Kirchhoff's laws in sinusoidal regime

Electric Dipole, Electrokinetic dipole, two-terminal element, two-port element

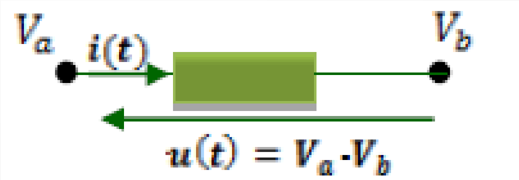
- An electrokinetic dipole is defined as any system connected to the outside by only two conductors. The behavior of a dipole is characterized by two dual electrical quantities: voltage and current.

Voltage / Current

- **Voltage or Potential Difference:**

The potential difference u(t) between points a and b is measured by the work required to move a unit charge from A to B. The unit is the volt (V),

1 volt = 1 joule/coulomb.



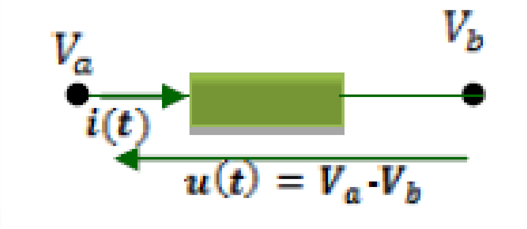
- **Electric Current:**

Electric current is the variation of electric charge as a function of time:

i(t) = dq/dt. The unit is the ampere (A): **1A = 1 coulomb/second**.

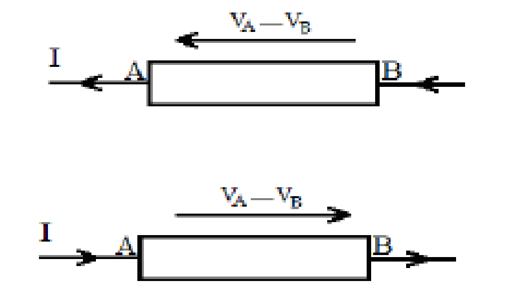
Electrical power

- **The instantaneous electrical power** absorbed / provided by the two-terminal element between terminals a and b is defined by: **p(t) = u(t) × i(t),** where u(t) and i(t) are the instantaneous voltage and current, respectively.

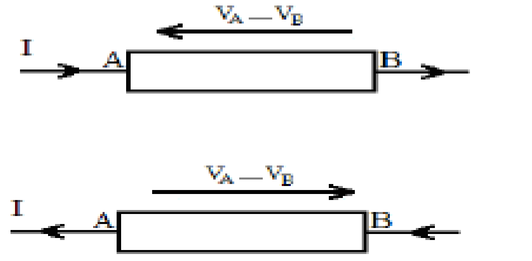


Generator / Load

- A **generator** is an active dipole that supplies electrical energy to a circuit. It converts other forms of energy (mechanical, chemical, solar, etc.) into electrical energy. Generators are characterized by their ability to maintain a voltage or deliver a current regardless of the external circuit conditions, within their operating limits.
- A **load** is a passive dipole that consumes electrical energy from a circuit. It converts electrical energy into other forms of energy (heat, light, mechanical work, etc.). Loads are characterized by their impedance, which determines the relationship between voltage and current.



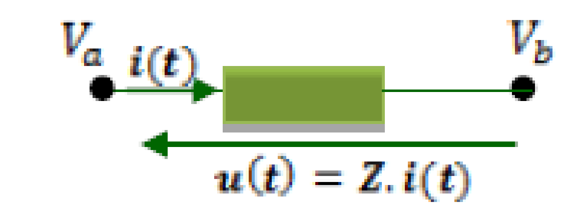
Generator case



Load Case

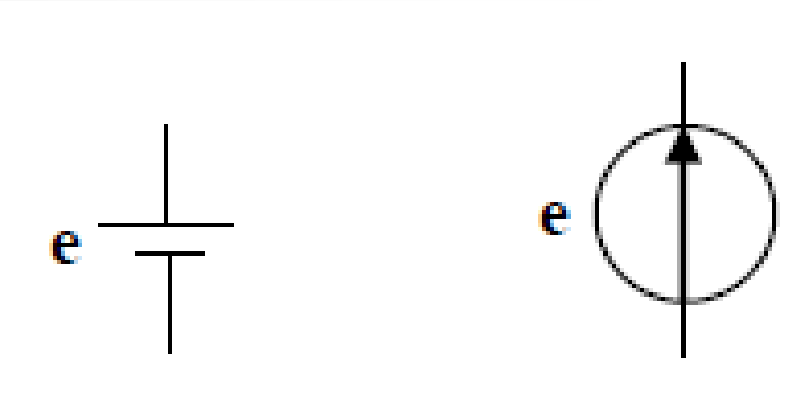
Impedance

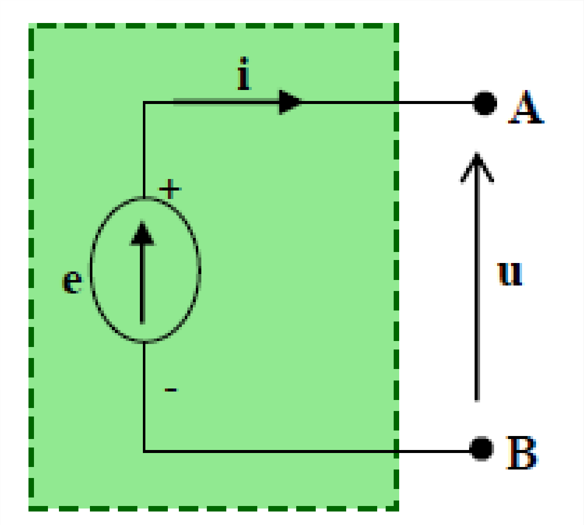
- For a load with impedance Z, Ohm's law states that the complex voltage across the load is equal to the product of the complex current and the impedance: **V = Z × I**



Types of Generators: Voltage source/ Current source

- An ideal voltage generator delivers a constant voltage form, independent of the delivered current. The internal resistance of an ideal voltage generator is zero. This generator is represented by the following symbol:





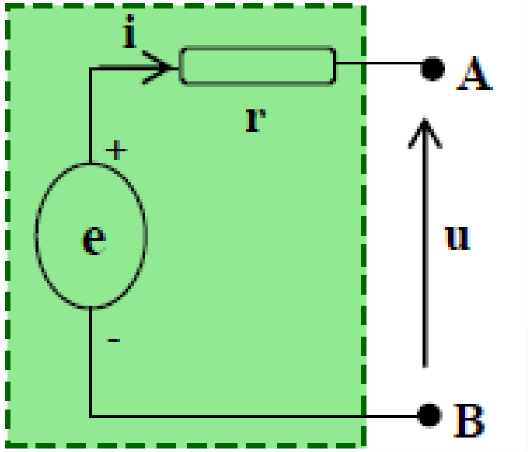
Types of Generators: Voltage source/ Current source

- **Properties of Ideal Voltage Sources:**
- **Voltage:** Constant form
- **Current:** Can vary from -∞ to +∞
- **Internal resistance:** Ri = 0 Ω
- **Power delivered:** P = V × I (depends on load)

Types of Generators: Voltage source/ Current source

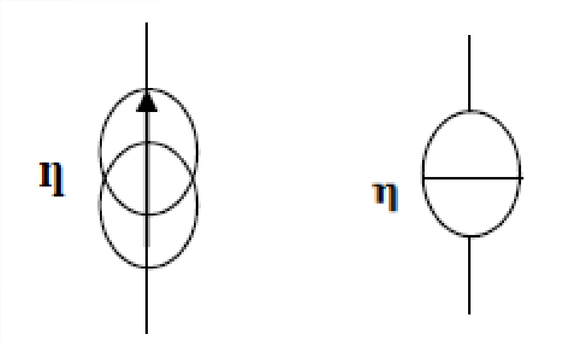
An ideal voltage generator does not exist in practice; the potential difference at its terminals decreases as a function of the output current.

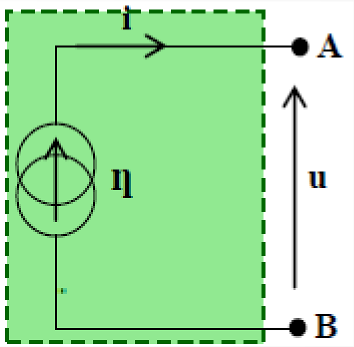
It can be modeled by an ideal generator in series with its internal resistance. The internal resistance causes a voltage drop, and therefore u = e - ri.



Types of Generators: Voltage source/ Current source

- An ideal current generator delivers a current independent of the load. This generator is represented by the following symbol:

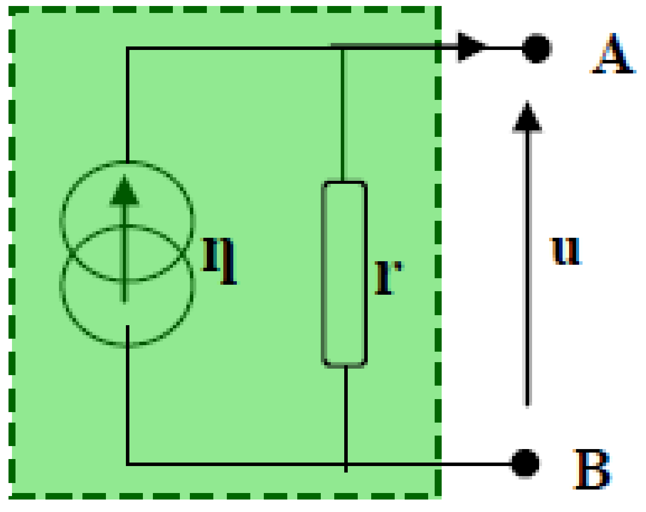




- Current: I = constant (regardless of load)
- Voltage: Adjusts automatically to maintain constant current
- Load independence: Current remains the same for any load impedance
- Internal resistance: Ri = ∞ Ω

Types of Generators: Voltage source/ Current source

- For a real current generator, we take into account its internal resistance; it is modeled by an ideal current source in parallel with its internal resistance r.



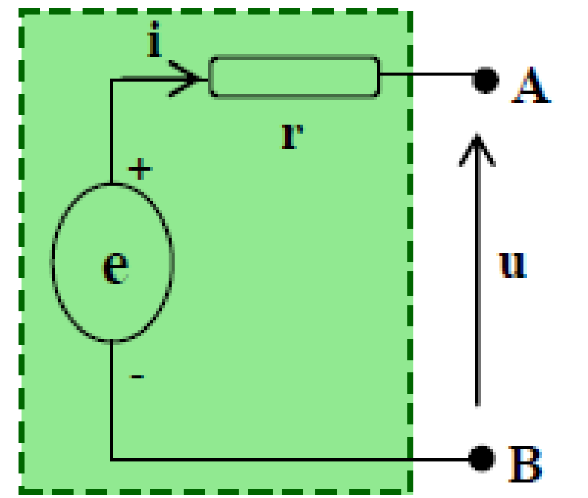
(e)

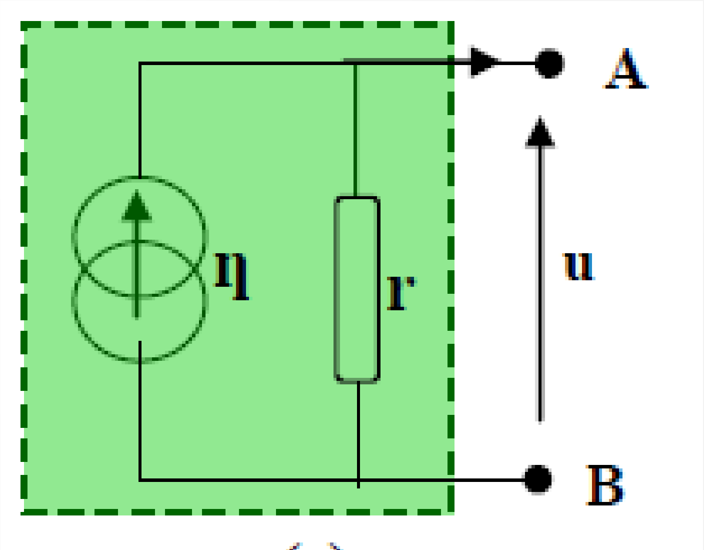
Types of Generators: Voltage source/ Current source

- **Generator Equivalence:**

A voltage generator and a current generator will be equivalent if they have the same internal impedance and the following relationship is satisfied:

$e = \eta \cdot r$





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Types of Generators: Independent Sources/ Dependent (Controlled) Sources

Definition: Sources whose output (voltage or current) is **not controlled** by any other variable in the circuit.

Characteristics:

- **Fixed value:** The source value is predetermined and constant
- **Autonomous operation:** Functions independently of circuit conditions
- **External control only:** Can only be changed by external means

Types:

- **Independent voltage source:** $v(t)$ = constant or $v(t) = f(t)$
- **Independent current source:** $i(t)$ = constant or $i(t) = f(t)$

Types of Generators: Independent Sources/ Dependent (Controlled) Sources

Definition: Sources whose output is **controlled** by another voltage or current elsewhere in the circuit.

Four types of controlled sources:

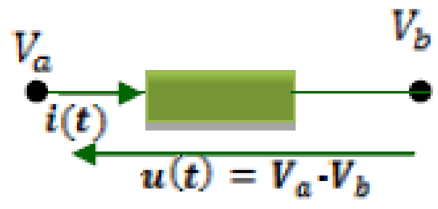
- **VCVS** - **Voltage Controlled Voltage Source** (Source de tension commandée en tension)
  - $v_2 = \mu \times v_1$ (where μ = voltage gain)
- **CCCS** - **Current Controlled Current Source** (Source de courant commandée en courant)
  - $i_2 = \beta \times i_1$ (where β = current gain)
- **VCCS** - **Voltage Controlled Current Source** (Source de courant commandée en tension)
  - $i_2 = g_m \times v_1$ (where g_m = transconductance)
- **CCVS** - **Current Controlled Voltage Source** (Source de tension commandée en courant)
  - $v_2 = r \times i_1$ (where r = transresistance)

Passive loads: *Resistors*

- The relationship that links voltage and current across a resistor is given by Ohm's law:

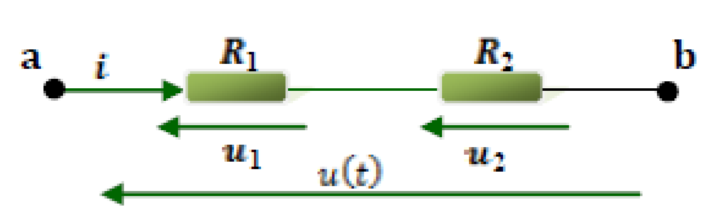
$u(t) = R \cdot i(t)$ or $i(t) = G \cdot u(t)$ with $G = 1/R$

G is the conductance and the relationship i = f(u) is called the characteristic of the resistor.



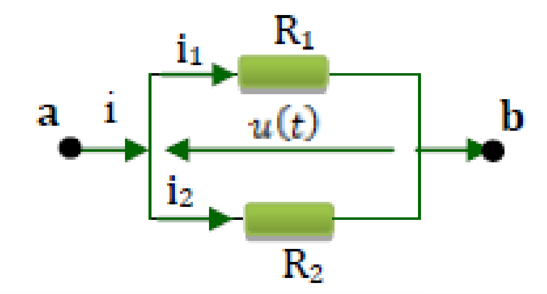
Passive loads: *Resistors*

- Series Association : For resistors in series



$$R_{eq} = R_1 + R_2 \rightarrow R_{eq} = \sum R_i$$
$$u(t) = u_1(t) + u_2(t)$$

- Parallel Association: For resistors in parallel



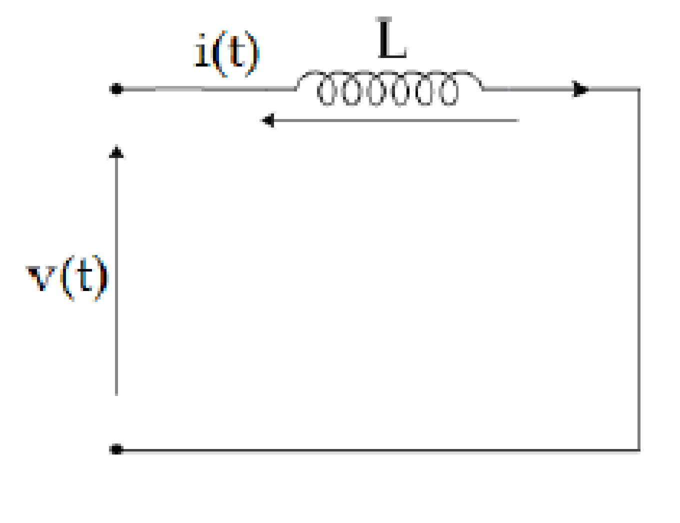
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow \frac{1}{R_{eq}} = \sum \frac{1}{R_i}$$
$$i(t) = i_1(t) + i_2(t)$$

Passive loads: *Inductor*

- v(t) = L .di(t)/dt
- In continuous (permanent) mode, the coil behaves like a short circuit (closed circuit) because $v(t) = 0$.
- Breaking electric current in a coil can lead to the production of very

high voltages, because :

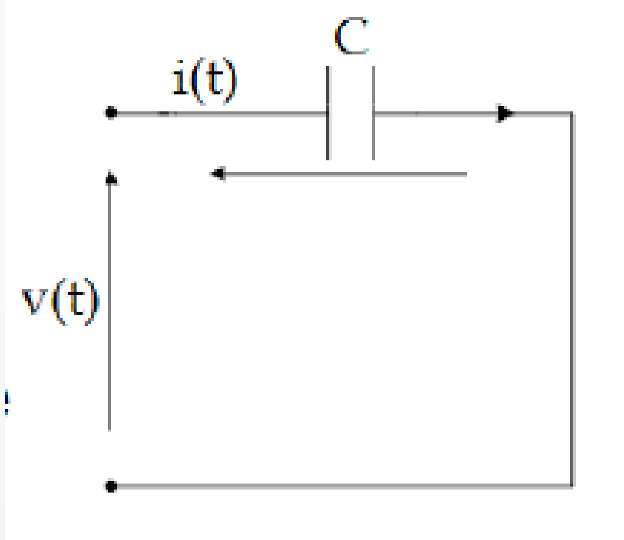
$$di(t) \neq 0 \text{ et } dt \to 0 \Rightarrow v(t) \to \infty$$



Passive loads: *capacitor*

- Governed by the fundamental laws studied in electrostatics:
- Relationship between potential and charge $Q=C.V$
- Relationship between load and current: $i(t)=dq(t)/dt$
- Hence : $i(t)=C.dv(t) / dt$
- A sudden variation in voltage produces an infinite current,

$$dv(t) \neq 0 \text{ et } dt \to 0 \Rightarrow i(t) \to \infty$$



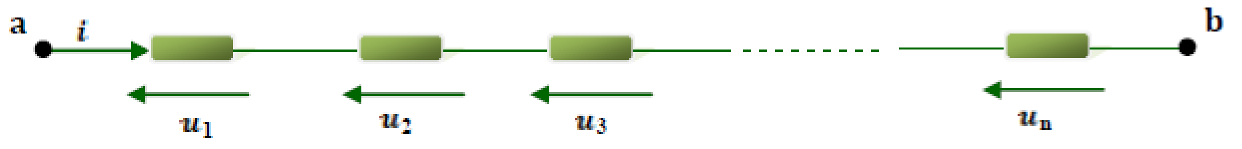
Node / Branch / mesh

- A node is a point where more than two conductors meet. This is not necessarily a point in the geometric sense; points in a circuit graph connected by conductors with zero impedance also form a node. A node is therefore the set of all points that have the same potential at any instant.
- A branch is a set of dipoles connected in series between two nodes.
- A mesh (loop) is a set of branches forming a closed loop that passes through each node only once.

Connection of circuit elements

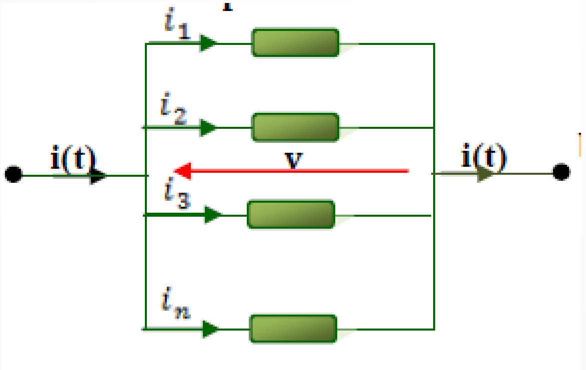
We distinguish between two types of dipole associations: series or parallel.

- **Series Association:** In this association, all dipoles carry the same current.



$$u(t) = \sum_{k=1}^{n} U_k(t)$$

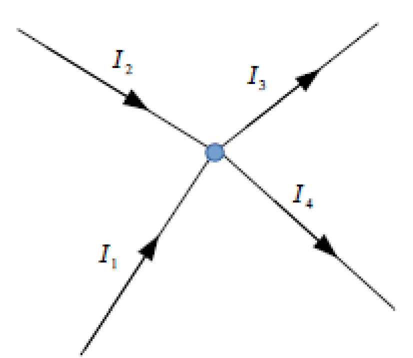
- **Parallel Association:** In this case, the voltage across the dipoles is the same, but the currents are different.



$$i(t) = \sum_{k=1}^{n} i_k(t)$$

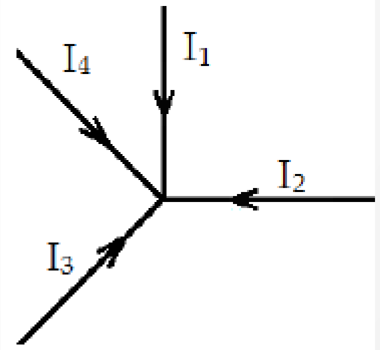
Kirchhoff's Current Law (KCL) / Nodal Law

- The sum of the intensities of the currents arriving at a node is equal to that of the currents leaving the node.



$I_1 + I_2 = I_3 + I_4$

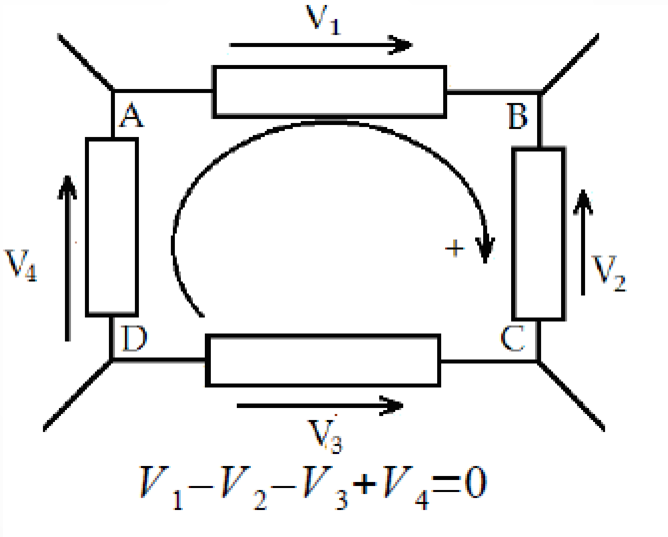
- The algebraic sum of currents at any node in an electrical circuit is zero at any instant.



$I_1 + I_2 + I_3 + I_4 = 0$

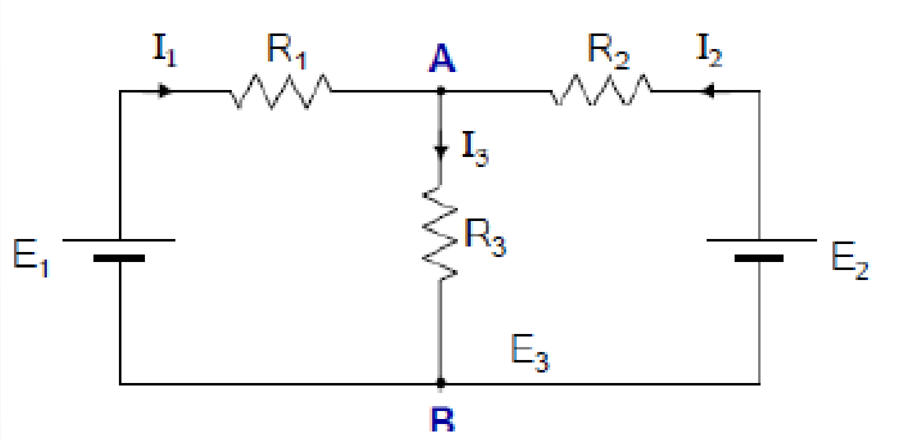
Kirchhoff's Voltage Law (KVL) / Loop Law

- The algebraic sum of voltages around any closed loop in an electrical circuit is zero at any instant.



Example

- We have to find the currents I1 , I2 and I3



Example

Step 1: Voltage Representation

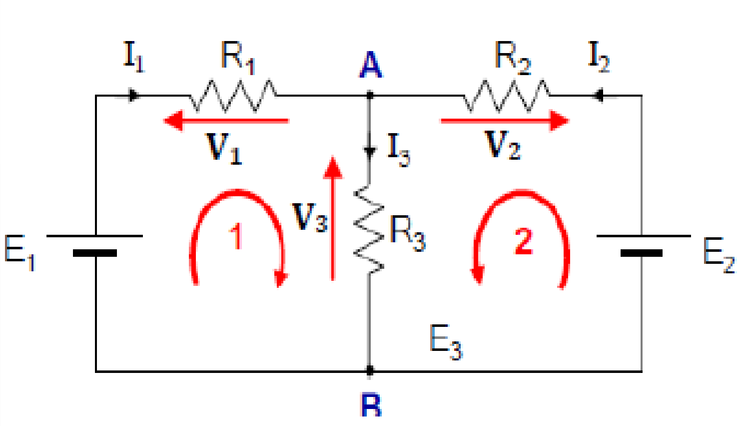
- **Label all voltages** across each element: $u_1(t)$, $u_2(t)$, $u_3(t)$...
- **Indicate voltage polarities** with + and - signs
- **Mark voltage phasors** in sinusoidal regime: $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$...

Step 2: Current Representation

- **Define current directions** for each branch: $i_1(t)$, $i_2(t)$, $i_3(t)$...
- **Use arrows** to show assumed current flow
- **Mark current phasors**: $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$...

Step 3: Loop Path Definition

- **Choose loop directions** (clockwise or counterclockwise)
- **Identify independent loops/meshes**
- **Mark the path** with curved arrows



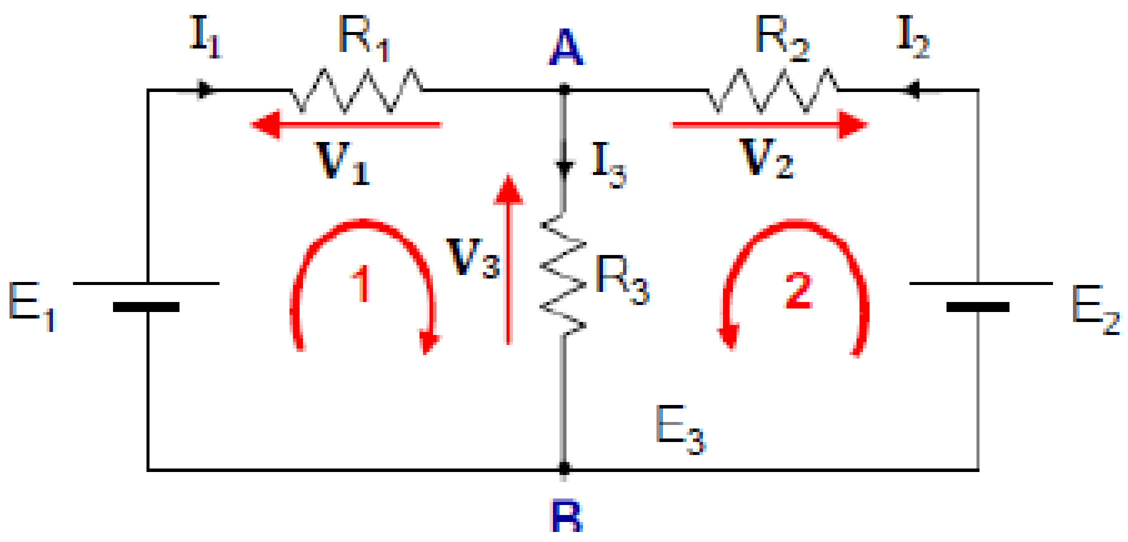
Example

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} (R_1 + R_3) & R_3 \\ R_3 & (R_2 + R_3) \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$I_1 = \frac{\Delta I_1}{\Delta} = \frac{(R_2 + R_3)E_1 - R_3 E_2}{R_3(R_1 + R_2) + R_1 R_2}$$

$$I_2 = \frac{\Delta I_2}{\Delta} = \frac{(R_1 + R_3)E_2 - R_3 E_1}{R_3(R_1 + R_2) + R_1 R_2}$$

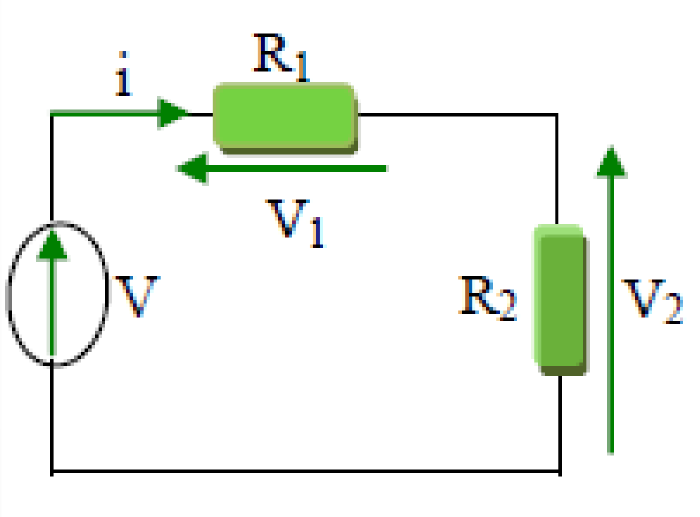
$$I_3 = I_1 + I_2 = \frac{R_2 E_1 - R_1 E_2}{R_3(R_1 + R_2) + R_1 R_2}$$



Voltage Divider / Current Divider

- If two resistances (or impedances) are placed in series across a voltage V, then we can directly express the voltage across one of the resistances as a function of V.

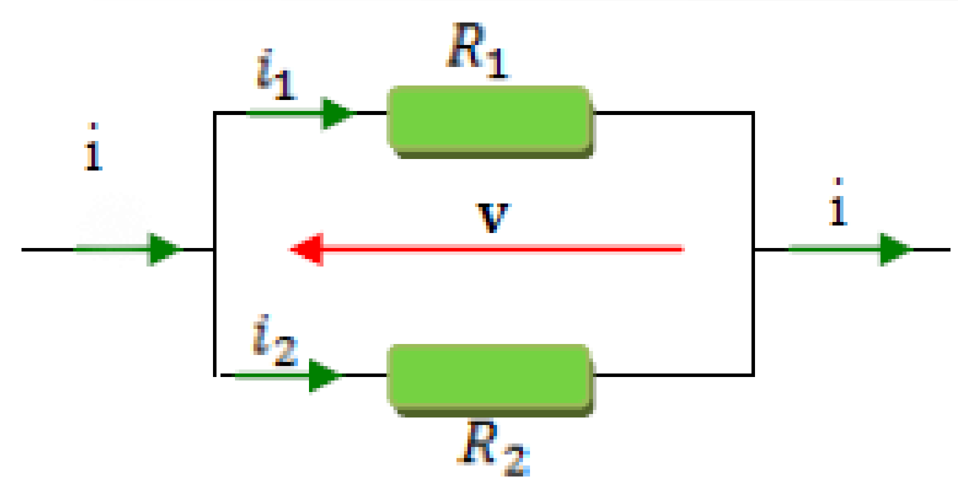
$$V_2 = \frac{R_2}{R_1 + R_2}.V;\ V_1 = \frac{R_1}{R_1 + R_2}.V$$



Voltage Divider / Current Divider

- If two resistances (or impedances) are placed in parallel and carry currents i_1 and i_2 from a total current i, then we can directly express one of the currents i_1 or i_2 as a function of the current i.

$$i_1 = \frac{R_2}{R_1 + R_2} . i \; ; \; i_2 = \frac{R_1}{R_1 + R_2} . i$$



Sinusoidal Steady State

The sinusoidal regime, often called harmonic regime, plays a considerable role in linear electronics, and more generally in linear system theory, for various reasons:

- The sinusoidal signal waveform is the only one that is preserved when passing through a linear system. Indeed, the integral or derivative of a sinusoid always remains a sinusoid with amplitude and phase that may vary.
- Fourier theory shows that any signal can be decomposed into an infinite sum of sinusoidal signals. We can therefore predict the response of a linear system to any signal by knowing its harmonic response.
- Finally, the sinusoidal signal is very widespread because it is easy to generate.

Generalities

- A physical quantity (current, voltage, etc.) is said to be periodic if it identically returns to the same value at equal time intervals.
- **Period T:** It is the minimum time required to return to the same value of the function.
- **Frequency F:** It is the inverse of the period; its unit is the HERTZ (Hz).
- **Instantaneous value** represents the actual value of a current or voltage at a given instant t.
- **Peak value or maximum value (Vm or Im)** is the largest instantaneous value encountered in a positive or negative alternation of a waveform. We then speak respectively of positive or negative peak.
- **The phase shift φ** reflects a time shift $\Delta t = \varphi/\omega$

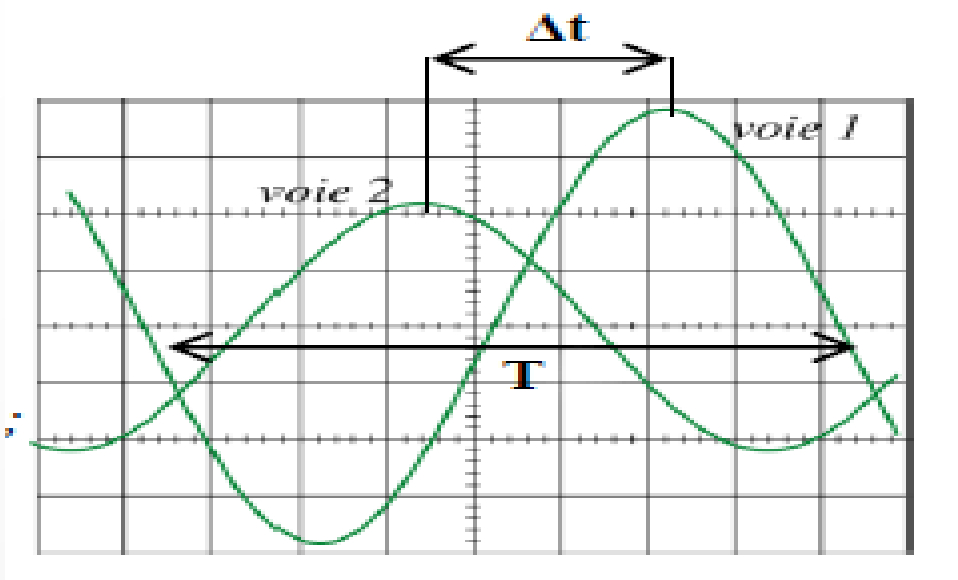
Notion of Phase Shift:

- **Definition:**

The phase shift is the phase difference between two sinusoidal signals of the same frequency.

For voltage and current:

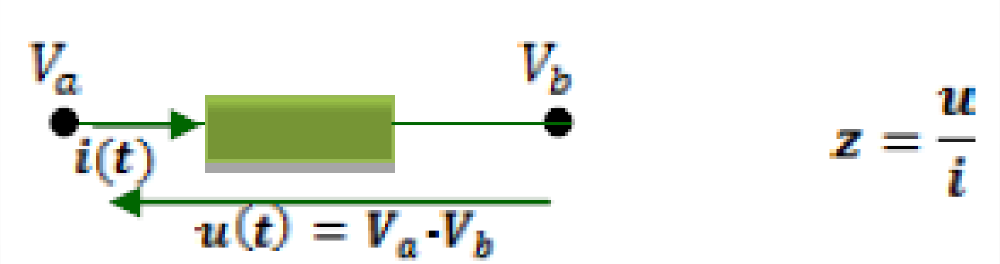
- **Current:** $I(t) = I_{max} \sin(\omega t + \varphi_i)$
- **Voltage:** $V(t) = V_{max} \sin(\omega t + \varphi_v)$
- **Phase shift of voltage with respect to current:** $\Delta\varphi = \varphi_v - \varphi_i$
- Signal 1 **lags** signal 2
- Signal 2 **leads** signal 1



Notion of Impedance:

The impedance **z** of an element, a circuit branch, or a complete circuit corresponds to the ratio of the voltage across the element to the current through the element:

- **z = u/i**
- For sinusoidal voltages and currents, this ratio called "complex impedance" z, has a magnitude and an argument.
- **|z| = Vmax / Imax**



Root Mean Square (RMS value)

- **The RMS value of a periodic current is the intensity of a direct current that would dissipate the same energy in a resistor during one period. The values indicated by measuring instruments such as voltmeters or ammeters are generally RMS values.**

$$I_{RMS}^2 = \frac{1}{T}\int_0^T i^2\,dt$$

Root Mean Square (RMS value)

- Consider a dipole subjected to a voltage u(t) and carrying a current i(t) such that:

I(t) = Imax sin(ωt) and **V(t) = Vmax sin(ωt + φ)**

- The RMS values of current and voltage are given by the following expressions:

$$I_{RMS}^2 = \frac{1}{T}\int_0^T i^2\,dt$$

$$I_{RMS}^2 = \frac{1}{T}\cdot I_{max}^2 \int_0^T (\sin(\omega t))^2\,dt = \frac{I_{max}^2}{T}\cdot\int_0^T \frac{1}{2}(1-\cos(2\omega t))\,dt\,;\ \int_0^T \cos(2\omega t)\,dt = 0$$

$$I_{RMS} = \frac{I_{max}}{\sqrt{2}}$$

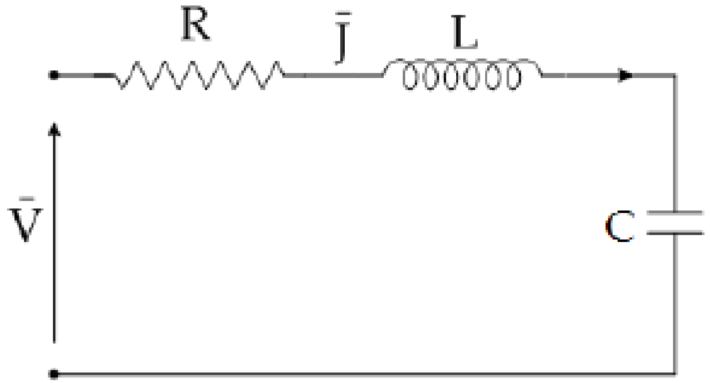
- **Average Values:** The average values of current and voltage are given by:

$$I_{avg} = \frac{1}{T}\int_0^T i(t)\,dt = \frac{1}{T}\int_0^T I_{max}\sin(\omega t)\,dt = 0 \quad \Rightarrow \quad I_{avg} = 0$$

$$V_{avg} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\int_0^T V_{max}\sin(\omega t + \varphi)\,dt = 0 \quad \Rightarrow \quad V_{avg} = 0$$

Fresnel Representation:

- Let us consider the determination of the currents and voltages of the RLC circuit using classical notions.



- The calculations are laborious

- Are there alternative calculation methods for determining **steady-state** currents and voltages when the source is **sinusoidal**?

- YES, two solutions:
    - Phasor (Fresnel diagram): limited to simple circuits,
    - Complex notation: much easier to use).

Fresnel Representation (Phasor):

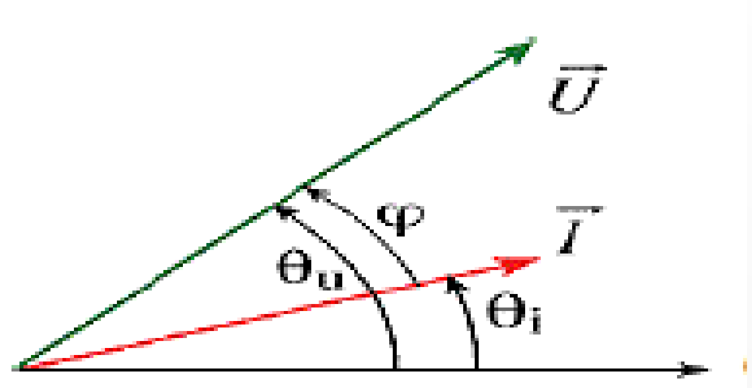
- Fresnel representation is a vector representation of sinusoidal quantities. An electric current (or voltage) is graphically represented by a vector whose **length equals the RMS value** of the quantity and whose angle represents its initial phase.
- Consider a dipole Z carrying a current *i* and having a voltage *u* across its terminals:

$$i(t) = I\sqrt{2}\,\sin(\omega t + \theta_i) \quad \Longleftrightarrow \quad \vec{I}(I, \theta_i)$$

$$u(t) = U\sqrt{2}\,\sin(\omega t + \theta_u) \quad \Longleftrightarrow \quad \vec{U}(U, \theta_u)$$

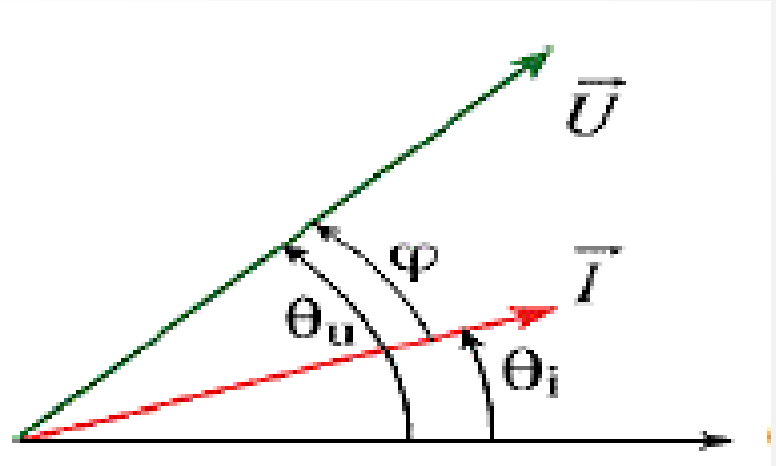
- where *U* and *I* are the RMS values of *u* and *i*, respectively.

$$\varphi = \theta_u - \theta_i$$



Fresnel Representation (Phasor):

- Hence the Fresnel representation or vector representation of u(t) and i(t).
- The origin of phases is the reference point (zero phase) chosen to express the phases of all sinusoidal quantities in a circuit.
- **The choice of phase origin is arbitrary**, but once chosen, it must be **consistent throughout the analysis**.

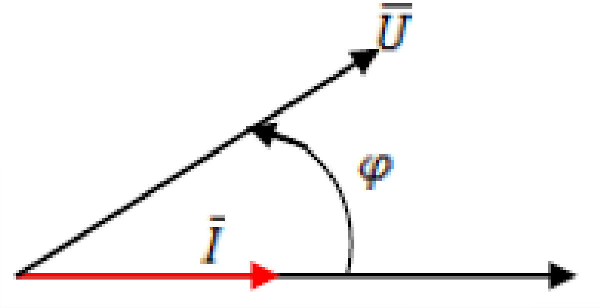


Fresnel Representation (Phasor):

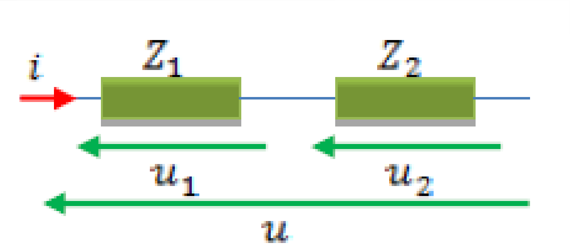
- If we take the current I as the origin of phases, the representation is simplified.

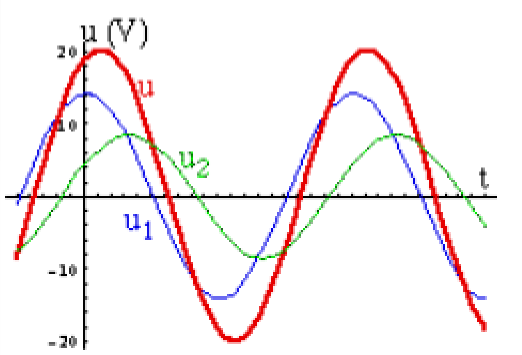
$$u(t) = U\sqrt{2}\,\sin(\omega t + \varphi) \quad \Longleftrightarrow \quad \overrightarrow{U}(U, \varphi)$$

$$i(t) = I\sqrt{2}\,\sin(\omega t) \quad \Longleftrightarrow \quad \overrightarrow{I}(I, 0)$$

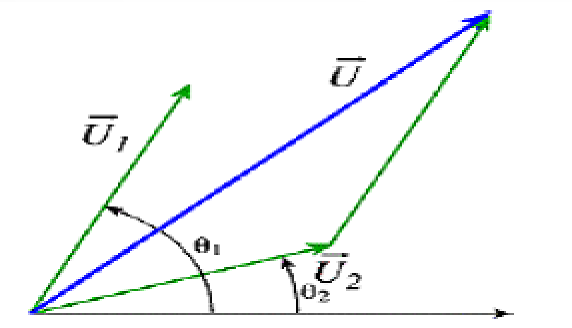


Kirchhoff's Voltage Law in Fresnel Representation





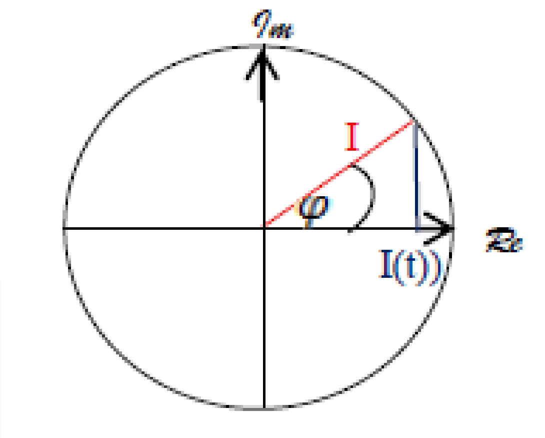
$$u(t) = u_1(t) + u_2(t)$$
$$u_1(t) = U_1\sqrt{2}\,\sin(\omega t + \theta_1)$$
$$u_2(t) = U_2\sqrt{2}\,\sin(\omega t + \theta_2)$$



Complex Representation

- We can notice that trigonometric functions are only the projections of the unit circle onto the real or complex axes.
- Let us call the complex current the vector Ī. When Ī traces the unit circle, then the projection of Ī onto the horizontal axis describes the real current I(t).
- It will often be more practical to manipulate the complex current $I = I_m e^{j(\omega t+\phi)}$

rather than the real current: $I(t) = I_m \cos(\omega t + \phi)$



Complex Representation

- Indeed, it is extremely easy to differentiate with respect to time a sinusoidal voltage written in its complex form $V = V_m e^{j(\omega t+\phi)}$ ative of v(t) with respect to time is written as:

$$\frac{dV}{dt} = j\omega \,.V_m e^{j(\omega t+\phi)} = j\omega V$$

- Moreover, the laws applicable in DC regime are also applicable to complex notation in sinusoidal regime. It is therefore possible, and often preferable, to represent alternating voltage and current in their complex forms

$$\underline{V} = V_m e^{j(\omega t+\psi)}$$

$$\underline{I} = I_m e^{j(\omega t+\phi)}$$

Complex Impedance

- We also define the notion of complex impedance as a generalization of the notion of resistance. The impedance of a passive linear dipole (resistor, capacitor, or inductor) is the complex quantity Z(jω) that relates, in complex representation, the potential difference to the current. Ohm's law in AC regime then becomes:

$$\underline{V} = \underline{Z}\,.\underline{I}$$

$$\underline{Z} = \frac{V_m . e^{j(\omega t+\psi)}}{I_m e^{j(\omega t+\phi)}} = \frac{V_m}{I_m} e^{j(\psi-\phi)} = \frac{V_m}{I_m} e^{j(\theta)} \; ; \; |Z| = \frac{V_m}{I_m} = \frac{V_{eff}}{I_{eff}}$$

$$\underline{Z} = R + jX = |Z| e^{j\theta}$$

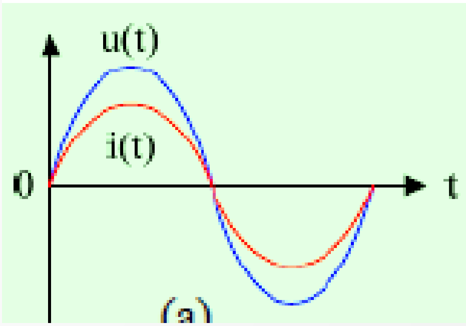
Complex Impedance

- The real part **R** is called **resistance** and the imaginary part **X** is called **reactance**.
- The **magnitude** of the impedance is therefore equal to the **ratio of the magnitudes of voltage and current**. Its **argument or phase shift is equal to the difference between the arguments of voltage and current**.
- We also define:
- The quantity $Q = |X|/R$ is called the quality factor of the dipole.
- The quantity $Y = 1/Z = G + j\,B$ is called the admittance of the dipole. The real part G is called conductance and the imaginary part B is called susceptance.

Complex Impedance

- **Resistance**

$$\boxed{\overline{Z_R} = R\ ;\quad |\bar{Z}_R| = R\ \ et\ \ \varphi = 0}$$

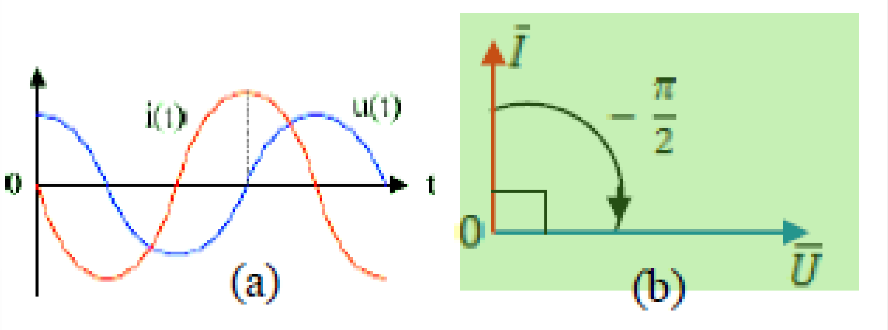


Complex Impedance

Capacity

$$\left.\begin{array}{l} i(t) = \frac{dq(t)}{dt} \\ q(t) = C.u(t) \end{array}\right\} \; i(t) = C\frac{du(t)}{dt}$$

$$\boxed{\overline{Z_c} = \frac{1}{jC\omega}\,; \quad |\overline{Z_c}| = \frac{1}{C\omega} \text{ et } \varphi = -\frac{\pi}{2}}$$

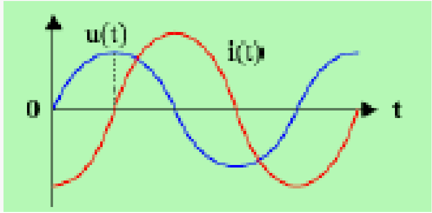


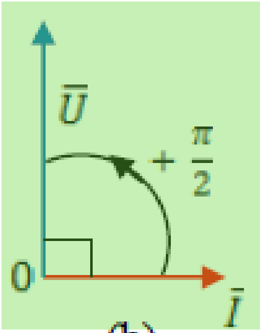
Complex Impedance

- **Inductance**

$$u_L(t) = L.\frac{di(t)}{dt}$$

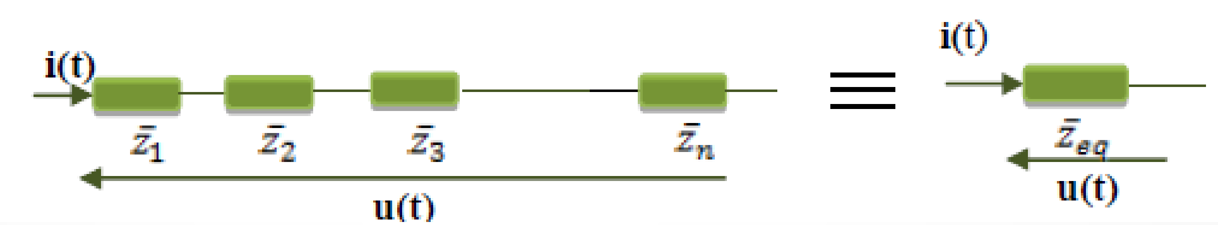
$$\boxed{\bar{z_L} = \mathrm{j}\,L\omega \;;\; \overline{|\bar{z}_L|} = L\omega \quad et \quad \varphi = +\frac{\pi}{2}}$$



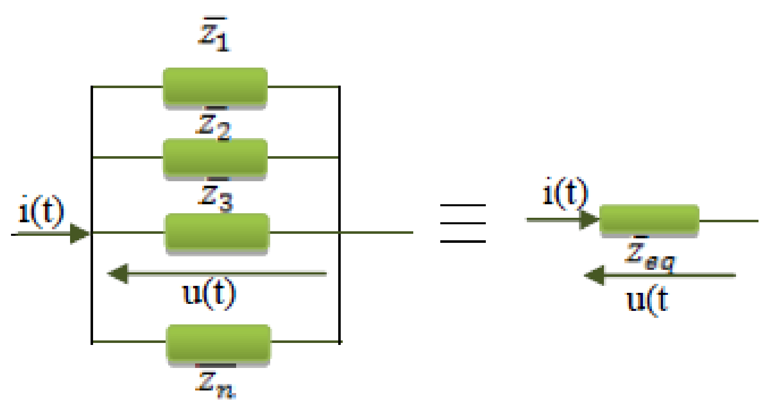


(b)

Complex Impedance



$$\boxed{\bar{z}_{eq} = \sum_{i=1}^{n} \bar{z}_i}$$



$$\boxed{\frac{1}{\bar{z}_{eq}} = \sum_{i=1}^{n} \frac{1}{\bar{z}_i}} \qquad \boxed{\bar{y}_{eq} = \sum_{i=1}^{n} \bar{y}_i}$$