Remark: For surfaces described by functions of x and z (or y and z), we can make the following adjustements ~~to~~ the definition above: If S is the graph of $y = g(x, z)$ and R is its projection onto the xz-plane, then

$$\iint_S f(x,y,z)\,d\sigma := \iint_R f(x, g(x,z), z) \sqrt{1 + \frac{\partial g}{\partial x}^2 + \frac{\partial g}{\partial z}^2}\, dx\, dz.$$

If S is the graph of $x = g(y,z)$ and R is its projection onto the yz-plane, then $\iint_S f(g(y,z), y, z) \sqrt{1 + \frac{\partial g}{\partial y}^2 + \frac{\partial g}{\partial z}^2}\, dy\, dz$. If $f = 1$ $\boxed{\iint_S d\sigma = \mathcal{A}(S)}$

Example ① Evaluate the surface integral

$\iint_S (y^2 + 2yz)\,d\sigma$, where S is the first-octant portion of the plane $2x + y + 2z = 0$

② A cone-sharped surface lamina S is given by: $z = 4 - 2\sqrt{x^2+y^2}$, $0 \le z \le 4$. At each point on S, the density is ~~propor~~ proportional to the distance between the point and the z-axis. Let us compute the mass of S

$\rho : (x,y) \mapsto k\sqrt{x^2+y^2}$; $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, $m = \frac{16\pi k\sqrt{5}}{3}$.

Surface integral over a parametric surface

Let $S : r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v) \in D$.

$$\iint_S f(x,y,z)\,d\sigma := \iint_D f(x(u,v), y(u,v), z(u,v))\, \| r_u(u,v) \wedge r_v(u,v) \|\, du\, dv.$$

Example: $S = \{(x,y,z) \in \mathbb{R}^3 : x \ge 0, y \ge 0, z \ge 0, \; y^2 + z^2 \le 9\}$

Def: Unit normal vector field on surface and orientation

S is orientable when a unit ~~vector~~ normal vector can be defined and continuous on each nonboundary point

~~Let $S : \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D$ and $z = g(x,y)\}$. We put $G(x,y,z)$~~

~~$G(x,y,z) = z - g(x,y)$~~ the unit normal vector field of a surface ~~$S : r(u,v)$ is $N(x,y,z)$~~. $S := \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, z = g(x,y)\}$

$S = G^{-1}(\{0\})$ where $G(x,y,z) = z - g(x,y)$.

The unit normal vector field of S is $N = \frac{\nabla G}{\|\nabla G\|}$

that is: ~~$N(x,y,z)$~~ $= N(x,y) = \frac{1}{\sqrt{1 + \frac{\partial g}{\partial x}^2 + \frac{\partial g}{\partial y}^2}} \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k} \right)$

If $S : r(u,v)$

$N = \frac{r_u \wedge r_v}{\|r_u \wedge r_v\|}$ or $N = \frac{r_v \wedge r_u}{\|r_u \wedge r_v\|}$.

~~So on~~ On each nonboundary point