FINAL EXAM CORRECTION

Exercice 1 : **6 pts=(2,5+3,5)**

1/ We put

$$\begin{cases} u = x + y \\ v = x - y \end{cases} \qquad \text{so} \qquad \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}.$$

The new domain of integration is the region of the plane delimited by the lines : $u = 1$, $v = u$ and $v = -u$. The jacocian is $\frac{1}{2}$. So

$$\begin{aligned} \iint_D \cos\left(\pi\left(\frac{x-y}{x+y}\right)\right) dxdy &= \frac{1}{2}\int_0^1 \left[\int_{-u}^{u} \cos\left(\frac{\pi v}{u}\right) dv\right] du = \frac{1}{2}\int_0^1 \left[\frac{u}{\pi}\sin\left(\frac{\pi v}{u}\right)\right]_{-u}^{u} du \\ &= \frac{1}{2}\int_0^1 \frac{u}{\pi}\left[\sin(\pi) - \sin(-\pi)\right]_{-u}^{u} du = \frac{1}{2}\int_0^1 0du = \mathbf{0} \end{aligned}$$

2/ We observe that

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x^2+y^2 \leq 3 \quad \text{and} \quad 2 - \sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}\right\}.$$

So, using cylindrical coordinates, we get

$$\begin{aligned} \iiint_\Omega zdxdydz &= \int_0^{2\pi}\left[\int_0^{\sqrt{3}} r\left[\int_{2-\sqrt{4-r^2}}^{\sqrt{4-r^2}} zdz\right] dr\right] d\theta \\ &= 2\pi\int_0^{\sqrt{3}}\left(-2r + 2r\sqrt{4-r^2}\right) dr = 2\pi\left[-r^2 - \frac{2}{3}\left(4-r^2\right)\sqrt{4-r^2}\right]_0^{\sqrt{3}} = \frac{\mathbf{10}}{\mathbf{3}}\boldsymbol{\pi}. \end{aligned}$$

Exercice 2 : **6,5 pts=(2,5+1+3)**

1/ *i)* If we denote by $\mathcal{A}(\Sigma)$ the area of the surface Σ, then

$$\mathcal{A}(\Sigma) := \iint_\Sigma d\sigma = \int_0^1\left[\int_0^{2\pi}\sqrt{1+u^2}dv\right] du = 2\pi\int_0^1 \sqrt{1+u^2}du.$$

Using the variable change $u = \sinh t$, we get

$$\begin{aligned} \mathcal{A}(\Sigma) &= 2\pi\int_0^{\ln(1+\sqrt{2})} \cosh^2 tdt = \pi\int_0^{\ln(1+\sqrt{2})} (1+\cosh 2t)\, dt \\ &= \pi\left[t + (\sinh t)\sqrt{1+\sinh^2 t}\right]_0^{\arg\sinh 1} = \boldsymbol{\pi}\left(\sqrt{\mathbf{2}} + \ln\left(\mathbf{1}+\sqrt{\mathbf{2}}\right)\right). \end{aligned}$$

ii) According to *i)*, we have

$$\begin{aligned} \iint_\Sigma \sqrt{x^2+y^2+1}d\sigma &= \int_0^1\left[\int_0^{2\pi}\sqrt{1+u^2}\sqrt{1+u^2}dv\right] du = 2\pi\int_0^1 \left(1+u^2\right) du \\ &= 2\pi\left[u + \frac{u^3}{3}\right]_0^1 = \frac{\mathbf{8}}{\mathbf{3}}\boldsymbol{\pi}. \end{aligned}$$

2/ If we deonte by D the disk centered at $(0,0)$ of radius 4 and $\phi_S(\mathbf{F}):$ the flux of $\mathbf{F}$ troughthout S, we have

$$\phi_S(\mathbf{F}) = \iint_D \left(2x^4 + 2y^4 + \left(4 - x^2 - y^2\right)^3\right) dxdy.$$

Using polar coordinates, we get

$$\phi_S(\mathbf{F}) = \int_0^{2\pi} \left[\int_0^2 \left(2r^5\left(\cos^4\theta + \sin^4\theta\right) + r\left(4 - r^2\right)^3\right) dr\right] d\theta.$$

As $\cos^4\theta + \sin^4\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta$, therefore

$$\begin{aligned} \phi_S(\mathbf{F}) &= \pi \int_0^2 \left(3r^5 + 2r\left(4 - r^2\right)^3\right) dr + \frac{1}{2}\left(\int_0^{2\pi} \cos 4\theta d\theta\right)\left(\int_0^2 r^5 dr\right) \\ &= \pi \left[\frac{1}{2}r^6 - \frac{1}{4}\left(4 - r^2\right)^4\right]_0^2 = \boldsymbol{96\pi}. \end{aligned}$$

Exercice 3 : **5 pts=(2+3)**

1/ Let $b_n = a_n + a_{n+1}$, for any $n \in \mathbb{N}$. Since $u_n = b_n - b_{n+1}$ then $U_n = b_0 - b_{n+1} = (a_0 + a_1) - (a_{n+1} + a_{n+2})$, for any $n \in \mathbb{N}$, which gives the requested equivalence. Furthermore, if $\sum\limits_{n\in\mathbb{N}} u_n$ converges and $L = \lim b_n$, then $U = a_0 + a_1 - L$.

2/ When $a_n = \frac{2}{2n-1}$ then $u_n = \dfrac{8}{4n^2 + 4n - 3}$, for any $n \in \mathbb{N}$, so

$$\sum_{n=0}^{+\infty} \frac{8}{4n^2 + 4n - 3} = -2 + 2 - 0 = \mathbf{0}.$$

In the same way : if $a_n = \frac{1}{n-1+\cos\theta}$ then $u_n = \dfrac{2}{n^2 + 2n\cos\theta - \sin^2\theta}$, for any $n \in \mathbb{N}$, so

$$\sum_{n=0}^{+\infty} \frac{2}{n^2 + 2n\cos\theta - \sin^2\theta} = \frac{1}{-1 + \cos\theta} + \frac{1}{\cos\theta}.$$

Exercice 4 : **2,5 pts=(1,5+1)**

1/ Because the function $t \longmapsto \sqrt{t}$ is continuous then

$$\lim_{n\longrightarrow+\infty} f_n(x) = \lim_{n\longrightarrow+\infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{\lim_{n\longrightarrow+\infty}\left(x^2 + \frac{1}{n}\right)} = \sqrt{x^2} = |x| := f(x),$$

for any $x \in \mathbb{R}$. Which shows that $(f_n)_{n\in\mathbb{N}^*}$ is pointwise convergent to f on $\mathbb{R}$.
Let $\alpha \in]0, +\infty[$, the function $t \longmapsto \left(\sqrt{t+\alpha} + \sqrt{t}\right)^{-1}$ is decreasing on the interval $[0, +\infty[$, then

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{1}{n} \sup_{x\in\mathbb{R}} \frac{1}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} = \frac{1}{\sqrt{n}},$$

for any $n \in \mathbb{N}^*$. Which shows that $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly on $\mathbb{R}$.
2/ If $(f'_n)_{n\in\mathbb{N}^*}$ converge uniformly on $\mathbb{R}$ then, by the derivability theorem, f is derivable on $\mathbb{R}$, what is wrong.