Numerical Analysis
Lab 1

Bisection Method and Fixed-Point Iteration

February 24, 2025

Objective

In this lab, we will implement a Matlab code that calculates the root of a nonlinear equation using the Bisection method and Fixed-Point iteration.

1 Graphical Method

Graphically, the root x of the equation $f(x) = 0$ is interpreted as the x-coordinate of the point of intersection between the curve representing $f(x)$ and the x-axis.

1.1 Plot a graph in Matlab

Create a script to plot the graph of the function $f(x)$.

1. Define the function $f(x) = x^2 - 5x + 3$ with x in the interval $[0, 1]$ with a 0.01 step.
2. Use the `plot` command to plot the graph of $f(x)$*
3. Add the grid.
4. Give the graph a title and $x - axis$ and $y - axis$ labels.
5. Repeat the same process for $f(x) = sin(x) + cos(2x) - 1$ on the interval $[2, 3]$ with a step of 0.1.

2 Bisection Method

The Bisection Method is a way to find the root of a function by repeatedly narrowing down an interval where the function changes signs. The steps are as follows :

1. **Choose an Interval**: Select two values, a and b, such that the function has opposite signs at these points, i.e., $f(a) \cdot f(b) < 0$. This ensures that there is a root between a and b (by the Intermediate Value Theorem).

*Graphs have many properties that you can explore by typing `help plot` in the MATLAB Command Window.

2. **Find the Midpoint**: Compute the midpoint of the interval:

$$m = \frac{a+b}{2}$$

3. **Evaluate the Function at the Midpoint**: Compute $f(m)$, the value of the function at the midpoint.

4. **Check for Convergence**:
   - If $f(m) = 0$ or if $|a - b| < \epsilon$ where ϵ is the tolerance, stop — m is the root.
   - Otherwise, continue to the next step.

5. **Update the Interval**:
   - If $f(a) \cdot f(m) < 0$, set $b = m$ (the root lies between a and m).
   - If $f(m) \cdot f(b) < 0$, set $a = m$ (the root lies between m and b).

6. **Repeat**: Repeat steps 2 to 5 until the interval becomes sufficiently small or the root is found with the desired accuracy.

2.1 Bisection Algorithm

- Start with $i = 0$
- If $f(a) == 0$:
  - Set x = a
  - Stop
- Else If $f(b) == 0$:
  - Set x = b
  - Stop
- Else If $f(a)$ and $f(b)$ have the same sign:
  - Display "Inappropriate interval" and stop the function
  - Stop
- End If
- While $|a - b| \geq$ tol (tolerance):
  - $x = \frac{a+b}{2}$
  - If $f(a) \cdot f(x) > 0$, then:
    - Set $a = x$
  - Else:
    - Set $b = x$
  - End If
  - $i = i + 1$
- End While

2.2 Coding the Bisection Method

1. First, we have to determine the function parameters, fill the table.

Inputs	Outputs

2. Create a MATLAB function based on the previous algorithm, save it as **bisection.m**.

3. Calculate the roots of the following functions using the bisection method program.

4. $f(x) = 2cos(x) - x$ in $[0, 2]$ with $\epsilon = 10^{-3}$

5. $f(x) = 2x^3 + x - 5$ in $[1, 3]$ with $\epsilon = 10^{-6}$

6. $f(x) = exp(x) + sin(x) - 10$ in $[1, 3]$ with $\epsilon = 10^{-5}$

2.3 Exercise

Consider the function

$$f(x) = x^3 - 6x^2 + 8x - 1$$

1. Draw the curve of $f(x)$ on the interval [0, 5], then, with a step size of 0.01, find suitable intervals to apply the bisection method.

2. For each interval (one for each root), apply the bisection method to $f(x)$. Consider the tolerance $\epsilon = 0.001$

3 Fixed-Point Iteration

The Fixed-Point Iteration method is a simple and powerful tool for solving nonlinear equations. However, its convergence depends on the nature of the function $g(x)$ and the choice of the initial guess.

- Let g be a continuous function on $[a, b]$. A fixed point of the function g is any point $x \in [a, b]$ that satisfies $g(x) = x$.

- Let $g : [a, b] \to [a, b]$ be a continuous function. Then the function $g(x)$ has at least one fixed point in $[a, b]$.

If an equation $f(x) = 0$ is equivalent to another equation of the form $g(x) = x$, then finding the zeros of f reduces to finding the fixed points of g: $g(\alpha) = \alpha$. Geometrically, we are then looking for the intersection of the graph of g with the graph of $x \mapsto x$, that is, with the first bisector $y = x$.

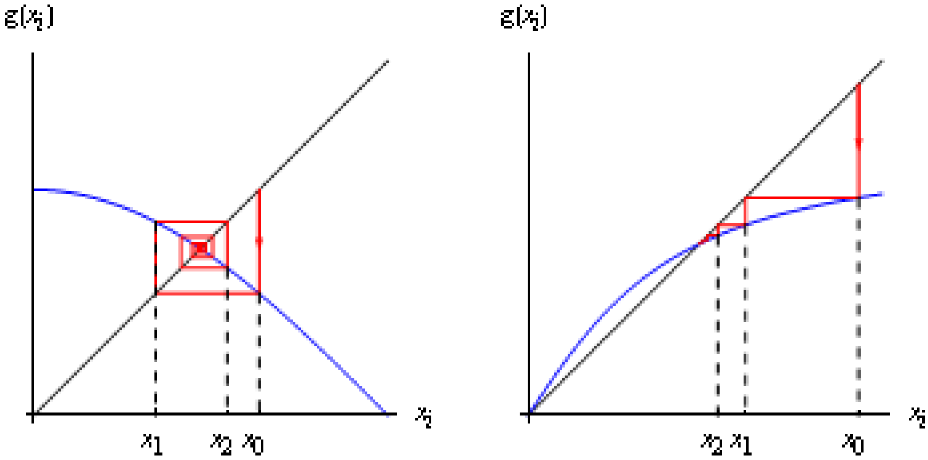

Figure 1: Principle of the Fixed-Point Iteration Method (Spiral and Staircase Convergence)

Steps

Start with the equation $f(x) = 0$. Rearrange it into the form $g(x) = x$, where $g(x)$ is a function that can be evaluated easily.

1. **Choose an initial guess:** Choose an initial guess $x_0 (n = 0)$.
2. **Iteration:** Compute the next approximation using the formula:

$$x_{n+1} = g(x_n)$$

3. **Convergence:** Continue iterating until the sequence x_n converges to a fixed point, meaning $|x_{n+1} - x_n| < tol$.
4. **Checking if maximum iterations are reached:** If the sequence does not converge within a set number of iterations N_{max}, stop the process.
5. Otherwise, proceed to step 2 for the next iteration $n + 1$ (where n becomes $n + 1$). Repeat this process iteratively to generate a sequence of approximations: $x_0, x_1, x_2, \ldots$.

Note: The success of the method and its convergence depend on the nature of the function $g(x)$ and the choice of the initial guess x_0. The function $g(x)$ should satisfy certain conditions for the method to guarantee convergence.

3.1 Coding the Fixed-point Iteration Method

1. First, we have to determine the function parameters, fill the table.

Inputs	**Outputs**

2. Create a MATLAB function based on the previous algorithm, save it as **fixedpoint.m**.

3.2 Exercise

Consider the function

$$f(x) = x^2 + x - 6$$

1. Find all possible ways to rewrite the function $f(x)$ in the form $g(x) = x$ such that it can be used in the fixed-point iteration method.
2. Apply the function **fixedpoint** to each of these functions with $x_0 = 5$, $\text{tol} = 10^{-5}$, and $n_{\max} = 40$.
3. Which function gives the fastest convergence, and explain why there is no root with one of them