Definition: Flux of vector field over an oriented surface

Let $F = M\vec{i} + N\vec{j} + P\vec{k}$, ~~who~~ be a class C^1 vector field over an oriented surface S by its unit normal vector field N. The flux of F across (à travers) S is given by: $\iint_S (F \cdot N)\, d\sigma$.

Geometrically, the flux of F is the integral surface over S of the normal component of F. If $\rho : S \to \mathbb{R}_+$ is the density of a fluid at, then $\iint_S \rho (F \cdot N)\, d\sigma$ represents the mass of the fluid flowing across S ~~pr~~ per unit of time.

th: If $S := G^{-1}(\{0\})$, where $G(x,y,z) = z - g(x,y) \,/\, (x,y) \in R$.

$$\iint_S (F \cdot N)\, d\sigma = \begin{cases} \iint_R \left(P - M \frac{\partial g}{\partial x} - N \frac{\partial g}{\partial y} \right)(x,y)\, dx\, dy & (\text{if } S \text{ is oriented upward}) \\ \iint_R \left(M \frac{\partial g}{\partial x} + N \frac{\partial g}{\partial y} - P \right)(x,y)\, dx\, dy & (\text{if } S \text{ " " downward}) \end{cases}$$

Example: $S = \{(x,y,z) \in \mathbb{R}^3 : z = 4 - x^2 - y^2 \,/\, (x^2+y^2) \le 4\}$

$F = x\vec{i} + y\vec{j} + z\vec{k}$; $\iint_S (F \cdot N)\, d\sigma = 24\pi$.

Remark: If $S : r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \,/\, (u,v) \in D$.

$$\iint_S (F \cdot N)\, d\sigma = \iint_D F\left(\frac{r_u \wedge r_v}{\| r_u \wedge r_v \|} \right) \cdot \| r_u \wedge r_v \|\, du\, dv = \iint_D F \cdot (r_u \wedge r_v)\, du\, dv$$

Example: $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = a^2\}$; $F(x,y,z) = \frac{kq \cdot r}{\|r\|^3}$ $(r = x\vec{i} + y\vec{j} + z\vec{k})$

$S : r(u,v) = a \sin u \cos v\, \vec{i} + a \sin u \sin v\, \vec{j} + a \cos u\, \vec{k}$; $(u,v) \in [0,\pi] \times [0,2\pi]$.

theorem: (The divergence theorem) (or Ostrogradsky theorem)

Let Ω be a solid region bounded by a closed smooth surface S oriented the unit normal vector N directed outward from Ω. ~~The fol~~ If F is of class C^1 over Ω then

$$\iint_S (F \cdot N)\, d\sigma = \iiint_\Omega \operatorname{div}(F)\, dx\, dy\, dz.$$

Example: $\Omega = \{(x,y,z) \in (\mathbb{R}_+)^3 : 2x + 2y + z \le 6\}$, $S = \partial\Omega$

$F = x\vec{i} + y^2\vec{j} + z\vec{k}$; $\operatorname{div}(F) = 2 + 2y$, So. $\iint_S (F \cdot N)\, d\sigma = \frac{63}{2}$.

theorem (Stockes's theorem): Let S oriented by N, $\partial S = C$ is a piecewise smooth simple closed curve with positive orientation. If F is a class over an open region $R \supset S$, then $\oint_C F \cdot dr = \iint_S (\operatorname{curl} F) \cdot N\, d\sigma$

Example: $S = \{2x + 2y + z = 6 ; x \ge 0, y \ge 0, z \ge 0\}$; $F = -y^2\vec{i} + z\vec{j} + x\vec{k}$.