Chapter 3.

Geometry of mass

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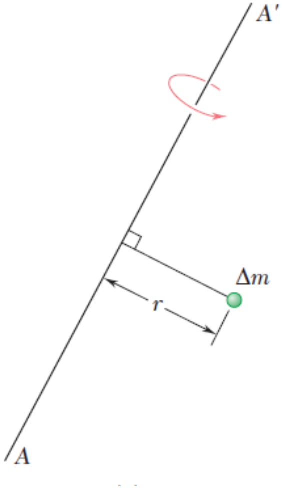
Moment of Inertia of a Simple Mass

Consider a small mass Δm mounted on a rod of negligible mass that can rotate freely about an axis AA'. If we apply a couple to the system, the rod and mass (assumed initially at rest) rotate about AA'.

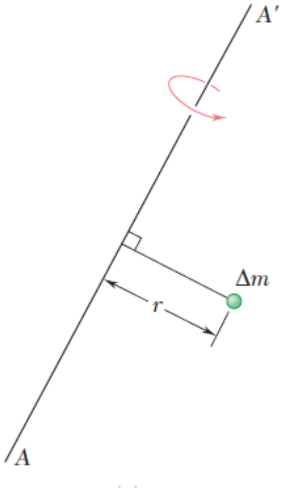


Moment of Inertia of a Simple Mass

The time required for the system to reach a given rotation speed is proportional to the mass Δm and the square of the distance r.

The product $r^2 \Delta m$ is called the moment of inertia of the mass Δm to axis AA'.

It provides a measure of the inertia of the system (i.e. a measure of the resistance the system offers when we try to set it in motion).



Moment of Inertia of a Simple Mass

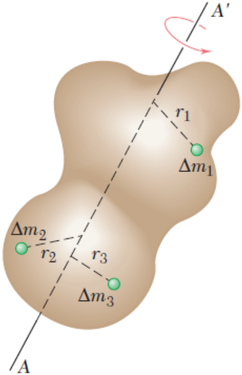
Let's consider a body of mass m rotating about an axis AA'.
The body is composed of elements of masses Δm_i.
The body's resistance to being

rotated is measured by the sum (the moment of inertia of the body to axis AA'):

$$\sum r_i^2.\Delta m_i$$

Increasing the number of elements, we find that the moment of inertia is equal, in the limit, to the integral:

$$I = \int r^2.dm$$

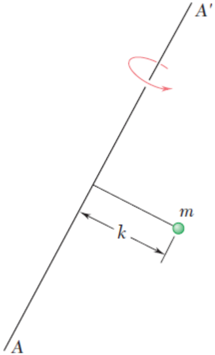


Radius of gyration of a mass

We define the radius of gyration k of the body with respect to axis AA' by the relation:

$$I = k^2.m \text{ or } k = \sqrt{\frac{I}{m}}$$

The radius of gyration k represents the distance at which the entire mass of the body should be concentrated if its moment of inertia to AA' is to remain unchanged.



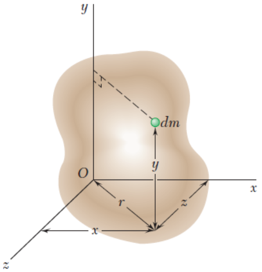
In SI units: the radius of gyration k is expressed in meters and the mass m in kilograms, so the unit for the moment of inertia of a mass is $kg.m^2$.

Moments of inertia to coordinate axes

We can express the moment of inertia of a body to a coordinate axis in terms of the coordinates x, y, and z of the element of mass dm.

For example, the moment of inertia of the body with respect to the y-axis is:

$$I_y = \int r^2 dm = \int (z^2 + x^2) dm$$



We obtain similar expressions for the moments of inertia to the x and z axes.

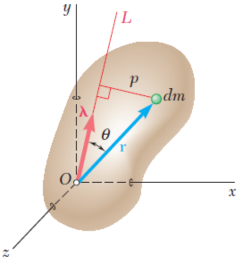
$$I_z = \int (y^2 + x^2) dm \text{ and } I_x = \int (y^2 + z^2) dm$$

Mass Products of Inertia

The moment of inertia I_{OL} of the body to OL is equal to $\int p^2 dm$, where p denotes the perpendicular distance from the element of mass dm to the axis OL.

Let's consider:
$\vec{\lambda}$ is the unit vector along OL.
$\vec{r}$ is the position vector of the element dm
Then, the perpendicular distance p is equal to:

$$p = r.sin(\theta) = \|\vec{\lambda} \wedge \vec{r}\|$$



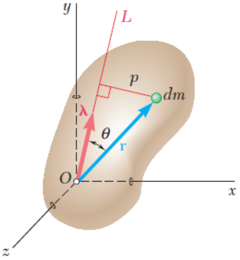
Mass Products of Inertia

We therefore have: $I_{OL} = \int p^2 dm = \int \|\vec{\lambda} \wedge \vec{r}\|^2 dm$

If we take:

$$\vec{\lambda}(\lambda_x, \lambda_y, \lambda_z)$$

$$\vec{r}(x, y, z)$$



we find:

$$I_{OL} = \lambda_x^2 \int (y^2 + z^2) dm + \lambda_y^2 \int (z^2 + x^2) dm + \lambda_z^2 \int (x^2 + y^2) dm$$

$$-2\lambda_x\lambda_y \int xydm - 2\lambda_y\lambda_z \int yzdm - 2\lambda_z\lambda_x \int zxdm$$

Mass Products of Inertia

The last three integrals which involve products of coordinates, are called the products of inertia of the body:

$$I_{xy} = \int xydm\,;\; I_{yz} = \int yzdm\,;\; I_{zx} = \int zxdm$$

Then:

$$I_{OL} = \lambda_x^2 I_x + \lambda_y^2 I_y + \lambda_z^2 I_z - 2\lambda_x\lambda_y I_{xy} - 2\lambda_y\lambda_z I_{yz} - 2\lambda_z\lambda_x I_{zx}$$

Or in matrix form:

$$I_{OL} = \begin{Bmatrix} \lambda_x & \lambda_y & \lambda_z \end{Bmatrix} \times \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \times \begin{Bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{Bmatrix}$$

$$I_{OL} = \vec{\lambda}^T \times I_{Oxyz} \times \vec{\lambda}$$

Product of inertia with respect to two perpendicular lines

Let Δ and Δ' be two straight lines with directions $\vec{n}$ and $\vec{t}$ (unit vectors).

$$I_{nt} = \int x_n x_t dm$$

x_n and x_t are the coordinates of P in the reference frame $(O, \vec{n}, \vec{t})$.

$$I_{nt} = -\vec{n}^T . I_{(Oxyz)} . \vec{t}$$

Change of reference frame in rotation

$I_{(Oxyz)}$is the inertia tensor with respect to a reference frame $R(Oxyz)$.

We can calculate the tensor in another frame $R'(Ox'y'z')$ defined by the rotation of R, using the following formulas:

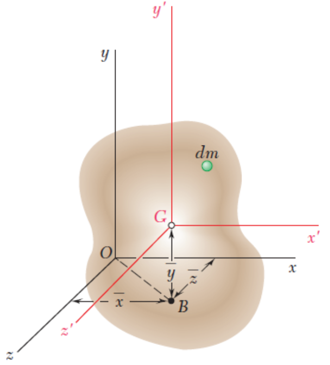
$$I_{(Ox'y'z')} = P_{(R'\rightarrow R)} I_{(Oxyz)} P_{(R\rightarrow R')}$$

Parallel-Axis Theorem for Mass Moments of Inertia

Consider a body of mass m and let $(Oxyz)$ be a system of rectangular coordinates whose origin is at the arbitrary point O. Let $(Gx'y'z')$ be a system of parallel axes (the origin is at the centre of gravity G).

We denote the coordinates :
$G(\bar{x}, \bar{y}, \bar{z})$ in $(Oxyz)$
$dm(x, y, z)$ in $(Oxyz)$
$dm(x', y', z')$ in $(Gx'y'z')$
Then :

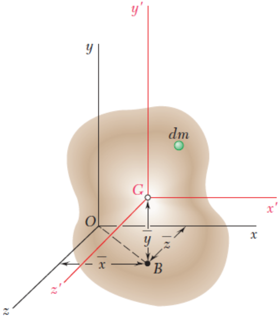
$$x = x' + \bar{x}; y = y' + \bar{y}; z = z' + \bar{z}$$



Parallel-Axis Theorem for Mass Moments of Inertia

We can express the moment of inertia of the body to the x-axis $I_x = \int (y^2 + z^2) dm$ as:

$$I_x = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2] dm$$



$$I_x = \int (y'^2 + z'^2) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + (\bar{y}^2 + \bar{z}^2) \int dm$$

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)$$

Parallel-Axis Theorem for Mass Moments of Inertia

Similarly,

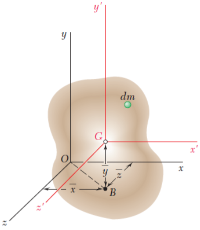
$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)$$

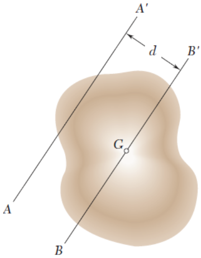
$$I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2)$$

$$I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2)$$

Parallel-axis theorem for mass moments of inertia

$$I = \bar{I} + md^2$$





Particular cases

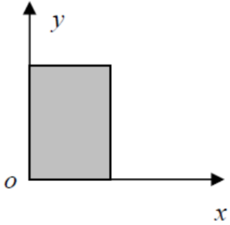
Case 1. Plane solids:
In the case of plane solids, one of the element's coordinates is zero. If the solid is in the plane (Oxy) then $z = 0$.
We immediately deduce that:

$$I_x = \int y^2 dm; \; I_y = \int x^2 dm$$

Hence:

$$I_z = \int (x^2 + y^2) dm = I_x + I_y$$

and $I_{xz} = I_{yz} = 0; I_{xy} = \int xydm$

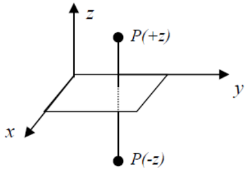


Particular cases

Case 2: plane of symmetry

If *oxy* is a plane of symmetry, then *oz* is a principal axis of inertia

$$\int xzdm = \int yzdm = 0 \rightarrow I_{xz} = I_{yz} = 0$$



$$I_{Oxyz}(S) = \begin{bmatrix} I_x & -I_{xy} & 0 \\ -I_{yx} & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

If *oyz* is a plane of symmetry then $I_{xy} = I_{xz} = 0$ (ox is a principal axis of inertia)

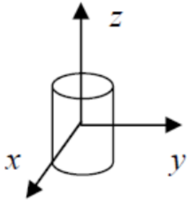
If *oxz* is a plane of symmetry then $I_{yz} = I_{xy} = 0$(oy is a principal axis of inertia)

Case 3: Axis of symmetry (revolution around an axis)
If the system is a body of revolution around *oz*, then any plane passing through *oz* is a plane of symmetry. Therefore:

$$I_{xy} = I_{xz} = I_{yz} = 0$$

and

$$I_x = I_y = \frac{I_z}{2} + \int z^2 dm$$



Particular cases

Case 4: Point of symmetry (spherical revolution)

If point O is a point of revolution then:

$$I_{xy} = I_{xz} = I_{yz} = 0$$

and

$$I_x = I_y = I_z = \frac{2}{3}\int (x^2 + y^2 + z^2) dm$$