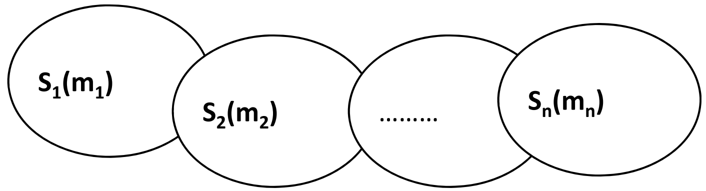
Chapter 3: Geometry of Masses

The objective of this chapter is to understand the geometric distribution of masses in order to describe the movements of material systems.

Mass of a material system

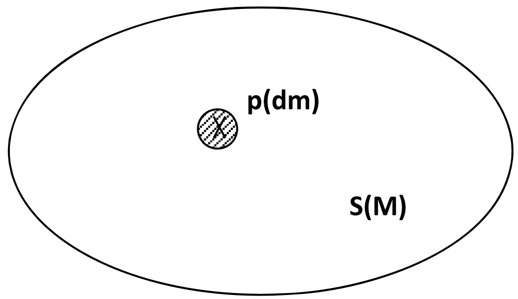
Discret System



Let (S) be a system composed of n solids (S_i) $(i = 1, n)$, each with mass m_i.The mass of the system (S) is given by:

$$m = \sum_{i=1}^{n} m_i$$

Continuous System



For a continuous system, the mass is given by:

$$M = \int_{p \in S} dm(p)$$

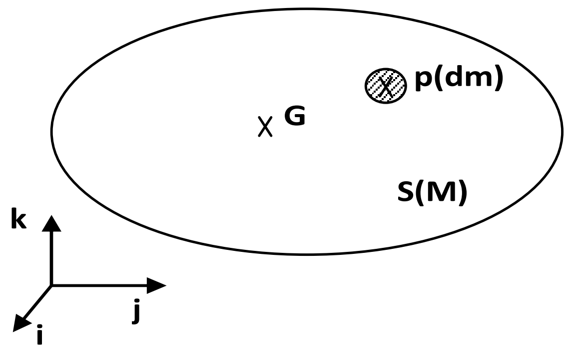
Center of Inertia (Center of Mass)

$R(O, i, j, k)$ is a direct orthonormal reference frame,

(S) is a solid with mass M,

$p(x, y, z)$ is a material point of mass dm belonging to (S),

$G(x_G, y_G, z_G)$ is the center of mass of the solid (S) such that:



$$\int_{p \in S} \overrightarrow{GP}\, dm = \vec{0}$$

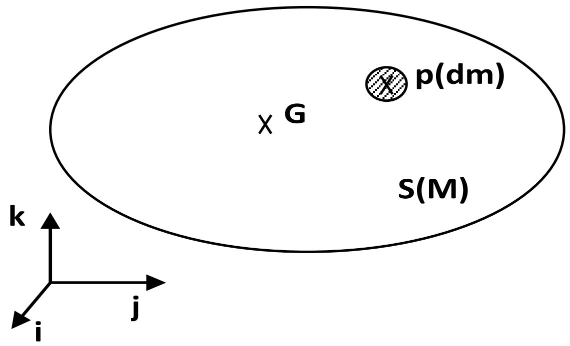
We are looking for the coordinates of the center of mass G:

We have:

$$\overrightarrow{OP} = \overrightarrow{OG} + \overrightarrow{GP}$$

Which gives:

$$\int_{p\in S} \overrightarrow{OP}\, dm = \int_{p\in S} \overrightarrow{OG}\, dm + \int_{p\in S} \overrightarrow{GP}\, dm = \overrightarrow{OG} \int_{p\in S} dm = \overrightarrow{OG}.M$$



$$\overrightarrow{OG} = \frac{1}{M} \int_{p \in S} \overrightarrow{OP}\, dm \Longrightarrow \begin{cases} x_G = \dfrac{1}{M} \displaystyle\int_{p \in S} x\, dm \\ y_G = \dfrac{1}{M} \displaystyle\int_{p \in S} y\, dm \\ z_G = \dfrac{1}{M} \displaystyle\int_{p \in S} z\, dm \end{cases}$$

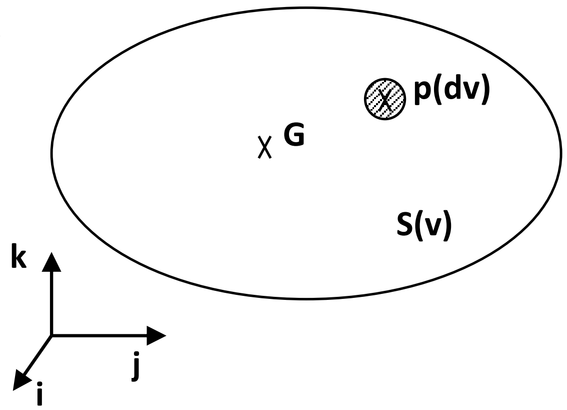
Remarks:

- For a homogeneous solid, the center of inertia coincides with the geometric center (centroid).
- If the solid has elements of symmetry (axes or planes), its center of inertia is necessarily located on these elements of symmetry.
- Let (S) be a discrete system consisting of several solids (S_i) with mass m_i. Knowing the centers of inertia G_i of the solids S_i, we can find the center G of (S) using the following formula (barycenter formula):

$$\overrightarrow{OG} = \frac{\sum_{i=1}^{n} \overrightarrow{OG_i}\, m_i}{\sum_{i=1}^{n} m_i}$$

CENTROIDS OF VOLUMES

If the body is made of a homogeneous material of specific weight $\boldsymbol{\rho}$, we can express the magnitude dm of the mass of an infinitesimal element in terms of the volume *dV* of the element and express the magnitude *m* of the total mass in terms of the total volume *V*. We obtain:



$$\boldsymbol{dm = \rho\, d}\mathbf{v}$$

where

$\boldsymbol{\rho}$ = specific mass (mass per unit volume) of the material

$\boldsymbol{dv}$ = volume of the element

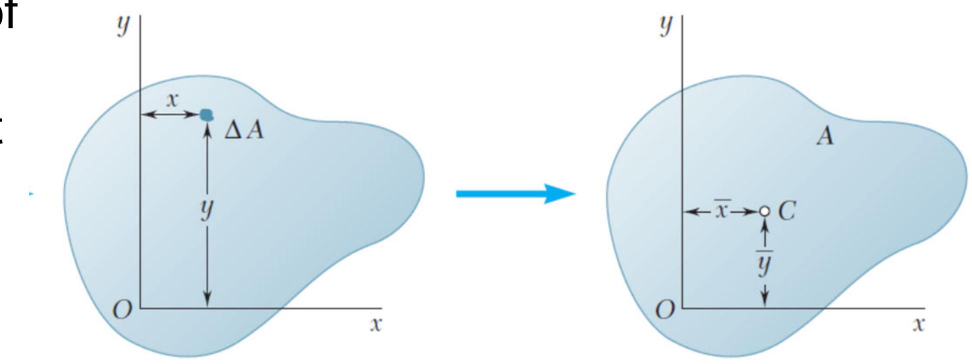
Centroid of a volume v

$$\overrightarrow{OC} = \frac{1}{v}\int_{p\in S} \overrightarrow{OP}\, dv \Longrightarrow \begin{cases} x_C = \dfrac{1}{v}\displaystyle\int_{p\in S} x\, dv \\ y_C = \dfrac{1}{v}\displaystyle\int_{p\in S} y\, dv \end{cases}$$

$\int x\, dV$ is known as the **first moment of the volume with respect to the *yz* plane**.

Centroids of Areas and Lines

In the case of a flat homogeneous plate of uniform thickness, we can express the magnitude $\boldsymbol{dm}$ of the mass of an element of the plate as:



$$\boldsymbol{dm = \rho\, t\, dA}$$

where

$\boldsymbol{\rho}$ = specific mass (mass per unit volume) of the material

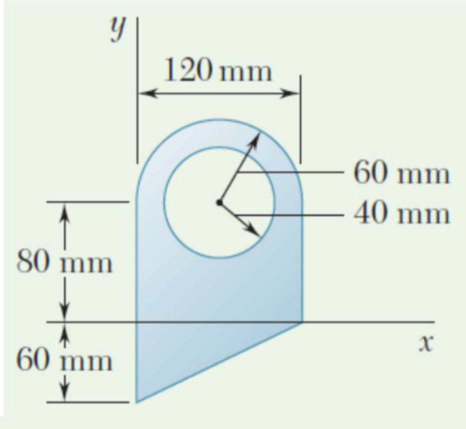
$\boldsymbol{t}$ = thickness of the plate

$\boldsymbol{dA}$ = area of the element

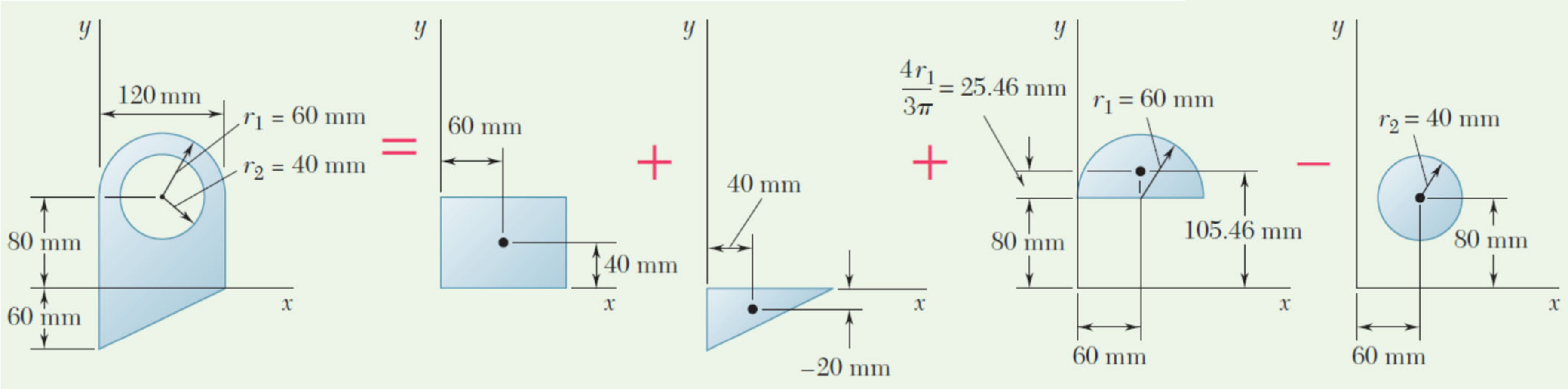
Centroid of an area *A*

$$\overrightarrow{OC} = \frac{1}{A}\int_{p\in S} \overrightarrow{OP}\, dA \Longrightarrow \begin{cases} x_C = \dfrac{1}{A}\displaystyle\int_{p\in S} x\, dA \\ y_C = \dfrac{1}{A}\displaystyle\int_{p\in S} y\, dA \end{cases}$$

Example: For the plane area shown, determine the location of the centroid.



STRATEGY: Break up the given area into simple components, find the centroid of each component, and then find the overall centroid.



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

$\bar{X}\Sigma A = \Sigma\bar{x}A$: $\quad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$

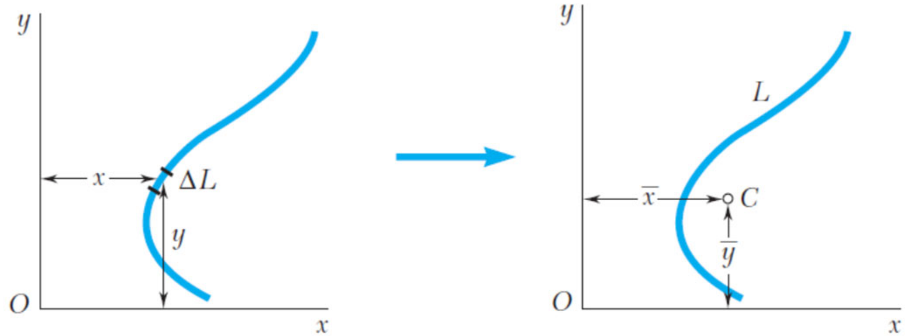
$$\bar{X} = 54.8 \text{ mm}$$

$\bar{Y}\Sigma A = \Sigma\bar{y}A$: $\quad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$

$$\bar{Y} = 36.6 \text{ mm}$$

Centroids of Areas and Lines

In the case of a homogeneous wire of uniform cross section, we can express the magnitude $\boldsymbol{dm}$ of the mass of an element of wire as



$$\boldsymbol{dm = \rho\, a\, dL}$$

Where:

$\boldsymbol{\rho}$ = specific mass of the material

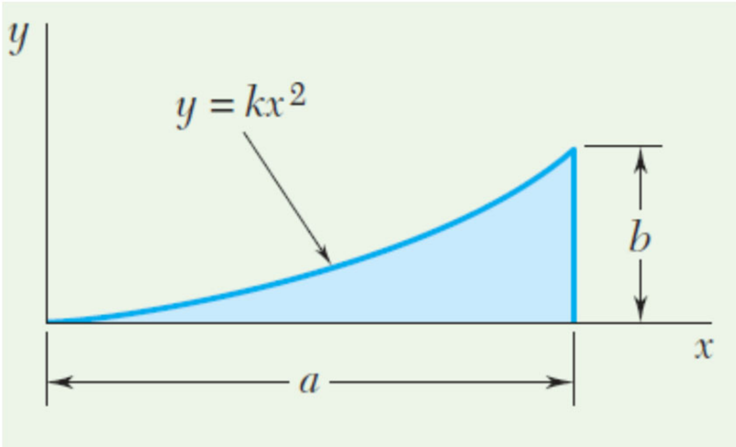
$\boldsymbol{a}$ = cross-sectional area of the wire

$\boldsymbol{dL}$ = length of the element

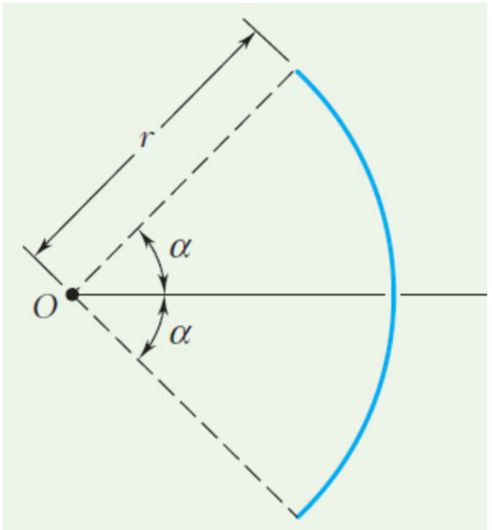
Centroid of a line *L*

$$\overrightarrow{OC} = \frac{1}{L} \int_{p \in S} \overrightarrow{OP}\, dL \Longrightarrow \begin{cases} x_C = \dfrac{1}{L} \displaystyle\int_{p \in S} x\, dL \\ y_C = \dfrac{1}{L} \displaystyle\int_{p \in S} y\, dL \end{cases}$$

Example: Determine the location of the centroid of a parabolic spandrel by direct integration.



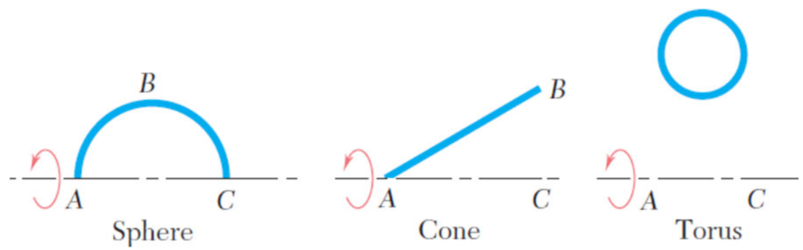
Example: Determine the location of the centroid of the circular arc shown.



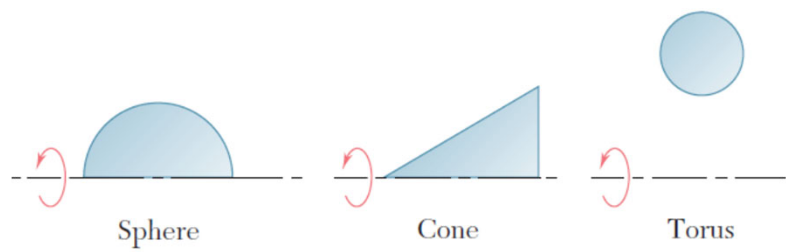
Theorems Of Pappus-Guldinus

These two theorems, which were first formulated by the Greek geometer Pappus during the third century C.E. and later restated by the Swiss mathematician Guldinus or Guldin (1577–1643), deal with surfaces and bodies of revolution.

A **surface of revolution** is a surface that can be generated by rotating a plane curve about a fixed axis. For example rotating about an axis *AC* :
a semicircular → surface of a sphere.
a straight line → surface of a cone.
circumference of a circle → surface of a torus or ring.



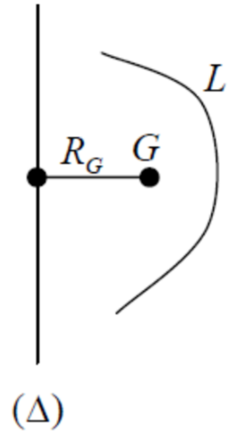
A **body of revolution** is a body that can be generated by rotating a plane area about a fixed axis. For example rotating the shapes shown opposite, about the indicated axis, generates a sphere, a cone, and a torus.



Theorems Of Pappus-Guldinus

Theorem I. *The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated.*

$$S_{/\Delta} = (2\pi R_G)L$$



Hence:

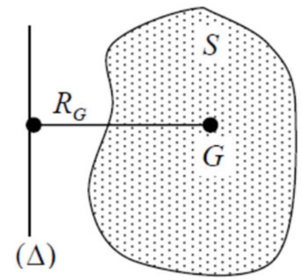
$$R_G = \frac{S_{/\Delta}}{2\pi L}$$

Note that the generating curve must not cross the axis about which it is rotated; if it did, the two sections on either side of the axis would generate areas having opposite signs, and the theorem would not apply.

In the case of a homogeneous system with multiple elements, we will have:

$$R_G = \frac{S_{totale\ /\Delta}}{2\pi L_{totale}}$$

Theorems Of Pappus-Guldinus



Theorem II. *The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.*

$$V_{/\Delta} = (2\pi R_G)S$$

Hence :

$$R_G = \frac{V_{/\Delta}}{2\pi S}$$

In the case of a homogeneous system with multiple surfaces, we will have:

$$R_G = \frac{V_{totale\ /\Delta}}{2\pi S_{totale}}$$

Again, note that the theorem does not apply if the axis of rotation intersects the generating area.

Example: Determine the area of the surface of revolution shown that is obtained by rotating a quarter-circular arc about a vertical axis.