Chapter 2

Fluid statics (hydrostatic)

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1- Introduction

This chapter focuses on the study of fluids at rest, when the fluid is not in motion. This falls within the scope of fluid statics or hydrostatics.
In this part of fluid mechanics, we will first define:

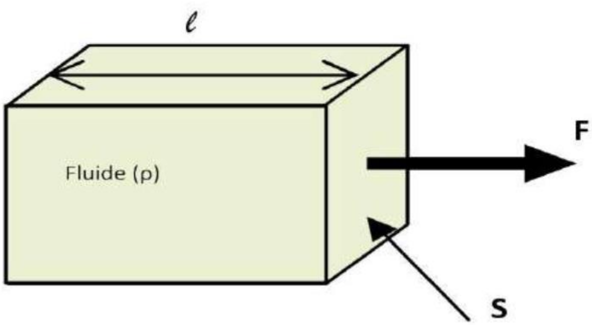
- **Pressure** and **pressure at a point** in a fluid. We will then derive the hydrostatic equation based on the fundamental principle of statics applied to a small element of the fluid's volume.
- **Pascal's theorem** will also be discussed. Instruments for measuring static pressure, the measurement of atmospheric pressure, the barometer, and **Torricelli's law** will be studied in this course.
- Finally, the **center of buoyancy** and the calculation of pressure forces will be performed for horizontal, vertical, and inclined flat plates.

The field of applications is vast, including, for example, the calculation of the resultant force applied to a dam and the calculation of pressure in reservoirs.

2- Notion of pressure

2-1 Definition

Pressure is a scalar quantity. It is defined as a force directed outward, acting perpendicularly to the surface of a wall. It is the ratio of force to the unit surface area.



$$P = \frac{F}{S}$$

Où

- P: Pression (Pa)
- F: Force (N)
- S: Surface (m^2)

Figure 1. Equilibrium fluid pressure

Note: In the International System (SI), pressures are measured in **(N/m^2)** or **Pascals (Pa)**.

However, other units of pressure also exist, including:

- **The bar:** 1 bar = 100,000 Pa = 1 kg/cm².

- **Atmosphere (atm):** 1 atm = 101,325 Pa
- **Meter of water column (mWC):** 1 mWC = 9,810 Pa
- **Millimeter of mercury (mmHg):** 1 mmHg = 133 Pa

2-2 Pressure at a Point in a Fluid

In a fluid at equilibrium, the pressure is independent of direction. To demonstrate this, we consider an element of fluid located at a certain depth within a reservoir filled with the same fluid and open to the atmosphere.

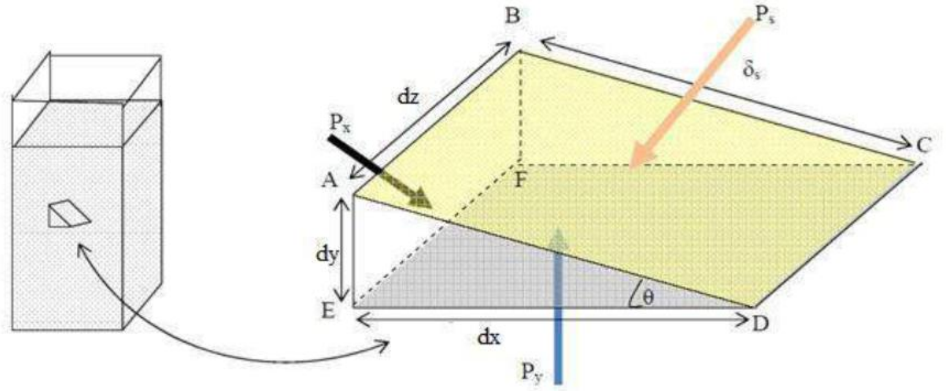


Figure .2 Pressure at a point of a liquid in equilibrium

Let us consider an element of fluid ABCDEF (a triangular prism), with P_x, P_y, and P_s being the pressures in the three directions x, y, and s, respectively.

Establishing the Relationship between Px, Py, and Ps:

In the x-direction:

- Force due to P_x :

$$F_{xx} = P_x .dydz$$

- Force due to P_y :

$$F_{yx} = 0$$

- Component due to P_s :

$$F_{sx} = -P_s(\sin\theta).dsdz \qquad , \quad (\sin\theta).ds=dy$$

Where :

$$F_{sx} = -P_s .dydz$$

And since the fluid is in equilibrium :

$$F_{xx} + F_{yx} + F_{sx} = 0$$

Where :

$$P_x.dydz - P_s.dydz = 0 \rightarrow \qquad P_x = P_s$$

In the y-direction:

- Force due to P_x :

$$F_{xy} = 0$$

- Force due to P_y :

$$F_{yy} = P_y \, .dxdz$$

- Component due to P_s :

$$F_{sy} = -P_s.(\cos\theta).dsdz \quad , \quad (\cos\theta).ds = dx$$

$$F_{sy} = -P_s \, .dxdz$$

And since the fluid is in equilibrium :

$$F_{yy} + F_{xy} + F_{sy} = 0$$

Where :

$$P_y \, .dxdz - P_s.dxdz = 0 \rightarrow \qquad P_y = P_s$$

Finally **:**

$$\mathbf{P_x = P_y = P_s}$$

The pressure of a fluid at one point is the same in all directions.

It can be verified that the pressure exerted within a liquid in equilibrium is:

- **Constant at all points of the same horizontal plane.**
- **Independent of the direction considered.**
- **Increases as one moves deeper from its free surface.**

2-3 Atmospheric and hydrostatic pressure

2-3.1. Atmospheric pressure

Each cm² of our skin supports approximately 1 kg of force, representing the weight of the atmosphere. This is the atmospheric pressure at sea level. We do not feel it because our body is incompressible and its cavities contain air at the same pressure.

If we ascend to 5,000 m, the atmospheric pressure is half that at sea level because the mass of air above our heads decreases by half. This highlights the necessity for pressurization in airplanes. At sea level: $P_{atm} = 1$ atm ≈ 1.013 bar = 760 mmHg

2-3.2. Hydrostatic Pressure

In scuba diving, hydrostatic pressure is usually measured in bars: 1 bar = 1 kg/cm². It's important to know that pressure increases as we descend deeper because we have to account for the weight of the water above us: at a depth of 10 meters, each cm² of our skin will bear a weight equal to: 1 cm² × 10 m (depth) = 1 cm² × 100 cm = 1000 cm³ = 1 liter of water = 1 kg of water. Therefore, the hydrostatic pressure due to water at a depth of 10 m is 1 kg/cm² = 1 bar.

Note: Hydrostatic pressure (like atmospheric pressure) acts in all directions (and not just from top to bottom).

3- *Fundamental Equation of Hydrostatics*

Let's consider a volume element of an incompressible fluid (a homogeneous liquid with volumetric weight ϖ). This volume element is shaped like a cylinder with axis $(G,\overrightarrow{u})$ that makes an angle α with the vertical axis $(O, \overrightarrow{Z})$ of a reference frame R $(O,\overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z})$. Let *l* be the length of the cylinder and dS its cross-sectional area.

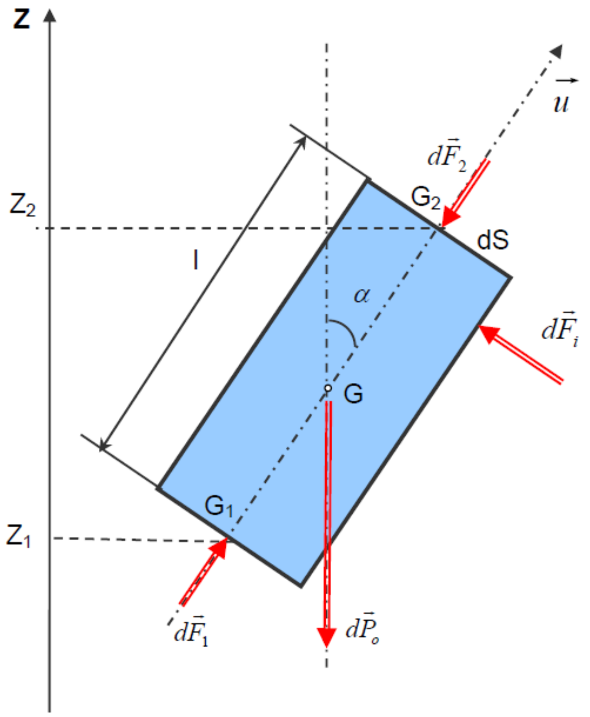


Figure 3. Forces acting on an inclined surface element

Let G_1 be at altitude Z_1 and G_2 at altitude Z_2, the centers of the extreme cross-sections.

Let's study the equilibrium of the elementary cylinder, which is subject to:

- Remote actions: its weight.

$$\overrightarrow{dp_0} = -\varpi \, l \, ds \, \overrightarrow{Z}$$

- **Contact actions:** pressure forces acting on:

The lateral surface: $\Sigma\,\overrightarrow{dF_i}$

The two extreme flat surfaces: $\overrightarrow{dF_1} = P_1.dS.\,\overrightarrow{u}$. et : $\overrightarrow{dF_2} = -P_2.dS.\,\overrightarrow{u}$. With P_1 and P_2 being the fluid pressures at G_1 and G_2 respectively.
Since the elementary cylinder is in equilibrium within the fluid, we can state that the resultant of the external forces applied to it is zero:

$$\overrightarrow{dp_0} + \Sigma\,\overrightarrow{dF_i} + \overrightarrow{dF_1} + \overrightarrow{dF_2} = \vec{0}$$

En projection sur l'axe de symétrie (G, $\overrightarrow{u}$) du cylindre : In projection onto the axis of symmetry (G, $\overrightarrow{u}$) of the cylinder:

$$-\varpi\, l\, ds\, .\cos\alpha + P_1.dS - P_2.dS = 0$$

Let's express the pressure difference $P_1 - P_2$ after dividing by dS and noting that $l\cos\alpha = Z_2 - Z_1$.

$$P_1 - P_2 = \varpi\,(Z_2 - Z_1) = \rho g\,(Z_2 - Z_1)$$

This is the fundamental equation of hydrostatics (Principle of Statics).

Another more general form: By dividing both sides of the previous equation by ϖ (volumetric weight):

$$\frac{P_1}{\varpi} + Z_1 = \frac{P_2}{\varpi} + Z_2 \,. \qquad \textbf{Or} \qquad \frac{P_1}{\rho g} + Z_1 = \frac{P_2}{\rho g} + Z_2$$

Since G_1 and G_2 were chosen arbitrarily within a fluid with volumetric weight ϖ, we can write, at any point at altitude Z, where pressure prevails:

$$\frac{P}{\varpi} + Z = \frac{P}{\rho g} + Z = Cte$$

On the same horizontal plane, the Z+P/ρg term remains constant.

By setting: $Z_2 - Z_1 = h$ et $P_2 = P_0$

We will have: $P_1 = P_0 + \rho g\, h$ Et si : $P_0 = 0$,

$$P_1 = \rho g\, h$$

The pressure therefore increases linearly with depth.

3-1 Equal pressure on the same horizontal plane

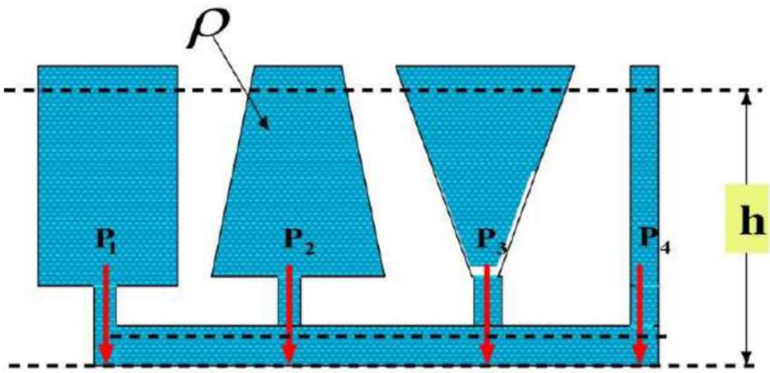


Figure 4: The pressure of a fluid is independent of the shape of the container.

$$P_1 = P_2 = P_3 = P_4 = \rho g h$$

- The free surfaces of the same fluid in the different tubes of the vessels are flat. For a given altitude, the pressure is the same in each shape of the tubes.
- The pressure in a homogeneous fluid, therefore, depends only on the height difference and the density; it is independent of the size or shape of the container.
- We can conclude that:

On the same horizontal plane, all pressures are equal (Isobaric Pressures).

3-2 Absolute Pressure and Effective Pressure

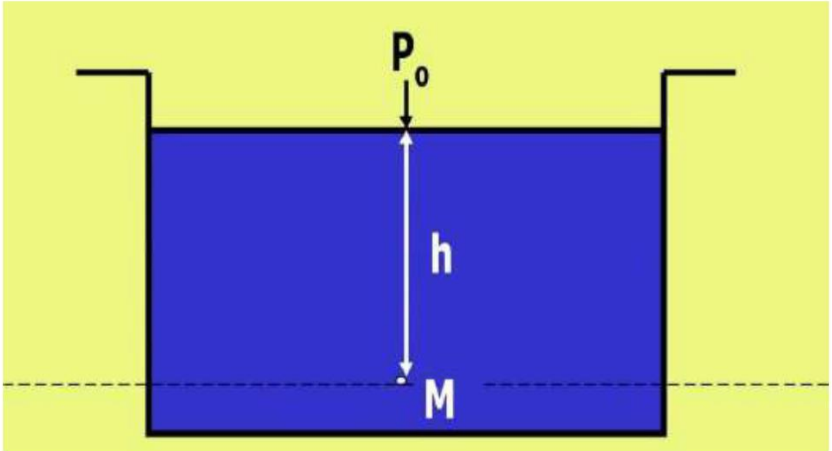


Figure 5: Absolute Pressure and Effective Pressure

Absolute pressure is defined relative to the pressure in a vacuum, which corresponds to zero pressure. The minimum possible absolute pressure is therefore zero. It is common to measure the pressure of a liquid relative to atmospheric pressure (air pressure).

In this case, we refer to relative pressure. In this way, the pressure at the free surface of a liquid is equal to zero.

We know that:

• At point M, the pressure is equal to: $P_M = P_0 + \rho\, g\, h$

• At the free surface of the fluid, the pressure is generally represented by the atmospheric pressure P_{atm}, hence:

$P_M = P_{atm} + \rho\, g\, h \rightarrow$ *Absolute pressure*

And if we neglect the influence of atmospheric pressure ($P_{atm} = 0$)

$P_M = \rho\, g\, h \rightarrow$ *Effective Pressure*

Remarque

In some cases, the absolute pressure is lower than the atmospheric pressure:

$P_M = P_{atm} + \rho\, g\, h < P_{atm}$

This creates a vacuum, the corresponding height of which is called the **height of the vacuum**:

$$h_{vacuum} = \frac{P_{atm} - P_{abs}}{\rho g}$$

Example: The balloon in the vacuum chamber

4- The Transmission of Pressures (Pascal's Principle)

4-1 Definition

Pascal's principle, or the principle of pressure transmission in fluids, is a principle of fluid mechanics which states that pressure exerted anywhere in a confined incompressible fluid is transmitted equally in all directions throughout the fluid.

4-2 Illustration

When a balloon is inflated, the pressure at the entrance increases. The air is distributed evenly inside the balloon. The pressure exerted by the air on the walls of the balloon will then be equal in all directions.

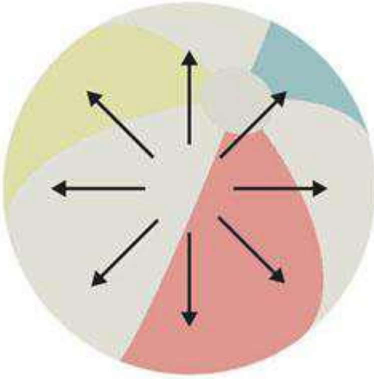


Figure 6. The distribution of pressure exerted by a fluid in a balloon.

4-3 Démonstration

Let's consider, for example, two cylinders with different cross-sections S and S', forming communicating vessels:

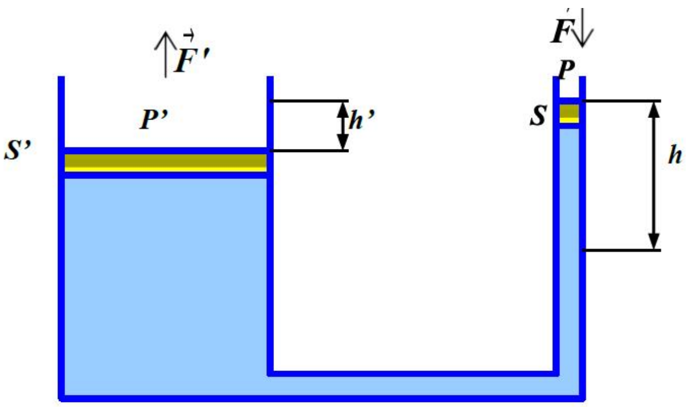


Figure 7. Pascal's Theorem Principle

Blaise Pascal : 1623- 1662

Let's apply a force F perpendicular to the surface of the small piston P, which creates an overpressure that is equal to:

$$\boldsymbol{Dp = F/S}.$$

By virtue of Pascal's theorem, the same pressure variation occurs on P′, which produces a force F′, and we can therefore write:

$$\boldsymbol{Dp = F/S = F'/S'}.$$

We can see that if S′>S, then F′>F, but the displacement of P′ is smaller than that of P. If P is pushed down by h, then P′ only rises by h′. Thus, there is conservation of work (or energy):

$$F.h = F'.h'.$$

5-Archimedes' Buoyancy

5-1 Definition

Anyone who has ever tried to swim to the bottom of a pool to retrieve their goggles has realized how difficult it is. In fact, a force tends to oppose the body descending to the bottom and pushes it toward the surface. This vertical upward force that applies to any object submerged in a fluid is called **Archimedes' buoyancy**.

5-2 Illustration

The formula for Archimedes' buoyancy is generally written in terms of g and the volume V_f of the object as follows:

$$F_{Archimede} = \rho_{fluid} \cdot g.\, V_{displaced\ fluid}$$

We note that the term $\rho_f\ V_f$ is simply the product of the fluid's density by the volume of fluid displaced, which corresponds to the mass of the displaced fluid. Therefore, we can replace the term $\rho_f\ V_f$ with m_f, the mass of the displaced fluid, which gives:

$$F_A = m_f \cdot g$$

It is clear that Archimedes' buoyancy is equal to the product of the displaced fluid's mass and the gravitational acceleration, which is essentially the weight of the displaced fluid! Thus, we can rewrite the formula as:

$$F_A = P_f$$

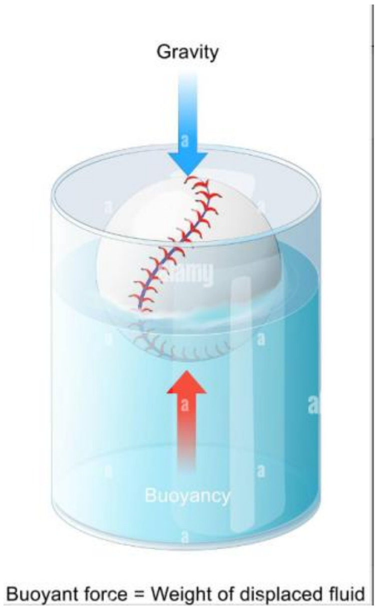


Figure 8. Archimedes' buoyancy

This equation illustrates **Archimedes' theorem**, which states the following: Any object submerged in a fluid experiences an upward vertical force, the magnitude of which is equal to the weight of the volume of displaced fluid. This force is called **Archimedes' buoyancy**.

To determine the value of Archimedes' buoyancy experienced by an object, one simply needs to calculate the weight of the fluid displaced by the object.

Note: The result depends on the magnitude of the buoyancy force and the weight of the object to determine whether:

- The object **sinks** (completely submerged and touches the bottom);

- The object **remains at a constant depth** (completely submerged and stays between two layers of liquid);
- The object **floats** (only part of the solid is submerged in the liquid).

5-2-1 The object is completely submerged but touches the bottom (it sinks)

The weight of the object is greater than the weight of the fluid it displaces (buoyancy). The downward force is therefore greater than the upward force. The resulting force is directed downward, and the object sinks to the bottom:

$$P_f > F_A$$

For the object to be completely submerged and touch the bottom, the condition is:

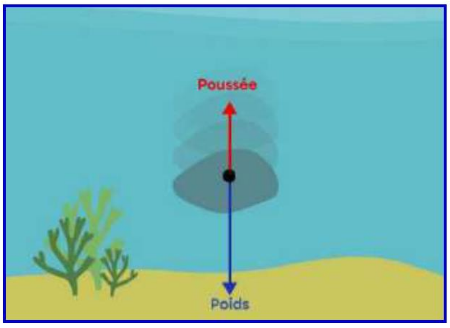


Figure 9. Forces on a sinking object

$$\rho_{objet} > \rho_{fluide}$$

5-2-2 The object is completely submerged and stays between two liquid layers

The weight of the object is equal to the weight of the fluid it displaces (buoyancy). The downward force and the upward force are of the same magnitude. They cancel each other out, and the object remains at the depth where it is:

$$P_f = F_A$$

For the object to remain at a constant depth, the condition is:

$$\rho_{object} = \rho_{fluid}$$

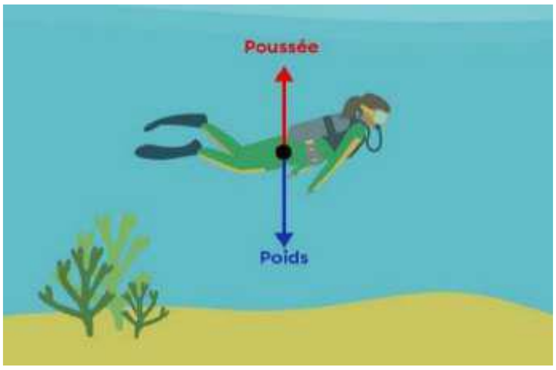


Figure 10: The distribution of forces on an object that remains at a constant depth

5-2-3 part of the solid is submerged in the liquid

The weight of the object is less than the weight of the fluid it displaces (buoyancy). The downward force is smaller than the upward force. The resulting force is directed upward, and the object rises to the surface.

$$P_f < F_A$$

For the object to float, the condition is:

$$\rho_{objet} < \rho_{fluide}$$

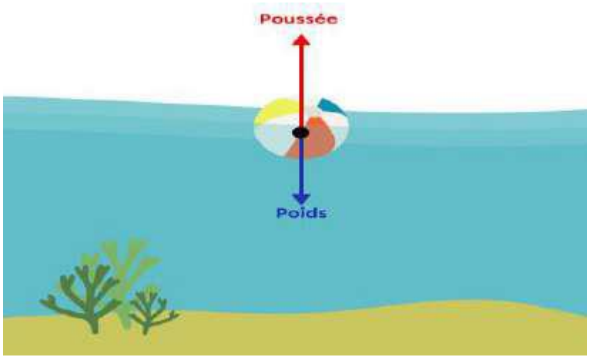


Figure 11. The distribution of forces on a floating

Note:

1. It is important to remember that the density ρ\rhoρ in the Archimedes' buoyancy formula: $FArchimede = \rho_{fluid}.g.V_{\text{displaced fluid}}$ refers to the density of the displaced fluid, not that of the submergeed object.
2. Keep in mind that the volume in the Archimedes' buoyancy formula refers to the volume of the displaced fluid (in other words, the volume of the submerged part of the object), and this volume is not necessarily the volume of the entire object.
3. There is a common misconception that the magnitude of Archimedes' buoyancy increases as the object sinks deeper into the fluid. However, buoyancy does not depend on depth. It only depends on the volume of displaced fluid $\boldsymbol{V}_{\textbf{displaced fluid}}$, the $\boldsymbol{\rho}_{fluid}$.and the acceleration due to gravity **g**.

6- Pressure Measuring Devices

There are two types of pressure measuring devices:

a- Manometric tubes: used for measuring relatively low pressures (in laboratories).

b- Mechanical manometers: used for measuring relatively higher pressures (1 to 2 kg/cm²).

6-1 The Simple Manometer or Piezometer

This is the simplest pressure measuring instrument. It consists of a tube connected to the point where the pressure is to be determined. The pressure is simply the height of the water rising in the tube. This device is used only for measuring the pressure of liquids, not gases.

Figure 12. Piezometer

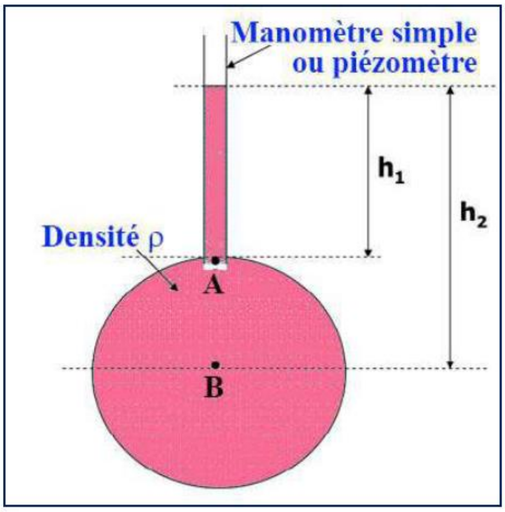


Figure 13. Simplified Manometer or Piezometer

- $P_A = \rho g h_1$
- $P_B = \rho g h_2 = P_A + \rho g (h_2 - h_1)$
- P_A and P_B are called **manometric pressures**.
- h_1 and h_2 are called **manometric heights**.

6-2. The U-shaped Manometric Tube

This is a device used to measure pressures in liquids and gases. It consists of a U-shaped tube, with one end connected to the measurement point and the other open to the air. The tube contains either mercury or another liquid denser than the fluid whose pressure is being measured, in order to measure manometric pressures.

Figure 14. U-shaped Manometric Tube

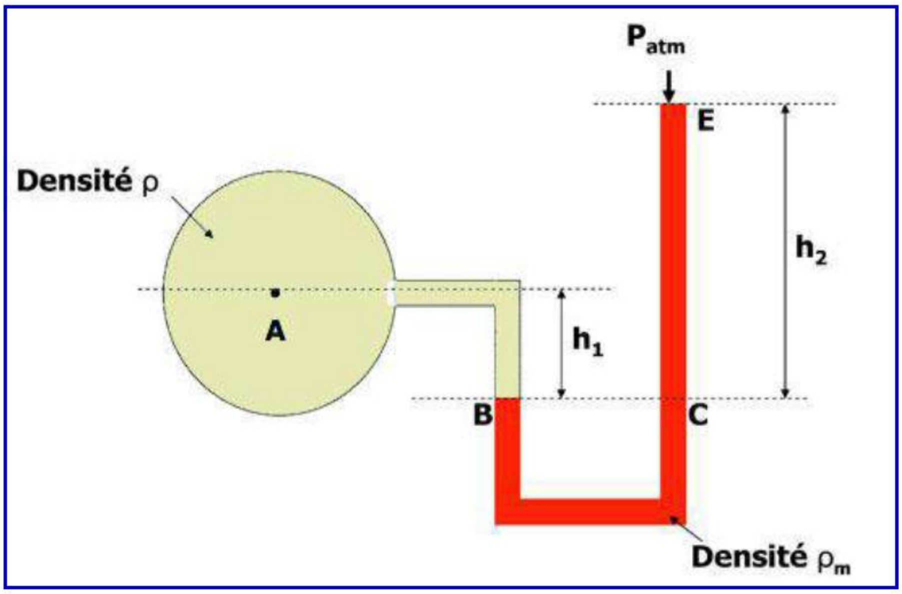


Figure 15. U-shaped Manometric Tube « U » schematic

According to the law of hydrostatics, we can write: $P_B = P_C$

Left side: $P_B = P_A + \rho\, g\, h_1$

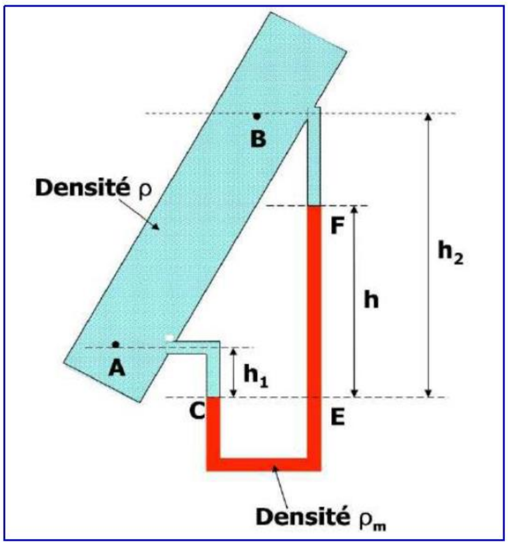
Right side: $P_C = P_E + \rho_m\, g\, h_2 = P_{atm} + \rho\, m\, g\, h_2$

And if we ignore P_{atm} :$P_C = \rho_m\, g\, h_2$

Therefore: : $P_B = P_C \rightarrow P_A + \rho g\, h_1 = \rho_m\, g\, h_2$

6-3 Measuring the pressure difference with a U-shaped manometer

We are asked to calculate $P_A - P_B$.



We can write: $P_C = P_E$

Left side: $P_C = P_A + \rho\, g\, h_1$

Right side: $P_E = P_B + \rho\, g(h_2 - h) + \rho_m\, g\, h$

Since $P_C = P_E$

$P_A + \rho g h_1 = P_B + \rho g(h_2 - h) + \rho_m\, g\, h$

Therefore:: $P_A - P_B = \rho g(h_2 - h_1) + (\rho_m - \rho)\, g\, h$

6.4. Manometer and Mercury Manometer

Water manometers are used to measure relatively low pressures because using them for high pressures would require excessively large tubes. For this reason, and due to its high density, mercury is preferred as the manometric fluid.

Figure 16. Water Manometer

Figure 17. Marcury Manometer

6-5 The Barometer

This is the instrument used to measure atmospheric pressure. The most common type is the mercury barometer, which was demonstrated by the Italian Evangelista Torricelli in 1644. He filled a one-meter-long glass tube, sealed at one end, with mercury (Figure 16). He then inverted the tube and submerged it in a mercury basin. He observed that the mercury level in the tube dropped, leaving a vacuum above it. He had just discovered atmospheric pressure. Atmospheric pressure is obtained by measuring the height h of the mercury column:

$$P_{Atm} = \rho_{Hg}\, g\, h$$

At sea level: P_{atm} =1 atm = 1,0133.105Pa, Or 762mm de Hg.

N.B.: A mechanical or electronic data recording system, known as a **barograph**, can be added to this barometer for automatic data recording. The highest pressures are recorded during cold weather.

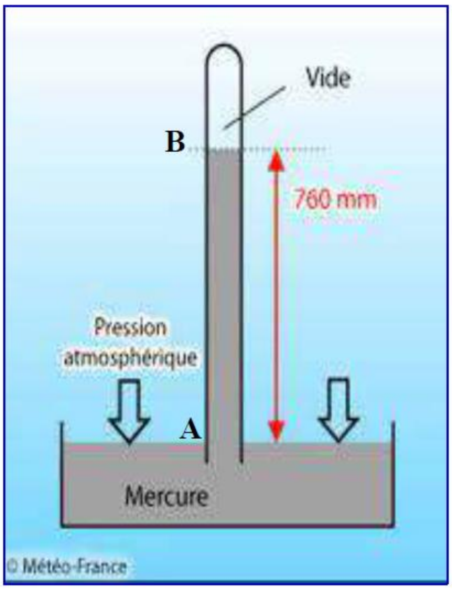


Figure 17. Mercury Barometer

Interpretation:

Let's determine the atmospheric pressure by applying the principle of statics between points A and B:

$P_A+\rho_m g z_A=P_B+\rho_m g z_B$

$P_{atm}=P_{vacuum}+\rho_m g(Z_A-Z_B)$

$P_{atm} = P_{vacuum} +\rho_m gh$

$Patm=0+13600\times9.8\times0.76$

$P_{atm}=101292$ Pa This is the value of the atmospheric pressure.