N.H.S.A.S.T **2024/2025**
Second year P.C - Analysis 4

SERIE N° 3 : FOURIER TRANSFORM

Exercice 1 :
1/ Proof that : if $f \in \mathcal{L}^1(\mathbb{R})$ and $\int_{\mathbb{R}} |tf(t)|\,dt$ conveges then $\mathcal{F}f$ is derivable, furthermore

$$(\mathcal{F}f)'(x) = -i\mathcal{F}(t \longmapsto tf(t))(x), \quad \text{for all } x \in \mathbb{R}.$$

2/ Let $\Gamma : t \longmapsto e^{-\frac{t^2}{2}}$, compute $\mathcal{F}(\Gamma)$.
3/ Let $s > 0$, we consider the function

$$\Gamma_s : t \longmapsto \frac{1}{\sqrt{s}} e^{-\frac{t^2}{2s}}.$$

i) Show that

$$\mathcal{F}(\Gamma_s) = \frac{1}{\sqrt{s}} \Gamma_{\frac{1}{s}}.$$

ii) Deduce $\mathcal{F}^2(\Gamma_s) := \mathcal{F}(\mathcal{F}(\Gamma_s))$.
iii) Let $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$, using the formula $\int_{\mathbb{R}} f\mathcal{F}(g) = \int_{\mathbb{R}} g\mathcal{F}(f)$ and the time shift property to establish a relationship between $\mathcal{F}^2(f * \Gamma_s)$ and $f * \Gamma_s$.
4/ For any $n \in \mathbb{N}^*$, we define $\gamma_n := \Gamma_{\frac{1}{2n}}$.
i) Show that the sequence $(\gamma_n)_{n\in\mathbb{N}^*}$ is an approximation of unity (that is : $\forall n \in \mathbb{N}^*, \int_{\mathbb{R}} \gamma_n = 1$ and $\forall \varepsilon > 0, \lim_{n\to\infty} \int_{\mathbb{R}-]-\varepsilon,\varepsilon[} \gamma_n = 0$).
ii) Using the fact that : for any $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$, the sequence $(f * \gamma_n)_{n\in\mathbb{N}^*}$ the sequence converges to f in $\mathcal{L}^1(\mathbb{R}, \mathbb{R})$, deduce (using the Riesz-Fischer theorem) the Fourier inversion formula for a continous function on $\mathcal{L}^1(\mathbb{R}, \mathbb{R})$ whose Fourier transform is also in $\mathcal{L}^1(\mathbb{R}, \mathbb{R})$.

Exercice 2 :
Let Π the function defined by

$$\Pi(t) = \begin{cases} 1 & \text{if} \quad 2|t| \leq 1 \\ 0 & \text{if} \quad 2|t| > 1 \end{cases}$$

1/ Compute the Fourier transform of Π.
2/ Using de properties of Fourier transform, calculate the Fourier transform of the following functions

$$i) \quad t \longmapsto \Pi\left(\frac{t-1}{2}\right), \qquad ii) \quad t \longmapsto t\Pi(t), \qquad iii) \quad t \longmapsto t^2\Pi(t).$$

Exercice 3 :
We consider the function

$$f(t) = \begin{cases} 1 - |t| & \text{if} \quad |t| \leq 1 \\ 0 & \text{if} \quad |t| > 1 \end{cases}$$

1/ Proof that $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$ and compute $\mathcal{F}f$.
2/ Apply the Fourier inversion formula to f at any point in $\mathbb{R}_+$.
3/ Deduce the value of the integral $\int_0^{+\infty} \frac{1 - \cos x}{x^2} \cos(\alpha x)\,dx, \alpha \in \mathbb{R}_+$.
4/ Give the value $\int_0^{+\infty} \frac{\sin^2 x}{x^2} dx$.

Exercice 4 :

For $\alpha > 0$, we put

$$f_\alpha : t \longmapsto e^{-\alpha|t|}.$$

1/ Compute de Fourier transform of f_α.

2/ Using the Fourier inversion formula, calculate the value of

$$\int_0^{+\infty} \frac{\cos(xt)}{x^2+\alpha^2} dx, \qquad \text{for any } t \in \mathbb{R}.$$

3/ Find a and b such that

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{a}{x^2+1} + \frac{b}{x^2+4}, \qquad \text{for any } x \in \mathbb{R}.$$

4/ Solve the differential equation

$$-y'' + y = f_2,$$

where y, y', y'' and y is derivable on $\mathbb{R}$.

Exercice 5 :

Let $\beta > 1$, solve in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R})$ the following integral equation

$$\int_{-\infty}^{+\infty} y(u)\, e^{-\beta(t-u)^2} du = e^{-t^2}, \qquad \text{for any } t \in \mathbb{R}.$$