System of linear equations

1 Notion of a Linear Algebraic System

Let $m, n \in \mathbb{N}^*$. A linear system of m equations and n unknowns with coefficients in a commutative field K is any system of equations of the form

$$\begin{cases} A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n = b_1 \\ A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n = b_2 \\ \cdots \\ A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n = b_m \end{cases}$$

where $(A_{i,j}) \in M_{(m,n)}(K)$ and $(b_i) \in M_{(m,1)}(K)$. Denoting this system by (S), we have:

- The $A_{i,j}$ are called coefficients, the x_j unknowns, and the b_i right-hand sides of the system (S).

- The matrix $A = (A_{i,j})$ is called the matrix of the system (S), and its rank is called the rank of the system (S).

1) The linear system (S) with matrix A can be written in matrix form as $AX = B$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

2) The n-tuple $U = (u_1, u_2, \ldots, u_n) \in K^n$ is a solution of the system (S) if it satisfies $AU = B$. Solving a system means determining the set of its solutions.

3) Two systems (S) and (S') are said to be equivalent if they have the same solution sets.

Example:

The system

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 0 \\ 2x_1 + 4x_2 - x_3 = 1 \end{cases}$$

is a linear system of 2 equations and 3 unknowns with real coefficients, and its matrix is

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \end{pmatrix}.$$

$t\left(\frac{2}{5}, 0, -\frac{1}{5}\right)$ is a solution of the system, whereas $t(0,0,0)$ is not a solution of the system.

Cramer's system

Definition 3. *Let A be the square matrix associated with a system (S). If matrix A is invertible, then the system (S) is called* **Cramer's system**.

Remark . *If the system (S) is Cramer's system (matrix A is invertible), then*

$$(S) \iff AX = b \iff X = A^{-1}b$$

In other words, calculate A^{-1}, the inverse of A, it suffices to solve the system (S).

Theorem 2 *If the system (S) is Cramer's system (matrix A is invertible), then (S) has a unique*

$$solution\ X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\ given\ by$$

$$x_k = \frac{1}{\det(A)} \det(A_k), \quad k = 1, \ldots, n$$

where A_k is the matrix obtained from A by replacing the k-th column with the vector

Examples

Solve the following linear systems:

$$\begin{cases} x + y + 2z &= 3 \\ x + 2y + z &= 1 \\ 2x + y + z &= 0 \end{cases} \qquad \begin{cases} x + 2z &= 1 \\ -y + z &= 2 \\ x - 2y &= 1 \end{cases}$$

Cramer, in the general case

Solution of a Linear Algebraic System Using Cramer's Method

Let (S) be the linear algebraic system $AX = B$, with unknown $X = (x_1, x_2, \ldots, x_n)$, right-hand side $B = (b_1, b_2, \ldots, b_m)$, and matrix $A = (A_{i,j}) \in M_{(m,n)}(K)$. Let

$$A' = \begin{pmatrix} A_{i_1,j_1} & A_{i_1,j_2} & \cdots & A_{i_1,j_r} \\ A_{i_2,j_1} & A_{i_2,j_2} & \cdots & A_{i_2,j_r} \\ \vdots & \ddots & & \vdots \\ A_{i_r,j_1} & A_{i_r,j_2} & \cdots & A_{i_r,j_r} \end{pmatrix}$$

be one of the square submatrices extracted from A with a non-zero determinant and of the largest possible order.

- The unknowns $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ become principal unknowns, and the unknowns $x_{j_{r+1}}, x_{j_{r+2}}, \ldots, x_{j_n}$ become parameters.

- The system (S') defined by

$$\begin{cases} A_{i_1,j_1}x_{j_1} + \cdots + A_{i_1,j_r}x_{j_r} = b_{i_1} - (A_{i_1,j_{r+1}}x_{j_{r+1}} + \cdots + A_{i_1,j_n}x_{j_n}) \\ \vdots \\ A_{i_r,j_1}x_{j_1} + \cdots + A_{i_r,j_r}x_{j_r} = b_{i_r} - (A_{i_r,j_{r+1}}x_{j_{r+1}} + \cdots + A_{i_r,j_n}x_{j_n}) \end{cases}$$

with matrix A' is a Cramer system, so it has a unique solution $X' = (x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ expressed in terms of the parameters $x_{j_{r+1}}, \ldots, x_{j_n}$.

- If the solution of (S') satisfies the remaining $m - r$ equations of (S):

$$\begin{cases} A_{i_{r+1},j_1}x_{j_1} + \cdots + A_{i_{r+1},j_r}x_{j_r} = b_{i_{r+1}} - (A_{i_{r+1},j_{r+1}}x_{j_{r+1}} + \cdots + A_{i_{r+1},j_n}x_{j_n}) \\ \vdots \\ A_{i_n,j_1}x_{j_1} + \cdots + A_{i_n,j_r}x_{j_r} = b_{i_n} - (A_{i_n,j_{r+1}}x_{j_{r+1}} + \cdots + A_{i_n,j_n}x_{j_n}) \end{cases}$$

then $X = (x_1, x_2, \ldots, x_n)$ are the solutions of (S). If any of the remaining $m - r$ equations is not satisfied by the solution of (S'), then the system (S) has no solution.

Example

2) Let (S_2) :
$$\begin{cases} x_1 - 3x_2 + 2x_3 = 0 \\ 2x_1 + 4x_2 - x_3 = 1 \\ 3x_1 + x_2 + x_3 = 2 \\ 5x_1 + x_3 = 0 \end{cases}$$

$$A_2 = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 3 & 1 & 1 \\ 5 & 0 & 1 \end{pmatrix}, \text{ is the matrix of the system } (S_2).$$

$$A_2' = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 5 & 0 & 1 \end{pmatrix} \text{ is a submatrix of } A_2 \text{ with non-zero determinant and of order 3, so the system } (S_2')$$

$$x_1 = \frac{1}{\det A_2'} \det \begin{pmatrix} 0 & -3 & 2 \\ 1 & 4 & -1 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{1}{5}, \quad x_2 = \frac{1}{\det A_2'} \det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix} = \frac{3}{5}$$

$$x_3 = \frac{1}{\det A_2'} \det \begin{pmatrix} 1 & -3 & 0 \\ 2 & 4 & 1 \\ 5 & 0 & 0 \end{pmatrix} = 1.$$

The solution $(x_1, x_2, x_3) = \left(-\frac{1}{5}, \frac{3}{5}, 1\right)$ does not satisfy the remaining equation $3x_1 + x_2 + x_3 = 2$, so the system (S) has no solutions.

- **Remark**

1- If the system has at least one solution then the equations are said to be consistent; otherwise, they are said to be inconsistent.

Augmented Matrix of Linear System

- If we consider a linear system AX=B, the augmented matrix of the system is the matrix obtained by adding to the matrix A the matrix B as the new and last column:

$$(A|B) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,p} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,p} & b_n \end{pmatrix}.$$
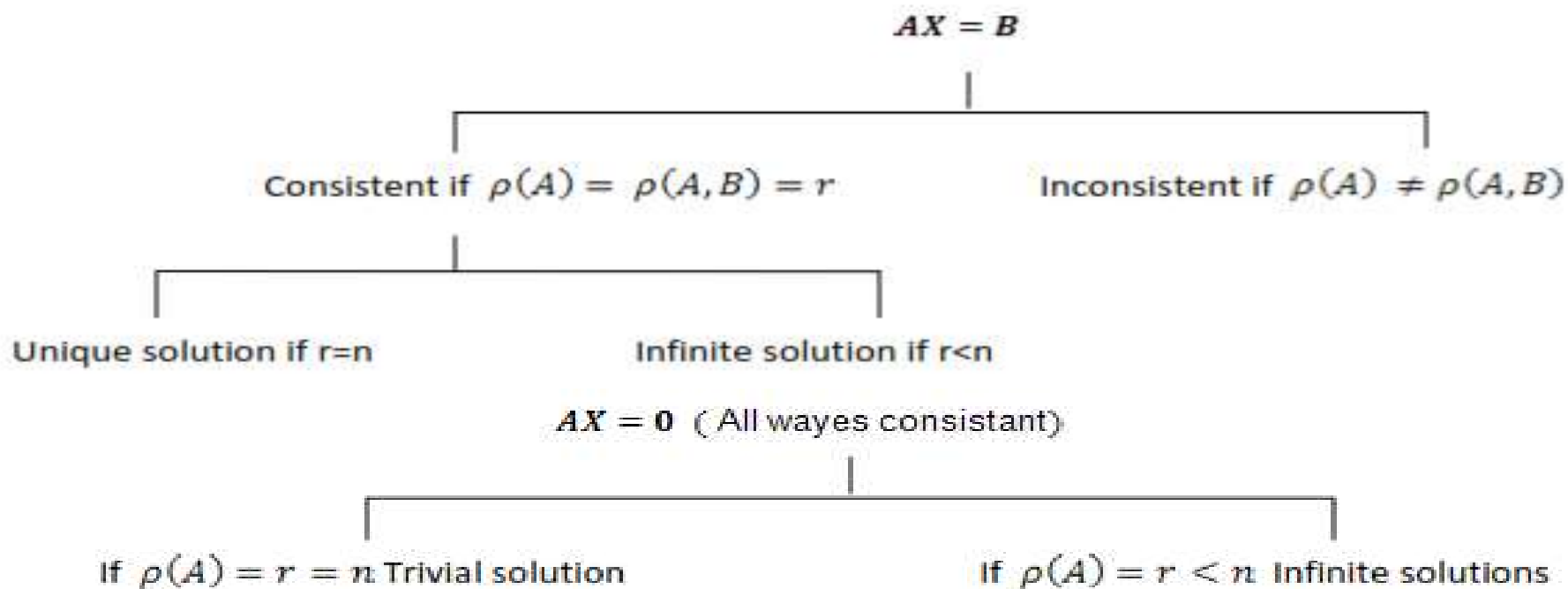
For example, for the system

$$\begin{cases} x_1 - x_2 + 2x_3 &=& 1 \\ 3x_1 + x_2 + x_3 &=& 4 \end{cases}$$

the augmented matrix is

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 1 & 4 \end{pmatrix}$$

According to the above result, we can use the rank of the matrix and the **augmented matrix** to determine if the system has solutions or not. This gives us the following result:

Theorem
A system of m equations in n unknowns represented by the matrix equation AX $=B$ is consistent if and only if r(A)=r(AI $B)=r$. That is the rank of matrix A is equal to the rank of the augmented matrix (AI B).

$$AX = B$$

Consistent if $\rho(A) = \rho(A, B) = r$

Inconsistent if $\rho(A) \neq \rho(A, B)$

Unique solution if r=n

Infinite solution if r<n

$$AX = 0 \quad (\text{All wayes consistant})$$

If $\rho(A) = r = n$ Trivial solution

If $\rho(A) = r < n$ Infinite solutions

Therefore, every system of linear equations has a solution under one of the following:
(i) There is no solution
(ii) There is a unique solution
(iii) There is more than one solution

- **Method Using Elementary row operations: (Gaussian Elimination)**

to solve a system of linear Equations:

- Suppose the coefficient matrix is of the type $m \times n$.

- That is, we have m equations in n unknowns. Write the matrix (A I B) and reduce it to Echelon form by applying elementary row transformations.

- Then we deduce the solutions to the system from the final equation in the Echelon form.

Example : Solve the following system of linear equations using Gaussian Elimination

(ii)

(i)

$$2x + y + -2z = 10$$

$$y + 10z = -28$$

$$x + 2y - 3z = -1$$

$$3x - y + 2z = 7$$

$$5x + 3y - 4z = 2$$