Examples: ① $D=\mathbb{R}^2$, $S: r(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}$

$N(1,2,5) = N(r(1,2)) = (\vec{i} + 2u\vec{k}) \times (\vec{j} + 2v\vec{k})$
$= \vec{k} - 2v\vec{j} - 2u\vec{i} = -2u\vec{i} - 2v\vec{j} + \vec{k} = -2\vec{i} - 4\vec{j} + \vec{k}$

② $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$; $N(x_0,y_0,z_0) = \cancel{r(u_0,v_0)}$.

Def: the tangent plane of a surface at a point

$T_{(x_0,y_0,z_0)} S = \{(x,y,z) \in \mathbb{R}^3 : \langle (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}, N(x_0,y_0,z_0)\rangle = 0\}$

Example: $T_{(1,2,5)} S = \{(x,y,z) \in \mathbb{R}^3 : -2(x-1) - 4(y-2) + z - 5 = 0\}$
$= \{(x,y,z) \in \mathbb{R}^3 : 2x + 4y - z - 5 = 0\}$

Definition: Area of a parametric Surface

Let S be a smooth parametric surface $r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$.
If $r: D \to S$ is bijective, then the surface area $\mathcal{A}(S)$ of S is given by: $\mathcal{A}(S) = \iint_S d\sigma = \iint_D \|r_u \times r_v\|(u,v)\, du\, dv$,
where $r_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$ and $r_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$.

Remark: If $S = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } z = f(x,y)\}$

$r: (u,v) \mapsto u\vec{i} + v\vec{j} + f(u,v)\vec{k} \Rightarrow N(x,y,z) = -\frac{\partial f}{\partial x}(u,v)\vec{i} - \frac{\partial f}{\partial y}(u,v)\vec{j} + \vec{k}$

$\mathcal{A}(S) = \iint_D \sqrt{1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}(x,y)\, dx\, dy$.

Example: $S = S^2$: $r(u,v) = \sin u \cos v\,\vec{i} + \sin u \sin v\,\vec{j} + \cos u\,\vec{k}$
$(u,v) \in [0,\pi] \times [0,2\pi] = D$.

$N(x,y,z) = \sin^2 u \cos v\,\vec{i} + \sin^2 u \sin v\,\vec{j} + \sin u \cos v\,\vec{k}$

$\|N(x,y,z)\| = \sin u \rightsquigarrow \mathcal{A}(S^2) = \int_0^{2\pi}\int_0^{\pi} \sin u\, du\, dv = \boxed{4\pi}$

II-3-2 Surface integrals

Definition:

Let S be a surface with equation $z = g(x,y)$ and R be its projection onto the xy-plane. If g is a class $\mathcal{C}^1$ on R and $f: \mathbb{R}^3 \to \mathbb{R}$ continuous on S, the the surface integrals of f over S is

$$\iint_S f(x,y,z)\, d\sigma := \iint_R f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$