

Eigenvalues and Eigenvectors

1

1-Eigenvalues and Eigenvectors

In the next, , K will denote R or C, and the space V is K^n .

Definition 1

Let $A \in M_n(K)$. If there exist $\lambda \in K$ and a nonzero column vector $x \in K^n$ satisfying $Ax = \lambda x$, we say that λ is an **eigenvalue** of A and x is an **eigenvector** of A corresponding to λ .

Example

Let

$$A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \quad \text{And} \quad X = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} \quad \text{then} \quad AX = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

So $AX = 10X$. 10 is an **eigenvalue** of A and X is an **eigenvector** of A

2

2-Characteristic Polynomial

Definition 2

The **characteristic polynomial** of the matrix A is defined by

$$P_A(t) = \det(A - tI).$$

Remark

The characteristic polynomial allows us to find all the eigenvalues of a given matrix, as the following result shows.

Proposition 1

A scalar $\lambda \in K$ is an eigenvalue of A if and only if $P_A(\lambda) = 0$.

3

Examples

Consider the matrix $A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$. We have

$$P_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 2 \\ 5 & -1-t \end{vmatrix} = t^2 - t - 12.$$

Then, the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 4$.

4

Similar Matrices

Definition 3

Let $A, B \in \mathcal{M}_n(K)$. We say that A is **similar** to B if there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such that $B = P^{-1}AP$.

Proposition 2

If $A, B \in \mathcal{M}_n(K)$ are similar, then they have the same characteristic polynomial and then the same eigenvalues with the same multiplicities.

5

Properties of Eigenvalues

Assume that the matrix $A \in \mathcal{M}_n(K)$ has n (non necessarily distinct) eigenvalues, then we have

- 1) the sum of the eigenvalues is the trace of A : $\lambda_1 + \dots + \lambda_n = \text{Tr}(A)$;
- 2) the product of the eigenvalues is the determinant of A : $\lambda_1 \cdots \lambda_n = \det(A)$;
- 3) if A is triangular, then its eigenvalues are the entries of the diagonal ;
- 4) if λ is an eigenvalue of an invertible matrix A , then λ^{-1} is an eigenvalue of A^{-1} ;
- 5) if λ is an eigenvalue of a matrix A , and m a positive integer, then λ^m is an eigenvalue of A^m .

6

Linear Independence of Eigenvectors

Proposition 4

If v_1, \dots, v_k are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of $A \in \mathcal{M}_n(K)$, then the set $\{v_1, \dots, v_k\}$ is linearly independent.

Eigenspace

Proposition 5

Let λ be an eigenvalue of A . Then the set

$$V_\lambda = \{v \in V : Av = \lambda v\}$$

is a subspace of V .

7

Definition 4

The subspace $V_\lambda = \{v \in V : Av = \lambda v\}$ is called **eigenspace** corresponding to λ .

Example

Consider the matrix of example 2, then

$$V_4 = \{v \in K^2 : Av = 4v\}.$$

Set $v = \begin{pmatrix} x \\ y \end{pmatrix}$, then we have

$$\begin{aligned} Av = 4v &\Leftrightarrow \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 2x + 2y = 4x \\ 5x - y = 4y \end{cases} \Leftrightarrow \begin{cases} -2x + 2y = 0 \\ 5x - 5y = 0 \end{cases} \Leftrightarrow x = y. \end{aligned}$$

Therefore V_4 is the subspace spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\dim(V_4) = 1$.

8

Diagonalization

Definition 5

A square matrix $A \in \mathcal{M}_n(K)$ is said to be **diagonalizable** if it is similar to a diagonal matrix, that is, there exists an invertible matrix $P \in \mathcal{M}_n(K)$ such $P^{-1}AP$ is a diagonal matrix.

Remark 1

Let f be the endomorphism of V represented by the matrix A . Then « A is diagonalizable » means that there exists a basis B of V consisting of eigenvectors such that $M(f, B)$ is diagonal. Moreover, we can take P as matrix whose columns are eigenvectors v_1, \dots, v_n to eigenvalues and we have corresponding respectively to eigenvalues $\lambda_1, \dots, \lambda_n$ $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$.

9

A Sufficient Condition for Diagonalization

Theorem 1

If a matrix $A \in \mathcal{M}_n(K)$ has n distinct eigenvalues, then it is diagonalizable.

A Necessary and Sufficient Condition Diagonalization

Theorem 2

Let $\lambda_1, \dots, \lambda_k$ the distinct eigenvalues of a matrix $A \in \mathcal{M}_n(K)$, and denote by m_1, \dots, m_k their respective multiplicities as roots of the characteristic polynomial of A . Then A is diagonalizable if and only if $\dim(V_{\lambda_i}) = m_i$ for all $i \in 1, \dots, k$.

10

Cayley-Hamilton theorem

Theorem Let $A \in M_n(K)$ and $P_A(t)$ its characteristic polynomial, then A satisfies the following equation.

$$P_A(A) = 0.$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$\begin{aligned} p(\lambda) &= \det(\lambda I_2 - A) = \det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} \\ &= (\lambda - 1)(\lambda - 4) - (-2)(-3) = \lambda^2 - 5\lambda - 2. \end{aligned}$$

11

then

$$p(A) = A^2 - 5A - 2I_2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

12

