National Higher School of Autonomous Systems Technology

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Vibrations and waves

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Waves

Chapter 1: General Aspects of Propagation Phenomena

Chapter 2: Vibrating Strings

Chapter 3: Acoustic Waves in Fluids and Solids

Chapter 4: Electromagnetic Waves

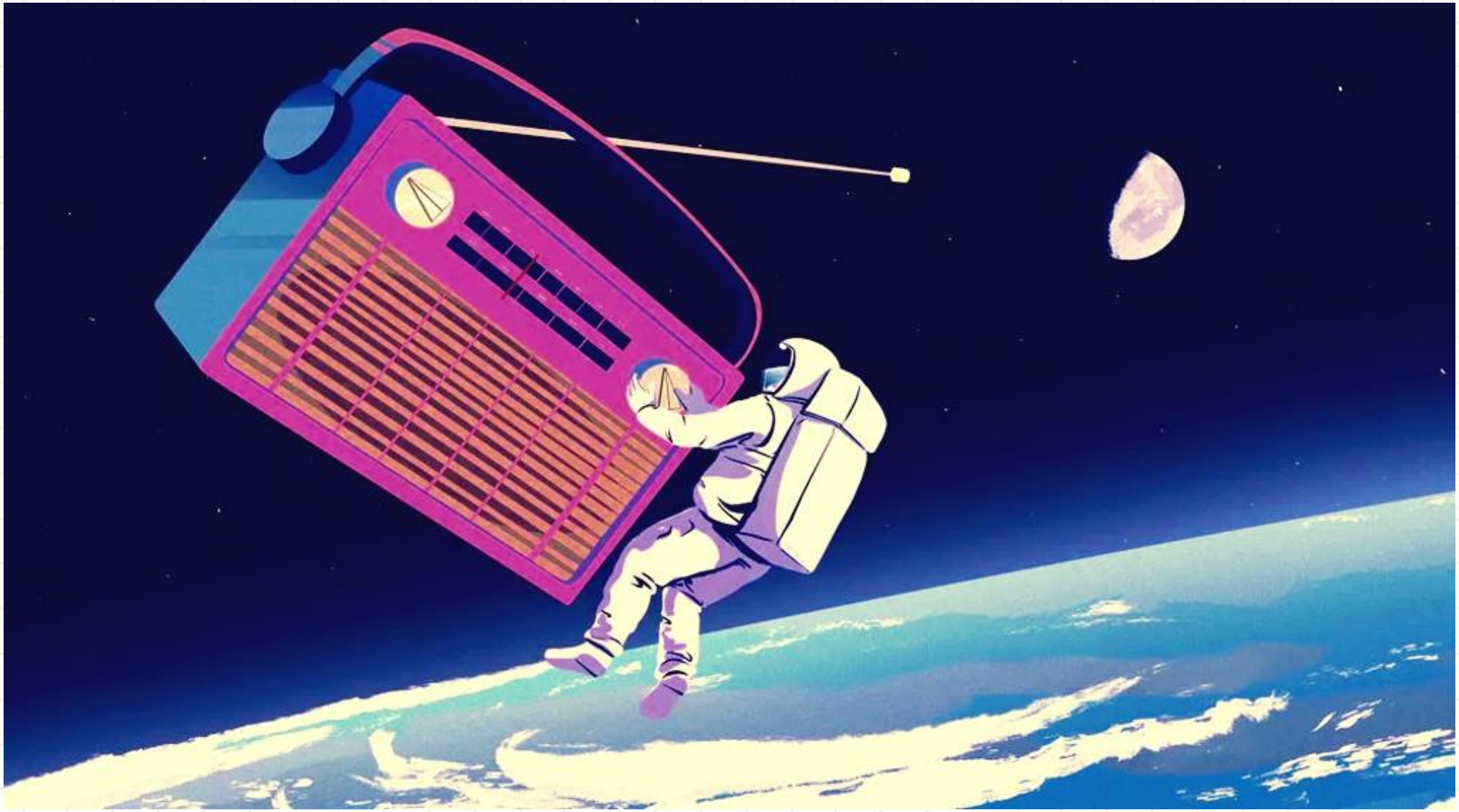
Generalities on Propagation Phenomena

contents

Generalities on Propagation Phenomena

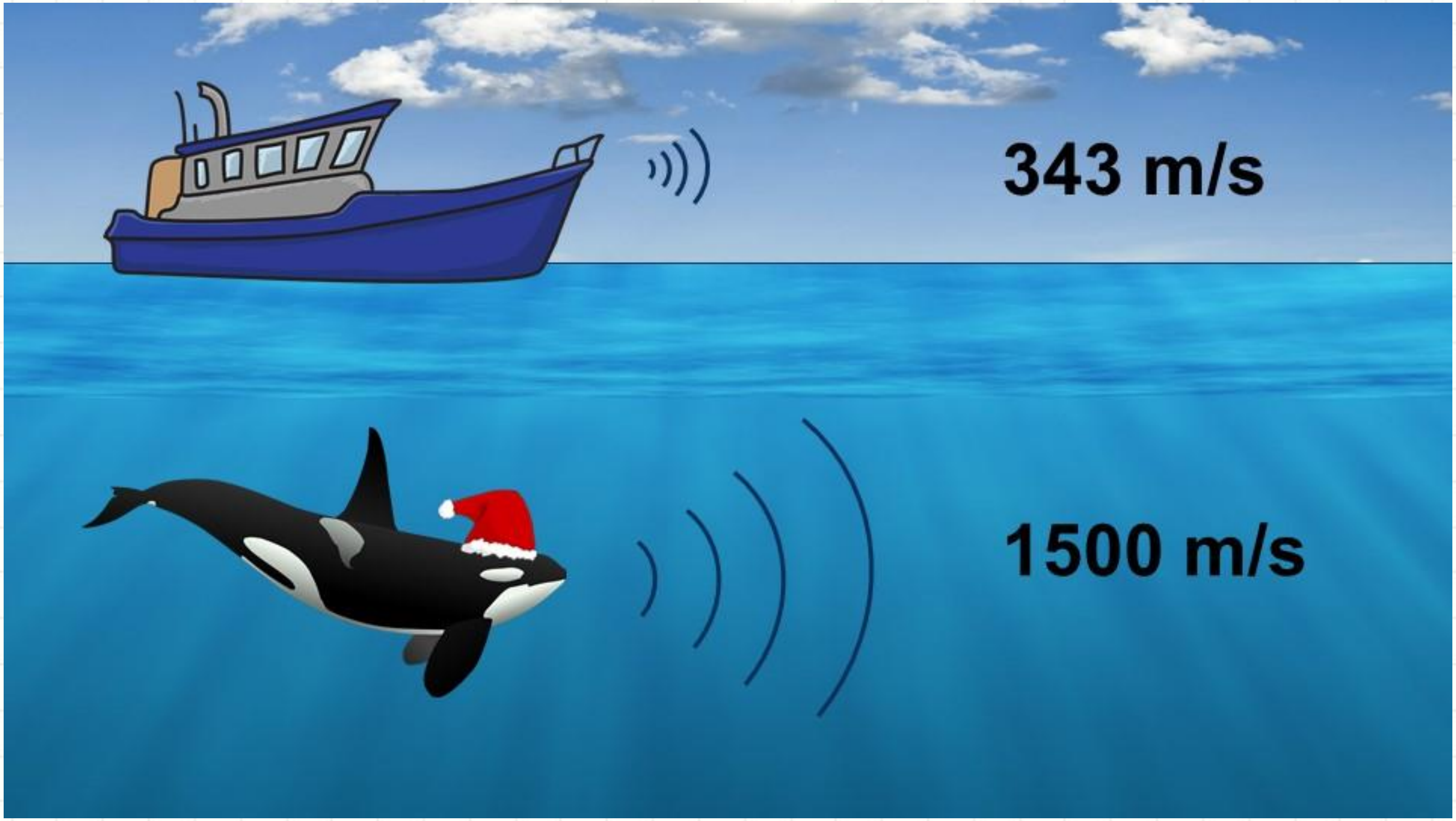
What can waves do?

Why is there no sound in space?



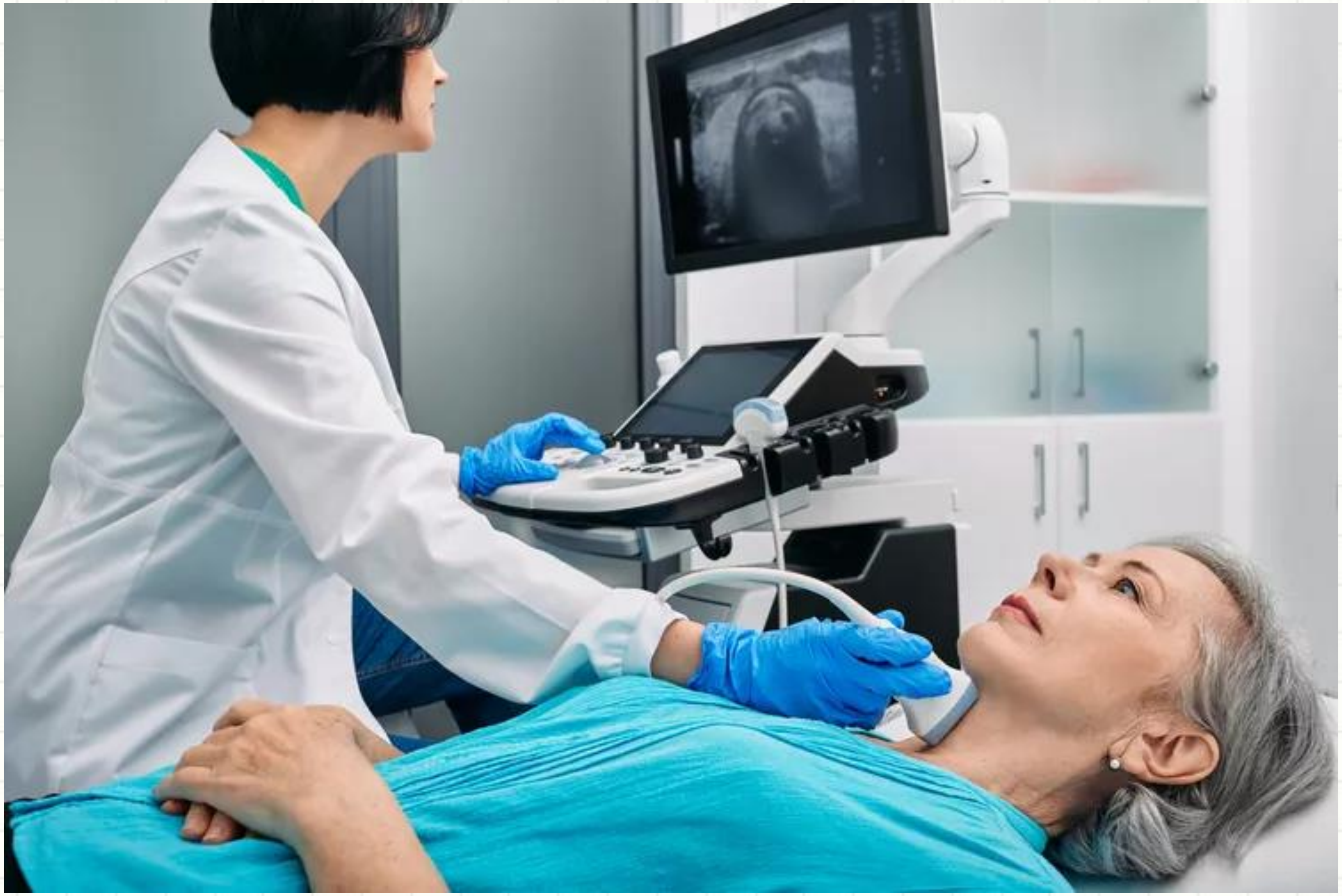
Generalities on Propagation Phenomena

Why are sound waves faster in water than in air?



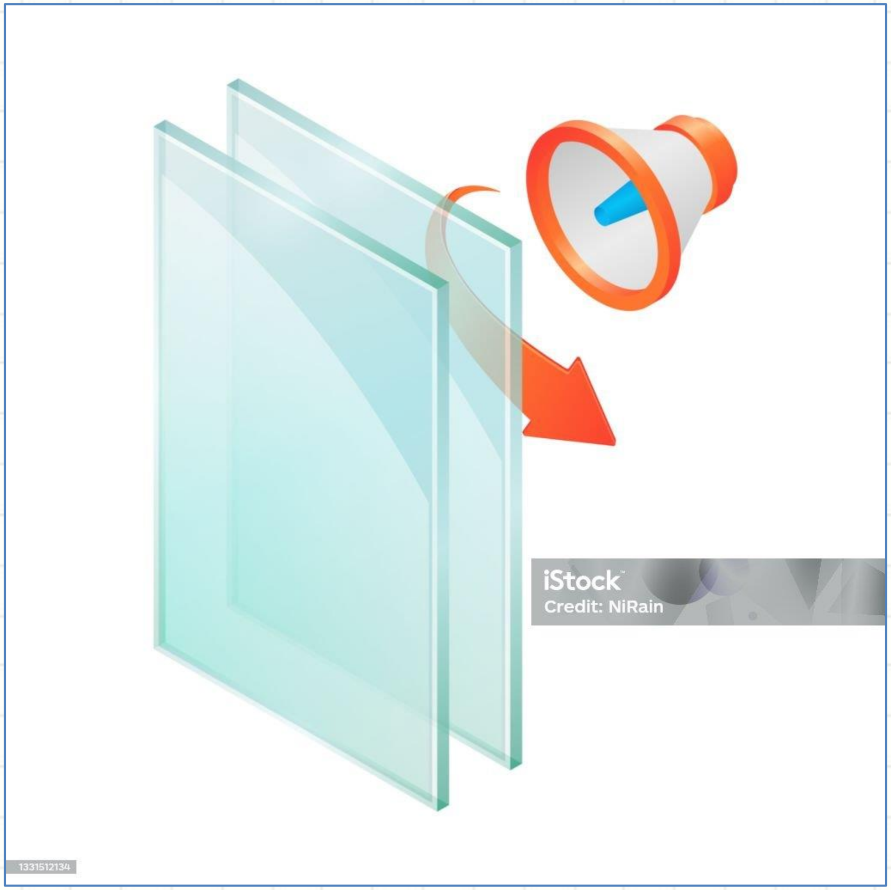
Generalities on Propagation Phenomena

How do doctors use waves to see inside your body? (Ultrasound, X-rays, …)



Generalities on Propagation Phenomena

Why can a light wave pass through glass, but not a sound wave?



Generalities on Propagation Phenomena

Why can you see lightning before hearing thunder?

Generalities on Propagation Phenomena

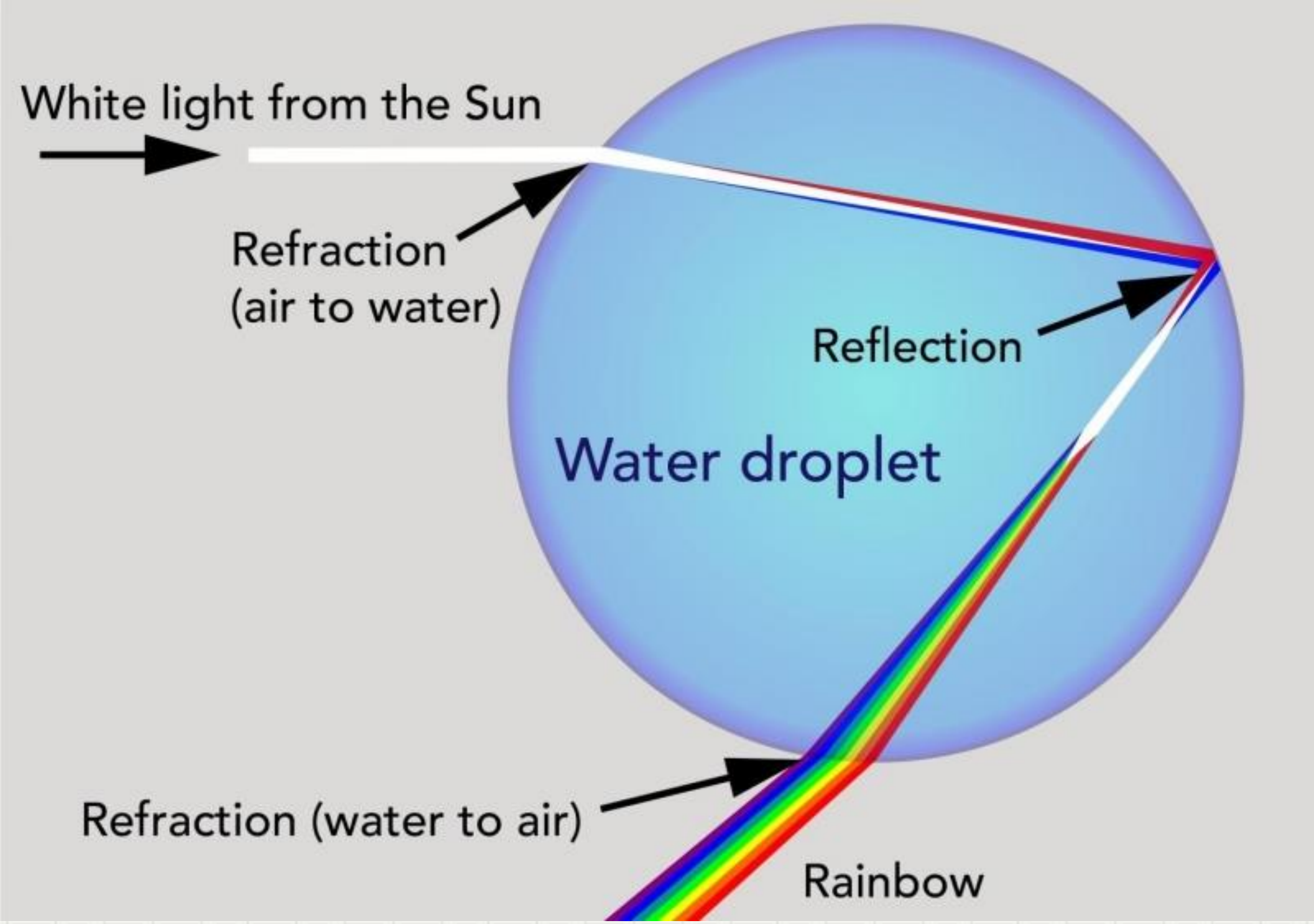
Why does the sound of a truck's horn change as it passes by you?

What can waves do?

Why don't mobile phones work in an elevator?

Generalities on Propagation Phenomena

Why do light waves bend and create rainbows after it rains?



Generalities on Propagation Phenomena

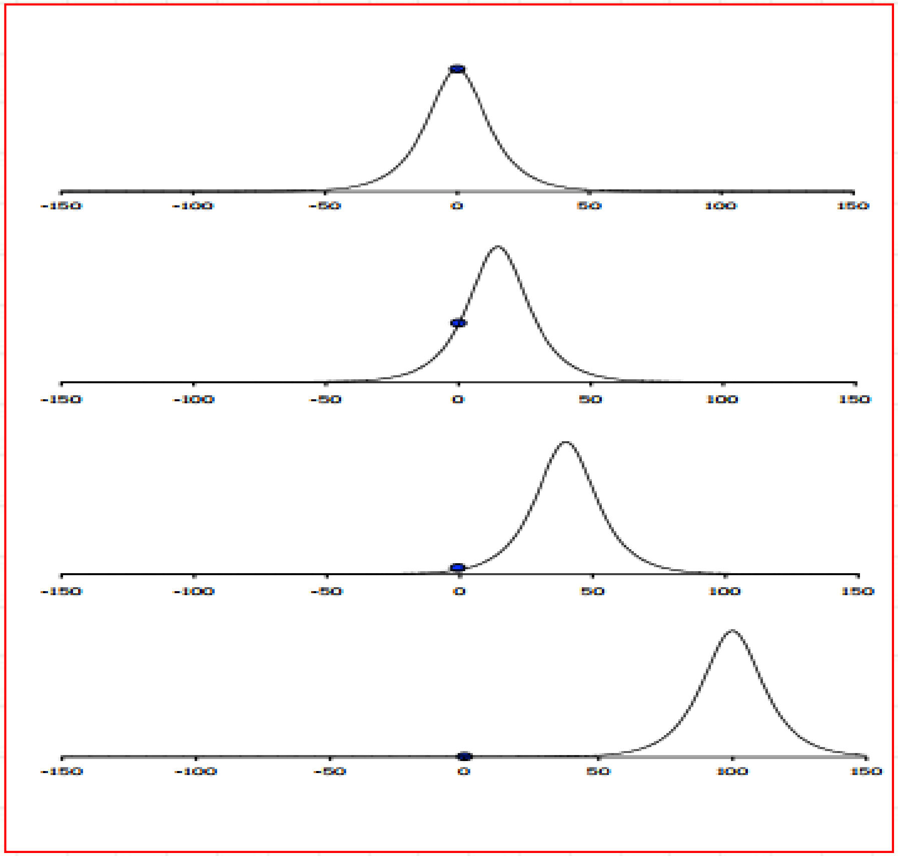
How are ultrasound waves used in medicine to visualize the inside of the body?

Generalities on Propagation Phenomena

How can waves be used to measure distances, as in the case of radar or sonar?

Generalities on Propagation Phenomena

Definition of a wave



A wave is a disturbance that transfers energy from one place to another. A wave does not carry matter with it; it only causes the matter to move as the wave passes through it

Generalities on Propagation Phenomena

Definition of a wave

<u>Note:</u> Standing waves do not transfer energy. The energy remains localized in specific areas of the vibration (nodes and antinodes). We will study this later.

Generalities on Propagation Phenomena

3- Types of waves

1-Based on the need for a medium:

a- **Mechanical Waves:** These waves need a medium to propagate (solid, liquid, or gas).

Examples: Sound waves, seismic waves, water waves…

b- **Electromagnetic Waves:** These waves do not need a medium and can travel through a vacuum.

Examples: Light, radio waves, X-rays, gamma rays…

Generalities on Propagation Phenomena

3- Types of waves

2- Based on the Direction of Particle Motion

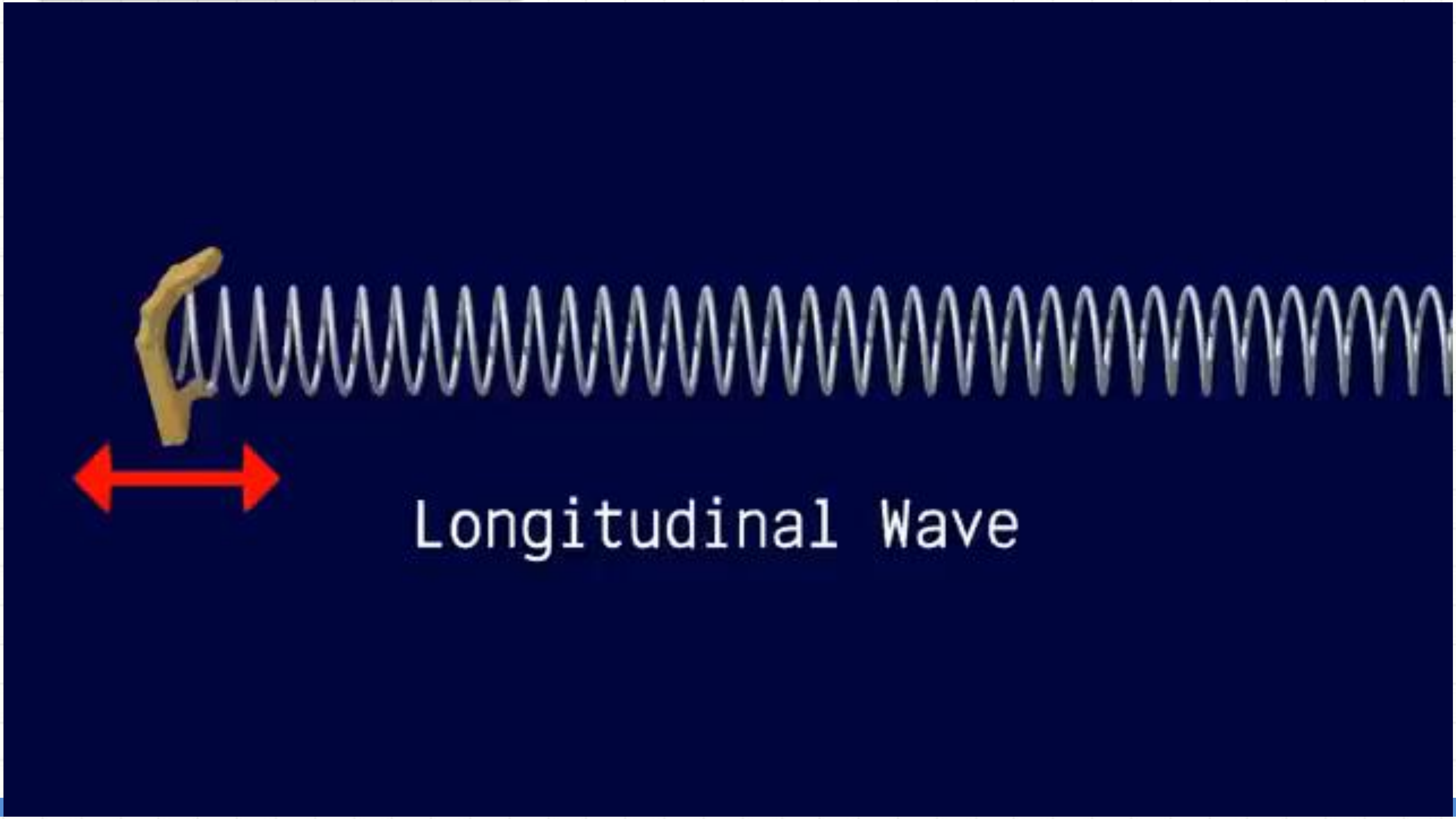
2-a- Transverse Wave: which moves the medium perpendicular to the wave motion

Examples: *water waves, wave in strings, electromagnetic waves*

2-b- Longitudinal Wave: *which moves the medium* **parallel** *to the wave motion.*

Examples: *sound waves, wave in pipes, waves in slinky springs.*

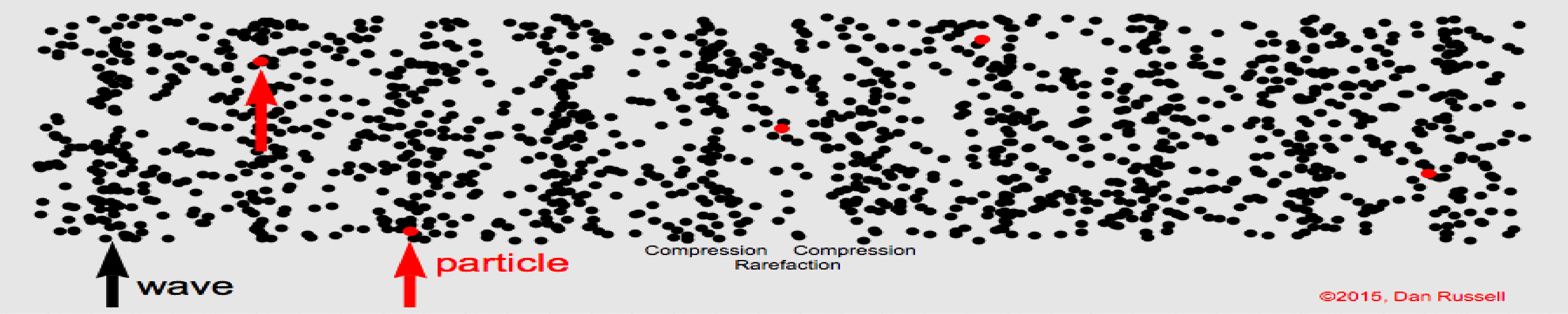
3- Types of waves



Generalities on Propagation Phenomena

3- Types of waves

2-b- Longitudinal Wave



Generalities on Propagation Phenomena

3- Types of waves

Note:

The waves emitted by an earthquake are of three types:

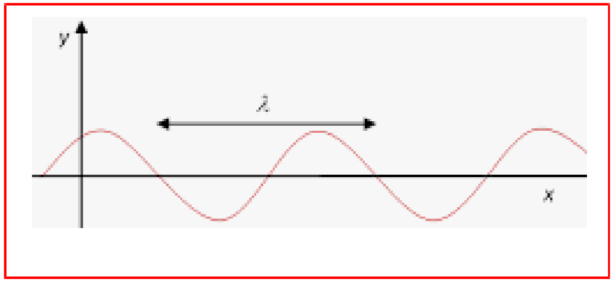
- **P waves** are longitudinal compression vibrations; they are the fastest, with propagation speeds reaching 3.5 to 14 km/s, depending on the type of rock and the depth of propagation.
- **S waves** are transverse shear waves, perpendicular to the direction of propagation; they are slower than P waves (the speed of P waves is approximately 1.7 times that of S waves).
- **L waves** are surface waves; they are even slower than S waves.

3- Types of waves

3- Based on the dimension of the medium

3-1 One-dimensional waves: Waves that propagate in only one direction of space (one dimension).

- **Examples**: A wave on a stretched string, Sound waves in a narrow tube.



Generalities on Propagation Phenomena

3- Types of waves

3- Based on the dimension of the medium

3-2 Two-dimensional waves: Waves that propagate in a plane (two dimensions).

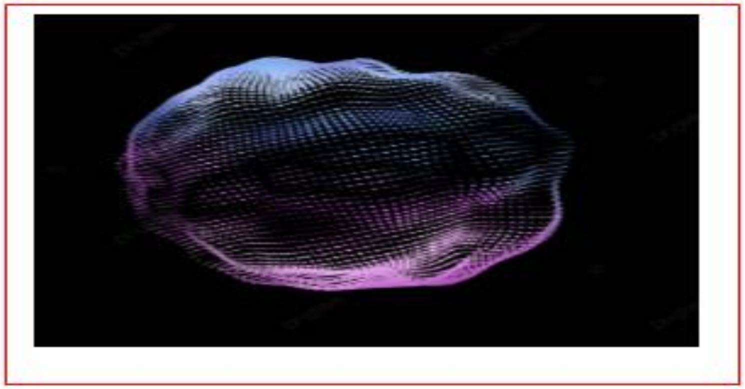
- **Examples**: Waves on the surface of water, Surface seismic waves.

3- Types of waves

3- Based on the dimension of the medium

3-3 Three-dimensional waves: Waves that propagate in all directions of space (three dimensions).

- **Examples**: Sound waves in air, Electromagnetic waves (light, radio waves, microwaves).



Sound Wave Three Dimensional

Generalities on Propagation Phenomena

3- Types of waves

4-Based on Energy Transfer:

4-1 Traveling Waves: Waves in which energy is transferred continuously from one point to another in the direction of wave propagation.

Examples

- Sunlight traveling through space
- Transmission of signals from a radio station
- Waves detected by a seismograph far from the earthquake's epicenter

Generalities on Propagation Phenomena

3- Types of waves

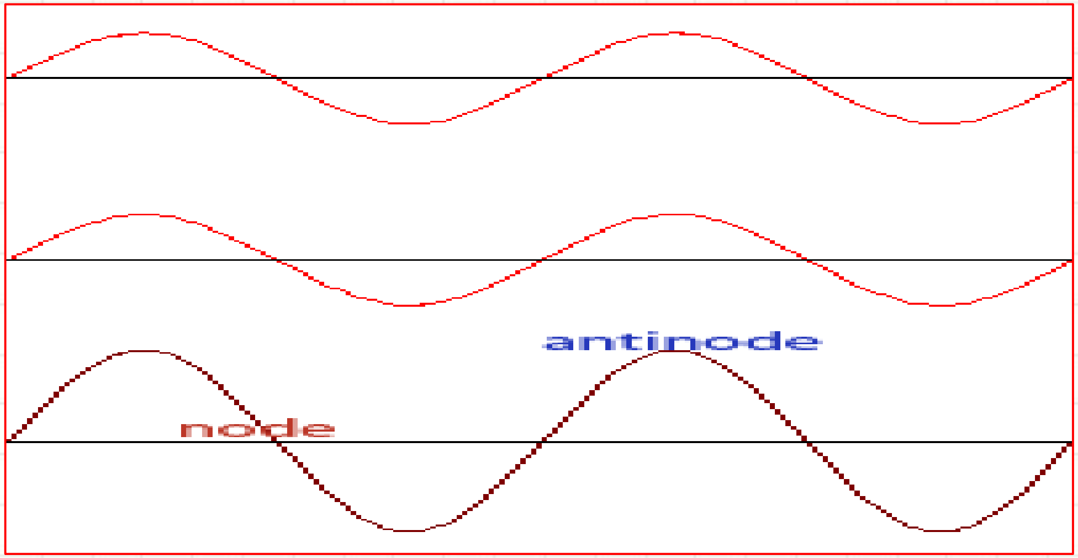
4-Based on Energy Transfer:

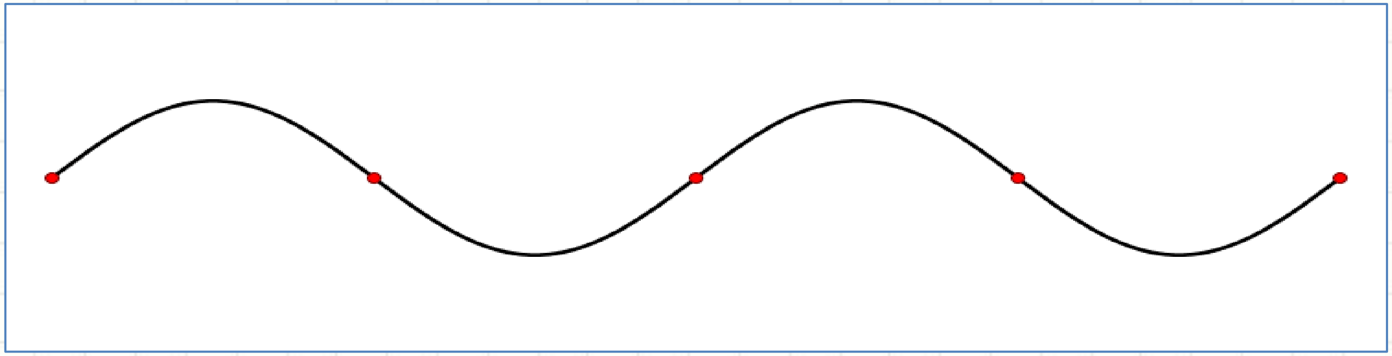
4-2 Standing Waves: Standing waves are waves that appear stationary, resulting from the interference of two waves traveling in opposite directions with the same frequency and amplitude. They are characterized by nodes (points of no motion) and antinodes (points of maximum motion). The energy does not propagate.

Examples

- Microwave oven cavity
- Laser cavity

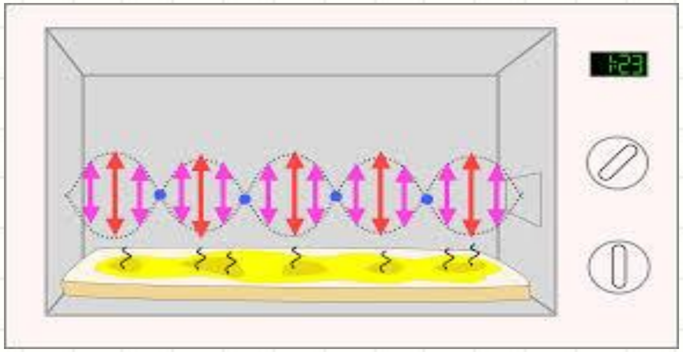
Generalities on Propagation Phenomena

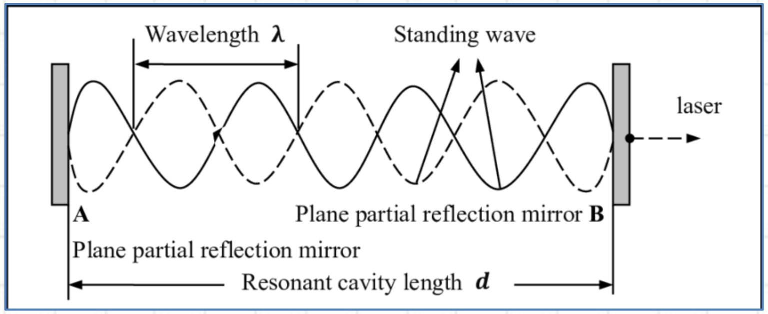


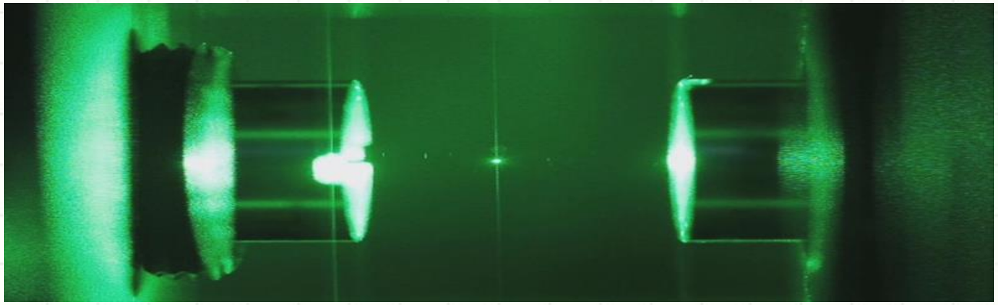


3- Types of waves

4-2 Standing Waves:







Generalities on Propagation Phenomena

3- Types of waves

4-Based on Energy Transfer:

Difference between Standing and Traveling Waves

Standing Waves

- The wave will not move
- It is a combination of two waves moving in opposite directions.
- Stores energy.
- Consists of nodes and antinodes.

Traveling Waves

- The wave will move.
- It consists of one wave moving in one direction.
- Transmits energy.
- All particles are vibrating.

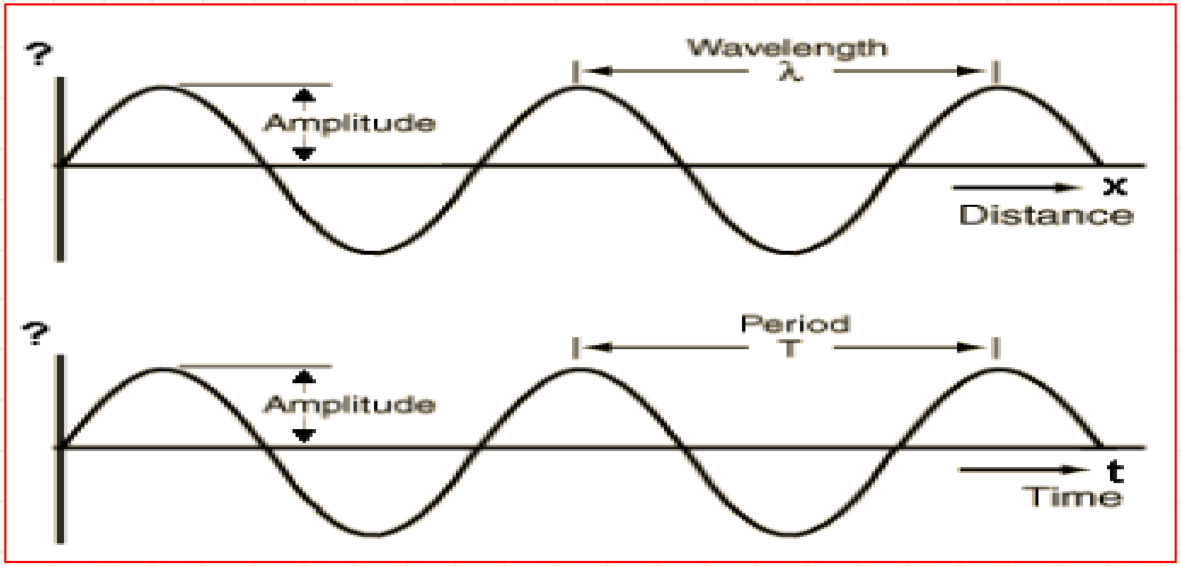
Generalities on Propagation Phenomena

4- Periodic waves

Double periodicity

✓**Spatial periodicity**

➤**Temporal periodicity**



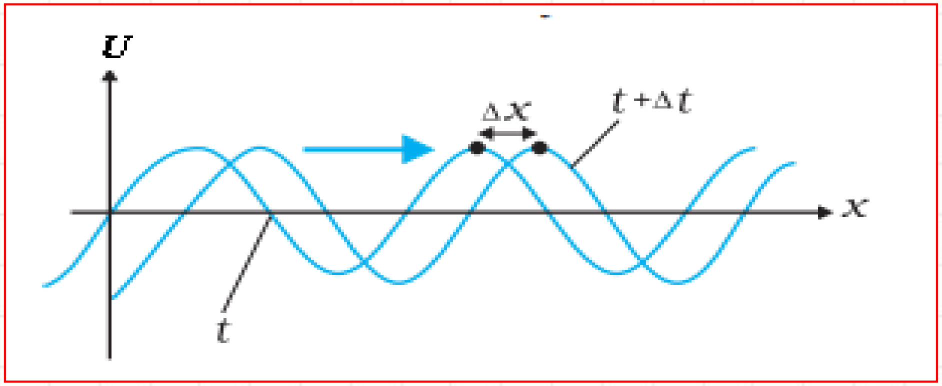
✓***Wavelength* λ :** The distance over which a wave repeats

✓***Period T***: The time taken by the wave to complete one cycle.

Generalities on Propagation Phenomena

4- Periodic waves

✓ ***Wave velocity* v :** is the speed at which the disturbance moves



➢ The shape of the wave at two instants of time, which differ by a small time internal Δt, the entire wave pattern is seen to shift to the right by a distance Δx. The speed of the wave is then $\boldsymbol{v = \frac{\Delta x}{\Delta t}}$

Generalities on Propagation Phenomena

4- Periodic waves

✓ Wave velocity v

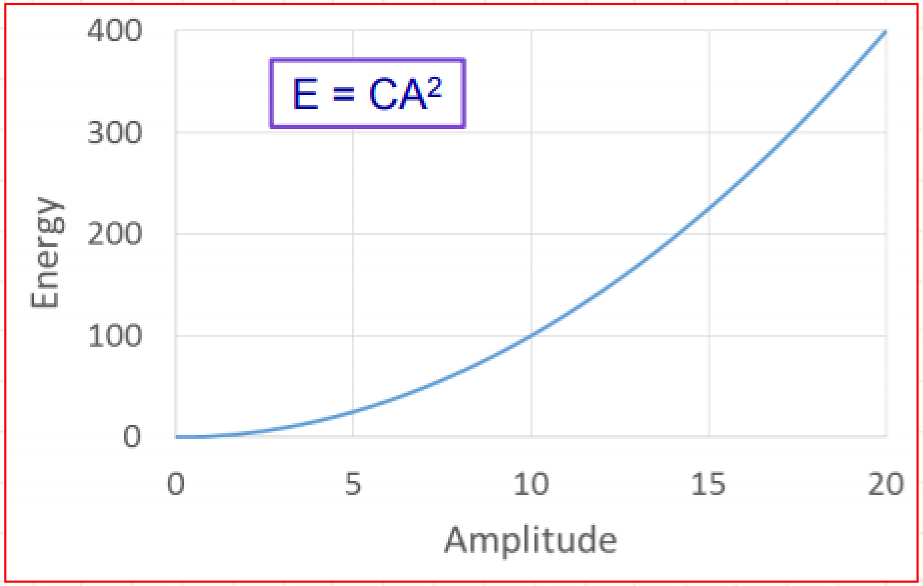
Note:

The wave speed is constant in a homogeneous, isotropic, and non-dispersive medium. It depends on the inertia (density), stiffness (Rigidity), and temperature of the medium. It varies from one medium to another.

✓ *The speed of a wave directly depends on the physical properties of the medium.*

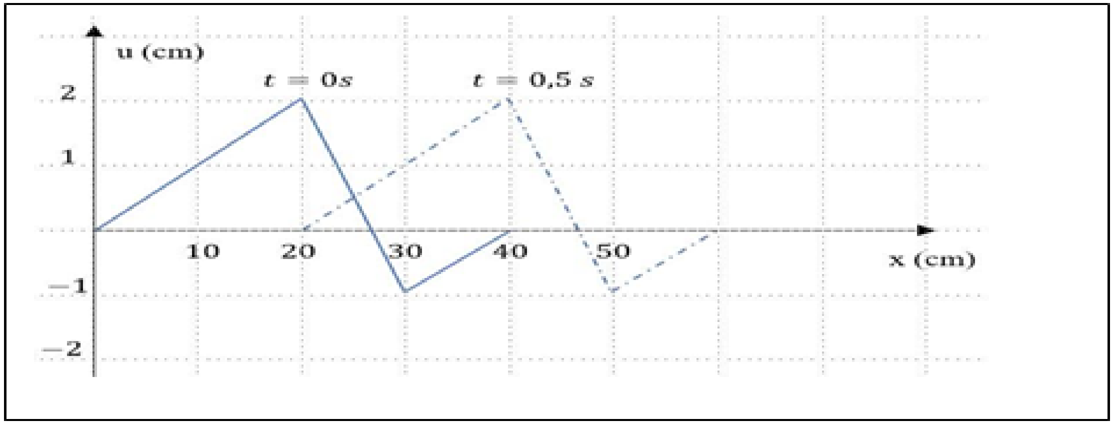
4- Periodic waves

✓ *Energy of a wave E:* is proportional to the square of its amplitude. Mathematically $E = C\ A^2$



Exercise

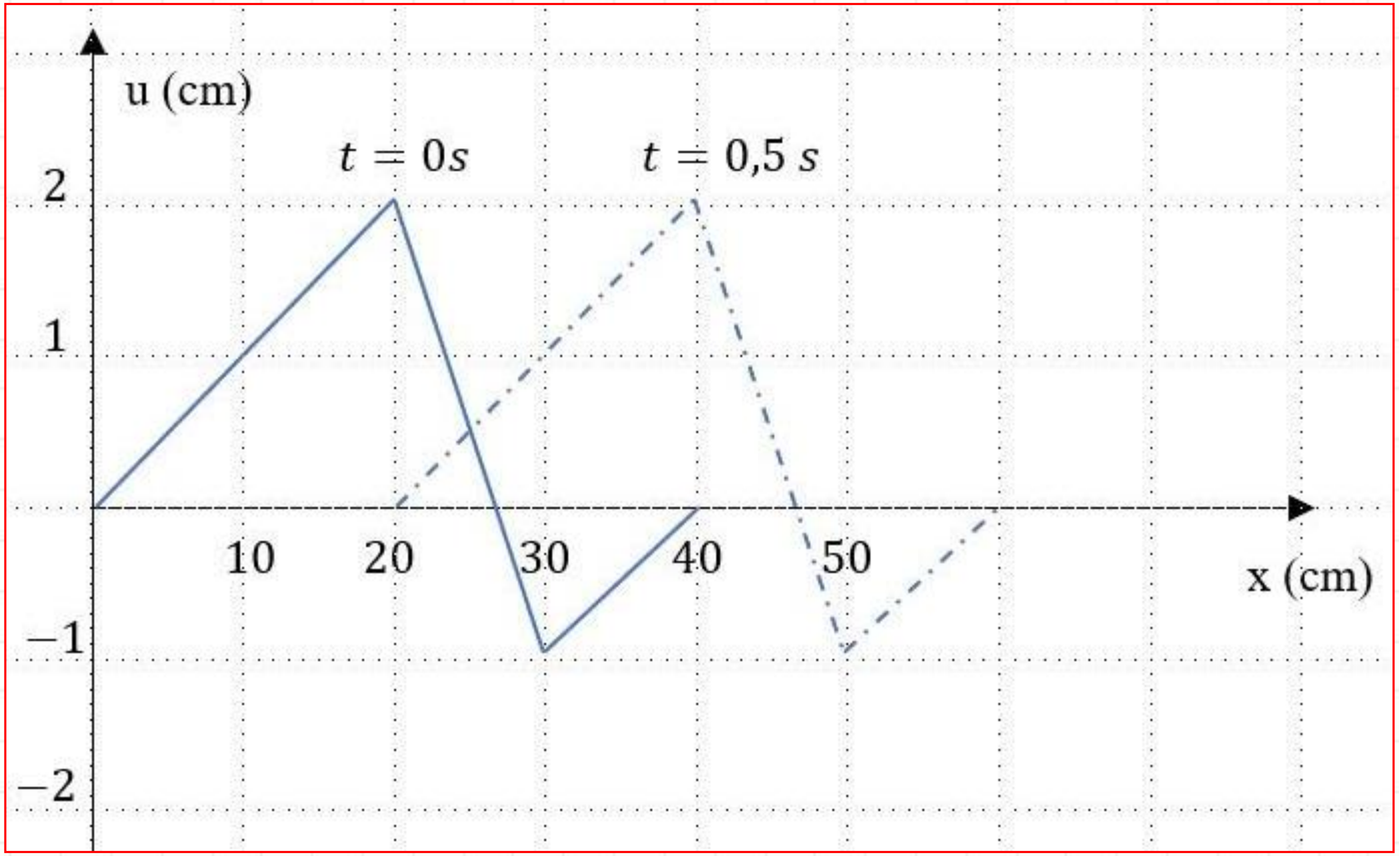
A string is excited by a transverse disturbance that propagates along Ox at the speed v. The shapes of the string at $t=0s$ and $t=0.5s$ are given in the figures below.



1. Determine the propagation speed of the wave.
2. Represent as a function of time the displacement $u(x_A, t)$ of point A, as well as the velocity of displacement $u'(x_A, t)$ of point A, where $x_A = 80\ cm$.

Note: It is recommended to solve the problem graphically.

Generalities on Propagation Phenomena

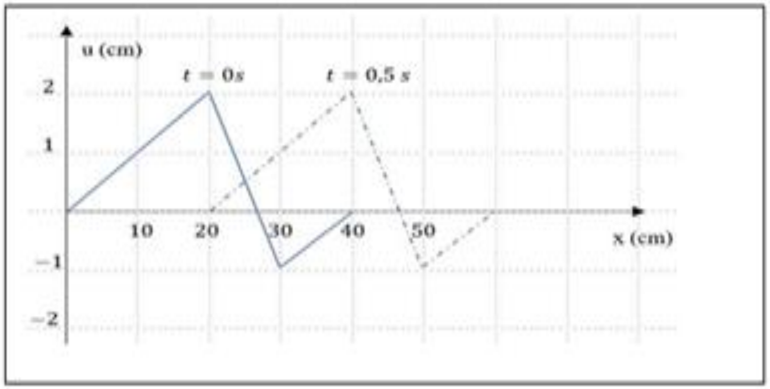


✓Solution

1. Determine the propagation speed of the wave.

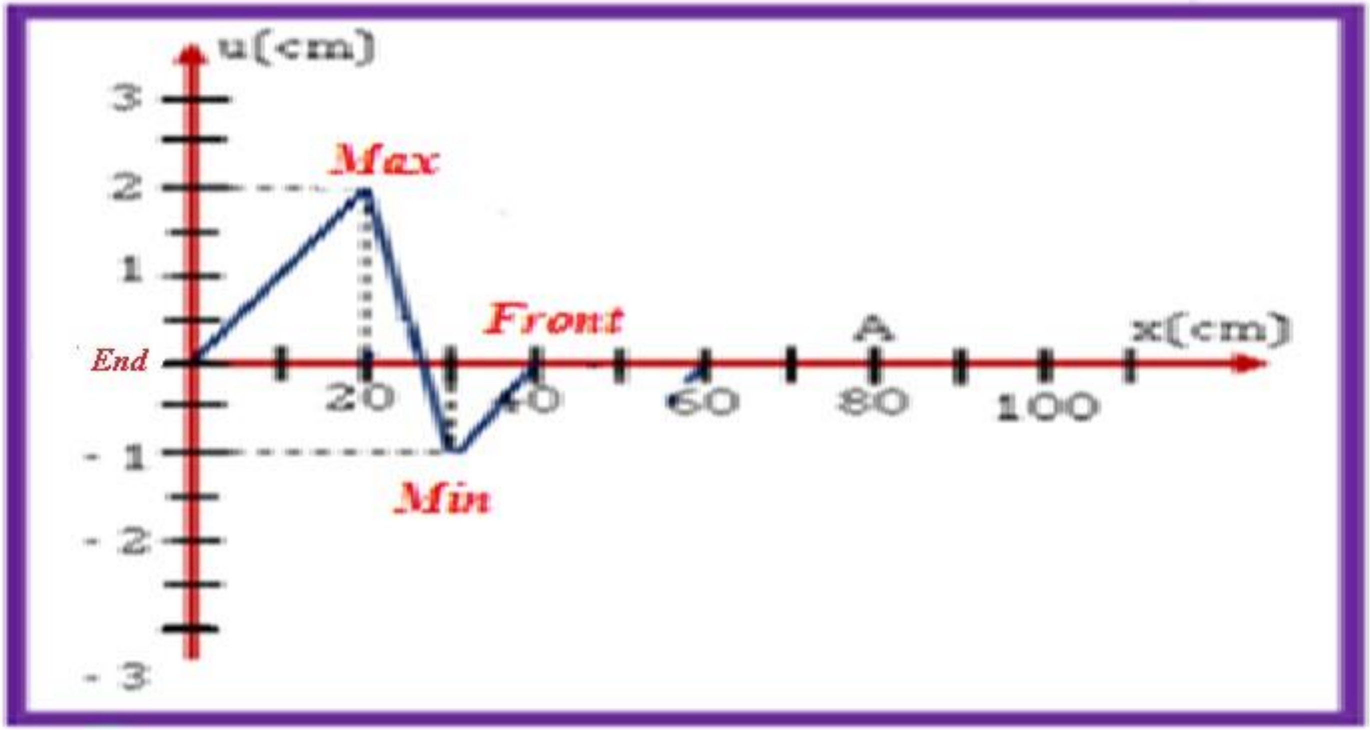
$$v = \frac{\Delta x}{\Delta t} \Rightarrow v = \frac{20}{0.5} = 40 cm/s$$

$$v = 0.4\ m/s$$



✓Solution

2. Represent as a function of time the displacement $u(x_A, t)$ of point A, as well as the velocity of displacement $u'(x_A, t)$ of point A, where $x_A = 80\ cm$.



Generalities on Propagation Phenomena

✓Solution

2. Represent as a function of time the displacement $u(x_A,t)$ of point A, as well as the velocity of displacement $u'(x_A,t)$ of point A, where $x_A = 80\ cm$.

$$t_F = \frac{x_A - x_F}{v} \Rightarrow \boxed{t_F = \frac{80-40}{40} = 1s}$$

$$t_{Max} = \frac{x_A - x_{Max}}{v} \Rightarrow t_{Max} = \frac{80-20}{40} = 1.5\ s$$

$$t_{Min} = \frac{x_A - x_{Min}}{v} \Rightarrow \boxed{t_{Min} = \frac{80-30}{40} = 1,25s}$$

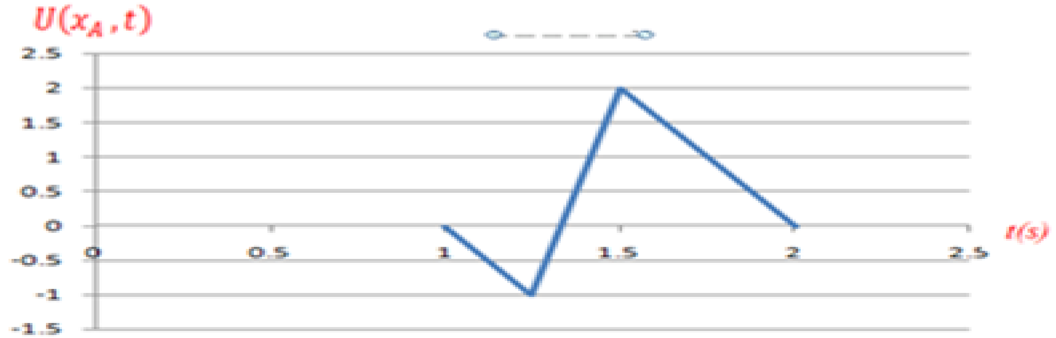
$$t_E = \frac{x_A - x_0}{v} \Rightarrow t_E = \frac{80-0}{40} = 2\ s$$

Generalities on Propagation Phenomena

✓Solution

2. Represent as a function of time the displacement $u(x_A, t)$ of point A, as well as the velocity of displacement $u'(x_A, t)$ of point A, where $x_A = 80\ cm$.

$t(s)$	$t_F = 1$	$t_{Min} = 1.25$	$t_{Max} = 1.5$	$t_E = 2$
$U(x_A, t)$	0	-1	2	0



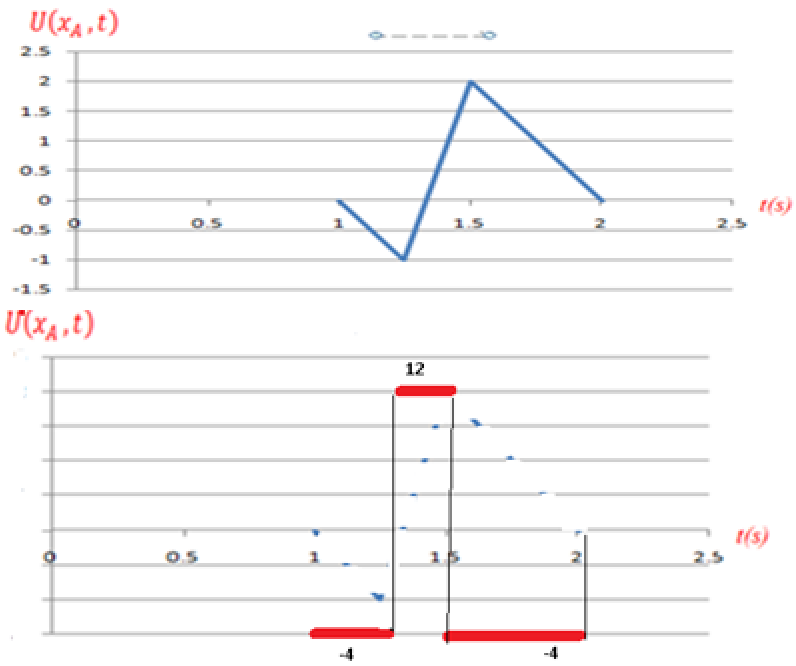
✓Solution

2. Represent as a function of time the displacement $u(x_A,t)$ of point A, as well as the velocity of displacement $u'(x_A,t)$ of point A, where $x_A = 80\ cm$.

$t \in [1,1.25]$ $v = -4\ cm/s$

$t \in [1.25,1.5]$ $v = 12\ cm/s$

$t \in [1.5,2]$ $v = -4\ cm/s$

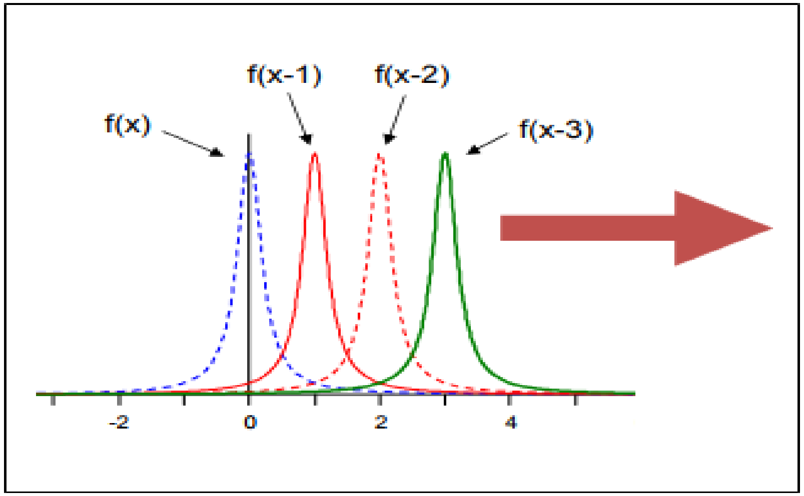


Generalities on Propagation Phenomena

4- Periodic waves

Mathematical description of a wave

- In the mathematical sense, a wave is any function that moves.
- To displace any function $f(x)$ to the right, just change its argument from x to $x - x_0$, where x_0 is a positive number.



Generalities on Propagation Phenomena

4- Periodic waves

Mathematical description of a wave

- If we let $x_0 = v\,t$, where v is positive and t is time, then the displacement increases with increasing time.
- So the wave function for a wave traveling in the $+x$ *direction* has the form:

$$u(x,t) = f(x - vt) \qquad \text{(Left–to–right)}$$

- Similarly the wave function for a wave traveling in the $-x$ *direction* has the form:

$$u(x,t) = f(x + vt) \qquad \text{(Right-to-left)}$$

Generalities on Propagation Phenomena

4- Periodic waves

✓ **For sinusoidal progressive wave**

➢ The wave carried by a **sinusoidal progressive wave** propagating in the direction of *increasing x (+x direction)* is expressed as:

$$u(x,t) = A\cos\,(wt - kx + \varphi)$$

Or

$$u(x,t) = A\sin\,(wt - kx + \varphi)$$

Or $$u(x,t) = Ae^{j(wt-kx+\varphi)}$$

Generalities on Propagation Phenomena

4- Periodic waves

Mathematical description of a wave

- We introduce the concept of a wave function $u(\vec{r},t)$ in order to quantify the properties of wave motion. Once we know the wave function, we can calculate the velocity, acceleration, and the amplitude of the wave at all times.

A sinusoidal travelling wave is described by:

$$u(\vec{r},t) = A\cos\left(wt - \vec{k}.\vec{r}\right)$$

$\vec{k}\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$ **Wave vector** $\vec{k}$: Describes the direction of wave

$\vec{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *Position vector*

Generalities on Propagation Phenomena

4- Periodic waves

Mathematical description of a wave

1D: $u(x,t) = A\cos\,(wt - kx)$

2D: $u(x,t) = A\cos\,(wt - k_x x - k_y y)$

3D: $u(x,t) = A\cos\,(wt - k_x x - k_y y - k_z z)$

u(x, t) : Displacement as a function of position x and time t

A: Amplitude of a wave (m)

ω : Angular frequency of the wave (rad)

k : Wave number (rad/m)

4- Periodic waves

Mathematical description of a wave

- Relationship between k, v and λ

$$v = \frac{\lambda}{T} \quad , \quad v = \lambda . f$$

$$k = \frac{2\pi}{\lambda} \quad , \quad |\vec{k}| = k$$

Generalities on Propagation Phenomena

4- Periodic waves

Mathematical description of a wave

<u>*Note*</u>

- Wave propagating in the direction of *increasing x*

1- If $u(0,t) = f(t)$

To find the expression for $u(x,t)$, substitute :

t with $t - \frac{x}{v}$

2- If $u(x,0) = f(x)$

To find the expression for $u(x,t)$, substitute :

x with $x - vt$

4- Periodic waves

Note

- Wave propagating in the direction of *decreasing x*

1- If $u(0,t) = f(t)$

$To \quad find \quad the\, expression\, for \quad u(x,t)$ substitute :

t with $t + \frac{x}{v}$

2- If $u(x,0) = f(x)$

$To \quad find \quad the\, expression\, for \quad u(x,t)$ substitute :

x with $x + vt$

4- Periodic waves

Example1:

1-Find the expression of $u(x,t)$ *for wave* propagating in the direction of *decreasing* x, if $u(x,0) = 2cos3x$

Solution

➢ $\boldsymbol{u(x,t)}$?

$$x \quad \text{with} \quad x + vt \quad , \quad (\text{wave in } decreasing\ x)$$

$$u(x,t) = 2\cos 3(x + vt)$$

Generalities on Propagation Phenomena

4- Periodic waves

Example1 :

2-*Find the expression of* $y(x,t)$ *for wave* propagating in the direction of *increasing* x , if $y(0,t) = 2e^{5t^2}$

Solution

➢ $\boldsymbol{u(x,t)}$?

t with $t - \frac{x}{v}$, (wave in *increasing* x)

$$y(x,t) = 2e^{5(t-\frac{x}{v})^2}$$

Generalities on Propagation Phenomena

4- Periodic waves

Example2:

A certain transverse wave is described by:

$$y(x,t) = 6.50 \ \ cos \ \ 2\pi\left(\frac{t}{0.036\ s} + \frac{x}{28\ cm}\right) \qquad in\ mm$$

Determine the wave's a) amplitude; b) wavelength;

c) frequency; d) speed of propagation; e) number k ;

f) direction of propagation.

4- Periodic waves

Solution

$$y(x,t) = A\cos(wt - kx)$$

-Amplitude: $A = 6.50\ mm$; $A = 6.50\ .10^{-3}$ m

-Frequency: $w = \frac{2\pi}{0.036}\ rad$; $f = \frac{w}{2\pi}\ Hz$

- Wave number: $k = \frac{2\pi}{28.10^{-2}}$, $k = ?\ rad/m$

- Wavelength: $k = \frac{2\pi}{\lambda}$; $\boldsymbol{\lambda = \frac{2\pi}{k}\ m}$

4- Periodic waves

Solution

-Speed of propagation: $v = \lambda . f$

or $k = \frac{w}{v}$ $v = \frac{w}{k}$ m/s

-Direction of propagation $\vec{k}\binom{k}{0}$ $\vec{k} = k\,\vec{\imath}$

✓ The wave propagates in the direction of increasing x $(+x)$.

Generalities on Propagation Phenomena

4- Periodic waves

The Principle of Superposition

A simple property of all of the waves in nature that we are likely to study is that they add together . More precisely, if one physical disturbance of a medium generates the wave $u_1(x,t)$ and another disturbance generates the wave $u_2(x,t)$ then if both effects act at the same time, the resultant wave will be:

$$\boldsymbol{u(x,t) = u_1(x,t) + u_2(x,t)}$$

- In this and the next section we give the results for the superposition of two harmonic waves which differ in only one respect; this allows us to understand the importance of the proprieties of wave play in the combining of waves.

The Principle of Superposition

a-Standing Waves :

We consider the result of the addition of two harmonic waves which have the same speed, frequency and amplitude but for opposite directions of propagation (In this case the phase constants for each wave won't matter.) We will add:

$$u(x,t) = A\sin(wt - kx) + A\sin(wt + kx)$$

$$\text{Using } \sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$

$$\Rightarrow u(x,t) = 2A\cos kx\,.\sin wt$$

$$\Rightarrow u(x,t) = f(x).\,g(t) \qquad \begin{cases} f(x) = 2A\cos kx \\ g(t) = \sin wt \end{cases}$$

The Principle of Superposition

a-Standing Waves :

- The resultant wave is a very interesting function of x and t, though it is not a travelling wave since it is not of the form $f(wt \mp kx)$. It is a sinusoidal function of the coordinate x, multiplied by a modulating factor $cos(\omega t)$. this wave of the form $u(x,t) = f(x).g(t)$ is called
- For this wave given there are points where there is no displacement:

$$cos(kx) = 0$$

$$cos(kx) = 0 \quad \Rightarrow \quad kx = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = (2n+1)\frac{\pi}{2k} \quad \Rightarrow \quad x = (2n+1)\frac{\lambda}{4}$$

✓ These points are the nodes of the standing wave pattern.

Chapter 1: Generalities of propagation phenomena

4- Periodic waves

The Principle of Superposition

a-Standing Waves :

- There are also points for which the displacement is a maximum:

$$\cos(kx) = \mp 1$$

$$\cos(kx) = \mp 1 \quad \Rightarrow \quad kx = n\pi$$

$$\Rightarrow \quad x = n\frac{\pi}{k} \quad \Rightarrow \quad x = n\frac{\lambda}{2}$$

- These points are called the antinodes of the standing wave pattern
- Consecutive nodes and antinodes are separated by $\frac{\lambda}{2}$
- A node and the closest antinode are separated by $\frac{\lambda}{4}$

4- Periodic waves

The Principle of Superposition

a-Standing Waves :

Note

The expression of the standing wave depends on the boundary conditions of the medium. For example, in the case of a string fixed at both ends, the boundary conditions impose that the wave amplitude is zero at the attachment points, leading to the formation of nodes at these positions. On the other hand, if one end is free, the standing wave will exhibit an antinode at that location, thereby modifying the natural modes of vibration. (Chapter 2)

- Example of expressions:

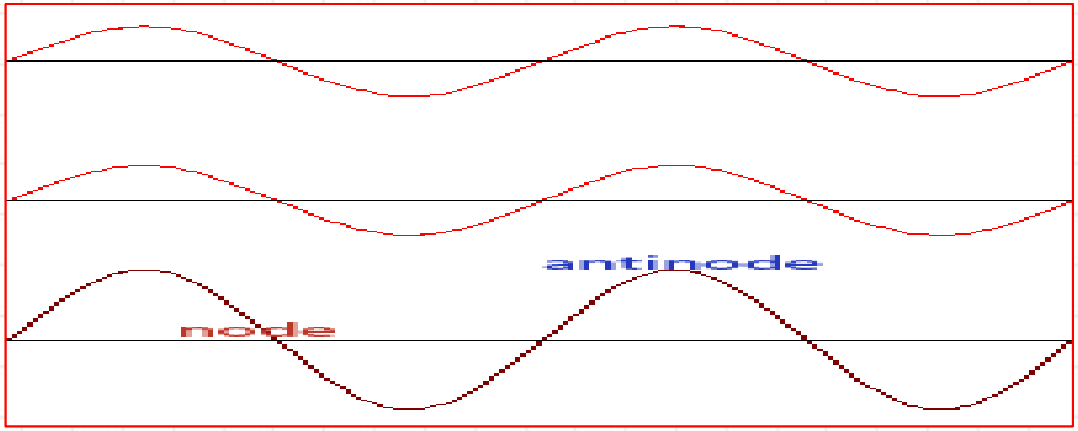
$$u(x,t) = 2A\, coskx\, .\, sin\, wt$$

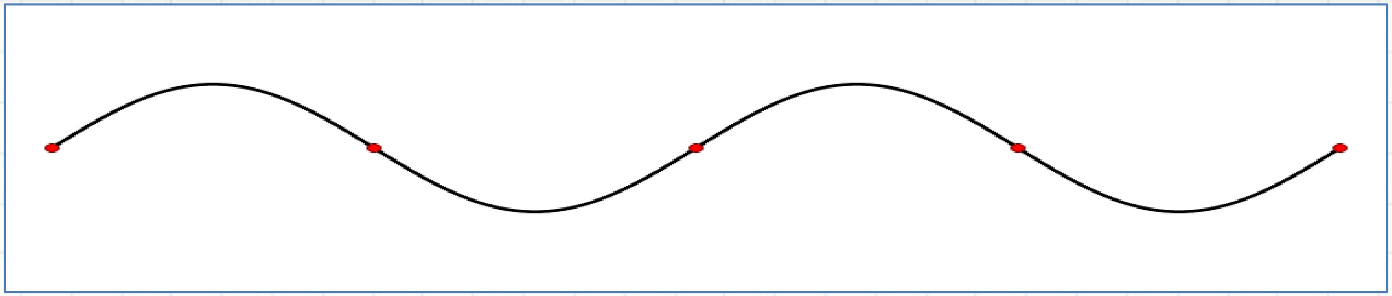
$$u(x,t) = 2A\, sinkx\, .\, cos\, wt$$ or authors' expressions

4- Periodic waves

The Principle of Superposition

a-Standing Waves :

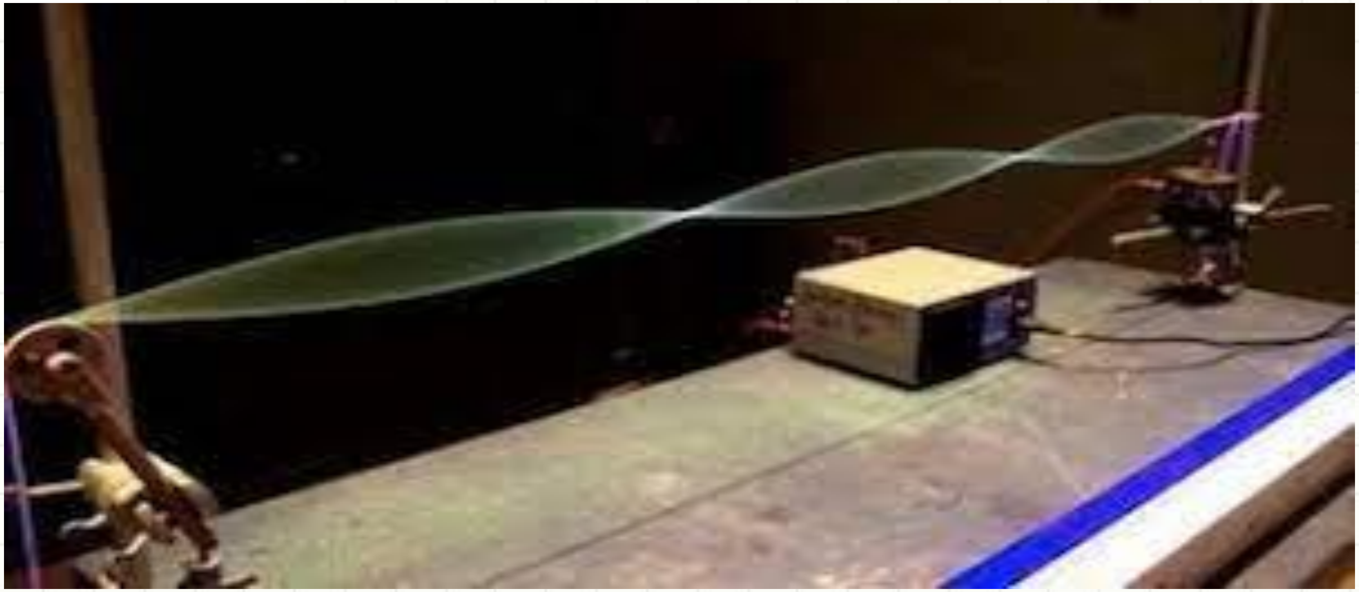




Generalities on Propagation Phenomena

The Principle of Superposition

a-Standing Waves :



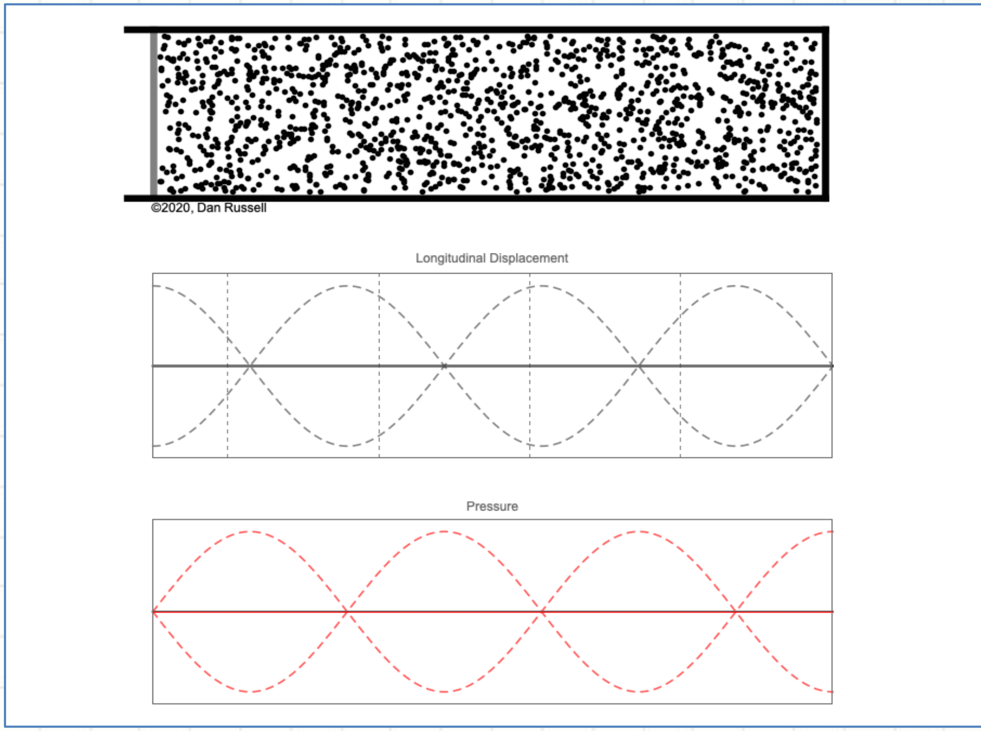
Chapter 2

Generalities on Propagation Phenomena

4- Periodic waves

The Principle of Superposition

a-Standing Waves :



Chapter 4

Chapter 3

4- Periodic waves

The Principle of Superposition

b-Interference of Waves

We take two waves with the same speed, frequency, amplitude and direction of motion but which differ by a phase constant. We will combine the two waves:

$$u(x,t) = A\sin(wt - kx) + A\sin(wt - kx + \varphi)$$

$$Using \ \sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$

$$\Rightarrow \ u(x,t) = 2A \ .\cos\frac{\varphi}{2}.\sin\left(wt - kx + \frac{\varphi}{2}\right)$$

4- Periodic waves

The Principle of Superposition

1-When $\varphi = 0 \quad \Rightarrow \quad \Delta\varphi = 0$

The amplitude is 2A; the waves are said to be completely in phase and that the addition is constructive. A maximum from wave $u_1(x,t)$ coincides with a maximum from wave $u_2(x,t)$ and a bigger wave is the result.

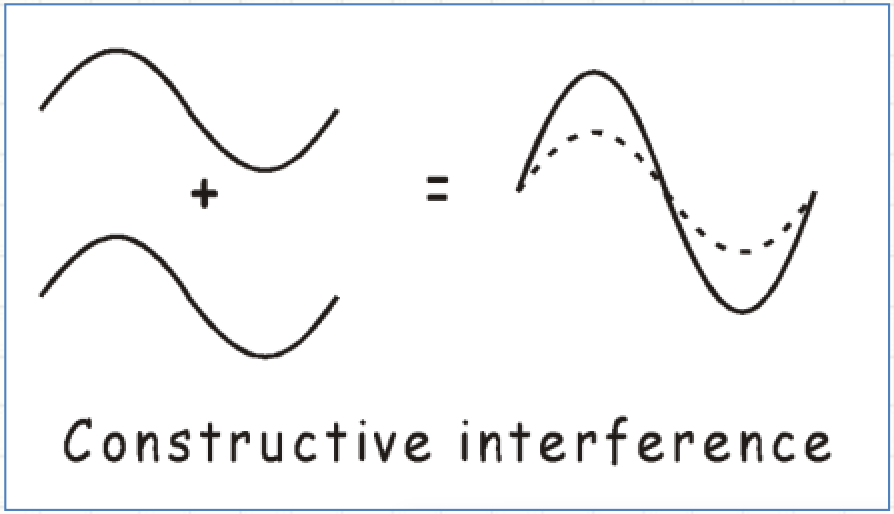
2-When $\varphi = \pi \quad \Rightarrow \quad \Delta\varphi = 0$

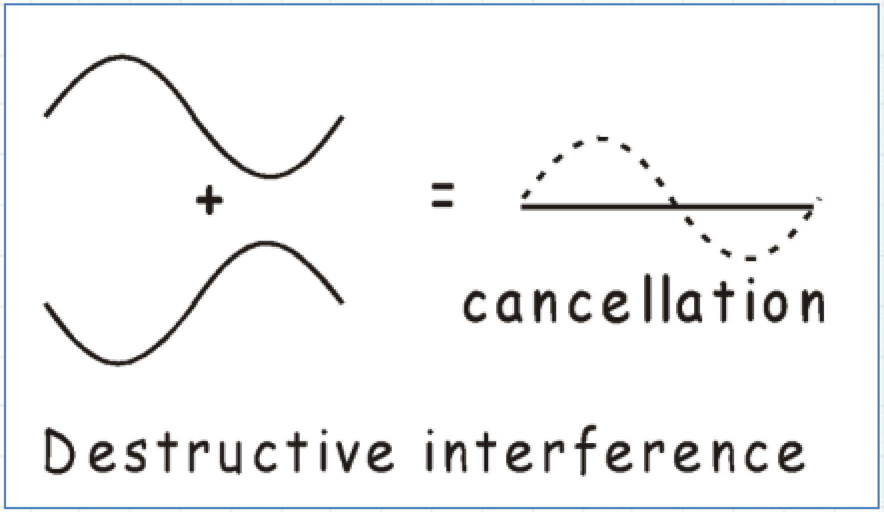
The amplitude is zero and the waves are said to be out of phase (in opposition phase) and that the addition is destructive. Here a maximum from wave $u_1(x,t)$ coincides with a minimum of wave $u_2(x,t)$ and the result is complete cancellation.

4- Periodic waves

The Principle of Superposition

b-Interference of Waves





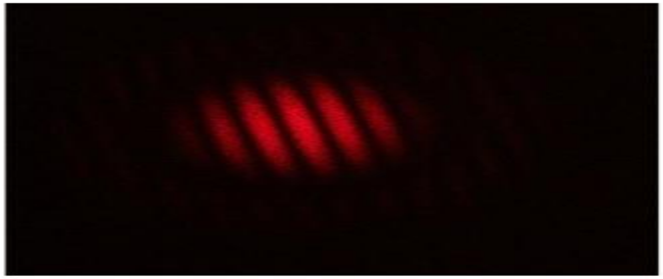
4- Periodic waves

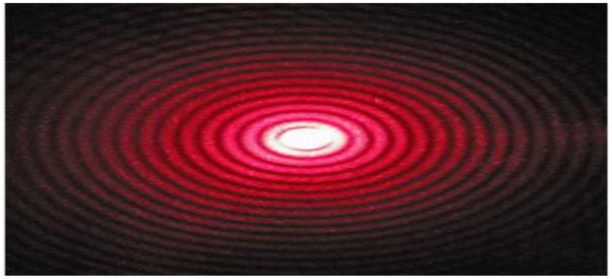
The Principle of Superposition

b-Interference of Waves

➢ **Conditions for Interference**

For two waves of the same nature to interfere effectively, they must be coherent, i.e., come from coherent sources that emit waves with the same frequency and a constant phase difference over time.





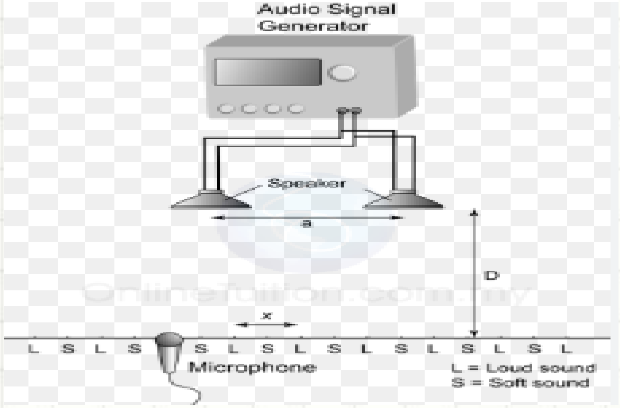
Interference laser

6- Plane wave

The Principle of Superposition

b-Interference of Waves

Interference of water wave



Interference Sound wave

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

The D'Alembert equation, also known as the "1D wave equation," describes waves in various physical systems: longitudinal vibrations of an elastic rod or a spring, transverse vibrations of a stretched string, sound waves in a tube, etc. Here, we focus on the general solutions to this partial differential equation in space and time, which involves a speed v. It is verified that the superposition of an arbitrary signal propagating to the right and another arbitrary signal propagating to the left is a solution to this linear equation.

French scientist Jean-Baptiste le Rond d'Alembert discovered the wave equation in one space dimension.[1]

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

General solutions of D'Alembert equation:

Let $\boldsymbol{u(x,t)}$ be a wave. $\boldsymbol{u(x,t)}$: $\quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$

$\boldsymbol{we\ put:} \begin{cases} \alpha = x - vt \\ \beta = x + vt \end{cases}$; $\boldsymbol{u(\alpha,\beta)}$: $\quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$

- $\frac{\partial^2 u}{\partial x^2} = ?$

$$\frac{\partial u(\alpha,\beta)}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} \quad \Rightarrow \quad \frac{\partial u(\alpha,\beta)}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}\right)$$

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

General solutions of D'Alembert equation:

- Then, by proceeding similarly :

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial \alpha}\left(\left(\frac{\partial u}{\partial \alpha}+\frac{\partial u}{\partial \beta}\right)\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\left(\frac{\partial u}{\partial \alpha}+\frac{\partial u}{\partial \beta}\right)\right)\frac{\partial \beta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial^2 \alpha}+\frac{\partial}{\partial \alpha}\frac{\partial u}{\partial \beta}+\frac{\partial}{\partial \alpha}\frac{\partial u}{\partial \beta}+\frac{\partial^2 u}{\partial^2 \beta} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial^2 \alpha}+2\frac{\partial^2 u}{\partial \alpha \partial \beta}+\frac{\partial^2 u}{\partial^2 \beta}$$

- $\frac{\partial^2 u}{\partial t^2}=?$

- $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} \quad \Rightarrow \quad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial \alpha}(-v)+\frac{\partial u}{\partial \beta}(v)$

$$\frac{\partial^2 u}{\partial t^2}=-v\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \alpha}-\frac{\partial u}{\partial \beta}\right) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial t^2}=v^2\left(\frac{\partial^2 u}{\partial^2 \alpha}-2\frac{\partial^2 u}{\partial \alpha \partial \beta}+\frac{\partial^2 u}{\partial^2 \beta}\right)$$

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

General solutions of D'Alembert equation:

The D'Alembert equation is written as:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial^2 \alpha} + 2\frac{\partial^2 u}{\partial\alpha\partial\beta} + \frac{\partial^2 u}{\partial^2 \beta} = \frac{1}{v^2}\, v^2\left(\frac{\partial^2 u}{\partial^2 \alpha} - 2\frac{\partial^2 u}{\partial\alpha\partial\beta} + \frac{\partial^2 u}{\partial^2 \beta}\right)$$

$$\Rightarrow \quad 4\frac{\partial^2 u}{\partial\alpha\partial\beta} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\beta}\right) = 0$$

This shows that the function $\frac{\partial u}{\partial\beta}$ is independent of ; it is therefore an arbitrary function of β , which can be written as:

$$\int \frac{\partial^2 u}{\partial\alpha\partial\beta}\, d\alpha = G(\beta) \quad \Rightarrow \quad \frac{\partial u}{\partial\beta} = G(\beta)$$

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

Then integrate with respect to β.

$$\Rightarrow \quad u(\alpha,\beta) = g(\beta) + f(\alpha)$$

Where $g(\beta)$ is the antiderivative of $G(\beta)$.

By substituting $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with their expressions, the result takes the final form:

$$u(x,y) = f(x - vt) + g(x + vt)$$

- This is referred to as d'Alembert's general solution to the wave equation

5- Equation of wave (D'Alembert equation)

Interprétation des solutions

- We study the case of the particular solution $f(x - vt)$. For this, we assume that the boundary conditions are such that $g(x + vt)$is constantly zero
- At time t_1, we consider a point with abscissa x_1. The value of the function at this point and at this time is $u(x_1, t_1)$. We are looking for the position x_2 at a later time t_2, such that the value of u is the same as it was at x_1 at time t_1.

$$u(x_1, t_1) = u(x_2, t_2) \quad \Rightarrow \quad f(x_1 - vt_1) = f(x_2 - vt_2)$$

$$x_1 - vt_1 = x_2 - vt_2 \quad \Rightarrow \quad x_2 - x_1 = v(t_2 - t_1)$$

$$t_2 > t_1 \quad \Rightarrow \quad x_2 > x_1$$

- The expression $f(x - vt)$ corresponds to a wave propagating in the direction of increasing x.

Generalities on Propagation Phenomena

5- Equation of wave (D'Alembert equation)

Interprétation des solutions

- We study now the case of the solution $g(x+vt)$. For this, we assume that the boundary conditions are such that $f(x-vt)$ is constantly zero
- At time t_1, we consider a point with abscissa x_1. The value of the function at this point and at this time is $u(x_1,t_1)$. We are looking for the position x_2 at a later time t_2, such that the value of u is the same as it was at x_1 at time t_1.

$$u(x_1,t_1)=u(x_2,t_2) \quad \Rightarrow \quad g(x_1+vt_1)=g(x_2+vt_2)$$

$$x_1+vt_1=x_2+vt_2 \quad \Rightarrow \quad x_1-x_2=v(t_2-t_1)$$

$$t_2>t_1 \quad \Rightarrow \quad x_1>x_2$$

- The expression $g(x+vt)$ corresponds to a wave propagating in the direction of decreasing x.

6- Plane wave

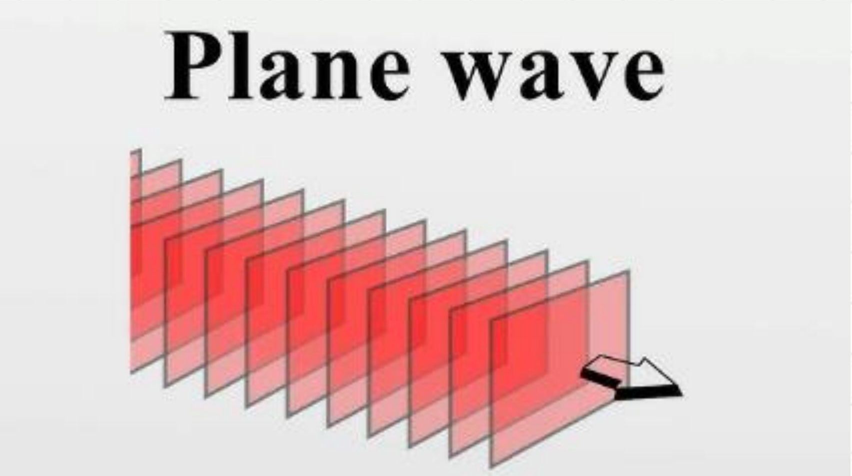
A plane wave is a type of wave that propagates in a single direction with constant amplitude; where the surfaces of constant phase (wave fronts) are infinite parallel planes perpendicular to the direction of wave propagation. It is an idealized concept often used in physics to simplify the study of wave.

Wave fronts = wave surface $\Rightarrow$ phase= constant $\Rightarrow$ Equation of plane

Plane wave

Constant amplitude

Wave front in the form of a plane.



Generalities on Propagation Phenomena

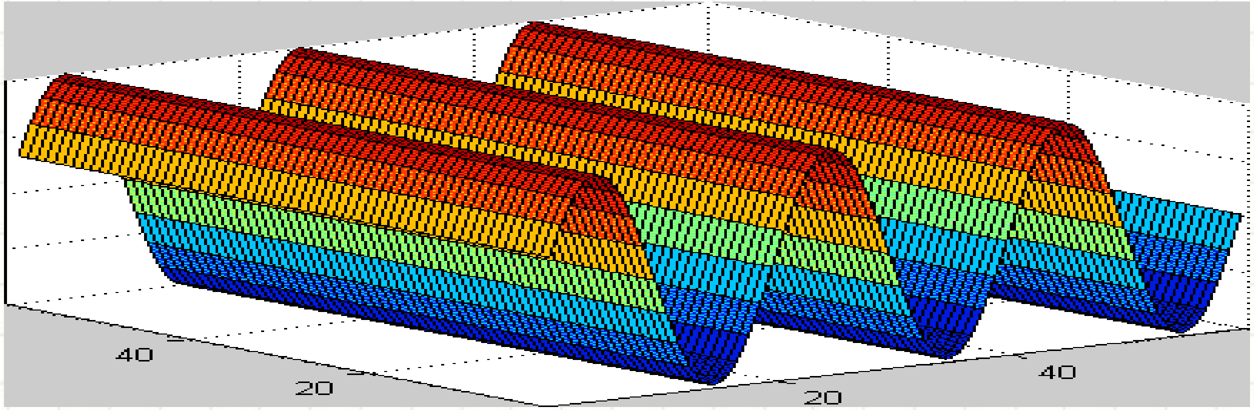
6- Plane wave

- A plane wave can be expressed as:

$$u(\vec{r},t) = A\cos\left(wt - \vec{k}.\vec{r}\right)$$

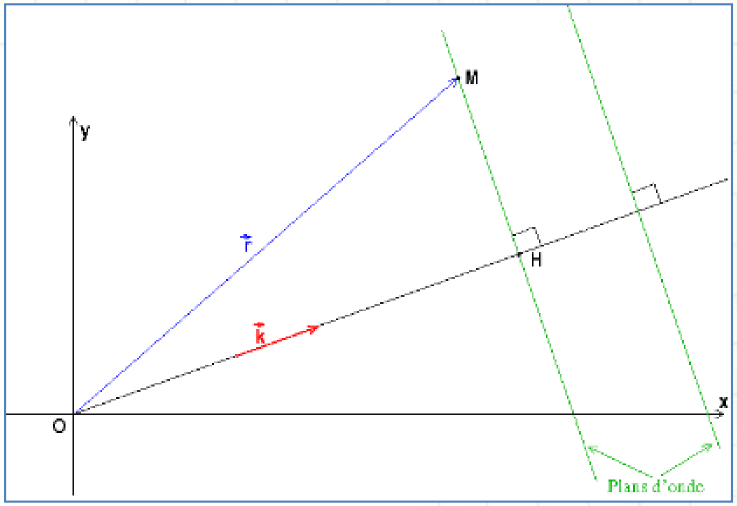
Or in complex form:

$$u(\vec{r},t) = Ae^{j((wt-\vec{k}.\vec{r})}$$



Generalities on Propagation Phenomena

6- Plane wave



Mathematical Demonstration

A plane wave is expressed as:

$$u(\vec{r},t) = A\cos\left(wt - \vec{k}.\vec{r}\right)$$

The phase is: $$phase = wt - \vec{k}.\vec{r}$$

For a fixed time t: $\vec{k}.\vec{r} = constant$ $\quad \vec{k}.\vec{r} = C$

- The condition $\vec{k}.\vec{r} = C$ represents the **equation of a plane** in three-dimensional space
- All points $\vec{k}.\vec{r} = C$ lie on a plane where the wave phase is identical. , implies that the points $\vec{r}$ with the same phase form a **plane** perpendicular to $\vec{k}$.. This is why the phase fronts in a plane wave are parallel planes.

7- Phase and group velocities

In a non-dispersive medium, all waves propagate at a single speed, independent of their wavelength or frequency. Sound in air is a typical example of a non-dispersive medium, where a single speed characterizes the propagation of sound waves. However, some media exhibit dispersive properties, where the propagation speed depends on the wavelength or frequency of the wave. This dispersion leads to distinct phenomena, such as phase velocity and group velocity, which are essential for understanding wave behavior in complex media.

Generalities on Propagation Phenomena

7- Phase and group velocities

Phase velocity

Phase velocity refers to the velocity of a monochromatic wave. Let us consider a sinusoidal progressive wave propagating in the direction of increasing x. A point at position x has, at time t, the displacement:

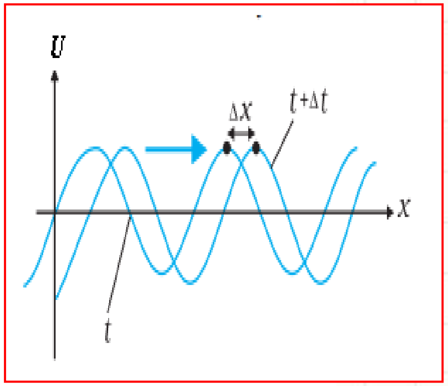
$$u(x,t) = A\cos\,(wt - kx)$$

After Δt, the wave progresses by Δx. At time $t + \Delta t$, the point at position $x + \Delta x$ has the same displacement as the point at position x. This can be expressed as: $u(x,t) = u(x+\Delta x, t+\Delta t)$

$$A\cos\,(wt - kx) = A\cos(w(t+\Delta t - k(x+\Delta x))$$

$$wt - kx = w(t+\Delta t - k(x+\Delta x)$$

$$w\Delta t = k\Delta x \qquad \Rightarrow \qquad v_p = \frac{\Delta x}{\Delta t} = \frac{w}{k} \qquad \Rightarrow \qquad v_p = \frac{w}{k}$$



- The phase velocity is given by the equation: $v_p = \frac{w}{k}$

Generalities on Propagation Phenomena

7- Phase and group velocities

Group velocity

When we superimpose waves of different frequencies we obtain wave packet. Which travels with the group velocity. We consider the superposition of two monochromatic waves:

$u_1(x.t) = A\cos(w_1 t - k_1 x)$ and $u_2(x.t) = A\cos(w_2 t - k_2 x)$

The superposition of $u_1(x.t)$ and $u_2(x.t)$ is given by the equation:

$$u(x;t) = u_1(x.t) + u_2(x.t)$$

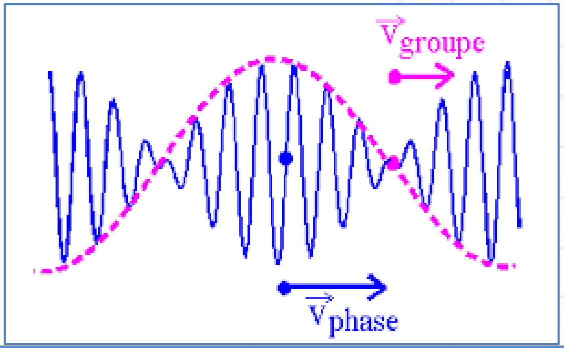
$$u(x;t) = 2A\cos\left(\frac{w_1 - w_2}{2}t - \frac{k_1 - k_2}{2}x\right).\cos\left(\frac{w_1 + w_2}{2}t - \frac{k_1 + k_2}{2}x\right)$$ Beat phenomena

$$\Rightarrow \quad u(x;t) = 2A\cos(w_B t - k_B x).\cos(w_m t - k_m x)$$

The envelope (modulate function) of the pack moves of group velocity v_g:

$$v_g = \frac{w_B}{k_B} = \frac{w_1 - w_2}{k_1 - k_2} \quad \Rightarrow \quad v_g = \frac{\Delta w}{\Delta k}$$

- k_1 very close to k_2 $\quad v_g = \frac{dw}{dk}$

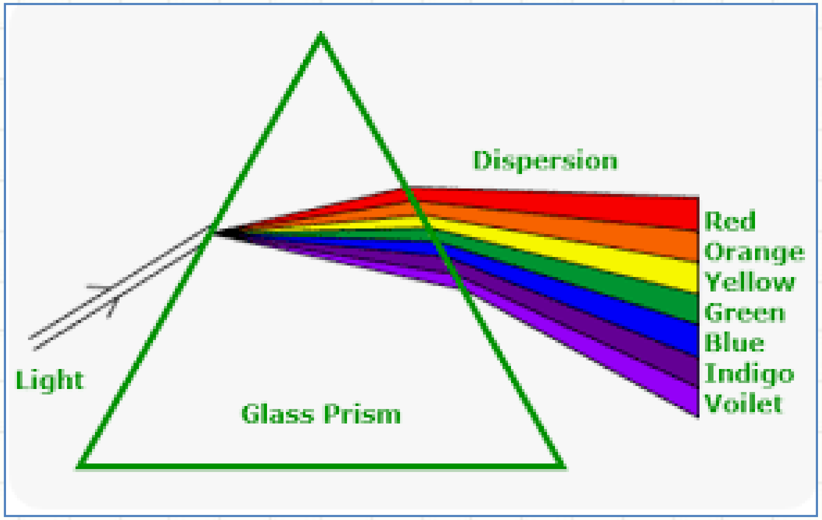


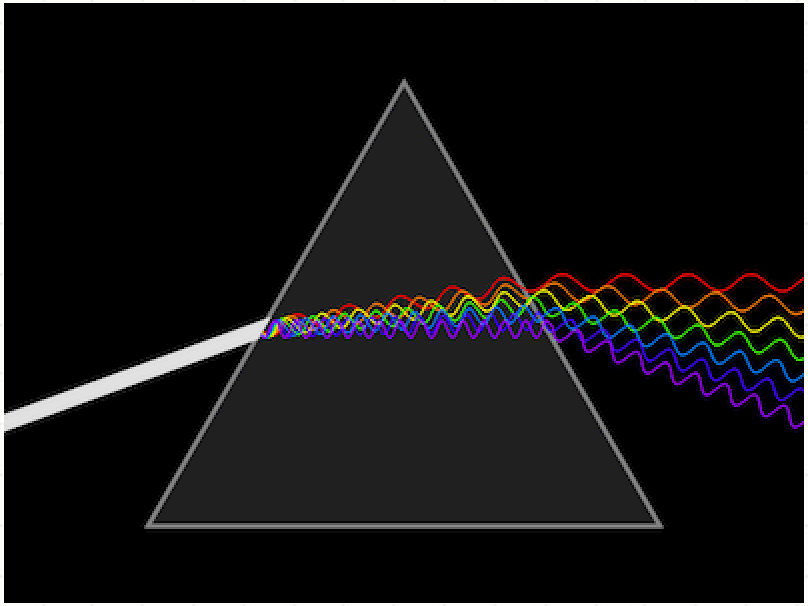
7- Phase and group velocities

- The media in which $v_g = v_p$ is called the non-dispersive medium
- The media in which $v_g \neq v_p$ is called the dispersive medium
- Generally, the function $\omega(k)$ is called the dispersion relation and indicates the dispersion properties of a medium.

Generalities on Propagation Phenomena

➢**Example of dispersive medium**





Generalities on Propagation Phenomena

7- Phase and group velocities