theorem: line integral ~~of~~ for area

If D is a plane region ~~simply connected~~ bounded by a piecewise simple closed curve C, the

$$\mathcal{A}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx .$$

C oriented ~~and~~ counterclockwise

Example: $D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$; $C = \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $0 \leq t \leq 2\pi$.

II-3- Surface integrals

II-3-1- Parametric surfaces

Definition:

Let x, y, z be functions of u and v that are continuous on a domain $D \subset \mathbb{R}^2$. the set S of points (x,y,z) given by:

$S: \quad r(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$ (parametric surface)

is called parametric surface. The following equations:

$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$; $(u,v) \in D$ are the parametric equations for the surface S.

Examples: (1) $S = S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

$r(\phi,\theta) = \cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}$; $D = [0,\pi] \times [0,2\pi]$

(2) $S: \; x = 3\cos u\,\vec{i} + 3\sin u\,\vec{j} + v\,\vec{k}$; $D = [0,2\pi] \times [0,4]$

(3) If $S = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } z = f(x,y)\}$

then $r(x,y) = x\,\vec{i} + y\,\vec{j} + f(x,y)\,\vec{j}$; $(x,y) \in D$, for example:

$S = \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } 0 \leq z \leq R\}$; $f : (x,y) \mapsto \sqrt{x^2+y^2}$; $D = C((0,0), R)$.

Definition: Normal vector of a smooth parametric surface

Let $S: r(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$; $(u,v) \in D$ be a ~~non~~ smooth parametric surface over an open subset $D \subset \mathbb{R}^2$. ~~The~~ normal vector field at the point $(x_0,y_0,z_0) = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)) \in S$, $(u_0,v_0) \in D$ is given by $N(x_0,y_0,z_0) = r_u(u_0,v_0) \times r_v(u_0,v_0)$ with $r_u = \frac{\partial r}{\partial u}$ and $r_v = \frac{\partial r}{\partial v}$.

Remark: $-N(x_0,y_0,z_0) = r_v(u_0,v_0) \times r_u(u_0,v_0)$ is also a normal vector of S at (x_0,y_0,z_0)