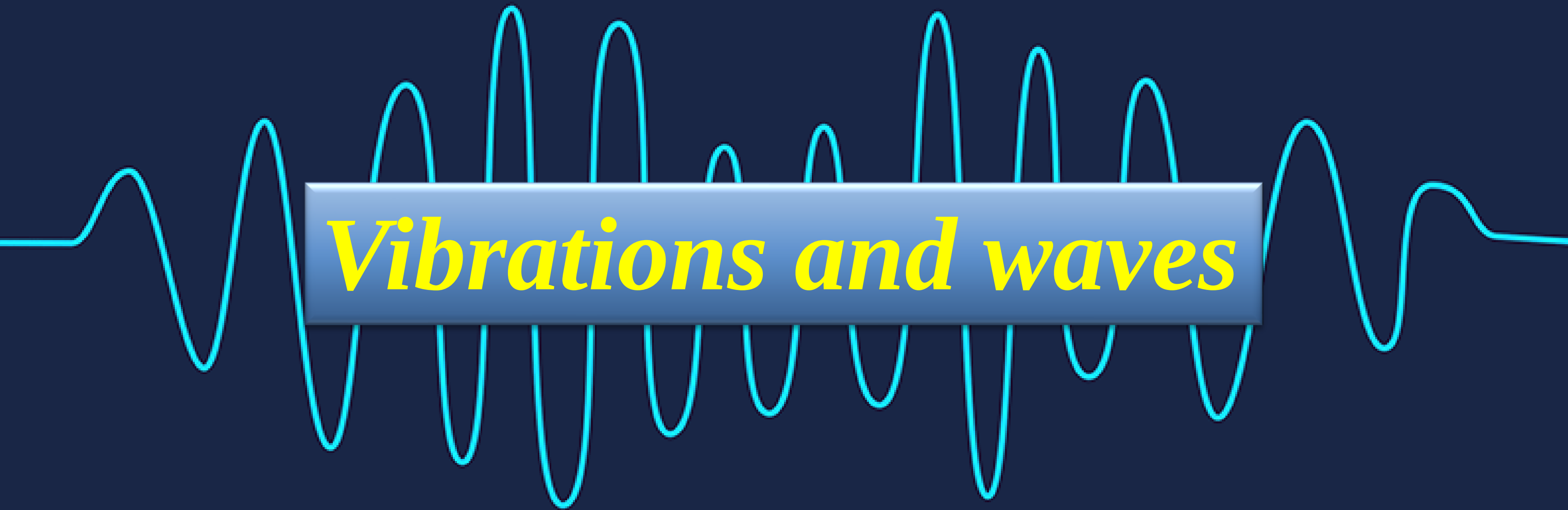
National Higher School of Autonomous Systems Technology

Academic year : 2024/2025



Vibrations and waves

By Dr. Malek ZENAD and Dr. Intissar DJOUADA

Chapter 4: Electromagnetic waves

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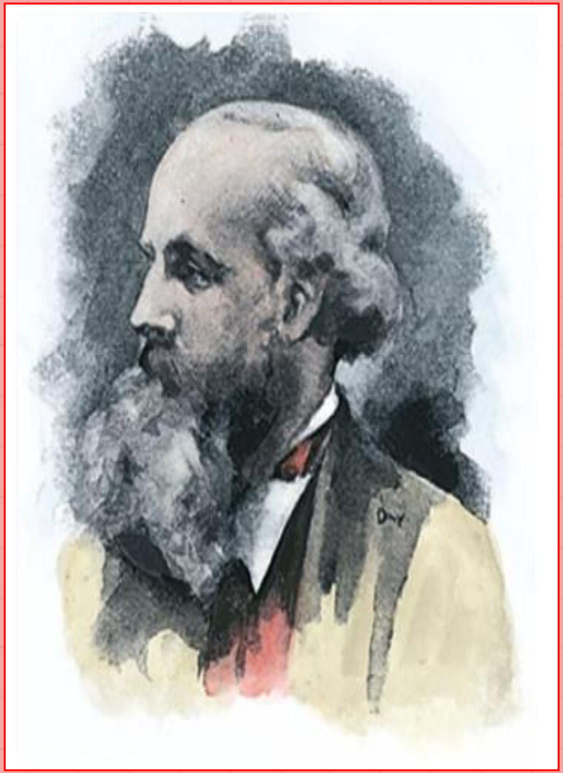


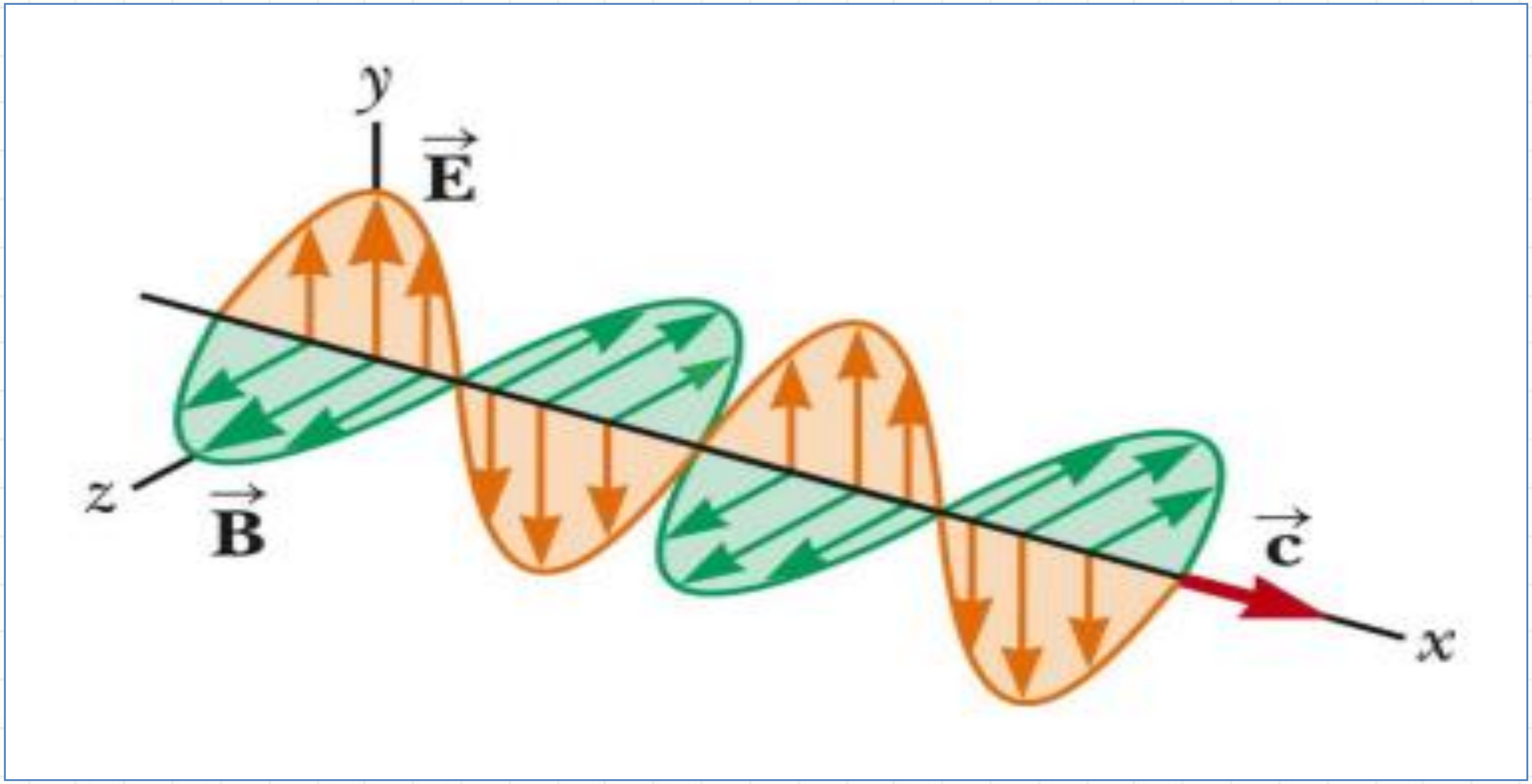
Chapter 4: Electromagnetic waves

1-Introduction

Electromagnetic waves play a fundamental role in modern physics and communication technologies. Present in our daily lives, from microwaves to radio waves, including visible light and X-rays.

They result from the coupled variation of electric and magnetic fields. The objective of this course is to understand the nature, origin, propagation, and properties of electromagnetic waves based on Maxwell's equations. We will explore how these waves propagate in vacuum and in material media, their main characteristics (direction of propagation, polarization, speed, energy, etc.), as well as their applications in various fields such as telecommunications, medical imaging, and radar systems.

Chapter 4: Electromagnetic waves



Chapter 4: Electromagnetic waves

1-Introduction

Before studying electromagnetic waves, it is essential to master some fundamental mathematical tools. Indeed, Maxwell's equations which govern all electromagnetic phenomena are expressed using vector operators such as the gradient, divergence, rotational , and the Laplacian.

Understanding the physical interpretation of these mathematical operations is therefore a crucial step. It not only helps in interpreting Maxwell's equations more clearly but also builds a strong physical intuition about how electric and magnetic fields behave in space.

So before we dive deeper into electromagnetic waves, we will review these vector operators, focusing on their physical meaning and their application in the context of electromagnetic fields.

Chapter 4: Electromagnetic waves

2- Mathematical reminder

$$\overrightarrow{\text{grad}}\, f = \vec{\nabla} f$$

$$\text{div}\, \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$\overrightarrow{\text{rot}}\, \vec{F} = \vec{\nabla} \times \vec{F}$$

Where

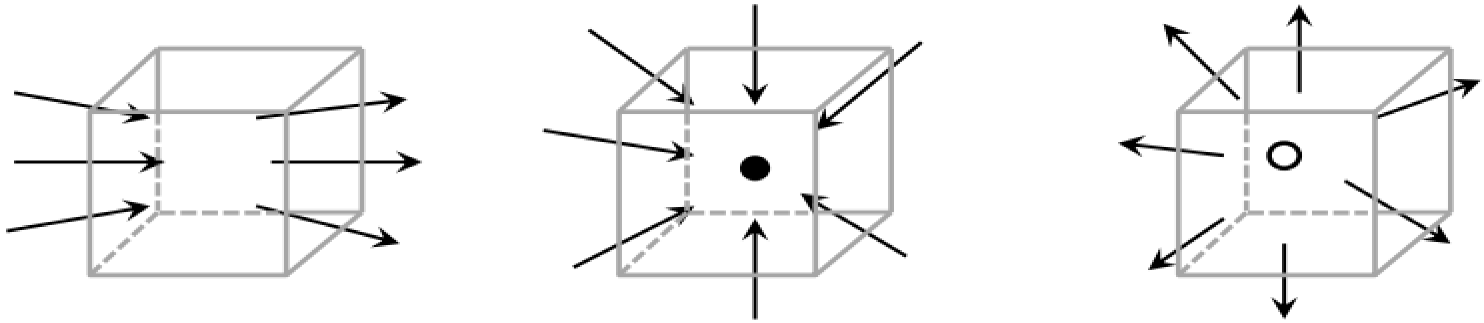
$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{u_x} + \frac{\partial}{\partial y}\vec{u_y} + \frac{\partial}{\partial z}\vec{u_z} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

2- Mathematical reminder

Divergence

$$\text{div}\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Applies to a vector field — Gives a scalar field



$\text{div}\,\vec{F} = 0$ — $\text{div}\,\vec{F} < 0$ — $\text{div}\,\vec{F} > 0$

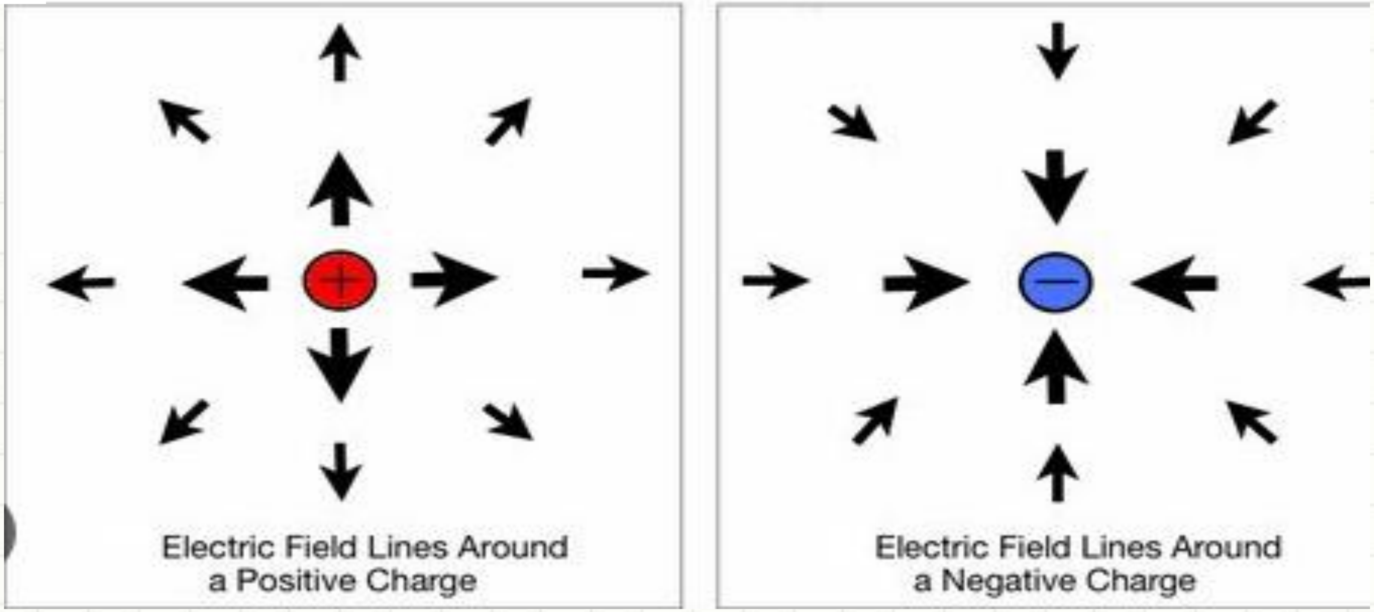
There is a source or a sink for the field.

(Il existe une source ou un puits pour le champ)

Chapter 4: Electromagnetic waves

2- Mathematical reminder

Divergence



The electric field originates from electric charges, and its divergence is non-zero if the charge is present.

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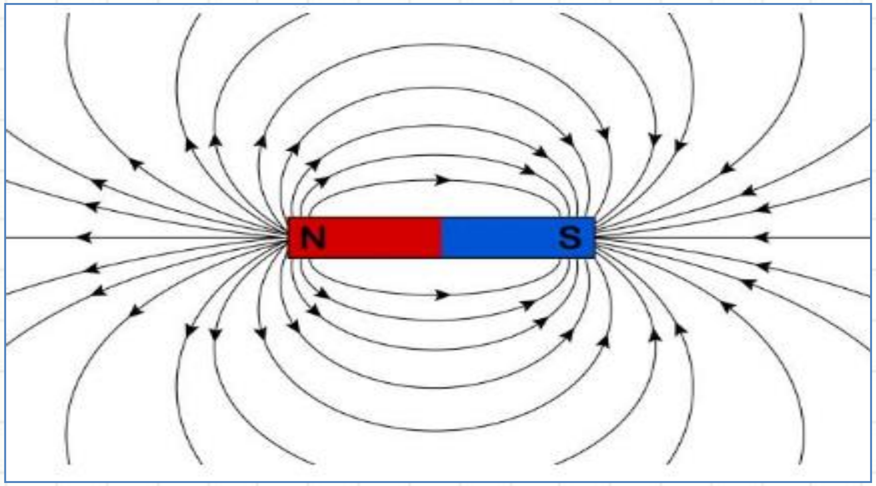
2- Mathematical reminder

Divergence



$$div\,\vec{B} = 0$$

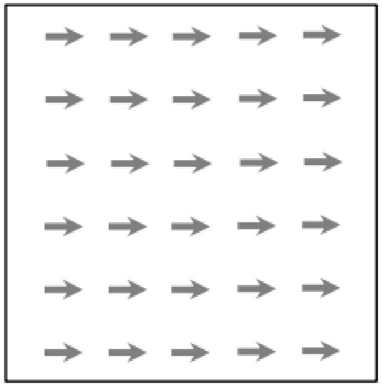
For a magnetic field, the divergence is always zero because there are no magnetic charges, meaning there are no magnetic monopoles. One can never separate the north pole from the south pole of a magnet.

2- Mathematical reminder

$$\overrightarrow{\mathrm{rot}}\,\vec{F} = \vec{\nabla} \times \vec{F}$$

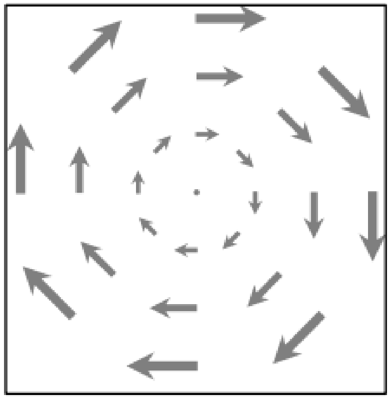
Rotationnel

Applies to a vector field gives a vector field



$\overrightarrow{\mathrm{rot}}\,\vec{F} = \vec{0}$

Irrotational field



$\overrightarrow{\mathrm{rot}}\,\vec{F} \neq \vec{0}$

Vortex field
Or Rotating field

2- Mathematical reminder

Rotationnel

Note: $\overrightarrow{rot}(\overrightarrow{grad}\, f) = \vec{0}$

$div(\overrightarrow{rot}\vec{A}) = 0$

Example

1- In electrostatics:

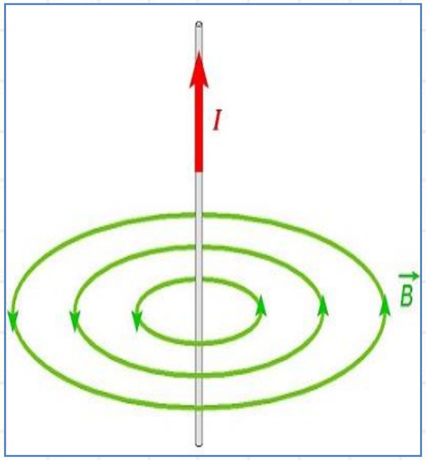
$\overrightarrow{rot}\vec{E} = \vec{0} \quad \Rightarrow \quad \vec{E} = -\overrightarrow{grad}\, v$

- So the electric field derives from a potential.

2- Infinite wire with current I

$\overrightarrow{rot}\vec{B} \neq \vec{0}$

- The magnetic field circulates around the wire

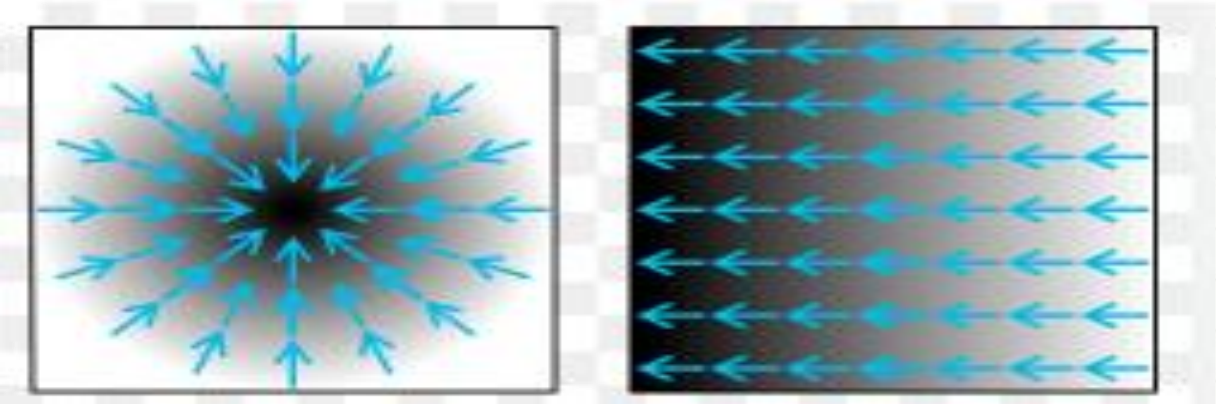


2- Mathematical reminder

Gradient $\overrightarrow{\text{grad}}\, f = \vec{\nabla} f$

$$\vec{\text{grad}}\ f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

Applies to a scalar field ⇒ gives a vector field



- The gradient transforms a scalar field into a vector field that describes how the scalar varies in space
- The gradient is directed toward areas of higher intensity of the scalar quantity

Chapter 4: Electromagnetic waves

2- Mathematical reminder

Laplacian

Applies to a scalar or vector field ⟹ gives a field of the same type

- Laplacian of a function ⟹ $$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplacian of a vector ⟹ $$\nabla^2 \vec{A} \begin{pmatrix} \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \\ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \\ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \end{pmatrix}$$

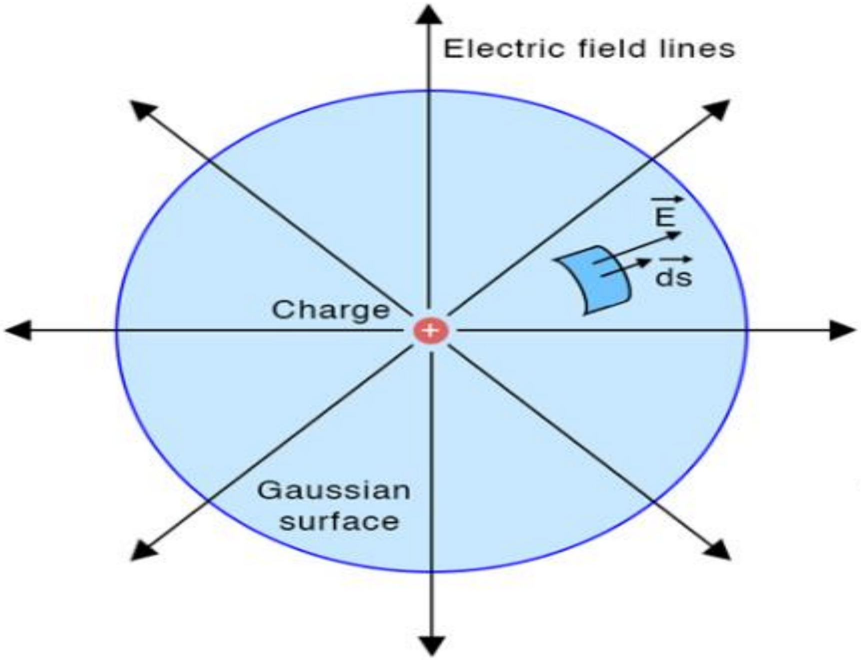
3- Electromagnetic fundamentals review

Gauss's law for electricity

Gauss's Law states that the electric flux through a closed surface (also known as a Gaussian surface) is directly proportional to the total charge enclosed within that surface. Mathematically, it is expressed as:

Gauss's Law

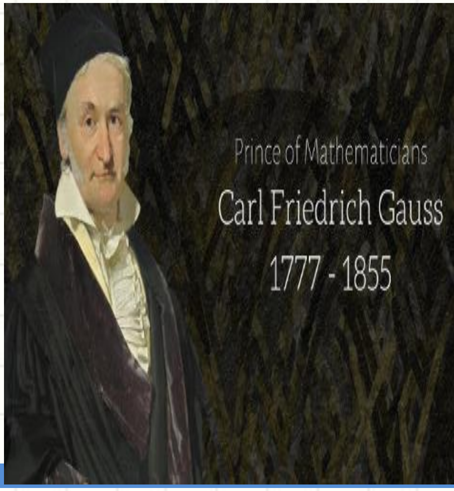
$$\phi = \oint \vec{E}.\vec{ds} = \frac{Q_{enc}}{\varepsilon_o}$$



$$\phi \propto Q_{enc}$$

$$\phi = \oint \vec{E}.\vec{ds} = \frac{Q_{enc}}{\varepsilon_o}$$

- ϕ : Electric flux
- $\vec{E}$: Electric field
- $\vec{ds}$: Infinitesimal surface area
- Q_{enc} : Charge enclosed
- ε_o : Permittivity of air



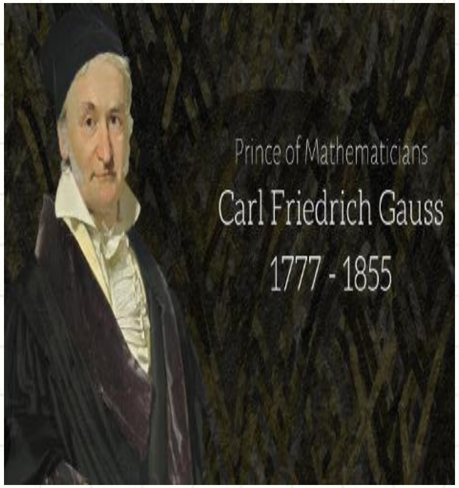
3- Electromagnetic fundamentals review

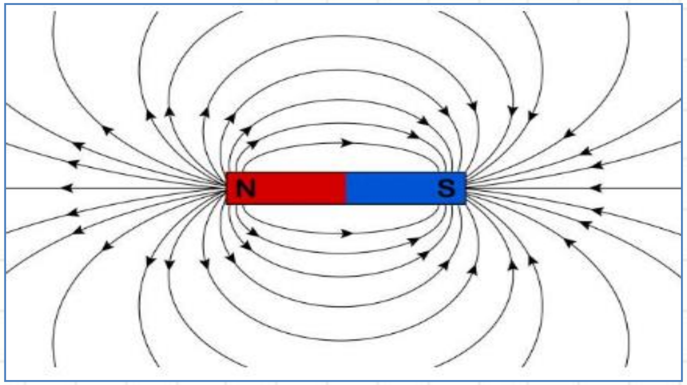
Gauss's Law for Magnetism

- We can write a similar relation for the magnetic case, however, there are *no* isolated magnetic charges (monopoles) exist. Therefore:

$$\boxed{\varnothing_{\vec{B}} = \oiint \vec{B}\,.\,\overrightarrow{ds} = 0}$$

- This is Gauss's Law for Magnetism





Chapter 4: Electromagnetic waves

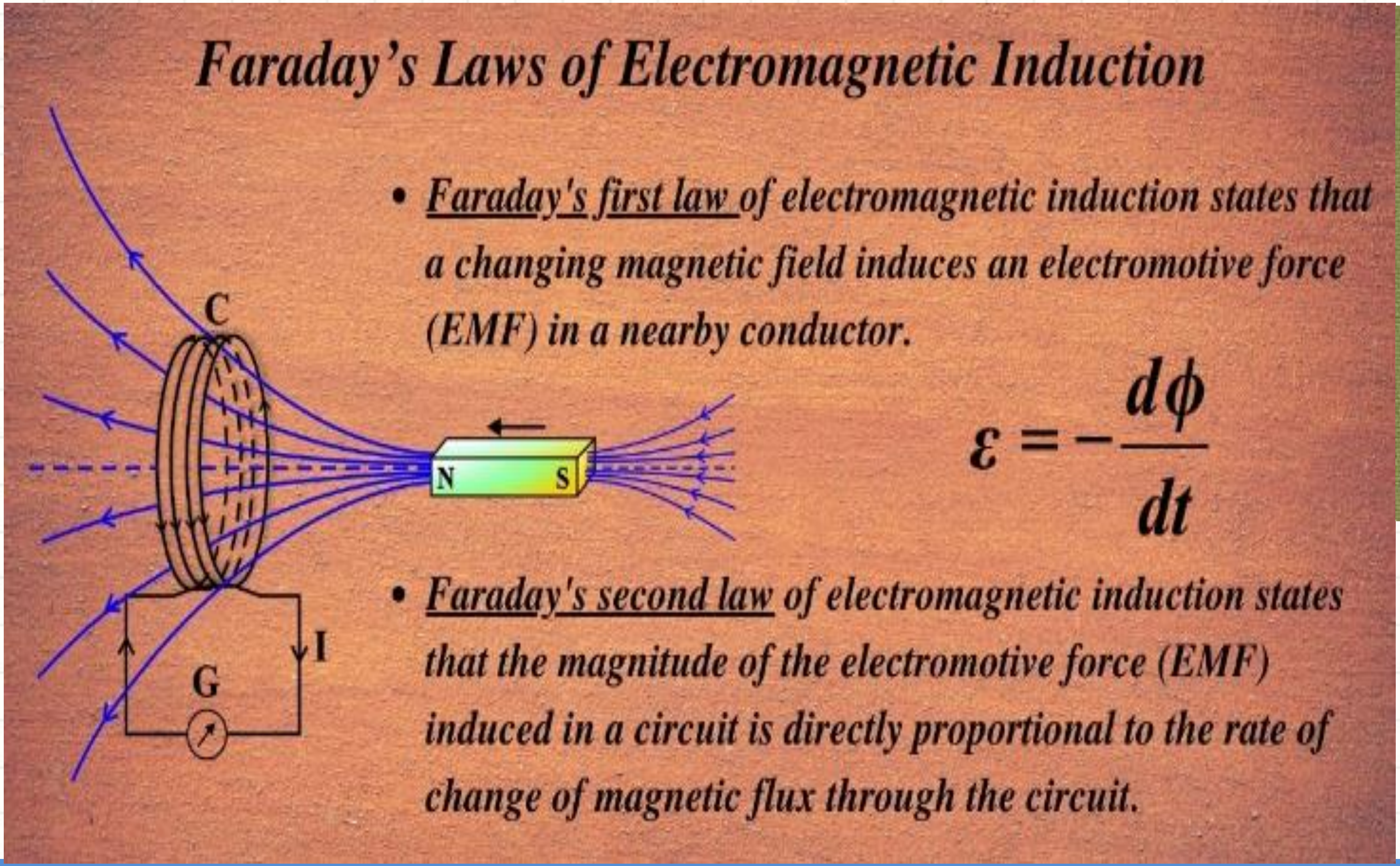
3- Electromagnetic fundamentals review

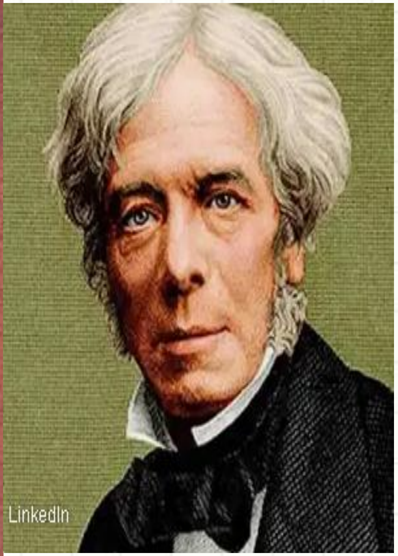
Gauss's Law for Magnetism

- Magnetic field lines are always closed loops: they emerge from the north pole and enter the south pole, but inside the magnet, they continue from the south pole back to the north pole.

- The magnetic flux through a closed surface is zero because the magnetic field has no point sources or sinks , in other words, there are no isolated magnetic charges (magnetic monopoles). Unlike the electric field, which originates from electric charges, the magnetic field is produced by electric currents or magnetic dipoles, such as magnets.

3- Electromagnetic fundamentals review





Michael Faraday

Physicien et chimiste britannique

3- Electromagnetic fundamentals review

An electromotive force (e.m.f.) may arise when a circuit is subject to a changing magnetic flux. This variation can occur in several ways:

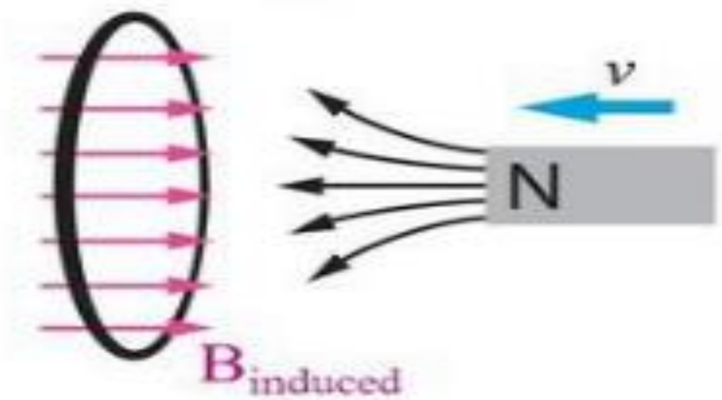
1. A stationary circuit placed in a time-varying magnetic field
2. A moving circuit within a constant magnetic field
3. A moving circuit in a time-varying magnetic field

3- Electromagnetic fundamentals review

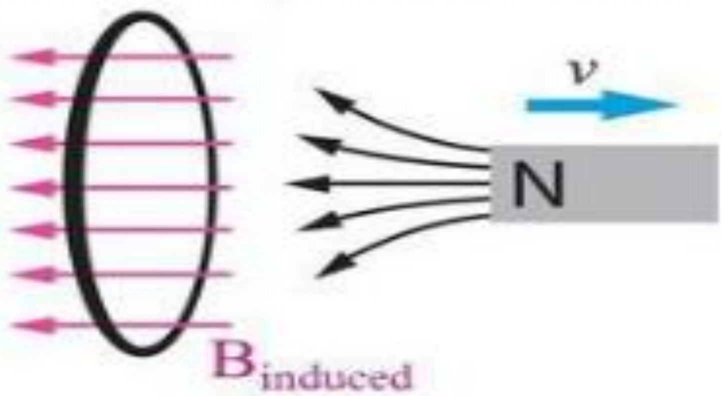
Lenz's Law

The *induced B field* in a loop of wire will **oppose the change in magnetic flux** through the loop.

If you try to **increase** the flux through a loop, the induced field will oppose that increase!



If you try to **decrease** the flux through a loop, the induced field will replace that decrease!



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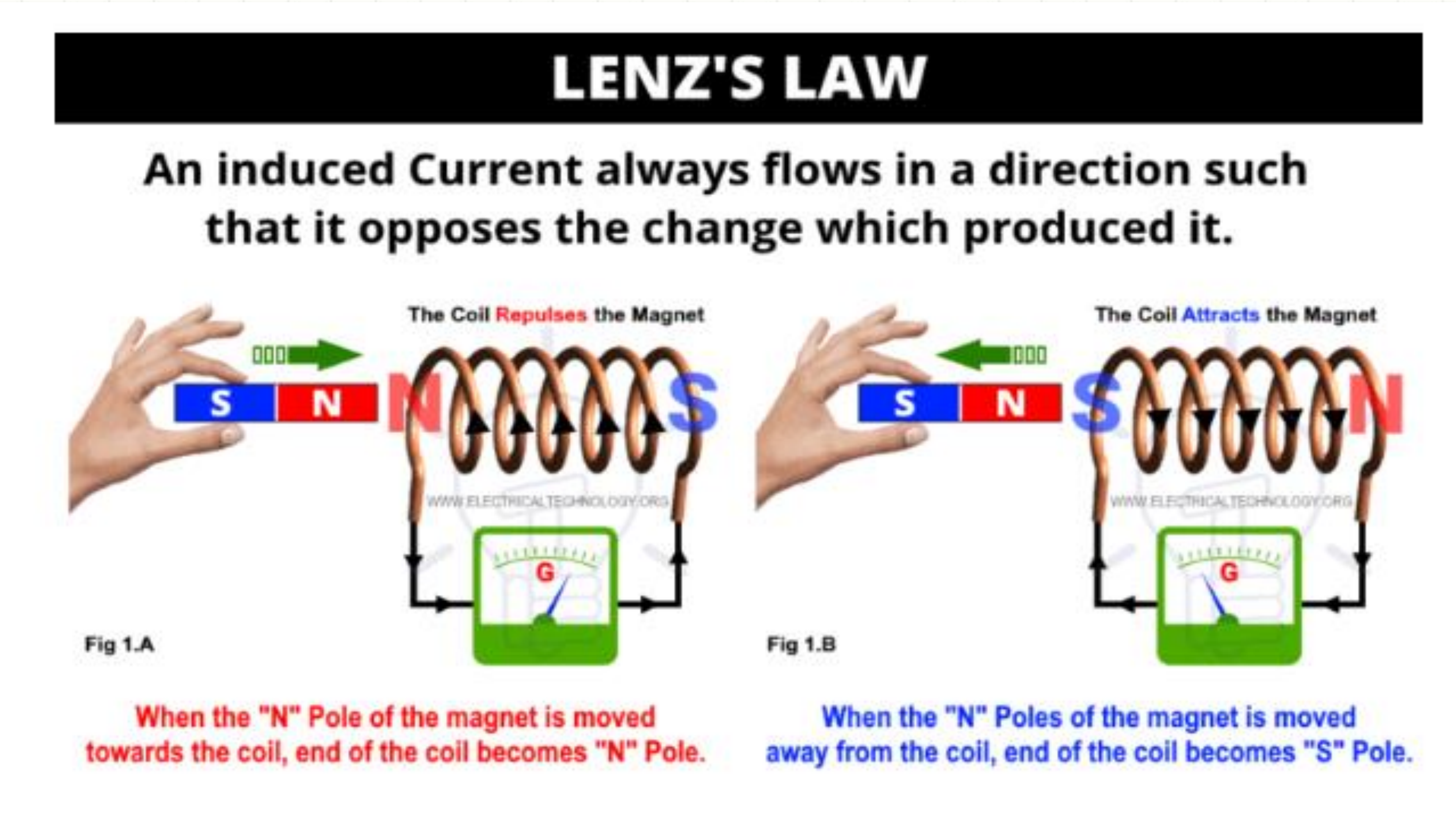
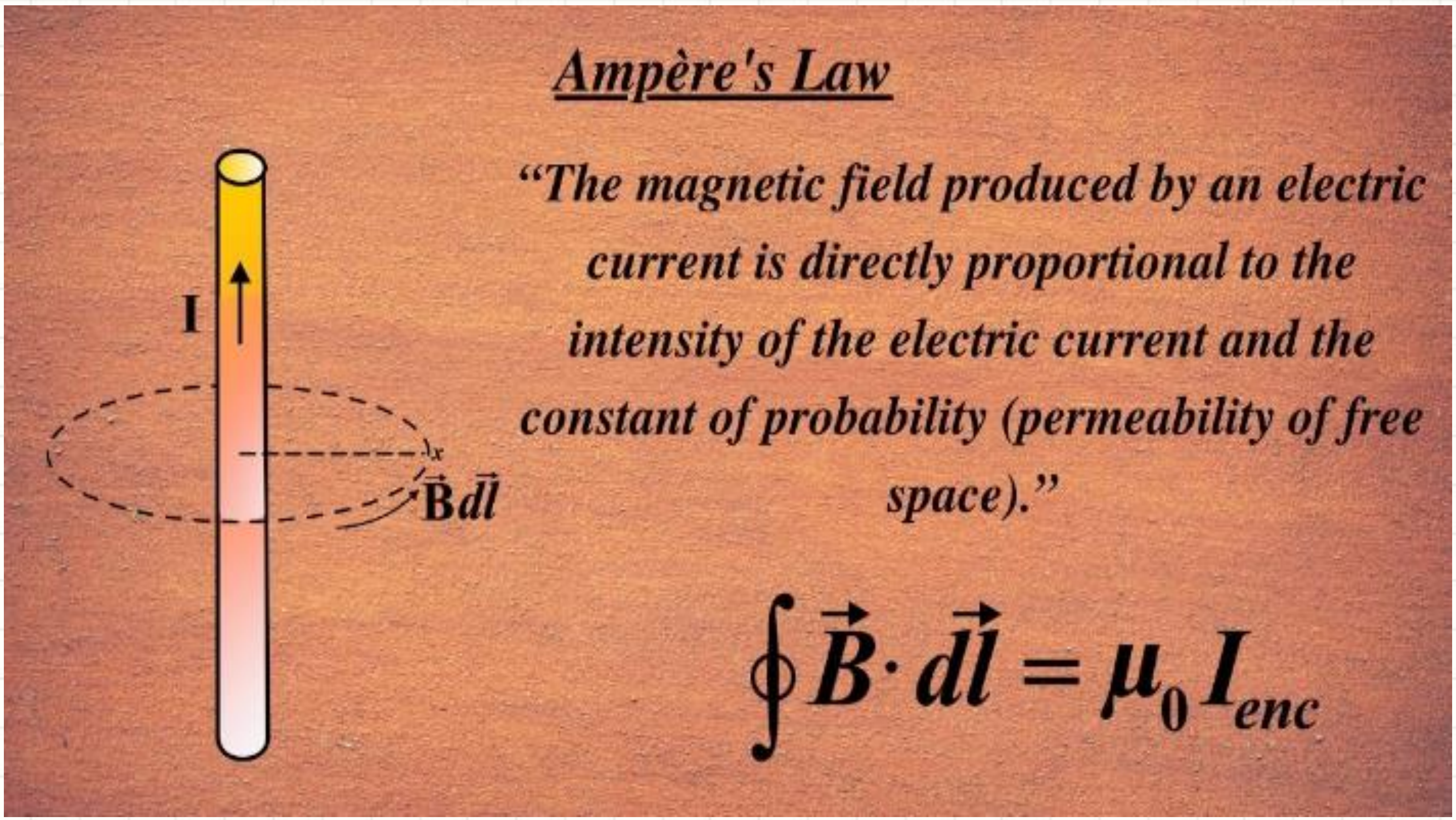
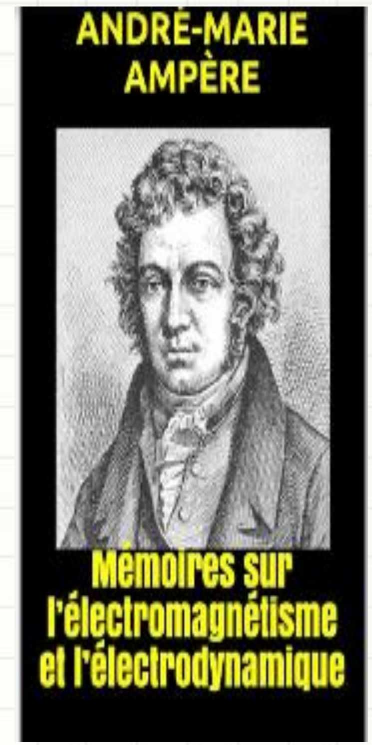
3- Electromagnetic fundamentals review



3- Electromagnetic fundamentals review





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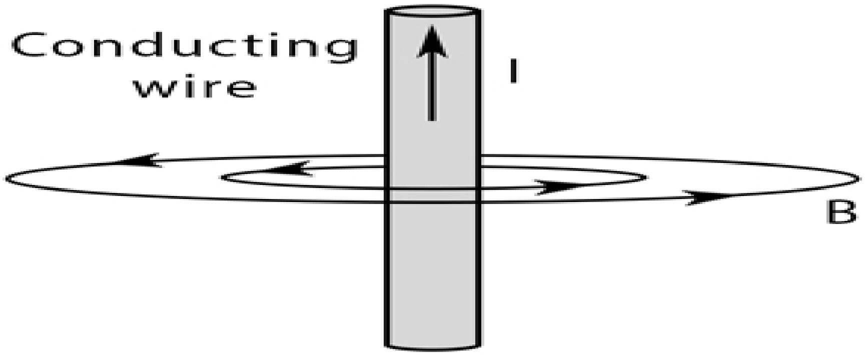
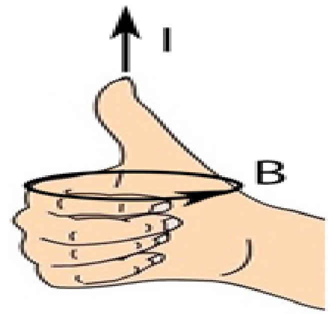
3- Electromagnetic fundamentals review

Ampere's Law



Right hand thumb rule



Integral form: $\oint \vec{B}.\vec{dl} = \mu_o I$

Differential form: $\vec{\nabla} X \vec{B} = \mu_o \vec{J}$

I : Electric curent

B : Magnetic field

μ_o: Permeability of free space

Thumb points in the direction of the electric current and fingers curl around the current indicating the direction of the magnetic field

3- Electromagnetic fundamentals review

Displacement Current and Maxwell-Ampère Correction

- The Problem with Classical Ampere's Law
- The classical Ampère's law relates the magnetic field to the electric current passing through a surface:

$$\oint \vec{B}\,.\,\overrightarrow{ds} = \mu_0 . I$$

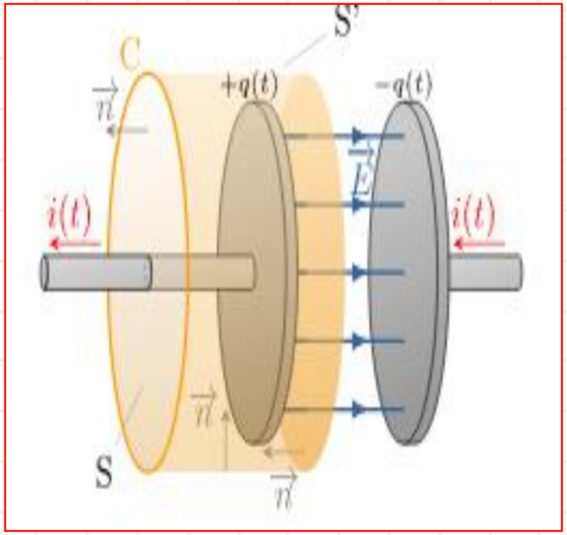
- This law works well for steady currents in conductors. However, in dynamic situations like a charging capacitor a paradox appears:
  - In the wires, a current I flows. But between the plates of the capacitor, no real current flows, so $I = 0$, which would imply a discontinuity in the magnetic field.

3- Electromagnetic fundamentals review

Let's consider a classic example in electrokinetics to show that the classical equations are not always consistent.

Take a charged capacitor discharging through a resistor.

- During discharge, an electric current flows through the circuit, generating a magnetic field $\vec{B}$ around the connecting wires.



Let's consider a circular loop C around the circuit

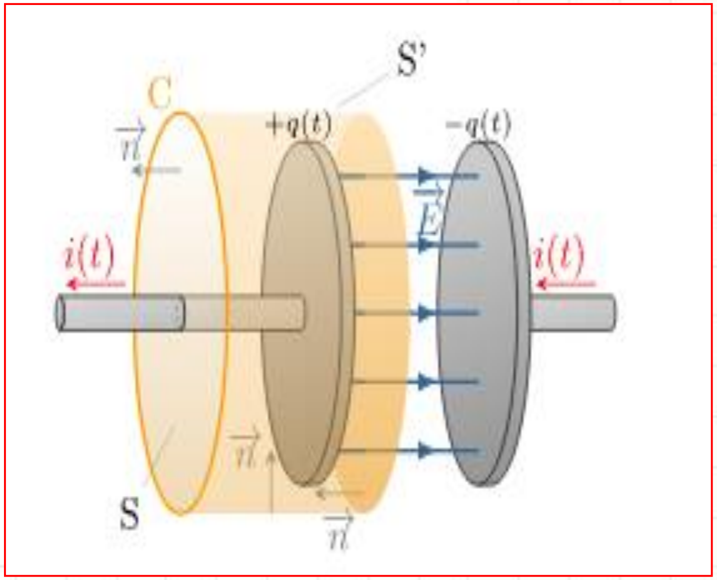
$\oint \vec{B}.\overrightarrow{ds} = \mu_0.I$ This is the Ampère's law through surface S, bounded by contour C.

3- Electromagnetic fundamentals review

However, we are free to choose any surface bounded by C (Stokes' theorem). Let's choose surface **S'** that lies between the plates of the capacitor. In this case, no conduction current passes through S', so:

$$\oint \vec{B} \cdot \overrightarrow{ds} = 0$$

This *contradicts* the previous result.



3- Electromagnetic fundamentals review

- To resolve this contradiction, Maxwell introduced the concept of displacement current.

Maxwell postulated that a time-varying electric field can also generate a magnetic field, even in the absence of moving charges.

He introduced a corrective term called the displacement current, defined as:

$$I_d = \epsilon_0 \frac{\partial \emptyset_{\vec{E}}}{\partial t}$$

where:

- ϵ_0: permittivity of free space,
- $\emptyset_{\vec{E}} = \oiint \vec{E}.\overrightarrow{ds}$ Electric flux through a surface.

- Maxwell suggested including an additional contribution, called the displacement current I_d , to the real current I,

$$\oint \vec{B}\,.\,\overrightarrow{dl} = \mu_0 (I + I_d)$$

4- Maxwell's equations

After reviewing the fundamental laws of electromagnetism namely:

- Gauss's law for electricity (charge distribution),
- Gauss's law for magnetism (absence of magnetic monopoles),
- Faraday–Lenz law (electromagnetic induction),
- and Ampère's law (relation between current and magnetic field)

it becomes possible to combine these results into a unified theoretical framework.

This was accomplished by James Clerk Maxwell in the 19th century, who formulated a set of four fundamental equations that fully describe the behavior of electric and magnetic fields, as well as their interaction with charges and currents. These equations are the foundation of classical electromagnetism and underpin many modern technologies (radio waves, motors, transformers, etc.).

Maxwell's equations also predict the existence of electromagnetic waves, of which light is a natural manifestation.

4- Maxwell's equations

$$\mathrm{div}(\vec{E}) = \frac{\rho}{\epsilon_0} \qquad \vec{\mathrm{rot}}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t}$$

(Maxwell-Gauss) (Maxwell-Faraday)

$$\mathrm{div}(\vec{B}) = 0 \qquad \vec{\mathrm{rot}}(\vec{B}) = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

(Maxwell-Flux) (Maxwell-Ampère)

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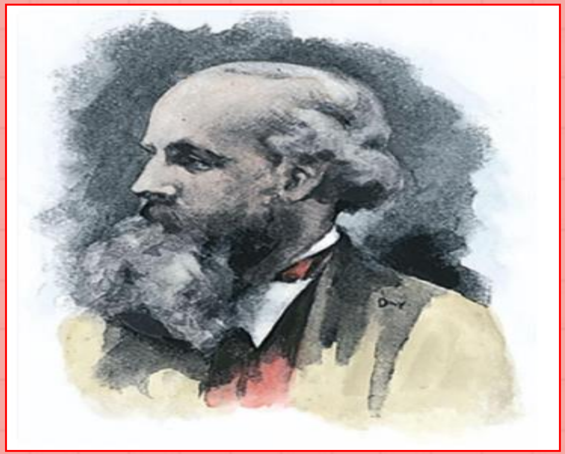
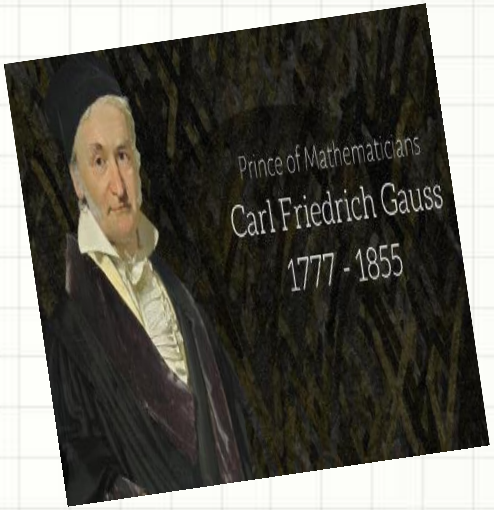
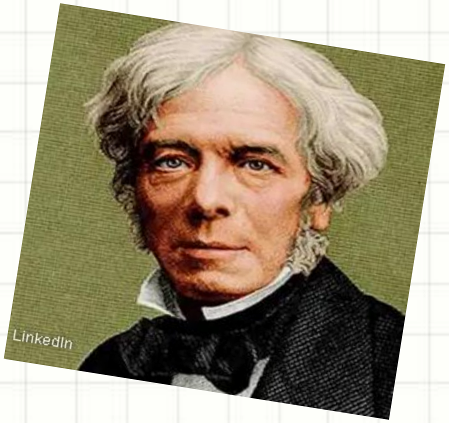






Chapter 4: Electromagnetic waves

4- Maxwell's equations

$1 - \quad div\vec{E} = \frac{\rho}{\varepsilon_0}$ Maxwell- Gauss ?

We begin with **Gauss's law for electricity** in its integral form:

$$\oiint \vec{E}.\overrightarrow{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

To express this law in differential form, we use the divergence theorem (also known as Gauss-Ostrogradsky's theorem), which allows us to convert a surface integral into a volume integral:

$$\oiint \vec{E}.\overrightarrow{ds} = \iiint div\vec{E}.dv$$

- ✓ This converts a surface integral into a volume integral over the divergence of the vector field.

Substituting this into Gauss's law gives: $\iiint div\vec{E}.dv = \frac{Q_{enc}}{\varepsilon_0}$

4- Maxwell's equations

$1 - \quad div\vec{E} = \frac{\rho}{\varepsilon_0}$ Maxwell- Gauss ?

The enclosed charge Q_{enc} can itself be written as a volume integral of the charge density:

$$Q_{enc} = \iiint \rho \, dv$$

Substituting this into the equation, we get: $\iiint div\vec{E}.\,dv = \iiint \frac{\rho}{\varepsilon_0}\, dv$

Since this equality holds for any arbitrary volume, we conclude that the integrands must be equal at every point. This leads to the differential form of Gauss's law:

$$div\vec{E} = \frac{\rho}{\varepsilon_0}$$

- ✓ This is the first of Maxwell's equations, describing how electric charges produce electric fields. (Describes how electric charges act as sources (or sinks) of the electric field).

4- Maxwell's equations

$2 - \ div\vec{B} = 0 \ \ Maxwell - Flux\,?$

We begin with Gauss's law for magnetism in its integral form: $\emptyset_{\vec{B}} = \oiint \vec{B}.\overrightarrow{ds} = 0$

This equation tells us that the net magnetic flux through any closed surface is zero.

we again use the divergence theorem

$$\oiint \vec{B}.\overrightarrow{ds} = \iiint div\vec{B}.dv = 0$$

Since this is true for any arbitrary volume, the integrand itself must be zero everywhere in space. Therefore, we obtain the differential form of Gauss's law for magnetism: $div\vec{B} = 0$

- ✓ This is the second of Maxwell's equations , expresses the fact that magnetic monopoles do not exist In other words: There are no isolated magnetic charges; every magnetic field line that enters a region also exits it magnetic field lines are always closed loops.

4- Maxwell's equations

3 − $\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ *Maxwell − Faraday ?*

Faraday's law states that the electromotive force (e.m.f.) around a closed loop is equal to the negative time rate of change of the magnetic flux through the surface it encloses: $e.m.f = -\frac{\partial \phi_{\vec{B}}}{\partial t}$

The electromotive force is also expressed as the line integral of the electric field around the closed loop: $e.m.f = \oint \vec{E}.\vec{dl}$

Combining the two expressions: $\oint \vec{E}.\vec{dl} = -\frac{\partial \phi_{\vec{B}}}{\partial t}$

Now, applying Stokes' theorem, which relates the circulation of a vector field around a closed loop to the curl of that field over the surface it encloses, we have:

$$\oint \vec{E}.\vec{dl} = \iint \overrightarrow{rot}\,\vec{E}.\vec{dS}$$

Substitute into Faraday's law: $\iint \overrightarrow{rot}\,\vec{E}.\vec{dS} = -\frac{\partial \phi_{\vec{B}}}{\partial t}$

4- Maxwell's equations

$3-\ \overrightarrow{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ *Maxwell – Faraday ?*

Assuming the surface is stationary in time, we can bring the time derivative inside the integral:

$$\iint \overrightarrow{rot}\ \vec{E}\,.\overrightarrow{dS} = -\frac{\partial(\oiint \vec{B}.\overrightarrow{ds})}{\partial t} \quad \Rightarrow \quad \overrightarrow{rot}\ \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- ✓ This is the third of Maxwell's equations , is the differential form **of** Faraday's law of induction. A time-varying magnetic field produces a circulating electric field. This is the principle behind electric generators, transformers…

Chapter 4: Electromagnetic waves

4- Maxwell's equations

$$4 - \ \overrightarrow{rot}\vec{B} = \mu_0\vec{J} + \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t} \quad Maxwell - Ampère\ ?$$

Maxwell introduced the concept of displacement current I_d , which accounts for the changing electric field:

$$\oint \vec{B}\,.\,\vec{dl} = \mu_0(I + I_d)$$

The displacement current is defined as: $I_d = \frac{\partial \emptyset_{\vec{E}}}{\partial t}$ with $\emptyset_{\vec{E}} = \oiint \vec{E}\,.\,\vec{ds}$

$$\oint \vec{B}\,.\,\vec{dl} = \mu_0(I + I_d) \quad \Rightarrow \quad \oint \vec{B}\,.\,\vec{dl} = \mu_0 I + \mu_0\,\epsilon_0 \frac{\partial \emptyset_{\vec{E}}}{\partial t}$$

We now use Stokes' theorem to convert the loop integral into a surface integral:

$$\oint \vec{B}\,.\,\vec{dl} = \iint \overrightarrow{rot}\,\vec{B}\,.\,\vec{dS}$$

Chapter 4: Electromagnetic waves

4- Maxwell's equations

$$4-\ \overrightarrow{rot}\vec{B} = \mu_0\vec{J} + \frac{1}{C^2}\frac{\partial \vec{E}}{\partial t}\quad Maxwell - Ampère\ ?$$

So the equation becomes

$$\oint \vec{B}\,.\,\vec{dl} = \mu_0 I + \mu_0\epsilon_0\frac{\partial \emptyset_{\vec{E}}}{\partial t} \quad\Rightarrow\quad \iint \overrightarrow{rot}\,\vec{B}\,.\,\overrightarrow{dS} = \mu_0 I + \mu_0\epsilon_0\frac{\partial(\oiint \vec{E}\,.\,\overrightarrow{ds})}{\partial t}$$

We have $\quad I = \iint \vec{J}\,.\,\overrightarrow{dS} \quad and \quad \emptyset_{\vec{E}} = \oiint \vec{E}\,.\,\overrightarrow{ds}$

Where $\vec{J}$ is the current density corresponding to the real (conduction) current I

We get $\quad \iint \overrightarrow{rot}\,\vec{B}\,.\,\overrightarrow{dS} = \mu_0 \iint \vec{J}\,.\,\overrightarrow{dS} + \mu_0\epsilon_0\frac{\partial(\oiint \vec{E}.\overrightarrow{ds})}{\partial t}$

$$\Rightarrow \qquad \overrightarrow{rot}\,\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

- ✓ This is the fourth of Maxwell's equations, This equation expresses that a magnetic field can be generated by: A conduction current $\vec{J}$ Or a time-varying electric field $\frac{\partial \vec{E}}{\partial t}$ (displacement current)

Chapter 4: Electromagnetic waves

4- Maxwell's equations

$$4-\ \overrightarrow{rot}\vec{B}=\mu_0\vec{J}+\frac{1}{C^2}\frac{\partial\vec{E}}{\partial t}\quad Maxwell-Ampère\ ?$$

✓ This unification by Maxwell is essential to describe electromagnetic waves and led to the prediction that light is an electromagnetic wave.

$$\overrightarrow{rot}\,\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\quad\Rightarrow\quad\overrightarrow{rot}\,\vec{B}=\mu_0\vec{J}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$c=\dfrac{1}{\sqrt{\mu_0\epsilon_0}}$ is the speed of light in vacuum.

Dielectric permittivity: $\varepsilon_0\approx 8.85\times 10^{-12}$ farad/meter

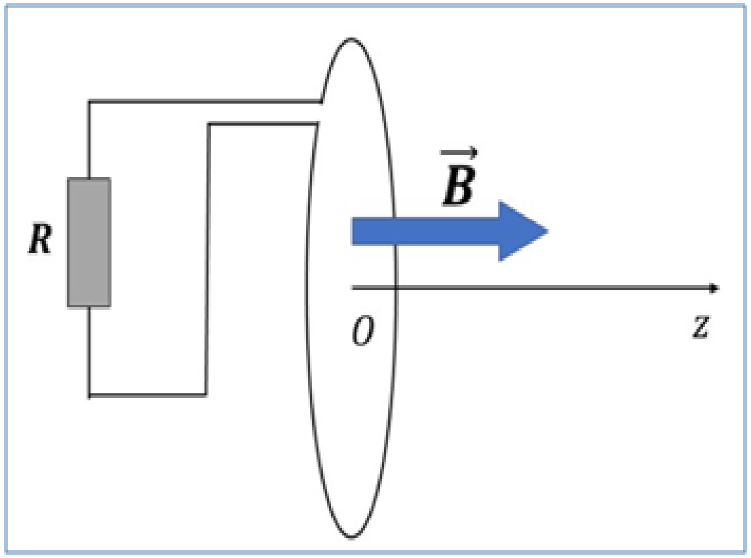
magnetic permeability: $\mu_0=4\pi\times 10^{-7}$ henry/meter

$c=3\times 10^8\ m/s$

Exercise 1:

Consider a circular conducting loop of radius "a" lying in the xOy plane, closed through a resistor R(the lengths of the connecting wires and the surface they enclose are neglected).

The loop is located in a region where there is a time-varying magnetic field that is parallel to the Oz axis (i.e., perpendicular to the plane of the loop).



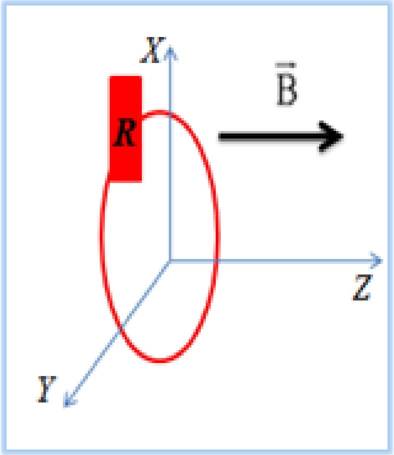
Given: $\vec{B} = B_0 \cos(\omega t)\vec{e_z}$

Calculate the current flowing through the circuit formed by the loop and the resistor.

Data: $a = 0.1\ m$, $R = 50\ \Omega$, $B_0 = 0.2\ T$, $\omega = 10^3 rad/s$

Exercice1

- Boucle conductrice (Rayon **a**, Résistance **R**) dans XOY



1-Calculer le courant induit i ?

$$\phi \rightarrow f.e.m \rightarrow i_{(\text{la loi d'Ohm})}$$

Circuit fixe plongé dans un champ magnétique variable

$$\emptyset_{\vec{B}} = \oiint \vec{B}\,.\,\overrightarrow{ds} \qquad = B_0 \cos wt \oiint \vec{e_z}\,.\,\overrightarrow{ds}$$

Exercice1

$$\vec{e_z}\cdot\vec{ds} = \vec{e_z}\cdot\vec{n}\cdot ds = ds \Rightarrow \vec{e_z} \parallel \vec{n}$$

$$\emptyset_{\vec{B}} = B_0 \cos wt \oiint ds \Rightarrow \boxed{\emptyset_{\vec{B}} = B_0\, a^2\, \pi \cos wt}$$

D'après la loi de Faraday $\Rightarrow$ $\boxed{e = -\dfrac{d\,\phi_{\vec{B}}}{d\,t}}$

avec $\Rightarrow$ e : *la force électromotrice (f.e.m)*

Exercice1

f.e.m *est une tension* *[f.e.m] =[U]*

$$e = -\frac{d\phi_{\vec{B}}}{dt} = -\pi a^2 \omega B \sin \omega t$$

L'intensité de i

$$i = \frac{f.e.m}{R} = \frac{e}{R} = \frac{\pi a^2 \omega B_0}{R} \sin \omega t$$

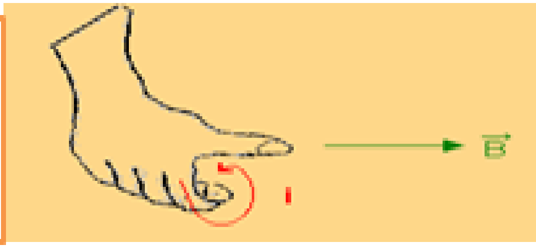
$$i = \frac{\pi a^2 \omega B_0}{R}$$

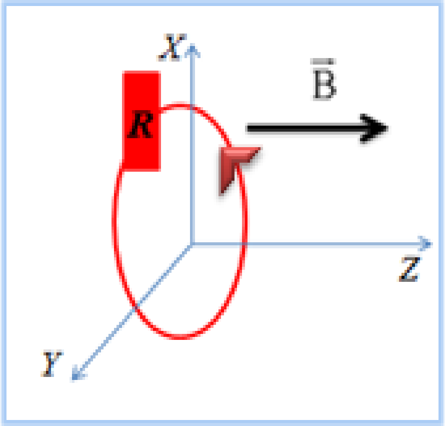
AN

$$i = \frac{\pi (0.1)^2 10^3 (0.2)}{50} = 0.13\text{A}$$

➤*Pour déterminer le sens de i on utilise la règle de la main droite pour le courant induit*

➤Le pouce de la main droite vers le champ B
➤les autres doigts indiquent le sens du courant induit i





Maxwell's Equations in Vacuum

In the absence of charges and currents (in vacuum), Maxwell's equations are:

$$\mathrm{div}\,\vec{E} = 0 \qquad\qquad \mathrm{div}\,\vec{B} = 0$$

$$\overrightarrow{\mathrm{rot}}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \overrightarrow{\mathrm{rot}}\,\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

5- Electromagnetic wave equation

Electromagnetic waves are all around us from visible light and radio waves to microwaves and X-rays. But how exactly do these waves travel through space? And what laws govern their motion? The answer lies in Maxwell's equations, which describe the behavior of electric and magnetic fields. When we apply these equations to empty space that is, a region with no electric charges or currents we discover something remarkable: both the electric field and the magnetic field follow a wave equation. This wave equation tells us that changes in the electric field create changes in the magnetic field, and vice versa, allowing the wave to move forward through space. Most importantly, it reveals that electromagnetic waves travel at a fixed speed the speed of light in vacuum.

In this course, we will derive the electromagnetic wave equation step by step, starting from Maxwell's equations. We'll see how these abstract laws lead directly to one of the most important phenomena in physics: the propagation of light.

Chapter 4: Electromagnetic waves

5- Electromagnetic wave equation

Electromagnetic Field ($\vec{E}, \vec{B}$)

When an electric field varies over time, it produces a magnetic field and vice versa. This mutual interaction allows the wave to propagate through space, even without a medium (in a vacuum).

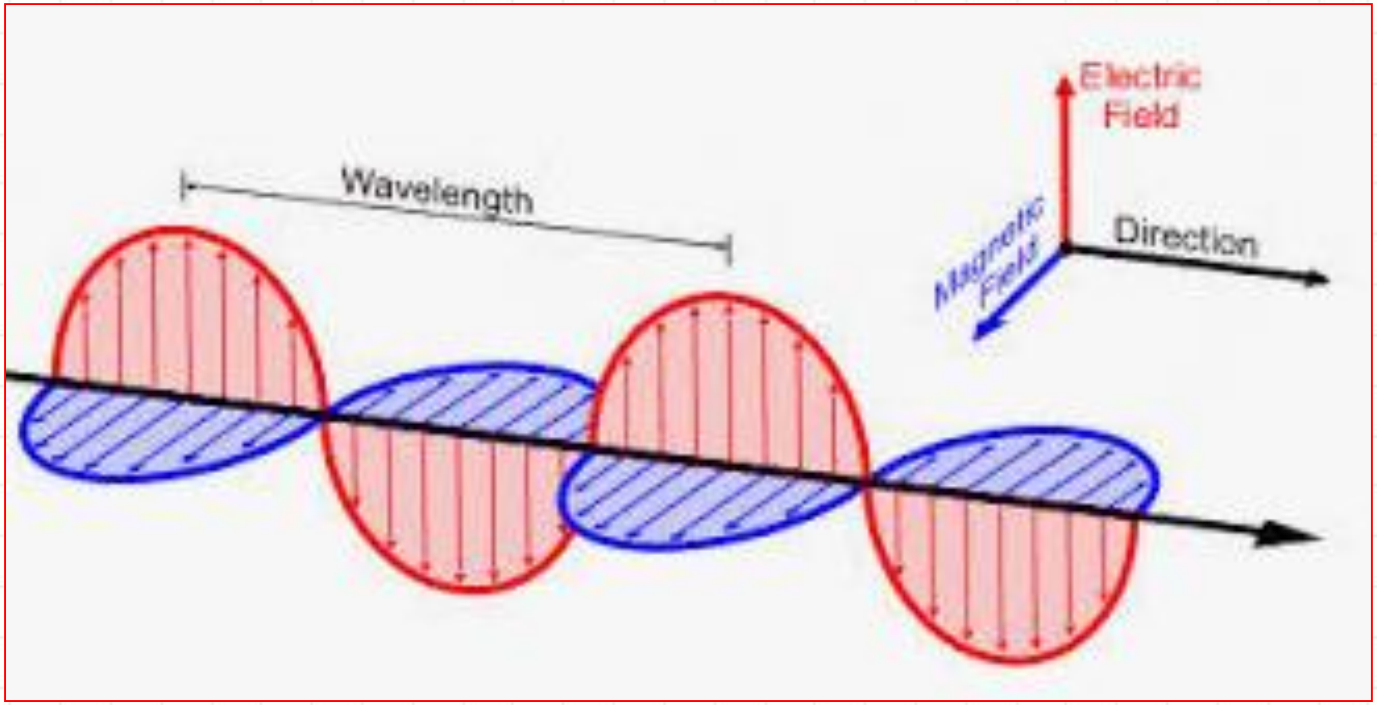
The two fields:

- are perpendicular to each other,
- are perpendicular to the direction of wave propagation,
- travel at the same speed c: $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, $c = 3 \times 10^8 m/s$

Chapter 4: Electromagnetic waves

5- Electromagnetic wave equation

Electromagnetic Field ($\vec{E}, \vec{B}$)



5- Electromagnetic wave equation

Wave Equation of Electromagnetic Waves in Vacuum

We start from $\overrightarrow{rot}\ \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and take the $\overrightarrow{rot}$ of both sides:

$$\overrightarrow{rot}\ (\overrightarrow{rot}\ \vec{E}) = \overrightarrow{rot}\left(-\frac{\partial \vec{B}}{\partial t}\right) \Rightarrow \overrightarrow{rot}\ (\overrightarrow{rot}\ \vec{E}) = -\frac{\partial}{\partial t}(\overrightarrow{rot}\ \vec{B})$$

Use the vector identity: $\overrightarrow{rot}\ (\overrightarrow{rot}\ \vec{A}) = \overrightarrow{grad}(div\vec{A}) - \Delta\vec{A}$

From Maxwell's equations:

$$\overrightarrow{rot}\ (\overrightarrow{rot}\ \vec{E}) = \frac{\partial}{\partial t}(\overrightarrow{rot}\ \vec{B}) \Rightarrow \overrightarrow{grad}(div\vec{E}) - \Delta\vec{E} = -\frac{\partial}{\partial t}(\overrightarrow{rot}\ \vec{B})$$

In vacuum: $div\vec{E} = 0$ and $\overrightarrow{rot}\ \vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$

$$-\Delta\vec{E} = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\partial \vec{E}}{\partial t}\right) \Rightarrow \Delta\vec{E} - \mu_0\epsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$$

$$\Rightarrow \Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad \text{With} \quad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

✓ This is the wave equation for electric fields

5- Electromagnetic wave equation

Wave Equation of Electromagnetic Waves in Vacuum

- Same for the Magnetic Field $\vec{B}$

From: $\overrightarrow{rot}\ \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ and take the $\overrightarrow{rot}$: $\overrightarrow{rot}\left(\overrightarrow{rot}\ \vec{B}\right) = \overrightarrow{rot}\left(\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

$$\Rightarrow \quad \overrightarrow{grad}\left(div\vec{B}\right) - \Delta\vec{B} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\overrightarrow{rot}\ \vec{E}\right)$$

In vacuum: $div\vec{B} = 0$ and $\overrightarrow{rot}\ \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

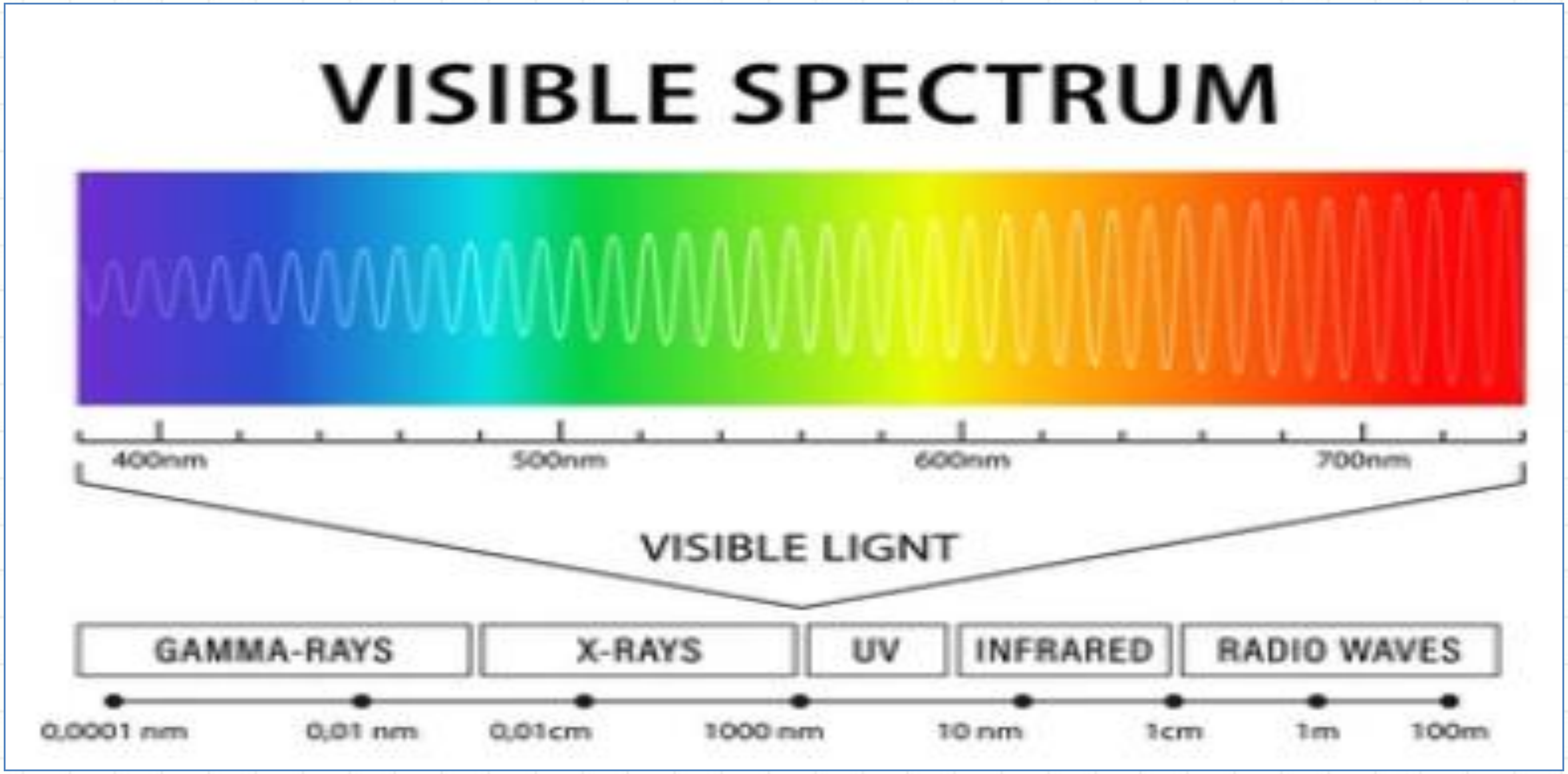
$$\overrightarrow{grad}\left(div\vec{B}\right) - \Delta\vec{B} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\overrightarrow{rot}\ \vec{E}\right) \quad \Rightarrow \quad \Delta\vec{B} - \mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0}$$

$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \vec{0} \qquad \text{With} \quad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

✓ This is the wave equation for magnetic fields

Chapter 4: Electromagnetic waves

5- Electromagnetic wave equation



5- Electromagnetic wave equation

Relationship between $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$

In a material medium, the electric field $\vec{\mathrm{E}}$ is related to the electric displacement field $\vec{\mathbf{D}}$ by:

$$\vec{D} = \varepsilon\,\vec{E}$$

Where:

- $\vec{E}$: electric field
- $\vec{D}$: electric displacement
- ε: permittivity of the medium

Chapter 4: Electromagnetic waves

5- Electromagnetic wave equation

Relationship between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$

Similarly, the magnetic flux density $\vec{\mathbf{B}}$ is related to the magnetic field intensity $\vec{\mathbf{H}}$ by:

$$\vec{B} = \mu \vec{H}$$

Where:

- $\vec{B}$: magnetic flux density
- $\vec{H}$: magnetic field intensity
- μ: permeability of the medium

6- Polarization

Polarization of Electromagnetic Waves

When an electromagnetic wave propagates through space, the electric field vector $\vec{E}$ oscillates in a direction perpendicular to the direction of propagation. The **polarization** of a wave refers to the trajectory described by the tip of the electric field vector in the transverse plane (the plane perpendicular to the direction of propagation).

Polarization describes how the electric field varies in time and space in the plane perpendicular to the wave's travel direction. While the wave's frequency, amplitude, and direction describe its energy and motion, the polarization provides critical information about its orientation and interaction with materials.

Chapter 4: Electromagnetic waves

6- Polarization

Let us consider an electromagnetic plane wave propagating in the z direction. The electric field has two components:

$$\begin{cases} E_x(z,t) = E_{0x}\cos\,(wt - kz + \varphi_1) \\ E_y(z,t) = E_{0y}\cos\,(wt - kz + \varphi_2) \end{cases}$$

We define a retarded time t' variable:

$$t' = (t - \frac{z}{w} + \frac{\varphi_1}{w})$$

Then the expressions become:

$$\begin{cases} E_x(z,t') = E_{0x}\cos\, wt' \\ E_y(z,t') = E_{0y}\cos(wt' + \Delta\varphi) \end{cases} \qquad \text{with} \qquad \Delta\varphi = \varphi = \varphi_2 - \varphi_1$$

6- Polarization

$$\Rightarrow \begin{cases} \dfrac{E_x}{E_{0x}} = \cos\ wt^{'} \\ \dfrac{E_y}{E_{0y}} = \cos(wt^{'} + \varphi) = \cos wt^{'} cos\varphi - \sin wt^{'} sin\ \varphi \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{E_x}{E_{0x}} = \cos\ wt \\ \dfrac{E_y}{E_{0y}} = \cos(wt + \varphi) = \cos wt cos\varphi - \sin wt sin\ \varphi \end{cases}$$

Expanding $\frac{E_y}{E_{0y}}$ using trigonometric identities:

$$\frac{E_y}{E_{0y}} = \cos(wt^{'} + \varphi) = \cos wt cos\varphi - \sin wt sin\ \varphi$$

6- Polarization

Substitute $\frac{E_x}{E_{0x}}$ into the expression of $\frac{E_y}{E_{0y}}$:

$$\frac{E_y}{E_{0y}} = \frac{E_x}{E_{0x}}.\,cos\varphi - \sqrt{1-(\frac{E_x}{E_{0x}})^2}.\,sin\varphi$$

Square both sides and rearrange to eliminate time:

$$(\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}.\,cos\varphi)^2 = (1-\left(\frac{E_x}{E_{0x}}\right)^2).\,sin^2\varphi$$

Finally, we get the general polarization equation:

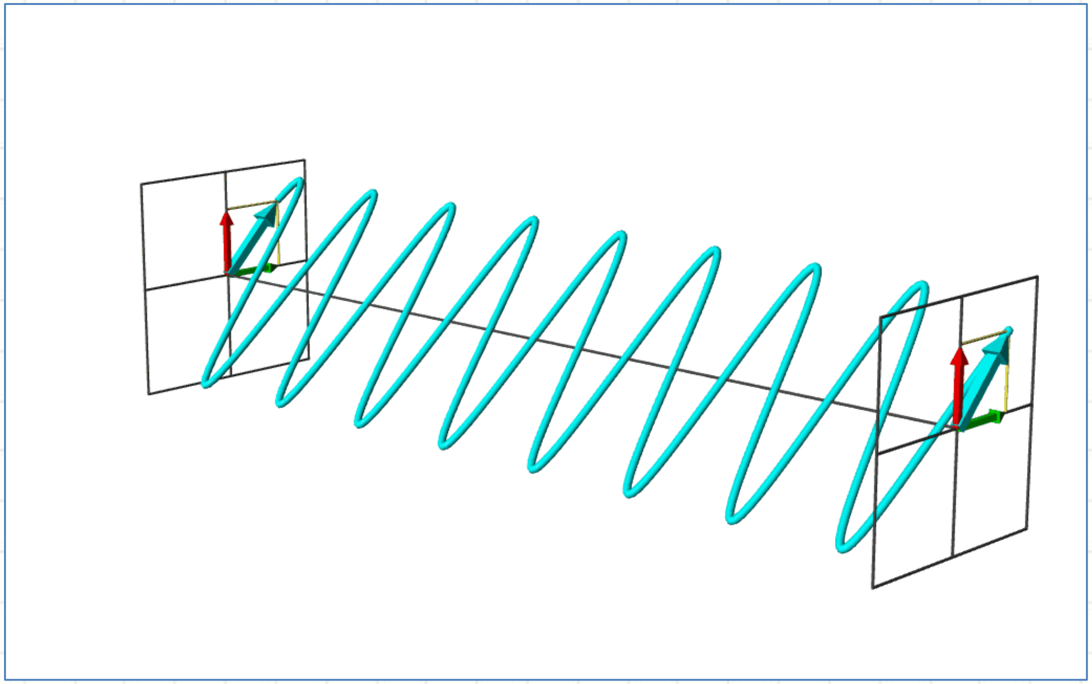
$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\frac{E_x}{E_{0x}}.\frac{E_y}{E_{0y}}cos\varphi = sin^2\varphi$$

- ✓ This equation describes the trajectory of the electric field in the XOY plane. The type of polarization depends on the phase difference $\Delta\,\varphi = \varphi$:

6- Polarization

1- If $\varphi = n\pi$ The equation becomes a straight line $\frac{E_x}{E_y} = cte$

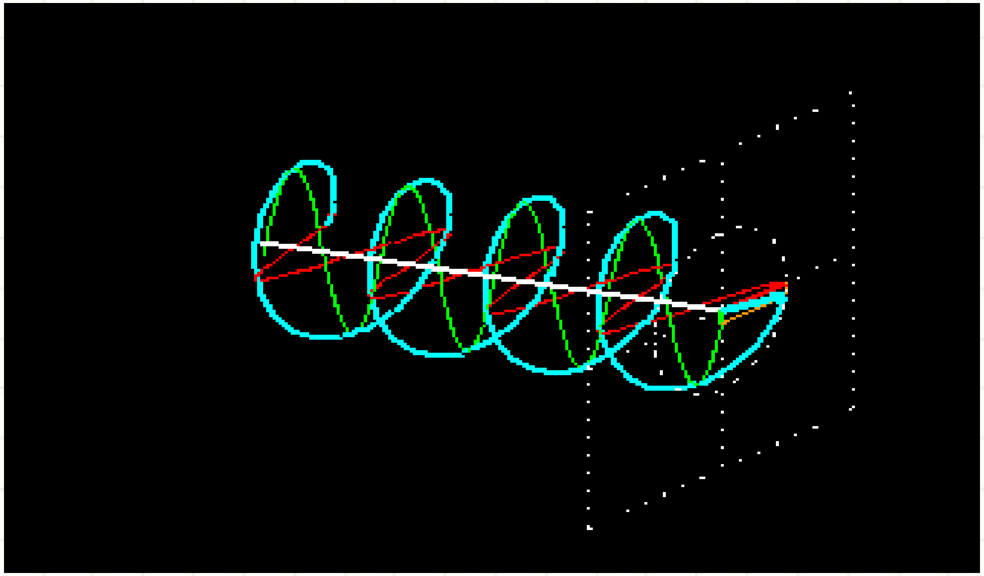
Linear polarization

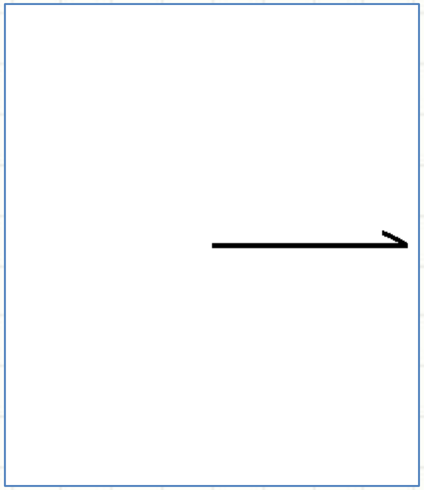


6- Polarization

2- If $\varphi = (2n+1)\frac{\pi}{2}$ and $E_{0x} = E_{0y}$ The equation becomes a circle

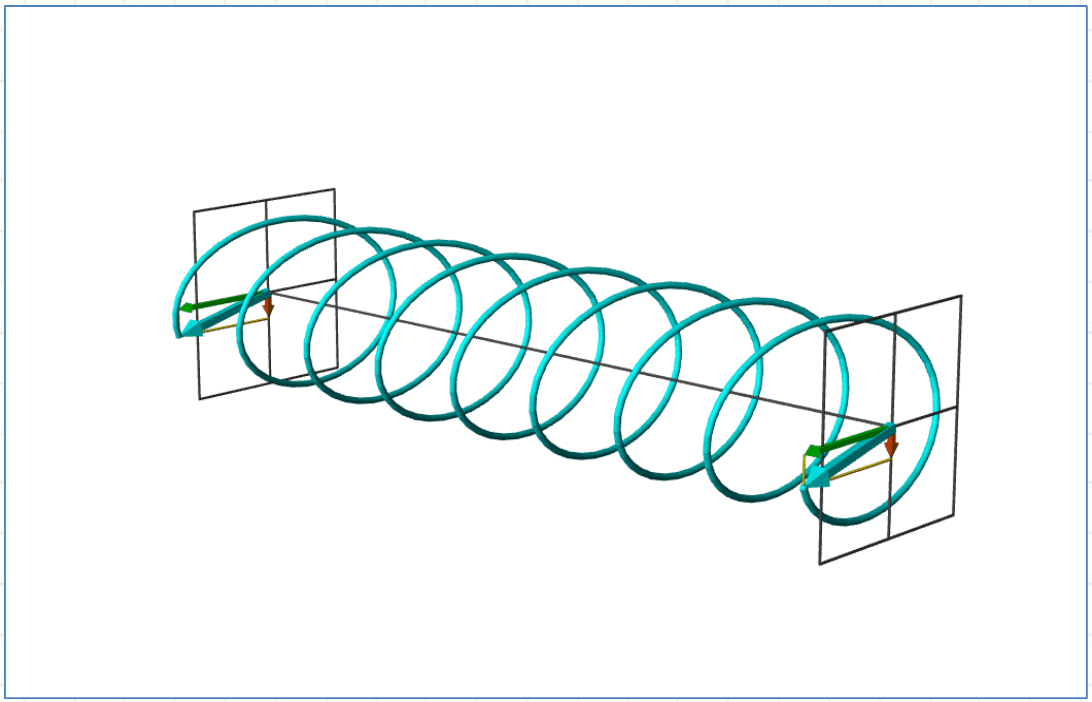
$E_x{}^2 + E_y{}^2 = R^2$ Circular polarization

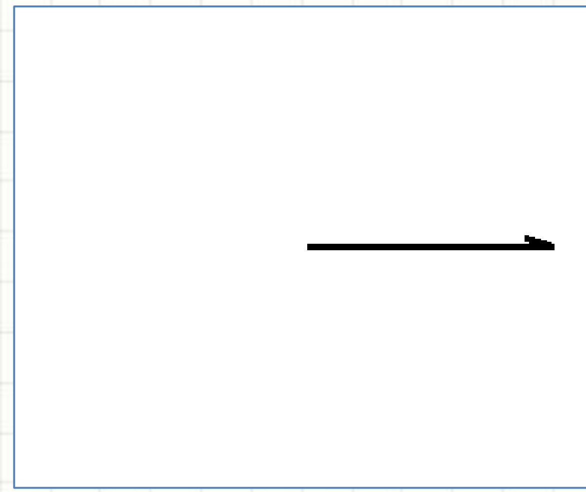




6- Polarization

3- For any other value of φ: The equation describes an ellipse
Elliptical polarization





6- Polarization

There are two types of circular polarization, depending on the direction in which the electric field rotates:

1. Right Circular Polarization (RCP)

In this case, the electric field rotates clockwise. The components of the electric field can be written as:

$$\begin{cases} E_x(z,t) = E_{0x}\cos\left(wt - kz + \varphi_1\right) \\ E_y(z,t) = E_{0y}\cos\left(wt - kz + \varphi_2\right) \end{cases}$$

The phase difference between the two components is:

$$\Delta\varphi = \varphi = \varphi_2 - \varphi_1$$

By choosing a reference frame where $\varphi_1 = 0$, the expressions simplify to:

6- Polarization

By choosing a reference frame where $\varphi_1 = 0$, the expressions simplify to:

$$\begin{cases} E_x(z,t) = E_{0x}\cos\,(wt - kz) \\ E_y(z,t) = E_{0y}\cos\,(wt - kz + \varphi) \end{cases}$$

- If $\varphi = \frac{\pi}{2}$ the wave is right circularly polarized.

2-Left circular polarization(LCP):

In this case, the electric field rotates counter clockwise.

If the phase difference is $\varphi = -\frac{\pi}{2}$, the wave exhibits left circular polarization.

6- Polarization

Important Note:

Using another notation where the phase shift is introduced as a negative term in the E_y component:

$$\begin{cases} E_x(z,t) = E_{0x}\cos\left(wt - kz\right) \\ E_y(z,t) = E_{0y}\cos\left(wt - kz - \varphi\right) \end{cases}$$

Then:

- If $\varphi = \frac{\pi}{2}$ this corresponds to left circular polarization.
- If $\varphi = -\frac{\pi}{2}$, it corresponds to right circular polarization.

6- Polarization

Polarization is important because it gives us extra information about electromagnetic waves. Understanding polarization helps us control light and signals better in science and technology and helps in many applications:

- Optics: Used in sunglasses, cameras, control light.
- Telecommunications: Helps antennas send and receive signals more efficiently, reducing noise.
- Material analysis: Reveals properties of materials like internal stress or structure.
- Astronomy: Helps study Earth's surface, atmosphere, and space (like stars and galaxies).

Impedance

The impedance of a plane progressive harmonic electromagnetic wave propagating in a medium (μ , ε) is defined as:

$$Z = \left|\frac{E}{H}\right| \qquad\qquad Z = \left|\frac{\mu\, E}{B}\right|$$

The characteristic impedance of the medium is given by the ratio of the electric field to the magnetic field amplitudes.

$$Z_c = \sqrt{\frac{\mu}{\varepsilon}}$$

- In vacuum, the impedance is:

$$Z_c = \sqrt{\frac{\mu_0}{\varepsilon_0}} \quad , \quad Z = 377\Omega$$

Chapter 4: Electromagnetic waves

Exercise 1:

1/ Determine the polarization state of the electromagnetic waves represented by their electric fields:
$\overrightarrow{E_1}$, $\overrightarrow{E_2}$, and $\overrightarrow{E_3}$.

$$\overrightarrow{E_1} = \begin{pmatrix} E_1\cos(\alpha)\cos(\omega t - kz) \\ E_1\sin(\alpha)\cos(\omega t - kz) \\ 0 \end{pmatrix}, \overrightarrow{E_2} = \begin{pmatrix} E_2\cos\omega(t - z/c) \\ E_2\sin\omega(t - z/c) \\ 0 \end{pmatrix}, \overrightarrow{E_3} = \begin{pmatrix} E_3\cos\omega(t - z/c) \\ -E_3\sin\omega(t - z/c) \\ 0 \end{pmatrix},$$

2/ What is the polarization state of the wave whose electric field is:
$\overrightarrow{E_2}$+$\overrightarrow{E_3}$., with $E_2 = E_3$?

7-Electromagnetic energy and Poynting vector

1- Electromagnetic Energy Density

The energy density of the electromagnetic wave is the energy per unit volume.

- ***Electric Field Energy Density:***

$$w_E = \frac{1}{2}\,\varepsilon_0 E^2$$

- ***Magnetic Field Energy Density:***

$$w_H = \frac{1}{2}\,\mu_0 H^2 \quad \text{Or} \quad w_B = \frac{1}{2\mu_0}\,B^2$$

- ***Total Energy Density:***

$$w_T = w_E + w_B$$

7-Electromagnetic energy and Poynting vector

2-Poynting vector

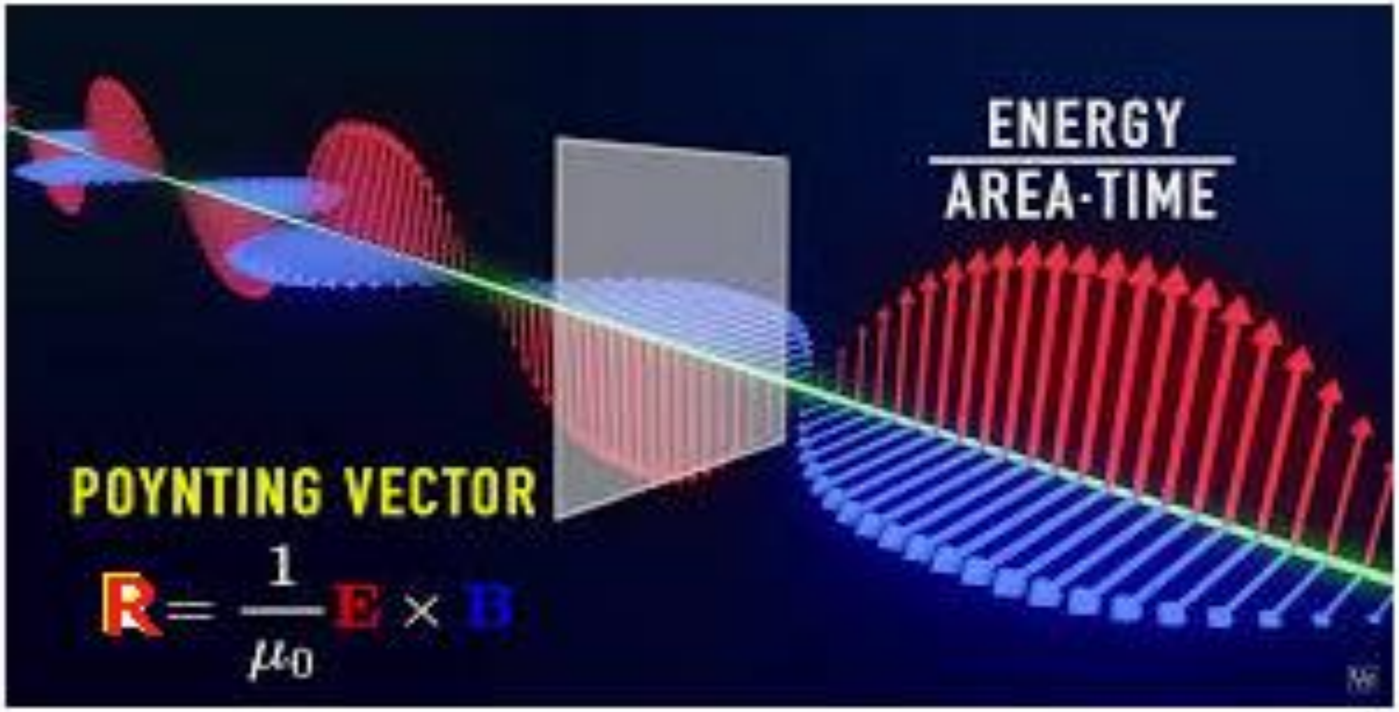
The Poynting vector, denoted $\vec{R}$, represents the electromagnetic power flux per unit area. It indicates the amount of electromagnetic energy transported through space, per unit time and per unit surface.

$$\vec{R} = \frac{\vec{E}\times\vec{B}}{\mu_0} \qquad \text{or} \qquad \vec{\pi} = \frac{\vec{E}\times\vec{B}}{\mu_0}$$

The direction of $\vec{R}$ is given by the cross product $\vec{E} \times \vec{B}$, which means:

- $\vec{R}$ is perpendicular to $\vec{E}$, perpendicular to $\vec{B}$, And parallel to the wave vector $\vec{k}$ (direction of propagation).
- Electromagnetic energy propagates in the same direction as the wave.
- The flux of the Poynting vector through a surface S gives the electromagnetic power P crossing that surface:

$$P = \emptyset_{\vec{R}} = \iint \vec{R}.\overrightarrow{ds}$$


ENERGY
AREA·TIME
POYNTING VECTOR
$\vec{R} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$

Exercise 2

1. Consider a plane electromagnetic wave propagating in vacuum, defined by:

$$\vec{E} = E_0 \cos(\omega t - kz)\overrightarrow{e_x}$$

(a) Specify:

- — the direction and speed of propagation,
- — the expression of k in terms of ω,
- — the nature of the polarization of this wave.

(b) Determine $\vec{B}$.

2. Consider a plane electromagnetic wave defined by:

$$\vec{E} = E_1 \cos(\omega t - kz)\overrightarrow{e_x} + E_1 sin(\omega t - kz)\overrightarrow{e_y}$$

(a) Specify:

- — the direction and speed of propagation,
- — the nature of the polarization of this wave.

(b) Determine $\vec{B}$.

Exercice 6 : *(Extrait de Examen final ENPA, 11/05/2017, 08points, durée de l'épreuve 2h)*

Une onde électromagnétique plane progressive sinusoïdale (de pulsation ω) polarisée rectilignement ($\vec{\mathbf{E}}$ parallèle à **Oz**) se propage dans le vide. L'espace est rapporté à un repère orthonormé de vecteurs de base $\vec{\mathbf{e}}_{\mathbf{x}}$, $\vec{\mathbf{e}}_{\mathbf{y}}$ et $\vec{\mathbf{e}}_{\mathbf{z}}$. La direction de propagation de cette onde est dans le plan xOy et fait un α avec Ox : l'angle $(\mathbf{Ox}, \vec{\mathbf{k}}) = \alpha$ $\vec{\mathbf{k}}$ désignant le vecteur d'onde.

1/ Ecrire les composantes du vecteur d'onde $\vec{k}$ puis celles du vecteur champ électrique $\vec{\mathbf{E}}(\vec{\mathbf{r}}, \mathbf{t})$ en un point quelconque de l'espace.

2/ Déduire les composantes du vecteur induction magnétique $\vec{\mathbf{B}}(\vec{\mathbf{r}}, \mathbf{t})$,

3/ Calculer la densité volumique d'énergie électromagnétique (énergie par unité de volume : $\mathbf{w = dW/dV}$) **w** puis déduire sa valeur moyenne dans le temps $\mathbf{<w>}$.

4/ Calculer le vecteur de Poynting $\vec{\mathbf{R}}$ puis son module et la valeur moyenne dans le temps de ce module $< |\vec{\mathbf{R}}| >$. Quelle relation existe-t-il entre les deux valeurs moyennes calculées ?

5/ On suppose que cette onde se propage dans le vide et transporte une puissance de **0,2 W/m²** mesurée à travers une surface disposée perpendiculairement à la direction

de propagation. Quelles sont les valeurs de $< |\vec{R}| >$, $<w>$? Calculer les amplitudes des vecteurs $\mathbf{E_0}$ et $\mathbf{B_0}$ des vecteurs champ électrique et induction magnétique.

On donne pour le vide : $\varepsilon_0 = 8{,}85.10^{-12}$ F/m et $\mu_0 = 1{,}26.10^{-6}$ H/m.

1- Les composantes du $\vec{k}$ et $\vec{E}$?

Onde plane dans le vide $(\vec{E} \,//\, \vec{e_z})$

$$\widehat{(Ox,\vec{k})} = \alpha \implies \boxed{\vec{k}\begin{pmatrix} k\cos\alpha \\ k\sin\alpha \\ 0 \end{pmatrix}}$$

$$E \text{ parallèle à } Oz \implies \boxed{\vec{E}\begin{pmatrix} 0 \\ 0 \\ E_0\cos(\omega t - k\cos(\alpha)x - k\sin(\alpha)y) \end{pmatrix}}$$

Polarisation rectiligne

2- les composantes du vecteur $\vec{B}(r,t)$?

on peut obtenir les composantes du vecteur $\vec{B}(r,t)$ *à partir de :*

1) $\overrightarrow{rot}\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Ou

2) $\vec{B} = \dfrac{\vec{k} \wedge \vec{E}}{w}$ (OPPH)

On obtient

$$\boxed{\vec{B}(r,t)\begin{pmatrix} \dfrac{E_0}{c}\sin(\alpha)\cos(\omega t - \vec{k}.\vec{r}) \\ -\dfrac{E_0}{c}\cos(\alpha)\cos(\omega t - \vec{k}.\vec{r}) \\ 0 \end{pmatrix}}$$

1)

$$\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Longrightarrow \vec{B} = -\int \overrightarrow{rot}\vec{E}\, dt$$

$$\overrightarrow{rot}\vec{E} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix}$$

2)

$$\vec{B} = \frac{\vec{k} \wedge \vec{E}}{w} \Longrightarrow \vec{B} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ k\, cos\alpha & k\, sin\alpha & 0 \\ 0 & 0 & E_z \end{vmatrix} \times \frac{1}{w}$$

3- La densité volumique d'énergie électromagnétique W ?

$$w_{em} = \frac{1}{2}\varepsilon_0\left|\vec{E}\right|^2 + \frac{1}{2}\mu_0\left|\vec{H}\right|^2 \Longrightarrow \boxed{w_{em} = \frac{1}{2}\varepsilon_0\, E^2 + \frac{1}{2\mu_0}B^2}$$

$$\Longrightarrow \boxed{w_{em} = \varepsilon_0 E_0^2 cos^2(\omega t - \vec{k}.\vec{r})}$$

$$< w_{em} >= \frac{1}{T}\int_0^T w_{em}\, dt \Longrightarrow \boxed{< w_{em} >= \frac{1}{2}\varepsilon_0 E_0^2}$$

4- Le vecteur de Poynting $\vec{R}$?

$$\vec{R} = \frac{\vec{E} \wedge \vec{B}}{\mu_0} \Longrightarrow \boxed{\vec{R}\begin{pmatrix} \frac{E_0^2}{\mu_0 c}\cos(\alpha)\cos^2(\omega t - \vec{k}.\vec{r}) \\ \frac{E_0^2}{\mu_0 c}\sin(\alpha)\cos^2(\omega t - \vec{k}.\vec{r}) \\ 0 \end{pmatrix}}$$

$\boxed{\vec{R}}$ *est parallèle à la direction de propagation*

- *La valeur moyenne* $\Longrightarrow < |\vec{R}| > = \frac{1}{2} c\varepsilon_0 E_0^2 \Longrightarrow \boxed{< |\vec{R}| > = c. < w_{em} >}$

5- Application numérique

$$< |\vec{R}| >= 0.2\ W/m^2$$

$$< W_{em} >= \frac{<|\vec{R}|>}{c} = 6{,}67.10^{-10} J/m^3$$

$$E_0 = \sqrt{\frac{2<|w|>}{\varepsilon_0}} = 12{,}28\ V/m$$

$$B_0 = \frac{E_0}{c} = 4{,}09.10^{-8} T$$