Example: $F: (x,y,z) \longmapsto x^3y^2z\,\vec{i} + 2x^2\,\vec{j} + yx^2\,\vec{k}$

$div(F): (x,y,z) \longmapsto (3x^2y^2z + 0) = 3zx^2y^2$.

Theorem: If F is a class $\mathcal{C}^1$ vector field in ~~the~~ space then $div(curl(F)) = 0$.

Def: Laplacian operator:

Let $f: \mathcal{D}\,\Omega \longrightarrow \mathbb{R}$, the Laplacian of f is ($\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$)

$$\Delta f = div(\nabla f)$$

If $\Omega \subset \mathbb{R}^2$, $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$; if $\Omega \subset \mathbb{R}^3$, $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

The scalar field f is called harmonic, if $\Delta f = 0$

Example: $f(x,y,z) = x^2 + y^2 + z^2$, $\Delta f(x,y,z) = 6$.

II-2- Line integrals

II-2-1- Line integral of a scalar field

Def: ~~Set~~ C : ~~[a,b]~~ $\subseteq \mathbb{R}^2$ (resp. $\mathbb{R}^3$) a smooth bounded curve parameterized by $t \overset{\gamma}{\longmapsto} (x(t), y(t))$ (resp. $(x(t), y(t), z(t))$), for $t \in [a,b]$.

① the line integral of a function $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$ (resp. $D \subset \mathbb{R}^3$) denoted by $\int_C f\,dl$ is defined by:

$$\int_C f\,dl := \int_a^b f(x(t), y(t)) \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\,dt}_{dl},$$

$$\left(\text{resp } \int_a^b f(x(t), y(t), z(t)) \underbrace{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}_{dl}\right).$$

this value is independent of the choice of parameterization

② the length of C, denoted by $\ell(C)$, is

$$\ell(C) = \int_C dl = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt \quad \left(\text{resp. } \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt\right)$$

Examples: ① $C = \{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = R^2\}$; $\ell(C) = 2\pi R$

② $\frac{\sqrt{6}}{6} = \int_C (x^2 - y + 3z)\,dl$, where C is the line segment $(0,2,0) \to (1,2,1)$

Remark: If $C = C_1 \cup C_2 \cup \dots \cup C_k$ / where $Card(C_i \cap C_j) \leq 1$, $\forall i \neq j$

$$\int_C f\,dl = \sum_{l=1}^{k} \int_{C_l} f\,dl \qquad f \text{ continuous on } C.$$