Linear maps

1. Linear map

In the following, K denotes field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. *Let V and W be K- vector spaces. A map $f : V \to W$ is said to be a linear map if:*

$$\forall x, y \in V, \forall \alpha, \beta \in K, f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

Proposition 1. *A map $f : V \to W$ is said to be a linear map if:*

$$\forall x, y \in V, \forall \lambda \in K, f(x + \lambda y) = f(x) + \lambda f(y).$$

f is also called a homomorphism of vector spaces.

Examples

1-The identity map

$$Id_V : V \to V$$
$$x \mapsto \mathrm{Id}_V(x) = x$$

is a linear map and also an automorphism of the vector space V.

2-The zero function

$$f : V \to W$$
$$x \mapsto f(x) = 0_W$$

is linear.

3-Consider the subspace

$$\mathcal{C}^\infty([a,b],\mathbb{R}) = \{f \in \mathcal{F}([a,b],\mathbb{R}) : f \text{ is infinitely differentiable } \},$$

and define

$$d : \mathcal{C}^\infty([a,b],\mathbb{R}) \to \mathcal{C}^\infty([a,b],\mathbb{R})$$
$$f \mapsto d(f).$$

where

$$d(f)(x) = f'(x), \forall x \in [a,b].$$

Then d is a linear map.

4-Consider the vector space $\mathbb{R}$ and

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto sh(x) + x,$$

Then f is not a linear map.

5-Consider the Two vector spaces $\mathbb{R}^3$ and $\mathbb{R}$.

$$f : \mathbb{R}^3 \to \mathbb{R}$$
$$(x,y,z) \mapsto x + y - z^2,$$

Then f is not a linear map.

Remark

-The set of linear maps of V in W is noted $L(V,W)$, and $L(V)$ if $V = W$.

-A linear map of V into V is also called an **endomorphism** of V.

Properties

(1) Any linear combination of linear maps is linear.

(2) We say that a linear map $f : V \to W$ is an isomorphism if it is bijective.

(3) The direct image of a vector subspace of V by a linear map is a vector subspace of W.

(4) The reciprocal image of a vector subspace of W by a linear map is a vector subspace of V.

1.1. **Composition of linear maps.**

Theorem 1. *Let $f : U \to V$ and $g : V \to W$ be two linear maps. Then the composed map $g \circ f : U \to W$ is a linear map.*

Theorem 2. *Let $f : U \to V$ be a linear map. If f is an isomorphism, then $f^{-1} : V \to U$ is also an isomorphism.*

Kernel and Image

Let $f \in L(V,W)$.

Definition 2. *We call the kernel of f the subset of V defined by:*

$$ker(f) = \{x \in V, f(x) = 0_W\}.$$

Definition 3. *We call the image of f the vector subspace of W defined by:*

$$\operatorname{Im}(f) = \{f(x), x \in V\}.$$

Properties

Let $f : V \to W$ be a linear map of K - vector spaces. Then we have :

1) $f(0_V) = 0_W$;
2) $\operatorname{Ker} f$ is a subspace of V;
3) $\operatorname{Im} f$ is a subspace of W;
4)If $\dim V < +\infty$ and $\{v_1, v_2, \cdots, v_n\}$ is a basis of V the

$$\operatorname{Im} f = \langle f(v_1), f(v_2), \cdots, f(v_n) \rangle$$

5) If $(x_i)_{i \in I}$ is a generating family of V, then $\operatorname{Im}(f) = span\{f(x_i), i \in I\}$.

Examples

1-Let

$$f : \mathbb{R}_3[x] \to \mathbb{R}_2[x]$$
$$f(P) = P'.$$

We have

$$\ker f = \{P \in \mathbb{R}_3[x] : f(P) = 0_V\} = \langle 1 \rangle,$$

then $\{1\}$ is a basis of $\ker f$, so $\dim_{\mathbb{R}}(\ker f) = 1$.

$$\operatorname{Im} f = \left\langle f(1), f(x), f\left(x^2\right), f\left(x^3\right) \right\rangle = \left\langle 1, 2x, 3x^2 \right\rangle,$$

then $1, 2x, 3x^2$ is a basis of $\operatorname{Im} f$, so $\dim_{\mathbb{R}} \operatorname{Im} f = 3$.

2-Let

$$\varphi : \mathcal{C}^\infty([a,b],\mathbb{R}) \to \mathcal{C}^\infty([a,b],\mathbb{R})$$
$$f \mapsto \varphi(f) = f''.$$

Then φ is a linear map, and we have

$$\ker\varphi = \{\alpha g + \beta : \alpha, \beta \in \mathbb{R}\},$$

where $g(x) = x$ for all $x \in \mathbb{R}$.

1.2. **Linear maps and basis.**

Properties

Let V, W be finite-dimensional vector spaces over K and let f be a linear map from V to W.

1- The map f is an isomorphism from V to W if, and only if, the image of a basis of V is a basis of W.

2-Two isomorphic finite-dimensional vector spaces over K have the same dimension.

3-Any vector space of finite dimension n over K is isomorphic to K^n.

Rank Theorem

Definition 4. *Let V and W be finite-dimensional vector spaces and $f : V \to W$ be a linear map. The rank of f, denoted by $r(f)$, is defined as the dimension of* $\operatorname{Im} f$.

Theorem 4. *Let $f : V \to W$ be a linear map between finite-dimensional K-vector spaces. Then*

$$r(f) = \dim_K(V) - \dim_K(\operatorname{Ker} f).$$

Characterization of injection, surjection and bijection.

Theorem 5. *Let $f : V \to W$ be a linear map between finite-dimensional K-vector spaces. Then we have:*

1) $r(f) \leq Min\,(\dim_K(V).\dim_K(W))$;

2) $r(f) = \dim_K(V) \Leftrightarrow f$ *is injective;*

3) $r(f) = \dim_K(W) \Leftrightarrow f$ *is surjective.*

If V and W have the same dimension, then for any a linear map f from V into W, the following properties are equivalents:

* f *injective.* * f *surjective.* * f *bijective.*