Sct n°0

Ex 1:

1) $Z = 1 + 2i \qquad |Z| = \sqrt{1+4} = \sqrt{5}$

$$\arg(Z) = \begin{cases} \cos\theta = \frac{1}{\sqrt{5}} \\ \sin\theta = \frac{2}{\sqrt{5}} \end{cases} \Rightarrow \theta = 1{,}107$$

$$\theta = \frac{1.107}{3{,}14}\pi = 0{,}35\pi$$

$$Z = \sqrt{5}\, e^{0{,}35\pi i}$$

$|Z| = \sqrt{1+1} = \sqrt{2}$

$Z = 1 - i$

$$\arg(Z) = \begin{cases} \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{cases}$$

$\theta = -\frac{\pi}{4} \qquad Z = \sqrt{2}\, e^{-\pi/4\, i}$

$Z = i \qquad |Z| = 1$

$$\arg(Z) = \begin{cases} \cos\theta = 0 \\ \sin\theta = 1 \end{cases} \Rightarrow \theta = \frac{\pi}{2}$$

$$Z = e^{\frac{\pi}{2} i}$$

2) $(1+2i)(1-i) = \sqrt{5}\, e^{0{,}35\pi i} \cdot \sqrt{2}\, e^{-\frac{\pi}{4} i}$

$= \sqrt{10}\, e^{i\pi(0{,}35 - \frac{1}{4})}$

$= \sqrt{10}\, e^{0{,}1\pi i}$

$$\frac{1-i}{1+2i} = \frac{\sqrt{2}\, e^{-\frac{\pi}{4} i}}{\sqrt{5}\, e^{0{,}35\pi i}} = \sqrt{\frac{2}{5}}\, e^{-0{,}55\pi i}$$

Ex 2:

$a\ddot{y} + b\dot{y} + c = f(t)$

$f(t) = 0$ Homogeneous Equation $a r^2 + br + c = 0$

$f(t) \neq 0$

$f(t) = c^{st}$

$f(t) = A\cos(\omega t) + B$ ... $f(t) = A\sin(\omega t + \varphi)$

$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$r_1 = \frac{-b-\sqrt{\Delta}}{2a}$, $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$	$r = \frac{-b}{2a}$	$r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$
$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	$y = (C_1 + C_2)e^{rt}$	$y = e^{\alpha t}[C_1\cos(\beta t) + C_2\sin(\beta t)]$

1) $\ddot{x} + 9x = 0$

$r^2 + 9 = 0 \qquad \Delta < 0$

$r_1 = 3i \qquad r_2 = -3i$

$x = 1\,[C_1 \cos(3t) + C_2\sin(3t)]$

2) $\ddot{x} - 9x = 0$

$r^2 - 9 = 0 \qquad \Delta > 0$

$r_1 = 3 \qquad r_2 = -3$

$x = C_1 e^{-3t} + C_2 e^{3t}$

3) $\ddot{\theta} - 3\dot{\theta} + 2\theta = 0$

$r^2 - 3r + 2 = 0 \qquad \Delta > 0$

$r_1 = 1 \qquad r_2 = 2$

$\theta = C_1 e^{t} + C_2 e^{2t}$

4) $\ddot{q} - 8\dot{q} + 16q = 0$

$r^2 - 8r + 16 = 0 \qquad \Delta = 0$

$r = 4 \qquad q = (C_1 + C_2)e^{4t}$

5) $\ddot{v} - 3\dot{v} + \frac{5}{2}v = A(t)$

$\ddot{v} - 3\dot{v} + \frac{5}{2}v = 0 \Rightarrow$ Homo solution

$r^2 - 3r + \frac{5}{2} = 0$

$\Delta < 0 \quad \Delta = -1$

$r_1 = \frac{3}{2} + \frac{1}{2}i$

$r_2 = \frac{3}{2} - \frac{1}{2}i$

$v = e^{\frac{3}{2}t}[C_1\cos(\frac{1}{2}t) + C_2\sin(\frac{1}{2}t)]$

Ex 3 :

average value :

$$m = \frac{1}{b-a}\int_a^b f(x)\,dx$$

1) $\langle \cos(\omega t+\varphi)\rangle = \frac{1}{T}\int_0^T \cos(\omega t+\varphi)dt$

$\omega = \frac{2\pi}{T}$

$$= \frac{1}{2\pi}\left[\sin(\omega t+\varphi)\right]_0^T$$

$$= \frac{1}{2\pi}\left[\sin(\omega T+\varphi) - \sin(\varphi)\right] = 0$$

2) $\langle A^2\cos^2(\omega t+\varphi)\rangle = \frac{A^2}{T}\int_0^T \cos^2(\omega t+\varphi)dt$

$$= \frac{A^2}{T}\int_0^T \frac{1+\cos(2(\omega t+\varphi))}{2}$$

$$= \frac{A^2}{2T}\left(\int_0^T 1\,dt + \int_0^T \cos(2(\omega t+\varphi))dt\right)$$

$$= \frac{A^2}{2T}\left[\left[t\right]_0^T + \frac{1}{2\omega}\left[\sin(2\omega t+2\varphi)\right]_0^T\right]$$

$$= \frac{A^2}{2T}\left[T + \frac{1}{2\omega}\left[\sin(2\omega T+2\varphi) - \sin(2\varphi)\right]\right] = \frac{A^2}{2}$$

3) $\langle A_1A_2\cos(\omega t)\sin(\omega t)\rangle$

$$= \frac{A_1A_2}{T}\int_0^T \cos(\omega t)\sin(\omega t)\,dt$$

$$= \frac{A_1A_2}{2T}\int_0^T \sin(2\omega t)\,dt$$

$$= -\frac{A_1A_2}{2T}\frac{1}{2\omega}\left[\cos(2\omega t)\right]_0^T$$

$A_3 = A_1A_2$

$$= -\frac{A_3}{4\omega T}(\cos 2\omega T - 1) = 0$$

Ex 4 :

1) finding A and φ

$\dot{x}(t) = A50\cos(50t+\varphi)$

$x(0) = A\sin(\varphi) = 3\times10^{-3}$ m

$\dot{x}(0) = A50\cos(\varphi) = 1$ m/s

$\tan(\varphi) = 150\times10^{-3} = 0,15$

$\varphi = 0,148$ Rad

$A\sin(\varphi) = 3\times10^{-3} \Rightarrow A = \frac{3\times10^{-3}}{\sin\varphi}$

$A = 0,02$ m $= 20$ mm

2) $x(t) = 20\sin(50t + 0,148)$

$\sin(50t+0,148) = \sin(50t)\cos(0,148) + \cos(50t)\sin(0,148)$

$A_2 = 20\cos(0,148) = 19,78$ mm

$A_1 = 20\sin(0,148) = 2,94$ mm

$x(t) = 2,94\cos(50t) + 19,78\sin(50t)$

Ex 5 :

1) $x_1(t) = 3\sin\left(2t+\frac{\pi}{4}\right)$

$x_2(t) = 6\sin\left(2t+\frac{\pi}{3}\right)$

$\overline{x_1(t)} = 3e^{j(2t+\frac{\pi}{4})}$

$\overline{x_2(t)} = 6e^{j(2t+\frac{\pi}{3})}$

$\overline{x(t)} = \overline{x_1(t)} + \overline{x_2(t)} = 3e^{j(2t+\frac{\pi}{4})} + 6e^{j(2t+\frac{\pi}{3})}$

$= e^{2jt}\left[3e^{j\frac{\pi}{4}} + 6e^{j\frac{\pi}{3}}\right]$

$= e^{2jt}\left[3\cos\left(\frac{\pi}{4}\right) + 3j\sin\left(\frac{\pi}{4}\right) + 6\cos\left(\frac{\pi}{3}\right) + 6j\sin\left(\frac{\pi}{3}\right)\right]$

$= e^{2jt}\left[\frac{3\sqrt{2}}{2} + j\frac{3\sqrt{2}}{2} + 3 + j3\sqrt{3}\right]$

$= e^{2jt}\left[\frac{3\sqrt{2}}{2} + 3 + j\left(\frac{3\sqrt{2}}{2} + 3\sqrt{3}\right)\right]$

$$r = \sqrt{\left(\frac{3\sqrt{2}}{2}+3\right)^2 + \left(\frac{3\sqrt{2}}{2}+3\sqrt{3}\right)^2}$$

$$\tan\varphi = \frac{\frac{3\sqrt{2}}{2}+3\sqrt{3}}{\frac{3\sqrt{2}}{2}+3}$$

$e^{2jt}\left[8{,}93\, e^{j\ldots}\right] = 8{,}93\, e^{\ldots}$

$x(t) = 8{,}93 \sin(2t + 0{,}96)$

2nd Method

$x(t) = 3\sin\left(2t + \frac{\pi}{4}\right) + 6\sin\left(2t + \frac{\pi}{3}\right)$

$3\left(\sin(2t)\cos\left(\frac{\pi}{4}\right) + \cos(2t)\sin\left(\frac{\pi}{4}\right)\right) +$

$6\left(\sin(2t)\cos\left(\frac{\pi}{3}\right) + \cos(2t)\sin\left(\frac{\pi}{3}\right)\right)$

$3\left[\frac{\sqrt{2}}{2}\sin(2t) + \frac{\sqrt{2}}{2}\cos(2t)\right]$

$+6\left[\frac{1}{2}\sin(2t) + \frac{\sqrt{3}}{2}\cos(2t)\right]$

$x(t) = \sin(2t)\left(\frac{3\sqrt{2}}{2} + 3\right) + \cos(2t)\left(\frac{3\sqrt{2}}{2} + 3\sqrt{3}\right)$

$x(t) = A\sin(2t + \varphi)$

$A\cos\varphi = \frac{3\sqrt{2}}{2} + 3$

$A\sin\varphi = \frac{3\sqrt{2}}{2} + 3\sqrt{3}$

$\tan\varphi = \dfrac{\frac{3\sqrt{2}}{2} + 3\sqrt{3}}{\frac{3\sqrt{2}}{2} + 3}$

$A = \sqrt{\left(\frac{3\sqrt{2}}{2} + 3\right)^2 + \left(\frac{3\sqrt{2}}{2} + 3\sqrt{3}\right)^2}$

$A = 8{,}93 \qquad \varphi = 0{,}96$

2) $x_1(t) = 10\cos(3t)$

$x_2(t) = 5\sin(12t) = 5\cos\left(12t - \frac{\pi}{2}\right)$

$\overline{X_1}(t) = 10\, e^{3jt}$

$\overline{X_2}(t) = 5\, e^{j\left(12t - \frac{\pi}{2}\right)}$

$\overline{X}(t) = \overline{X_1}(t) + \overline{X_2}(t) = e^{3tj}\left(10 + 5e^{j\left(9t - \frac{\pi}{2}\right)}\right)$

$= e^{3tj}\left[10 + 5\cos\left(9t - \frac{\pi}{2}\right) + 5j\sin\left(9t - \frac{\pi}{2}\right)\right]$

$X_0(t) = \sqrt{\left(10 + 5\cos\left(9t - \frac{\pi}{2}\right)\right)^2 + \left(5\sin\left(9t - \frac{\pi}{2}\right)\right)^2}$

$\tan\varphi = \dfrac{5\sin\left(9t - \frac{\pi}{2}\right)}{10 + 5\cos\left(9t - \frac{\pi}{2}\right)}$

$\overline{X}(t) = x_0(t)\, e^{j(3t + \varphi(t))}$

$x(t) = x_0(t)\cos(3t + \varphi(t))$

$X_0(t) = \sqrt{10^2 + (5\cos(\,))^2 + 2\cdot 5\cos(\,) + (5\sin(\,))^2}$

$X_0(t) = \sqrt{10^2 + 5^2 + 2\cdot 5\cos\left(9t - \frac{\pi}{2}\right)}$

$x_{0\,min} = \sqrt{(10-5)^2} = 10 - 5$

$X_{0\,max} = \sqrt{(10+5)^2} = 10 + 5$

3) $x_1(t) = 2\sin(25t) \qquad x_2(t) = 2\sin(24t)$

$\overline{X_1}(t) = 2e^{25jt} \qquad \overline{X_2}(t) = 2e^{24jt}$

$\overline{X}(t) = 2\left(e^{25jt} + e^{24jt}\right) = 2e^{j24{,}5t}\left(e^{\frac{1}{2}jt} + e^{-\frac{1}{2}jt}\right)$

$= 4e^{24{,}5jt}\cos\left(\frac{1}{2}t\right)$

$x(t) = 4\sin(24{,}5t)\cos\left(\frac{1}{2}t\right)$

$\frac{\omega_1 + \omega_2}{2} \qquad \frac{\omega_1 - \omega_2}{2}$

Ex 6: $x_1(t) = A\cos(\omega_1 t + \varphi_1)$

$x_2(t) = B\sin(\omega_2 t + \varphi_2)$

$A + B = 4$
$A - B = 0$ $\Rightarrow A = B = 2$

$x_1(t) = 2\cos(\omega_1 t)$

$x_2(t) = 2\sin(\omega_2 t)$

$x_1(t) + x_2(t) = 2\cos(\omega_1 t) + 2\cos(\omega_2 t)$

$= 2\left[2\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)\right]$

$= 4\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)$

fast oscillation $\qquad$ slow

$\frac{\omega_1 + \omega_2}{2} = \frac{2\pi}{T_F} \qquad T_F = \frac{2{,}3}{2} = 1{,}15$

$\frac{\omega_2 - \omega_1}{2} = \frac{2\pi}{T_S} \qquad T_S = 6{,}3 \times 2 = 12{,}6$

$$\omega_1 + \omega_2 = \frac{4\pi}{1,15} = 10,92 \quad (1)$$

$$\omega_2 - \omega_1 = \frac{4\pi}{12,6} = 0,99 \quad (2)$$

$(1) + (2) \Rightarrow 2\omega_1 = 11,91 \Rightarrow \omega_1 = 5,9$

$\omega_1 + 5,96 = 10,92 \Rightarrow \omega_2 = 4,96$