N.S.A.S.T **2025/2026**
Second year P.C - Analysis 3

SERIE N° 1 : DOUBLE AND TRIPLE INTEGRALS

Exercice 1 :
Using Riemann sums, calculate $\iint_D (x+2y)\,dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : \quad x \geq 0, \quad y \geq 0, \quad x+y \leq 1\}$.

Exercice 2 :
1/ Evaluate the following integrals :

$$i)\quad I_1 = \iint_D \frac{y}{\sqrt{1+xy+y^2}}dxdy, \qquad D = \left\{(x,y) \in \mathbb{R}^2 : \quad -2 \leq x \leq 2, \quad 3 \leq y \leq 7\right\},$$

$$ii)\quad I_2 = \iint_D (x+3y)\,dxdy, \qquad D = \left\{(x,y) \in \mathbb{R}^2 : \quad x \geq 0, \quad x^2+y^2 \leq 1\right\}.$$

2/ Let **I** be the integral :

$$\mathbf{I} = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx.$$

i) Show that for any real number $x > -1$, we have : $\ln(1+x) = \int_0^1 \frac{x}{1+xy}dy$, deduce that

$$\mathbf{I} = \iint_{[0,1]^2} \frac{x}{(1+xy)(1+x^2)}dxdy.$$

ii) By reversing the roles of x and y, show that

$$2\mathbf{I} = \iint_{[0,1]^2} \frac{x+y}{(1+y^2)(1+x^2)}dxdy.$$

iii) Deduce the value of **I**.

Exercice 3 :
i) Find the volume of the solid in the first octant (i.e. $(\mathbb{R}^+)^3$) bounded by the graphs of the equations :

$$\mathbf{1/}\qquad 1 = x^2+y^2, \quad x^2+z^2 = 1,$$
$$\mathbf{2/}\qquad z = x+y, \quad x^2+y^2 = 4.$$

ii) The same question for the solid Ω, in each of the following cases :

$$\mathbf{1/}\qquad z = 1+x+x^2, \quad x+y = 1, \quad x = 0, \quad y = 0,$$
$$\mathbf{2/}\qquad 4z = 16-4x^2-y^2, \quad x^2+y^2 = 2x.$$

Exercice 4 :
Find the area of the surface given by the equation $z = f(x,y)$ over D, in the following :

1/ $\quad f : (x,y) \longmapsto 8+2x-3y, \qquad D = \left\{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leq 9\right\}$

2/ $\quad f : (x,y) \longmapsto -\ln|\sin x|, \qquad D = \left\{(x,y) \in \mathbb{R}^2 : \quad 0 \leq x \leq \frac{\pi}{4}, \quad 0 \leq y \leq \tan x\right\}.$

Exercice 4 :
Use a change of variables to find the volume of the solid region lying below the surface $z = f(x,y)$ and above the plane region D.

1/ $f \ : \ (x,y) \longmapsto (3x+2y)^2 \sqrt{2y-x}$, D is the parallelogram with vertices $(0,0), (-2,3), (2,5), (4,2)$

2/ $f \ : \ (x,y) \longmapsto \dfrac{xy}{1+x^2y^2}$, D is bounded by the graphs of $xy = 1$, $xy = 4$, $x = 1$, $x = 4$.

Exercice 5 :
Find the mass, the center of mass and the moments of inertia I_x, I_y of the lamina bounded by the graphs of the equations with given density ρ :

1/ $y = \dfrac{4}{x}, \quad y = 0, \quad x = 1, \quad x = 4, \qquad \rho(x,y) = kx^2,$

2/ $y = 4 - x^2, \quad y = 0, \qquad \rho(x,y) = ky,$

3/ $y = \sqrt{a^2 - x^2}, \quad 0 \le y \le x, \qquad \rho(x,y) = k.$

Exercice 6 :
i) Integrate the given function f over the indicate region D.

1/ $f(x,y,z) = 8xyz$, D is bounded by $y = x^2$, $y + z = 9$, $z = 0$,

2/ $f(x,y,z) = yz + x^2$, D is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + 2z \le 1$.

3/ $f(x,y,z) = 1 - z^2$, D is the tetrahedron with vertices $(0,0,0), (1,0,0), (0,2,0)$ and $(0,0,3)$.

ii) Using an appropriate change of variables, calculate $\iiint_D f(x,y,z)\, dxdydz$ in the following cases :

1/ $f(x,y,z) = x^2 + y^2$, $D = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x^2 + y^2 \le (z-1)^2, \quad 0 \le z \le 1\right\}$.

2/ $f(x,y,z) = z\cos\left(x^2 + y^2\right)$, $D = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 \le 1, \quad z \ge 0\right\}$.

Exercice 7 :
Use a triple integral to find the volume of the solid Ω bounded by the graphs of the equations :

1/ $z = 9 - x^3, \quad y = 2 - x^2, \quad y = 0, \quad z = 0, \quad x \ge 0,$

2/ $z = 2 - y, \quad z = 4 - y^2, \quad x = 0, \quad x = 3, \quad y = 0.$

Exercice 8 :
Find the center of mass $(\overline{x}, \overline{y}, \overline{z})$, the moments of inertia about the y- and the z-axes, of Ω with the density ρ, where

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x^2 + y^2 \le z \le \sqrt{4 - x^2 - y^2}\right\}, \qquad \rho(x,y,z) = kz.$$

Additional exercises

Exercice 9 :

i) Following the indicated change of variables, calculate $\iint_D xydxdy$, where

$$D=\left\{(x,y)\in\mathbb{R}^2:\quad x\geq 0,\ y\geq 0,\quad x^{\frac{2}{3}}+y^{\frac{2}{3}}\leq 1\right\}\quad \text{and}\quad x=r\cos^3\theta,\quad y=r\cos^3\theta.$$

ii) The same question for $\iint_D \left(x^2+y^2\right)\left(y^2-x^2\right)^{xy}dxdy$, where

$$D=\left\{(x,y)\in\mathbb{R}^2:\quad 0<x<y,\ 0<a<xy<b,\quad y^2-x^2<1\right\}\quad \text{and}\quad u=xy,\quad v=y^2-x^2.$$

Exercice 10 :

i) Integrate the given function f over the indicate region D.

1/ $f(x,y,z) \;=\; 2x-y+z,$ D is bounded by $z=x^2+y^2,\quad z=0,\quad x=0,\quad x=1,\quad y=-2,\quad y=2.$

2/ $f(x,y,z) \;=\; \dfrac{z}{\sqrt{x^2+y^2}},$ $D=\left\{(x,y,z)\in\mathbb{R}^3:\quad x^2+y^2+z^2\leq 1,\ y^2-2xz\leq 0,\ 4z^2\leq x^2+y^2,\ z\geq 0\right\}$

ii) Let $\Omega=\{(x,y,z)\in\mathbb{R}^3:\quad x^2+y^2+z^2\leq 1\}$ and $a>1$, calculate

$$\iiint_\Omega \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}dxdydz.$$