N.H.S.A.S.T **2024/2025**
Second year P.C - Analysis 4

SERIE N° 1 : POWER SERIES

Exercice 1 :

1/ Write an equivalent series with the index of summation beginning at $n = 1$:

$$\sum_{n\in\mathbb{N}} \frac{x^{2n+1}}{(2n+1)!}, \qquad \sum_{n\in\mathbb{N}^*} (-1)^n (n+1) x^n, \qquad \sum_{n\in\mathbb{N}} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

2/ Determine the radius of convergence of the following power series

$$\sum_{n\in\mathbb{N}} (\sin n) x^n, \qquad \sum_{n\in\mathbb{N}} \frac{x^n}{(n+1)^{\alpha+1} 3^n}, \qquad \sum_{n\in\mathbb{N}} \arccos\left(1 - \frac{1}{n^2}\right) x^n, \qquad \sum_{n\in\mathbb{N}} \frac{n^n}{n!} x^n.$$

3/ Find the interval of convergence of the following power series

$$\sum_{n\in\mathbb{N}} \frac{(-1)^n}{2^n} (x+1)^n, \qquad \sum_{n\in\mathbb{N}^*} \frac{x^n}{n}, \qquad \sum_{n\in\mathbb{N}} \frac{(-1)^n n!}{3^n} (x-5)^n.$$

Exercice 2 :

Determine the radius of convergence as well as the sum of the following power series

$$\sum_{n\in\mathbb{N}} \frac{x^n}{(n+1)(n+3)}, \quad \sum_{n\in\mathbb{N}} \left(n^2 - n - 3\right) 3^{n-1} x^n, \quad \sum_{n\in\mathbb{N}} \cosh(na) x^n, \quad \sum_{n\in\mathbb{N}} \frac{n^2 - n + 4}{n+1} x^n, \quad \sum_{n\in\mathbb{N}} \frac{x^n}{2n-1}.$$

Exercice 3 :

Expand the following functions into power series at the origin (Maclaurin series)

$$f_1 : x \longmapsto \frac{1}{(3+x)^2}, \qquad f_2 : x \longmapsto x \ln\left(x + \sqrt{x^2+1}\right), \qquad f_3 : x \longmapsto \arctan\left(\frac{1-x^2}{1+x^2}\right).$$

Exercice 4 :

Use the definition of Taylor series to find the Taylor series, centered at x_0 for the function f in the following cases :

$$f_1 : x \longmapsto \cos x, \quad x_0 = \frac{\pi}{4}; \qquad f_2 : x \longmapsto e^x, \quad x_0 = 1; \qquad f_3 : x \longmapsto \sqrt{x}, \quad x_0 = 4.$$

Exercice 5 :

1/ Solve in $\mathbb{R}$ the equation $\sum_{n=0}^{+\infty} (3n+1)^2 x^n = 0$.

2/ Using integration by parts to establish equality

$$\iint_{[0,1]^2} xye^{xy} dxdy = e - 1 - \sum_{n=1}^{+\infty} \frac{1}{n(n!)}.$$

Exercice 6 :

Let $(a_n)_{n\in\mathbb{N}}$ the sequence defined by $a_0 = 1$ and $2a_{n+1} = \sum_{k=0}^{n} C_n^k a_k a_{n-k}$, we put $f : x \longmapsto \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n$.

1/ Show that $a_n \leq n!$, for any $n \in \mathbb{N}$; deduce that the radius of convergence of $\sum_{n\in\mathbb{N}} \frac{a_n}{n!} x^n$ is strictly positive.

2/ Calculate f' as a function of f, deduce f.

3/ Deduce a_n as function of n.