Chapter 3

Perfect incompressible fluid dynamics

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1- *Introduction*

Fluid dynamics (hydrodynamics) is the science concerned with the behavior of fluids in motion.

In this chapter, we will study moving fluids by neglecting friction, meaning we will consider them as perfect (having zero viscosity) and incompressible (constant density). The focus is primarily on the fundamental equations governing the dynamics of incompressible perfect fluids, which are:

• The continuity equation (conservation of mass).

• Bernoulli's equation (conservation of energy).

• Euler's theorem (conservation of momentum).

2- *Context*

a- Velocity

A property of a fluid particle that gives the velocity and direction of the particle's movement at a given moment. The mathematical definition of velocity is:

$$V_A = \frac{dr_A}{dt}$$

Where:

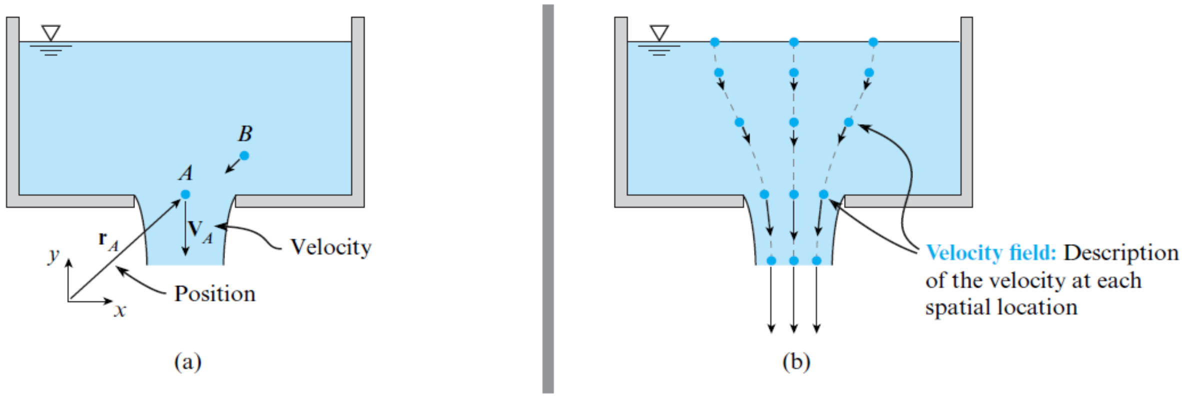
V_A: Velocity of particle A (a vector quantity, including both magnitude and direction).

r_A: Position vector of particle A at time t.

b- Velocity Field

A field is a mathematical or visual description of a variable as a function of position and time. A description of the velocity of each fluid particle in a flow is called a velocity field. In general, each fluid particle in a flow has a different velocity.

For example, particles A and B in the following figure have different velocities. Thus, the velocity field describes how the velocity varies with position.



c- Acceleration

Acceleration is a property of a fluid particle that characterizes the change in its velocity, meaning:

- The variation in speed (the magnitude of velocity).
- The variation in the direction of movement at a given moment.

The mathematical definition of acceleration is:

$$a = \frac{dV}{dt}$$

Where:

- a: Acceleration.
- V: Velocity of the fluid particle.

3- Different Flow Regimes

The state of a fluid, whether at rest or in motion, can be described by various physical quantities. These quantities can vary in space and time, leading to different types of flow. The classification of these flows can be made based on several criteria.

a- Based on Temporal Dependence

The flow can either be steady (stationary, permanent) or unsteady (transient) depending on how the fluid properties change over time.

Steady Flow:

The properties of the fluid (such as velocity, pressure) do not change with time at any point, meaning that the flow conditions remain constant as the fluid moves (e.g., a fluid flowing at a constant velocity in a straight pipe.

Unsteady Flow:

The properties vary with time, and the flow conditions are different at different moments. This type of flow is observed in natural systems like rivers, where the velocity changes over time.

b- Based on Uniformity

Uniform Flow:

A flow is considered uniform (or homogeneous) if the physical properties (pressure, temperature, velocity, density, etc.) are the same at every point in space at a given moment (e.g., an idealized flow where conditions like velocity and pressure are the same everywhere)

Non-Uniform Flow:

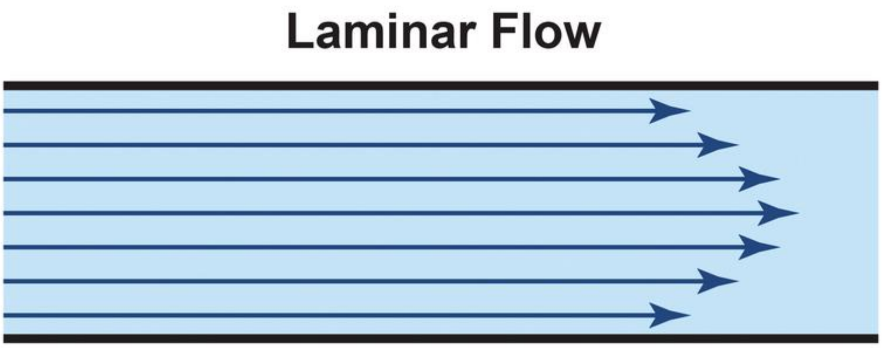
A non-uniform flow is one in which the physical properties vary depending on the position at a given time. Non-uniform flow is common in situations where the flow passes through regions with varying geometries, such as pipes with changing cross-sectional areas.

c- Based on the Nature of the Flow

The nature of the flow can either be laminar or turbulent, depending on the fluid's motion.

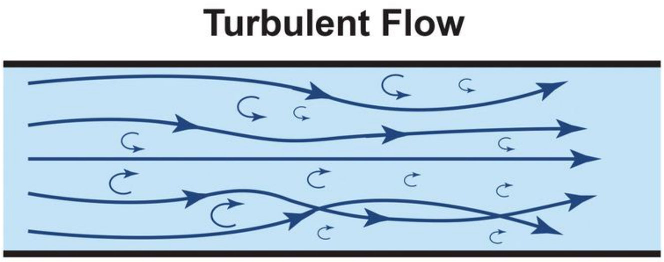
Laminar Flow:

A laminar flow is smooth and orderly, where the fluid moves in parallel layers with minimal mixing. It occurs at lower speeds or in highly viscous fluids.



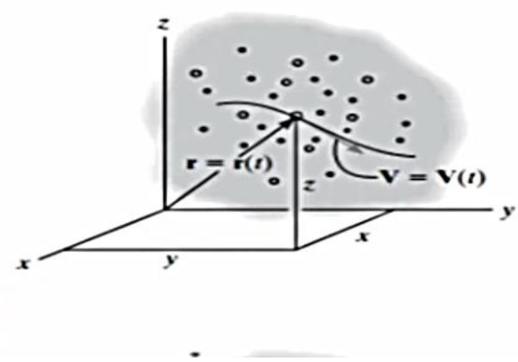
Turbulent Flow:

On the other hand, a turbulent flow is unpredictable and chaotic. This type of flow is characterized by high velocities and is common in fluids moving rapidly, such as rivers or airflows around vehicles.

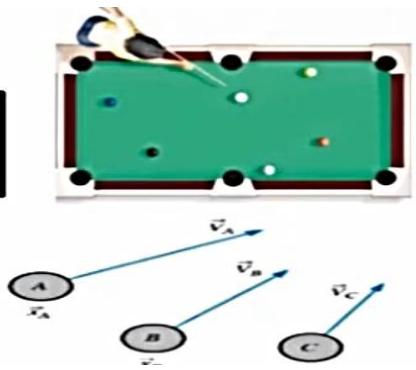


4- *Lagrangian and Eulerian Descriptions*

There are two ways to describe the motion of fluid particles in a flow. A Lagrangian description has limited utility because it requires tracking the position and velocity of each fluid particle as a function of time.

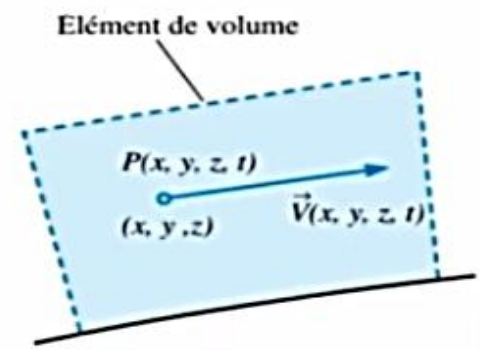


$$\boxed{\mathbf{V} = \mathbf{V}(t) = \frac{d\mathbf{r}(t)}{dt}}$$



An Eulerian description considers a volume element in the flow and measures the movement or properties of the fluid of all particles passing through that region.

Inside the volume element, we define a field variable, which is a function of both space and time. For example,



the pressure field is a scalar field variable for non-steady fluid flows in Cartesian coordinates.

- **Pressure field:** P=P(x,y,z,t)
- **Velocity field (vector field)**: $\vec{V}=\vec{V}$ (x,y,z,t)
- **Acceleration field (vector field) :** $\vec{a}=\vec{a}$ (x,y,z,t)

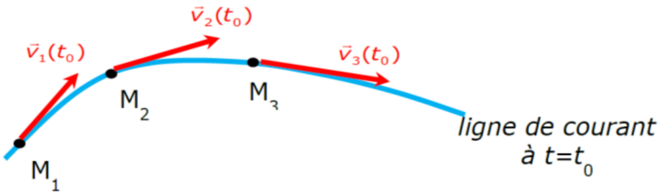
These field variables define the flow field. The velocity field can be developed in Cartesian coordinates using unit vectors (x,y,z) $(\vec{i},\vec{j},\vec{k})$

$$\vec{V}=(u,v,w) = u\ (x,y,z,t)\ \vec{i} + v\ (x,y,z,t)\ \vec{j} + w\ (x,y,z,t)\ \vec{k}$$

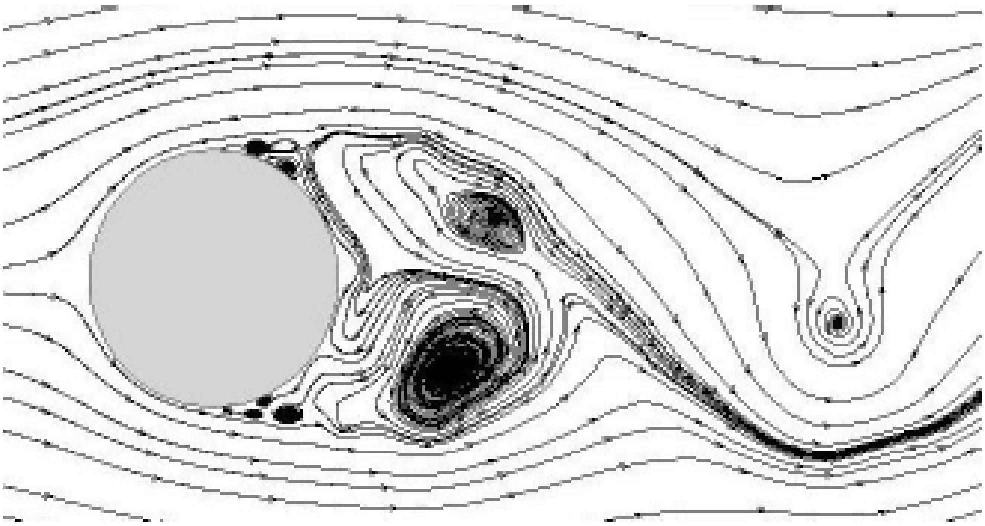
5- *Streamline, Stream tube and pathline*

a- *Streamline*

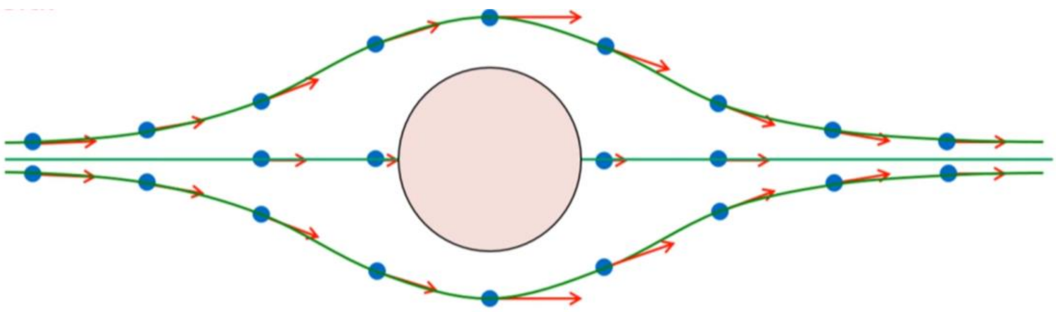
A **streamline** is a curve where the tangent at each point is, at every instant and locally, collinear with the velocity vector of the flow field. It is a curve that derives from the Eulerian description. In other words, the streamline represents the path followed by fluid particles at any given moment, with the direction of the flow being tangent to the curve at every point.

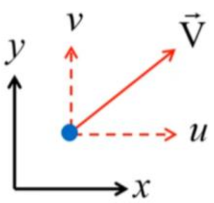


The **streamlines** evolve over time, just like the velocity vector field. To trace the streamlines, a snapshot of the flow is taken, and the tangents to the velocity vectors are drawn at each point. This gives the instantaneous directions of the flow, which may change as time progresses. Essentially, streamlines represent the flow pattern at a particular moment, but they can shift as the flow conditions change.



This figure illustrates the flow of a fluid around a circular obstacle, highlighting the streamlines. Before encountering the obstacle, the flow is uniform and regular. As the fluid flows around the cylinder, it experiences separation at the edges of the obstacle, forming recirculation zones and vortices behind it, which characterize a low-pressure region. These vortices, typical of unsteady and turbulent flow, result from the interaction between opposing currents. Further away from the obstacle, the flow gradually returns to regularity. This representation is essential for analyzing the forces exerted on objects in a fluid and understanding phenomena such as drag or vibrations induced by the vortices.

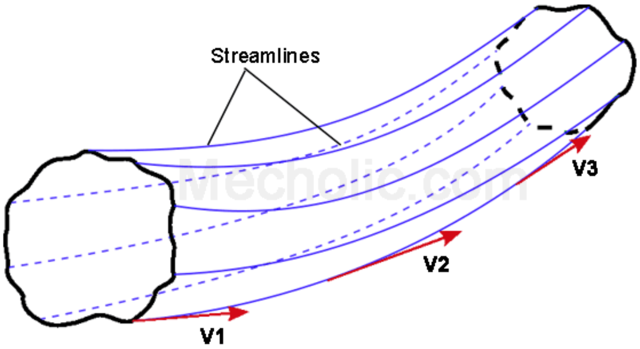


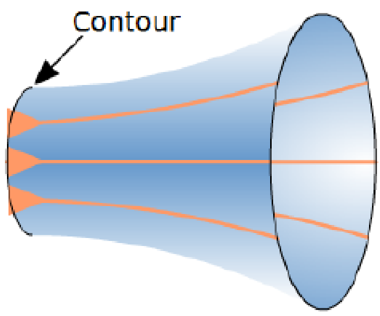


$$\frac{v}{u} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}$$

b- Stream tube

A stream tube is defined as the set of streamlines that are bounded by a closed contour. It represents a "tube" through which the fluid flows, with the streamlines acting as the boundaries of this tube.

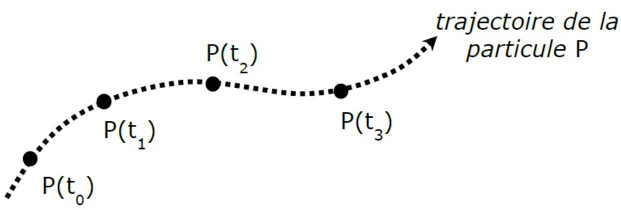




Tube de courant relatif au contour

c- Pathline

A **pathline** of a fluid particle is the curve traced by the particle over time. It is the geometric locus of the successive positions occupied by a particle as time progresses. Pathlines result from the Lagrangian description, as they track the movement of individual fluid particles, showing the actual path each particle follows throughout the flow. Unlike streamlines, which represent the flow direction at an instant, pathlines represent the trajectory of a specific particle over time.



It is important not to confuse **streamlines** and **pathlines**. They are fundamentally different concepts. For **steady flows**, streamlines coincide with pathlines. In steady flow, the flow conditions do not change with time, meaning the trajectory of a fluid particle (pathline) will always align with the streamline, as both represent the same direction of flow at any given moment. However, in **unsteady flows**, streamlines and pathlines will generally differ, as the direction of flow may change over time.

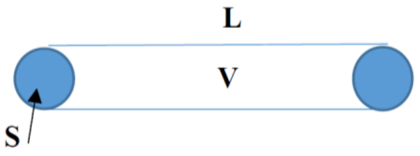
6- Flow Rate Concept

Flow rate refers to the amount of fluid that flows during a given time t. This quantity can be defined either by volume or mass

a- Volumetric Flow Rate

The volumetric flow rate is the rate at which the volume of fluid passes through a cross-sectional area per unit of time:

$$Q_V = \frac{V}{t} = \frac{S.L}{t} = S.v$$



Where :

Q_V: Volumetric flow rate (m³/s)

S: Cross-sectional area of the fluid's path (m²)

V: Volume of the fluid passing through the section (m³)

L: Length of the fluid path (m)

v: Average velocity of the fluid through the section SS (m/s)

t: Time of flow (s)

b- Mass Flow Rate

It is the ratio of the vein's mass per unit of time.:

$$Q_m = \frac{m}{t} = \frac{\rho.V}{t} = \rho.S.v = \rho.Q_v$$

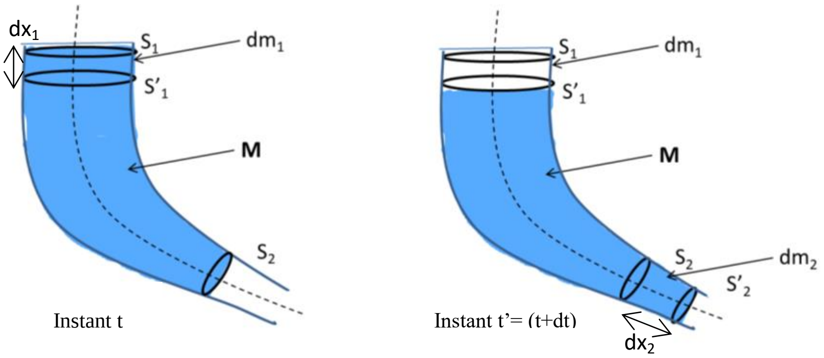
Q_m: Mass flow rate (kg/s)

Q_V: Volumetric flow rate (m³/s)

ρ: Fluid density (kg/m³)

7- Continuity Equation

The continuity equation represents the principle of mass conservation: The change in mass over a time dt of a fluid volume element must equal the sum of the mass of fluid entering, minus the mass of fluid exiting. Consider a stream of an incompressible fluid with density ρ, undergoing steady flow.



We denote:

- S_1 and S_2 as the fluid entry and exit sections at time *t*.
- S'_1 and S'_2 as the fluid entry and exit sections at time
- $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ as the velocity vectors of the fluid flow through sections S_1 et S_2 respectively.
- dx_1 et dx_2 as the displacements of sections S_1 and S_2 during the time interval dt.
- dm_1 : as the mass element entering between sections S_1 and S'_1 ;
- dm_2 : as the mass element exiting between sections S_2 et S'_2 ;
- M : as the mass contained between S'_1 and S_2 ;
- dV_1 : as the elemental volume entering between sections S_1 and S'_1 ;
- dV_2 : as the elemental volume exiting between sections S_2 et S'_2.

At time t: the fluid between S_1 and S_2 has a mass equal to (dm_1+ M):

At time t+dt: the fluid between S'_1 and S'_2 has a mass equal to (M+ dm_2).

By mass conservation $dm_1+M = M+dm_2$

Simplifying by M we have : $dm_1= dm_2$

Thus : $\rho_1.dV_1= \rho_2.dV_2$

Or : $\rho_1.S_1.dx_1 = \rho_2.S_2.dx_2$

Dividing by dt, we get:

$$\rho_1.\ S_1.\frac{dx_1}{dt} = \rho_2.\ S_2.\ \frac{dx_2}{dt} \leftrightarrow \rho_1.S_1.V_1 = \rho_2.S_2.V_2$$

Since the fluid is incompressible: $\rho_1 = \rho_2 = \rho$

This equation represents the law of mass conservation:

$$S_1.V_1 = S_2.V_2$$

The continuity equation represents the law of mass conservation.

8- Energy Conservation for an Ideal Fluid: Bernoulli's Theorem

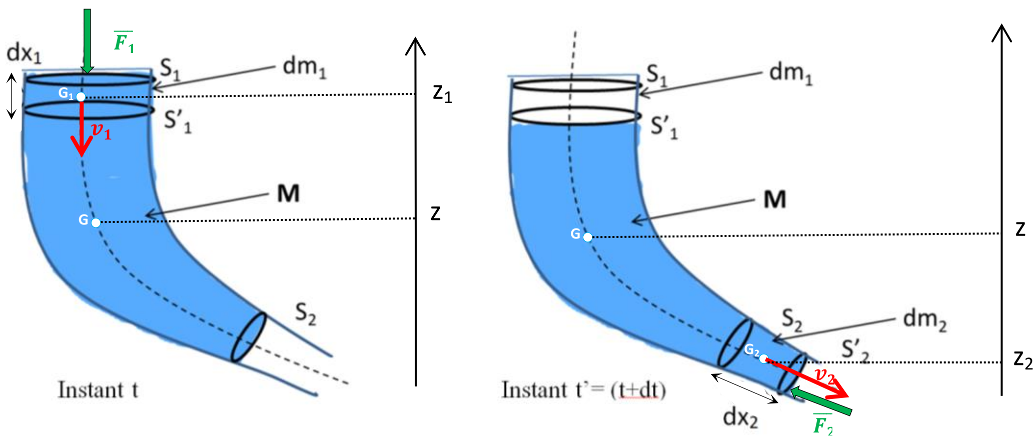
a) Without Energy Transfer to the Outside of the Pipe

Let us consider the previous diagram with the same notations and the following assumptions:

- The fluid is ideal and incompressible;
- The flow is steady;
- The flow is in a perfectly smooth pipe.

Let's consider a vertical z-axis pointing upwards, where z_1, z_2 and z represent the respective altitudes, and G, G_1 and G_2 the centers of inertia of the centers of gravity of the masses M, dm_1 and dm_2, and of the respective volumes $dV_1=S_1dx_1$ and $dV_2=S_2dx_2$, where pressures p_1 and p_2 prevail.

Let's denote by F_1 and F_2 the norms of the fluid pressure forces acting at sections S_1 and S_2.



Applying the principle of conservation of energy between time t and (t+dt):

$E_{initial} + W_{provided} - |W_{ceded}| = E_{final}$

$Ec_1 + E_{pp1} + W(F_1) - |W(F_2)| = Ec_2 + E_{pp2}$

At time « t » we have :	**At time « t+dt » we have :**
$E_{c1} = \frac{1}{2} dm_1 v_1^2 + \int_{S'_1}^{S_2} \frac{1}{2} dm\, v$	$E_{c2} = \frac{1}{2} dm_2 v_2^2 + \int_{S'_1}^{S_2} \frac{1}{2} dm\, v$
$E_{\,pp1} = dm_1 g z_1 + Mgz$	$E_{pp2} = dm_2 g z_2 + Mgz$
$W(F_1) = F_1.dx_1$	$W(F_2) = - F_2.dx_2$

The result is:

$$\frac{1}{2}dm_1v_1^2 + \int_{S_1'}^{S_2}\frac{1}{2}dm\,v + dm_1gz_1 + Mgz + F_1.\mathrm{dx}_1 - F_2.\mathrm{dx}_2 = \frac{1}{2}dm_2v_2^2 + \int_{S_1'}^{S_2}\frac{1}{2}dm\,v + dm_2gz_2 + Mgz$$

$$\frac{1}{2}dm_1v_1^2 + dm_1gz_1 + F_1.\mathrm{dx}_1 - F_2.\mathrm{dx}_2 = \frac{1}{2}dm_2v_2^2 + dm_2gz_2$$

$$F_1.\mathrm{dx}_1 = P_1\,S_1\,dx_1 = P_1\,dV_1 = P_1\,\frac{dm_1}{\rho_1} \quad \text{et} \quad F_2.\mathrm{dx}_2 = P_2\,S_2\,dx_2 = P_2dV_2 = P_2\,\frac{dm_2}{\rho_2}$$

This gives:

$$\frac{1}{2}dm_1v_1^2 + dm_1gz_1 + P_1\,\frac{dm_1}{\rho_1} - P_2\,\frac{dm_2}{\rho_2} = \frac{1}{2}dm_2v_2^2 + dm_2gz_2$$

By conservation of mass: $dm_1 = dm_2 = dm$

Since the fluid is incompressible: $\rho_1 = \rho_2 = \rho$ We arrive at Bernoulli's equation:

$$\frac{1}{2}v_1^2 + gz_1 + \frac{P_1}{\rho} - \frac{P_2}{\rho} = \frac{1}{2}v_2^2 + gz_2$$

$$\boxed{\frac{P_2}{\rho} + \frac{1}{2}v_2^2 + gz_2 = \frac{P_1}{\rho} + \frac{1}{2}v_1^2 + gz_1 = \frac{P_i}{\rho} + \frac{1}{2}v_i^2 + gz_i = cst}$$

Other form :

$$\boxed{\frac{P_2 - P_1}{\rho} + \frac{v_2^2 - v_1^2}{2} + g(z_2 - z_1) = 0}$$

Bernoulli's theorem without energy transfer

The terms in this expression are homogeneous in J/kg :

$\frac{1}{2}v_i^2$ in $\frac{m^2}{s^2} = \mathrm{m}.\frac{m}{s^2} = \mathrm{m}\,.\frac{N}{kg} = \frac{j}{kg}$

$\frac{P_i}{\rho}$ in $\frac{pa}{kg/m^3} = \frac{N/m^2}{kg/m^3} = \frac{N.m}{kg} = \frac{j}{kg}$

gz_i in $\frac{m}{s^2}\,.m = \frac{N}{kg}.\mathrm{m} = \frac{j}{kg}$

Notes: Between two points in a pipe with no exchange of energy with the outside, Bernoulli's theorem can be written as follows:

$$\boxed{\mathrm{P}_2 + \rho.\mathrm{g}.\mathrm{z}_2 + \rho\,\frac{v_2^2}{2} = \mathrm{P}_1 + \rho.\mathrm{g}.\mathrm{z}_1 + \rho\,\frac{v_1^2}{2}}$$

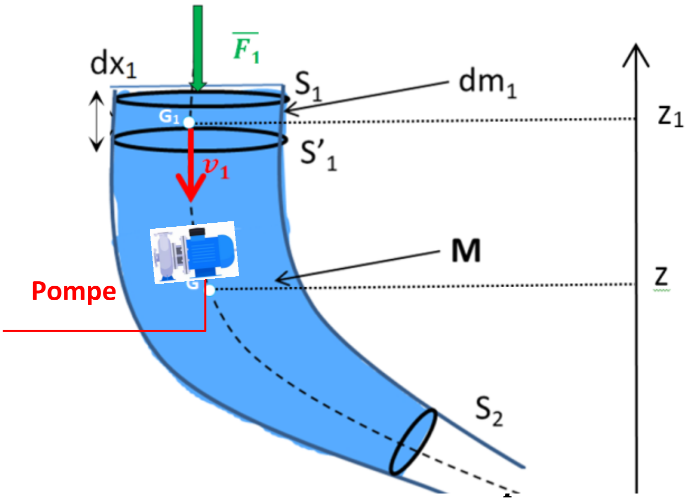
b) With energy transfer to/from the outside of the pipe

The Bernoulli theorem is a fundamental law of fluid mechanics that describes the behavior of a fluid in motion along a streamline. It is based on the conservation of energy and allows the connection of pressure, velocity, and height (potential energy) in a moving fluid.

In flow systems, such as pumps and turbines, there is an exchange of work between the energy supplied to the fluid (mechanical or energetic work) and the energy that the fluid itself can transmit. We will apply the Bernoulli theorem in two practical cases: a pump and a turbine in a pipeline.

1)Case of a Pump in a Pipeline

A pump in a pipeline is used to increase the pressure of the fluid, leading to an increase in the internal energy of the fluid. When the pump operates, it performs work (positive work) on the fluid to raise its pressure or to move it against resistance (such as a height difference or a smaller pipe section).



The efficiency of a pump is a measure of its effectiveness in converting the electrical energy it consumes into energy transferred to the fluid in the form of pressure or flow rate (mechanical energy). It is defined as the ratio of the net power delivered to the fluid to the electrical power absorbed by the pump.

The efficiency of the pump, η_{pump}, is given by the following expression:

$$\eta_{pompe} = \frac{P_{mechanical}}{P_{electrical}}$$

Where: $p_{mecanical} = \frac{W_{mecanical}}{dt}$ $\longrightarrow$ $W_{mecanical} = p_{mecanical}$.dt

where :

- $P_{mecanical}$: The net power exchanged between the machine (pump) and the fluid, i.e., the energy per unit time (measured in watts, W).
- $P_{elactrical}$: The power absorbed by the pump, or the energy it consumes to perform work on the fluid.
- $W_{mecanical}$: The network exchanged between the machine and the fluid, which is the total energy transferred during a given time interval.
- Δt : The time interval during which this energy exchange occurs.

- ***Net Work Exchange with the Machine (Application of Bernoulli's Theorem)***

By applying the principle of conservation of energy between the instant t and (t+Δt):

$E_{initial} + W_{provided} - |W_{ceded}| = E_{final}$

$Ec_1 + E_{pp1} + W(F_1) - |W(F_2)| + W_{net} = Ec_2 + E_{pp2}$

$$\frac{1}{2}dm_1 v_1^2 + dm_1 g z_1 + P_1\frac{dm_1}{\rho_1} - P_2\frac{dm_2}{\rho_2} + P_{net}.dt = \frac{1}{2}dm_2 v_2^2 + dm_2 g z_2$$

$$\frac{1}{2}dm_2 v_2^2 + dm_2 g z_2 - \frac{1}{2}dm_1 v_1^2 - dm_1 g z_1 - P_1\frac{dm_1}{\rho_1} + P_2\frac{dm_2}{\rho_2} = P_{net}.dt$$

By conservation of mass:$dm_1 = dm_2 = dm$ and since the fluid is incompressible:$\rho_1 = \rho_2 = \rho$, We end up with Bernoulli's equation with exchange of work:

$$\frac{1}{2}v_2^2 + gz_2 - \frac{1}{2}v_1^2 - gz_1 - \frac{P_1}{\rho} + \frac{P_2}{\rho} = P_{net}.dt$$

Deviating on dm and simplifying, we obtain :

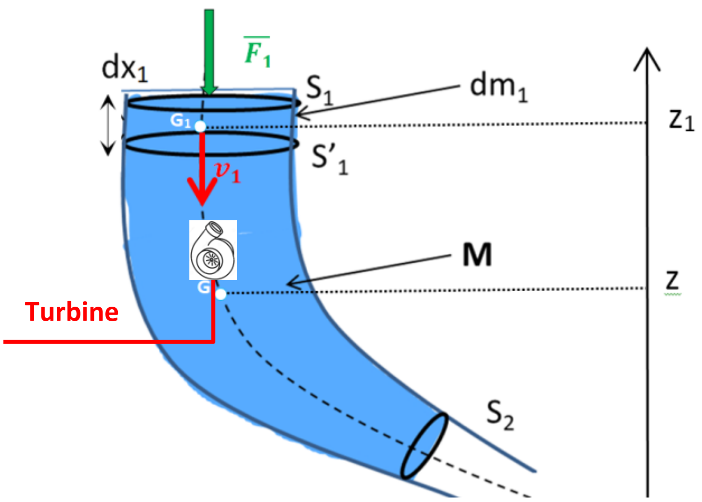
$\frac{P_2 - P_1}{\rho} + \frac{v_2^2 - v_1^2}{2} + g(z_2 - z_1) = \frac{P_{net}.dt}{dm}$ with : $\frac{dt}{dm} = \frac{1}{Q_m}$ we obtain :

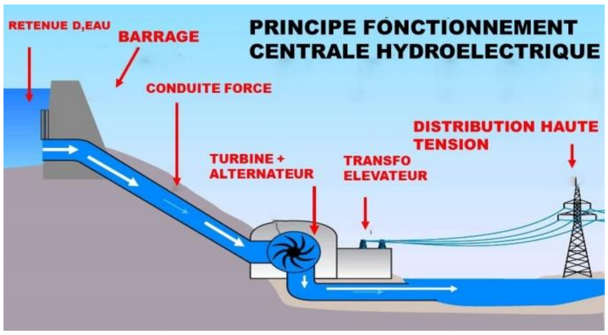
$$\boxed{\frac{P_2-P_1}{\rho}+\frac{v_2^2-v_1^2}{2}+g(z_2\ -\ z_1)=\frac{P_{net}}{Q_m}}$$

This is Bernoulli's equation in the case of a flow with exchange of work (the case of a pump).

2) Case of a Turbine in a Pipeline

A turbine in a pipeline is used to recover the kinetic energy and pressure of the moving fluid (mechanical energy) in order to produce electrical work. Unlike a pump, which does work on the fluid, the turbine extracts energy from the fluid. This energy recovered by the turbine can then be converted into electrical energy (for example, to drive an electrical generator). In other words, in the case of the turbine, the work is transferred from the fluid to the system (negative work).





In the case of a turbine: efficiency is given by the following expression:

$$\eta_{Turbine} = \frac{P_{electrical}}{P_{mecanical}}$$

Between times t and t'=(t+dt), the fluid has exchanged net electrical work with the hydraulic machine.

$$W_{electrical} = P_{elctricale}.dt$$

In the case of a turbine, since it extracts energy from the fluid (the fluid transfers work to the turbine), the work is considered negative.

Applying Bernoulli's law, we obtain:

$$\boxed{\frac{P_2-P_1}{\rho} + \frac{v_2^2-v_1^2}{2} + g(z_2 - z_1) = -\frac{P_{electrical}}{Q_m}}$$

This is Bernoulli's equation in the case of a flow with exchange of work (case of a Turbine).

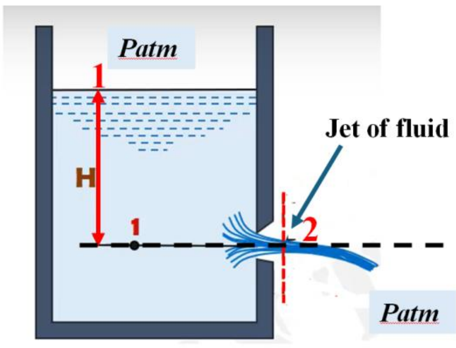
9- *Applications of Bernoulli's theorem*

a- Emptying a reservoir (Torricelli's theorem)

One of the simplest applications of Bernoulli's theorem is that which leads to the drainage velocity of a tank with a free surface through a small orifice compared to the surface area of the tank.

Let's apply Bernoulli's equation between points 1 and 2:

$$\frac{P_2}{\rho} + \frac{1}{2}v_2^2 + gz_2 = \frac{P_1}{\rho} + \frac{1}{2}v_1^2 + gz_1$$



We have :

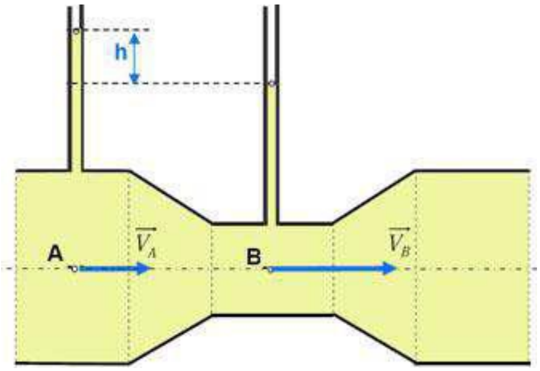
$P_1=P_2=P_{atm}$ et $Z_1-Z_2=h$

Since the reservoir is large, the downward velocity of the free surface level can be considered negligible compared to that of the fluid flowing in the jet: $V_1 << V_2$ Hence Torricelli's formula relating the outlet velocity to the height h of liquid above the orifice:

$$\boxed{v_2 = \sqrt{2gh}}$$

b- Venturi tube

The purpose of the Venturi tube is to measure flow by determining the pressure difference. This device consists of passing a flow through a contraction to obtain a pressure reduction.



The Bernoulli equation between A and B is :

$$\frac{P_A}{\rho} + \frac{1}{2}v_A^2 + gz_A = \frac{P_B}{\rho} + \frac{1}{2}v_B^2 + gz_B$$

we have :

$Z_A = Z_B$ (same level)

$V_A\ S_A = V_B\ S_B$ (Continuity equation)

$$\mathbf{P_A + \rho.g.z_A + \rho\,\frac{v_A^2}{2} = P_B + \rho.g.z_B + \rho\,\frac{v_B^2}{2}}$$

$\mathbf{P_A = P_{atm} + \rho.g.h_A}$

$\mathbf{P_B = P_{atm} + \rho.g.h_B}$ **so** $\mathbf{P_A - P_B = \rho.g.h}$ (hydrostatic equation)

Bernoulli's relation then becomes :

$$\mathbf{g.h = \frac{1}{2}\left(v_B^2 - \left(\frac{v_B S_B}{S_A}\right)^2\right)} \longrightarrow \mathbf{2g.h = v_B^2\left(1 - \left(\frac{S_B}{S_A}\right)^2\right)}$$

$$\mathbf{2g.h} = \boldsymbol{v_B^2}\left(\mathbf{1} - \left(\frac{S_B}{S_A}\right)^2\right)$$

we have:

$$\boxed{\boldsymbol{v_B} = \sqrt{\frac{\mathbf{2g.h}}{\mathbf{1} - \left(\frac{S_B}{S_A}\right)^2}}}$$

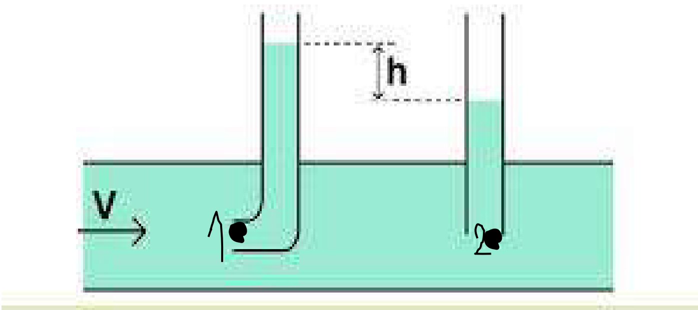
Hence the flow rate: = $V_B.\ S_B$

This flow rate is considered a theoretical flow rate, as the fluid is assumed to be perfect. The actual flow rate is obtained by multiplying the theoretical flow rate by a correction factor that takes into account the energy loss in the Venturi. This factor is called the discharge coefficient, C_d.

$Q = C_d\ .V_B.\ S_B$

c- Pitot tube

A Pitot tube, often simply called 'Pitot', is the most commonly used device for measuring velocity in various flow situations. The device is named after its inventor, Henri de Pitot, who first tested it in the Seine River in August 1732.



The principle is based on measuring both the static pressure and the dynamic pressure at a point in a flow. The Bernoulli equation between points 1 and 2 is written as:

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2 + gz_2$$

we have :

$Z_1 = Z_2$ (same level)

$V_1 = 0$ (point 1 is a stopping point i.e. is an obstacle)

The hydrostatic equation gives: $P_1 - P_2 = \rho g h$

Hence the expression for fluid velocity in the pipe, known as the Torricelli relation: :

$$\frac{P_1}{\rho} - \frac{P_2}{\rho} = \mathrm{gh} = \frac{1}{2} v_2^2 \qquad \boxed{v_2 = \sqrt{2gh}}$$

10- *Euler's Theorem*

The theorem of momentum, or Euler's theorem, is one of the most important theorems in fluid mechanics. It can be summarized as follows: The net force applied to a moving fluid is equal to the change in momentum.

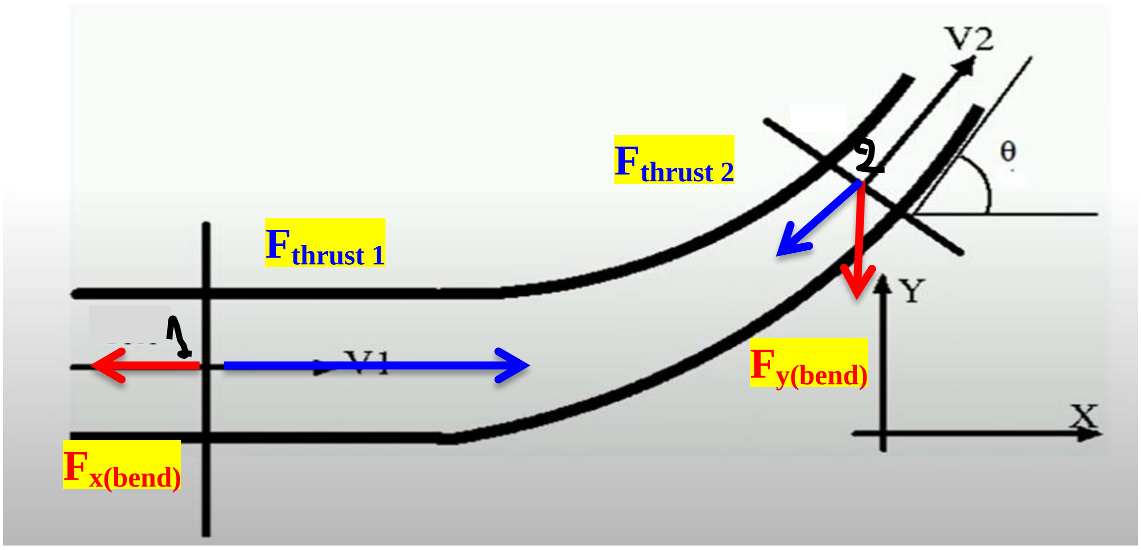
- **Force applied to a change in section and direction:**

  According to Newton's principle:

$$\sum F_{ext} = m\frac{dv}{dt} = \frac{dmv}{dt} = \frac{m}{\Delta t}\Delta v = \rho Q \Delta v$$

Thus, this is the equation of momentum, which is known as the equation of impulse.

- Let there be a converging elbow with an angle θ: P_1S_1.



If we denote F_x and F_y as the forces exerted by the liquid on the elbow, then $-F_x$ and $-F_y$ are the reaction forces exerted by the elbow on the liquid, and these are the ones that are involved according to Euler's theorem.

- The other external forces are the thrust forces P_1S_1 and P_2S_2. By applying Euler's theorem, we obtain:

$$\sum F_{ext} = \overrightarrow{F_{x(bend)}} + \overrightarrow{F_{y(bend)}} + \overrightarrow{F_{thrust\ 1}} + \overrightarrow{F_{trusth\ 2}} = \rho Q \Delta v$$

- F_{ext} represents the total external force.
- $F_{x(bend)}$ and $F_{y(bend)}$ are the components of the force at the elbow in the x and y directions.
- F_{thrust} represents the thrust forces exerted by the fluid at points 1 and 2.
- ρ is the density.
- Q is the volumetric flow rate
- Δv est la variation de vitesse.

We project the forces along the X-axis and the Y-axis:

Along the X-axis

$$-F_{x(bend)} + F_{(thrust1)} - F_{(trust2)}\cos\theta + 0 = \rho Q(v_2 2cos\theta - v_1)$$

$$-F_{x(bend)} + P_1S_1 - P_2S_2\cos\theta = \rho Q(v_2 2cos\theta - v_1)$$

$$\boxed{F_{x(bend)} = P_1S_1 - P_2S_2\cos\theta + \rho Q(v_1 - v_2 cos\theta)}$$

Along the Y-axis

$$0 - F_{(bend\ 2)}\sin\theta - F_Y = \rho Q(v_2 2sin\theta)$$

$$\boxed{F_{Y} = \rho Q(-v_2 sin\theta) - P_2S_2\sin\theta}$$

The resultant (ΣFext)of the external mechanical forces exerted on an isolated fluid (contained within the boundary limited by S_1 and S_2) is equal to the rate of change of the momentum, which enters at S_1 with a velocity V_1 and exits at S_2 with a velocity V_2.

11- CONCLUSIION

The laws and equations established in this chapter, particularly Bernoulli's equation, have considerable practical significance as they help to understand the operating principles of many flow measurement instruments such as the Pitot tube, the Venturi tube, etc.

These laws and equations, which are applicable to incompressible fluids, can also be used in certain specific cases for compressible fluids with low pressure variation. Such variations occur in many practical situations.

However, when compressibility needs to be taken into account in calculations, it is necessary to use the appropriate formulas.