N.H.S.A.S.T **2024/2025**
Second year P.C - Analysis 4

SERIE N° 4 : PARTIAL DIFFERENTIAL EQUATIONS

Exercice 1 : **First-order linear PDEs**.

1/ Let $a, b \in \mathbb{R}$ such that $a^2+b^2 \neq 0$ and $u \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, we consider the homogeneous equation

$$\text{(HE)} \qquad a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0.$$

i) Let the following change of variables

$$s = bt - ax \qquad \text{and} \qquad y = at + bx.$$

Show that any solution to equation (HE) is of the form

$$u : (t, x) \longmapsto u(t, x) = \varphi(bt - ax),$$

where $\varphi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$.

ii) Show that, under the initial condition : $u(0, x) = h(x)$ for any x, with $a \neq 0$, the equation (HE) admits a unique solution.

2/ Under the same assumptions and notations above, let $f \in \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ and we consider the equation

$$\text{(E)} \qquad a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = f.$$

We note $g : (s, y) \longmapsto f(t, x)$ and G is a function such that $g = \frac{\partial G}{\partial y}$.

i) Proof that any solution to equation (E) is of the form

$$u : (t, x) \longmapsto u(t, x) = \frac{1}{a^2+b^2} G(bt - ax, at + bx) + \varphi(bt - ax).$$

ii) Show that, under the initial condition : $u(0, x) = \psi(x)$ for any x, with $a \neq 0$, the equation(E) admits a unique solution.

3/ Applications :

i) $f = 0$, $a = 4$, $b = -3$ and $h : x \longmapsto x^3$.

ii) $f : (t, x) \longmapsto \rho(4t + x)$, $a = 1$, $b = 1$ and $\psi : x \longmapsto x^2$.

Exercice 2 :

Using the method of separation of variables, solve the following boundary value problem

$$\begin{cases} 4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} & \text{on} \quad \mathbb{R}_+ \times [0, \pi] \\ u(t, 0) = u(t, \pi) = 0 & \text{for any} \quad t > 0, \\ u(0, x) = \pi - x & \text{for any} \quad 0 < x < \pi, \end{cases}$$

Exercice 3 :

Solve the the following problem on $\mathbb{R}_+ \times \mathbb{R}$ using the Fourier transform

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \\ u(0, x) = e^{-x^2} \end{cases}$$