# <u>Chapitre 1</u>

# Kirchhoff's laws in sinusoidal regime

# Introduction

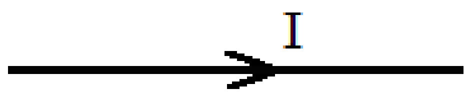
- Why a sinusoidal regime?
  - Electrical energy is produced, distributed, and consumed in sinusoidal alternating form,
  - Raising the voltage and/or lowering it is done by simple devices: **transformers.**

- Purpose of this chapter:
  - Basic notions of alternating current (single phase),
  - Reminder of **Kirchhoff's laws** and alternating current application,
  - Simplification of these laws in steady state thanks to the use of complex notation,
  - Concept of impedance.

# Direct Current (DC) reminders

## *Current direction rules*

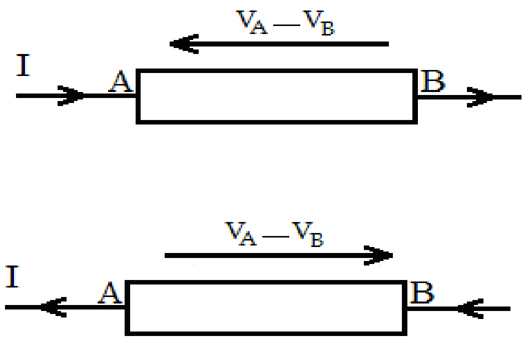
- Let us not forget that the **conventional** direction of electric current is the **opposite** of the direction of **movement of electrons**.
- Electrons move from **lower** potential to **higher** potential.
- The current is symbolized by an arrow going from the **highest** potential to the **lowest** potential,
- If the direction of the current is not known, then an arbitrary direction is chosen,
- If, after calculation, we obtain a negative current, this means that the current flows in the opposite direction:



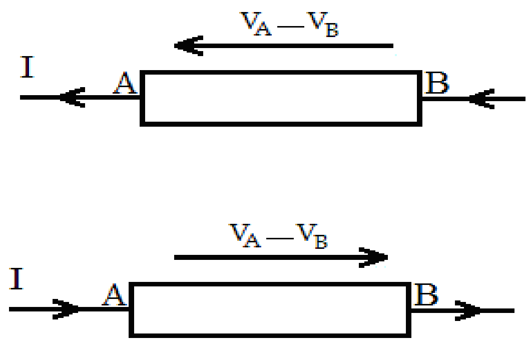
# Direct Current (DC) reminders

## *Voltage direction rules*

- An arrow near the latter symbolizes a voltage across a dipole.
- The tip of the arrow **generally** symbolizes the highest potential if the latter is known,
- otherwise, the orientation of the arrow is done arbitrarily.
- The direction of the voltage/current pair is, however, dependent: the same direction for a generator and the opposite direction for a load.




**Load Case**

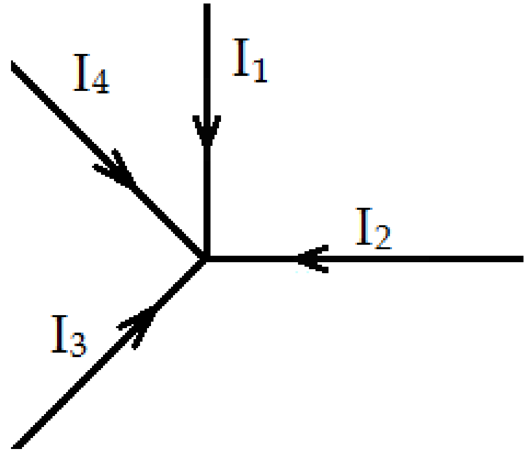



**Generator case**

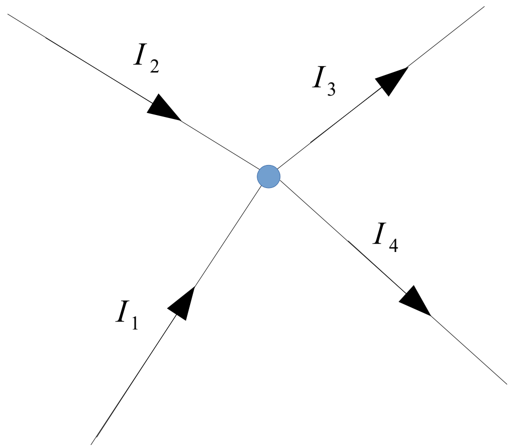
# Kirchhoff's Laws

## *Junction rule*

- Law connecting the currents of several branches (wires) interconnected into a single node.
- A node: junction point between several conductors:
- he sum of the intensities of the currents arriving at a node is equal to that of the currents leaving the node.




$$I_1 + I_2 + I_3 + I_4 = 0$$

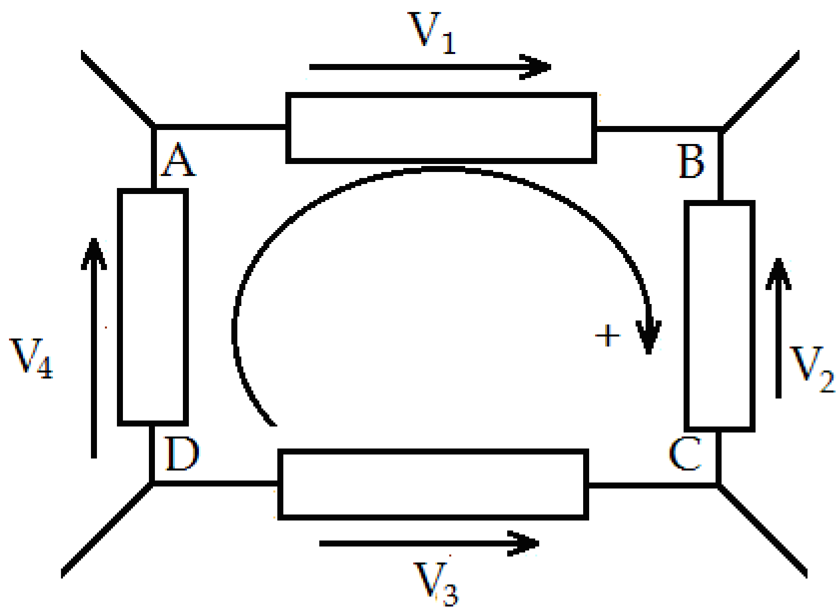



$$I_1 + I_2 = I_3 + I_4$$

# Kirchhoff’s Laws

## *Loop rule*

- **Loop:** a set of conductors (wires) and components starting from one point and returning to the same point (forming a loop).
- The algebraic sum of the tensions along the mesh is zero.
- ***Example :***

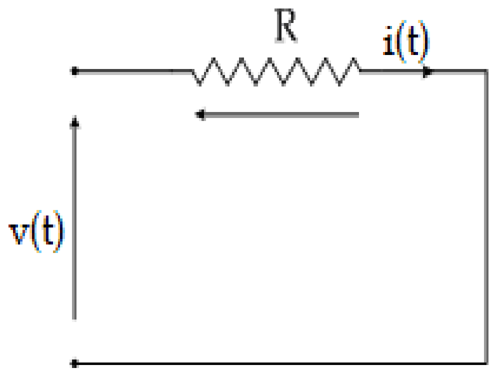



$$V_1 - V_2 - V_3 + V_4 = 0$$

# Usual dipoles in variable regime

## *Resistance :*

- Governed by Ohm's law:




$$v(t)=R.i(t)$$

- Voltage is always proportional to current.

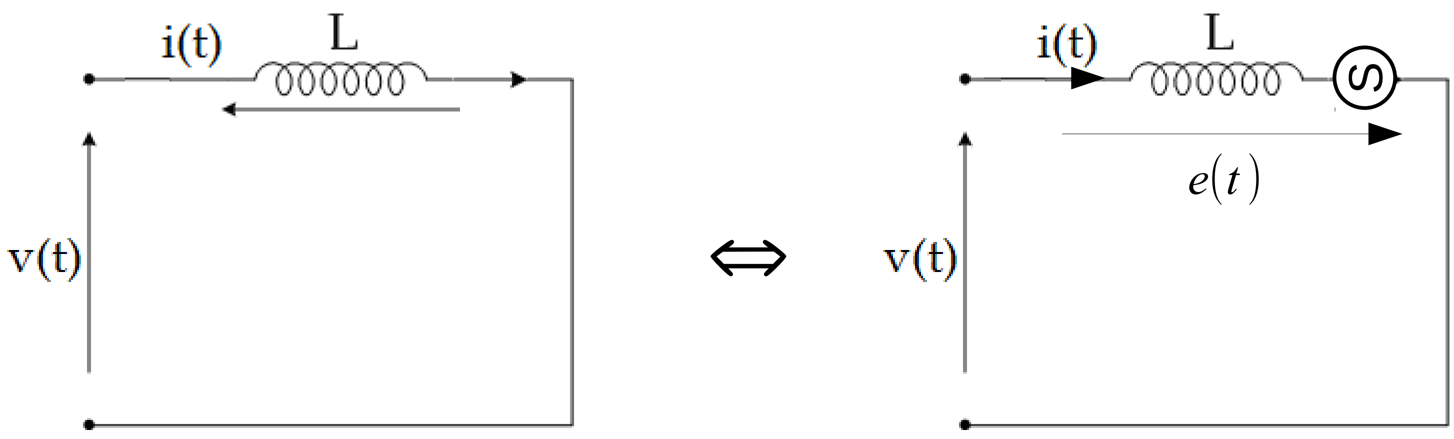
- A resistance does not create a time lag between voltage and current.

# Usual dipoles in variable regime

## *Inductance (1/2) :*

- **Governed by Lenz's law:** A potential difference across an inductor induces a (counter) electromotive force that opposes the electric current. This electromotive force is:

$$e(t) = \frac{-d\phi_B(t)}{dt} \quad \text{where} \quad \phi_B(t) = L.i(t) \quad \text{and then} \quad v(t) = -e(t) = L\frac{di(t)}{dt}$$



# Usual dipoles in variable regime

## *Inductance (2/2) :*

We deduce that:

- The voltage is always ahead of the current in the coil because the **Lenz** effect states that there is opposition to establishing an electric current.

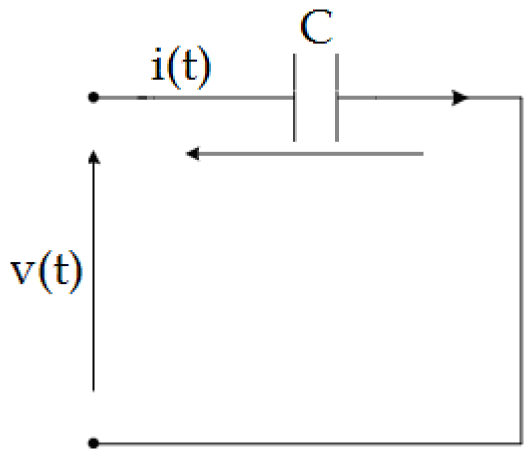
- In continuous (permanent) mode, the coil behaves like a short circuit (closed circuit) because $v(t)=0$.

- Breaking electric current in a coil can lead to the production of very high voltages, because :

$$di(t)\neq 0 \quad \text{et} \quad dt \rightarrow 0 \quad \Rightarrow \quad v(t) \rightarrow \infty$$

# Usual dipoles in variable regime

## *Capacitor (1/2) :*




- Governed by the fundamental laws studied in electrostatics:
  - Relationship between potential and charge

$$Q = C.V$$

  - Relationship between load and current:

$$i(t) = dq(t)/dt$$

  - Hence :

$$i(t) = C.\frac{dv(t)}{dt}$$

# Usual dipoles in variable regime

## *Capacitor (2/2) :*

We deduce the following:

- The capacitor reacts like a coil except that voltage and current roles must be reversed, and therefore,

- The current is ahead of the voltage,

- A sudden variation in voltage produces an infinite current,

$$dv(t) \neq 0 \quad \text{et} \quad dt \rightarrow 0 \quad \Rightarrow \quad i(t) \rightarrow \infty$$

- In continuous operation, the capacitor behaves like an open circuit because zero current passes through it. $i(t) = 0$ .

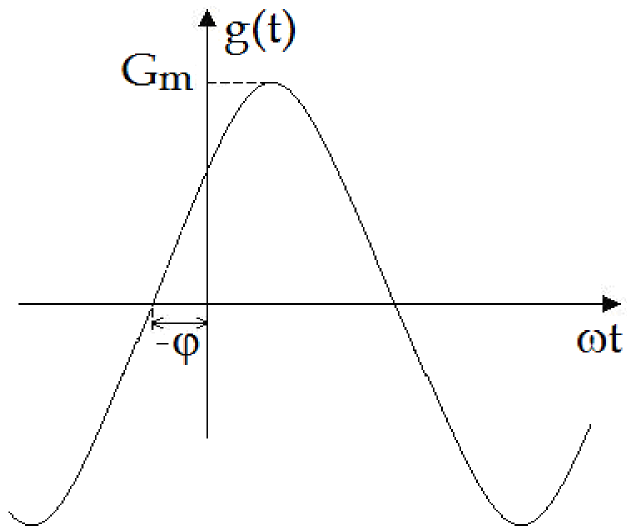
# Sinusoidal alternating quantities

## *Generalities :*

- The quantity G varies as a function of time according to the law:

  $g(t)=G_m.\sin(\omega.t+\varphi)$ or even $g(t)=G_m.\cos(\omega.t+\varphi')$

- Conventions :
  - Variable quantities such as $g(t)$ are noted in lowercase,
  - $\omega$ is the pulsation with $\omega=2\pi f$ where $f$ is the frequency.
  - $G_m$ is the maximum value of g(t).
  - $\varphi$ is called the phase shift of g relative to an established reference.




  - The phase shift $\varphi$ reflects a time shift $\Delta t=\frac{\varphi}{\omega}$

# Sinusoidal alternating quantities

## *Valeur efficace (RMS) :*

RMS or root mean square current of the alternating current (La valeur efficace, notée $I$ (en majuscule)) of current $i(t)$ represents the DC current (I) that dissipates the same amount of power as the average power dissipated by the alternating current $i(t)$.

- Let us calculate the energy dissipated in AC over a period:

$$W(T)=\int_0^T P(t)dt=\int_0^T v(t).i(t)dt=\int_0^T Ri^2(t)dt$$

- Let us calculate the energy dissipated in the case of a direct current I

$$W(T)=\int_0^T P(t)dt=\int_0^T V.I\,dt=\int_0^T RI^2dt=RI^2T$$

- By equalizing the two dissipated energies, we obtain:

$$I=\sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \quad \text{or in general} \quad G_{eff}=G=\sqrt{\frac{1}{T}\int_0^T g^2(t)dt}$$

# Sinusoidal alternating quantities

## *RMS in the case of sinusoidal signal :*

**Problem :**

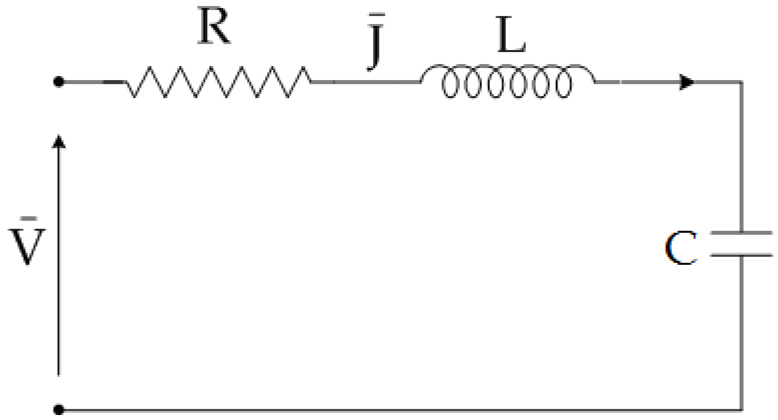
Show that if g(t) is sinusoidal, then the effective value of g(t) is related to the maximum value as follows:

$$G = G_m / \sqrt{2}$$

# Phasor Representation (Fresnel)

## *Problem statement :*

- Let us consider the determination of the currents and voltages of the RLC circuit using classical notions.




- The calculations are laborious

- Are there alternative calculation methods for determining **steady-state** currents and voltages when the source is **sinusoidal**?

- YES, two solutions:
    - Phasor (Fresnel diagram): limited to simple circuits,
    - Complex notation: much easier to use).

# Phasor (Fresnel diagram)

## *Solution with respect to time (1/4) :*

Let us determine current $i(t)$ knowing that $v(t)=V_m\cos(\omega t)$

- Loop law :

$$v(t)=v_R(t)+v_L(t)+v_C(t)$$

- Let us express each potential difference :

$$v(t)=Ri(t)+L\frac{di(t)}{dt}+\frac{1}{c}\int_0^t i(t)dt$$

- We derive

$$\dot{v}(t)=R\dot{i}(t)+L\ddot{i}(t)+\frac{1}{c}i(t)$$

- or else :

$$C.\dot{v}(t)=LC\ddot{i}(t)+RC.\dot{i}(t)+i(t)$$

# Phasor (Fresnel diagram)

## *Solution with respect to time (2/4) :*

We have to solve the following differential equation :

$$C.\dot{v}(t)=LC\ddot{i}(t)+RC.\dot{i}(t)+i(t)$$

- Homogeneous solution for v(t)=0
  - Solution of the form $e^{\alpha t}$. As *R* and *C > 0*, we show that this solution tends to 0 with time (transitory : pre-steady state).

- Particular solution :
  - as $v(t)=V_m\cos(\omega t)$ then, we suppose $i(t)=I_m\cos(\omega t+\varphi)$

    or $i(t)=I_m\cos(\varphi).\cos(\omega t)-I_m\sin(\varphi).\sin(\omega t)$

    then, $i(t)=i_1\cos(\omega t)+i_2\sin(\omega t)$

    with $I_m=\sqrt{i_1^2+i_2^2}$ and $\varphi=-arctg(i_2/i_1)$
  - Let us determine $i_1$ and $i_2$.

# Phasor (Fresnel diagram)

## *Solution with respect to time (3/4):*

- Replacing this, in the differential equation, we obtain :

$$-C.V_m\omega.\sin(\omega t)=LC(-i_1\omega^2\cos(\omega t)-i_2\omega^2\sin(\omega t))$$
$$+RC.(-i_1\omega\sin(\omega t)+i_2\omega\cos(\omega t))+i_1\cos(\omega t)+i_2\sin(\omega t)$$

- then :

$$-CV_m\omega\sin(\omega t)=\sin(\omega t)\{-i_2\omega^2 LC-i_1 RC\omega+i_2\}$$
$$+\cos(\omega t)\{-i_1\omega^2 LC+RC\omega i_2+i_1\}$$

- We obtain the two following equations :

$$\begin{cases} i_1(LC\omega^2-1)-RC\omega i_2=0 \\ i_1(RC\omega)+i_2(Lc\omega^2-1)=C\omega V_m \end{cases}$$

# Phasor (Fresnel diagram)

## *Solution with respect to time (3/4): (4/4):*

- The system's solution :

$$i_1 = \frac{-R(C\omega)^2}{(LC\omega^2-1)^2+(RC\omega)^2}.V_m$$

$$i_2 = \frac{-C\omega(LC\omega^2-1)}{(LC\omega^2-1)^2+(RC\omega)^2}.V_m$$

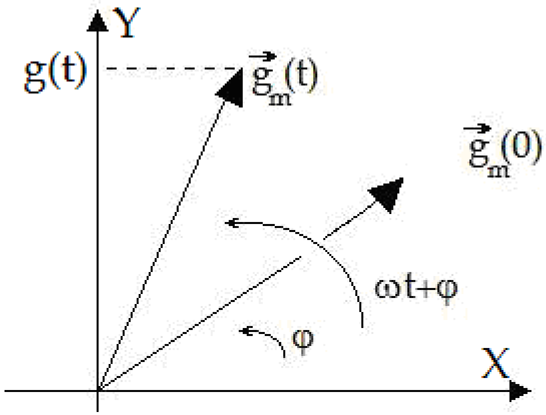
- and then :

$$I_m = \frac{C\omega}{\sqrt{(LC\omega^2-1)^2+(RC\omega)^2}}.V_m \quad \text{et la phase} \quad \varphi = arctg\left(\frac{1-LC\omega^2}{RC\omega}\right)$$

- Resolution process too complicated for such a simple circuit => ANOTHER METHOD : ***Phasor (Fresnel diagram)***

# Phasor (Fresnel diagram)

## *A sinusoidal function as a vector :*

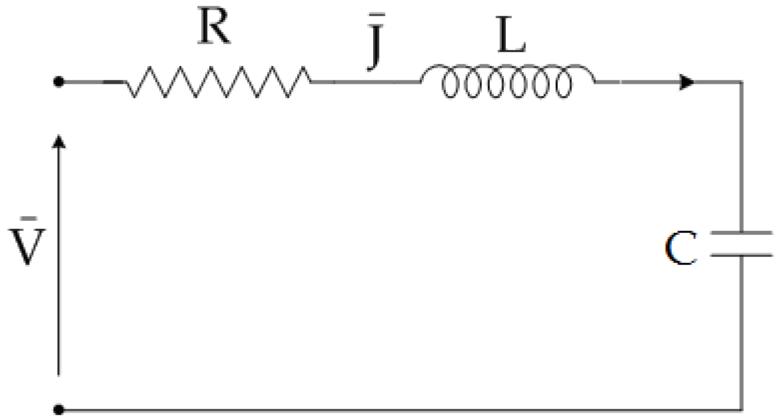



- Alternative solution for **the steady state,**
- A representation in the *(OXY) which means* in the complex plan.
- The time is frozen (atemporal representation. We take $t=0$ ),
- The sinusoidal is represented by a vector starting at O,
- The **module** of the vector represents the **RMS (la valeur efficace)**
- **The angle** between the vector and *(OX)* represents **the phase**.
- In fact, the vector turns with an angular speed $\omega$
- The projection of the vector on ***(OX)*** or ***(OY)*** allows finding values in **cosinus** or in **sinus**. In general, the **cosinus** version is the one used (à $\sqrt{2}$ près).

# Phasor (Fresnel diagram)

## *RLC circuit : graphic solution (1/4)*

- Let us consider the determination of currents and voltages of the RLC circuit using the Fresnel diagram.




- Loop Law :
  - $v(t)=v_R(t)+v_L(t)+v_C(t)$
- Let us express each potential difference :
  - $v(t)=Ri(t)+L\dfrac{di(t)}{dt}+\dfrac{1}{c}\displaystyle\int_0^t i(t)dt$
- As $v(t)=V_m\cos(\omega t)$ then, we suppose $i(t)=I_m\cos(\omega t+\varphi)$
- let us take the current as the phase origin. Then :

$i(t)=I_m\cos(\omega t)$ et $v(t)=V_m\cos(\omega t+\phi)$

# Phasor (Fresnel diagram)

## *RLC circuit : graphic solution (2/4)*

Then : $V_m \cos(\omega t + \phi) = (R)(I_m \cos(\omega t)) - (L\omega).(I_m \sin(\omega t)) + (\frac{1}{C\omega})(I_m \sin(\omega t))$

$$V_m \cos(\omega t + \phi) = (R)(I_m \cos(\omega t)) + (L\omega).(I_m \cos(\omega t + \frac{\pi}{2})) + (\frac{1}{C\omega})(I_m \cos(\omega t - \frac{\pi}{2}))$$

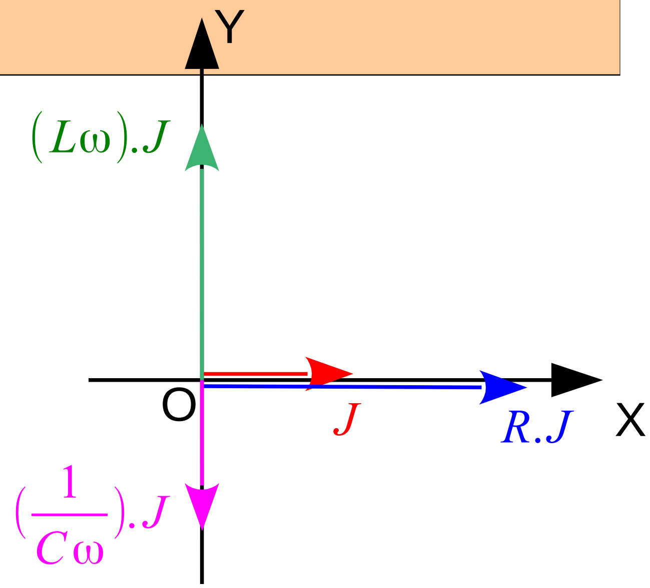
- In the phasor form :

$$\vec{V} = \vec{V}_R + \vec{V}_L + \vec{V}_C \quad \text{with} \quad \begin{cases} \vec{V} \Leftrightarrow V_m \cos(\omega t + \phi) \\ \vec{V}_R \Leftrightarrow (R)(I_m \cos(\omega t)) \\ \vec{V}_L \Leftrightarrow (L\omega).(I_m \cos(\omega t + \frac{\pi}{2})) \\ \vec{V}_C \Leftrightarrow (\frac{1}{C\omega})(I_m \cos(\omega t - \frac{\pi}{2})) \\ \vec{I} \Leftrightarrow I_m \cos(\omega t) \end{cases}$$

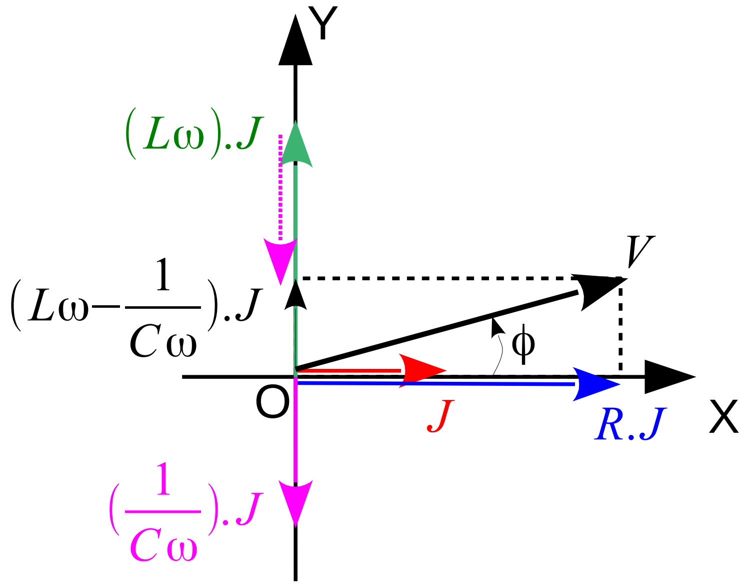
# Phasor (Fresnel diagram)

## *RLC circuit : graphic solution(3/4)*

- We represent the current $\vec{I}$ on ($Ox$) ,
- Then $\vec{V}_R$ parallel to $\vec{I}$ ,
- Then $\vec{V}_L$ on ($Oy$)
- Finally $\vec{V}_C$ on ($Oy$') opposed to $\vec{V}_L$




- We draw the sum $\vec{V}=\vec{V}_R+\vec{V}_L+\vec{V}_C$
- We, then, deduce the module and the phase of the result.



# Phasor (Fresnel diagram)

## *RLC circuit : graphic solution (4/4)*

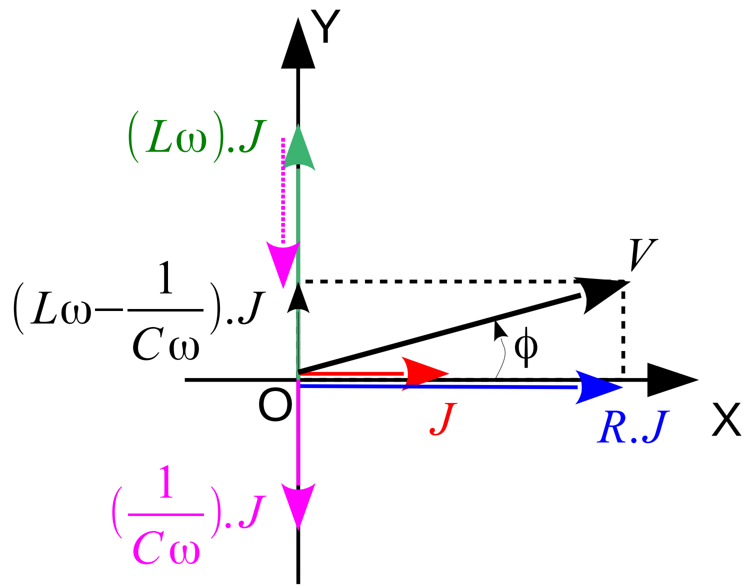
- The module :

$$V^2 = R^2 I^2 + \left(L\omega - \frac{1}{C\omega}\right)^2 I^2$$

- Then RMS and max of curren :

$$I = \frac{C\omega}{\sqrt{(LC\omega^2 - 1)^2 + (RC\omega)^2}} . V$$

$$I_m = \frac{C\omega}{\sqrt{(LC\omega^2 - 1)^2 + (RC\omega)^2}} . V_m$$




- Phase of the voltage in respect to the current : $\phi = arctg\left(\frac{LC\omega^2 - 1}{RC\omega}\right)$

- Phase of the current in respect to the voltage $\varphi = arctg\left(\frac{1 - LC\omega^2}{RC\omega}\right)$

# Phasor (Fresnel diagram)

## *RLC circuit : graphic solution Remarks + Conclusions*

**Remarks**

- The voltage $v_R(t)$ is in phase with the current $i(t)$
- The voltage $v_L(t)$ is ahead of current $i(t)$. The phase shift is of $+\pi/2$. We say : they are in quadrature.
- The voltage $v_C(t)$ has a lag towards current $i(t)$. The phase shift is of $-\pi/2$. they are in quadrature.

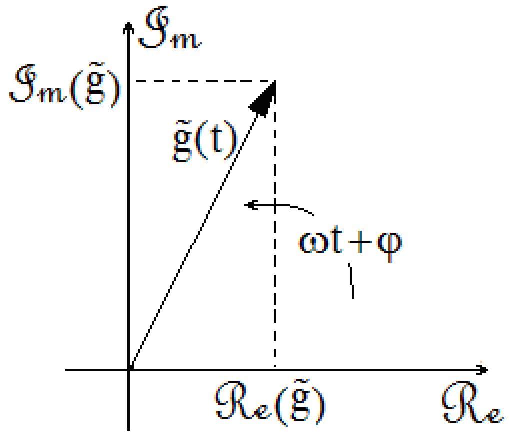
**Conclusions :**

- More simple than the time solution, however still complex,
- Is there any simpler method ?
- ***YES*** : Use complex numbers .

# Complex Representation

## *Principle*

- A physical value $g(t)=G_m \cos(\omega t+\varphi)$ such as a voltage signal,
- We right it in the complex form :
  - $g(t) \quad \Leftrightarrow \quad \tilde{g}(t)=G_m e^{j(\omega t+\varphi)}=\tilde{G}_m e^{j\omega t}$
  - $\tilde{G}_m=G_m e^{j\varphi}$ is the complex amplitude
  - We define too $\underline{G}=G e^{j\varphi}=\tilde{G}_m/\sqrt{2}$.




- We do all our processing (linear calculations) using complex numbers.
- The result of our calculations is another quantity $\tilde{h}(t)$ such as current for example. o get the true value of $h(t)$, , we return to the real domain : $h(t)=\Re(\tilde{h}(t))$.
- Nonlinear calculations such as power calculations are not feasible in this way !

# Complex Representation

## *Derivation and integration rules :*

- As

$$\tilde{g}(t) = \tilde{G}_m e^{j\omega t}$$

- the derivation gives :

$$\frac{\partial \tilde{g}(t)}{\partial t} = j\omega \tilde{g}(t)$$

- For the integration we get :

$$\int \tilde{g}(t) dt = \frac{1}{j\omega} \tilde{g}(t)$$

# Complex Representation – case study

## *Resistance case :*

- Ohm's Law in real domain :

$$v(t)=R.i(t)$$

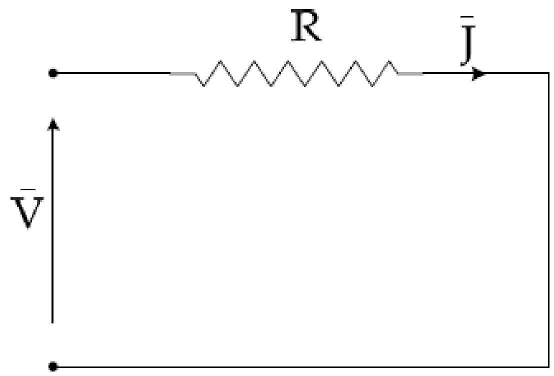
- In the complex domain

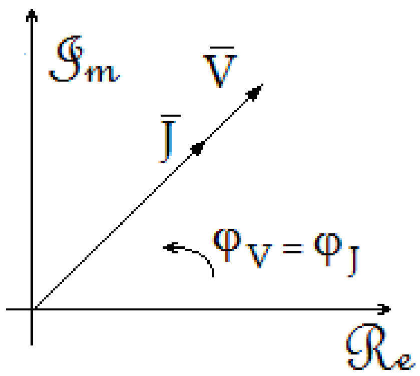
$$\tilde{v}(t)=R.\tilde{i}(t)$$

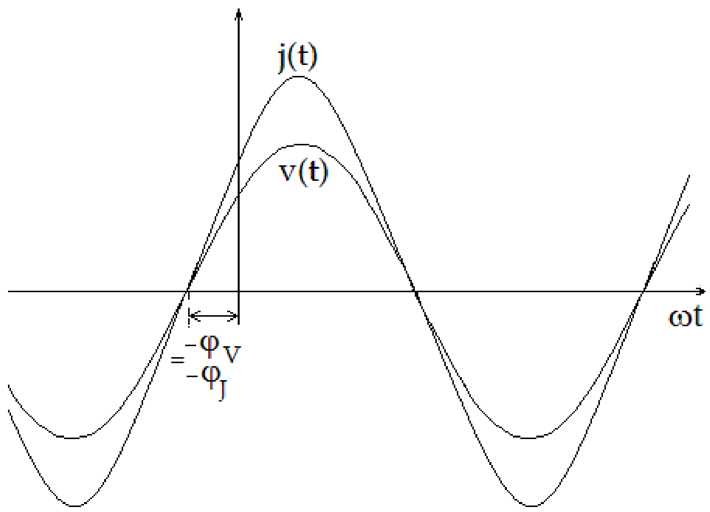
- In other words only the module is affected :

$\tilde{V}=R\tilde{I}$ and $\varphi_V=\varphi_R+\varphi_I=\varphi_I$ because $\varphi_R=0$

- Voltage and current are in phase ( $\Delta\varphi=0$ )



# Complex Representation – case study

## *Inductance case (1/2) :*

- ***Lenz*** Law: $$v(t) = L\frac{di(t)}{dt}$$
- In the complex domain $\tilde{v}(t) = L\frac{d\tilde{i}(t)}{dt} = jL\omega\tilde{i}(t)$
- Which we can write in the form of generalized Ohm's law :

$\tilde{v}(t) = Z_L\tilde{i}(t)$ with $Z_L = jL\omega$ impedance of an inductance.

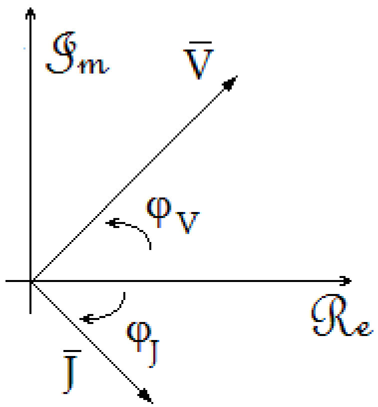
- Voltage across the inductance is ahead of the current and the phase shift is $\pi/2$.
- Indeed :

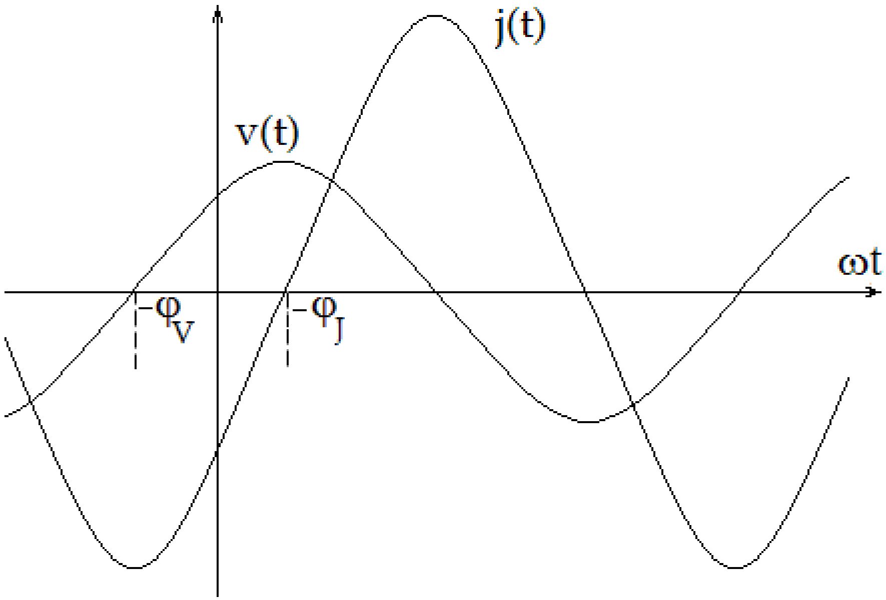
$\varphi_{\tilde{v}} = \varphi_{\tilde{i}} + \varphi_j + \varphi_{L\omega}$ and then $\varphi_{\tilde{v}} = \varphi_{\tilde{i}} + \pi/2$ because $\varphi_{L\omega} = 0$

# Complex Representation – case study

## *Inductance case (2/2):*

- Phasor and temporal diagrams :




- Current lags behind voltage

# Complex Representation – case study

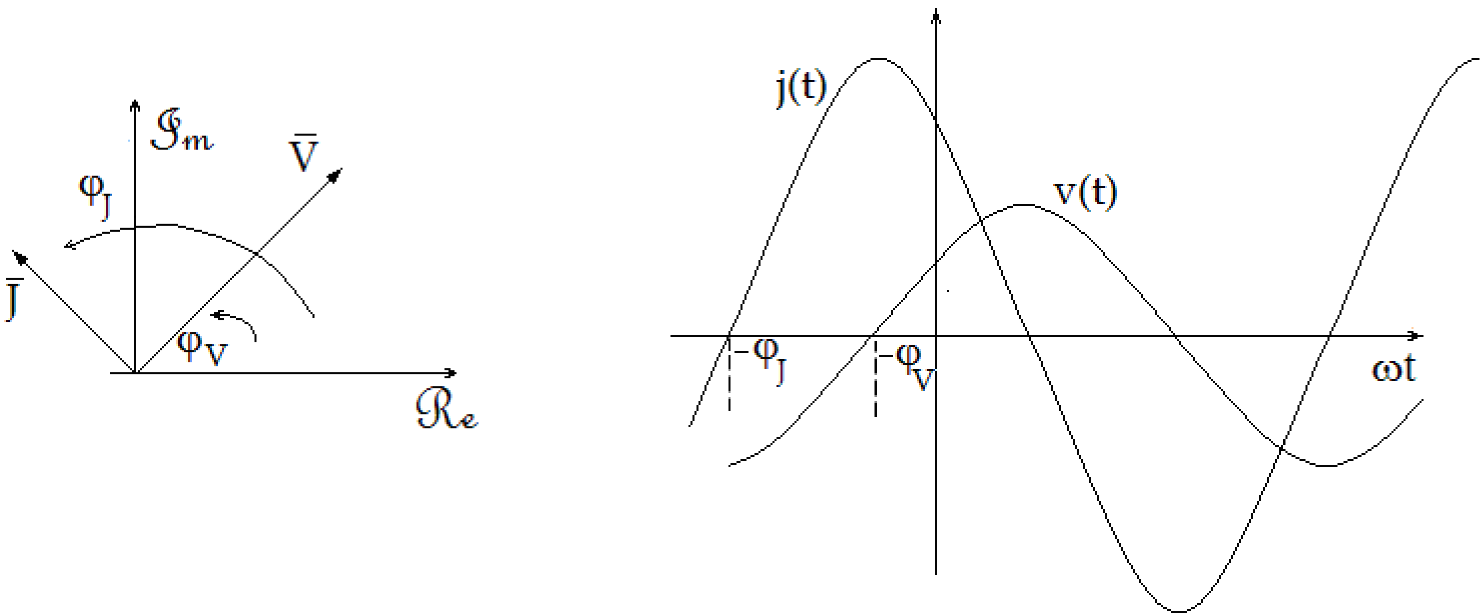
## *Capacitor case (1/2) :*

- Current-voltage relation : $i(t)=C.\frac{dv(t)}{dt}$ then $v(t)=\frac{1}{C}\int i(t)dt$
- In the complex domain $\tilde{v}(t)=\frac{1}{C}\int \tilde{i}(t)dt=\frac{1}{jC\omega}\tilde{i}(t)$
- Which we can write in the form of generalized Ohm's law :
  - $\tilde{v}(t)=Z_C\tilde{i}(t)$ with $Z_C=\frac{1}{jC\omega}=\frac{-j}{C\omega}$ the capacitor impedance.
- The voltage across the capacitor is lagged by the current and the phase shift is $-\pi/2$.
- Indeed :
  - $\varphi_{\tilde{v}}=\varphi_{\tilde{i}}+\varphi_{-j}-\varphi_{C\omega}$ then $\varphi_{\tilde{v}}=\varphi_{\tilde{i}}-\pi/2$ because $\varphi_{C\omega}=0$

# Complex Representation – case study

## *Capacitor case (2/2) :*

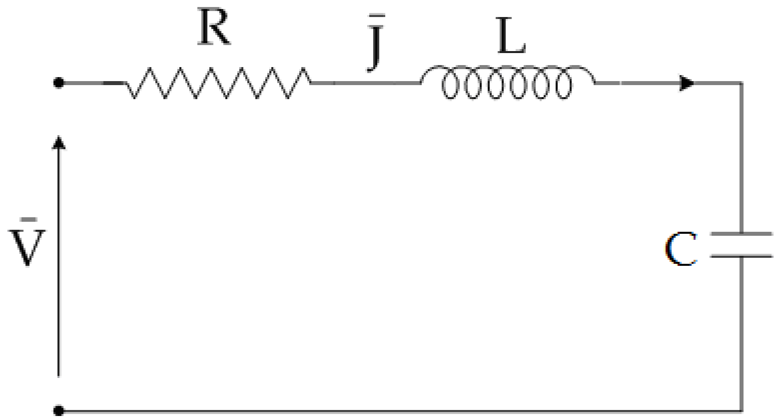
- Phasor and temporal diagrams :




- The current is ahead of the voltage.

# Complex Representation

## *RLC Circuit :*




- Let us consider determining the currents and voltages of the RLC circuit using complex notation.
- Loop law :

$$v(t)=v_R(t)+v_L(t)+v_C(t)$$

- In complex notation :

$$\tilde{v}(t)=\tilde{v}_R(t)+\tilde{v}_L(t)+\tilde{v}_C(t)$$

- Let us express the voltages as a function of the current :

$$\tilde{v}(t)=R\,\tilde{i}(t)+Z_L\,\tilde{i}(t)+Z_C\,\tilde{i}(t)=\{R+jL\omega-\frac{j}{C\omega}\}\tilde{i}(t)=Z\,\tilde{i}(t)$$

- $Z$ represents the impedance of the circuit. $Z=R+j(L\omega-\frac{1}{C\omega})$

# Complex Representation

## *RLC circuit : Definitions and remarks*

**Définitions :**

- We will generally write Z in the form :

$$Z = R + jX$$

- The real part of Z is R. It is called **« resistance »**,
- *The imaginary part of Z is X.* It is called **« reactance »**.

**Remarks :**

- In our circuit : $X = L\omega - \frac{1}{C\omega}$
- $X > 0$ indicates a dominant inductive effect,
- $X < 0$ indicates a dominant capacitive effect
- $X = 0$ indicates a resonance state.

# Complex Representation

## *RLC circuit : calculations*

- We found :

$$\tilde{v}(t)=R\tilde{i}(t)+Z_L\tilde{i}(t)+Z_C\tilde{i}(t)=\{R+jL\omega-\frac{j}{C\omega}\}\tilde{i}(t)$$

- Hence the value of $\tilde{i}(t)$ :

$$\tilde{i}(t)=\frac{\tilde{v}(t)}{R+j(L\omega-\frac{1}{C\omega})}$$

- The complex amplitude of $\tilde{i}(t)$

$$\tilde{I}_m e^{j\omega t}=\frac{\tilde{V}_m e^{j\omega t}}{R+j(L\omega-\frac{1}{C\omega})}$$ or else $$\tilde{I}_m=\frac{\tilde{V}_m}{R+j(L\omega-\frac{1}{C\omega})}$$

# Complex Representation

## *RLC circuit*

- We can now deduce the amplitude and the phase of the current in relation to those of the voltage. Indeed :

$$\tilde{I}_m = \frac{\tilde{V}_m}{R + j\left(L\omega - \frac{1}{C\omega}\right)} \quad \text{then} \quad I_m e^{j\varphi_I} = \frac{V_m e^{j\varphi_V}}{R + j\left(L\omega - \frac{1}{C\omega}\right)}$$

- We now just need to compare the modulus and phase of the complex numbers located on either side of the equality :

$$\left|I_m e^{j\varphi_I}\right| = \frac{\left|V_m e^{j\varphi_V}\right|}{\left|R + j\left(L\omega - \frac{1}{C\omega}\right)\right|} \quad \text{and then} \quad I_m = \frac{V_m}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

# Complex Representation

## *RLC circuit*

- For the phase :

$$\varphi\{I_m e^{j\varphi_I}\}=\varphi\{\frac{V_m e^{j\varphi_V}}{R+j(L\omega-\frac{1}{C\omega})}\}$$ and then $$\varphi_I=\varphi_V-\varphi\{R+j(L\omega-\frac{1}{C\omega})\}$$

- and then :

$$\varphi_I=\varphi_V-arctg\{\frac{(L\omega-\frac{1}{C\omega})}{R}\}=\varphi_V-arctg\{\frac{LC\omega^2-1}{RC\omega}\}$$

- finally :

$$\Delta\varphi=\varphi_V-\varphi_I=arctg\{\frac{LC\omega^2-1}{RC\omega}\}$$

# Complex Representation

## *RLC circuit : Conclusions*

- The generalization of Ohm's law in the complex domain makes it possible to take into account more simply inductors and capacitors knowing the expression of their impedance.
- The laws of loops and junctions are used exactly as in DC.
- It is faster to work directly with complex amplitudes $\tilde{G}_m$ or $\bar{G}$ because the term « phasor » $e^{j\omega t}$ is omitted.
- To obtain the effective values, it will suffice to divide the modules of the complex amplitudes by $\sqrt{2}$. which means $G=|\tilde{G}_m|/\sqrt{2}=|\bar{G}|$ .
- We retain that $Z_L=jL\omega$ , $Z_c=\dfrac{1}{jC\omega}$ and $Z_R=R$ .
- The association of impedances is identical to that of resistances.