$$= \frac{[(n+2)(n+3)\cdots(2n)(2n+1)(2n+2)]\, n^n}{(n+1)(n+2)\cdots(2n)\,(2n+2)^n\,(2n+2)}$$

$$= \frac{2n+1}{n+1} \times \frac{(2n)^n}{(2n+2)^n} = \frac{2n+1}{n+1} \times \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\longrightarrow \frac{2}{e} < 1$$

So $\sum u_n$ CV

Série 4 :

Ex 1 :

$$f_n(x) = \ln\left(x + \frac{1}{n}\right) \qquad D = \mathbb{R}^+$$

If $x \neq 0$: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \ln\left(x+\frac{1}{n}\right)$

$= \ln\left(\lim\limits_{n\to\infty}\left(x+\frac{1}{n}\right)\right) = \ln x$

If $x = 0$: $\lim\limits_{n\to+\infty} f_n(0) = \lim\limits_{n\to\infty} \ln\frac{1}{n} = -\lim \ln n$

$= -\infty$

So $(f_n) \xrightarrow[\,]0,+\infty[\,]{C.S} f$ simply con pointwise

Uniform convergence:

$\|f_n(x) - f(x)\|_\infty = \sup \ln\left(1 + \frac{1}{nx}\right)$

$= +\infty$

because: $\lim\limits_{x\to 0^+} \ln\left(1+\frac{1}{nx}\right) = +\infty$

So (f_n) does not cv uniformly on $]0, +\infty[$

we can take: $x_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} \left(f_n(x_n) - f(x_n)\right) =$

$= \lim\limits_{n\to+\infty} \ln\left(1 + \frac{1}{n}\right)^n$ sequence $\to$ C.N, C.V, C.A, C.S (series)

$= \ln 2 \neq 0$

So (f_n) does not CV uniformly on $]0, +\infty[$

$$f_n(x) = \frac{\ln(1+nx)}{1+nx}, \quad D = \mathbb{R}^+$$

A is not an upper bound $\Rightarrow \exists (x_n) \subset A$, $\lim x_n = +\infty$

A is not a lower bound $\Rightarrow \exists (x_n) \subset A$, $\lim x_n = -\infty$

$\lim\limits_{n\to\infty} f_n(0) = 0$, $\forall x > 0$, $\lim\limits_{n\to+\infty} f_n(x) = 0$

So $(f_n) \xrightarrow[{[0,+\infty[}]{C.S} 0$

$$t \overset{\varphi}{\longmapsto} \frac{\ln t}{t}, \quad t \geq 1$$

$$\varphi'(t) = \frac{\frac{1}{t}t - \ln t}{t^2} = \frac{1-\ln t}{t^2}$$

t	1		e		$+\infty$
$\varphi'(t)$		$+$	0	$-$	
$\varphi(t)$	0	$\nearrow$	$\varphi(e)$	$\searrow$	0
$\varphi(1+nx)$	0	$\nearrow$	$1/e$	$\searrow$	0

$\sup\limits_{x\geq 0} |f_n(x) - f(x)|$

$= \sup\limits_{x\geq 0} \varphi(1+nx) = \varphi(e) = \frac{1}{e} \underset{n\to\infty}{\not\to} 0$

so (f_n) does not uniformly converge on $\mathbb{R}^+$

we can take $x_n = \frac{1}{n}$

$\lim\left[f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right] = \lim \frac{\ln 2}{2} \neq 0$

So (f_n) does not Cv unif on $[0,+\infty[$

$f_n(x) = \dfrac{x}{n(1+x^2)}$ $D = \mathbb{R}$

$= \dfrac{1}{n} \times \varphi(x)$

If φ is bounded on $\mathbb{R}$ then

$|f_n(x)| \leq \dfrac{M}{n}$, where $M = \sup\limits_{x \in \mathbb{R}} |\varphi(x)|$

So, $(f_n) \xrightarrow[\mathbb{R}]{C.V} 0$

that is $(|x|-1)^2 \geq 0$

$\Longleftrightarrow |x|^2 - 2|x| + 1 \geq 0$

$\Longleftrightarrow x^2 + 1 \geq 2|x|$

$\Longleftrightarrow \left|\dfrac{x}{1+x^2}\right| \leq \dfrac{1}{2} = M$

Ex2 :

1) $f_n(x) = n^2 x^n (1-x^n)$

$D = [0,1]$

- $f_n(0) = f_n(1) = 0$, $\forall n \in \mathbb{N}$

- $x \in]0,1[$, $\lim\limits_{n \to +\infty} (1-x^n) = 1$

$\lim\limits_{n \to +\infty} n^2 x = 0$

So, $(f_n) \xrightarrow[[0,1]]{C.S} 0$

2) $\int_0^1 \left(\lim\limits_{n \to +\infty} f_n(x)\right) dx = 0$

$\lim\limits_{n \to +\infty} \int_0^1 f_n(x)\, dx = \lim\limits_{n \to +\infty} \boxed{\int_0^1 n^2(x^n - x^{2n})\, dx}$

$= \lim\limits_{n \to +\infty} n^2 \left[\dfrac{x^{n+1}}{n+1} - \dfrac{x^{2n+1}}{2n+1}\right]_0^1$

$= \lim\limits_{n \to +\infty} n^2 \left(\dfrac{1}{n+1} - \dfrac{1}{2n+1}\right)$

$= \lim\limits_{n \to +\infty} \dfrac{n^3}{(n+1)(2n+1)} = +\infty$

$\neq \int_0^1 \lim\limits_{n \to +\infty} f(x)\, dx$

So f_n does not CV uniformly on $[0,1]$.

3) $\forall r \in]0,1[$; $(f_n) \xrightarrow[[0,r]]{C.V} 0$

because :

$|f_n(x)| = n^2 x^n (1-x^n) \leq n^2 x^n$

$\leq n^2 r^n \xrightarrow[n \to \infty]{} 0$

Ex3 :

1) $|f_n(x)| = \dfrac{1}{n} \left|\dfrac{nx}{1+n^2x^2}\right| \leq \dfrac{1}{2n}$

$\forall x \in [-1,1]$ so $(f_n) \xrightarrow[[-1,1]]{C.V} 0$

2) $f_n'(x) = \dfrac{1 \cdot (1+n^2x^2) - 2n^2 x \cdot x}{(1+n^2x^2)^2}$

$= \dfrac{1-n^2x^2}{(1+n^2x^2)}$

$\lim\limits_{n \to +\infty} f_n'(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0 \end{cases} = g(x)$

So, $(f_n') \xrightarrow[[-1,1]]{C.S} g$

3) since g is not cont then f_n does not converge uniformly on $[-1,1]$

C.N → C.S

C.N → C.A → C.S

Ex4:

$$f_n = \frac{x}{n(1+nx^2)} \qquad \left|\frac{t}{1+t^2}\right| \leq \frac{1}{2}$$

$$f_n(x) = \frac{1}{n\sqrt{n}} \times \frac{\sqrt{n}\,x}{1+(\sqrt{n}\,x)^2} \qquad D = [0,+\infty[$$

$$|f_n(x)| \leq \frac{1}{2n\sqrt{n}}$$

$\sum \frac{1}{2n\sqrt{n}}$ cv $(\alpha = \frac{3}{2})$

then $\sum f_n$ is normally convergent

$$|f_n(x)| = \frac{e^{-nx}}{1+n^2} \leq \frac{1}{1+n^2} \leq \frac{1}{n^2}$$

$\forall x \geq 0$

Since $\sum \frac{1}{n^2}$ cv $(\alpha = 2)$

then $\sum f_n$ is normally convergent

Ex5:

1/ $f_n : x \longrightarrow \frac{x\sin(nx)}{2n^{1/2}+\cos x}$

$D =]0,\pi[$

$F : x \longrightarrow \sum_{n=1}^{+\infty} f_n(x)$

F is defined on the convergence domain of the series $\sum_{n\in\mathbb{N}^*} f_n$

to prove that F is cont we can prove that $\sum f_n$ c.v. on $]0,\pi[$; because, f_n is continuous on $]0,\pi[$, for any $n \in \mathbb{N}^*$

So, F is cont on $]0,\pi[$ because

$$|f_n(x)| = \left|\frac{x\sin nx}{2n\sqrt{n}+\cos x}\right| \leq \frac{|x|}{2n\sqrt{n}+\cos x} \leq \frac{\pi}{2n\sqrt{n}+\cos x}$$

$$\leq \frac{\pi}{2n\sqrt{n}-1} = U_n$$

Since $\sum U_n$ cv because $U_n \sim \frac{\pi}{2}\cdot\frac{1}{n^{3/2}}$, then $\sum f_n$ cv on $]0,\pi[$, so it is uniformly convergent

2/ $$f_n'(x) = \frac{(x\sin(nx))'(2n\sqrt{n}+\cos x) + \sin x\, f_n}{(2n\sqrt{n}+\cos x)^2}$$

$$= \frac{(xn\cos nx)(2n\sqrt{n}+\cos x) + x\sin x\sin nx}{(2n\sqrt{n}+\cos x)^2}$$

$$= \underbrace{\frac{2x\,n^2\sqrt{n}\cos nx}{(2n\sqrt{n}+\cos x)^2}}_{\text{doesn't cv unifo}} + \underbrace{g_n(x)}_{\sum g_n \text{ cv}}$$

when f_n' doesn't cv uniformly

$$F(x) = x\,G(x)$$

where $G(x) = \sum_{n=1}^{+\infty} \frac{\sin nx}{2n^{3/2}+\cos x}$

$$\sum_{n=1}^{+\infty} g_n(x)$$

$$g_n'(x) = \frac{n\cos nx\,(2n\sqrt{n}+\cos x) + \sin x\sin nx}{(2n\sqrt{n}+\cos x)^2}$$

$$= \frac{2n^2\sqrt{n}\cos nx}{(2n\sqrt{n}+\cos x)^2} + \frac{n\cos nx\cos x + \sin x\sin nx}{(2n\sqrt{n}+\cos x)^2}$$

$$\leq \frac{n+2}{(2n\sqrt{n}-1)^2} \sim \frac{n}{\ldots} = \frac{1}{\ldots} \quad \text{cv}$$

$$x_n = \frac{1}{n} \to \frac{2n^2\sqrt{n}\cos 1}{(2n\sqrt{n}+\cos\frac{1}{n})^2} \sim \frac{2\cos 1}{8}\,\frac{n^2\sqrt{n}}{n^3} \ldots$$

Since $\sum \frac{1}{\sqrt{n}}$ dv then

$\sum g_n'$ doesn't cv uniformly So

G is not derivable

F .. .. ~

Ex6 :

$f_n : x \longrightarrow e^{-nx} \sin(2nx)$.

$D = [0, +\infty[$

1/ $f_n(0) = 0 \Longrightarrow \sum f_n(0)$ c.v

$\forall x > 0$, $|f_n(x)| \leqslant e^{-nx}$

since the geometric series $\sum (e^{-x})^n$ cv

then $\sum f_n$ cv on $]0, +\infty[$

conclusion : $\sum f_n$ is pointwise convergent on $[0, +\infty[$

2/ Let $a > 0$: $[a, +\infty[$

$|f_n(x)| \leqslant e^{-nx} \leqslant e^{-na}$

Since $\sum e^{-na}$ cv

then $\sum f_n$ C.N on $[a, +\infty[$

So, it converges uniformly on $[a, +\infty[$

3/ on $[0, a]$ $x_n = \frac{1}{n}$

$\sum f_n\left(\frac{1}{n}\right) = \sum e^{-1} \sin 2$

doesn't cv

So $\sum f_n$ doesn't cv uniformly on $[0, a]$