National Higher School of Autonomous Systems Technology

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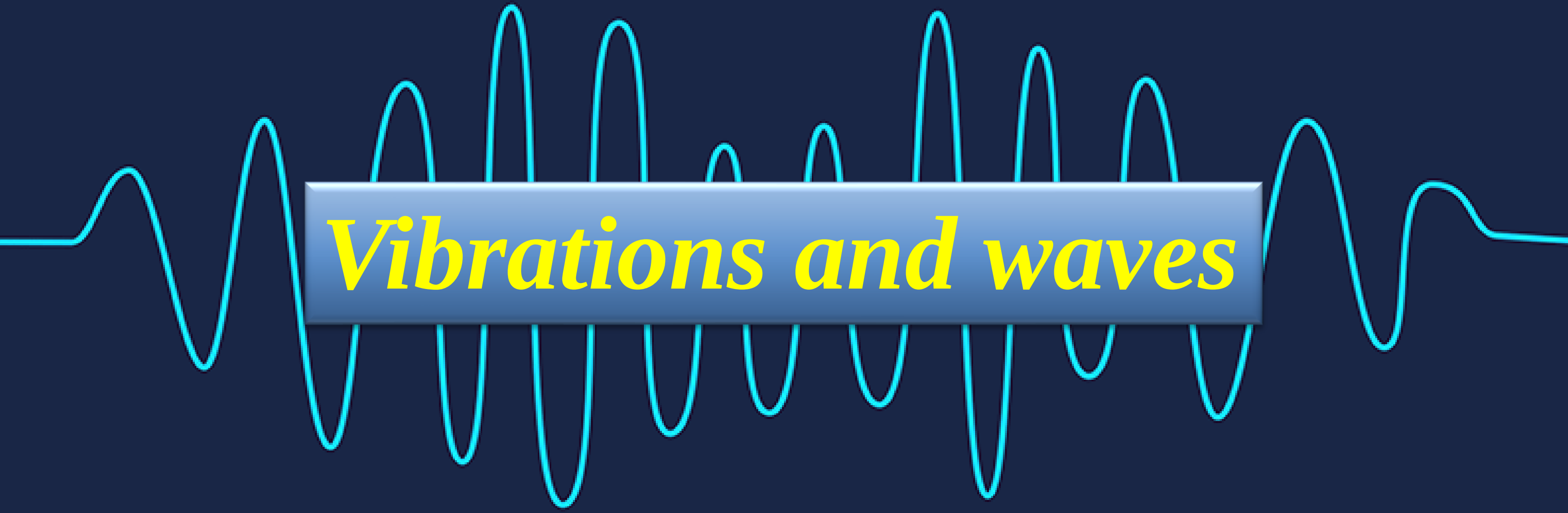


Vibrations and waves

By Dr. Malek ZENAD and Dr. Intissar DJOUADA

Chapter 2: Vibrating strings

1-Introduction

- Wave propagation phenomena are common in physics, whether in acoustics, optics, or structural mechanics. Among the fundamental models for studying these phenomena, the vibrating string plays a crucial role.
- The vibrating string equation is a wave equation that describes the propagation of disturbances in a stretched string. It is a fundamental case of wave equations and appears in various fields, such telecommunications (signal transmission), and structural engineering.

Chapter 2: Vibrating strings

1-Introduction

- Studying the vibrating string not only helps to understand mechanical vibrations but also introduces key concepts such as standing waves, natural frequencies, and normal modes. This course aims to explore these aspects in detail through both analytical and experimental approaches.

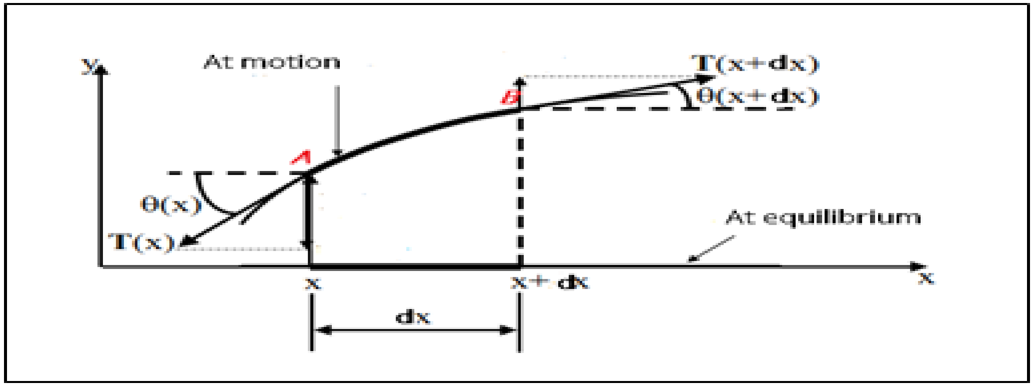
Example1: The analysis of vibrations in suspension bridges (such as the Tacoma Narrows Bridge) relies on equations similar to those of a vibrating string.

Example2: Radio waves and electrical signals in cables follow principles similar to mechanical waves in a vibrating string.

Chapter 2: Vibrating strings

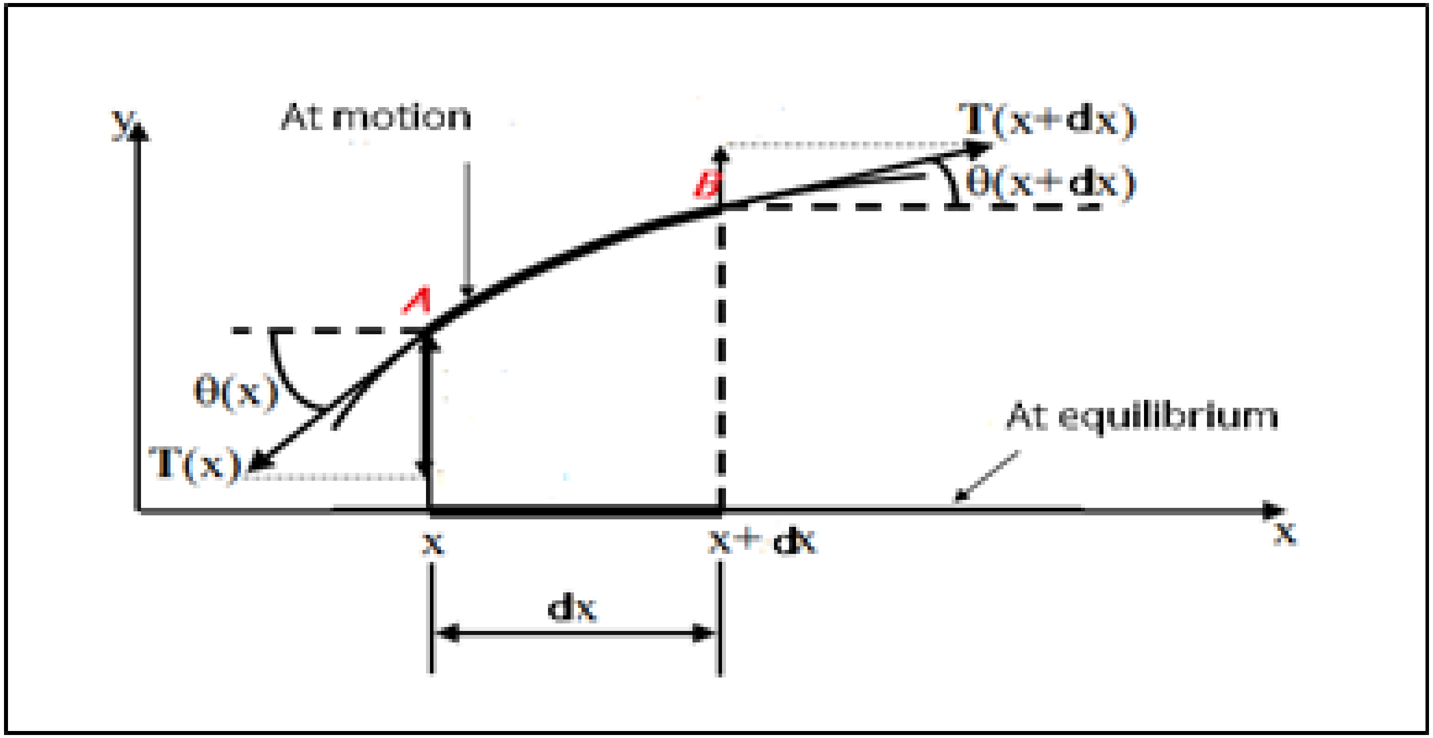
2- Wave equation for vibrating string

Let's consider a homogeneous and inextensible string with a linear mass density μ (mass per unit length). At equilibrium, the string takes a straight-line shape. The string is subjected to a tension $T = T_A$ applied at point A. The tension T is constant and significantly greater than gravitational forces ($T \gg P$). Now, consider a small element of the string $dx = AB$. Under the applied tension T, this element experiences a tension $T = T_B$ at its endpoint B, which results from the tension exerted by the right-hand side of the string.



Chapter 2: Vibrating strings

2- Wave equation for vibrating string



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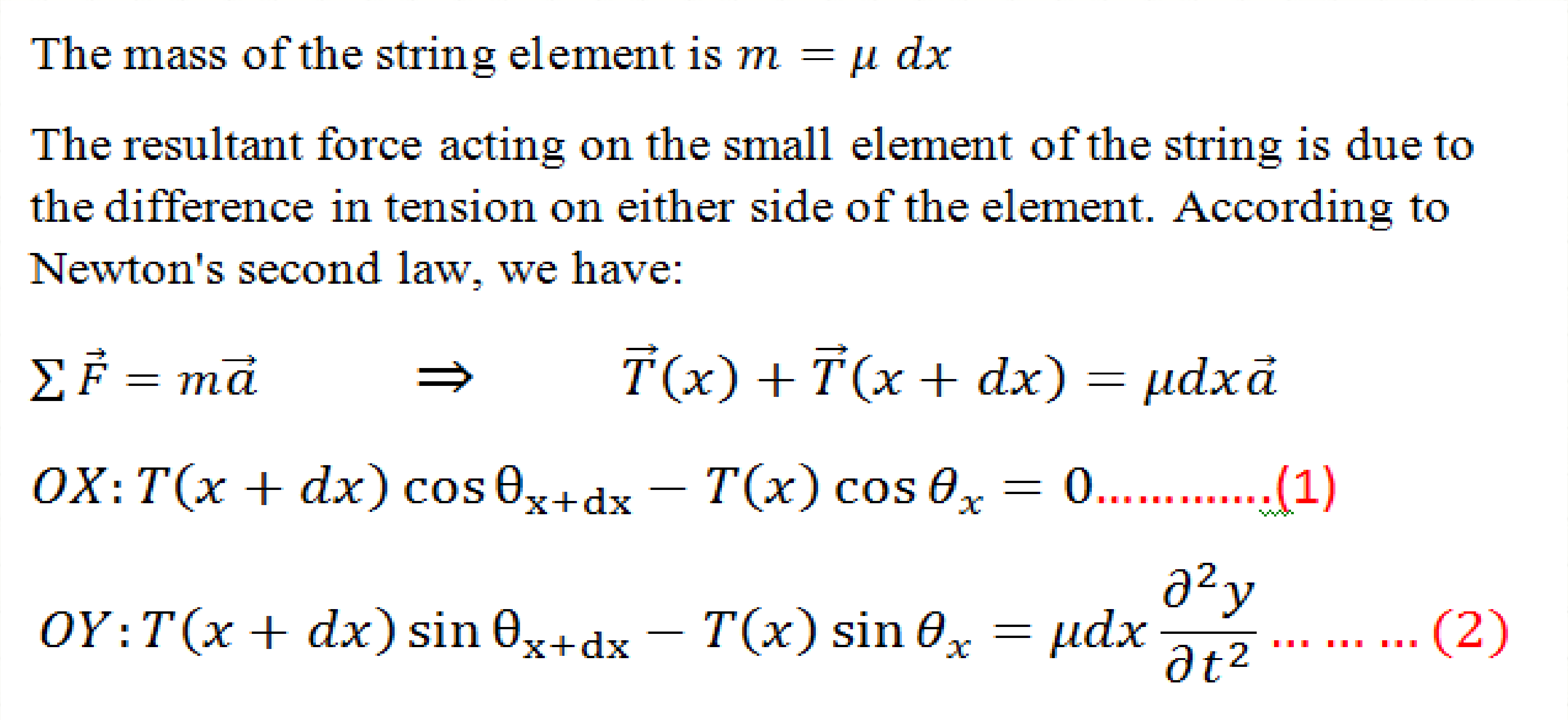
2- Wave equation for vibrating string



The mass of the string element is $m = \mu\, dx$

The resultant force acting on the small element of the string is due to the difference in tension on either side of the element. According to Newton's second law, we have:

$$\sum \vec{F} = m\vec{a} \qquad \Rightarrow \qquad \vec{T}(x) + \vec{T}(x+dx) = \mu dx \vec{a}$$

$$OX: T(x+dx)\cos\theta_{x+dx} - T(x)\cos\theta_x = 0 \text{............}(1)$$

$$OY: T(x+dx)\sin\theta_{x+dx} - T(x)\sin\theta_x = \mu dx \frac{\partial^2 y}{\partial t^2} \text{ ... ... }(2)$$

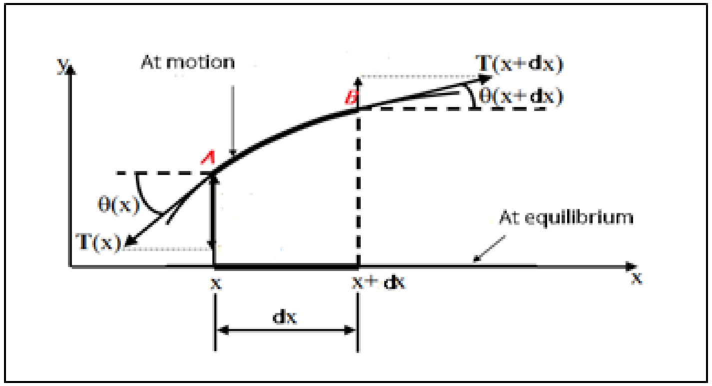
Chapter 2: Vibrating strings

2- Wave equation for vibrating string

- Using an approximation for small angle variations:

$$For\ \theta \ll 1 \Rightarrow \begin{cases} cos\theta \approx 1 \\ sin\theta \approx tg\theta \end{cases} \Rightarrow \quad sin\theta_x \approx tg\theta_x = \frac{\Delta y}{\Delta x} \approx \frac{\partial y}{\partial x}$$

(1) $$T(x+dx) = T(x) = T$$



(2) $$\Rightarrow \quad T\left(\frac{\partial y}{\partial x}\Big|_x - \frac{\partial y}{\partial x}\Big|_{x+dx}\right) = \mu dx \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad T \frac{\left(\frac{\partial y}{\partial x}\Big|_x - \frac{\partial y}{\partial x}\Big|_{x+dx}\right)}{dx} = \mu \frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad T\frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}$$

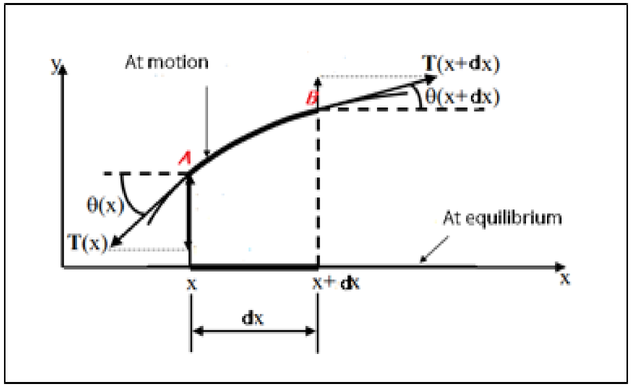
Chapter 2: Vibrating strings

2- Wave equation for vibrating string

$$\Rightarrow \quad \frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad or \quad \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

Where v is the wave propagation speed, given by:

$$v = \sqrt{\frac{T}{\mu}}$$



- μ is in $\boldsymbol{kg/m}$ (mass per unit length),
- T is in $\boldsymbol{N}$ (tension in Newtons),
- $u(x,t)$ is the displacement of the string, measured in **m** (meters).

Chapter 2: Vibrating strings

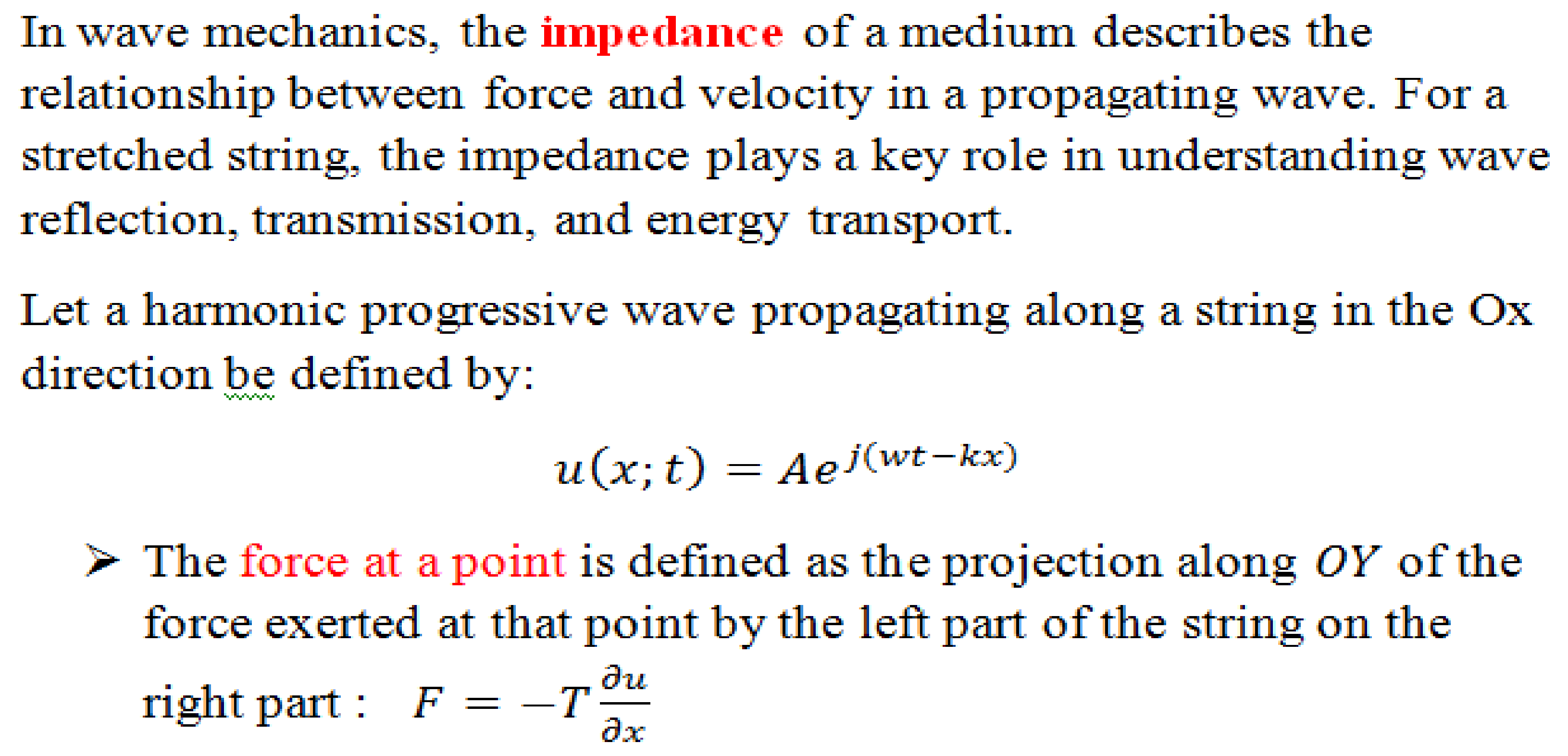
2- Wave equation for vibrating string

Note:

- Vibrating string represents a transverse wave, as the oscillations of the string occur in a direction perpendicular to the direction of wave propagation
- One should not confuse the wave propagation speed (the speed at which the wave travels along the string $v = \sqrt{\frac{T}{\mu}}$) with the vertical displacement speed of the string element (the speed at which a specific point on the string moves in the perpendicular direction to the string $\dot{y}(x,t) = \frac{dy(x,t)}{dt}$).

Chapter 2: Vibrating strings

3- Impedance



In wave mechanics, the **impedance** of a medium describes the relationship between force and velocity in a propagating wave. For a stretched string, the impedance plays a key role in understanding wave reflection, transmission, and energy transport.

Let a harmonic progressive wave propagating along a string in the Ox direction be defined by:

$$u(x;t) = Ae^{j(wt-kx)}$$

- The force at a point is defined as the projection along OY of the force exerted at that point by the left part of the string on the right part : $F = -T\frac{\partial u}{\partial x}$

Chapter 2: Vibrating strings

3- Impedance

- The impedance at a point is defined as the ratio of the complex amplitude of the force to the complex amplitude of the particle velocity

$$Z(x) = \frac{F_y}{\dot{u}(x,t)}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = -jkAe^{j(wt-kx)} \\ \dot{u}(x,t) = jwAe^{j(wt-kx)} \end{cases} \Rightarrow \quad z(x) = -T\,\frac{-jkAe^{j(wt-kx)}}{jwAe^{j(wt-kx)}}$$

Chapter 2: Vibrating strings

3- Impedance

$$Z(x) = \frac{kT}{w} \qquad \Rightarrow \qquad Z_c = \frac{kT}{w}$$

Using the wave velocity $v = \sqrt{\frac{T}{\mu}}$, we get:

$$Z_c = \frac{kT}{w} \qquad \Rightarrow \qquad Z_c = \mu v \quad \Rightarrow \qquad or \qquad \boldsymbol{Z_c = \sqrt{\mu T}}$$

Where μ is the linear mass density.

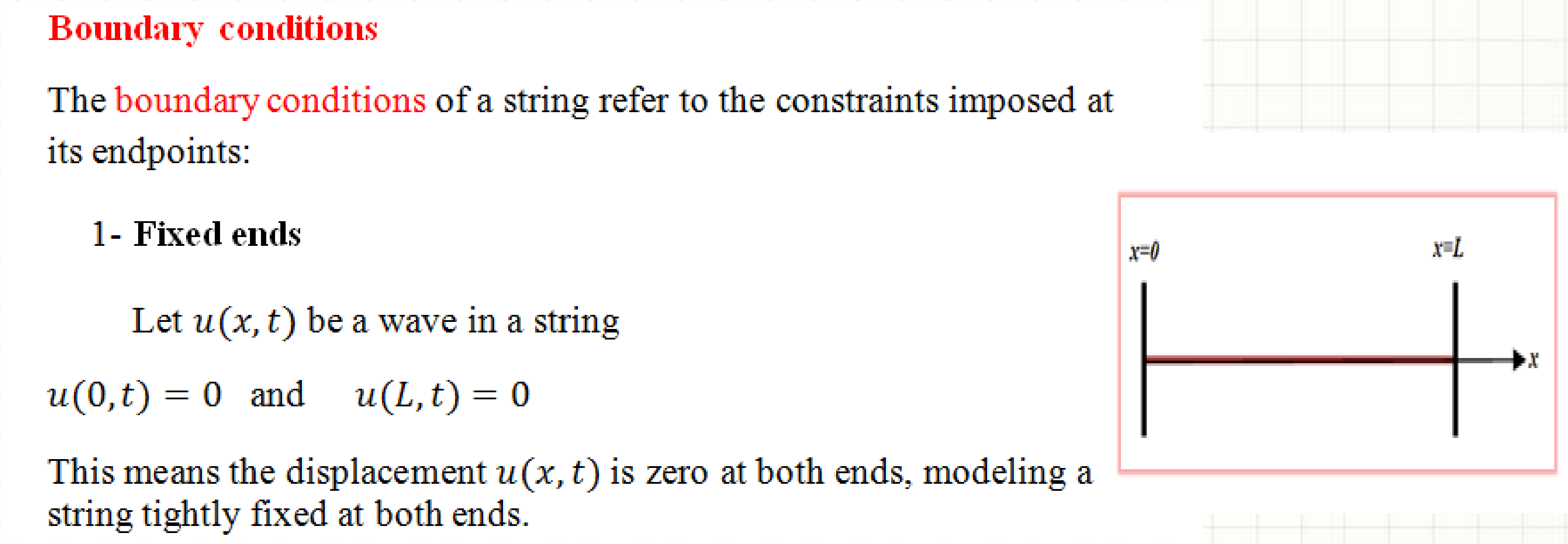
- Z_c Represents how the string resists transverse motion due to tension and mass distribution.
- A property of the plane progressive wave is obtained:

$$\forall x \qquad\qquad Z(x) = Z_c = \sqrt{\mu T}$$

Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

Boundary conditions



Boundary conditions

The boundary conditions of a string refer to the constraints imposed at its endpoints:

1- Fixed ends

Let $u(x,t)$ be a wave in a string

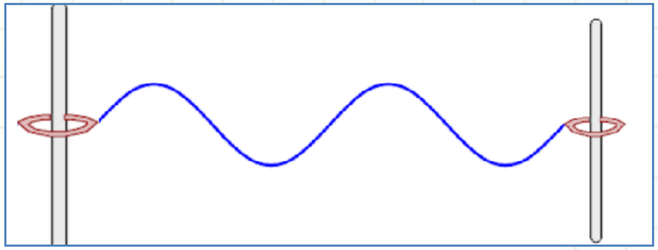
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

This means the displacement $u(x,t)$ is zero at both ends, modeling a string tightly fixed at both ends.

Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

Boundary conditions

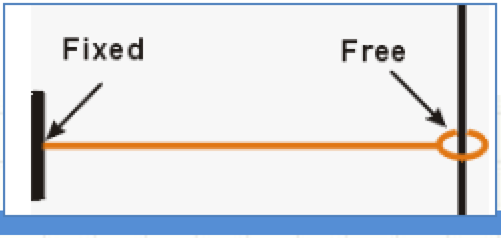


2- Free ends

$$\frac{\partial U(0,t)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial U(l,t)}{\partial x} = 0 \quad \text{(PFD)}$$

This implies that there is no force (zero slope) at the endpoints, modeling a freely vibrating string.

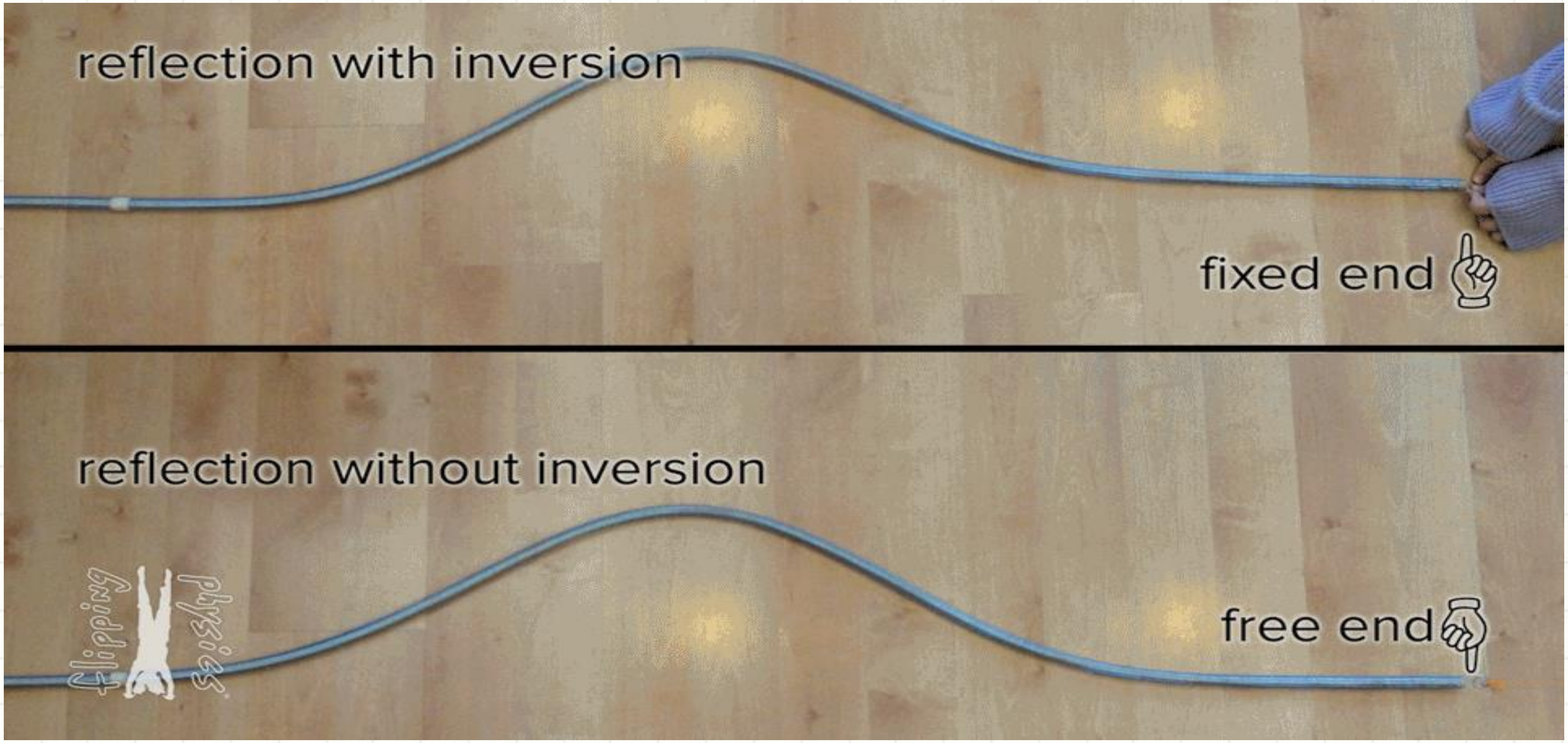
3-Mixed conditions: One end could be fixed while the other is free, depending on the physical setup.



Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

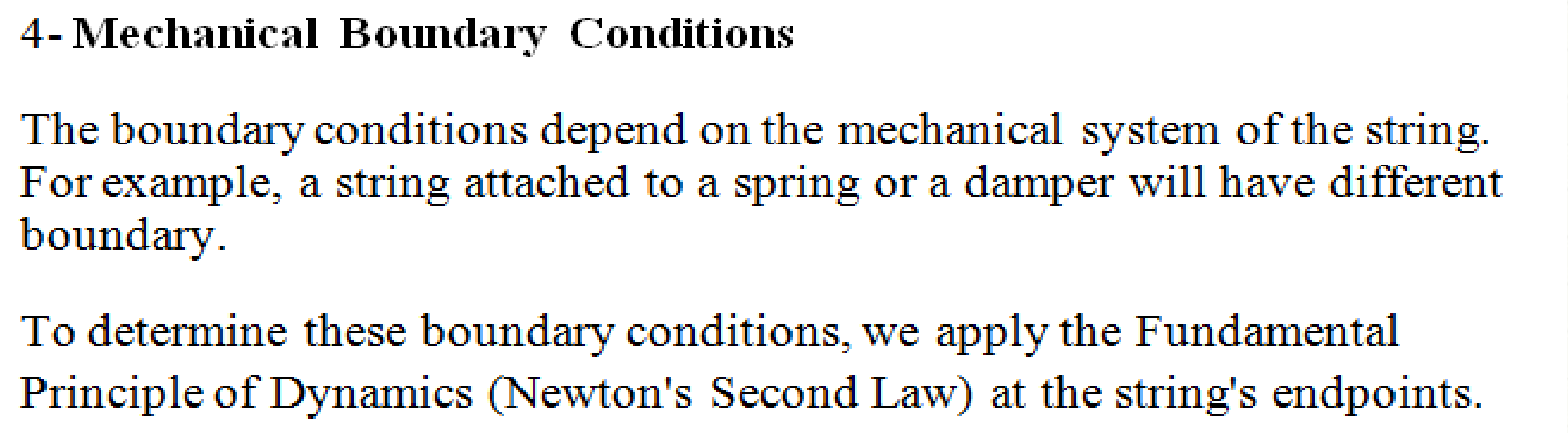
Boundary conditions



Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

Boundary conditions

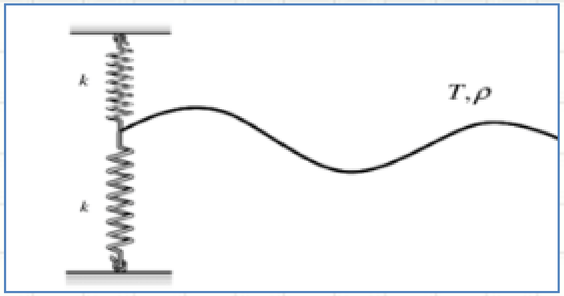


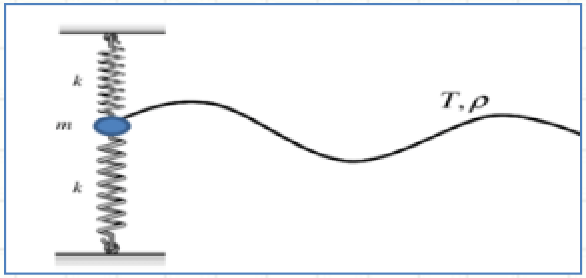
4- Mechanical Boundary Conditions

The boundary conditions depend on the mechanical system of the string. For example, a string attached to a spring or a damper will have different boundary.

To determine these boundary conditions, we apply the Fundamental Principle of Dynamics (Newton's Second Law) at the string's endpoints.

- **Example**





Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

Let us consider a string of length L fixed at the points $x = 0$ and $x = L$.

Let us look for a solution of the wave equation in the form:

$$u(x,t) = f(x - vt) + g(x + vt)$$

Or $$u(x,t) = f\left(t - \frac{x}{}\right) + g\left(t + \frac{x}{}\right)$$

Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

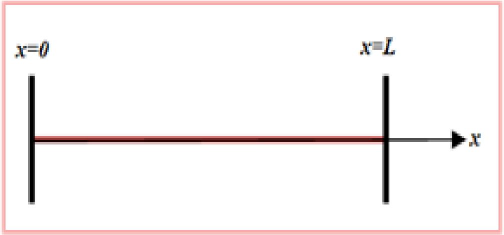
The **boundary conditions** of a string refer to the constraints imposed at its endpoints. For a string of length L fixed at x=0 and x=L.

- Using the boundary conditions:

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \Rightarrow \begin{cases} f(t) + g(t) = 0 \ldots\ldots\ldots (1) \\ f\left(t - \frac{L}{v}\right) + g\left(t + \frac{L}{v}\right) = 0 \ldots\ldots (2) \end{cases}$$

$$\begin{cases} g(t) = -f(t) \\ f\left(t - \frac{L}{v}\right) + g\left(t + \frac{L}{v}\right) = 0 \end{cases} \Rightarrow g\left(t + \frac{L}{v}\right) = -f\left(t + \frac{L}{v}\right) \ldots . (3)$$

$$(3) \Rightarrow f\left(t - \frac{L}{v}\right) - f\left(t + \frac{L}{v}\right) = 0$$



Chapter 2: Vibrating strings

4- Wave propagation in a finite-length string

We put $\tau = t - \frac{L}{v} \qquad \Rightarrow$

$$f(\tau) - f\left(t + \frac{L}{v} + \frac{L}{v} - \frac{L}{v}\right) = 0 \quad \Rightarrow \quad f(\tau) - f\left(\tau + \frac{2L}{v}\right) = 0$$

The function f is periodic with period $T=\frac{2L}{v}$. The corresponding angular frequency (pulsation) is given by:

$$\omega = \frac{2\pi}{T} = \frac{\pi v}{L}$$

We define the fundamental angular frequency as , so that the general angular frequencies are given by: $\omega_0 = \frac{\pi v}{L}$

✓ $\omega_n = n\omega_0$, with n being a positive integer.

$$\omega_n = n\omega_0 \quad \Rightarrow \quad \omega_n = n\,\frac{\pi v}{L}$$

4- Wave propagation in a finite-length string

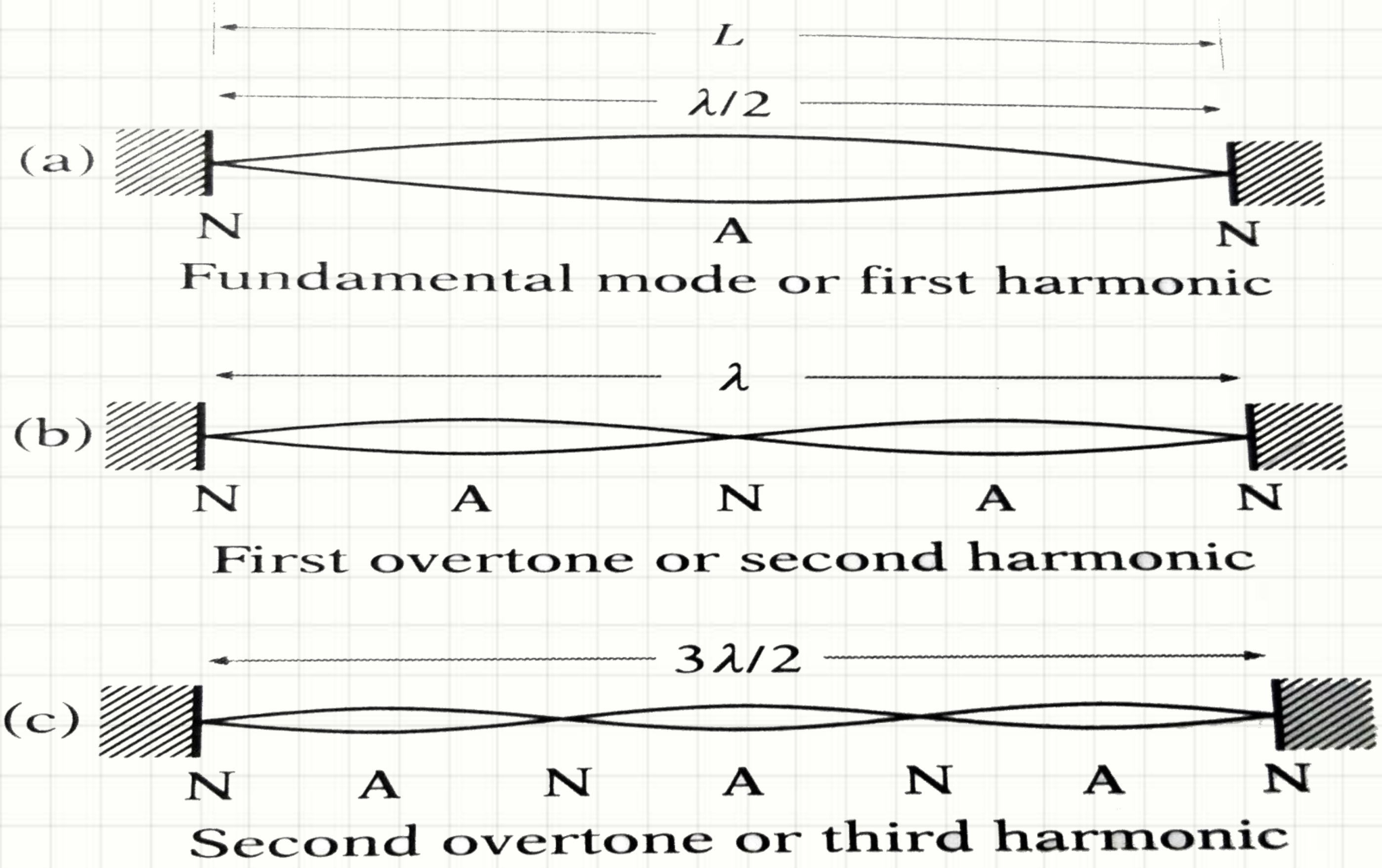
The natural modes of a vibrating string of length L are given by standing wave solutions that satisfy the boundary conditions. The displacement $u(x,t)$ can be expressed as:

$$u(x;t) = A_n\ sin\frac{n\pi x}{L}\cos(\omega_n t + \varphi)$$

Each mode corresponds to a standing wave with n half-wavelengths fitting within the string length L. $L = n\frac{\lambda}{2}$

Chapter 2: Vibrating strings

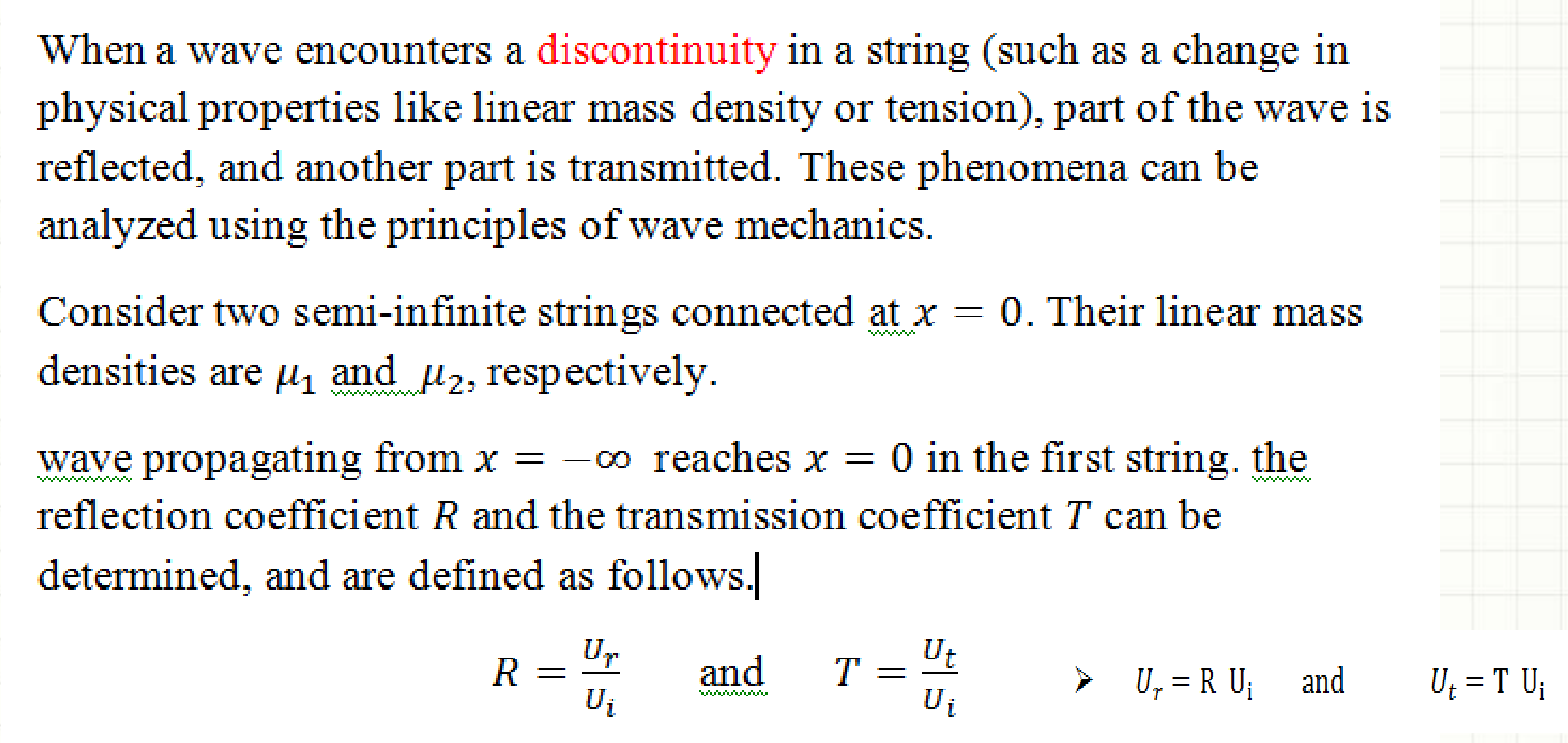
4- Wave propagation in a finite-length string



Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings



When a wave encounters a discontinuity in a string (such as a change in physical properties like linear mass density or tension), part of the wave is reflected, and another part is transmitted. These phenomena can be analyzed using the principles of wave mechanics.

Consider two semi-infinite strings connected at $x = 0$. Their linear mass densities are μ_1 and μ_2, respectively.

wave propagating from $x = -\infty$ reaches $x = 0$ in the first string. the reflection coefficient R and the transmission coefficient T can be determined, and are defined as follows.

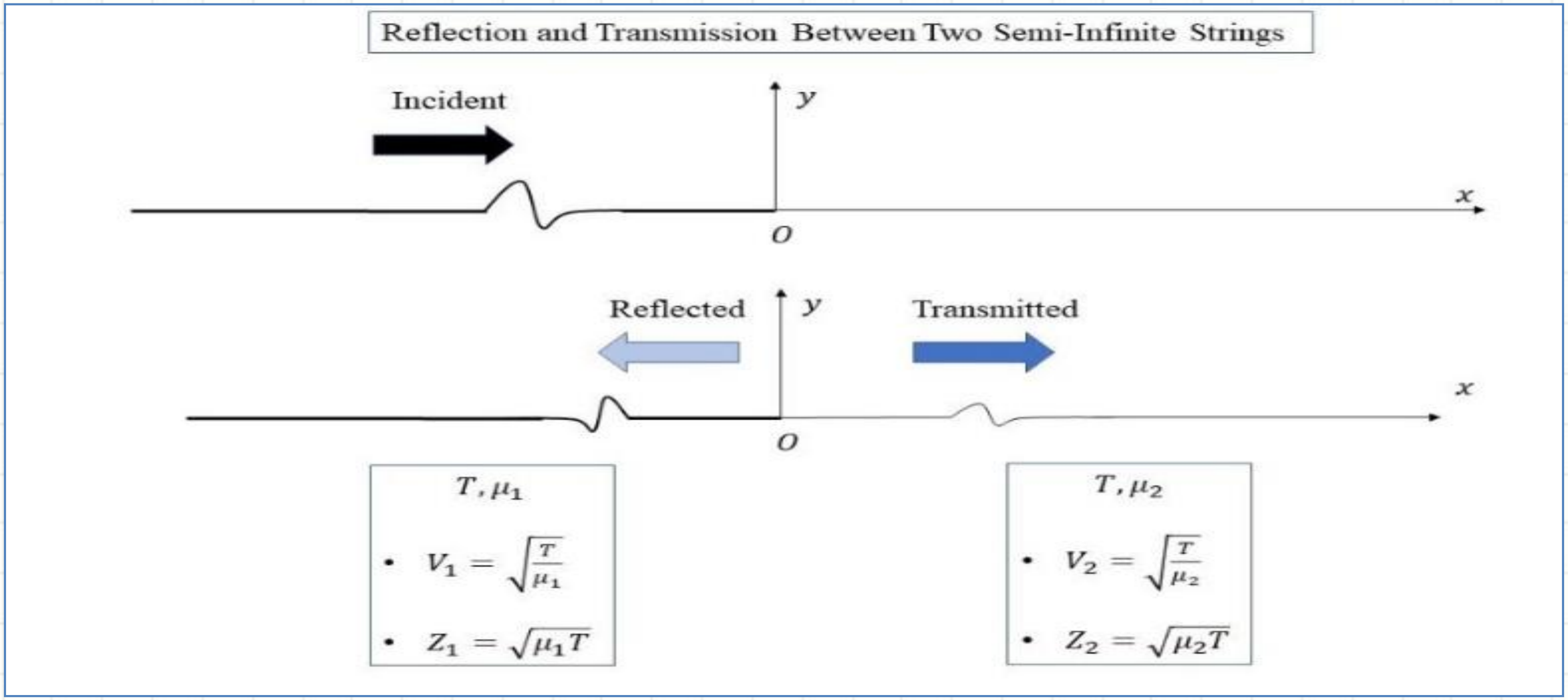
$$R = \frac{U_r}{U_i} \quad \text{and} \quad T = \frac{U_t}{U_i}$$

➤ $U_r = R\ U_i$ and $U_t = T\ U_i$

Chapter 2: Vibrating strings

5- Reflection and transmission

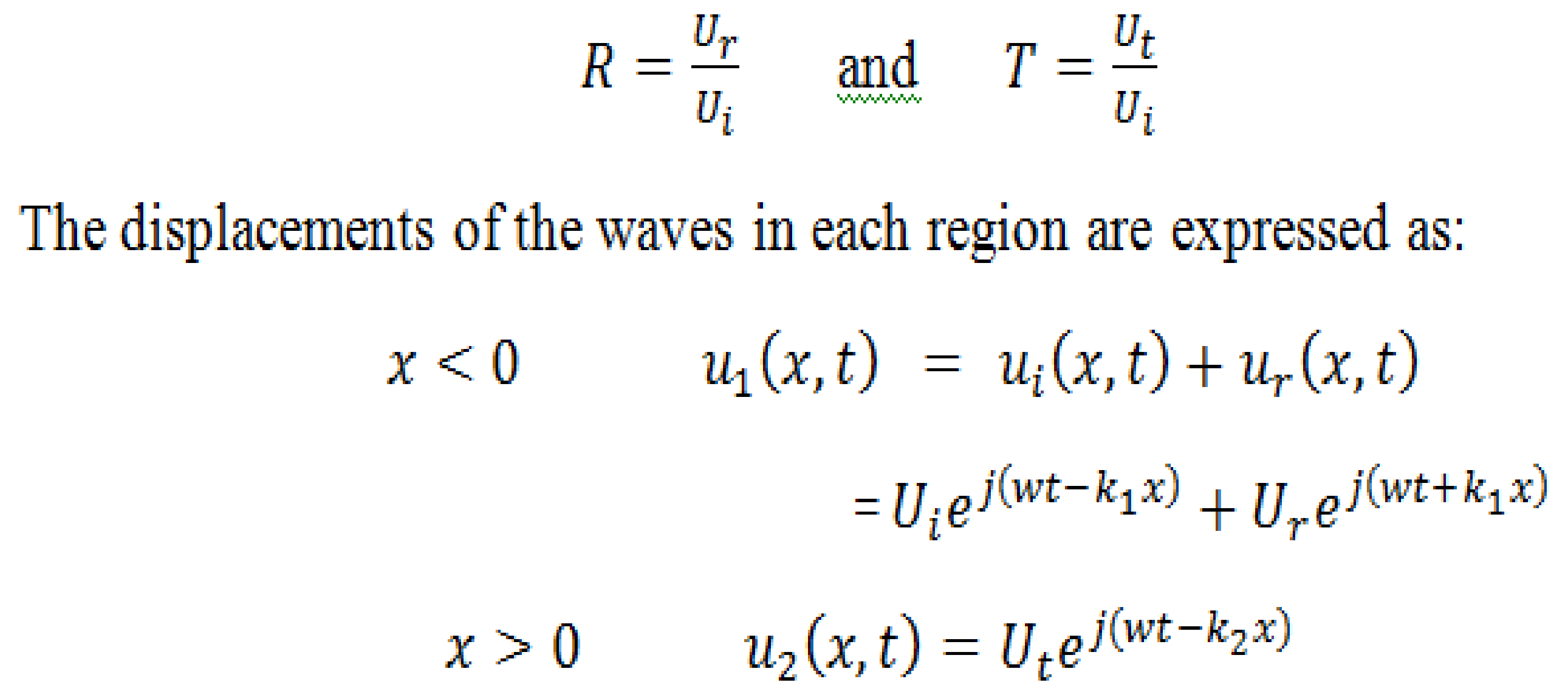
1-Case of two semi-infinite strings



Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings



$$R = \frac{U_r}{U_i} \quad \text{and} \quad T = \frac{U_t}{U_i}$$

The displacements of the waves in each region are expressed as:

$$x < 0 \qquad u_1(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt-k_1x)} + U_r e^{j(wt+k_1x)}$$

$$x > 0 \qquad u_2(x,t) = U_t e^{j(wt-k_2x)}$$

Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings

Using Boundary Conditions

To determine the reflection and transmission coefficients, two conditions are applied at x=0:

- ✓ **Continuity of displacement:**

  $u_1(0,t) = u_2(0,t)$, This ensures the string remains connected.

  $\Rightarrow \quad U_i + U_r = U_t \ldots\ldots\ldots.(1) \quad \Rightarrow \quad 1 + R = T$

- ✓ **Continuity of force (PFD at x=0):**

  This represents the continuity of mechanical tension at the junction.

  Oy: $$-T\frac{\partial u_1}{\partial x}\bigg|_{x=0} + T\frac{\partial u_2}{\partial x}\bigg|_{x=0} = m.\frac{\partial^2 u}{\partial t^2}$$

Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings

- m is very small

$$\Rightarrow \quad -T\frac{\partial u_1}{\partial x}\bigg|_{x=0} + T\frac{\partial u_2}{\partial x}\bigg|_{x=0} = 0$$

$$-T\left(-jk_1 U_i e^{j(wt-k_1.0)} + jk_1 U_r e^{j(wt+k_1.0)}\right) - Tjk_2 U_t e^{j(wt+k_2.0)} = 0$$

$$k_1(U_i - U_r) = k_2 U_t \qquad \Rightarrow \qquad U_i - U_r = \frac{k_2}{k_1} U_t \;\dots\dots\dots.(2)$$

Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings

$$\Rightarrow \qquad \begin{cases} U_i + U_r = U_t \dots\dots\dots (1) \\ U_i - U_r = \frac{k_2}{k_1} U_t \dots\dots . (2) \end{cases}$$

$$(1) + (2) \quad and \ (1) - (2) \ \Rightarrow \begin{cases} 2U_i = U_t(1 + \frac{k_2}{k_1}) \\ 2U_r = U_t(1 - \frac{k_2}{k_1}) \end{cases}$$

$$R = \frac{U_r}{U_i} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \text{or} \qquad R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$T = \frac{U_t}{U_i} = \frac{2k_1}{k_1 + k_2} \qquad \text{or} \qquad T = \frac{2Z_1}{Z_1 + Z_2}$$

Chapter 2: Vibrating strings

5- *Reflection and transmission*

1-Case of two semi-infinite strings

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad \Rightarrow \quad R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

$$T = \frac{2 Z_1}{Z_1 + Z_1} \quad \Rightarrow \quad T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

- In the general case $R = |R| . e^{j\,\theta}$

Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings

- There are various cases to consider:

1- Uniform string:

$$\mu_2 = \mu_1 \quad \Rightarrow \quad Z_2 = Z_1 \quad \Rightarrow \quad R = 0 \quad and \quad T = 1$$

$$R = |R|.e^{j\,\theta} \quad \Rightarrow \quad R = 0$$

- No reflection occurs, and the wave passes through without any change.

2- Brick wall on the right (rigid wall):

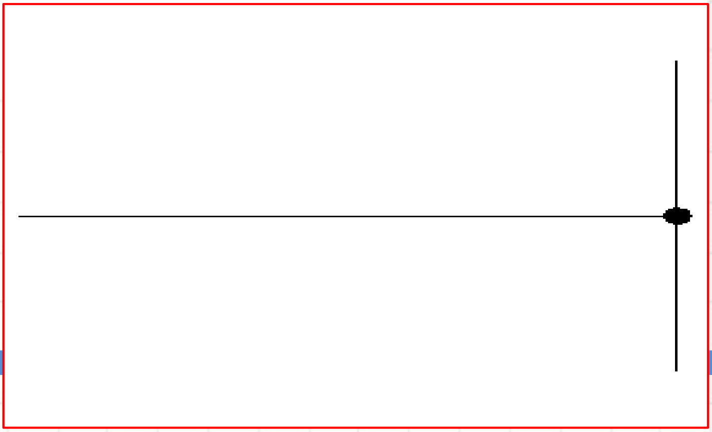
$$\mu_2 = \infty \quad \Rightarrow \quad Z_2 = \infty$$

$$R = -1 \quad and \quad T = 0$$

- Nothing is transmitted $T = 0$, but the reflected wave has the same amplitude as the incident wave and is inverted R = -1.

$$R = |R|.e^{j\,\theta} \quad \Rightarrow \quad R = -1 \quad \Rightarrow \quad \begin{cases} |R| = 1 \\ \theta = \pi \end{cases}$$

- Full reflection **with inversion**.



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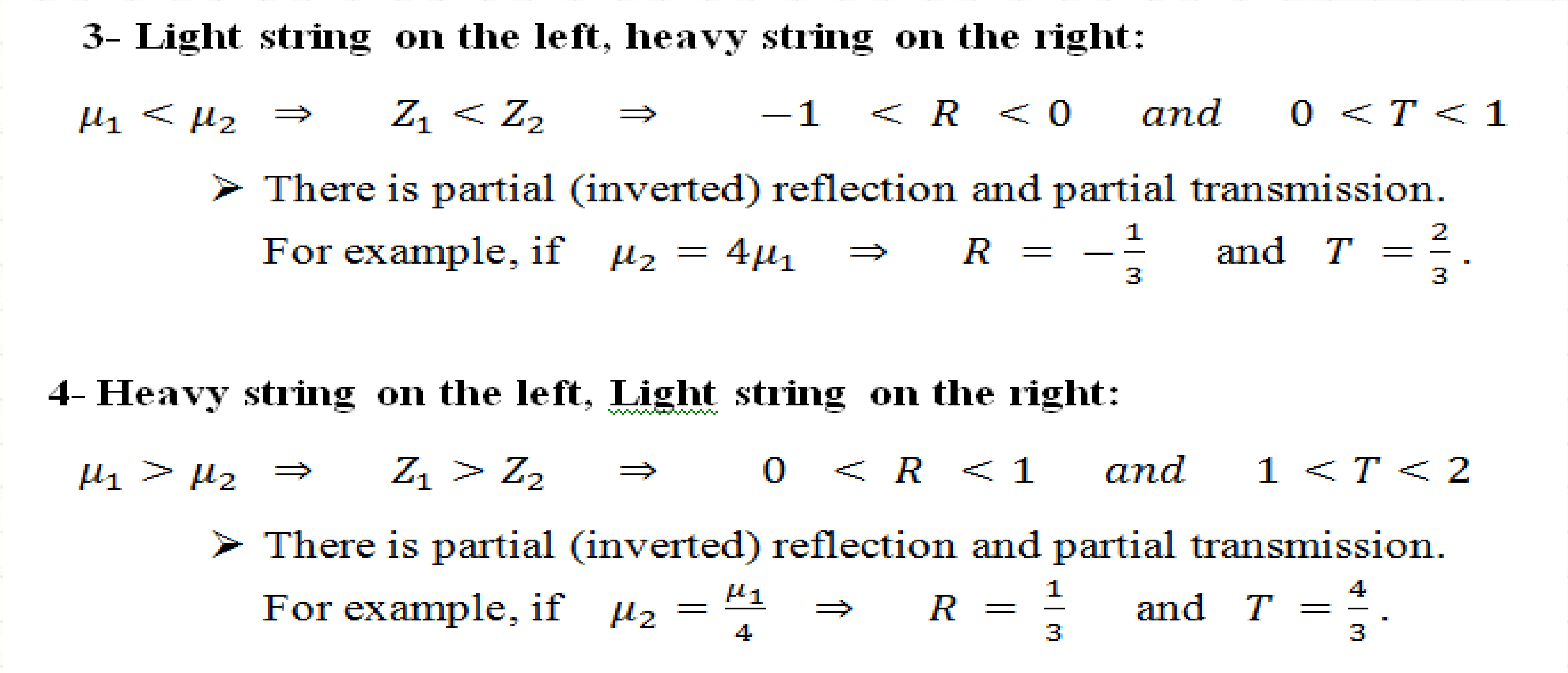
5- Reflection and transmission

1-Case of two semi-infinite strings



3- Light string on the left, heavy string on the right:

$$\mu_1 < \mu_2 \quad \Rightarrow \quad Z_1 < Z_2 \quad \Rightarrow \quad -1 < R < 0 \quad and \quad 0 < T < 1$$

- ➢ There is partial (inverted) reflection and partial transmission.
  For example, if $\mu_2 = 4\mu_1 \quad \Rightarrow \quad R = -\frac{1}{3}$ and $T = \frac{2}{3}$.

4- Heavy string on the left, Light string on the right:

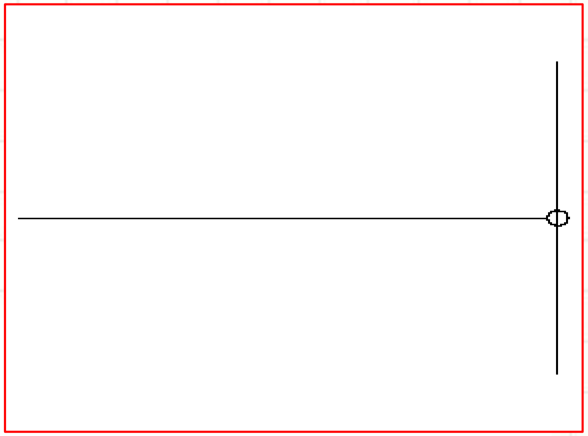
$$\mu_1 > \mu_2 \quad \Rightarrow \quad Z_1 > Z_2 \quad \Rightarrow \quad 0 < R < 1 \quad and \quad 1 < T < 2$$

- ➢ There is partial (inverted) reflection and partial transmission.
  For example, if $\mu_2 = \frac{\mu_1}{4} \quad \Rightarrow \quad R = \frac{1}{3}$ and $T = \frac{4}{3}$.

Chapter 2: Vibrating strings

5- Reflection and transmission

1-Case of two semi-infinite strings



5- Zero-Mass String on Right $\equiv$ Free end

$$\mu_2 = 0 \quad \Rightarrow \quad Z_2 = 0 \quad \Rightarrow \quad R = 1 \quad \Rightarrow \quad T=2$$

$$R = |R|.\,e^{j\theta} \quad \Rightarrow \quad R = 1 \quad \Rightarrow \quad \begin{cases} |R| = 1 \\ \theta = 0 \end{cases}$$

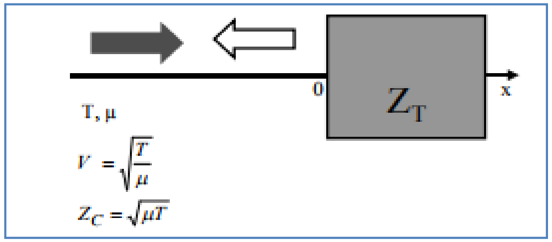
- The string on the right has no mass, so it can’t carry energy.
- Full reflection **without inversion**. The reflected wave has the **same phase** as the incident wave.

Chapter 2: Vibrating strings

5- Reflection and transmission

3- Reflection on an impedance.

Consider a semi-infinite string of linear mass density μ, stretched horizontally with a tension T, and terminated at $x = 0$ by a mechanical impedance Z_T. When a harmonic wave propagates along the string from $x = -\infty$ towards $x = 0$, it undergoes a reflection at this point. Knowing that the particle displacement is expressed as:



$$u(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)}$$

The impedance at a point x is defined as:

$$Z(x) = \frac{F_y}{\dot{u}(x,t)} \quad \Rightarrow \quad Z(x) = \frac{-T\frac{\partial u(x,t)}{\partial x}}{\frac{\partial u(x,t)}{\partial t}}$$

Chapter 2: Vibrating strings

5- Reflection and transmission

2-Reflection on an impedance

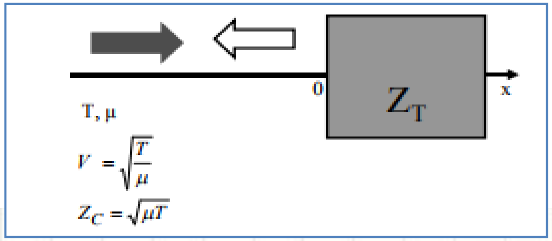
$$Z(x) = \frac{-T(-jkU_i e^{j(wt-kx)} + jkU_r e^{j(wt+kx)})}{jw(U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)})}$$

$$\Rightarrow \quad Z(x) = \frac{Tk}{w} \cdot \frac{U_i\, e^{-jkx} - U_r e^{jkx}}{U_i\, e^{-jkx} - U_r e^{jkx}} \quad \Rightarrow \quad Z(0) = Z_C \frac{1-R}{1+R}$$

$$At\ x = 0 \quad \Rightarrow \quad Z(0) = Z_T \quad \Rightarrow \quad Z_T = Z_C \frac{1-R}{1+R}$$

From this, we deduce the reflection coefficient R as a function of the characteristic impedance Z_c of the string and the impedance Z_T at the string's extremity:

$$R = \frac{Z_c - Z_T}{Z_c + Z_T}$$



Chapter 2: Vibrating strings

5- Reflection and transmission

2-Reflection on an impedance

Reflection coefficients R for a string terminated by different elements:

1. String Terminated by a Mass m:

The mechanical impedance of the mass is given by $Z_T = Z_m = jmw$

The reflection coefficient is:

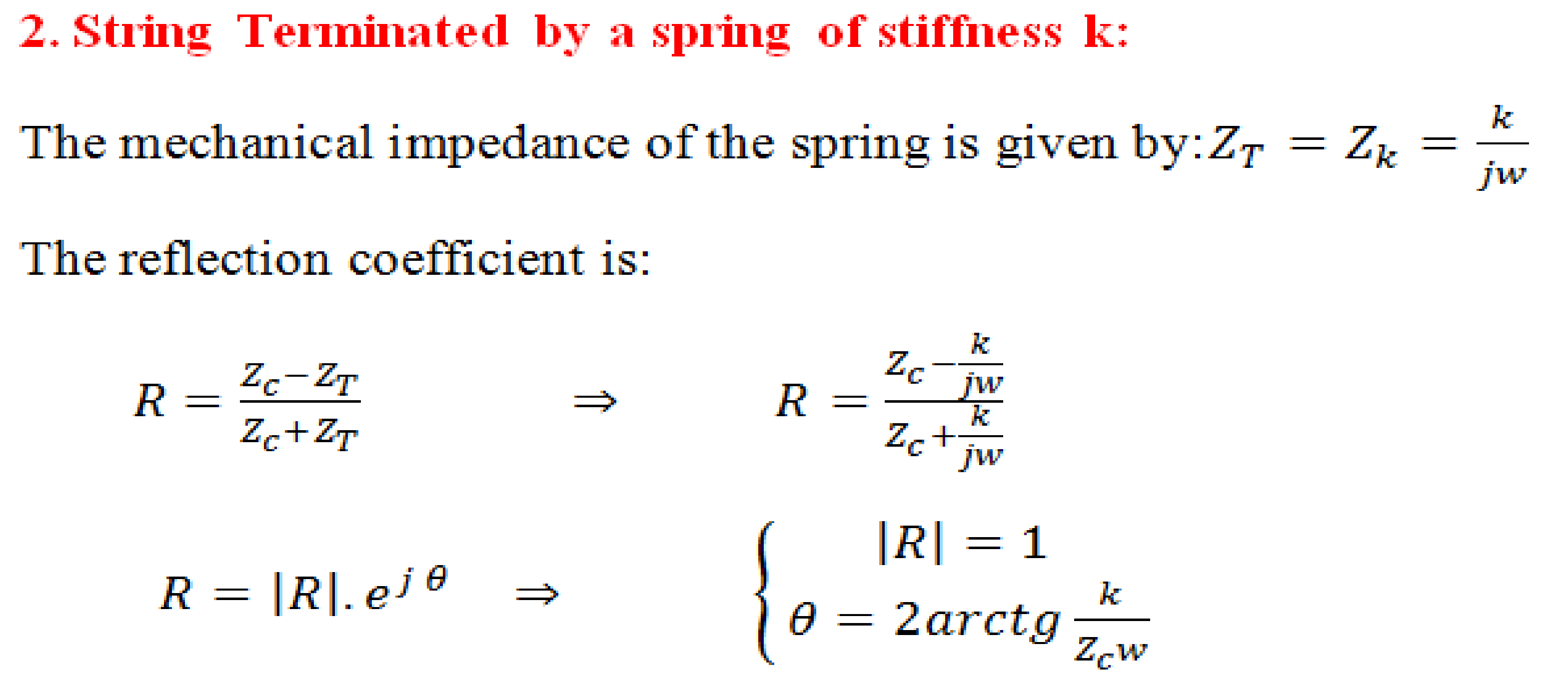
$$R = \frac{Z_c - Z_T}{Z_c + Z_T} \quad \Rightarrow \quad R = \frac{Z_c - jmw}{Z_c + jmw}$$

$$R = |R|.e^{j\,\theta} \quad \Rightarrow \quad \begin{cases} |R| = 1 \\ \theta = -2arctg\,\frac{mw}{Z_c} \end{cases}$$

Chapter 2: Vibrating strings

5- Reflection and transmission

2-Reflection on an impedance



2. String Terminated by a spring of stiffness k:

The mechanical impedance of the spring is given by: $Z_T = Z_k = \frac{k}{jw}$

The reflection coefficient is:

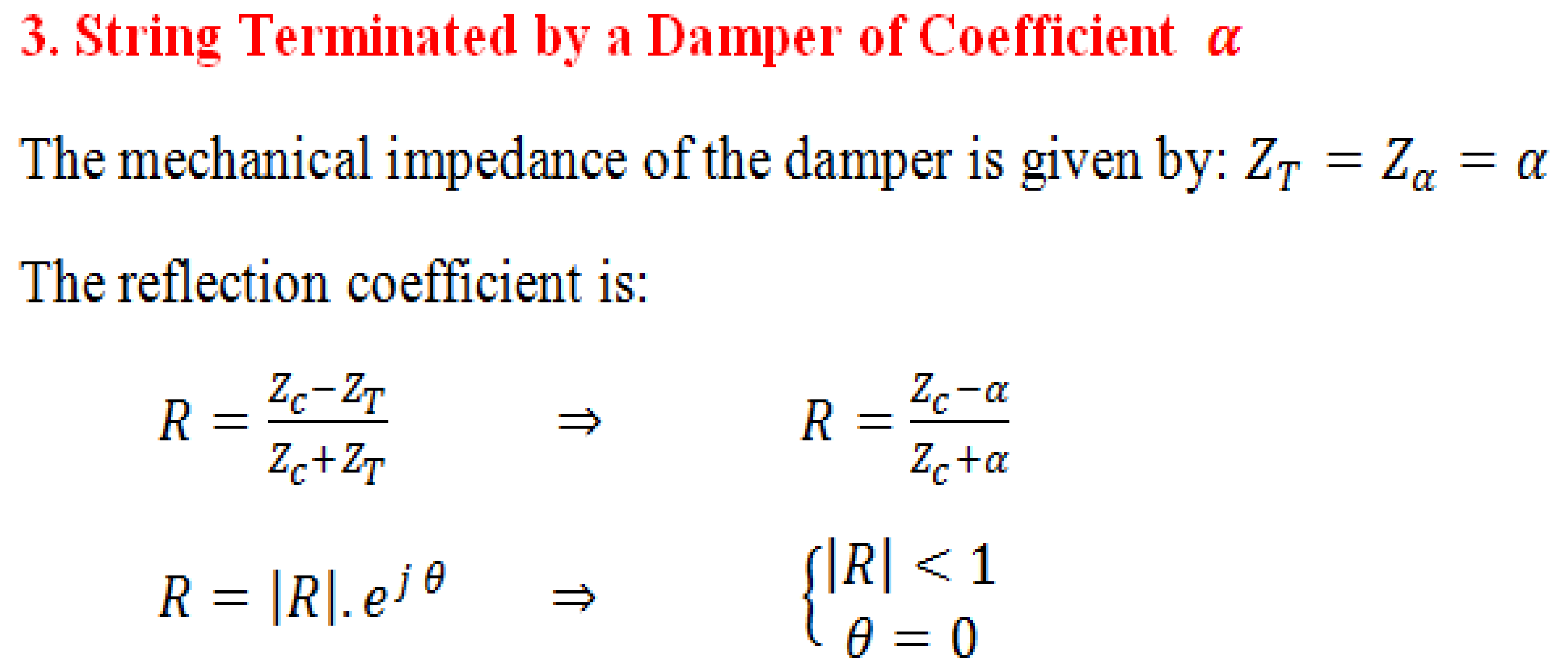
$$R = \frac{Z_c - Z_T}{Z_c + Z_T} \quad \Rightarrow \quad R = \frac{Z_c - \frac{k}{jw}}{Z_c + \frac{k}{jw}}$$

$$R = |R|.\, e^{j\,\theta} \quad \Rightarrow \quad \begin{cases} |R| = 1 \\ \theta = 2arctg\,\frac{k}{Z_c w} \end{cases}$$

Chapter 2: Vibrating strings

5- Reflection and transmission

2-Reflection on an impedance



3. String Terminated by a Damper of Coefficient α

The mechanical impedance of the damper is given by: $Z_T = Z_\alpha = \alpha$

The reflection coefficient is:

$$R = \frac{Z_c - Z_T}{Z_c + Z_T} \quad \Rightarrow \quad R = \frac{Z_c - \alpha}{Z_c + \alpha}$$

$$R = |R|.e^{j\,\theta} \quad \Rightarrow \quad \begin{cases} |R| < 1 \\ \theta = 0 \end{cases}$$

Chapter 2: Vibrating strings

6-Energy Density

Consider a little segment of the string between x and $x + dx$. In general, this segment of mass dm has both kinetic and potential energy .The total energy is the sum of these two components:

1- Kinetic energy

The kinetic energy comes from the transverse motion (the longitudinal motion is negligible), so it equals:

$$dE_k = \frac{1}{2} dm V_y{}^2 \quad \Rightarrow \quad dE_k = \frac{1}{2} \mu dx \left(\frac{\partial y}{\partial t}\right)^2$$

- Thus the kinetic energy per unit length (density of kinetic energy) is:

$$e_k = \frac{dE_k}{dx} \quad \Rightarrow \quad e_k = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2$$

Or $\quad e_k = \frac{1}{2} \mu \left(\frac{\partial u}{\partial t}\right)^2 \quad , \quad u(x,t) \equiv y(x;t)$

Where μ: Linear mass density of the string kg/m.
$y(x,t)$: Displacement of the string

Chapter 2: Vibrating strings

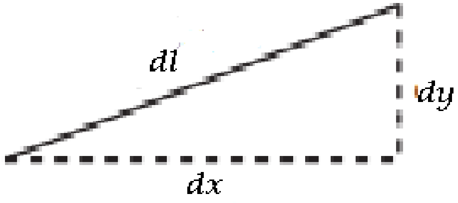
6-Energy Density

2- Potential energy

The potential energy depends on how stretched the string is. Of course, having a string with some tension T automatically has some potential energy due to the stretching:

dx : The length of the string segment at equilibrium,

dl : The length of the string segment during motion.



$$dl = \sqrt{dx^2 + dy^2} = \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right].\mathrm{dx} \quad \Rightarrow \quad dl = \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx$$

$$dE_p = T(dl - dx) = T\left[\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx - dx\right] \quad \Rightarrow \quad dE_p = \frac{1}{2} T\left(\frac{dy}{dx}\right)^2 \mathrm{dx}$$

Chapter 2: Vibrating strings

6-Energy Density

- Thus the potential energy per unit length (density of potential energy) is:

$$e_p = \frac{dE_p}{dx} \qquad \Rightarrow \qquad e_p = \frac{1}{2}T(\frac{dy}{dx})^2$$

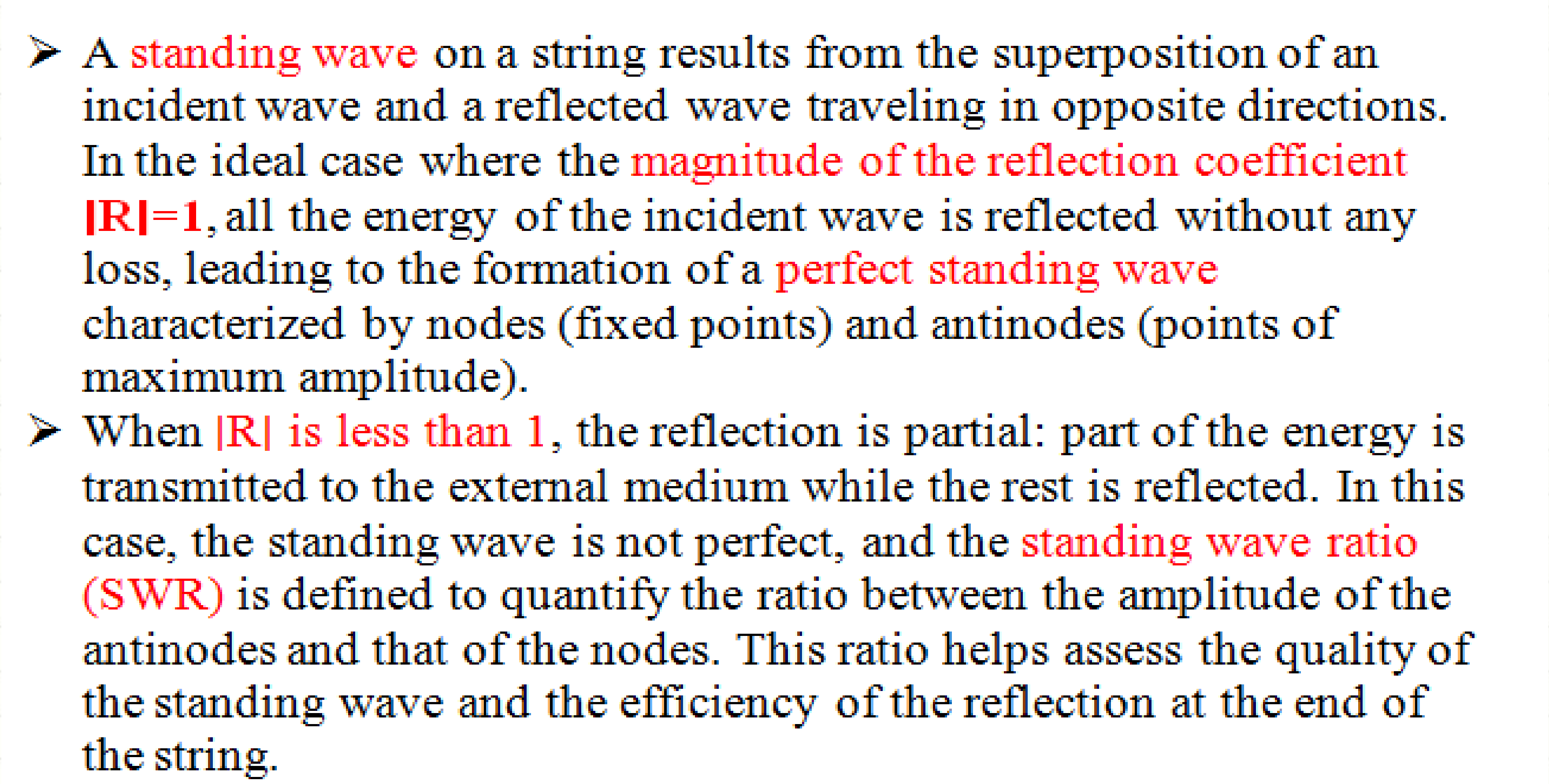
T: Tension in the string (N)

- The total energy density is the sum of the kinetic and potential energy densities:

$$e_T = e_p + e_k \qquad e_T = \frac{1}{2}\mu(\frac{\partial y}{\partial t})^2 + e_p = \frac{1}{2}T(\frac{dy}{dx})^2$$

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

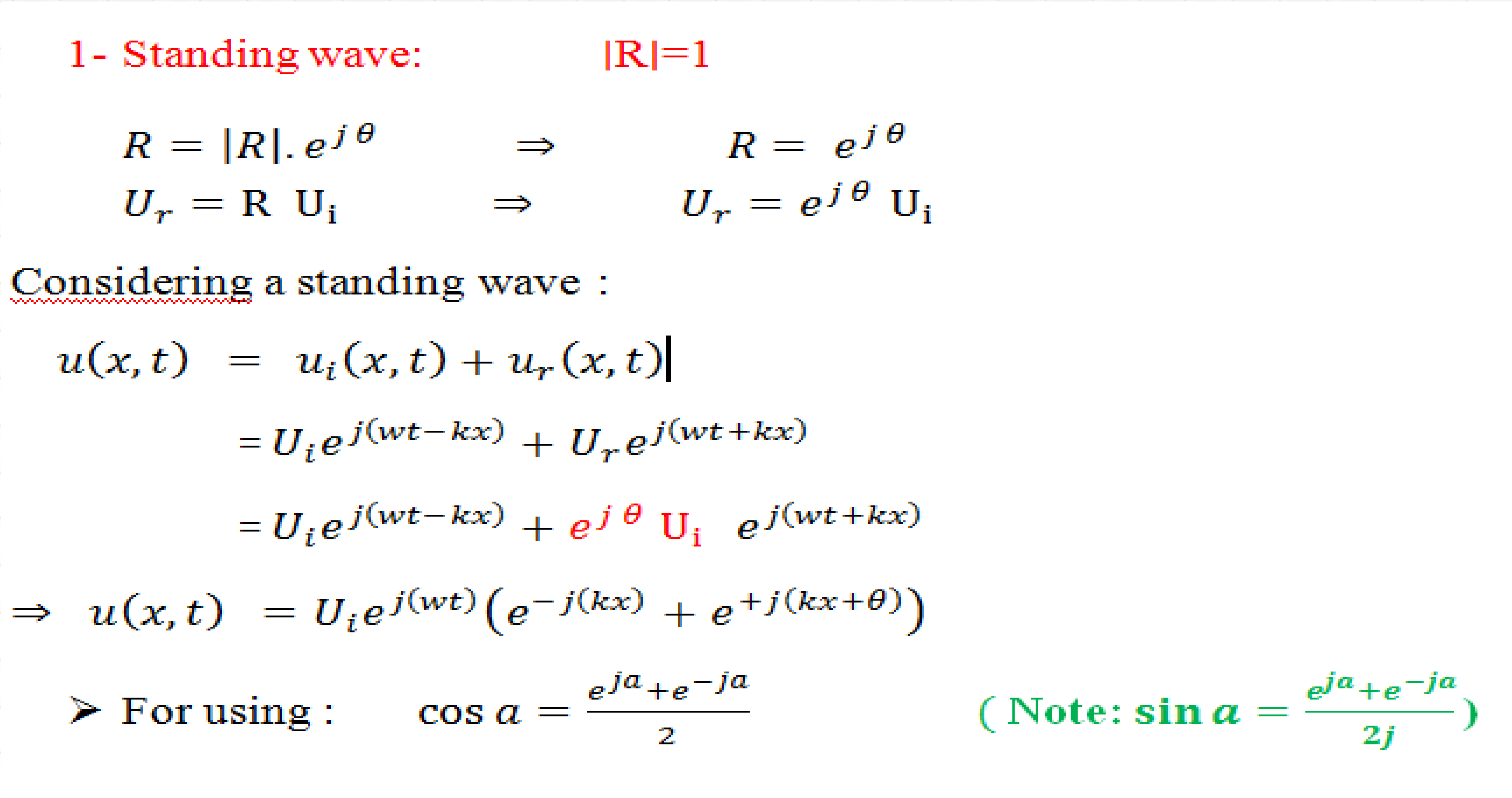


- A standing wave on a string results from the superposition of an incident wave and a reflected wave traveling in opposite directions. In the ideal case where the magnitude of the reflection coefficient **$|R|=1$**, all the energy of the incident wave is reflected without any loss, leading to the formation of a perfect standing wave characterized by nodes (fixed points) and antinodes (points of maximum amplitude).
- When $|R|$ is less than 1, the reflection is partial: part of the energy is transmitted to the external medium while the rest is reflected. In this case, the standing wave is not perfect, and the standing wave ratio (SWR) is defined to quantify the ratio between the amplitude of the antinodes and that of the nodes. This ratio helps assess the quality of the standing wave and the efficiency of the reflection at the end of the string.

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

1-Standing wave



1- Standing wave: |R|=1

$$R = |R|.\, e^{j\,\theta} \quad \Rightarrow \quad R = e^{j\,\theta}$$

$$U_r = \mathrm{R}\ \mathrm{U_i} \quad \Rightarrow \quad U_r = e^{j\,\theta}\ \mathrm{U_i}$$

Considering a standing wave :

$$u(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)}$$

$$= U_i e^{j(wt-kx)} + e^{j\,\theta}\ \mathrm{U_i}\ \ e^{j(wt+kx)}$$

$$\Rightarrow \ u(x,t) = U_i e^{j(wt)}\left(e^{-j(kx)} + e^{+j(kx+\theta)}\right)$$

- For using : $\cos a = \frac{e^{ja}+e^{-ja}}{2}$ (Note: $\sin a = \frac{e^{ja}+e^{-ja}}{2j}$)

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

1-Standing wave

$$u(x,t) = U_i e^{j(wt)}\left(e^{-j(kx)} + e^{+j(kx+\theta)}\right) \qquad \times \frac{e^{-j\frac{\theta}{2}}}{e^{-j\frac{\theta}{2}}}$$

$$\Rightarrow u(x,t) = U_i e^{j\left(wt+\frac{\theta}{2}\right)}\left(e^{-j\left(kx+\frac{\theta}{2}\right)} + e^{+j\left(kx+\frac{\theta}{2}\right)}\right)$$

The real part of the function u(x,t): $u(x,t) = 2U_i e^{j\left(wt+\frac{\theta}{2}\right)} \cos\left(kx+\frac{\theta}{2}\right)$

$$u(x,t) = 2U_i.\cos\left(kx+\frac{\theta}{2}\right) . \cos\left(wt+\frac{\theta}{2}\right)$$

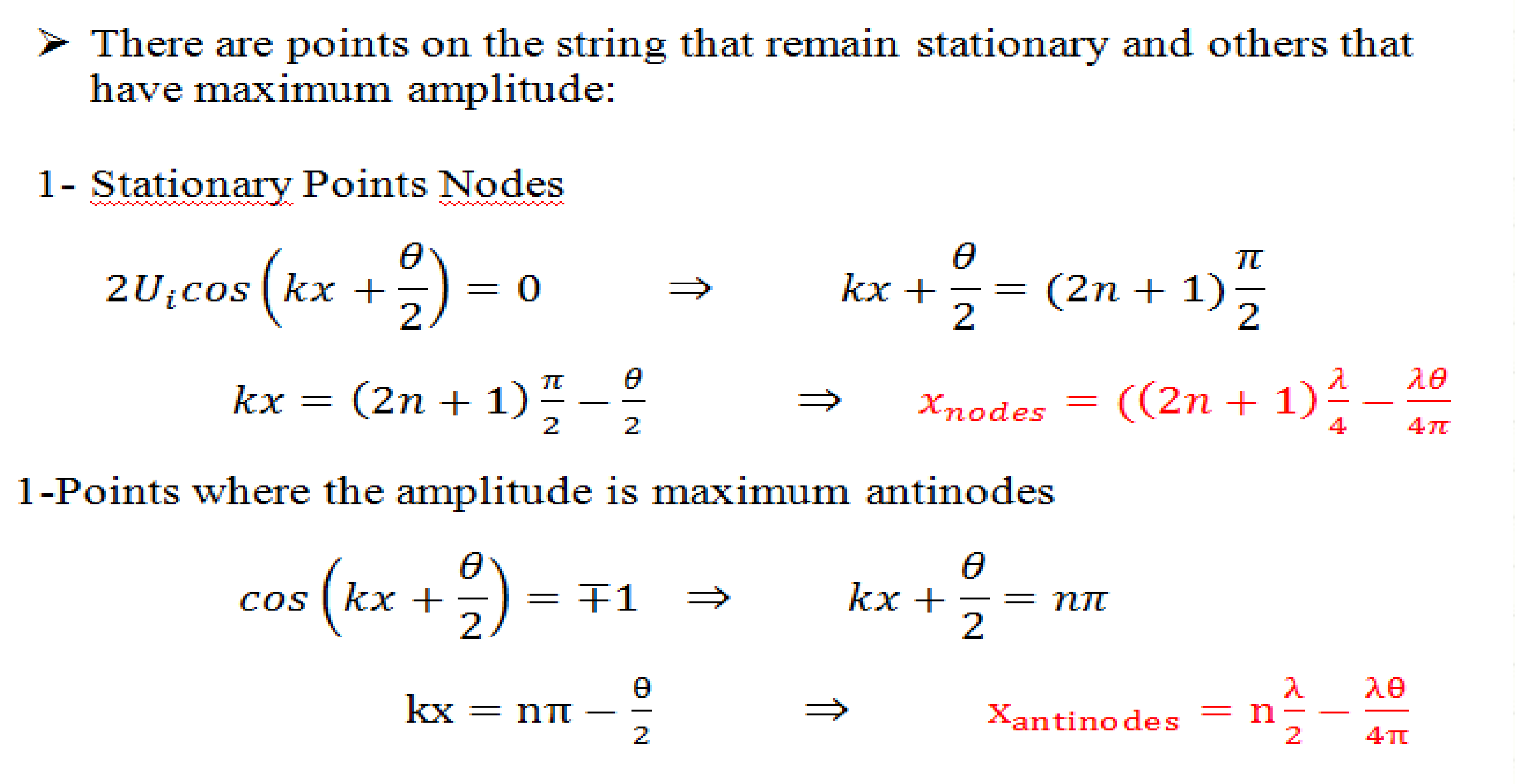
$$u(x,t) = f(x).\ g(t) \quad , \qquad \text{where: } \begin{cases} f(x) = 2U_i.\cos\left(kx+\frac{\theta}{2}\right) \\ g(t) = \cos\left(wt+\frac{\theta}{2}\right) \end{cases}$$

$$\boxed{-2U_i \leq Amplitude \leq 2U_i}$$

Chapter 2: Vibrating strings

7- *Standing wave and standing wave ratio*

1-Standing wave



- There are points on the string that remain stationary and others that have maximum amplitude:

1- Stationary Points Nodes

$$2U_i cos\left(kx+\frac{\theta}{2}\right)=0 \quad \Rightarrow \quad kx+\frac{\theta}{2}=(2n+1)\frac{\pi}{2}$$

$$kx=(2n+1)\frac{\pi}{2}-\frac{\theta}{2} \quad \Rightarrow \quad x_{nodes}=((2n+1)\frac{\lambda}{4}-\frac{\lambda\theta}{4\pi}$$

1-Points where the amplitude is maximum antinodes

$$cos\left(kx+\frac{\theta}{2}\right)=\mp 1 \quad \Rightarrow \quad kx+\frac{\theta}{2}=n\pi$$

$$kx=n\pi-\frac{\theta}{2} \quad \Rightarrow \quad x_{antinodes}=n\frac{\lambda}{2}-\frac{\lambda\theta}{4\pi}$$

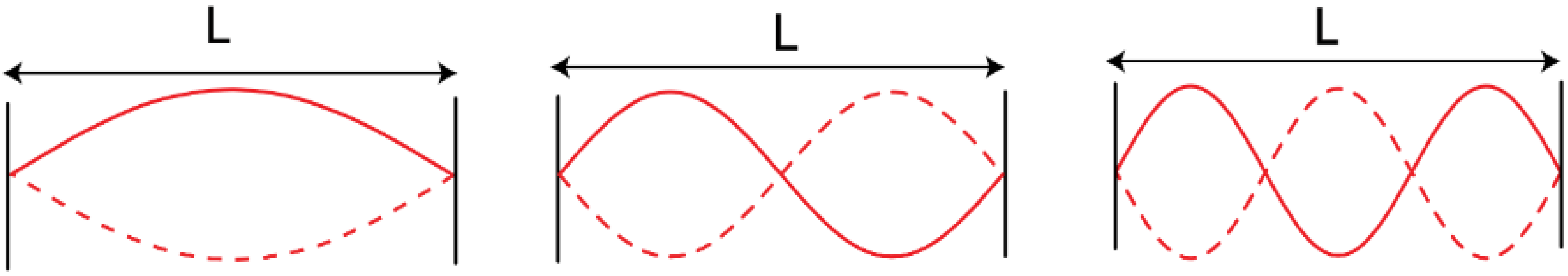
Chapter 2: Vibrating strings

7- *Standing wave and standing wave ratio*

1-Standing wave

- **Harmonics for Two Fixed Ends**

$$f_n = \frac{v}{\lambda} = \left(\frac{v}{2L}\right) n = f_1 n;\ n = 1, 2, 3, \ldots$$



1st harmonic

$$\lambda = 2L \rightarrow f = \left(\frac{v}{2L}\right)$$

2nd harmonic

$$\lambda = L \rightarrow f = 2\left(\frac{v}{2L}\right)$$

3rd harmonic

$$\lambda = \frac{2L}{3} \rightarrow f = 3\left(\frac{v}{2L}\right)$$

$$\lambda_n = \frac{2L}{n};\ n = 1, 2, 3, \ldots$$

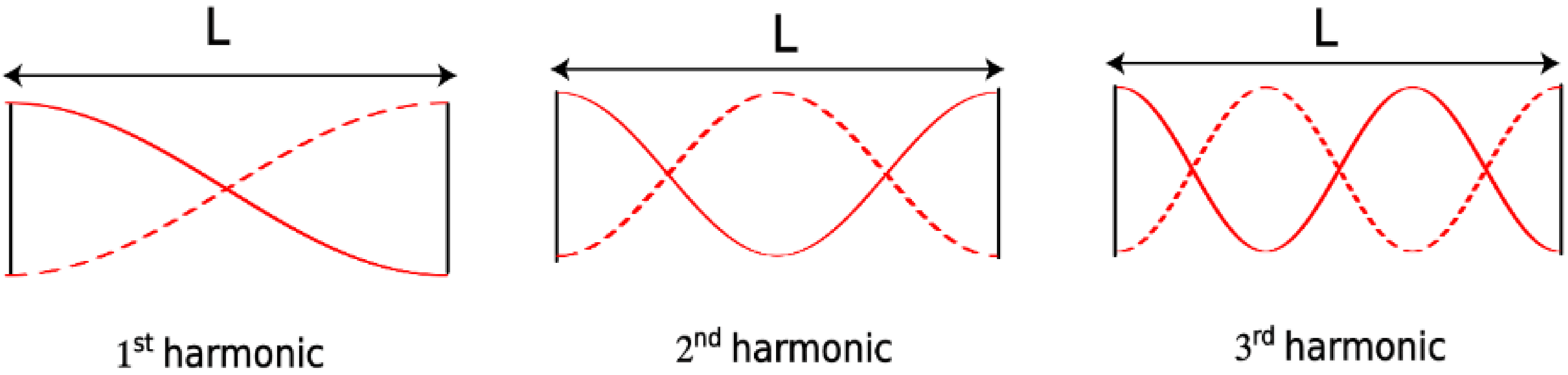
Where: n is a positive integer

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

1-Standing wave

2-Harmonics for Two Free Ends



$$\lambda = 2L \rightarrow f = \left(\frac{v}{2L}\right)$$

$$\lambda = L \rightarrow f = 2\left(\frac{v}{2L}\right)$$

$$\lambda = \frac{2L}{3} \rightarrow f = 3\left(\frac{v}{2L}\right)$$

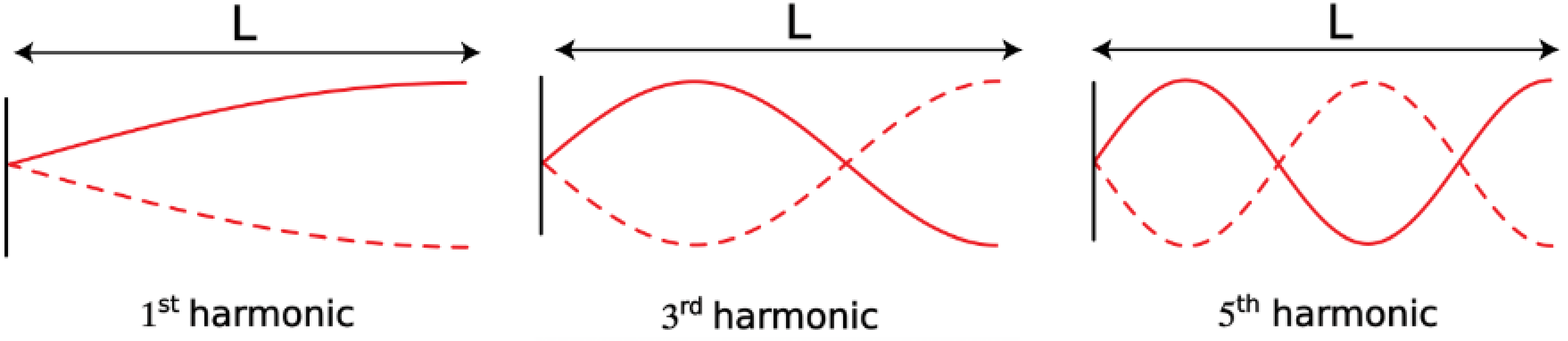
Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

1-Standing wave

$$f_n = \frac{v}{\lambda} = \left(\frac{v}{4L}\right) n = f_1 n;\ n = 1, 3, 5, \ldots$$

- **Harmonics for One Fixed One Free End**



1st harmonic 3rd harmonic 5th harmonic

$$\lambda = 4L \rightarrow f = \left(\frac{v}{4L}\right) \qquad \lambda = \frac{4L}{3} \rightarrow f = 3\left(\frac{v}{4L}\right) \qquad \lambda = \frac{4L}{5} \rightarrow f = 5\left(\frac{v}{4L}\right)$$

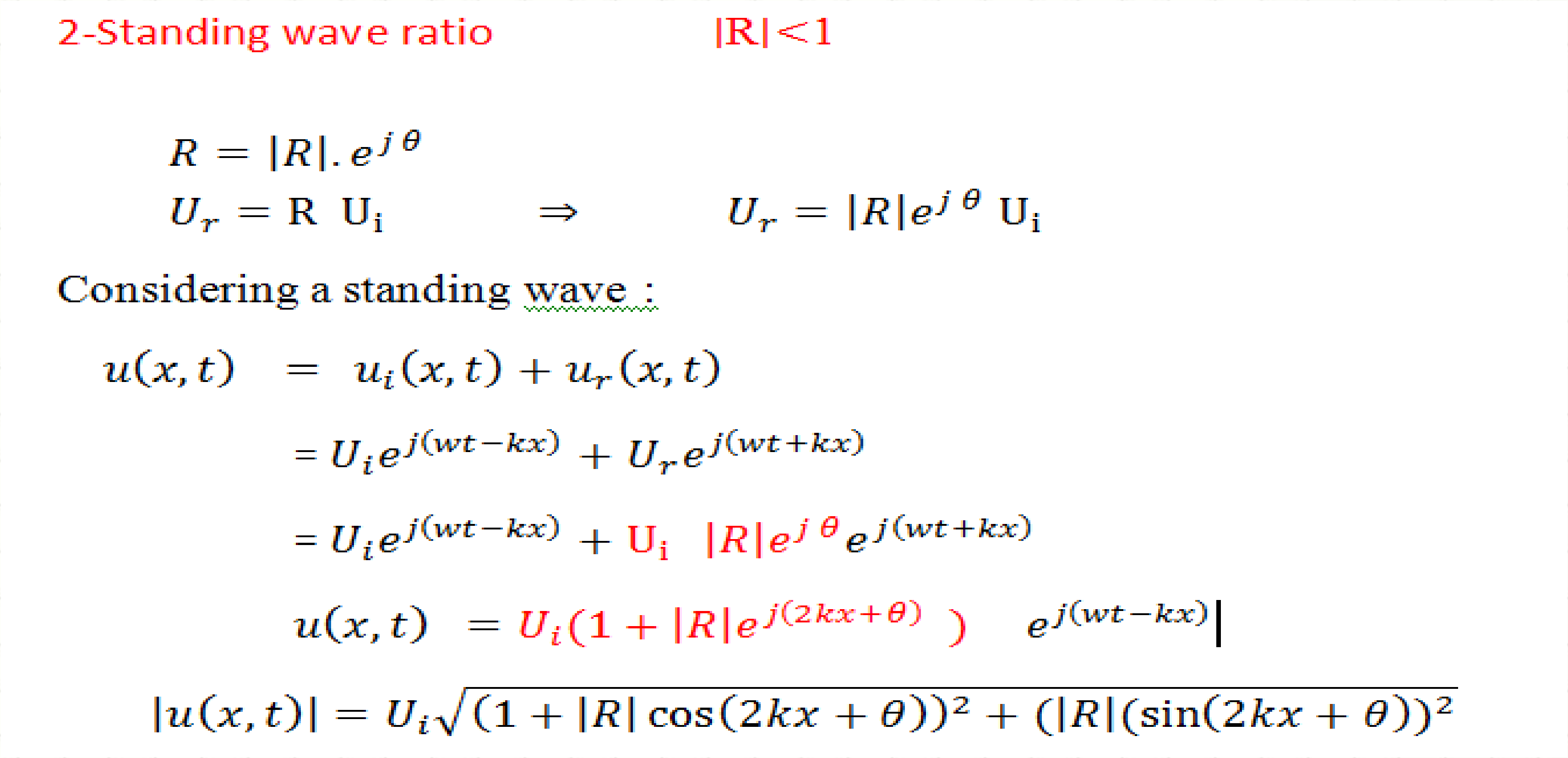
$$\lambda_n = \frac{4L}{n};\ n = 1, 3, 5, \ldots$$

where : n is a positive odd integer

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

2-Standing wave ratio



2-Standing wave ratio |R|<1

$$R = |R|.e^{j\,\theta}$$

$$U_r = \mathrm{R}\ \ \mathrm{U_i} \qquad \Rightarrow \qquad U_r = |R|e^{j\,\theta}\ \mathrm{U_i}$$

Considering a standing wave :

$$u(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)}$$

$$= U_i e^{j(wt-kx)} + \mathrm{U_i}\ \ |R|e^{j\,\theta} e^{j(wt+kx)}$$

$$u(x,t) = U_i(1 + |R|e^{j(2kx+\theta)}\ )\quad e^{j(wt-kx)}$$

$$|u(x,t)| = U_i\sqrt{(1 + |R|\cos(2kx + \theta))^2 + (|R|(\sin(2kx + \theta))^2}$$

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

2-Standing wave ratio

$$|u(x,t)| = U_i\sqrt{1 + R^2 + 2|R|(\cos(2kx + \theta)}$$

$$U_{min} \leq |u(x,t)| \leq U_{max}$$

- $U_{min} = (1 - |R|)U_i$ and $U_{max} = (1 + |R|)U_i$
- Standing wave ratio : $SWR = \frac{U_{max}}{U_{min}} \Rightarrow SWR = \frac{1+|R|}{1-|R|}$

Chapter 2: Vibrating strings

7- Standing wave and standing wave ratio

2-Standing wave ratio