Example: $F(x,y) = 2xy\,\vec{i} + (x^2 - y)\,\vec{j}$, $M(x,y) = 2xy$ and $N(x,y) = x^2 - y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then F is conservative. Let us compute a potential f of F.

Since $\frac{\partial f}{\partial x} = M \Rightarrow f(x,y) = \int 2xy\,dx = \boxed{yx^2 + \ell(y)}$

$N = \frac{\partial f}{\partial y} \Leftrightarrow x^2 + \ell'(y) = x^2 - y \Rightarrow \ell'(y) = -y \rightsquigarrow \ell(xy) = -\frac{y^2}{2}$

So, $f(x,y) = yx^2 - \frac{y^2}{2}$.

Def: Curl of a vector field in space

The curl of $F = M\vec{i} + N\vec{j} + P\vec{k}$ is:

~~So~~ $\mathrm{curl}(F)$ ~~$(x,y,z) = \nabla \times F$~~ $= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$

~~We call that~~

~~The~~ Where $\mathrm{curl}(F) = 0$, F is said to be irrotationnal

If we denote by ∇ the operator $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial y}\vec{k}$

then, $\mathrm{Curl}(F) = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$.

Example: $F(x,y,z) = 2xy\,\vec{i} + (x^2 + z^2)\vec{j} + 2yz\,\vec{k} \rightsquigarrow \mathrm{Curl}(F) = 0$

So, F is irrotationnal.

Theorem: Let $M, N, P : \Omega \to \mathbb{R}$ of class C^1 over ~~an~~ an open sphere Ω in space. The vector field F is conservative if, and only if, $\mathrm{curl}(F) = 0$.

Example: Finding a potential of $F(x,y,z) = 2xy\,\vec{i} + (x^2 + z^2)\vec{j} + 2yz\,\vec{k}$

$f(x,y,z) = \int 2xy\,dx + g(y,z) = x^2 y + g(y,z)$

$x^2 + \frac{\partial g}{\partial y}(y,z) = x^2 + z^2 \rightarrow g(y,z) = yz^2 + h(z)$

$\rightsquigarrow 2yz + h'(z) = 2yz \longrightarrow h'(z) = 0 \Longrightarrow f(x,y,z) = y(x^2 + z^2)$

Def: Divergence of a vector field

$F = P\vec{i} + Q\vec{j} \longrightarrow \mathrm{div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$; $\mathrm{div}(F) = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$.

If $\mathrm{div}(F) = 0$, then F is said to be divergence free.