National Higher School of Autonomous Systems Technology

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Vibrations and waves

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Chapter 3 : Damped free vibration of single degree of freedom systems

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

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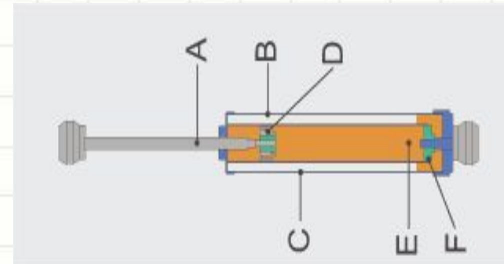
Chapter 3 : Damped free vibration of single degree of freedom systems

1-Definition



Viscous damper

✔ A **viscous damper** the piston moves, inside a cylinder filled with viscous fluid, the fluid flows through small openings, creating resistance. This resistance converts the motion energy into heat, dissipating vibrations and helping the system return to equilibrium

1-Definition

Viscous damper

In the case of viscous friction, the friction force is proportional to the velocity but acts in the opposite direction.

$$\vec{F}_{\alpha} = -\alpha\,\vec{v}$$

Where α is the coefficient of viscous friction and $\vec{v}$ is the velocity of the particle

- Dissipation function $D = \frac{1}{2}\alpha\,\dot{q}^2$

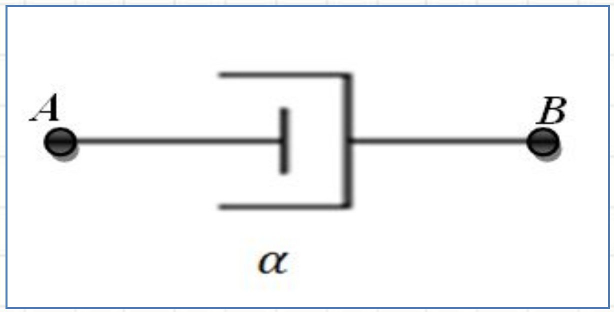
$$f = -\frac{\partial D}{\partial \dot{q}}$$

Chapter 3 : Damped free vibration of single degree of freedom systems

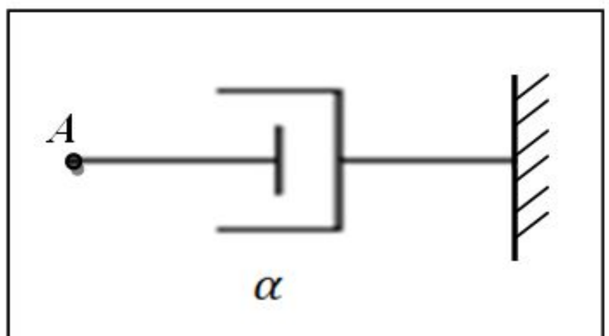
1-Definition

Viscous damper

$$D = \frac{1}{2}\,\alpha\,(\vec{v}_A - \vec{v}_A)^2$$



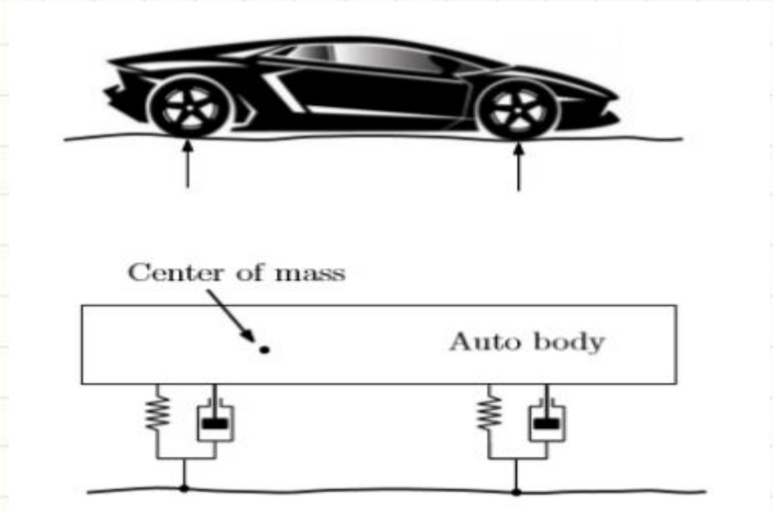
$$D = \frac{1}{2}\,\alpha\, v_A{}^2$$

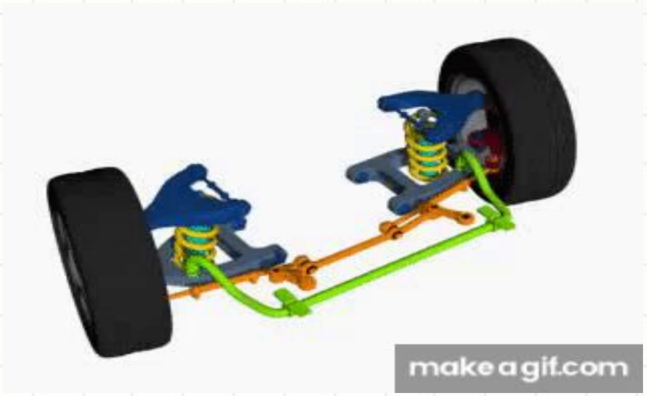


2- Applications

Example 1

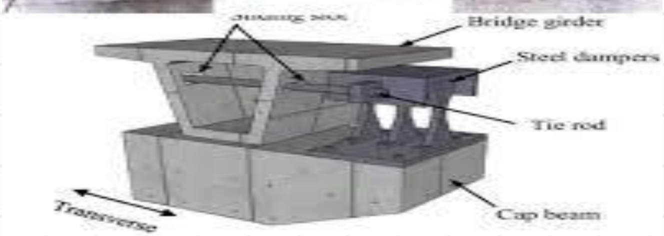
The suspension is a system in vehicles designed to absorb shocks caused by the road. It makes driving more comfortable and improves vehicle stability by keeping the wheels in contact with the ground.

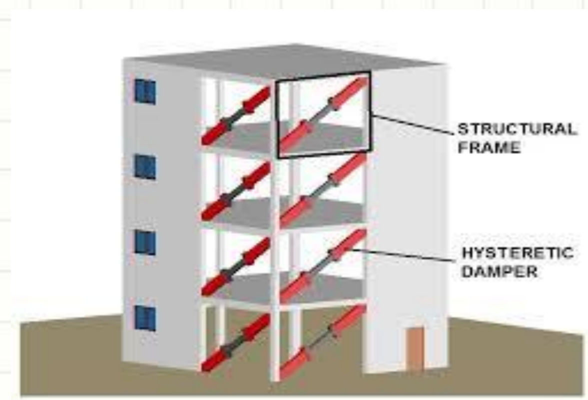




2- Applications

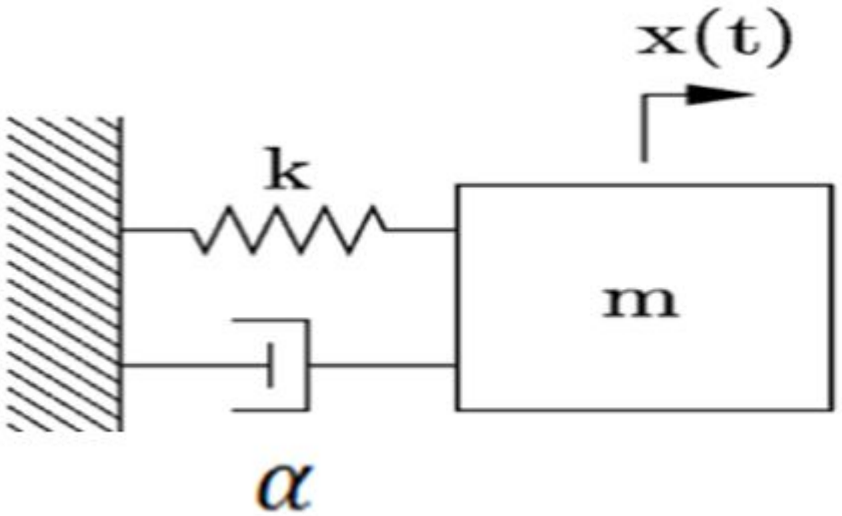
Buildings and bridges: Dampers are installed to reduce vibrations caused by wind or earthquakes, thus protecting the structures.





2- Differential equation:

Consider the single degree of freedom model with viscous damping shown in Figure. The viscous damper with coefficient α.



- **Using Lagrange's equation:**

$$T = \frac{1}{2}m\dot{x}^2 \quad , \quad U = \frac{1}{2}kx^2$$

- Dissipation function : $D = \frac{1}{2}\alpha\dot{x}^2$
- *Lagrangian of the system:*

$$L = T - U \quad \Rightarrow \quad L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}}\right) - \frac{\partial(L)}{\partial q} + \frac{\partial D}{\partial\dot{q}} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{x}}\right) - \frac{\partial(L)}{\partial x} + \frac{\partial D}{\partial\dot{x}} = 0$$

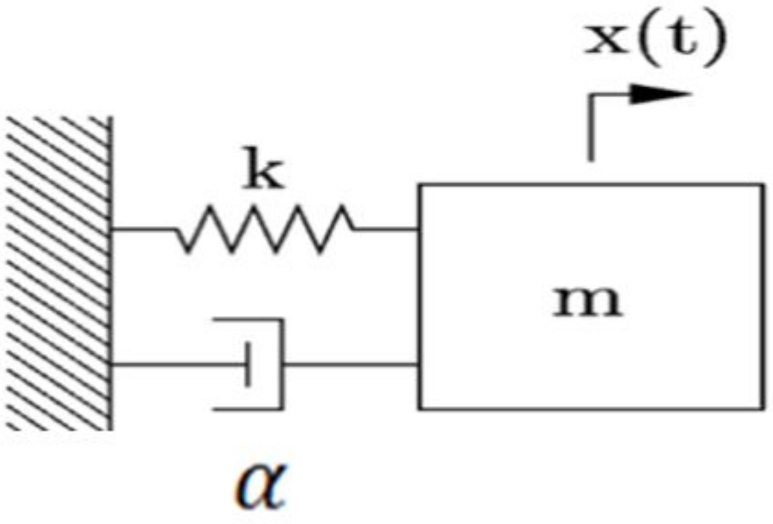
2- Differential equation:

$$\Rightarrow \quad m\ddot{x} + \alpha\dot{x} + kx = 0 \qquad \Rightarrow \quad \ddot{x} + \frac{\alpha}{m}\dot{x} + kx = 0 \qquad \Rightarrow \quad \ddot{x} + 2\delta\dot{x} + w_n{}^2 x = 0$$

Where:

- $\omega_n = \sqrt{\frac{k}{m}}$ The natural frequency of the system.
- $2\delta = \frac{\alpha}{m} \quad \Rightarrow \quad \delta = \frac{\alpha}{2m}$

δ: Damping factor or damping ratio



3- Solution of differential equation :

- The equation of motion here is like that of an undamped system, with the addition of an extra term to account for the damping in this model
  It is a second-order linear differential equation with constant coefficients.

$$\ddot{x} + 2\delta\dot{x} + {w_n}^2 x = 0$$

The characteristic equation of this equation is written as:

$$r^2 + 2\delta r + {w_n}^2 = 0$$

- We distinguish three regimes based on the sign of the reduced discriminant:

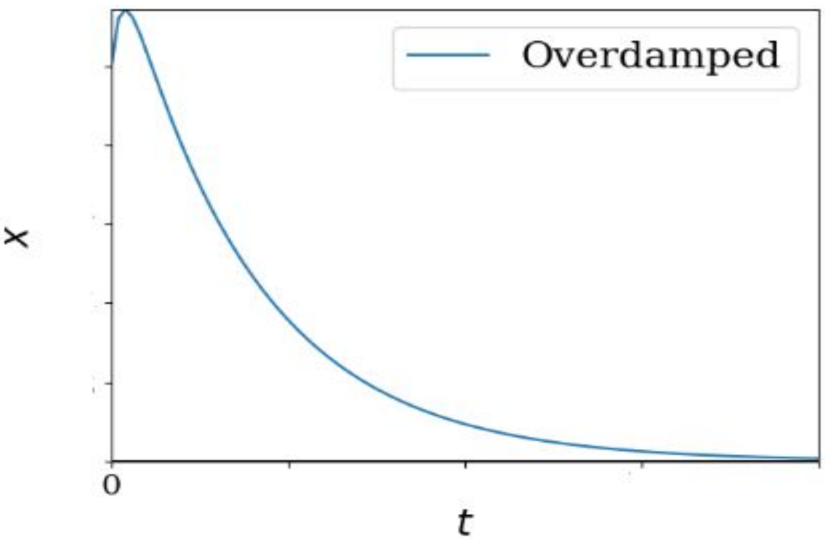
$$\Delta = \sqrt{\delta^2 - {w_n}^2}$$

3- Solution of differential equation :

1- Overdamped response $\Delta > 0$:

$$\Delta > 0 \quad \Rightarrow \quad \delta > w_n$$

✓ The characteristic equation has two real solutions



The solution is written as:

$$x(t) = e^{-\delta t}\left(Ae^{-(\sqrt{(\delta^2 - w_n{}^2)}t} + Be^{\sqrt{(\delta^2 - w_n{}^2)}t}\right)$$

✓ In an overdamped system, the return to equilibrium occurs without oscillation and more slowly, due to significant damping

3- Solution of differential equation :

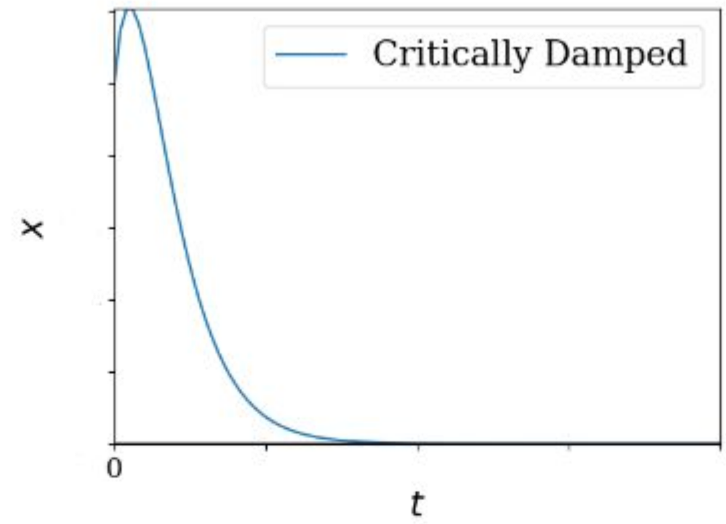
2- Critically damped response $\Delta = 0$:

$\Delta = 0 \quad \Rightarrow \quad \delta = w_n$

- ✓ The characteristic equation has double solution

The solution is written as:

$$x(t) = e^{-\delta t}(A + Bt)$$

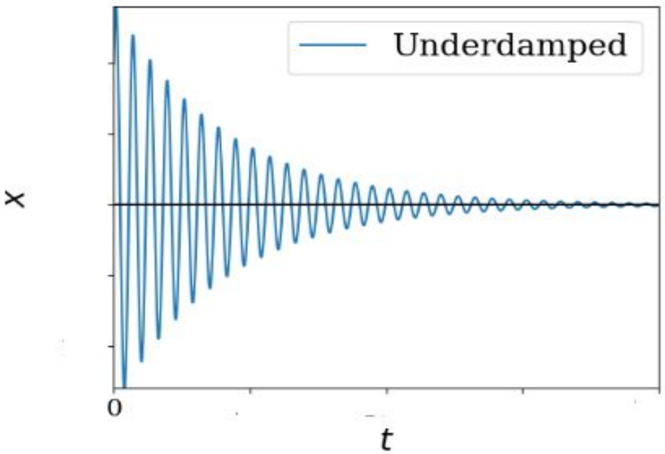


- ✓ The critically damped system returns to its equilibrium state as quickly as possible without oscillating. In this regime, the damping is just sufficient to prevent oscillations, and the system reaches equilibrium in minimal time

3- Solution of differential equation :

3- Underdamped response $\Delta < 0$:

$$\Delta < 0 \quad \Rightarrow \quad \delta < w_n$$



✓ The characteristic equation has two complex roots

The solution is written as:

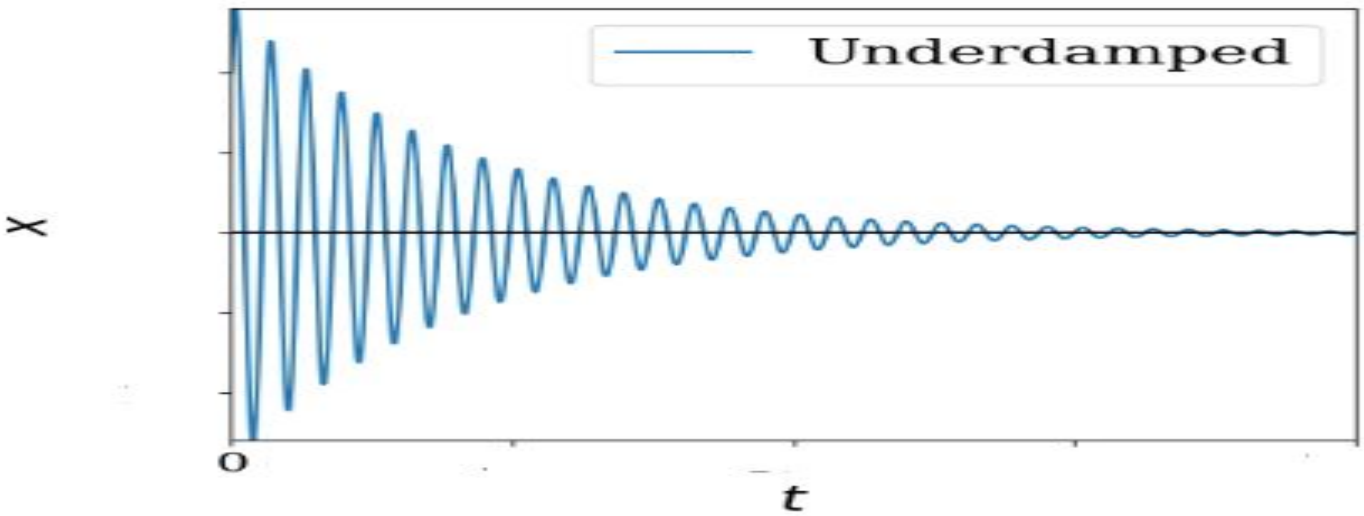
$$x(t) = Ae^{-\delta t}\cos\left(w_d t + \varphi\right)$$

✓ Where w_d : pseudo frequency or damped natural frequency

$$w_d = \sqrt{{w_n}^2 - \delta^2}$$

✓ The pseudo period or damped period $T_d = \frac{2\pi}{w_d} \quad \Rightarrow \quad T_d = \frac{2\pi}{\sqrt{{w_n}^2 - \delta^2}}$

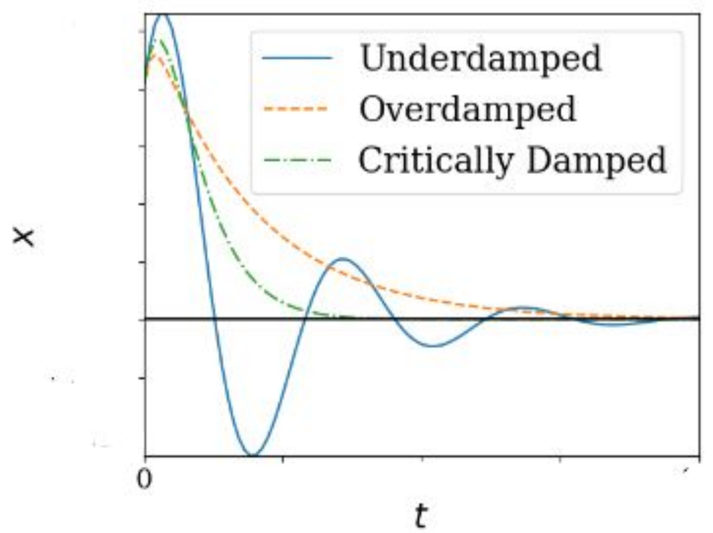
3- Solution of differential equation :



- ✓ In the underdamped regime, the system oscillates around its equilibrium position with an amplitude that gradually decreases over time due to damping

3- Solution of differential equation :

✓ $\begin{cases} T_n = \dfrac{2\pi}{w_n} \\ T_d = \dfrac{2\pi}{w_d} = \dfrac{2\pi}{\sqrt{{w_n}^2 - \delta^2}} \end{cases} \Rightarrow \quad T_d > T_n$

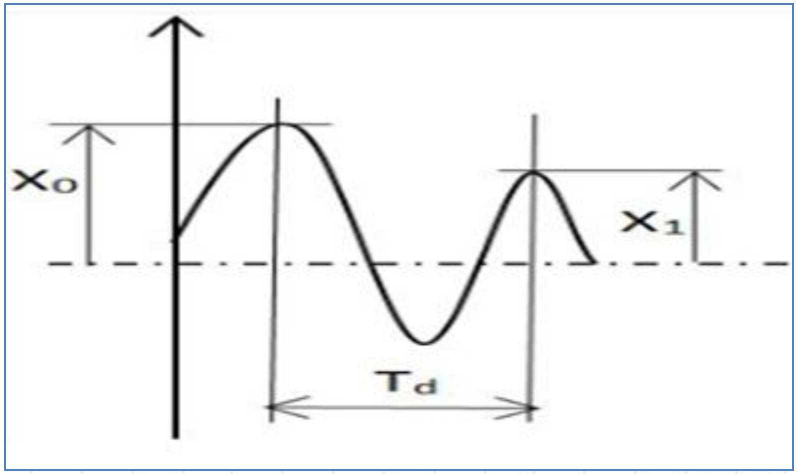


4-Logarithmic decrement

The logarithmic decrement is a measure used in system dynamics to quantify the decay of a damped oscillation. It represents the logarithmic ratio between two successive oscillation amplitudes. it indicates how quickly a system loses energy and dampens over time.

➤ In a general way $$D = \frac{1}{n} ln \frac{q(t)}{q(t+nT_d)}$$

- $q(t)$ is the amplitude at time t
- $q(t+nT_d)$ is the amplitude after nT_d period



4-Logarithmic decrement

- Underdamped response

$$D = \frac{1}{n} ln \frac{x(t)}{x(t+nT_d)} \quad \Rightarrow \quad D = \frac{1}{n} ln \frac{Ae^{-\delta t} \cos(w_d t+\varphi)}{Ae^{-\delta(t+nT_d)} \cos(w_d(t+nT_d)+\varphi)}$$

$$\Rightarrow \quad D = \frac{1}{n} ln \frac{Ae^{-\delta t} \cos(w_d t+\varphi)}{Ae^{-\delta t}.e^{-n\delta T_d} \cos(w_d t+\varphi)} \quad \Rightarrow \quad D = \frac{1}{n} ln\, e^{n\delta T_d} \quad \Rightarrow \quad D = \delta T_d$$

5-Quality factor

The quality factor Q of a system is a dimensionless measure of the damping rate of an oscillator.

- The lower the damping, the higher the quality of the system.

- There are three definitions of Q:

5-Quality factor

1. Related to energy:

$$Q = 2\pi \frac{E_{Sys}}{E_{Diss}}$$

Where

- E_{Sys} is the maximum energy contained in the system
- E_{Diss} is the energy dissipated by the system over a pseudo-period

5-Quality factor

2-Related to differential equation of the system:

$$Q = \frac{w_n}{2\delta}$$

$$\ddot{x} + \overbrace{2\delta}^{Q}\dot{x} + w_n{}^2 x = 0$$

- The quality factor is used to determine the type of transient regime in an oscillatory system (**Overdamped**, **Critically damped**, or **Underdamped**)

$$\ddot{x} + \frac{w_n}{Q}\dot{x} + w_n{}^2 x = 0$$

5-Quality factor

1-Overdamped response $\Delta > 0$:

$$\Delta = \sqrt{\delta^2 - w_n^{\,2}}$$

$$\Delta > 0 \quad \Rightarrow \quad \delta > w_n \quad \Rightarrow \quad \frac{\delta}{w_n} > 1$$

$$\frac{w_n}{2\delta} < \frac{1}{2} \quad \Rightarrow \quad Q < \frac{1}{2}$$

2- Critically damped response $\Delta = 0$:

$$\delta = w_n \quad \Rightarrow \quad Q = \frac{1}{2}$$

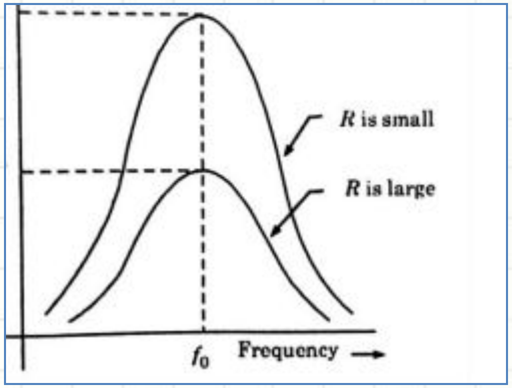
3- Underdamped response $\Delta < 0$:

$$\delta < w_n \quad \Rightarrow \quad Q > \frac{1}{2}$$

5-Quality factor

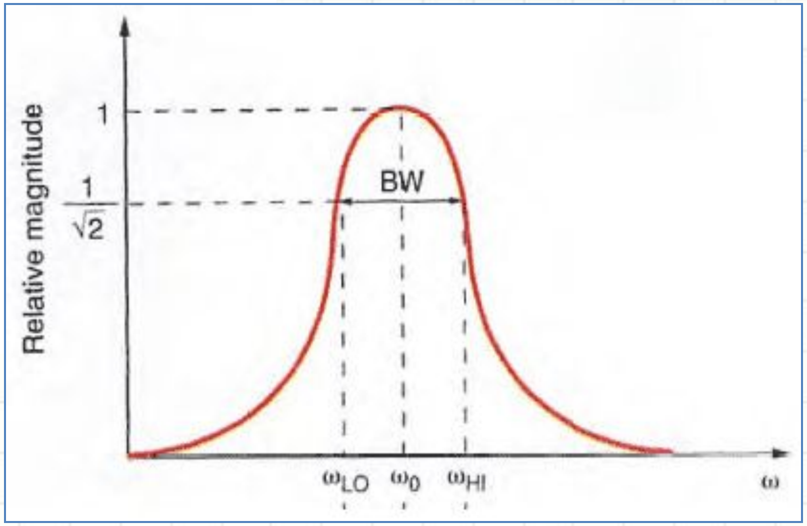
3-Related to the bandwidth :

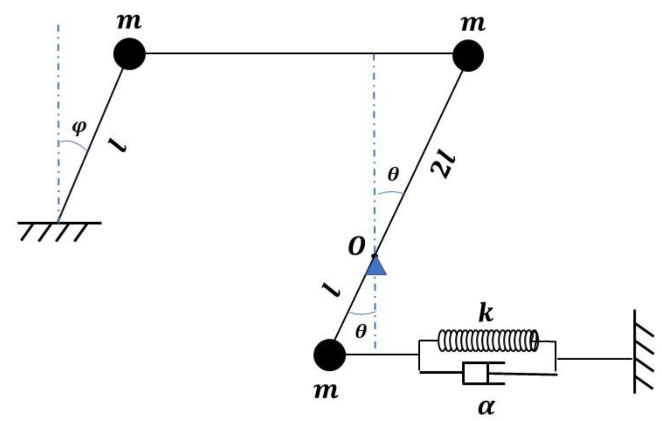
$$Q = \frac{w_r}{B} \qquad \text{(Chapter 4)}$$



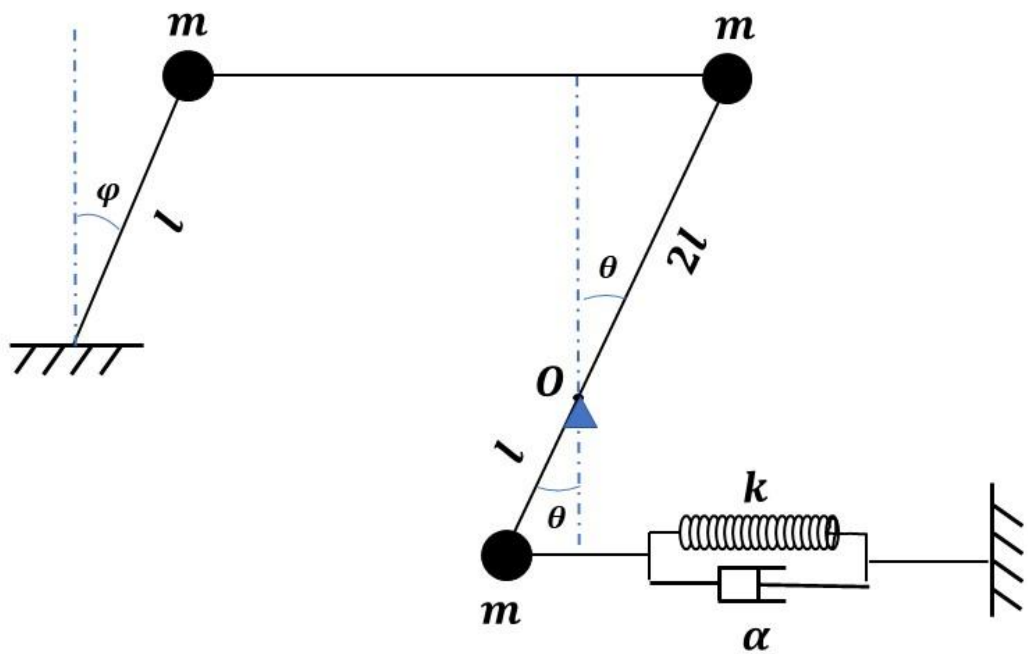
- The bandwidth is the range of frequencies over which a system responds effectively to vibrations. (forced systems)
- The higher the quality factor, the smaller the bandwidth
- The higher the Q, the more selective the filter

(Filter electronic, acoustic, optical....)





1. Determine the potential energy of the system and the condition for oscillation in the absence of friction.
2. Determine the kinetic energy as well as the dissipation function, and deduce the equation of motion for the damped system.
3. What are the values of α required to maintain the system in oscillation?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