$(79)_{10} = (1001111)_2$

convert from base 10 . 2

$(7/8)_{10} = (0,111)_2$

$(35/16)_{10} = (10.0011)_2$

$(5/7)_{10} = (0,101101 \ldots)_2$

$(0.129)_{10} = (0,\overline{0621505.0220563743463})_7$

$(0,\overline{1011})_2 = \frac{11}{15}$

Ex 1:

① $a = 11011$ $b = 10101$

$$\begin{array}{r} 11011 \\ 10101 \\ \hline 10000 \end{array}$$

② $a = 1,0011 \times 2^3$ $b = 11,111 \times 2^2$

$(a)_8 = (1257)_8$

Ex 2: $(b)_8 = (1346)_8$

I) find the opposite of x, 4 digits in each case

decimal	Binary
7	0111
6	0110
5	0101
4	0100
3	0011
2	0010
1	0001
0	0000
-1	1111
-2	1110
-3	1101
-4	1100

hum Ang

$a + x = 0$

$-1 = ?$

$$\begin{array}{r} 0001 \\ + \; d_1 d_2 d_3 d_4 \\ \hline 0000 \end{array}$$

$d_4 = 1$

$d_3 = 1$

$d_2 = 1$

$d_1 = 1$

$-1 = 1111$

$-2 = ?$

$$\begin{array}{r} 0010 \\ + \; d_1 d_2 d_3 d_4 \\ \hline 0000 \end{array}$$

$d_4 = 0$

$d_3 = 1$

$d_2 = 1$

$d_1 = 1$

II) Evaluate $a - b$:

$110111 - 011011 =$

$110111 + (-011011) =$

$-(011011)_2 = (100100)_2$

$$\begin{array}{r} 100100 \\ + \; 000001 \\ \hline 100101 \end{array}$$

The sum:

$$\begin{array}{r} 110111 \\ + \; 100101 \\ \hline 1011100 \end{array}$$

step 1 = invert

" 2 = +1

-n

Ex 3:

Evaluate the multiplication of a and b in the following cases:

① $a = 11011$, $b = 10101$

② $a = 10111$, $b = 11101$

① $\times\ 11011$
$\quad 10101$

$(1+2^1+0\times2^2+1\times2^3+2^4)\cdot(10101)$

$$\begin{array}{r} 10101 \\ +\ 101010 \\ 000000 \\ 10101000 \\ 101010000 \\ \hline \end{array}$$

② 10111
$\times\ 11101$

$(1+0\times2^1+1\times2^2+1\times2^3+1\times2^4)\cdot(11101)$

$$\begin{array}{r} 11101 \\ +\ 000000 \\ 1110100 \\ 11101000 \\ 111010000 \\ \hline 1 \end{array}$$

$x = 0.\overline{1304}$

$= \frac{0}{5} + \frac{1}{5^2} + \frac{3}{5^3} + \frac{0}{5^4} + \frac{4}{5^5}$

$+ \frac{0}{5^6} + \frac{1}{5^7} + \frac{3}{5^8} + \frac{0}{5^9} + \frac{4}{5^{10}}$

$+ \frac{0}{5^{11}} + \frac{1}{5^{12}} + \frac{3}{5^{13}} + \frac{0}{5^{14}} + \frac{4}{5^{15}} + \dots$

$= \left(\frac{1}{5^2} + \frac{1}{5^7} + \frac{1}{5^{12}} + \dots\right) + \left(\frac{3}{5^3} + \frac{3}{5^8} + \frac{3}{5^{13}} + \dots\right)$

$+ \left(\frac{4}{5^5} + \frac{4}{5^{10}} + \frac{4}{5^{15}} + \dots\right)$

$= \frac{1}{5^2}\left(1 + \frac{1}{5^5} + \frac{1}{5^{10}} + \dots\right)$

$+ \frac{3}{5^3}\left(1 + \frac{1}{5^5} + \frac{1}{5^{10}} + \dots\right)$

$+ \frac{4}{5^5}\left(1 + \frac{1}{5^5} + \frac{1}{5^{10}} + \dots\right)$

$= \left(\frac{1}{5^2} + \frac{3}{5^3} + \frac{4}{5^5}\right)\left[\sum_{k=0}^{\infty} \left(\frac{1}{5^5}\right)^k\right]$

$\left(\frac{125+75+20}{5^5}\right)\left(\frac{1}{1-\frac{1}{5^5}}\right)$

$= \frac{220\times5}{3124} =$

Ex 3: $(27)_{10} = (1000)_B$

$B^0\times0 + B^1\times0 + B^2\times0 + B^3\times1 = 27$

$B^3 = 27 \Rightarrow \boxed{B=3}$

$(545)_{10} = (1406)_B$

$6B^0 + 0B^1 + 4B^2 + 1B^3 = 545$

$6 + 4B^2 + B^3 = 545$

$B^2(4+B) = 539$

if $B = 7 \Rightarrow B^2 = 49$

$\Rightarrow \frac{539}{49} = 11 = 4+7$ ✓

$(5125)_{10} = (12005)_B$

$(12005)_B = 5B^0 + 2B^3 + 1B^4$

$5125 = 5 + B^4 + 2B^3$

$B^4 + 2B^3 = 5120$

$B^3(B+2) = 5120$

if $B = 2^3 = 8 \Rightarrow 512 = B^3$

$8 + 2 = 10$

$B = \sqrt[3]{512} = 8$ ✓

$[0,1]$

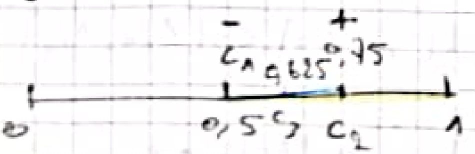
Ex1. $f(x) = \sqrt{x} - \cos x$

1) $C_3 \Leftrightarrow n = 3$

$f(0)\, f(1) < 0 \,?\ \begin{cases} f(0) < 0 \\ f(1) > 0 \end{cases}$

$c_1 = \frac{a+b}{2} = \frac{0+1}{2} = 0,5$

$f(c_1) =$



$c_3 = 0,625$

$[-2; 1,5]$

2) $f(x) = 3(x^2 - 1)\left(x - \frac{1}{2}\right)$

$f(-2) < 0 \quad / \ f(1,5) > 0$

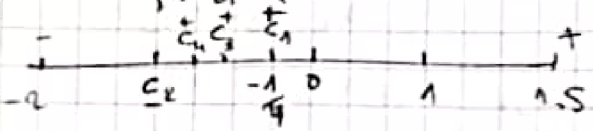
$f(-2)\, f(1,5) < 0$

$c_1 = \frac{-2+1,5}{2} = \frac{-0,5}{2} = -\frac{1}{4}$

$f(c_1) > 0$

$c_2 = \frac{-2 - 0,25}{2} = -1,125$

$f(c_2) < 0$



$c_3 = \frac{-1,25}{2} = -0,6875$

$f(c_3) < 0$

$c_4 = \frac{-0,6875 - 1,125}{2}$

$c_4 = -0,90625$

$f(c_4) > 0$

$c_5 = \frac{-0,90625 - 1,125}{2}$

$c_5 = -1,015625$

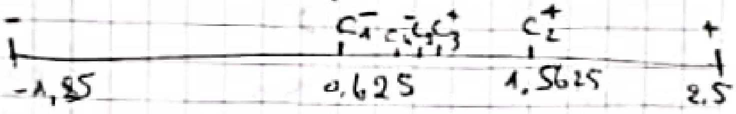
$f(c_5) < 0$

$[-1,25 : 2,5]$

3) $f(x) = x^2 |\sin x| - 4,1$

$f(-1,25) < 0 \qquad f(2,5) > 0$

< 0



$c_1 = 0,625 \ / \ f(c_1) < 0$

$c_2 = 1,5625 \ / \ f(c_2) > 0$

$c_3 = 1,09375 \ / \ f(c_3) > 0$

$c_4 = 0,859375 \ / \ f(c_4) < 0$

$c_5 = 0,9765625 \ / \ f(c_5) <$

Ex2.

$\frac{|b-a|}{2^n} \leq \varepsilon = 10^{-2}$

$\ln|b-a| - n \ln 2 \leq \ln \varepsilon$

$\ln\left(\frac{|b-a|}{\varepsilon}\right) \leq n \ln 2$

$\frac{\ln\left(\frac{|b-a|}{\varepsilon}\right)}{\ln 2} \leq n$

first, we should determine the number of iteration n

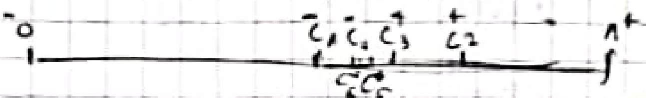
Case 1: $x^3 - 7x^2 + 14x - 6 \quad [0,1]$

$n \geq -\ln\varepsilon / \ln 2 = 2\ln 10 / \ln 2 = 6,6$

$\boxed{n = 7}$

$f(0) = -6 < 0 \qquad f(1) = 2 > 0$

$c_1 = 0,5 \quad / \ f(c_1) < 0$



$c_2 = 0,75 \ / \ f(c_2) > 0$

$c_3 = 0,625 \ / \ f(c_3) > 0$

$c_4 = 0,5625 \ / \ f(c_4) < 0$

$c_5 = 0,59375 \ / \ f(c_5) > 0$

$c_6 = 0,578125 \ / \ f(c_6) < 0$

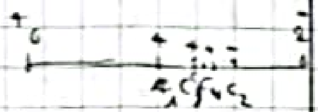
$c_7 = 0,5859375$

$f(c_7) \ 0$

$x^4 - 2x^3 - 4x^2 + 4x + 4 \quad [0,2]$

$n = 2$

$f(0) \cdot f(2) < 0$



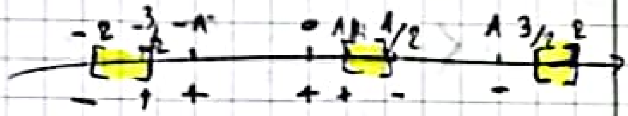
$c_1 = 1$	$f(c_1) \geq 0$
$c_2 = 1.5$	$f(c_2) < 0$
$c_3 = 1.25$	$f(c_3) > 0$
$c_4 = 1,375$	$f(c_4) > 0$
$c_5 = \frac{23}{16} = 1,4375$	$f(c_5) < 0$
$c_6 = \frac{45}{32} = 1,40625$	$f(c_6) > 0$
$c_7 = \frac{91}{64} = 1,421875$	$f(c_7) < 0$
$c_8 = \frac{181}{128} = 1,4140625$	

Ex 3 :

$x^3 - 3x + 1 = 0$

$3x^2 - 3 = 0 \Rightarrow 3(x-1)(x+1) = 0$

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	3	$\searrow$	-1	$\nearrow$	$+\infty$



$f(x) = 0 \iff x = g(x)$

$x = \frac{x^3+1}{3} = g_1(x)$

$x = \sqrt[3]{3x-1} = g_2(x)$

$x^2(x-3) = -1 \Rightarrow x = \frac{-1}{x^2-3} = g_3(x)$

$x^2 - 3 = -\frac{1}{x} \Rightarrow x = \pm\sqrt{-\frac{1}{x} + 3} = g_4(x)$

$g : [a,b] \longrightarrow \mathbb{R}$

$g([a,b]) \subset [a,b] \Rightarrow \exists$ fixed point

$|g'(x)| < 1 \Rightarrow$ unique

$[\frac{1}{4}, \frac{1}{2}]$: $g_1(1/2) = 0,375$

$g_1(1/4) = 0,338$

$\Rightarrow g_1([1/4, 1/2]) \subset [1/4, 1/2]$

$[1/4, \frac{1}{2}]$: $g_2(1/2) = 0,79$

$g_2(1/4) = -0,63$ $\not\subset [1/4, 1/2]$

$|g_1'(x)| < 1 \qquad x_{n+1} = g_1(x_n)$

n	x_n
0	$x_0 = 0,35$
1	$x_1 = 0,34763$
2	$x_2 = 0,34734$
3	$x_3 = 0,34730$
4	$x_4 = 0,3473$

$[\frac{1}{4}, \frac{1}{2}]$: $g_3(\frac{1}{2}) = 0,36$

$g_3(1/4) = 0,34$ $\subset [1/4, 1/2]$

n	x_n
0	$x_0 = 0,35$
1	$x_1 = 0,34752$
2	$x_2 = 0,34732$
3	$x_3 = 0,3473$

g_4 ✓ $[\frac{3}{2}, 2]$

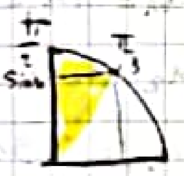
g_2 on $[-2, \frac{-3}{2}]$

$g_2(-2) = -1,91$ / $g_2(\frac{-3}{2}) = -1,765$

n	x_n
0	$x_0 = -1,31$
1	$x_1 = -1,885$
2	$x_2 = -1,881$
3	$x_3 = -1,8798$
4	$x_4 = -1,8795$
5	$x_5 = -1,8794$
6	$x_6 = -1,87938$
7	$x_7 = -1,87939$

Ex 4. $g(x) = \sin x + \frac{1}{4}$

$g : [\frac{\pi}{3}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$



$\frac{\sqrt{3}}{2} \leqslant \sin x \leqslant 1$

$\frac{\sqrt{3}}{2} + \frac{1}{4} \leqslant \sin x + \frac{1}{4} \leqslant \frac{5}{4}$

$\underset{\frac{\pi}{3}}{\overset{\parallel}{x_0}} < 1,11 = \frac{2\sqrt{3}+1}{4} \leqslant g(x) \leqslant \frac{5}{4} = 1,25 < \frac{\pi}{2} = 1,5$

$g(x) \in [\frac{\pi}{3}, \frac{\pi}{2}]$

$|g'(x)| = |\cos x| \leqslant \frac{1}{2} < 1$

$x_{n+1} = g(x_n)$, $x_0 = \frac{3\pi}{8} \in [\frac{\pi}{3}, \frac{\pi}{2}]$

$\Rightarrow (x_n) \longrightarrow p = g(p)$

n	x_n
0	$x_0 = 1,1781$
1	$x_1 = 1,17388$
2	$x_2 = 1,17226$
3	$x_3 = 1,17163$
4	$x_4 = 1,171385$
5	$x_5 = 1,171290$
6	$x_6 = 1,171253$
7	$x_7 = 1,171239$
8	$x_8 = 1,171233$
9	$x_9 = 1,171231$
10	$x_{10} = 1,171230$

Ex 5. $e^x + 10x - 2 = 0$

~~by~~ bisection $\begin{cases} f(0) = -1 < 0 \\ f(1) = e + 8 > 0 \end{cases}$

$x_0 = 0$, $x_1 = 1$

bisection	$f(x_i)$	$x_{n+1} = \frac{1}{2}[2 - e^{x_n}]$
$x_1 = 0,5$	+	1/2
$x_2 = \frac{1}{4}$	+	0,176
$x_3 = \frac{1}{8}$	+	0,403
$x_4 = \frac{1}{16}$	−	0,251
$x_5 = \frac{3}{32}$	+	~~0,485~~ 0,357
$x_6 = \frac{5}{64}$	−	~~0,486~~ 0,285

Ex 6.

$f(x) = x^4 + 2x^2 - x - 3$

$x^4 + 2x^2 - x - 3 = 0$

$x = (3 + x - 2x^2)^{1/4} = g_1(x)$

$g_2(x) = x \iff \left(\frac{x + 3 - x^4}{2}\right)^{1/2} =$

$\iff x^2 = \frac{x + 3 - x^4}{2}$

$2x^2 = x + 3 - x^4$

$\iff x^4 + 2x^2 - x - 3 = 0 \iff f(x)$

$g_3(x) = x$

$$\left(\frac{x+3}{x^2+2}\right)^{1/2} = x \Leftrightarrow x^2 = \frac{x+3}{x^2+2}$$

$$\Leftrightarrow x^4 + 2x^2 = x + 3$$

$$\Leftrightarrow x^4 + 2x^2 - x - 3 = 0$$

$$\Leftrightarrow f(x) = 0$$

$$g_4(x) = x \Leftrightarrow x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$$

$$4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3$$

$$x^4 - 2x^2 - x - 3 = 0$$

$$f(x) = 0$$

n \ x_{n+1}	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$
0	$P_0 = 1$	$P_0 = 1$	$P_0 = 1$	$P_0 = 1$
1	$P_1 = 1,50$	$P_1 = 1,22$	$P_1 = 1,15$	$P_1 = 1,12$
2	$P_2 = 1,03$	$P_2 = 1,004$	$P_2 = 1,12$	$P_2 = 1,12$
3	$P_3 = 1,15$	$P_3 = 1,22$	$P_3 = 1,105$	$P_3 = 1,12$
4	$P_4 = 1,11$	$P_4 = 1,004$	$P_4 = 1,124$	$P_4 = 1,12$
5	$P_5 = 1,13$			

Ex 07 :

$$x = \sqrt[k]{N} = N^{1/k} \quad (N \in \mathbb{R}^+)$$

$$x^k = N \Rightarrow x^k - N = 0 = f(x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{k x_n^{k-1}}$$

$$\text{thus : } x_{n+1} = \frac{k x_n^k - x_n^k + N}{k x_n^{k-1}}$$

$$\boxed{x_{n+1} = \frac{(k-1)x_n^k + N}{k x_n^{k-1}}} \quad \sqrt{2} = 1,4142$$

I) $k = 2$, $N = 2 \Rightarrow$

$$x_{n+1} = \frac{x_n^2 + 2}{2 x_n} \qquad x_0 = 1$$

$x_1 = 3/2$

$x_2 = 17/12$

$x_3 = \dots 577/408 = 1,4142$

II) $k = 3$ $N = 7$

$$x_{n+1} = \frac{2x_n^3 + 7}{3 x_n^2} \qquad P_0 = 2$$

$$P_1 = \frac{23}{12}$$

$$\sqrt[3]{7} = 1,91129$$

$P_2 = 1,9129$

$x_1 = 3$ $P_0 = 1$

$x_2 = 2,26$ $x_3 = 1,963$

$x_4 = 1,914$ $x_5 = 1,91129$

Ex 08 :

$$x = \frac{1}{R} \Rightarrow x - \frac{1}{R} = 0 = f(x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - 1/R}{1} = \frac{1}{R}$$

$$x = \frac{1}{R} \Rightarrow \frac{1}{x} = R \Rightarrow \frac{1}{x} - R = 0 = f(x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1/x_n - R}{-\frac{1}{x_n^2}}$$

$$= x_n + \left(\frac{1 - R x_n}{x_n}\right)(x_n^2)$$

$$x_{n+1} = x_n(2 - R x_n$$

$r=3 \quad x_0=0{,}5$

$x_1 = 0{,}5\,(2-1{,}5) = 1/4$

$x_2 = 5/16 = 0{,}3125$

$x_3 = \frac{85}{256} = 0{,}3320$

$x_4 = 0{,}3333..$

$r = \frac{1}{3}, \quad x_0 = 2, \quad x_1 = \frac{8}{3}$

$x_2 = 2{,}96, \quad x_3 = 2{,}999$

Ex 9: $a = t - \varepsilon \sin(t)$

$0<\varepsilon<1, \quad a \in [0,\pi]$

$f(x)=0 \iff x = g(x)$

① $|g'(x)|<1$

② $|g(r)-g(s)| \leq k|r-s|$ $\quad k<1$

$a + \varepsilon \sin t = t \iff g(t)$

① $g'(t) = \varepsilon \cos(t)$

$|\cos t| < 1$

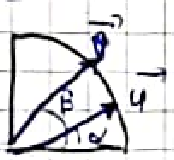
$\varepsilon|\cos t| < \varepsilon < 1$

② is g a contraction

$|g(r)-g(s)| = |a+\varepsilon\sin(r) - a - \varepsilon\sin(s)|$

$= \varepsilon|\sin(r)-\sin(s)|$ — (F)

$\cos(\alpha-\beta)$



$\vec{u}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \quad \vec{v}\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$

$\vec{u}\cdot\vec{v} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$-\beta = \gamma \Rightarrow \beta = -\gamma$

$\cos(\alpha+\beta) = \cos\alpha\cos(-\gamma) + \sin\alpha\sin(-\gamma)$

$= \cos\alpha\cos\gamma - \sin\alpha\sin\gamma$

$\sin\alpha = \cos\left(\frac{\pi}{2}-\alpha\right)$

$\cos\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$

$\gamma = \frac{\pi}{2} - \theta$

$\cos\left(\alpha+\left(\frac{\pi}{2}-\theta\right)\right) = \cos\alpha\cos\left(\frac{\pi}{2}-\theta\right)$

$-\sin\alpha\sin\left(\frac{\pi}{2}-\theta\right)$

$\cos\left(\frac{\pi}{2}-(\theta-\alpha)\right) = \cos\alpha\sin\theta - \sin\alpha\cos\theta$

(*) $\sin(\theta-\alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$

$\alpha = -\eta \Rightarrow -\alpha = \eta$

$\sin(\theta+\eta) = \sin\theta\cos(-\eta) - \cos\theta\sin(-\eta)$

(*) $\sin(\theta+\eta) = \sin\theta\cos\eta + \cos\theta\sin\eta$

$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$= \dfrac{\dfrac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$\sin(\beta+\alpha) + \sin(\beta-\alpha) = 2\sin\beta\cos\alpha$

$\sin(\beta+\alpha) - \sin(\beta-\alpha) = 2\cos\beta\sin\alpha$