N.S.A.S.T **2024/2025**
Second year P.C - Analysis 3

SERIE N° 2 : VECTOR ANALYSIS

Exercice 1 :

1/ Find the curl of $\mathbf{F}$, in each of the following cases :

$$i) \quad \mathbf{F} : (x,y,z) \longmapsto \frac{yz}{y-z}\overrightarrow{\mathbf{i}} + \frac{zx}{x-z}\overrightarrow{\mathbf{j}} + \frac{xy}{x-y}\overrightarrow{\mathbf{k}}$$

$$ii) \quad \mathbf{F} : (x,y,z) \longmapsto \sin(x-y)\overrightarrow{\mathbf{i}} + \sin(y-z)\overrightarrow{\mathbf{j}} + \sin(z-x)\overrightarrow{\mathbf{k}}.$$

2/ In the following, determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function f for $\mathbf{F}$

$$i) \quad \mathbf{F} : (x,y,z) \longmapsto xy^2z^2\overrightarrow{\mathbf{i}} + yx^2z^2\overrightarrow{\mathbf{j}} + zx^2y^2\overrightarrow{\mathbf{k}}$$

$$ii) \quad \mathbf{F} : (x,y,z) \longmapsto \sin z\overrightarrow{\mathbf{i}} + \sin x\overrightarrow{\mathbf{j}} + \sin y\overrightarrow{\mathbf{k}}.$$

3/ In the following, find the divergence of the vector field $\mathbf{F}$:

$$i) \quad \mathbf{F} : (x,y) \longmapsto xe^x\overrightarrow{\mathbf{i}} + ye^y\overrightarrow{\mathbf{j}}$$

$$ii) \quad \mathbf{F} : (x,y,z) \longmapsto \ln\left(x^2+y^2\right)\overrightarrow{\mathbf{i}} + xy\overrightarrow{\mathbf{j}} + \ln\left(y^2+z^2\right)\overrightarrow{\mathbf{k}}.$$

Exercice 2 :

Let $\mathbf{F}$ and $\mathbf{G}$ two vector fields and f a scalar field, prove the following properties :

1/ $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = (\operatorname{curl}\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\operatorname{curl}\mathbf{G})$,

2/ $\operatorname{div}(f\mathbf{G}) = f\operatorname{div}(\mathbf{G}) + (\operatorname{grad} f)\cdot\mathbf{G}$, **3/** $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$,

4/ $\operatorname{curl}(f\mathbf{G}) = f\operatorname{curl}(\mathbf{G}) + \operatorname{grad} f\times\mathbf{G}$, **5/** $\operatorname{curl}(\operatorname{grad} f) = 0$,

where $\cdot$ denotes the scalar product and $\times$ denotes the vector product.

Exercice 3 :

1/ Evaluate $\int_C (x^2+y^2)\,d\gamma$, where :
i) C : the line segment from $(0,0)$ to $(1,1)$.
ii) C : the line segment from $(0,0)$ to $(2,4)$.
iii) C : the counterclockwise arournd the circle $x^2+y^2=1$ from $(1,0)$ to $(0,1)$.
2/ Evaluate $\int_C (2x+y^2-z)\,dr$, where :
i) C : the broken (or polygonal) line $(0,0,0)\longrightarrow(1,0,0)\longrightarrow(1,0,1)\longrightarrow(1,1,1)$.
ii) C : the triangle of vertices $(0,0,0),(0,1,0),(0,1,1)$.

Exercice 4 :

Evaluate $\int_C \mathbf{F}\cdot dr$, where :
i) $\mathbf{F} : (x,y) \longmapsto xy\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}}$ *and* $C = \left\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+ : x^2+y^2=4\right\}$.
ii) $\mathbf{F} : (x,y,z) \longmapsto xy\overrightarrow{\mathbf{i}} + xz\overrightarrow{\mathbf{j}} + yz\overrightarrow{\mathbf{k}}$ *and* $C : t \overset{r}{\longmapsto} t\overrightarrow{\mathbf{i}} + t^2\overrightarrow{\mathbf{j}} + 2t\overrightarrow{\mathbf{k}}$, $t\in[0,1]$.
iii) $\mathbf{F} : (x,y,z) \longmapsto x^2\overrightarrow{\mathbf{i}} + y^2\overrightarrow{\mathbf{j}} + z^2\overrightarrow{\mathbf{k}}$ *and* $C : t \overset{r}{\longmapsto} 2\sin t\overrightarrow{\mathbf{i}} + 2\cos t\overrightarrow{\mathbf{j}} + \frac{1}{2}t^2\overrightarrow{\mathbf{k}}$, $t\in[0,\pi]$.

Exercice 5 :

1/ Evaluate $\int_C (x^2+y^2)\,dx + 2xydy$, where :

i) $C : t \overset{r}{\longmapsto} t^3\,\overrightarrow{\mathbf{i}} + t^2\,\overrightarrow{\mathbf{j}}$, $t \in [0,2]$. *ii)* $C : t \overset{r}{\longmapsto} 2\cot t\,\overrightarrow{\mathbf{i}} + 2\sin t\,\overrightarrow{\mathbf{j}}$, $t \in \left[0, \frac{\pi}{2}\right]$

2/ Evaluate $\int_C yzdx + xzdy + xydz$, where :

i) $C : t \overset{r}{\longmapsto} t\,\overrightarrow{\mathbf{i}} + 2\,\overrightarrow{\mathbf{j}} + t\,\overrightarrow{\mathbf{k}}$, $t \in [0,4]$. *ii)* $C : t \overset{r}{\longmapsto} t^2\,\overrightarrow{\mathbf{i}} + t\,\overrightarrow{\mathbf{j}} + t^2\,\overrightarrow{\mathbf{k}}$, $t \in [0,2]$.

Exercice 6 :

Use Green's theorem to evaluate the indicate line integral :

1/ $\int_C 2xydx + (x+y)\,dy$, where C is the boundary of the region lying $y=0$ and $y = 1-x^2$.

2/ $\int_C 2\arctan\frac{y}{x}dx + \ln(x^2+y^2)\,dy$, where C is the the ellipse $(x-4)^2 + 4(y-4)^2 = 4$.

3/ $\int_C (x^2-y^2)\,dx + 2xydy$, where C is the circle $x^2+y^2=16$.

Exercice 7 :

1/ Find the area of the torus $\mathbf{T} : (u,v) \overset{r}{\longmapsto} (2+\cos u)\cos v\,\overrightarrow{\mathbf{i}} + (2+\cos u)\sin v\,\overrightarrow{\mathbf{j}} + \sin u\,\overrightarrow{\mathbf{k}}$.

2/ Find a vector field $\mathbf{F}$ whose graph is the indicated surface :

i) The plane $z = y$. *ii)* The cone $z = \sqrt{4x^2+9y^2}$.

iii) The cylinder $4x^2+y^2 = 16$. *iv)* The ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

Exercice 8 :

1/ Evaluate $\iint_S f(x,y,z)\,d\sigma$, where :

i) $f : (x,y,z) \longmapsto x^2+y^2+z^2$ and $S = \{(x,y,z) \in \mathbb{R}^3 : z = x+y, x^2+y^2 \leq 1\}$.

ii) $f : (x,y,z) \longmapsto \sqrt{x^2+y^2+z^2}$ and $S = \left\{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{x^2+y^2}, (x-1)^2 + y^2 \leq 1\right\}$.

2/ Find the flux of the vector field $\mathbf{F}$ through S, where S is oriented upward by the unit normal vector field $\mathbf{N}$:

i) $\mathbf{F} : (x,y,z) \longmapsto 3z\,\overrightarrow{\mathbf{i}} - 4\,\overrightarrow{\mathbf{j}} + y\,\overrightarrow{\mathbf{k}}$ and $S = \left\{(x,y,z) \in (\mathbb{R}^+)^3 : z = 1-x-y\right\}$.

ii) $\mathbf{F} : (x,y,z) \longmapsto x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}$ and $S = \{(x,y,z) \in \mathbb{R}^3 : z = 1-x^2-y^2, z \geq 0\}$.

Exercice 9 :

Use the divergence theorem to compute the flux of $\mathbf{F} : (x,y,z) \longmapsto x^2\,\overrightarrow{\mathbf{i}} - 2xy\,\overrightarrow{\mathbf{j}} + xyz^2\,\overrightarrow{\mathbf{k}}$ through the surface $S = \left\{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{a^2-x^2-y^2}\right\}$.

Exercice 10 :

Check the Stockes's theorem for : $\mathbf{F} : (x,y,z) \longmapsto x^2\,\overrightarrow{\mathbf{i}} + y^2\,\overrightarrow{\mathbf{j}} + z^2\,\overrightarrow{\mathbf{k}}$ and

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : z = y^2, 0 \leq x \leq a, 0 \leq y \leq a\right\}.$$