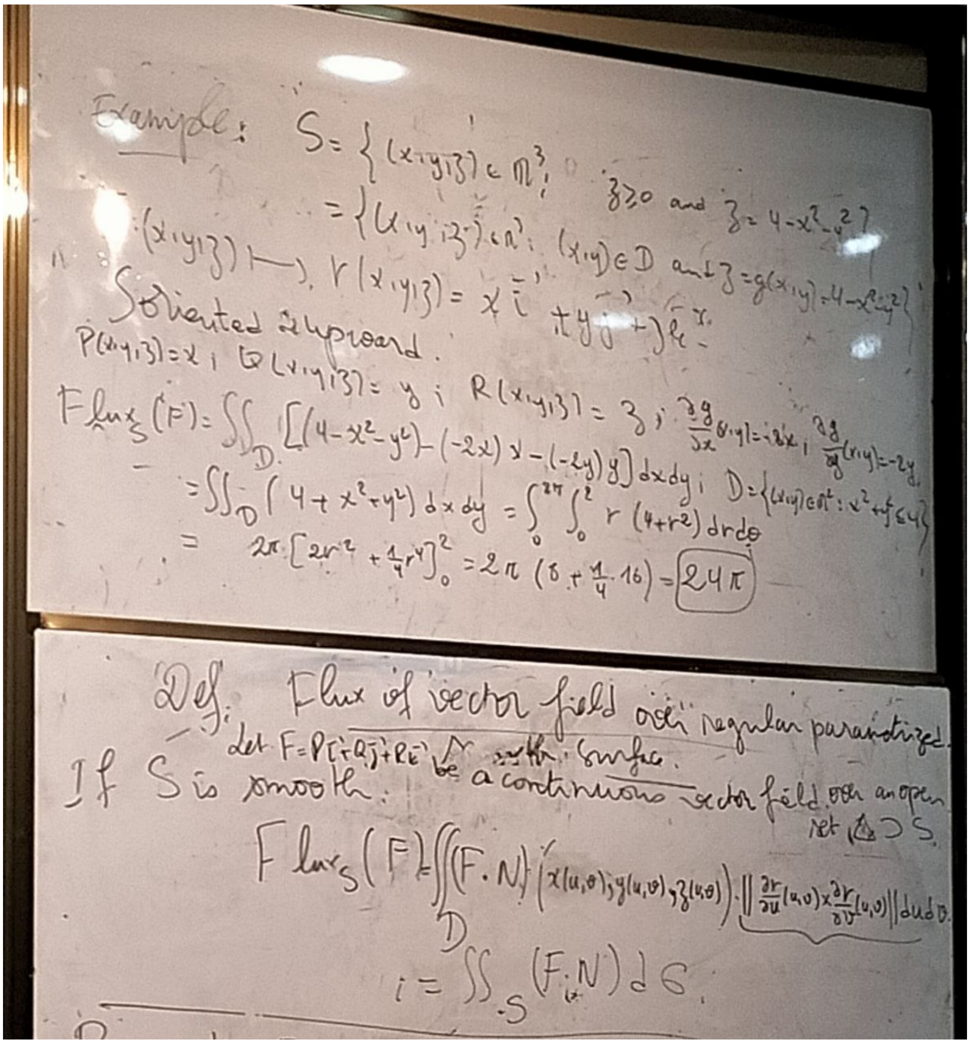
If S is smooth

$$Flux_S(F) = \iint_D (F \cdot N)(x(u,v), y(u,v), z(u,v))||\frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}||dudv$$
$$= \iint_S (F.N)d\sigma$$

Remark $F = P\vec{i} + Q\vec{j} + R\vec{k}$ $S = Gr_f = \{(x,y,z) \in \mathbb{R}^3 \quad : (x,y) \in D \quad ; z = g(x,y)\}$

$$Flux_S(f) = \begin{cases} \iint_D \left[R - \dfrac{\partial g}{\partial x}(x,y)P - \dfrac{\partial g}{\partial y}(x,y)Q\right](x,y,g(x,y))dxdy \\ \qquad \text{if S is oriented upward} \\ \iint_D \left[\dfrac{\partial g}{\partial x}(x,y)P((x,y,g(x,y)) + \dfrac{\partial g}{\partial y}(x,y)Q(x,y,g(x,y)) - R(x,y,g(x,y)\right] dxdy \\ \qquad \text{if S is oriented downward} \end{cases}$$

Example 16. *1)*

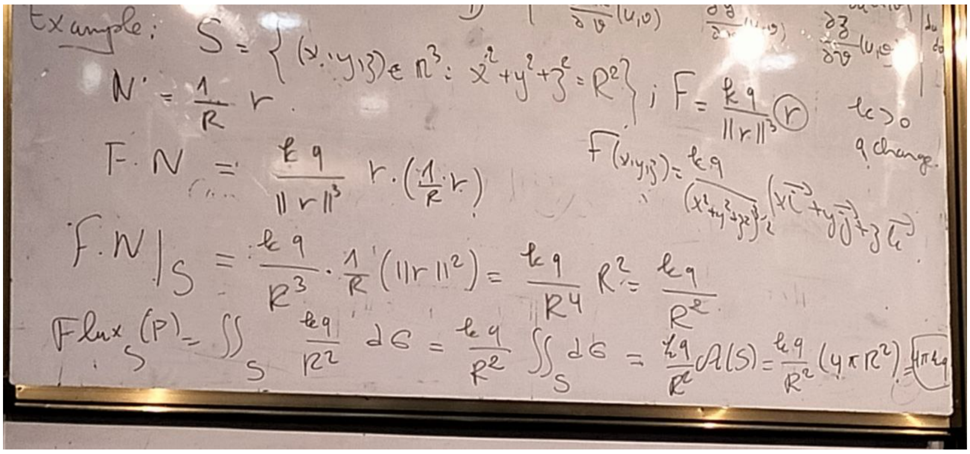


Remark

$$\iint_S (F.N)d\sigma = \iint_D F(x(u,v), y(u,v), z(u,v)).(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}(u,v))dudv$$

$$\iint_D \begin{vmatrix} P(x(u,v),y(u,v),z(y,v)) & Q(x(u,v),y(u,v),z(u,v)) & R(x(u,v),y(u,v),z(u,v)) \\ \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) & \frac{\partial z}{\partial u}(u,v)) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) & \frac{\partial z}{\partial v}(u,v) \end{vmatrix} dudv$$

Example 17. *1 ..*



There are two methods to compute the flux of vector field over a surface S .

1) if S is boundary of a solid domain $\Omega \subset \mathbb{R}^3, (i-eS=\partial D)$,

(S is closed. without boundary $\partial S=\phi$) we can use the divergence theorem

2) If S is not closed, $C=\partial S$ is piece-wise curve we can use Stokes's theorem

Theorem 6. ***Divergence theorem (Ostrograsky's theorem)***

Let Ω be a closed solid bounded region in $\mathbb{R}^3$ by

closed smooth piece-wise regular surface $S=\partial\Omega$ oriented outward from Ω

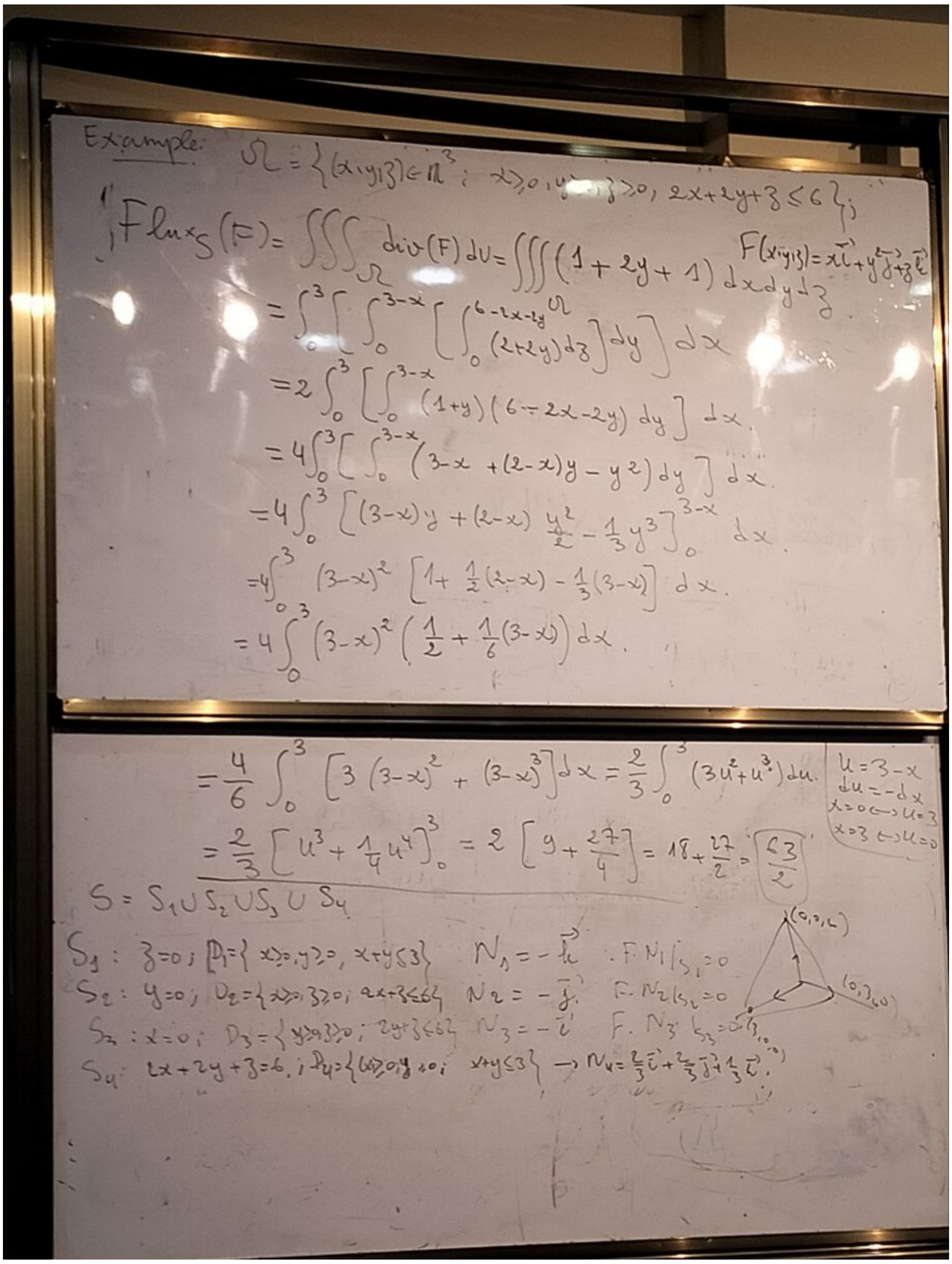
If $F=P\vec{i}+Q\vec{j}+R\vec{k}$ is of class C^1 over Ω then;

$$Flux_S(F)=\iiint div(F)dv=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial Z}\right)(x,y,z)dxdydz$$

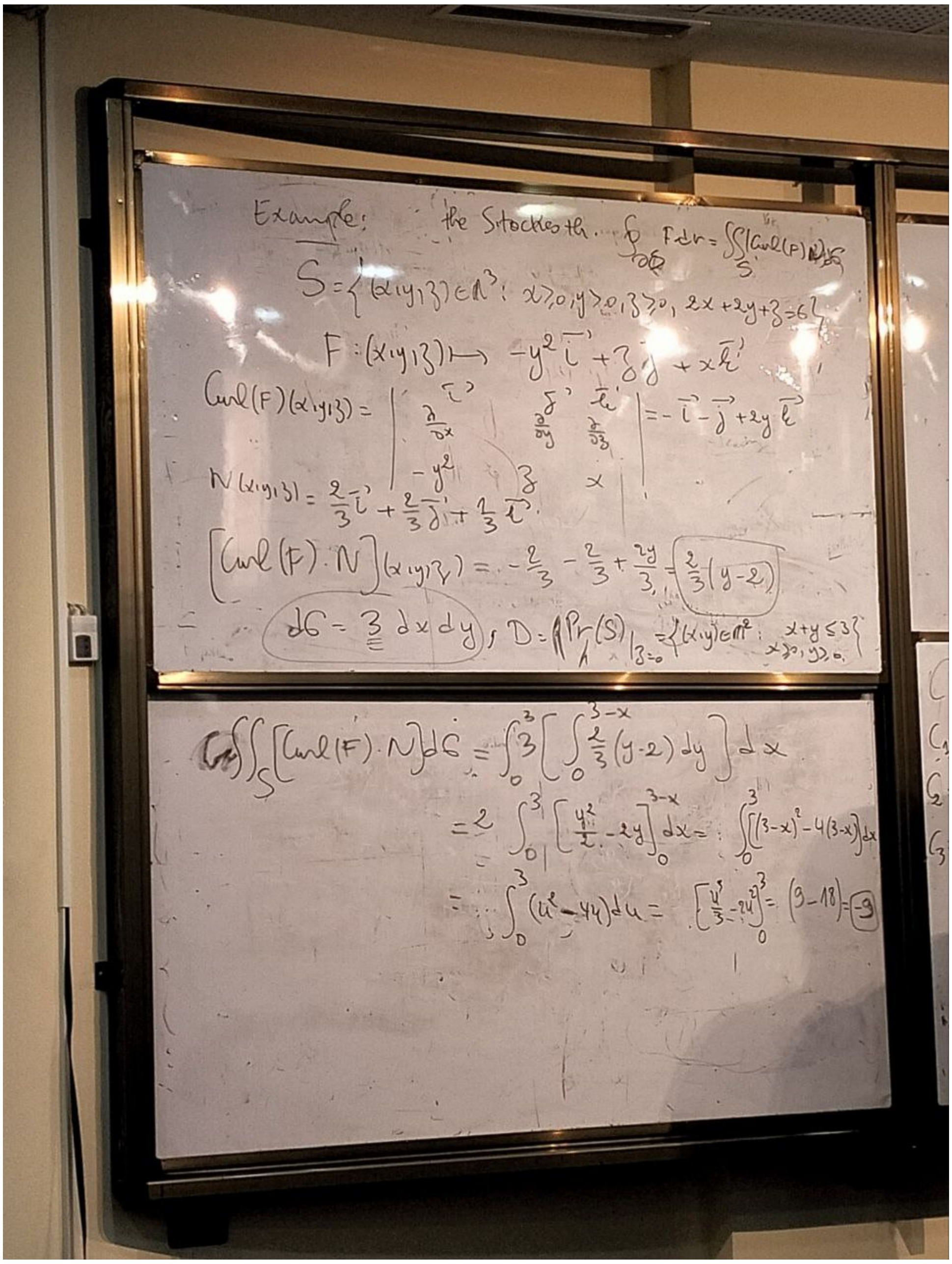
Theorem 7. ***Stockes's theorem*** *Let S be orented by N ; $C=\partial S$ is a piece-wise smooth simple closed curve with positive orientation. If $F=P\vec{i}+Q\vec{j}+R\vec{k}$ is C^1 vector field over a open region $\Delta\subset S$, then:*

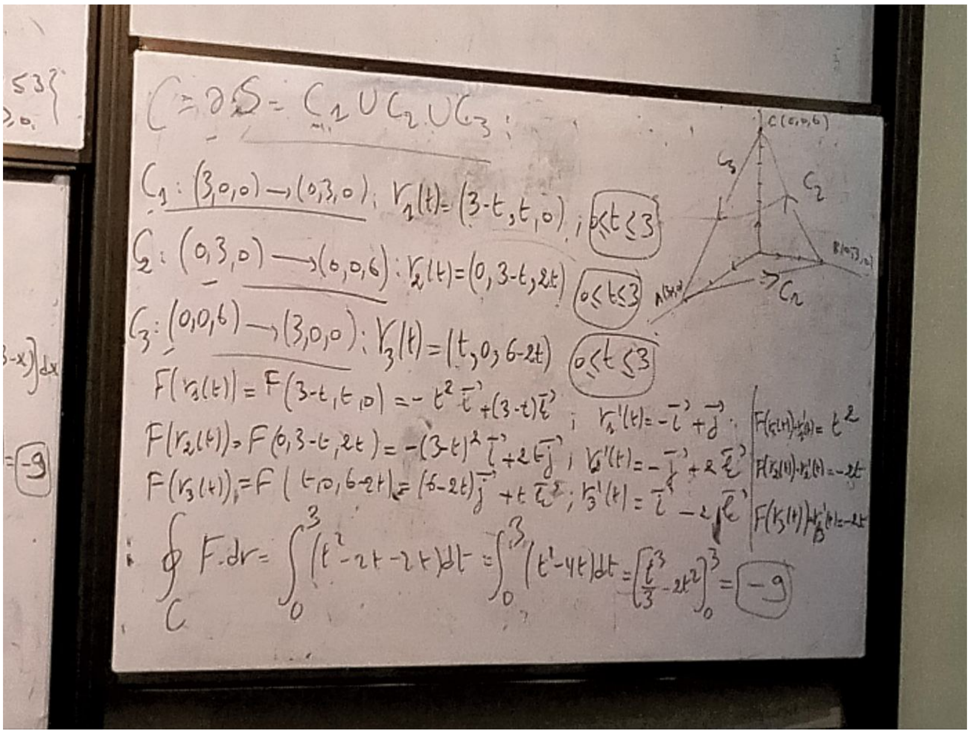
$$\oint_{C=\partial S} F\cdot dr=\iint_S(Curl(F)\cdot N)d\sigma=Flux_S(Curl(F))$$

Example 18. *1)*



Example 19. *stokes example*


Example: the Stockes th. $\oint_{\partial S} F\cdot dr = \iint_S [\mathrm{Curl}(F)\cdot N]\, d\sigma$
$S = \{(x,y,z)\in\mathbb{R}^3 : x\geq 0, y\geq 0, z\geq 0, 2x+2y+z=6\}$
$F : (x,y,z) \longmapsto -y^2\vec{i} + z\vec{j} + x\vec{k}$
$\mathrm{Curl}(F)(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = -\vec{i} - \vec{j} + 2y\vec{k}$
$N(x,y,z) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$
$[\mathrm{Curl}(F)\cdot N](x,y,z) = -\frac{2}{3} - \frac{2}{3} + \frac{2y}{3} = \frac{2}{3}(y-2)$
$d\sigma = 3\, dx\, dy$, $D = \mathrm{Pr}(S)|_{z=0} = \{(x,y)\in\mathbb{R}^2 : x+y\leq 3, x\geq 0, y\geq 0\}$
$\iint_S [\mathrm{Curl}(F)\cdot N]\, d\sigma = \int_0^3 3\left[\int_0^{3-x} \frac{2}{3}(y-2)\, dy\right] dx$
$= 2\int_0^3 \left[\frac{y^2}{2} - 2y\right]_0^{3-x} dx = \int_0^3 [(3-x)^2 - 4(3-x)]\, dx$
$= \int_0^3 (u^2 - 4u)\, du = \left[\frac{u^3}{3} - 2u^2\right]_0^3 = (9-18) = \boxed{-9}$


$C = \partial S = C_1 \cup C_2 \cup C_3$:
$C_1 : (3,0,0) \to (0,3,0)$; $r_1(t) = (3-t, t, 0)$; $0 \le t \le 3$
$C_2 : (0,3,0) \to (0,0,6)$: $r_2(t) = (0, 3-t, 2t)$ $0 \le t \le 3$
$C_3 : (0,0,6) \to (3,0,0)$: $r_3(t) = (t, 0, 6-2t)$ $0 \le t \le 3$
$F(r_1(t)) = F(3-t, t, 0) = -t^2\vec{i} + (3-t)\vec{k}$; $r_1'(t) = -\vec{i} + \vec{j}$; $F(r_1(t))\cdot r_1'(t) = t^2$
$F(r_2(t)) = F(0, 3-t, 2t) = -(3-t)^2\vec{i} + 2t\vec{j}$; $r_2'(t) = -\vec{j} + 2\vec{k}$; $F(r_2(t))\cdot r_2'(t) = -2t$
$F(r_3(t)) = F(t, 0, 6-2t) = (6-2t)\vec{j} + t\vec{k}$; $r_3'(t) = \vec{i} - 2\vec{k}$; $F(r_3(t))\cdot r_3'(t) = -2t$
$\therefore \oint_C F\cdot dr = \int_0^3 (t^2 - 2t - 2t)\,dt = \int_0^3 (t^2 - 4t)\,dt = \left[\frac{t^3}{3} - 2t^2\right]_0^3 = \boxed{-9}$
C(0,0,6)
B(0,3,0)
A(3,0,0)
C_1
C_2
C_3