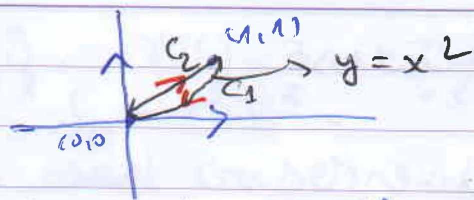
Example: $C_1: y=x^2$, $C_2: y=x$



$C_1: \gamma: [0,1] \longrightarrow \mathbb{R}^2$
$t \longmapsto (t,t)$

$C_2: \gamma_2: [0,1] \longrightarrow \mathbb{R}^2$
$t \longmapsto (1-t, (1-t)^2)$

$C = C_1 \cup C_2$ is a path.

$$\int_C x\, dl = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{3/2}-1) \simeq 1{,}56.$$

II-2-2- Line integral of a vector field.

Def: Let $t \xrightarrow{\gamma} \gamma(t)$ is a parameterization of a curve C in $\mathbb{R}^2$ ~~($\mathbb{R}^3$)~~ (resp $\mathbb{R}^3$) and F is continuous vector field in the plane (resp. in space). The line ~~(or~~ integral (or the circulation of F along C) of F on C is given by

$$\int_C F\cdot d\gamma = \int_a^b F(x(t), y(t))\cdot(x'(t)\vec{i} + y'(t)\vec{j})\,dt$$

$$\left(\text{resp} = \int_a^b F(x(t), y(t), z(t))\cdot(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k})\,dt\right)$$

. Where C is a path, we denote $\oint_C F\cdot d\gamma$, or $\oint_C F$.

. ~~If $F = M\vec{i} + N\vec{j} + P\vec{k}$, $\int_C (M\,dx + N\,dy + P\,dz) = \int_C F\cdot d\gamma$.~~

Example: Work done by a force

$F(x,y,z) = -\frac{1}{2}x\vec{i} - \frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k}$; $\gamma(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$
~~From~~ $t \in [0, 3\pi]$.

$$W = \int_0^{3\pi} \left[\left(-\frac{1}{2}\cos t\right)(-\sin t) - \frac{1}{2}\sin t\cdot\cos t + \frac{1}{4}\right]dt = \boxed{\frac{3\pi}{4}}$$

Remark: The differential form of line integrals

$$d\gamma = \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)\cdot dt \quad ; \quad F = M\vec{i} + N\vec{j} + P\vec{k}$$

$$\int_C F\cdot d\gamma = \int_C \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)\cdot dt$$

$$= \int_C (M\,dx + N\,dy + P\,dz).$$

~~If $P=0$ in the case~~ In $\mathbb{R}^2$

~~and~~ we get: $\int_C F\,d\gamma = \int_C (M\,dx + N\,dy)$