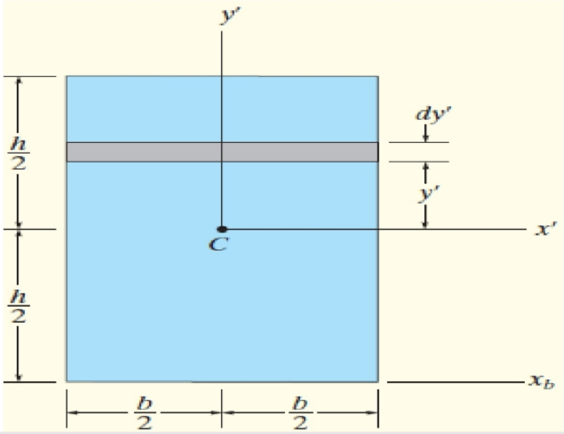
Problem 1 : (rectangular area)

1) Determine the moment of inertia for the rectangular area shown below with respect to :
(a) the centroidal x' axis using direct integration of an horizontal strip,
(b) the axis x_b passing through the base of the rectangle using direct integration,
2) Determine the polar moment of inertia about the centroid C. (the pole or z' axis is perpendicular to the $x' - y'$ plane and passing through the centroid C).



1) a) -

$$I_{x'} = \int_A y'^2 \, dA \quad \Big\backslash \quad dA = b \, dy'$$

$$I_{x'} = \int_{-\frac{h}{2}}^{\frac{h}{2}} y'^2 \, b \, dy' = b \cdot \frac{1}{3} \left[y'^3 \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \boxed{\frac{b h^3}{12}}$$

(b) - $$I_{x_b} = \int_A y'^2 dA = \int_0^h y'^2 b \, dy' = \int_0^h y'^2 b \, dy'$$

$$= \frac{b}{3} \left[y'^3 \right]_0^h = \boxed{\frac{b h^3}{3}}$$

2) $I_p = I_{x'} + I_{y'}$

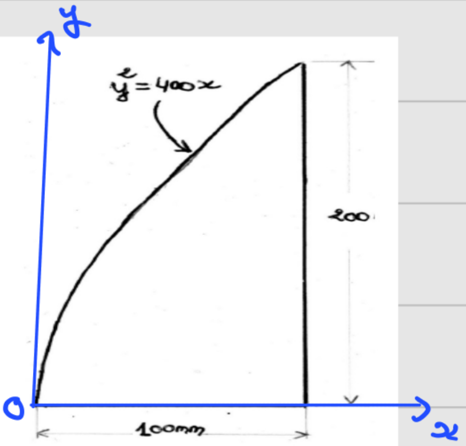
$$I_{y'} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} x'^2 \, dx' \, dy' = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{3} \left[x'^3 \right]_{-\frac{b}{2}}^{\frac{b}{2}} dy'$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{b^3}{12} dy' = \frac{b^3}{12} \left[y' \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{b^3 h}{12}$$

$$I_p = I_{x'} + I_{y'} = \frac{bh^3}{12} + \frac{b^3h}{12}$$

Problem 2 : (irregular area)

Determine the moment of inertia I with respect to the x axis lying on its base.(base = 100 mm, h = 200 mm, $y^2 = 400.x$)
Choose the appropriate rectangular differential element (strip)



$$y = 20\sqrt{x}$$

$$D : \left\{ \begin{array}{l} 0 : x : 100 \\ 0 : y : 20\sqrt{x} \end{array} \right\}$$

$$I_x = \iint y^2\,dy\,dx = \int_0^{100} \left(\int_0^{20\sqrt{x}} y^2\,dy \right) dx$$

$$= \int_0^{100} \frac{1}{3} \left[y^3 \right]_0^{20\sqrt{x}} dx = \int_0^{100} \frac{1}{3} \left(20\sqrt{x}\right)^3 dx$$

$$= \frac{1}{3} \times (20)^3 \times \frac{2}{5} \times \left[x^{5/2} \right]_0^{100} = 107.10^6 \text{ mm}^4$$

Problem 3 : (Parabolic Spandrel)

For the parabolic spandrel used in Problem 3 (TD 2). Using the appropriate differential element for each case, determine :
a) The moment of inertia about the x axis (I_x).
b) The moment of inertia about the y axis (I_y).

(a)- $I_x = \int_A y^2\,dA = \iint y^2\,dy\,dx = \int_0^1 \int_0^{x^2} y^2\,dy\,dx$

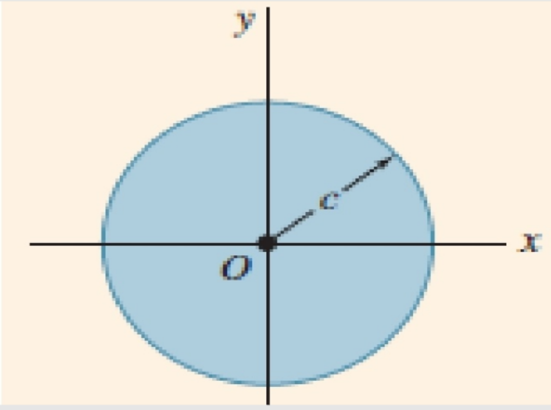
$$\boxed{I_x = \frac{1}{21} \text{ m}^4}$$

$$I_y = \int_A x^2\, dA = \iint x^2\, dx\, dy =$$

$$= \int_0^1 x^2 \times \left(\int_0^{x^2} dy\right) dx = \int_0^1 x^4\, dx = \frac{1}{5}\ m^4$$

Problem4: (circular area)

For this circular area, determine:
(a) the polar moment of inertia I_p or J_O,
(b) the rectangular moments of inertia I_x and I_y.



ⓐ $I_p = \int \rho^2\, dA = \int_0^c \rho^2\, 2\pi\rho\, d\rho = 2\pi \frac{1}{4}\left[\rho^4\right]_0^c$

$$\boxed{I_p = \frac{c^4\pi}{2}}$$

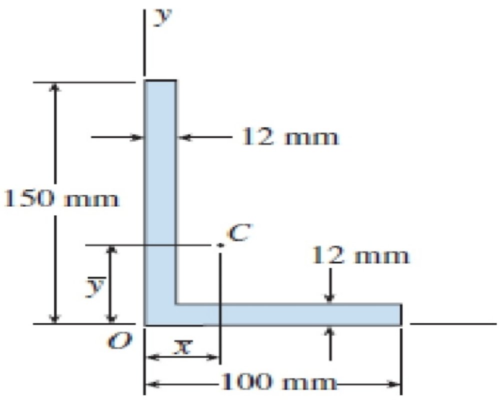
ⓑ par symétrie : $I_x = I_y$

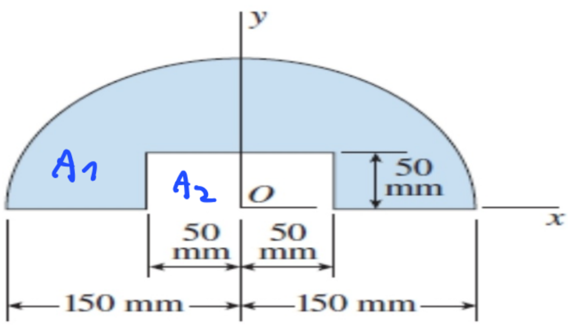
et $I_x + I_y = 2I_x = 2I_y = I_p$

$$I_x = I_y = \frac{1}{2} I_p = \frac{c^4\pi}{4}$$

Problem 6: (composite area)

A semicircular area of radius 150 mm has a rectangular cutout of dimensions (see figure). Calculate the moments of inertia I_x and I_y with respect to the x and y axes.





A_1 : demi cercle :

$$I_x = I_x(A_1) - I_x(A_2)$$

$$I_x(A_1) = \iint y^2 \, dx \, dy$$

using polar coordinate:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$D : \left\{ \begin{array}{l} 0 : r : 150 \\ 0 : \theta : \pi \end{array} \right\}$$

$$I_x(A_1) = \iint_D r^2 \sin^2\theta \, r \, dr \, d\theta$$

$$= \int_0^{150} r^3 \, dr \int_0^{\pi} \frac{1 - \cos(2\theta)}{2} \, d\theta$$

$$= \frac{1}{4} 150^4 \cdot \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi}$$

$$= \frac{150^4}{8} \pi \ mm^4$$

$$I_x(A_2) = \int_0^{50} \int_{-50}^{50} y^2 \, dx \, dy = \int_0^{50} 100 \, y^2 \, dy = \frac{100}{3} \cdot 50^3 \ mm^4$$

$$I_x = \frac{150^4}{8}\pi - \frac{100}{3} \cdot 50^3$$

$$8 \qquad\qquad 3$$

$$= 1{,}946 \times 10^{-4}\ m^4$$

$$I_y = I_y(A_1) - I_y(A_2)$$

$$I_y(A_1) = \iint_D x^2\, dy\, dx$$

using polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$I_y(A_1) = \iint_D r^2\cos^2\theta\, r\, dr\, d\theta$$

$$= \int_0^{150} r^3\, dr \times \int_0^{\pi} \frac{\cos(2\theta)+1}{2}\, d\theta$$

$$= \frac{1}{4} \cdot 150^4 \cdot \frac{1}{2}\left[\frac{1}{2}\sin(2\theta) + \theta\right]_0^{\pi} = \frac{150^4}{8}(\pi)\ mm^4$$

$$I_y(A_2) = \int_{-50}^{50}\int_0^{50} x^2\, dy\, dx$$

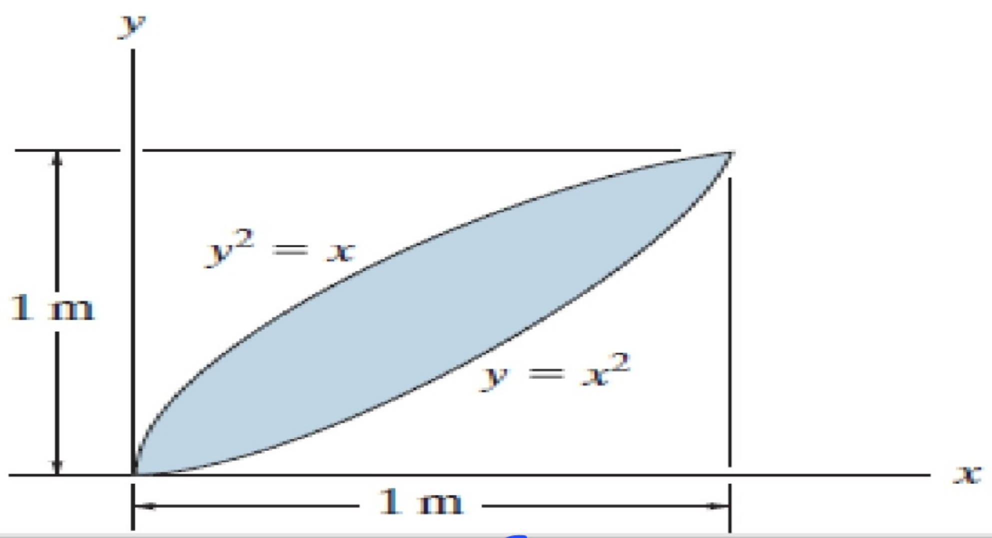
$$= \int_{-50}^{50} x^2 \cdot 50\, dx = \frac{50}{3}\left[50^3 + 50^3\right)$$

$$= \frac{2 \times 50^4}{3}\ mm^4$$

$$I_y = \frac{1\,50^4}{8}\pi - \frac{2}{3}\cdot 50^4$$

$$I_y = 1{,}946 \times 10^{-4}\ m^4$$

Problem 1 : Locate the centroid of this area.



$$\bar{x} = \frac{Q_y}{A} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^1\int_{x^2}^{\sqrt{x}} x\,dy\,dx}{\iint dy\,dx}$$

$$\bar{x} = \frac{\int_0^1 x(\sqrt{x}-x^2)\,dx}{\int_0^1(\sqrt{x}-x^2)\,dx}$$

$$= \frac{\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1}{\left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1} = 0{,}45\ m$$

$$\bar{y} = \frac{Q_x}{A} = \frac{\int y\,dA}{\int dA}$$

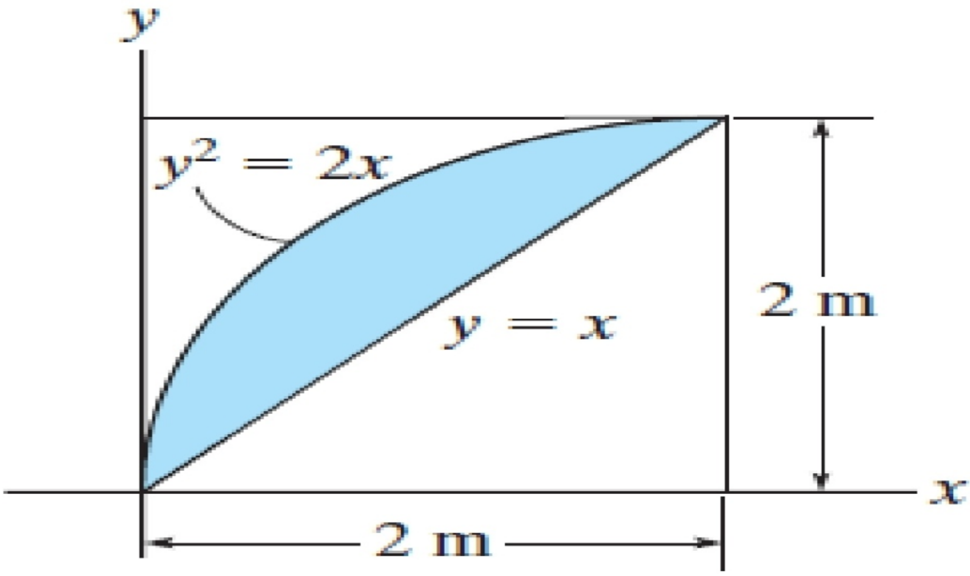
$$\int dA = \frac{1}{3}\ m^2$$

$$\int y\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} y\,dy\,dx = \int_0^1 \frac{1}{2}(x - x^4)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{20}\,m$$

$$\bar{y} = \frac{\frac{3}{20}}{\frac{1}{3}} = 0{,}45\,m$$

Problem 2 : Calculate the moments of inertia I_x and I_y



$$I_x = \int y^2\,dA = \int_0^2 \int_x^{\sqrt{2}\sqrt{x}} y^2\,dy\,dx$$

$$= \int_0^2 \frac{1}{3}\left(\left(\sqrt{2}\sqrt{x}\right)^3 - x^3\right) dx = \frac{1}{3}\int_0^2 \left(2\sqrt{2}\,x^{3/2} - x^3\right) dx$$

$$= \frac{1}{3}\left[2\sqrt{2}\cdot\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^2 = 0{,}8\,m^4$$

$$I_y = \int_A x^2\,dA = \iint_{0\ x}^{2\ \sqrt{2}\sqrt{x}} x^2\,dy\,dx = \int_0^2 x^2\left(\sqrt{2}\sqrt{x} - x\right)dx$$

$$= \left[\sqrt{2}\,\frac{2}{7}x^{7/2} - \frac{x^4}{4}\right]_0^2 = 0{,}571\,m^4$$