Set N°2

Ex1:

1) Potential energy

$U = U_{m_1} + U_{m_2} + U_{k_1} + U_{k_2}$

$= \frac{1}{2} k (x_1 + \Delta l)^2 + k (x_2 + \Delta l_2)^2$

$+ 2 l m_1 g \underset{\theta}{\underset{\wr}{\sin\theta}} - m_2 g y$

$U_{m_1} = -\int_0^\theta W_{m_1} \cdot d\vec{r}_m \quad \vec{W}_{m_1} = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$

$\vec{r}_m \begin{pmatrix} -2l \cos\theta \\ 2l \sin\theta \end{pmatrix} \quad d\vec{r}_m \begin{pmatrix} 2l \sin\theta \\ 2l \cos\theta \end{pmatrix}$

$\int_0^\theta 2 m_1 g \, l \cos\theta d\theta = 2 m_1 g l \sin\theta$

$\simeq 2 m_1 g l \theta$

$U = \frac{1}{2} k (2l\theta + \cancelto{0}{\Delta l_1})^2 + k (l\theta + \Delta l_2)^2$

$+ 2 l m_1 g \theta - m_2 g l \theta$

$U = 2kl^2\theta^2 + kl^2\theta^2 + k\Delta l_2^2 + 2kl\theta\Delta l_2$

$+ 2 l m_1 g \theta - m_2 g l \theta$

$U = 3kl^2\theta^2 + k\Delta l_2^2 + \theta(2kl\Delta l_2 + 2lm_1 g - m_2 g l)$

2) at equilibrium

$\left.\frac{dU}{d\theta}\right|_{\theta_{eq}} = 0$

$6kl^2\theta + 2kl\Delta l_2 + 2lm_1 g - m_2 g l$

$\cancel{\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}}$ at $\theta = 0$:

$2kl\Delta l_2 + 2lm_1 g - m_2 g l = 0$

$\Delta l_2 = \frac{m_2 g l - 2lm_1 g}{2kl} = \frac{g(m_2 - 2m_1)}{2k}$

$\Rightarrow U = 3kl^2\theta^2 + k\left(\frac{g(m_2 - 2m_1)}{2k}\right)^2$

$U = 3kl^2\theta^2 + \frac{g^2(m_2 - 2m_1)^2}{4k}$

3) $\Delta l = 0 \iff m_2 = 2m_1$

4) $m_2 = 2m_1$

$U = 3kl^2\theta^2$

Kinetic energy:

$T = T_R + T_{m_1} + T_{m_2}$

$= \frac{1}{2} I \dot\theta^2 + \frac{1}{2} m_1 \dot x_1^2 + \frac{1}{2} m_2 \dot y^2$

$= \frac{1}{4}(4m_1 l^2)\dot\theta^2 + \frac{1}{2} m_1 (2l\dot\theta)^2 + \frac{m_1}{2}(l\dot\theta)^2$

$= m_1 l^2 \dot\theta^2 + 2 m_1 l^2 \dot\theta^2 + m_1 l^2 \dot\theta^2$

$T = 4 m_1 l^2 \dot\theta^2$

$L = T - U = 4m_1 l^2\dot\theta^2 - 3kl^2\theta^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$8 m_1 l^2 \ddot\theta + 6kl^2\theta = 0$

$\ddot\theta + \frac{3}{4}\frac{k}{m_1}\theta = 0$

$\omega_L = \sqrt{\frac{3}{4}\frac{k}{m_1}}$

$\theta(t) = \theta_{max}\cos(\omega_L t + \varphi)$

Ex 2:

Kinetic energy

$T = T_T + T_R = \frac{1}{2} M (R\dot\theta)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot\theta^2$

$T = \frac{3}{4} M R^2 \dot\theta^2$

potential energy

$U = U_{K_1} + U_{K_2} = \frac{1}{2}\Big[K_1(\Delta l_1 + R\theta + a\theta)^2$

$+ K_2(\Delta l_2 + R\theta + a\theta)^2\Big]$

$$\frac{\partial U}{\partial \theta} = \frac{1}{2}\Big[2k_1(a+R)(\Delta l_1+(R+a)\theta)$$

$$+2k_2(a+R)(\Delta l_2+(R+a)\theta)\Big]$$

$$=(R+a)\Big[k_1(\Delta l_1+(R+a)\theta)+k_2(\Delta l_2+(R+a)\theta)\Big]$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0}=0$$

$$=(R+a)(k_1\Delta l_1+k_2\Delta l_2)=0$$

$$k_1\Delta l_1+k_2\Delta l_2=0$$

$$U=\frac{1}{2}\big[k_1(\Delta l_1^2+(a+R)^2\theta^2+2\Delta l_1(a+R)\theta)$$

$$+k_2(\Delta l_2^2+(a+R)^2\theta^2+2\Delta l_2(a+R)\theta)$$

$$U=\frac{1}{2}k_1\Delta l_1^2+\frac{1}{2}k_1(a+R)^2\theta^2+\frac{1}{2}k_2\Delta l_2^2+\frac{1}{2}k_2(a+R)^2\theta^2$$

$$U=\frac{1}{2}(a+R)^2(k_1+k_2)\theta^2+C$$

$$L=T-U$$

$$L=\frac{3}{4}MR^2\dot{\theta}^2-\frac{1}{2}(a+R)^2(k_1+k_2)\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)-\frac{\partial L}{\partial \theta}=0$$

$$0=\frac{3}{2}MR^2\ddot\theta+(a+R)^2(k_1+k_2)\theta$$

$$\ddot\theta+\frac{2}{3}\frac{(a+R)^2(k_1+k_2)\theta}{MR^2}$$

Ex 3:

(2) $x=\frac{x_1}{2}+\frac{x_2}{4}$

$2T_1=W$

$2k_1x_1=W$

$2T_2=\frac{W}{2}$

$T_2=\frac{W}{4}$

$k_2x_2=\frac{W}{4}$

$2k_1x_1=W$

$4k_2x_2=W$ $\Rightarrow$ (1) $k_1x_1=2k_2x_2$

$$U_{k_1}=\frac{1}{2}k_1(x_1+\Delta l_1)^2$$

$$U_{k_2}=\frac{1}{2}k_2(x_2+\Delta l_2)^2$$

$$T=\frac{1}{2}m\dot{x}^2$$

$$U_T=U_{k_1}+U_{k_2}+U_m$$

from (1) $x_2=\frac{k_1x_1}{2k_2}$

(2) $\to$ $x=\frac{x_1}{2}+\frac{k_1x_1}{8k_2}$

$$x=x_1\left(\frac{1}{2}+\frac{k_1}{8k_2}\right)$$

$$x=x_1\left(\frac{4k_2+k_1}{8k_2}\right)\Rightarrow \boxed{x_1=\left(\frac{8k_2}{4k_2+k_1}\right)x}$$

$$U_{k_1}=\frac{1}{2}k_1\left(\frac{8k_2}{4k_2+k_1}x+\Delta l_1\right)^2$$

from (1) $x_1=\frac{2k_2x_2}{k_1}$

(2) $\to$ $x=\frac{k_2x_2}{k_1}+\frac{x_2}{4}$

$$x=x_2\left(\frac{k_2}{k_1}+\frac{1}{4}\right)=x_2\left(\frac{4k_2+k_1}{4k_1}\right)$$

$$x_2=x\left(\frac{4k_1}{4k_2+k_1}\right)$$

put $k_1=9k$

$$U_{k_2}=\frac{1}{2}k_2\left(x\left(\frac{4k_1}{4k_2+k_1}\right)+\Delta l_2\right)^2$$

$$U_{k_1} = \frac{1}{2} K_1 \left( \frac{4}{3} x + \Delta l_1 \right)^2$$

$$U_{k_2} = \frac{1}{2} K_2 \left( \frac{4}{3} x + \Delta l_2 \right)^2$$

$$U_m = -mgx$$

$$U_T = K_2 \left( \frac{4}{3} x + \Delta l_1 \right)^2 + \frac{1}{2} K_2 \left( \frac{4}{3} x + \Delta l_2 \right)^2 - mgx$$

$$\frac{dU_T}{dx} = 2K_2 \left( \frac{4}{3} \right) \left( \frac{4}{3} x + \Delta l_1 \right) + K_2 \left( \frac{4}{3} \right) \left( \frac{4}{3} x + \Delta l_2 \right) - mg$$

$$\left. \frac{dU_T}{dx} \right|_{x=0} = \frac{8}{3} K_2 \Delta l_1 + \frac{4}{3} K_2 \Delta l_2 - mg = 0$$

$$U_T = K_2 \left[ \frac{16}{9} x^2 + \Delta l_1^2 + \frac{8}{3} x \Delta l_1 \right] + \frac{1}{2} K_2 \left[ \frac{16}{9} x^2 + \Delta l_2^2 + \frac{8}{3} x \Delta l_2 \right] - mgx$$

$$= K_2 \frac{16}{9} x^2 + K_2 \Delta l_1^2 + \frac{16}{18} x^2 K_2 + \Delta l_2^2$$

$$\boxed{U_T = \frac{8}{3} K_2 x^2 + C}$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{8}{3} K_2 x^2$$

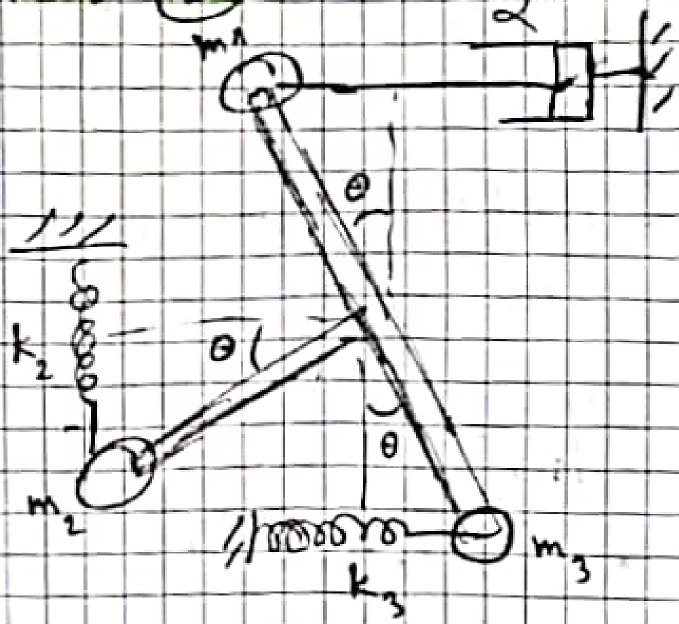
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = m \ddot{x}$$

$$-\frac{\partial L}{\partial x} = \frac{16}{3} K_2 x$$

$$m \ddot{x} + \frac{16}{3} K_2 x = 0 \qquad \omega_n = \sqrt{\frac{16}{3m} K_2}$$

$$\boxed{\ddot{x} + \frac{16}{3m} K_2 x = 0}$$

Ex5 : (D)




$$U = U_{m_1} + U_{m_2} + U_{m_3} + U_{K_2} + U_{K_3}$$

$$U_{m_2} = -mgl\theta$$

$$U_{K_2} = \frac{1}{2} K_2 (l\theta + \Delta l_2)^2$$

$$U_{K_3} = \frac{1}{2} K_3 (l\theta + \Delta l_3)^2$$

$K_2 = K_3 = K$

$$U = \frac{1}{2} K (l\theta + \Delta l_2)^2 + \frac{1}{2} (l\theta + \Delta l_3)^2 - mgl\theta$$

$$\frac{dU}{d\theta} = Kl(l\theta + \Delta l_2) + Kl(l\theta + \Delta l_3) - mgl$$

$$\left. \frac{dU}{d\theta} \right|_{\theta=0} = Kl\Delta l_2 + Kl\Delta l_3 - mgl = 0$$

$$U = \frac{1}{2} K (l^2\theta^2 + 2l\Delta l_2 \theta + \Delta l_2) + \frac{1}{2} K (l^2\theta^2 + 2l\Delta l_3 \theta + \Delta l_3) - mgl\theta$$

$$\boxed{U = Kl^2\theta^2 + C}$$

$$T = \frac{3}{2} m (l\dot{\theta})^2 = \frac{3}{2} m l^2 \dot{\theta}^2$$

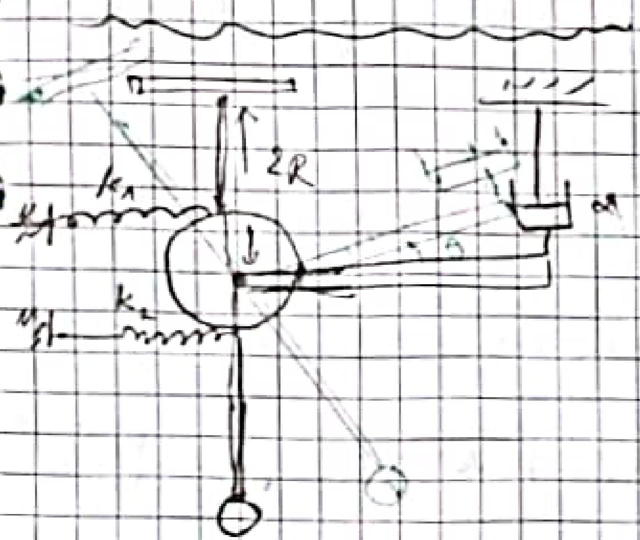
$$D = \frac{1}{2} \alpha (l\dot{\theta})^2$$

$$L = \frac{3}{2} m l^2 \dot{\theta}^2 - Kl^2\theta^2 - C$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot\theta} = 0$$

$$3ml^2\ddot\theta + 2Kl^2\theta + \alpha l^2\dot\theta = 0$$

$$\boxed{\ddot\theta + \underbrace{\frac{\alpha}{3m}}_{2\delta}\dot\theta + \underbrace{\frac{2K}{3m}}_{\omega_n^2}\theta = 0}$$




$$U = U_{m_1} + U_{m_3} + U_{m_4} + U_{K_1} + U_{K_2}$$

$$U_{K_1} = \frac{1}{2}K_1(R\theta + \Delta l_1)^2$$

$$U_{K_2} = \frac{1}{2}K_2(R\theta + \Delta l_2)^2$$

$$\vec{r}_{m_1}\begin{pmatrix}3R\sin\theta\\ -3R\cos\theta\end{pmatrix} \quad \vec{dr}_{m_1}\begin{pmatrix}3R\cos\theta\, d\theta\\ 3R\sin\theta\, d\theta\end{pmatrix}$$

$$\vec{W}_{m_1} = \begin{pmatrix}0\\ -mg\end{pmatrix}$$

$$\vec{W}_{m_1}\cdot\vec{dr}_{m_1} = -mg\,3R\sin\theta\, d\theta$$

$$U_{m_1} = -\int \vec{W}_{m_1}\,\vec{dr}_{m_1} = m_1 g 3R\int_0^\theta \sin\theta\, d\theta$$

$$= -m_1 g 3R\,[\cos\theta]_0^\theta$$

$$= (\cos\theta - 1)$$

$$(1 - \theta^2 - 1)$$

$$U_{m_1} = \frac{3}{2}m_1 gR\theta^2$$

$$U_{m_3} = -m\,gR\theta^2$$

$$U_{m_4} = \frac{1}{2}m_4 g\,R\theta = mg\,R\theta$$

$$U = \frac{1}{2}m g R\theta^2 + mgR\theta + \frac{1}{2}K_1(-R\theta + \Delta l_1)^2$$
$$+ \frac{1}{2}K_2(R\theta + \Delta l_2)^2$$

$$\left.\frac{dU}{d\theta}\right|_{\theta=0} = 0$$

$$\frac{dU}{d\theta} = mgR\theta + mgR - K_1R(R\theta + \Delta l_1)$$
$$+ K_2R(R\theta + \Delta l_2)$$

$$\left.\frac{dU}{d\theta}\right|_{\theta=0} = mgR - K_1R\Delta l_1 + K_2R\Delta l_2 = 0$$

$$U = \frac{1}{2}mgR\theta^2 + K(R\theta)^2 + \underbrace{\frac{1}{2}K(\Delta l_1^2 + \Delta l_2^2)}_{C}$$

$$T = T_{m_1} + T_{m_2} + T_{m_3} + T_{m_4}$$

$$T_{m_1} = \frac{1}{2}m(3R\dot\theta)^2 = \frac{9}{2}mR^2\dot\theta^2$$

$$T_{m_2} = \frac{1}{2}I_{/0}\dot\theta^2 = \frac{1}{4}MR^2\dot\theta^2$$

$$T_{m_3} = T_{Rm_3} + T_{Tm_3} = \frac{1}{2}\left(\frac{1}{12}mR^2\right)\dot\theta^2 + \frac{1}{2}m(2R\dot\theta)^2$$

$$T_{m_3} = \frac{49}{24}mR^2\dot\theta^2$$

$$T_{m_4} = \frac{1}{2}\left(\frac{1}{3}m_4R^2\right)\dot\theta^2 = \frac{1}{3}mR^2\dot\theta^2$$

$$T = \frac{57}{3} m R^2 \dot{\theta}^2$$

$$D = \frac{1}{2} \alpha (R\dot{\theta})^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = 0$$

$$L = \frac{57}{8} m R^2 \dot{\theta}^2 - \left(\frac{1}{2} mg R\theta^2 + k R^2\theta^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{57}{4} m R^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -(mg R + 2kR^2)\theta$$

$$\frac{57}{4} m R^2 \ddot{\theta} + (mg R + 2kR^2)\theta + \alpha R^2 \dot{\theta} = 0$$

$$\ddot{\theta} + \underbrace{\frac{4}{57}\frac{\alpha}{m}}_{2\delta}\dot{\theta} + \frac{4}{57}\underbrace{\left(\frac{g}{R} + \frac{2k}{m}\right)}_{\omega_n^2}\theta = 0$$