

Chapter 02: Vector Analysis

Notes from prof zeglaoui course

December 21, 2025

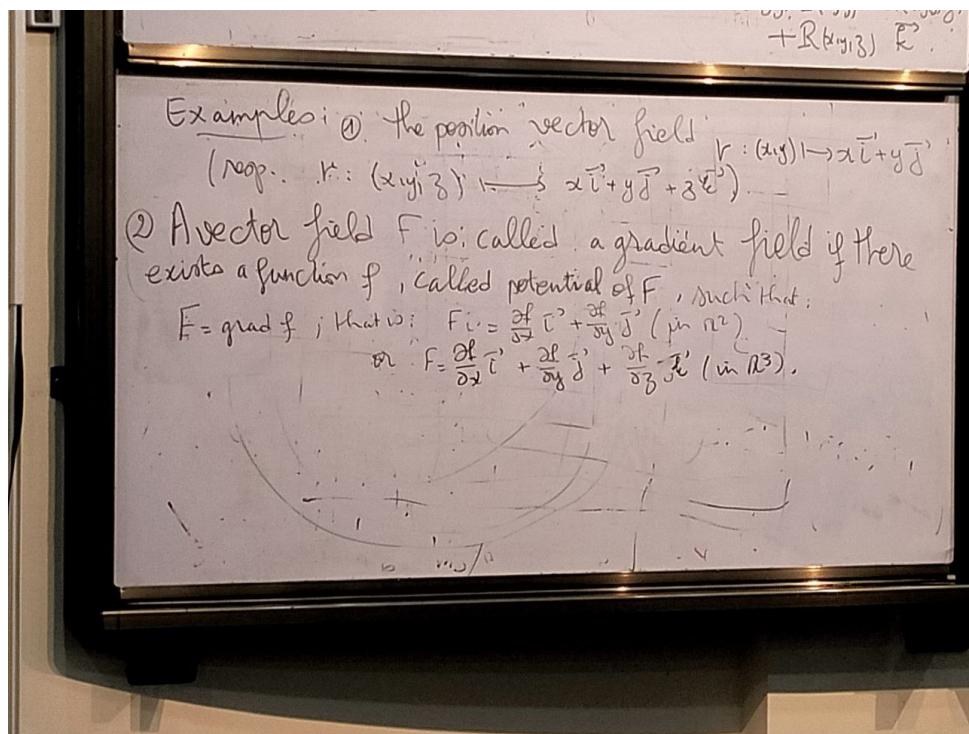
1 vector fields and scalar fields

Definition 1. 1) We call a scalar field defined on a region $D \subset \mathbb{R}^2$ (respectively on $\Omega \subset \mathbb{R}^3$), a real function defined on D (respectively Ω)

2) a vector field F over $D \subset \mathbb{R}^2$ (resp Ω) is a function F that assigns to any $(x, y) \in D$ a vector $F(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$ where $M, N \in \mathcal{F}(D, \mathbb{R})$

3) a vector field F over $\Omega \subset \mathbb{R}^3$ is a function F that assignes to any $(x, y, z) \in \Omega$ a vector $F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$

Example 1. 1) the position vector field



Definition 2. A vector field F is conservative if there exists a

potential function f , $F = \text{grad}(f)$

Example 2. ...

Example: $F: (x, y, z) \mapsto \frac{k}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \vec{v} (x, y, z)$

$\vec{v} (x, y, z) = \langle x^2 + y^2 + z^2 \rangle^{\frac{1}{2}}$

$F = \text{grad } f$ where $f: (x, y, z) \mapsto -\frac{k}{\sqrt{x^2 + y^2 + z^2}} = -k(x^2 + y^2 + z^2)^{-\frac{1}{2}}$

because:

$$\frac{\partial f}{\partial x} (x, y, z) = (-k)(2x) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} = \frac{kx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial y} (x, y, z) = (-k)(2y) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} = \frac{ky}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial z} (x, y, z) = (-k)(2z) \left(-\frac{1}{2}\right) (x^2 + y^2 + z^2)^{-\frac{3}{2}} = \frac{kz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$F = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \leq \text{grad } f$$

Theorem 1. A vector field $F = M\vec{i} + N\vec{j}$ over a convex region $D \subset \mathbb{R}^2$ is conservative if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof. if F is conservative

$$F = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \quad \text{where } f \text{ is of class } C^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0$$

Shwarz' theorem

\Leftrightarrow

$$f(x, y) = \int_0^1 [(x - x_0)M(x_0 + t(x - x_0), y_0 + t(y - y_0)) + (y - y_0)N(x_0 + t(x - x_0), y_0 + t(y - y_0))] dt$$

$$\begin{aligned}
 \frac{\partial I}{\partial x}(x, y) &= \int_0^1 M(x(t), y(t)) dt \\
 &\quad + \int_0^1 \left[t(x-x_0) \frac{\partial M}{\partial x}(x(t), y(t)) + t(y-y_0) \frac{\partial N}{\partial x}(x(t), y(t)) \right] dt \\
 &= \int_0^1 M(x(t), y(t)) dt + \int_0^1 t \left[\left(x-x_0 \right) \frac{\partial M}{\partial x}(x(t), y(t)) + \left(y-y_0 \right) \frac{\partial M}{\partial y}(x(t), y(t)) \right] dt \\
 &= \int_0^1 \Psi(t) dt \quad ; \text{ where } \Psi(t) = t M(x_0 + t(x-x_0), y_0 + t(y-y_0)) \\
 &= \Psi(1) - \Psi(0) = M(x_0 + (x-x_0), y_0 + (y-y_0)) - 0 \\
 &= \boxed{M(x, y)} \quad \boxed{\frac{\partial f}{\partial y} = N}
 \end{aligned}$$

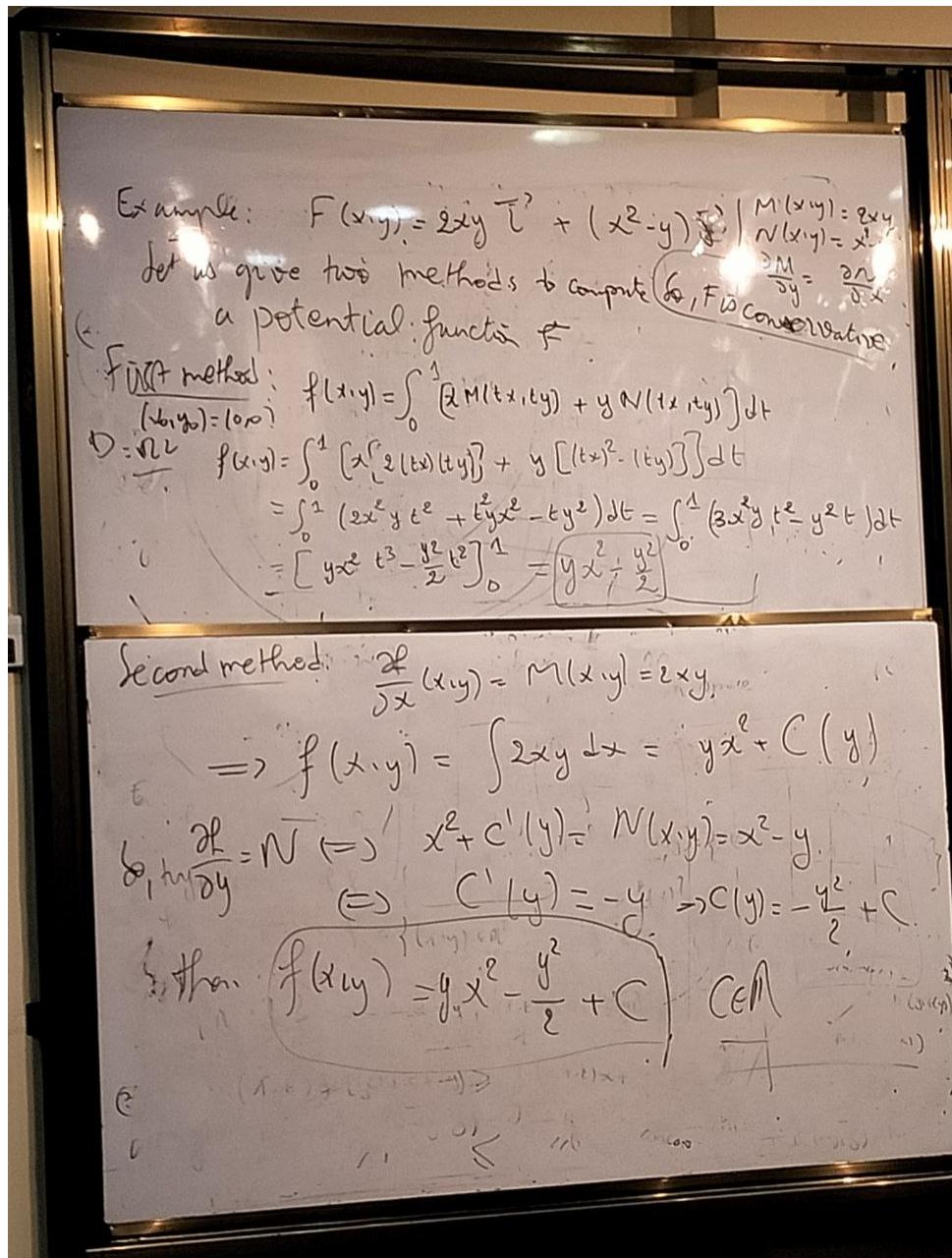
In the same way we prove that

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \text{by} \quad \text{Schwarz's theorem.}$$

(\Leftarrow) $f(x, y) = \int_{(x_0, y_0)}^1 \left[(x-x_0) M \underbrace{(x_0 + t(x-x_0), y_0 + t(y-y_0))}_{\Psi(t)} + (y-y_0) N \right] dt$

Example 3.

□



Definition 3. Let $F : (x,y,z) \rightarrow P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be a vector field over $\Omega \subset \mathbb{R}^3$. Such that the first partial derivatives of F exists

the curl of F , denoted by $\text{Curl}(F)$ (Or $\nabla \times F$) is the vector fields over Ω defined by :

$$\text{Curl}(F) = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

We say that F is irrotational vector field if $\text{Curl}(F) = 0$

Example 4.

Example: $\vec{F} : (x, y, z) \mapsto 2xy \vec{i} + (x^2 + z^2) \vec{j} + 2yz \vec{k}$

 $P(x, y, z) = 2xy; Q(x, y, z) = x^2 + z^2; R(x, y, z) = 2yz$

$\text{Curl}(F) = 0$ because: $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

So, F is irrotational.

Theorem 2. Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be vector field over a convex domain $\Omega \subset \mathbb{R}^3$ of class C^1 then F is conservative $\Leftrightarrow \text{Curl}(F) = 0$

proof

Proof (\Rightarrow): If F is conservative, there exists $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ of class C^2 on \mathcal{U} such that:

$$P = \frac{\partial f}{\partial x}; Q = \frac{\partial f}{\partial y}; R = \frac{\partial f}{\partial z} \quad (F = \text{grad } f),$$

$$\text{curl}(F) = \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial y} \right) \hat{i} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \hat{k}$$

Schwarz's
Thm

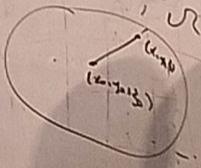
$$= 0 \hat{i} + 0 \hat{j} + 0 \hat{k} = \mathbf{0}$$

(\Leftarrow): Since \mathcal{U} is convex; let $(x_0, y_0, z_0) \in \mathcal{U}$,

$\forall (x, y, z) \in \mathcal{U}, \forall t \in [0, 1]; (x(t), y(t), z(t)) \in \mathcal{U}$.

where

$$\begin{cases} x(t) = x_0 + t(x - x_0) \\ y(t) = y_0 + t(y - y_0); \quad t \in [0, 1] \\ z(t) = z_0 + t(z - z_0) \end{cases}$$



$$\text{curl } f(x, y, z) = \int_0^1 \left[(x-x_0) \frac{\partial P}{\partial z} (x(t), y(t), z(t)) + (y-y_0) \frac{\partial Q}{\partial z} (x(t), y(t), z(t)) \right. \\ \left. + (z-z_0) \frac{\partial R}{\partial z} (x(t), y(t), z(t)) \right] dt = 0$$

$\rightarrow \text{curl}(F) = 0$

$$\frac{\partial f}{\partial x}(x, y, z) = \int_0^1 \varphi'(t) dt \text{ where } \varphi(t) = t P(x, y, z(t))$$

$$= \varphi(1) - \varphi(0) = P(x, y, z) - 0$$

$$= P(x, y, z).$$

In the same way, we prove that $\frac{\partial f}{\partial y} = Q$ and $\frac{\partial f}{\partial z} = R$.

Example 5.

Example: $F(x, y, z) \mapsto 2xy \hat{i} + (x^2 + z^2) \hat{j} + 2yz \hat{k}$, $V^3 = \mathbb{R}^3$

$$f(x, y, z) = \int_0^1 [x P(tx, ty, tz) + y Q(tx, ty, tz) + z R(tx, ty, tz)] dt$$

$$= \int_0^1 [x(2(tx)(ty)) + y((tx)^2 + (tz)^2) + z(2(ty)(tz))] dt$$

$$f(x, y, z) = \int_0^1 [2x^2y t^2 + y(x^2 + z^2)t^2 + 2xyz t^2] dt$$

$$= \int_0^1 3y(x^2 + z^2)t^2 dt = \left[y(x^2 + z^2)t^3 \right]_0^1$$

$f(x, y, z) = y(x^2 + z^2)$

$$\frac{\partial f}{\partial x} = P, \frac{\partial f}{\partial y} = Q, \frac{\partial f}{\partial z} = R$$

Definition 4. Divergence of vector field-Laplacian of scalar field

$$1) \operatorname{div}(F) = \nabla \cdot F \rightarrow \begin{cases} F = M\vec{i} + N\vec{j} \implies \operatorname{div}(F) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \\ F = P\vec{i} + Q\vec{j} + R\vec{k} \implies \operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \end{cases}$$

2) If $F = \operatorname{grad}(f) \implies \operatorname{div}(F) = \operatorname{div}(\operatorname{grad}(f)) = \Delta(f)$ the Laplacian of f

3) F is called divergence free if $\operatorname{div}(F) = 0$

$$\text{if is called harmonic function if } \Delta(f) = 0 \rightarrow \begin{cases} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \end{cases}$$

Remark: If F is of class C^2 vector field, then $\operatorname{div}(\operatorname{curl}(F)) = 0$, using Schwarz's theorem

2 Line integrals

2.1 Line integral of scalar field

Let $f : D \rightarrow \mathbb{R}$ a continuous function over $D \subset \mathbb{R}^2$ (*resp* : $D \subset \mathbb{R}^3$), let

$$\Gamma : \begin{cases} [a, b] \rightarrow D \\ t \mapsto \gamma(t) \end{cases}$$

of C^1 curve on D

the line integral of f over Γ denoted by $\int_{\Gamma} f dl$ is a real number

$$\int_{\Gamma} f dl = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt \quad \text{where } \|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

in particular if $f \equiv 1$;

$$\int_{\Gamma} dl = l(\Gamma) \quad \text{length of } \Gamma = \int_a^b \|\gamma'(t)\| dt$$

$\int_{\Gamma} f dl$ is independant to parametrization $([a, b], \gamma)$ of Γ

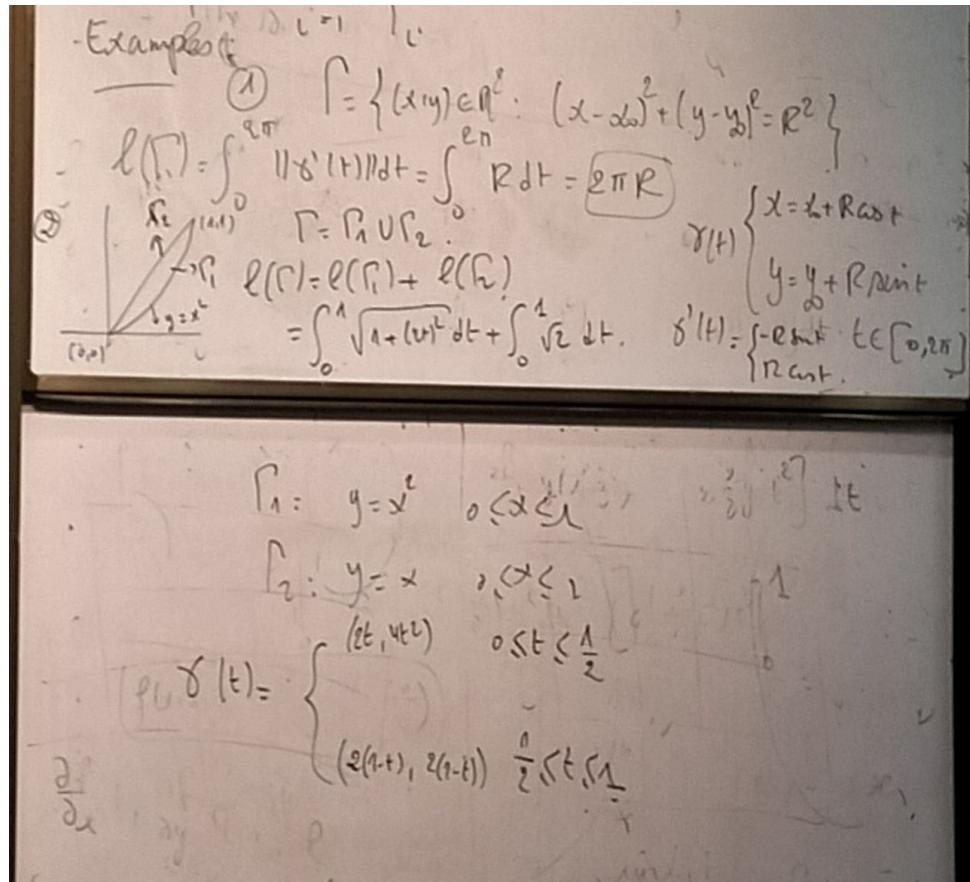
2) If Γ is piece-wise C^1 ; i.e $\exists k \geq 2$

$$\forall i \in \{1, \dots, k\}; \Gamma_i \text{ is } C^1; \Gamma_i \cap \Gamma_{i+1} = \{pt\} \quad \text{and } \Gamma_i \cap \Gamma_j = \phi \quad \text{if } |i - j| \geq 2$$

then:

$$\int_{\Gamma} f dl = \sum_{i=1}^k \int_{\Gamma_i} f dl$$

Example 6.



2.2 Line integral of a vector field

Let Γ a curve parametrized by $[a, b] \xrightarrow{\gamma} \gamma(t)$ and F a continuous vector field

the line integral of F over Γ (or the circulation of F to Γ) is :

$$\int_{\Gamma} F \cdot dr = \int_a^b [F(\gamma(t)) \cdot \gamma'(t)] dt$$

which is independant of the choice of parametrization

in \mathbb{R}^2 if $\gamma(t) = (x(t), y(t))$ and $F = M\vec{i} + N\vec{j}$

$$\int_{\Gamma} F \cdot dr = \int_a^b [x'(t)M(x(t), y(t)) + y'(t)N(x(t), y(t))] dt$$

In \mathbb{R}^3 if $\gamma(t) = (x(t), y(t), z(t))$ and $F = P\vec{i} + Q\vec{j} + R\vec{k}$

then:

$$\int_{\Gamma} F \cdot dr = \int_a^b [x'(t)P(x(t), y(t), z(t)) + y'(t)Q(x(t), y(t), z(t)) + z'(t)R(x(t), y(t), z(t))] dt$$

Remark:

We denoted by $\int_{\Gamma} M dx + N dy = \int_{\Gamma} F \cdot dr$ int \mathbb{R}^2 and by

$$\int_{\Gamma} P dx + Q dy + R dz = \int_{\Gamma} F \cdot dr \text{ in } \mathbb{R}^3$$

this is the differential representation of the integral of F along to Γ

Remark: If $F = \text{grad}(f)$ is conservative vector field then:

$$\begin{cases} df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = F \cdot dr & | dr = dxi + dyj \\ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = F \cdot dr & | dr = dxi + dyj + dzk \end{cases}$$

Example 7. ...

Example: $F: (\alpha, \beta) \rightarrow \mathbb{R}^3$

$$F(\alpha, \beta) = \left(\frac{-1}{2}x\vec{i} - \frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k} \right)$$

$$\gamma[0, 2\pi] \rightarrow \mathbb{R}^3$$

$$t \rightarrow \gamma(t) = (\cos t, \sin t, t)$$

$$\gamma'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$$

$$F(\gamma(t)) = -\frac{1}{2}\cos(t)\vec{i} - \frac{1}{2}\sin(t)\vec{j} + \frac{1}{4}\vec{k}$$

$$\Rightarrow F(\gamma(t)) \cdot \gamma'(t) = \left(-\frac{1}{2}\cos(t) \right) (-\sin t) + \left(\cos(t) \right) \left(\frac{-1}{2}\sin(t) \right) + \frac{1}{4}$$

$$= \frac{1}{4}$$

$$\int_{\Gamma} F \cdot dr = \int_0^{2\pi} \frac{1}{4} dt = \boxed{\frac{\pi}{2}}$$

Example 8. ...

Example 02:

$$\gamma(t) = 3 \cos(t) \vec{i} + 3 \sin(t) \vec{j}$$

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$$

$$\int_{\Gamma} (y^3 dx + (x^3 + 2xy^2) dy) = \int_{\Gamma} M dx + N dy \quad \left| \begin{array}{l} M(x, y) = y^3 \\ N(x, y) = x^3 + 2xy^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} x(t) = 3 \cos t \Rightarrow x'(t) = -3 \sin t \\ y(t) = 3 \sin t \Rightarrow y'(t) = 3 \cos t \end{array} \right.$$

$$\begin{aligned} & \rightarrow \int_0^{2\pi} \left[x'(t) M(x(t), y(t)) + y'(t) N(x(t), y(t)) \right] dt \\ &= \int_0^{2\pi} \left[-3 \sin t (3 \sin t)^3 + 3 \cos t ((3 \cos t)^3 + 2(3 \cos t)(3 \sin t)^2) \right] dt \\ &= \int_0^{2\pi} \left[-81 \sin^4 t + 81 \cos^4 t + 162 \cos^3 t \sin^2 t \right] dt \\ &= 81 \int_0^{2\pi} (\cos^4 t - \sin^4 t + 2 \cos^2 t \sin^2 t) dt \\ &= 81 \int_0^{2\pi} \left[\cos(4t) + \frac{1}{2} (\sin(4t))^2 \right] dt \\ &= \frac{81}{4} \int_0^{2\pi} [4 \cos(4t) + 1 - \cos(4t)] dt = \frac{81}{2} \pi \end{aligned}$$

the parametrization
of a curve
each point $(x, y(t))$

Theorem 3. Let $F = \text{grad}(f)$ be a conservative vector field over $D : \Gamma \subset D$, parametrized by $([a, b], \gamma)$, then:

$$\int_{\Gamma} F \cdot dr = f(\gamma(b)) - f(\gamma(a))$$

in particular, if Γ is a closed path ($\gamma(a) = \gamma(b)$) then : $\oint F \cdot dr = 0$

Proof

$$F = \text{grad}(f) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}; \quad \left\{ \begin{array}{l} \gamma(t) = x(t) \vec{i} + y(t) \vec{j} \\ \gamma'(t) = x'(t) \vec{i} + y'(t) \vec{j} \end{array} \right.$$

$$F(\gamma(t)) \cdot \gamma'(t) = x'(t) \frac{\partial f}{\partial x}(x(t), y(t)) + y'(t) \frac{\partial f}{\partial y}(x(t), y(t)) = (f \circ \gamma)' t \quad | \quad (f \circ \gamma)(t) = f(x(t), y(t))$$

$$\begin{aligned}\int_{\Gamma} F \cdot dr &= \int_{\Gamma} \text{grad}(f) \cdot dr = \int_a^b (f \circ \gamma)'(t) dt = [f(\gamma(t))]_a^b \\ &= f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

Theorem 4. (Green's theorem)

Let D be a simply connected and connected region of \mathbb{R}^2 with piece-wise C^1 boundary simple curve $\Gamma = \partial D$

and $F = M\vec{i} + N\vec{j}$ be a C^1 vector field over D , Then :

$$\oint_{\Gamma} F \cdot dr = \oint_{\Gamma} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) (x, y) dxdy$$

Example 9. ...

Example:

$$\begin{aligned}M(x, y) &= y^3 & N(x, y) &= x^3 + 2x \\ r(t) &= 3 \cos t & & \\ y(t) &= 3 \sin t & &\end{aligned}$$

$$\begin{aligned}\oint_{\Gamma} M dx + N dy &= \iint_D (3x^2 + 2y^2 - 3y^2) dxdy \\ &= \iint_D (3x^2 - y^2) dxdy \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \\ &= \int_0^{2\pi} \int_0^3 r (3r^2 \cos^2 \theta - r^2 \sin^2 \theta) dr d\theta \\ &\quad \begin{matrix} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{matrix} \\ &= \left(\int_0^3 r^3 dr \right) \left(\int_0^{2\pi} (3 \cos^2 \theta - 1 + \cos^2 \theta) d\theta \right) \\ &= \left[\frac{r^4}{4} \right]_0^3 \int_0^{2\pi} (4 \cos^2 \theta - 1) d\theta \\ &= \frac{81}{4} \int_0^{2\pi} (1 + \cos 2\theta) d\theta \\ &= \frac{81}{4} \cdot 2\pi = \frac{81}{2} \pi\end{aligned}$$

Theorem 5. Corollary Line Integral area

if D is a plane region bounded by a piece-wise simple closed curve $\Gamma = \partial D$ oriented counterclockwise, then $A(D) = \frac{1}{2} \oint_{\partial D} (xdy - ydx)$

Proof

$$M(x, y) = -\frac{1}{2}y; \quad N(x, y) = \frac{1}{2}x$$

by Green's theorem;

$$\oint_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) (x, y) dxdy$$

Example 10. .

Example: $D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, $\Gamma = \partial D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$

$\delta: [0, 2\pi] \rightarrow \mathbb{R}^2$, $t \mapsto (x(t), y(t)) = (a \cos t, b \sin t)$

$A(D) = \frac{1}{2} \int_0^{2\pi} [(a \cos t)(b \cos t) - (b \sin t)(-a \sin t)] dt = \frac{ab}{2} \int_0^{2\pi} (a^2 \cos^2 t + b^2 \sin^2 t) dt = \boxed{\pi ab}$

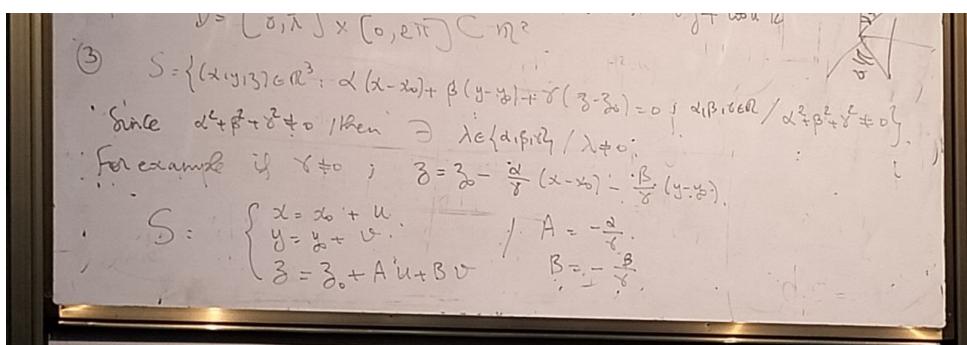
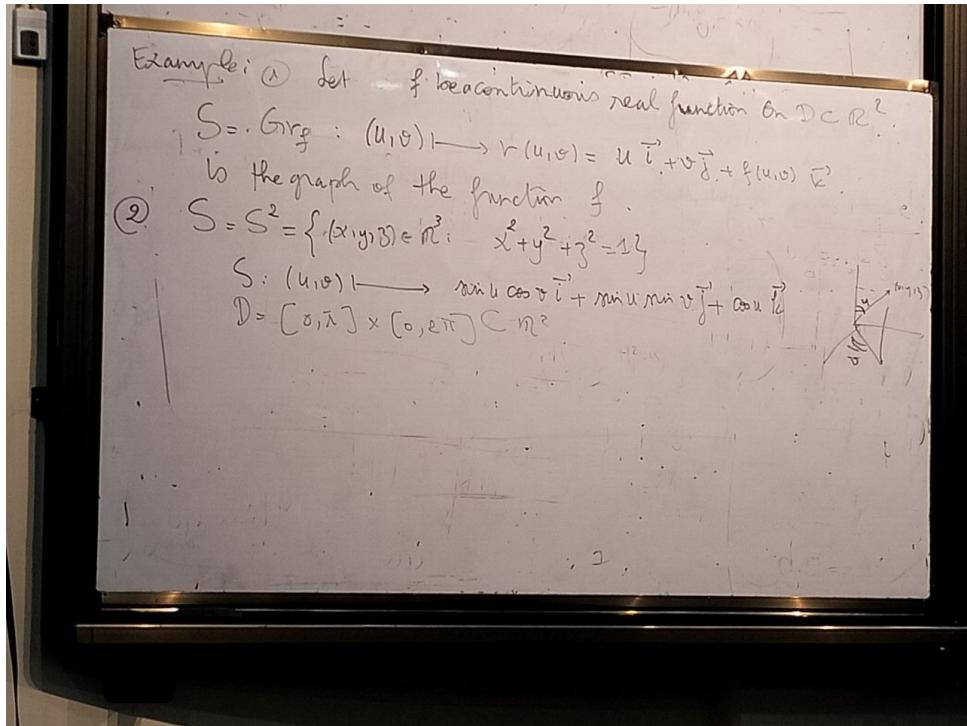
3 Surface integral

3.1 Parametric surface

Definition 5. Let x, y, z be three functions of u and v , that are continuous on $D \subset \mathbb{R}^2$. The set S of points (x, y, z) given by : $r(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ be a parametrized surface. The following equations:

$$S: \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \text{ are called the parametric equation of } S$$

Example 11. .



Definition 6. Let (D, r) be a parametrization of the surface S .

We say that S is smooth (or C^1) if x, y, z are C^1 or the vector field V is C^1 on D

we say that S is piece-wise smooth. if $\exists k \in \mathbb{N}^*; S = S_1 \cup S_2 \cup \dots \cup S_k \mid S_i \cup S_j$ is at most a curve and S_i is smooth

Let S be smooth surface. The tangent plane of S at a point $(x_0, y_0, z_0) \in S$ is the affine space

$$T_{(x_0, y_0, z_0)}S : \begin{cases} x = x_0 + (u - u_0) \frac{\partial x}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial x}{\partial v}(u_0, v_0) \\ y = y_0 + (u - u_0) \frac{\partial y}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial y}{\partial v}(u_0, v_0) \\ z = z_0 + (u - u_0) \frac{\partial z}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial z}{\partial v}(u_0, v_0) \end{cases}$$

$$: r : (u, v) \mapsto r(u_0, v_0) + (u - u_0) \frac{\partial r}{\partial u}(u_0, v_0) + (v - v_0) \frac{\partial r}{\partial v}(u_0, v_0)$$

$$r(u_0, v_0) = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k} ; \quad \frac{\partial r}{\partial u} = r_u ; \quad \frac{\partial r}{\partial v} = r_v$$

the normal vector field of S ; denoted by N is :

$$N = r_u \times r_v = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$$

$$N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\text{So;} \quad T_{(x_0, y_0, z_0)} S \perp N(x_0, y_0, z_0), \quad \|N(x_0, y_0, z_0)\| = \left\| \frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0) \right\|$$

Example 12. ...

Example 12. $S = \{G_{uv}\}$

$r(u, v) = u \vec{i} + v \vec{j} + f(u, v) \vec{k}$

$r_u(u, v) = \frac{\partial r}{\partial u}(u, v) = \vec{i} + \frac{\partial f}{\partial u}(u, v) \vec{k}$, $r_v(u, v) = \frac{\partial r}{\partial v}(u, v) = \vec{j} + \frac{\partial f}{\partial v}(u, v) \vec{k}$

$N(\vec{i}, \vec{j}, \vec{k}) = N(u, v) = r_u(u, v) \times r_v(u, v) = \left(\vec{i} + \frac{\partial f}{\partial u}(u, v) \vec{k} \right) \times \left(\vec{j} + \frac{\partial f}{\partial v}(u, v) \vec{k} \right)$

$= \vec{i} \times \vec{j} + \frac{\partial f}{\partial u}(u, v) \vec{i} \times \vec{k} + \frac{\partial f}{\partial v}(u, v) \vec{j} \times \vec{k}$

$= \vec{k} - \frac{\partial f}{\partial v}(u, v) \vec{j} - \frac{\partial f}{\partial u}(u, v) \vec{i} = -\frac{\partial f}{\partial u}(u, v) \vec{i} - \frac{\partial f}{\partial v}(u, v) \vec{j} + \vec{k}$

$\|N(u, v)\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v) \right)^2 + \left(\frac{\partial f}{\partial v}(u, v) \right)^2}$.

② $S = S^2$

$N(u_0, v_0) = V(u_0, v_0) = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$

Definition 7. the area of a smooth surface

Let $S : r : (u, v) \mapsto r(u, v)$ over $D \subset R$ be a smooth parametrized surface the area of S b

$$\begin{aligned} A(S) &= \iint_D \|r_u(u, v) \times r_v(u, v)\| = \iint_{\sigma} d\sigma \\ &= \|N(u, v)\| du dv \end{aligned}$$

Example 13. ...

Example: $S = \iint_D ||N(u,v)|| dudv$

$$S = \{(x,y,z) \in \mathbb{R}^3 : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2\}, R > 0$$

$$r(u,v) = \sqrt{x_0^2 + R^2 \sin u \cos v} \vec{i} + \sqrt{y_0^2 + R^2 \sin u \sin v} \vec{j} + \sqrt{z_0^2 + R^2 \cos u} \vec{k}$$

$$\mathcal{D} = [0, \pi] \times [0, 2\pi]$$

$$N(x,y,z) = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ R \sin u \cos v & R \sin u \sin v & -R \cos u \\ -R \sin u \sin v & R \sin u \cos v & 0 \end{pmatrix} = R^2 \sin u \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \cos u \cos v & 0 \end{pmatrix}$$

$$dS = (R^2 \sin u) dudv$$

$$A(S) = \iint_{[0,\pi] \times [0,2\pi]} dS = R^2 \left(\int_0^\pi \sin u du \right) \left(\int_0^{2\pi} dv \right) = R^2 \left[-\cos u \right]_0^\pi \left[v \right]_0^{2\pi} = 4\pi R^2$$

Remark:

$$S = Gr_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \subset D \text{ and } z = f(x, y)\}$$

$$S : r(u, v) = u\vec{i} + v\vec{j} + f(u, v)\vec{k}$$

$$d\sigma = \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v) \right)^2 + \left(\frac{\partial f}{\partial v}(u, v) \right)^2}$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial u}(u, v) \right)^2 + \left(\frac{\partial f}{\partial v}(u, v) \right)^2} dudv$$

3.2 Surface integral

Definition 8. Let $S : r(u, v) \mapsto x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ be a parametrized smooth surface and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ a function continuous over S , denoted by $\int_S f d\sigma$ is the real number

$$\iint f d\sigma = \iint_D f(r(u, v)) ||r_u(u, v) \times r_v(u, v)|| dudv$$

$$\iint f d\sigma = \iint_D f(x(u, v) + y(u, v) + z(u, v)) \left| \left| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right| \right| dudv$$

Remark:

1) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : z = \varphi(x, y)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(x, y, \varphi(x, y)) \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x, y)\right)^2 + \left(\frac{\partial \varphi}{\partial y}(x, y)\right)^2} dx dy \end{aligned}$$

2) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : y = \varphi(x, z)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(x, \varphi(x, z), z) \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}(x, z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(x, z)\right)^2} dx dz \end{aligned}$$

3) If

$$\begin{aligned} S &= \{(x, y, z) \in \mathbb{R}^3 : x = \varphi(y, z)\} \rightarrow \iint_S f d\sigma \\ &= \iint_D f(\varphi(y, z), y, z) \sqrt{1 + \left(\frac{\partial \varphi}{\partial y}(y, z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(y, z)\right)^2} dy dz \end{aligned}$$

4)

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = c, (x, y) \in D\}; A(S) = A(D) = \iint_D dx dy$$

Example 14. 1)...

Examples: ① $\iint_S (y^2 + 2yz) d\sigma$, where S is the first octant of the plane $x+y+z=2$.

$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 2\}$

$$\begin{aligned} \iint_S f d\sigma &= \iint_D f(x, y, 2-x-y) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy \\ &= \iint_D (y^2 + 2yz) \sqrt{\frac{1}{4}} dy dx = \frac{3}{2} \int_0^2 \left[\int_{-\frac{y}{2}}^{\frac{2-y}{2}} (y^2 + 2yz) dz \right] dy \\ &= \frac{3}{2} \int_0^2 \left(1 - \frac{y}{2} \right) \left(y^2 + 2y(1 - \frac{y}{2}) \right) dy = \frac{3}{2} \int_0^2 \left(1 - \frac{y}{2} \right) \left(y + \frac{y^2}{2} \right) dy \\ &= \frac{3}{2} \int_0^2 y \left(1 - \frac{y}{2} \right) \left(1 + \frac{y}{2} \right) dy = \frac{3}{2} \int_0^2 y \left(1 - \frac{y^2}{4} \right) dy = \frac{3}{2} \int_0^2 \left(\frac{y^2}{2} - \frac{y^4}{4} \right) dy \\ &= \frac{3}{2} \left(\frac{4}{2} - \frac{16}{30} \right) = \boxed{\frac{3}{2}} \end{aligned}$$

2)..

② A cone-shaped surface lamina is given by $z = 4 - 2\sqrt{x^2 + y^2}$. At each point of S , the mass density is proportional to the distance between this point and the z -axis.

The mass of S , i.e., $m = \iint_S \rho d\sigma = \iint_D k\sqrt{x^2 + y^2} \sqrt{1 + \left(\frac{-2x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{-2y}{\sqrt{x^2 + y^2}}\right)^2} dxdy$

$$\begin{aligned} &\text{where } 0 \leq 3 \leq 4 \\ &0 \leq 4 - 2\sqrt{x^2 + y^2} \leq 4 \\ &-4 \leq -2\sqrt{x^2 + y^2} \leq 0 \\ &0 \leq \sqrt{x^2 + y^2} \leq 2 \end{aligned}$$

$$\begin{aligned} &= k \iint_D \sqrt{x^2 + y^2} \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} dxdy \\ &= k \sqrt{5} \iint_D \sqrt{x^2 + y^2} dxdy = k \sqrt{5} \int_0^{2\pi} \int_0^2 r^2 dr = 2\pi k \sqrt{5} \left[\frac{r^3}{3} \right]_0^2 = \boxed{\frac{16\pi k \sqrt{5}}{3}} \end{aligned}$$

3.3 Flux of vector field over surface

Definition 9. Unit normal vector fields - Oriented surfaces

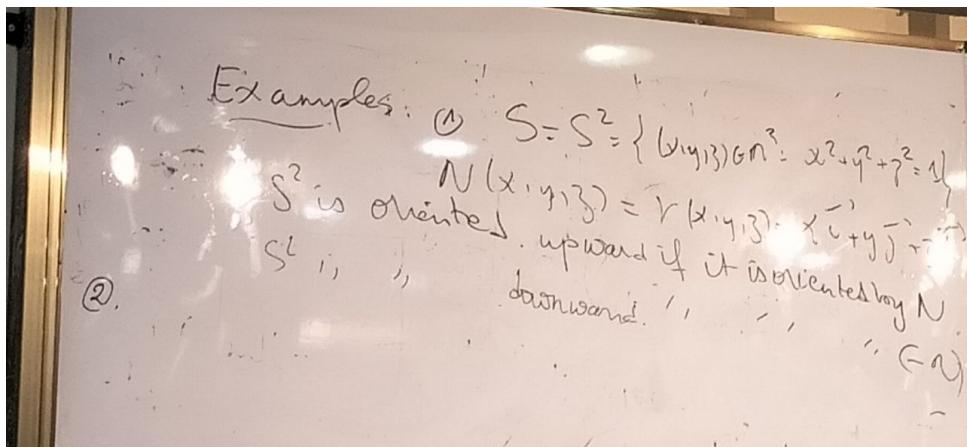
Let S be a smooth parametrized surface

S is regular if $\|r_u \times r_\theta\|(u, v) \neq 0, \forall (u, v) \in D \mid D$ connected

$$\text{So, } S \text{ can be oriented by } N(x, y, z) = \frac{1}{\|r_u \times r_\theta\|(u, v)} ((r_u \times r_\theta)(u, v)) \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

$$\text{or, } S \text{ can be also oriented by } (-N)(x, y, z) = \frac{1}{\|r_\theta \times r_u\|(v, u)} ((r_\theta \times r_u)(v, u)) \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

Example 15. 1)



2)

$$2. S = \{(x, y, z) \in \mathbb{R}^3 : z = g(x, y), (x, y) \in D\} \text{ where } D \subset \mathbb{R}^2 \text{ is an open set. } N(x, y, z) = N(x, y, g(x, y)) = \frac{1}{\sqrt{1 + (g_x)^2 + (g_y)^2}} \left[-\frac{\partial g}{\partial x}(x, y) \hat{i} - \frac{\partial g}{\partial y}(x, y) \hat{j} + \hat{k} \right]$$

Definition 10. Flux of vector field over regular parametrized

Let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuous vector field over an open set $\Delta \subset S$

If S is smooth

$$\begin{aligned} \text{Flux}_S(F) &= \iint_D (F \cdot N)(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v} \right\| dudv \\ &= \iint_S (F \cdot N) d\sigma \end{aligned}$$

Remark $F = P\vec{i} + Q\vec{j} + R\vec{k}$ $S = Gr_f = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ ; } z = g(x, y)\}$

$$Flux_S(f) = \begin{cases} \iint_D \left[R - \frac{\partial g}{\partial x}(x, y)P - \frac{\partial g}{\partial y}(x, y)Q \right] (x, y, g(x, y)) dx dy \\ \quad \text{if } S \text{ is oriented upward} \\ \iint_D \left[\frac{\partial g}{\partial x}(x, y)P((x, y, g(x, y))) + \frac{\partial g}{\partial y}(x, y)Q(x, y, g(x, y)) - R(x, y, g(x, y)) \right] dx dy \\ \quad \text{if } S \text{ is oriented downward} \end{cases}$$

Example 16. 1)

Example: $S = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0 \text{ and } z = 4 - x^2 - y^2\}$
 $= \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } z = g(x, y) = 4 - x^2 - y^2\}$
 $(x, y, z) \mapsto, r(x, y) = x\vec{i} + y\vec{j} + z\vec{k}$
 $\text{S oriented upward.}$
 $P(x, y, z) = x, Q(x, y, z) = y, R(x, y, z) = z$
 $\frac{\partial g}{\partial x}(x, y) = 2x, \frac{\partial g}{\partial y}(x, y) = 2y$
 $\text{Flux}_S(F) = \iint_D \left[(4 - x^2 - y^2) - (-2x)x - (-2y)y \right] dx dy, D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$
 $= \iint_D (4 + x^2 + y^2) dx dy = \int_0^{2\pi} \int_0^2 r(4 + r^2) dr d\theta$
 $= 2\pi [2r^2 + \frac{1}{4}r^4]_0^2 = 2\pi (8 + \frac{1}{4} \cdot 16) = 24\pi$

Def: Flux of vector field over regular parameterized surface.
If S is smooth: let $F = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuous vector field over an open set $(D) \subset S$.

$\text{Flux}_S(F) = \iint_S (F \cdot N) d\sigma$

Remark

$$\iint_S (F \cdot N) d\sigma = \iint_D F(x(u, v), y(u, v), z(u, v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \times \frac{\partial r}{\partial v}(u, v) \right) du dv$$

$$\iint_D \begin{vmatrix} P(x(u,v), y(u,v), z(u,v)) & Q(x(u,v), y(u,v), z(u,v)) & R(x(u,v), y(u,v), z(u,v)) \\ \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) & \frac{\partial z}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) & \frac{\partial z}{\partial v}(u,v) \end{vmatrix} dudv$$

Example 17. 1 ..

There are two methods to compute the flux of vector field over a surface S .

1) if S is boundary of a solid domain $\Omega \subset \mathbb{R}^3$, ($i - eS = \partial D$),

(S is closed. without boundary $\partial S = \phi$) we can use the divergence theorem

2) If S is not closed, $C = \partial S$ is piece-wise curve we can use Stokes's theorem

Theorem 6. Divergence theorem (Ostrogradsky's theorem)

Let Ω be a closed solid bounded region in \mathbb{R}^3 by

closed smooth piece-wise regular surface $S = \partial\Omega$ oriented outward from Ω

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is of class C^1 over Ω then;

$$\text{Flux}_S(F) = \iiint \text{div}(F) dv = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) (x, y, z) dx dy dz$$

Theorem 7. Stockes's theorem Let S be oriented by N ; $C = \partial S$ is a piece-wise smooth simple closed curve with positive orientation. If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is C^1 vector field over a open region $\Delta \subset S$, then:

$$\oint_{C=\partial S} F \cdot dr = \iint_S (\text{curl}(F) \cdot N) d\sigma = \text{Flux}_S(\text{curl}(F))$$

Example 18. 1)

Example: $S = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, 2x + 2y + z \leq 6\}$

$$\text{Flux } F = \iiint_S \text{div}(F) dV = \iiint_S (1 + 2y + 1) dxdydz$$

$$= \int_0^3 \left[\int_0^{3-x} \left[\int_0^{6-2x-2y} (1 + 2y + 1) dy \right] dz \right] dx$$

$$= 2 \int_0^3 \left[\int_0^{3-x} (1+y)(6-2x-2y) dy \right] dx$$

$$= 4 \int_0^3 \left[\int_0^{3-x} (3-x + (2-x)y - y^2) dy \right] dx$$

$$= 4 \int_0^3 \left[(3-x)y + (2-x)\frac{y^2}{2} - \frac{1}{3}y^3 \right]_0^{3-x} dx$$

$$= 4 \int_0^3 (3-x)^2 \left[1 + \frac{1}{2}(2-x) - \frac{1}{3}(3-x) \right] dx$$

$$= 4 \int_0^3 (3-x)^2 \left(\frac{1}{2} + \frac{1}{6}(3-x) \right) dx$$

$$= \frac{4}{6} \int_0^3 [3(3-x)^2 + (3-x)^3] dx = \frac{2}{3} \int_0^3 (3u^2 + u^3) du$$

$$= \frac{2}{3} \left[u^3 + \frac{1}{4}u^4 \right]_0^3 = 2 \left[9 + \frac{27}{4} \right] = 18 + \frac{27}{2} = \frac{63}{2}$$

$S = S_1 \cup S_2 \cup S_3 \cup S_4$

$S_1: z=0; D_1 = \{x \geq 0, y \geq 0, x+2y \leq 3\} \quad N_1 = -\vec{i}$
 $S_2: y=0; D_2 = \{x \geq 0, 3x \geq 0, 2x+2y \leq 6\} \quad N_2 = -\vec{j}$
 $S_3: x=0; D_3 = \{y \geq 0, 3y \geq 0, 2y+3 \leq 6\} \quad N_3 = -\vec{k}$
 $S_4: 2x+2y+z=6; D_4 = \{6 \geq x \geq 0, 6 \geq y \geq 0, x+y \leq 3\} \rightarrow N_4 = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$

Example 19. stokes example

Example: the Stokes theorem $\oint_{\partial S} F \cdot dr = \iint_S [\text{curl}(F) \cdot N] dS$

$S = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0, 2x + 2y + z = 6\}$

$F : (x, y, z) \mapsto -y^2 \vec{i} + 3z \vec{j} + x \vec{k}$

$\text{curl}(F)(x, y, z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & 3z & x \end{vmatrix} = -\vec{i} - \vec{j} + 2y \vec{k}$

$N(x, y, z) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$

$[\text{curl}(F) \cdot N](x, y, z) = -\frac{2}{3} - \frac{2}{3} + \frac{2y}{3} = \frac{2}{3}(y-2)$

$dS = \sqrt{1 + 4} dx dy, D = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x + y \leq 3\}$

$\iint_S [\text{curl}(F) \cdot N] dS = \int_0^3 \left[\int_0^{3-x} \frac{2}{3}(y-2) dy \right] dx$

$= 2 \int_0^3 \left[\frac{y^2}{2} - 2y \right]_0^{3-x} dx = \int_0^3 [(3-x)^2 - 4(3-x)] dx$

$= \int_0^3 (u^2 - 4u) du = \left[\frac{u^3}{3} - 2u^2 \right]_0^3 = (3-18)-(-9)$

