

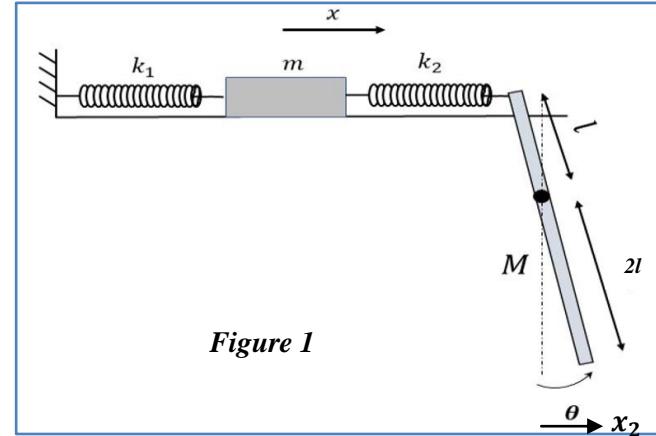
## Set 5

### Free and forced vibration of two degrees of freedom systems

#### **Exercise 1:**

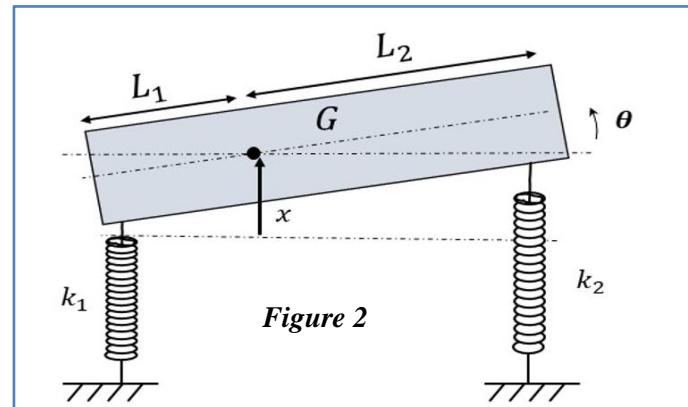
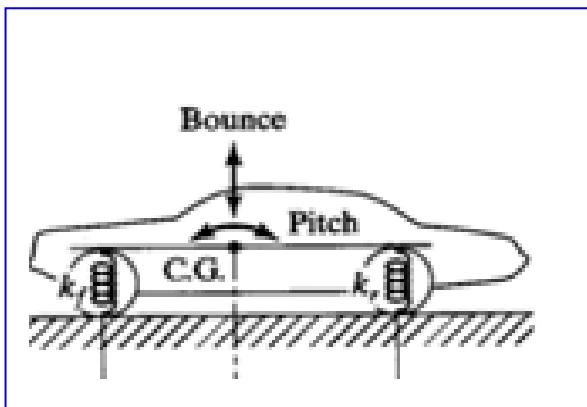
In the system shown in Figure 1, the homogeneous rod of mass  $m$  and total length  $3L$  can oscillate without friction around point O, located at a distance  $L$  from one of its ends. The upper end of the rod is connected to a mass via a spring with stiffness  $k_2$ . This mass is also connected to a fixed wall through a spring with stiffness  $k_1$ . The variables  $x_1$  and  $x_2 = 2l\theta$  are used to analyze this system.

1. Provide the expressions for the kinetic and potential energies of the system;
2. Derive the differential equations of the system;
3. Calculate the natural angular frequencies by assuming  $k_1 = k_2 = k$  and  $m_1 = m_2 = m$



#### **Exercise 2:**

In Figure 2, we have outlined a vehicle with its suspension (without dampers). We assume that the springs remain vertical. The mass of the vehicle is  $m$ , and its moment of inertia relative to a horizontal axis D passing through the center of gravity G and perpendicular to the plane of the figure is  $I_0$ . The displacement of the center of gravity relative to the equilibrium position is denoted by  $x$  (bouncing; pompage). The angle  $\theta$  (pitch; tangage) made by the chassis with the ground due to rotation around D is assumed to be small. The tilt on the sides (roll) is assumed to be zero.



The following values are provided:

- Moment of inertia of the vehicle:  $I_0 = mr^2$ ,  $r = 0.9 \text{ m}$ , vehicle mass:  $m = 1000 \text{ Kg}$ ,  $L_1 = 1 \text{ m}$ ,  $L_2 = 1.5 \text{ m}$   
 $k_1 = 18 \text{ KN/m}$     $k_2 = 18 \text{ KN/m}$

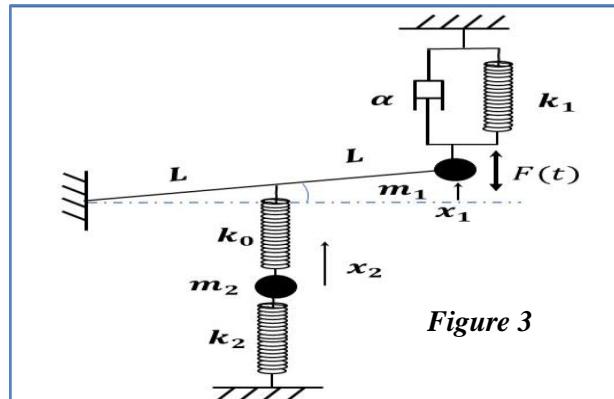
Determine the natural frequencies of the system as well as the ratio of amplitudes in each mode.

- Write the solutions of  $x(t)$  and  $\theta(t)$
- What condition must be satisfied to achieve decoupling between  $x$  and  $\theta$ ?
- What are the natural frequencies of bouncing  $f_b$  and pitching  $f_p$  under this condition?

#### **Exercise 3:**

We consider a mechanical system with two degrees of freedom, consisting of two masses  $m_1$  and  $m_2$ , identified by their respective displacements  $x_1(t)$  and  $x_2(t)$  relative to their equilibrium positions. The mass  $m_1$  is connected to a fixed frame through a spring with stiffness  $k_1$  and a damper with a damping coefficient  $\alpha$ . It is subjected to a vertical sinusoidal force with amplitude  $F_0$  and angular frequency  $\Omega$ . The mass  $m_2$  is connected to two springs: One, with stiffness  $k_0$ , has one end welded to the middle of a massless rod of negligible length

2L. The other, with stiffness  $k_2$ , is attached to a fixed frame. The rod can rotate frictionlessly around one of its extremities attached to the frame.



We consider small-amplitude motions and assume:  $m_1 = \frac{m_2}{2} = m$  and  $k_1 = \frac{k_2}{2} = \frac{k_0}{4} = k$

1. Show that the equations of motion can be expressed as follows:

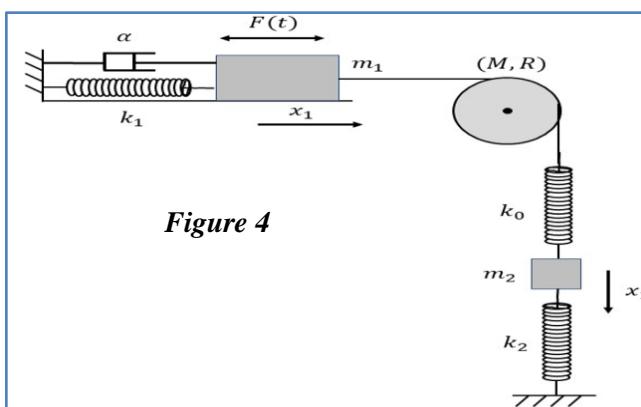
$$\begin{cases} \frac{M}{2}\ddot{x}_1 + \alpha x_1 + K(x_1 - x_2) = F(t) \\ M\ddot{x}_2 + 2Kx_2 + K(x_2 - x_1) = 0 \end{cases}$$

Determine  $M$  and  $K$  in terms of  $m$  and  $k$ .

3. a) Write the integro-differential equations in terms of the velocities  $\dot{x}_1$  and  $\dot{x}_2$ .  
b) Derive the corresponding electrical equations using the Force-Voltage analogy. Specify the correspondence between the mechanical and electrical elements.  
c) Provide the equivalent electrical circuit for the studied mechanical system.
4. a) Calculate the mechanical input impedance  $Z_i = \frac{F}{\dot{x}}$   
b) Determine the angular frequency for which the mass  $m_1$  remains stationary.

#### Exercise 4:

In the mechanical system shown in Figure 4, the pulley with mass  $M$  and radius  $R$  can oscillate frictionlessly around its axis passing through point O.



#### Part 1: Undamped free oscillations ( $\alpha = 0$ and $F(t) = 0$ ).

1. Derive the differential equations of motion.
2. Deduce the equation for natural angular frequencies.

#### Part 2: Forced Oscillations ( $F(t) = F_0 \cos(\Omega t)$ and $\alpha \neq 0$ )

Assume the following:  $M = 2(m_2 - m_1)$ ,  $m_2 = m$ ,  $k_0 = k_1 = k_2 = k$ .

- Derive the new differential equations of motion.
- Calculate the input impedance.
- Determine the velocities  $\dot{x}_1(t)$  and  $\dot{x}_2(t)$ .
- At what angular frequency does mass  $m_1$  remain stationary? What is the amplitude of the oscillations of mass  $m_2$  in this case?