National Higher School of Autonomous Systems Technology

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# *Vibrations and waves*

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# Chapter 5 : Free and forced vibration of two degree of freedom systems

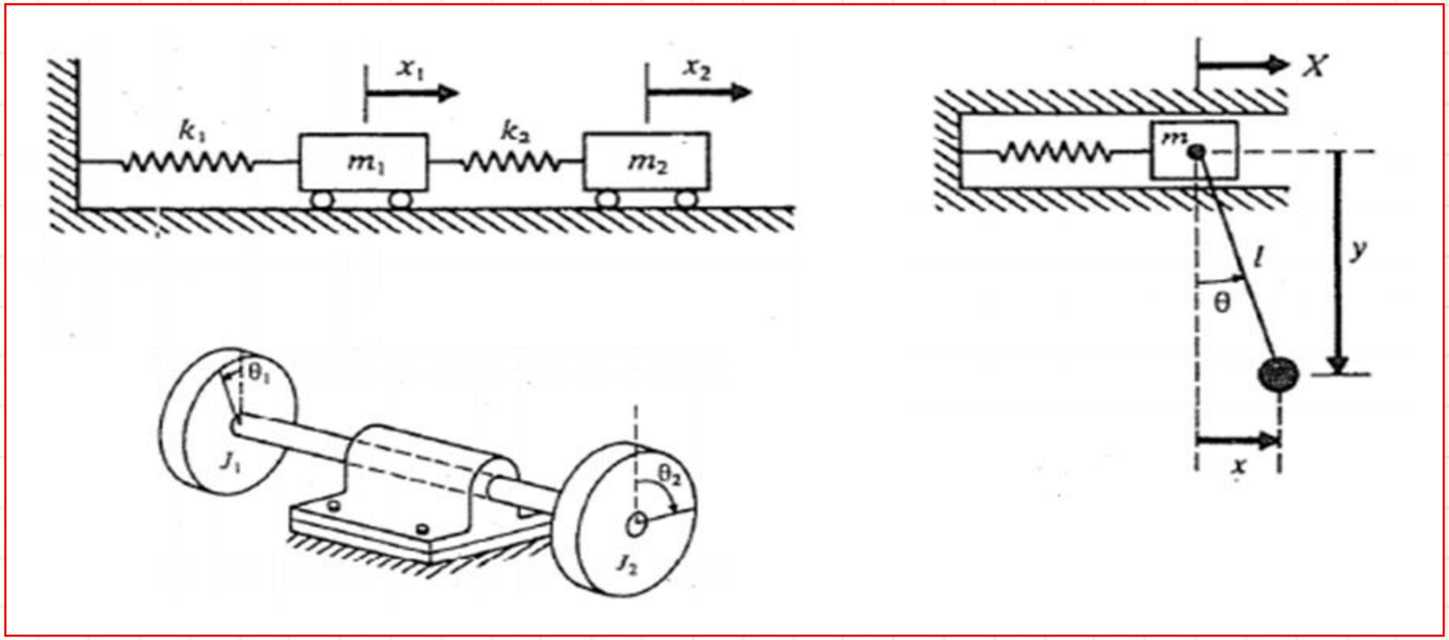
$$\begin{cases} \frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}_1}\right) - \frac{\partial(L)}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}_2}\right) - \frac{\partial(L)}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = 0 \end{cases} \quad \textit{and} \quad \begin{cases} \frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}_1}\right) - \frac{\partial(L)}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = F_{eq_1} \\ \frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}_2}\right) - \frac{\partial(L)}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = F_{eq_2} \end{cases}$$

## *contents*

# Free and forced vibration of two degree of freedom systems

## 1-Introduction

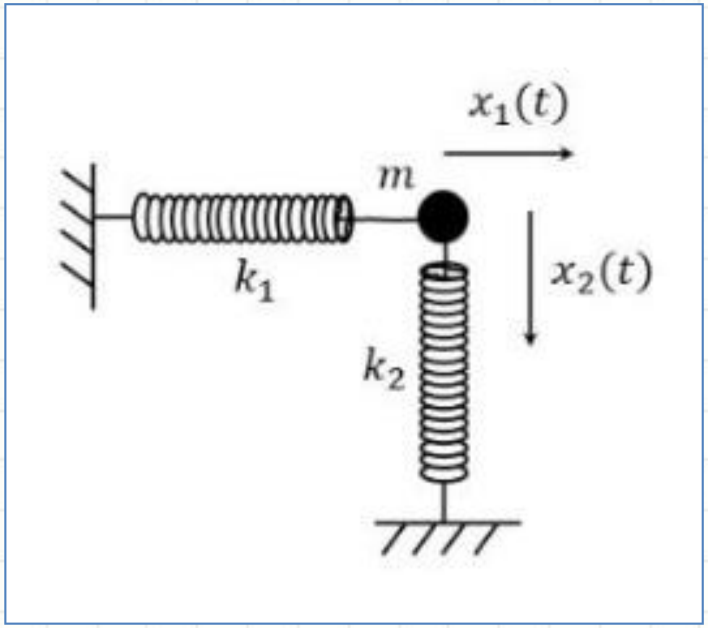
In mechanical systems, vibrations with two degrees of freedom play a significant role, as they represent a generalization of single-degree-of-freedom vibrations. Unlike simple systems, where a single parameter is sufficient to fully describe the motion (such as the displacement of a mass on a spring), systems with two degrees of freedom require two independent variables to characterize their dynamic behavior.



# Free and forced vibration of two degree of freedom systems

## 1-Introduction

### ✓*Uncoupled system*



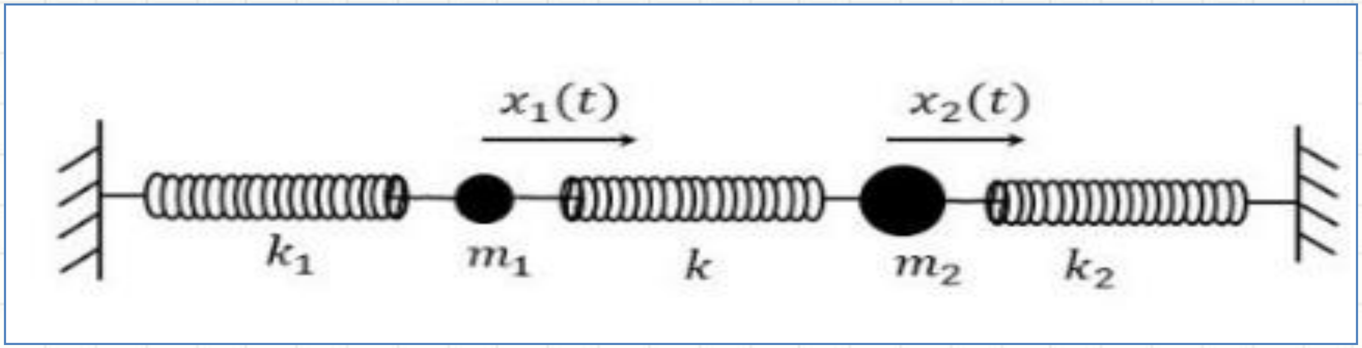

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_1} - \dfrac{\partial L}{\partial x_1} = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_2} - \dfrac{\partial L}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} m\ddot{x}_1 + k_1 x_1 = 0 \\ m\ddot{x}_2 + k_2 x_2 = 0 \end{cases}$$

$$\begin{cases} x_1(t) = A\cos(\omega_{n_1} t + \varphi_1) \\ x_2(t) = B\cos(\omega_{n_2} t + \varphi_2) \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 1-Introduction

✓*Coupled system*

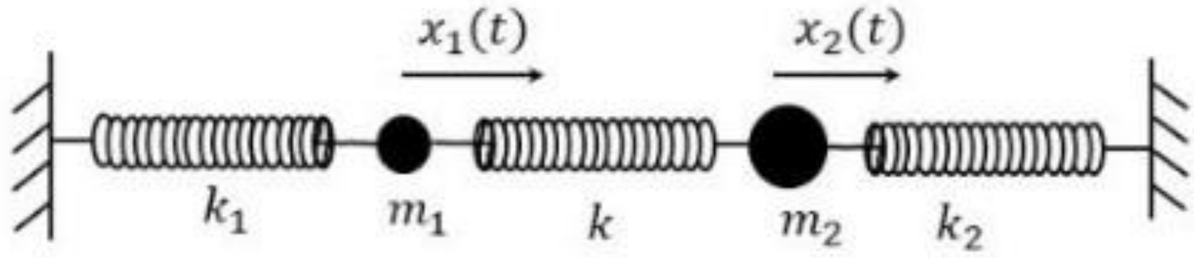



$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + (k_2 + k)x_2 - kx_1 = 0 \end{cases}$$

$$\begin{cases} x_1(t) = a_{11}\cos(w_1 t + \varphi_1) + a_{12}\cos(w_2 t + \varphi_2) \\ x_1(t) = a_{22}\cos(w_1 t + \varphi_1) + a_{21}\cos(w_2 t + \varphi_2) \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*




$$T = \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2$$

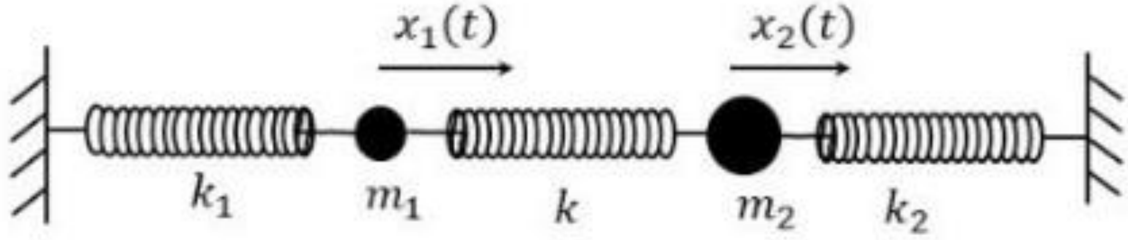
$$U = \frac{1}{2}k_1x_1{}^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k_2x_2{}^2$$

$$L = T - U \quad \Rightarrow \quad L = \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2 - \left(\frac{1}{2}k_1x_1{}^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k_2x_2{}^2\right)$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial(\mathrm{L})}{\partial\dot{\mathrm{q}}_1}\right) - \frac{\partial(\mathrm{L})}{\partial\mathrm{q}_1} = 0 \\ \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial(\mathrm{L})}{\partial\dot{\mathrm{q}}_2}\right) - \frac{\partial(\mathrm{L})}{\partial\mathrm{q}_2} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial(\mathrm{L})}{\partial\dot{x}_1}\right) - \frac{\partial(\mathrm{L})}{\partial x_1} = 0 \\ \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial(\mathrm{L})}{\partial\dot{x}_2}\right) - \frac{\partial(\mathrm{L})}{\partial x_{21}} = 0 \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## *2- Free vibration of 2DOF*




$$\begin{cases} m_1\ddot{x}_1 + k_1x_1 - k(x_2 - x_1) = 0 \\ m_2\ddot{x}_2 + k_2x_2 + k(x_2 - x_1) = 0 \end{cases} \Rightarrow \begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + (k_2 + k)x_2 - kx_1 = 0 \end{cases}$$

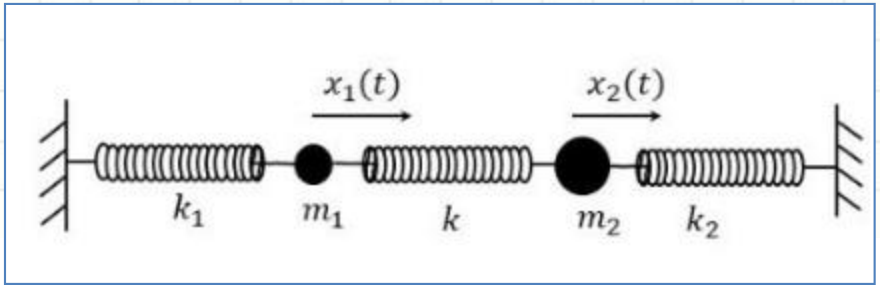
- Using complex representation

$$\begin{cases} x_1(t) = A_1\cos(wt + \varphi_1) \\ x_2(t) = A_2\cos(wt + \varphi_2) \end{cases} \Rightarrow \begin{cases} \overline{x_1(t)} = \overline{A_1}e^{jw} \\ \overline{x_2(t)} = \overline{A_2}e^{jw} \end{cases}$$

$$\Rightarrow \begin{cases} (-m_1w^2 + k_1 + k)\overline{A_1} - k\overline{A_2} = 0 \\ (-m_2w^2 + k_2 + k)\overline{A_2} - k\overline{A_1} = 0 \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*




$$\begin{cases} (-m_1w^2 + k_1 + k)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + (-m_2w^2 + k_2 + k)\overline{A_2} = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

- Which represents two simultaneous homogeneous algebraic equations in the unknowns $A_1$ and $A_2$.
- It can be seen that the above equation can be satisfied by the trivial solution $A_1 = A_2 = 0$, which implies that there is no vibration.
- For a nontrivial solution of $A_1$ and $A_2$, the determinant must be zero.

$$\det(w) = 0 \qquad \begin{vmatrix} -m_1w^2 + k_1 + k & -k \\ -k & -m_2w^2 + k_2 + k \end{vmatrix} = 0$$

## 2- Free vibration of *2DOF*

$$(-m_1w^2 + k_1 + k)(-m_2w^2 + k_2 + k) - k^2 = 0 \quad \Rightarrow \quad w^4 + C_1w^2 + C_2 = 0$$

- ✓ We only take the positive values of w

- ➢ ***For*** $\boldsymbol{k_2 = k_1 = k}$ ***and*** $\boldsymbol{m_2 = m_2 = m}$

$$(-mw^2 + 2k)^2 - k^2 = 0 \quad \Rightarrow \quad (-mw^2 + 2k - k)(-mw^2 + 2k + k) = 0$$

$$(-mw^2 + k)(-mw^2 + 3k) = 0 \quad \Rightarrow \quad \begin{cases} w_1 = \sqrt{\dfrac{k}{m}} \\ w_2 = \sqrt{\dfrac{3k}{m}} \end{cases}$$

## 2- Free vibration of *2DOF*

✓ $w_1$ *and* $w_2$ *are the natural frequencies of the system*

- The lowest frequency term called fundamental
- The other term is called harmonic

In this case $\Rightarrow$

$$\begin{cases} w_1 = \sqrt{\dfrac{k}{m}} & \textit{Fundamental} \\ w_2 = \sqrt{\dfrac{3k}{k}} & \textit{Harmonic} \end{cases}$$

## 2- Free vibration of *2DOF*

- The solution is the combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$x(t) \quad \Rightarrow \quad \begin{cases} x_1(t) = a_{11}\cos(w_1 t + \varphi_1) + a_{12}\cos(w_2 t + \varphi_2) \\ x_2(t) = a_{22}\cos(w_1 t + \varphi_1) + a_{21}\cos(w_2 t + \varphi_2) \end{cases}$$

- For $w = w_1 = \sqrt{\frac{k}{m}}$

$$\begin{cases} (-m_1 w^2 + k_1 + k)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + (-m_2 w^2 + k_2 + k)\overline{A_2} = 0 \end{cases} \dots\dots\dots(1) \Rightarrow \begin{cases} (-mw^2 + 2k)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + (-mw^2 + 2k)\overline{A_2} = 0 \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*

$$\begin{cases} \left(-m\left(\sqrt{\frac{k}{m}}\right)^2 + 2k\right)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + \left(-m\left(\sqrt{\frac{k}{m}}\right)^2 + 2k\right)\overline{A_2} = 0 \end{cases} \Rightarrow \begin{cases} k\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + k\overline{A_2} = 0 \end{cases}$$

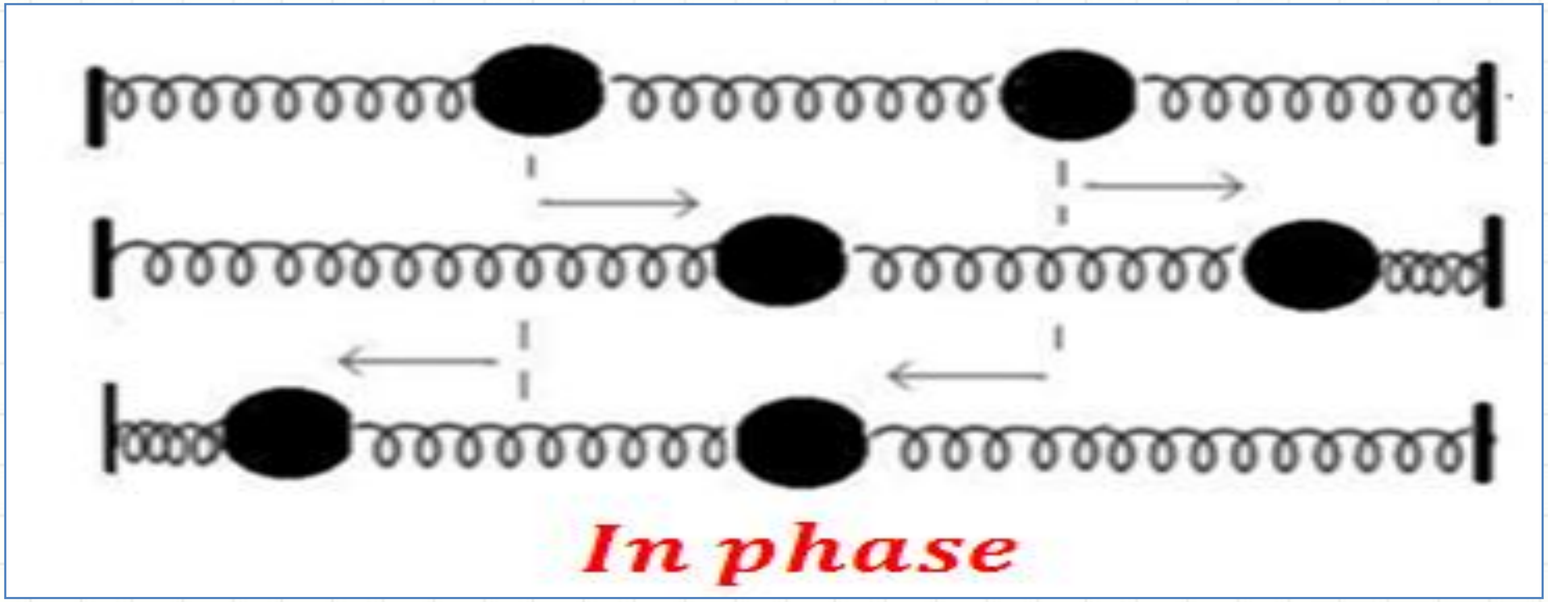
- To study the motion of mass $m_1$ (with coordinate $x_1$) relative to the motion of mass $m_2$ (with coordinate $x_2$), we calculate the ratio $\frac{\overline{A_1}}{\overline{A_2}}$ :

$$\frac{\overline{A_1}}{\overline{A_2}} = 1 > 0 \quad \Rightarrow \quad \frac{\overline{A_1}}{\overline{A_2}} = \frac{A_1 e^{j\varphi_1}}{A_2 e^{j\varphi_2}} = 1 \quad \Rightarrow \quad \frac{A_1}{A_2} e^{j(\varphi_1 - \varphi_2)} = 1e^{j0}$$

$\Delta\varphi = 0 \quad \Rightarrow$ The two masses oscillate in phase

# Chapter 5 : Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*



# Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*

- For $w = w_2 = \sqrt{\frac{3k}{m}}$

$$\begin{cases} (-mw^2 + 2k)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + (-mw^2 + 2k)\overline{A_2} = 0 \end{cases} \Rightarrow \begin{cases} \left(-m\left(\sqrt{\frac{3k}{m}}\right)^2 + 2k\right)\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} + \left(-m\left(\sqrt{\frac{3k}{m}}\right)^2 + 2k\right)\overline{A_2} = 0 \end{cases}$$

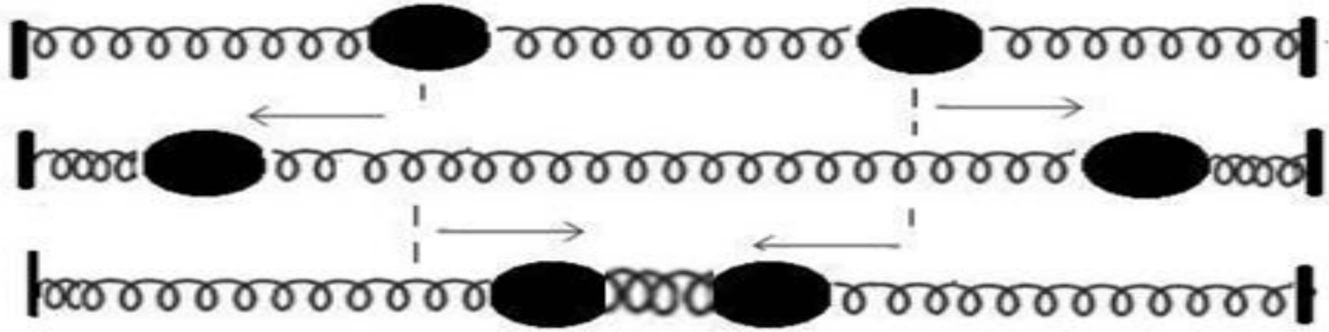
$$\Rightarrow \quad \begin{cases} -k\overline{A_1} - k\overline{A_2} = 0 \\ -k\overline{A_1} - k\overline{A_2} = 0 \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 2- Free vibration of *2DOF*

$$\frac{\overline{A_1}}{\overline{A_2}} = -1 < 0 \quad \Rightarrow \quad \frac{\overline{A_1}}{\overline{A_2}} = \frac{A_1 e^{j\varphi_1}}{A_2 e^{j\varphi_2}} = -1 \quad \Rightarrow \quad \frac{A_1}{A_2} e^{j(\varphi_1 - \varphi_2)} = 1 e^{j\pi}$$

$\Delta\varphi = \pi \quad \Rightarrow$ The two masses oscillate in opposite phase

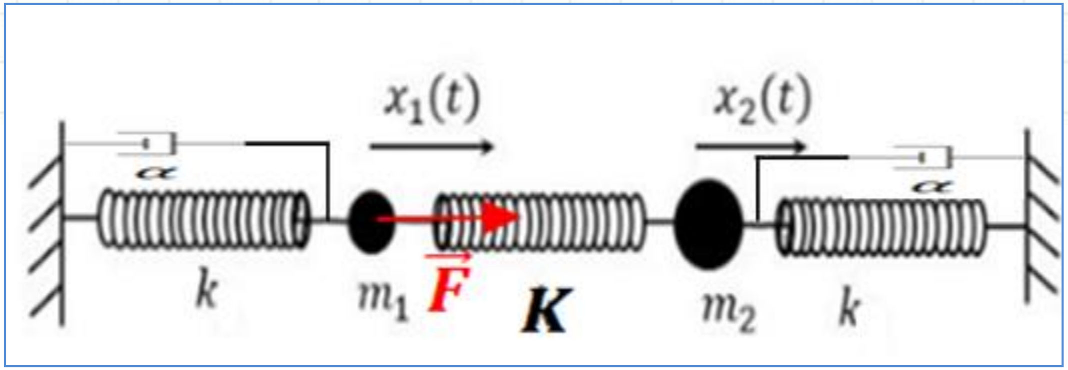


*In opposite phase*

The solution $x(t) \quad \Rightarrow \quad \begin{cases} x_1(t) = a_{12}\cos(w_2 t + \varphi_2) \\ x_1(t) = a_{21}\cos(w_2 t + \varphi_2) \end{cases}$

# Free and forced vibration of two degree of freedom systems

## *3*- Forced vibration of *2DOF*




$$F_{ext} = F_0 \cos\Omega t$$

$$T = \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2$$

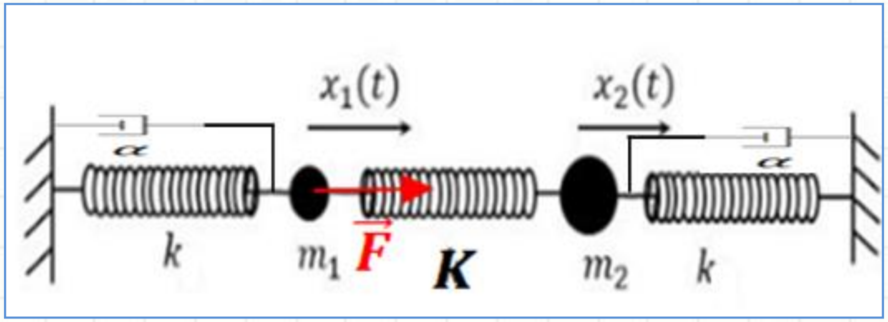
$$U = \frac{1}{2}kx_1{}^2 + \frac{1}{2}K(x_2 - x_1)^2 + \frac{1}{2}kx_2{}^2$$

$$D = D_1 + D_2 \quad \Rightarrow \quad D = \frac{1}{2}\alpha\dot{x}_1{}^2 + \frac{1}{2}\alpha\dot{x}_2{}^2$$

$$L = T - U \quad \Rightarrow \quad L = \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2 - \left(\frac{1}{2}kx_1{}^2 + \frac{1}{2}K(x_2 - x_1)^2 + \frac{1}{2}kx_2{}^2\right)$$

# Free and forced vibration of two degree of freedom systems

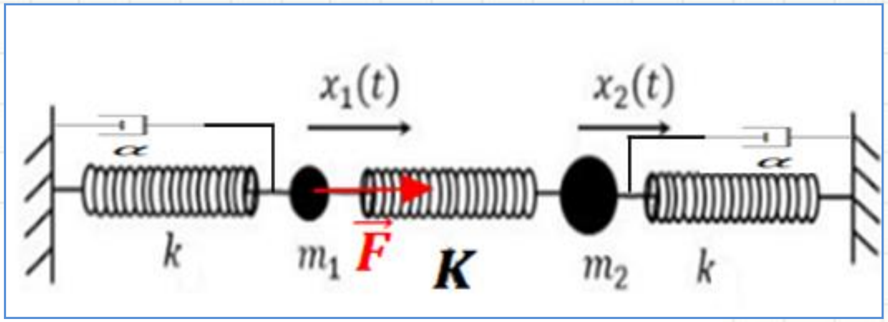
## 3- Forced vibration of *2DOF*




$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial(L)}{\partial \dot{q}_1}\right) - \dfrac{\partial(L)}{\partial q_1} + \dfrac{\partial D}{\partial \dot{q}_1} = F_{eq1} \\ \dfrac{d}{dt}\left(\dfrac{\partial(L)}{\partial \dot{q}_2}\right) - \dfrac{\partial(L)}{\partial q_2} + \dfrac{\partial D}{\partial \dot{q}_2} = F_{eq2} \end{cases} \Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial(L)}{\partial \dot{x}_1}\right) - \dfrac{\partial(L)}{\partial x_1} + \dfrac{\partial D}{\partial \dot{x}_1} = F_{ext} \\ \dfrac{d}{dt}\left(\dfrac{\partial(L)}{\partial \dot{x}_2}\right) - \dfrac{\partial(L)}{\partial x_2} + \dfrac{\partial D}{\partial \dot{x}_2} = 0 \end{cases}$$

$$\begin{cases} m_1\ddot{x}_1 + kx_1 - K(x_2 - x_1) + \alpha\dot{x}_1 = F_0 \cos\Omega t \\ m_2\ddot{x}_2 + kx_2 + K(x_2 - x_1) + \alpha\dot{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} m_1\ddot{x}_1 + \alpha\dot{x}_1 + (k + K)x_1 - Kx_2 = F_0 \cos\Omega t \\ m_2\ddot{x}_2 + \alpha\dot{x}_2 + (k+K)x_2 - Kx_1 = 0 \end{cases}$$

## 3- Forced vibration of *2DOF*




- ✓ Steady state (permanent) response

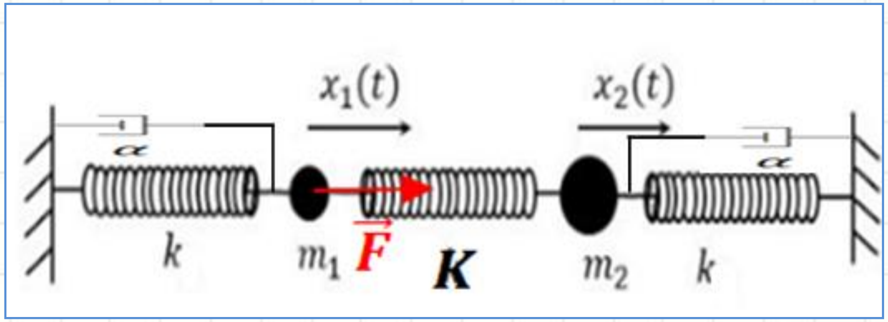
- ➢ Using complex representation

$$\begin{cases} x_1(t) = A_1 \cos(\Omega t + \varphi_1) \\ x_2(t) = A_2 \cos(\Omega t + \varphi_2) \end{cases} \Rightarrow \begin{cases} \overline{x_1(t)} = \overline{A_1} e^{j\Omega} \\ \overline{x_2(t)} = \overline{A_2} e^{j\Omega} \end{cases}$$

$$\mathrm{F} = \mathrm{F}_0 \cos\Omega \mathrm{t} \quad \Rightarrow \quad \overline{F(t)} = \mathrm{F}_0 e^{j\Omega}$$

$$\Rightarrow \begin{cases} \left(-m_1\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_1} - K\overline{A_2} = \mathrm{F}_0 \\ \left(-m_2\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_2} - K\overline{A_1} = 0 \end{cases}$$

## 3- Forced vibration of *2DOF*

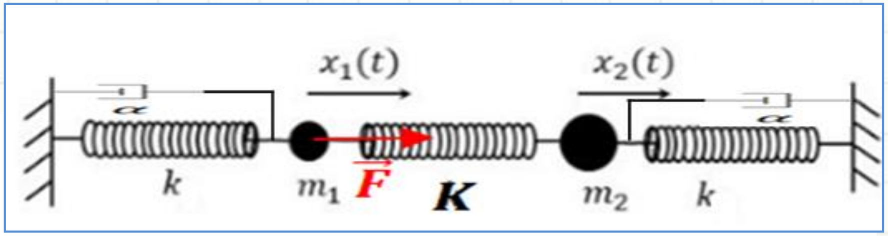



$$\begin{cases} \left(-m_1\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_1} - K\overline{A_2} = F_0 \\ -K\overline{A_1} + \left(-m_2\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_2} = 0 \end{cases} \quad \text{..........................(1)}$$

***1- Case $\alpha = 0$*** , $m_1 = m_2 = m$ *and* $K = k$

$$\begin{cases} \left(-m_1\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_1} - K\overline{A_2} = F_0 \\ -K\overline{A_1} + \left(-m_2\Omega^2 + j\Omega\alpha + K + k\right)\overline{A_2} = 0 \end{cases} \Rightarrow \begin{cases} \left(-m\Omega^2 + 2k\right)\overline{A_1} - k\overline{A_2} = F_0 \\ -k\overline{A_1} + \left(-m\Omega^2 + 2k\right)\overline{A_2} = 0 \end{cases}$$

## 3- Forced vibration of *2DOF*




- $\overline{A_1}$ *and* $\overline{A_2}$ ?

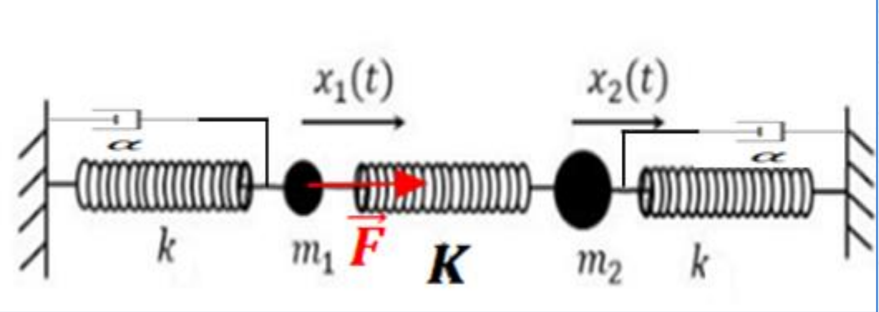
$$\begin{cases} \left(-m\Omega^2+2k\right)\overline{A_1}-k\overline{A_2}=\mathrm{F}_0 \ldots\ldots\ldots\ldots(1) \\ -k\overline{A_1}+\left(-m\Omega^2+2k\right)\overline{A_2}=0\ldots\ldots\ldots\ldots(2) \end{cases}$$

$$(2)\quad \ldots\ldots\ldots\quad \overline{A_2}=\frac{k\overline{A_1}}{\left(-m\Omega^2+2k\right)}$$

$$(2)\ and\ (1)\ \Rightarrow\ \left(-m\Omega^2+2k\right)\overline{A_1}-k\frac{k\overline{A_1}}{\left(-m\Omega^2+2k\right)}=\mathrm{F}_0\ \Rightarrow\ \overline{A_1}=\frac{\mathrm{F}_0\left(-m\Omega^2+2k\right)}{\left(-m\Omega^2+2k\right)^2-k^2}$$

$$\overline{A_2}=\frac{k\overline{A_1}}{\left(-m\Omega^2+2k\right)}\ \Rightarrow\ \overline{A_2}=\frac{\mathrm{F}_0\left(-m\Omega^2+2k\right)}{\left(-m\Omega^2+2k\right)^2-k^2}\cdot\frac{k}{\left(-m\Omega^2+2k\right)}\ \Rightarrow\ \overline{A_2}=\frac{\mathrm{F}_0 k}{\left(-m\Omega^2+2k\right)^2-k^2}$$

## *3-* Forced vibration of *2DOF*



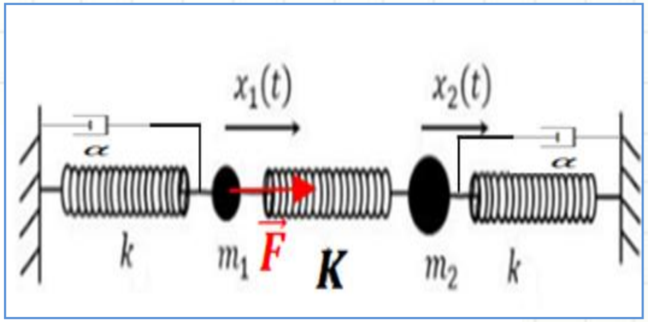

- $\overline{A_1} = \dfrac{F_0(-m\Omega^2+2k)}{(-m\Omega^2+2k+K)(-m\Omega^2+2k-k)}$

$$\overline{A_1} = \frac{F_0(-m\Omega^2+2k)}{(-m\Omega^2+3k)(-m\Omega^2+k)} \Rightarrow \overline{A_1} = \frac{F_0\left(-\Omega^2+\frac{2k}{m}\right)}{m\left(-\Omega^2+\frac{3k}{m}\right)\left(-\Omega^2+\frac{k}{m}\right)}$$

- $\overline{A_1} = \dfrac{F_0\left(\frac{2k}{m}-\Omega^2\right)}{m\left({w_2}^2-\Omega^2\right)\left({w_1}^2-\Omega^2\right)}$

# Free and forced vibration of two degree of freedom systems

## *3*- Forced vibration of *2DOF*



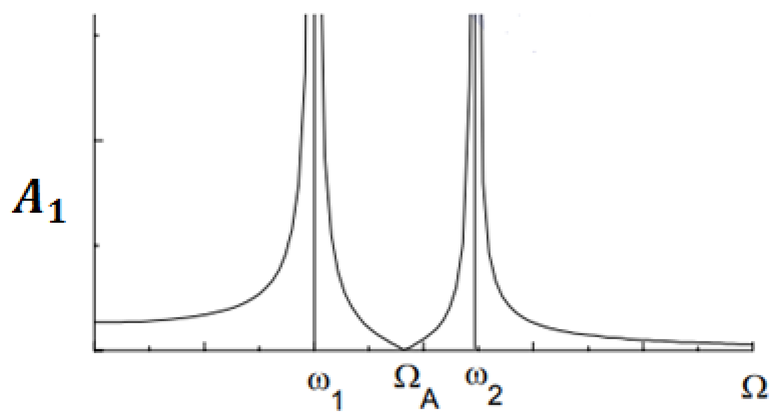

- $\overline{A_2} = \dfrac{F_0 k}{(-m\Omega^2+2k)^2-k^2} \quad \Rightarrow \quad \overline{A_2} = \dfrac{F_0 k}{m\left(w_2{}^2-\Omega^2\right)\left(w_1{}^2-\Omega^2\right)}$

- The phenomenon of resonance occurs for $\overline{A_1}$ as well as for $\overline{A_2}$ when the excitation pulsation $\Omega$ is equal to one of the natural pulsations $w_1$ or $w_2$ of the system.

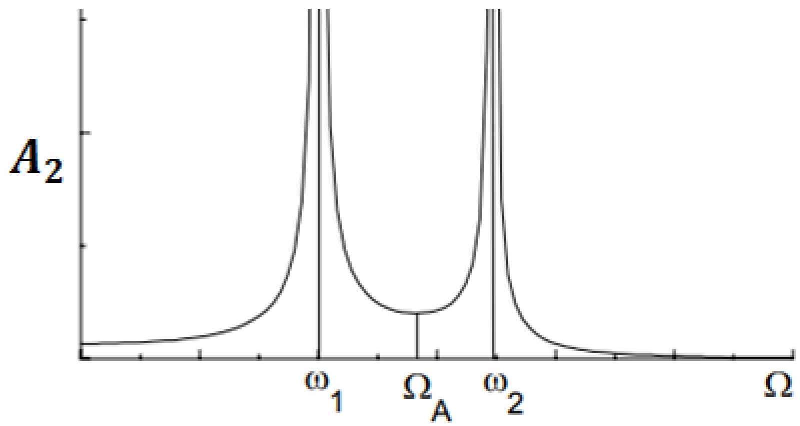
- $\overline{A_1} = 0 \quad for \quad \Omega = \Omega_A = \sqrt{\dfrac{2k}{m}}$ , $\Omega_A$: called the antiresonance pulsation

# Free and forced vibration of two degree of freedom systems

## *3*- Forced vibration of *2DOF*




Variation of $A_1$ as a function of $\Omega$




Variation of $A_2$ as a function of $\Omega$

# Free and forced vibration of two degree of freedom systems

## *3- Forced vibration of 2DOF*

### *3-1- Mechanical impedance*

Is a physical quantity that characterizes a mechanical system's resistance to motion when subjected to an oscillatory force. It is defined as the complex ratio between the applied force F(t) and The velocity v(t) of the point of application of this force:

$$\overline{Z_m} = \frac{\overline{F}}{\overline{v}}$$

- In harmonic motion, Z can be expressed as a complex function dependent on the frequency Ω.

$$\overline{Z_m} = R_e + j\ Im$$

$R_e$: The real part, called mechanical resistance (energy dissipated)

$Im$: The imaginary part, called mechanical reactance (energy stored as kinetic or potential energy

# Free and forced vibration of two degree of freedom systems

## 3- Forced vibration of *2DOF*

### *3-1- Mechanical impedance*

- **Damper**

$$F = \alpha v \quad \Rightarrow \quad Z = \alpha$$

- **Mass**

$$F = m\frac{dv}{dt} \quad \Rightarrow \quad Z = jm\,\Omega \quad \Rightarrow \quad Z = m\Omega e^{j\frac{\pi}{2}}$$

- **Spring**

$$F = kx \quad \Rightarrow \quad Z = k\int vdt \quad \Rightarrow \quad Z = \frac{1}{j\Omega}k \quad Z = \frac{k}{\Omega}e^{-j\frac{\pi}{2}}$$

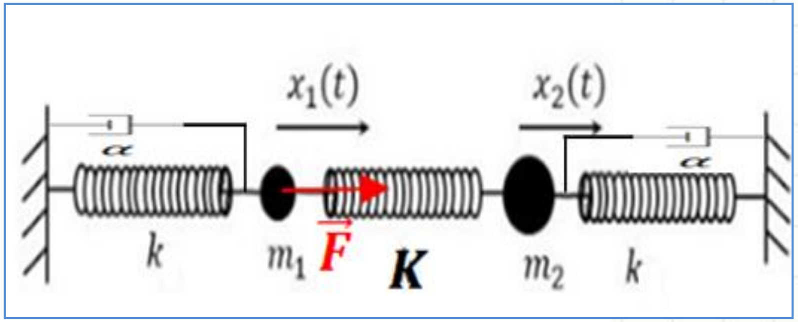
# Free and forced vibration of two degree of freedom systems

## *3*- Forced vibration of *2DOF*

### *3-1- Mechanical impedance*

- Consider the two-degree-of-freedom system studied in the previous System

  - ✓ Mechanical input impedance $Z_i = \frac{F(t)}{\dot{x}_1}$

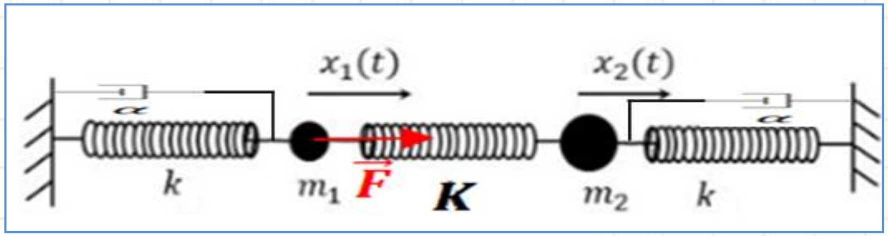



- Integro-differential equations $\Rightarrow \begin{cases} \text{Mechanical input impedance} \\ Mechanical - Electrical\ analogy \end{cases}$

# Free and forced vibration of two degree of freedom systems

## 3- Forced vibration of *2DOF*

### *3-1- Mechanical impedance*




$$\begin{cases} m_1\ddot{x}_1 + \alpha\dot{x}_1 + (k + K)x_1 - Kx_2 = F(t) \\ m_2\ddot{x}_2 + \alpha\dot{x}_2 + (k+K)x_2 - Kx_1 = 0 \end{cases} \Rightarrow \begin{cases} \overline{x_1(t)} = \overline{A_1}e^{j\Omega} \\ \overline{x_2(t)} = \overline{A_2}e^{j\Omega} \end{cases}$$

$$\Rightarrow \begin{cases} m_1\dfrac{d\dot{x}_1}{dt} + \alpha\dot{x}_1 + (k + K)\displaystyle\int \dot{x}_1 dt - K\int \dot{x}_2 dt = F(t) \\ m_2\dfrac{d\dot{x}_2}{dt} + \alpha\dot{x}_2 + (k + K)\displaystyle\int \dot{x}_2 dt - K\int \dot{x}_1 dt = 0 \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## *3*- Forced vibration of *2DOF*

### *3-1- Mechanical impedance*

$$\begin{cases} j\Omega m_1\overline{\dot{x}_1} + \alpha\overline{\dot{x}_1} + (k+K)\dfrac{\overline{\dot{x}_1}}{j\Omega} - K\dfrac{\overline{\dot{x}_2}}{j\Omega} = \overline{F} \\ j\Omega m_2\overline{\dot{x}_2} + \alpha\overline{\dot{x}_2} + (k+K)\dfrac{\overline{\dot{x}_2}}{j\Omega} - K\dfrac{\overline{\dot{x}_1}}{j\Omega} = 0 \end{cases} \Rightarrow \begin{cases} \left(j\Omega m_1 + \alpha + (k+K)\dfrac{1}{j\Omega}\right)\overline{\dot{x}_1} - K\dfrac{\overline{\dot{x}_2}}{j\Omega} = \overline{F} \\ \left(j\Omega m_2 + \alpha + (k+K)\dfrac{1}{j\Omega}\right)\overline{\dot{x}_2} - K\dfrac{\overline{\dot{x}_1}}{j\Omega} = 0 \end{cases} \quad \ldots(3)$$

- **For undamped oscillations $\alpha = 0$ ,** $m_1 = m_2 = m$ *and* $K = k$

$$(3) \Rightarrow \begin{cases} \left(j\Omega\, m - j\dfrac{2k}{\Omega}\right)\overline{\dot{x}_1} + j\dfrac{k}{\Omega}\overline{\dot{x}_2} = \overline{F} \\ \left(j\Omega\, m - j\dfrac{2k}{\Omega}\right)\overline{\dot{x}_2} + j\dfrac{k}{\Omega}\overline{\dot{x}_1} = 0 \end{cases} \Rightarrow \begin{cases} \left(j\Omega\, m - j\dfrac{2k}{\Omega}\right)\overline{\dot{x}_1} + j\dfrac{k}{\Omega}\overline{\dot{x}_2} = \overline{F} \\ \overline{\dot{x}_2} = -\dfrac{j\frac{K}{\Omega}}{j\Omega\, m - j\frac{2k}{\Omega}}\overline{\dot{x}_1} \end{cases}$$

# Free and forced vibration of two degree of freedom systems

## 3- Forced vibration of *2DOF*

### *3-1- Mechanical impedance*

$$\begin{cases} \left(j\Omega\, m - j\,\frac{2k}{\Omega}\right)\overline{\dot{x}_1} - j\,\frac{(\frac{k}{\Omega})^2}{\Omega\, m - \frac{2k}{\Omega}})\,\overline{\dot{x}_1} = \bar{F} \\ \overline{\dot{x}_2} = -\frac{j\frac{k}{\Omega}}{j\Omega\, m - j\frac{2k}{\Omega}}\,\overline{\dot{x}_1} \end{cases} \Rightarrow \begin{cases} (j\Omega\, m - j\,\frac{2k}{\Omega} - j\,\frac{(\frac{k}{\Omega})^2}{\Omega\, m - \frac{2k}{\Omega}})\,\overline{\dot{x}_1} = \bar{F} \\ \overline{\dot{x}_2} = -\frac{\frac{k}{\Omega}}{\Omega\, m - \frac{2k}{\Omega}}\,\overline{\dot{x}_1} \end{cases}$$

- ✓ Mechanical input impedance $Z_i = \frac{F(t)}{\dot{x}}$

$$Z_i = \mathrm{j}\left[(\Omega\, m - \frac{2k}{\Omega}) - \frac{(\frac{k}{\Omega})^2}{\Omega\, m - \frac{2k}{\Omega}})\right]$$

$$Z_i = j\,\frac{(\Omega\, m - \frac{2k}{\Omega})^2 - (\frac{k}{\Omega})^2}{\Omega\, m - \frac{2k}{\Omega}} \Rightarrow Z_i = j\,\frac{m}{\Omega}\,\frac{\left(\Omega^2 - \frac{2k}{m}\right)^2 - (\frac{k}{m})^2}{\Omega^2 - \frac{2k}{m}} \Rightarrow Z_i = j\,\frac{m}{\Omega}\,\frac{(\Omega^2 - \frac{k}{m})(\Omega^2 - \frac{3k}{m})}{\Omega^2 - \frac{2k}{m}}$$

## *3-* Forced vibration of *2DOF*

### *3-1- Mechanical impedance*

$$\Rightarrow \quad Z_i = j\frac{m}{\Omega}\,\frac{(\Omega^2-\frac{k}{m})(\Omega^2-\frac{3k}{m})}{\Omega^2-\frac{2k}{m}}$$

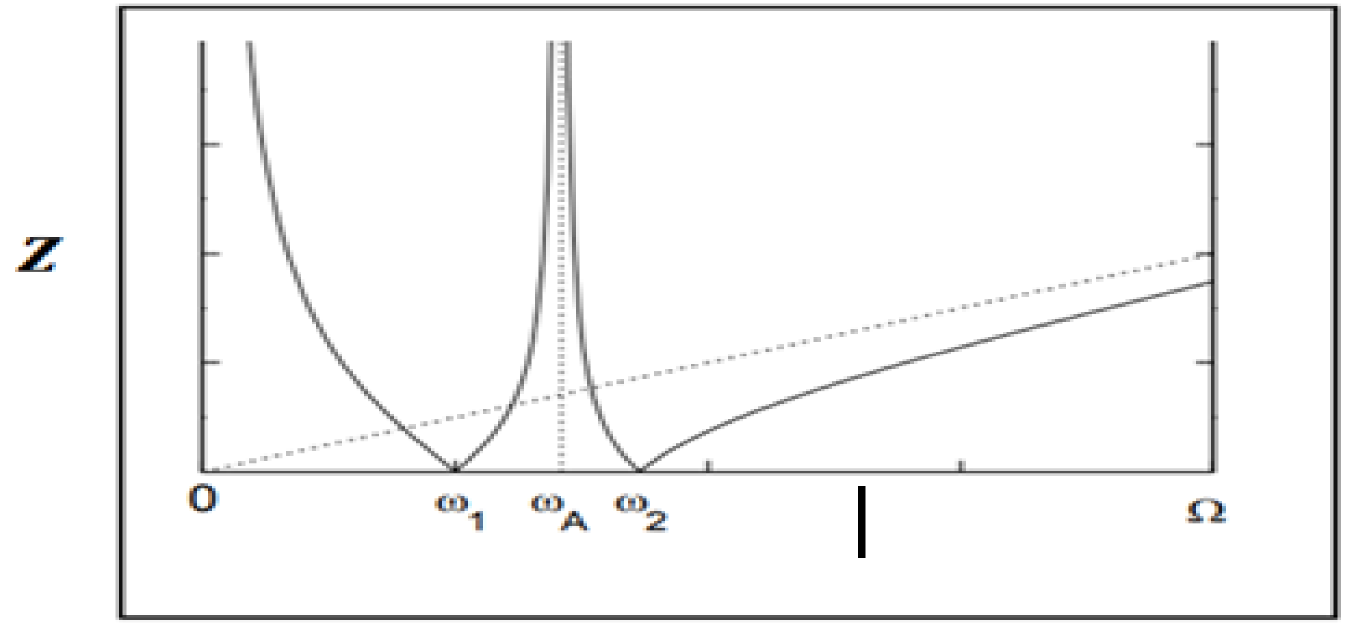
$$we\ have \quad \begin{cases} w_1 = \Omega_1 = \sqrt{\dfrac{k}{m}} \\ w_2 = \Omega_2 = \sqrt{\dfrac{3k}{m}} \\ \Omega_A = \sqrt{\dfrac{2k}{m}} \end{cases} \quad \Rightarrow \quad \boldsymbol{Z_i = j\frac{m}{\Omega}\,\frac{(\Omega^2-\Omega_1{}^2)(\Omega^2-\Omega_2{}^2)}{\Omega^2-\Omega_A{}^2}}$$

- $Z_i = 0$ for $\Omega = \Omega_1$ *and* $\Omega = \Omega_2$
- $Z_i \to \infty$ *for* $\Omega = \Omega_A$

# Free and forced vibration of two degree of freedom systems

## *3-* Forced vibration of *2DOF*

### *3-1- Mechanical impedance*




Variation of $Z$ as a function of $\Omega$

## 5- Mechanical-Electrical analogy

- Integro-differential equations $\Rightarrow \begin{cases} \text{Mechanical input impedance} \\ \textit{Mechanical} - \textit{Electrical analogy} \end{cases}$

| Mechanical system | Electrical system |
|---|---|
| $m\ddot{x} + \alpha\dot{x} + kx = F$ | $L\ddot{q} + R\dot{q} + \frac{q}{c} = e(t)$ |
| Elongation $\boldsymbol{x}$ | Charge $\boldsymbol{q}$ |
| Velocity $\dot{\boldsymbol{x}}$ | Current $\boldsymbol{i}$ |
| Mass $\boldsymbol{m}$ | Inductance $\boldsymbol{L}$ |
| Spring $\boldsymbol{k}$ | Inverse of capacitance $\frac{1}{C}$ |
| Coefficient $\alpha$ | Resistance R |
| F | $e(t)$ |

$$m \equiv L$$
$$k \equiv \frac{1}{C}$$
$$x \equiv q$$
$$\dot{x} \equiv i$$
$$\alpha \equiv R$$
$$F(t) = e(t)$$

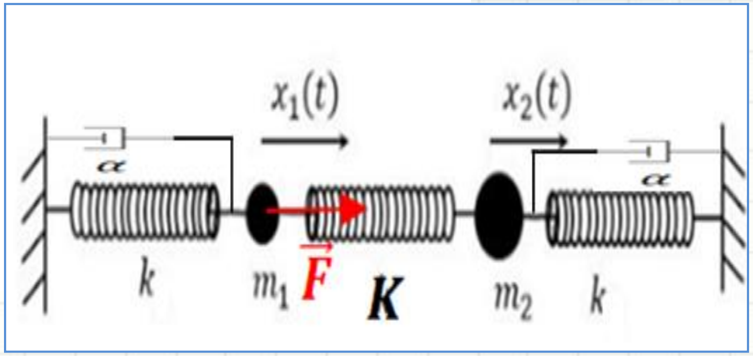
# Free and forced vibration of two degree of freedom systems

## 5- Mechanical-Electrical analogy

- Integro-differential equations $\Rightarrow$ $\begin{cases} \text{Mechanical input impedance} \\ \textit{Mechanical} - \textit{Electrical analogy} \end{cases}$

- Consider the two-degree-of-freedom system studied in the previous System

- $$\begin{cases} m_1\ddot{x}_1 + kx_1 - K(x_2 - x_1) + \alpha\dot{x}_1 = F_0 \cos\Omega t \\ m_2\ddot{x}_2 + kx_2 + K(x_2 - x_1) + \alpha\dot{x}_2 = 0 \end{cases}$$



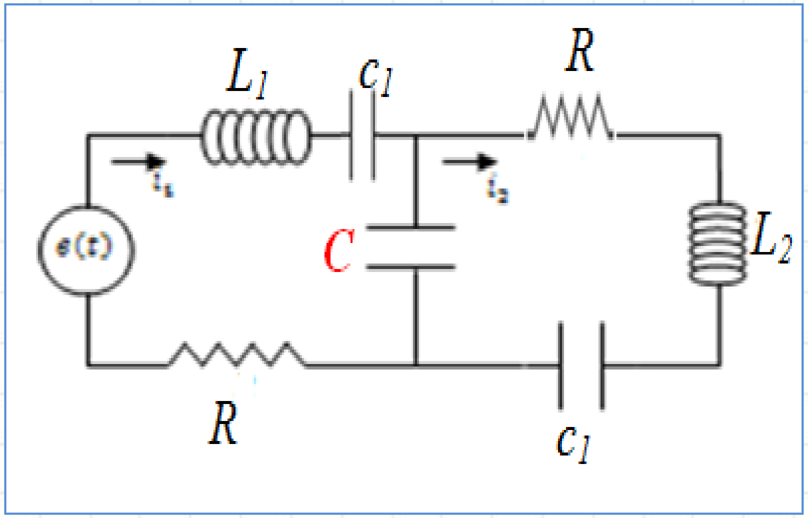
## 5- Mechanical-Electrical analogy

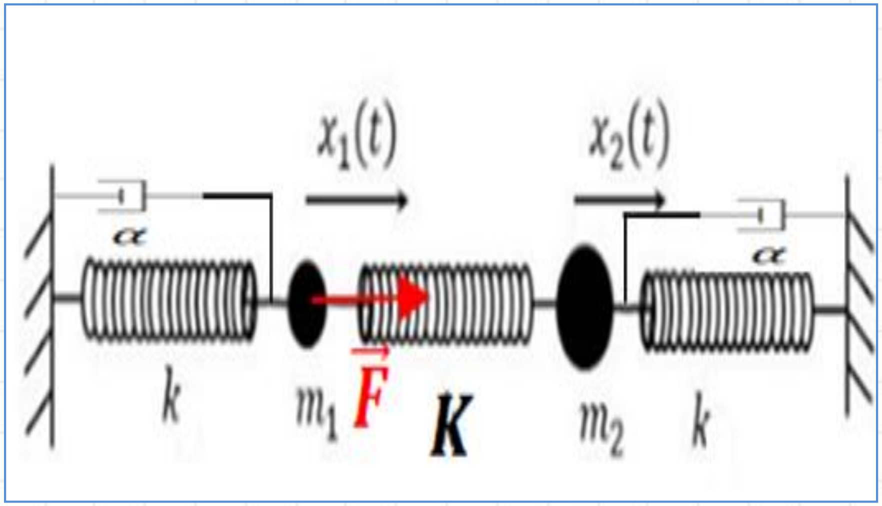
$$\Rightarrow \begin{cases} m_1 \dfrac{d\dot{x}_1}{dt} + \alpha \dot{x}_1 + K\displaystyle\int (\dot{x}_1 - \dot{x}_2)dt + k\int \dot{x}_1 dt = \mathrm{F(t)} \\ m_2 \dfrac{d\dot{x}_2}{dt} + \alpha \dot{x}_2 + K\displaystyle\int (\dot{x}_2 - \dot{x}_1)dt + k\int \dot{x}_2 dt = 0 \end{cases}$$

$m_1 \equiv L_1$ ; $m_2 \equiv L_2$ ; $\alpha \equiv R$ ; $K \equiv \frac{1}{C}$ ; $k \equiv \frac{1}{C_1}$ ; $\mathrm{F(t)} \equiv e(t)$ ; $\dot{x}_1 \equiv i_1$; $\dot{x}_2 \equiv i_2$

$$\Rightarrow \begin{cases} L_1 \dfrac{di_1}{dt} + R\dot{x}_1 + \dfrac{1}{C}\displaystyle\int (i_1 - i_2)dt + k\int i_1 dt = e(t) \\ L_2 \dfrac{di_2}{dt} + R\dot{x}_2 + \dfrac{1}{C}\displaystyle\int (i_2 - i_1)dt + k\int i_2 dt = 0 \end{cases}$$

## *5- Mechanical-Electrical analogy*



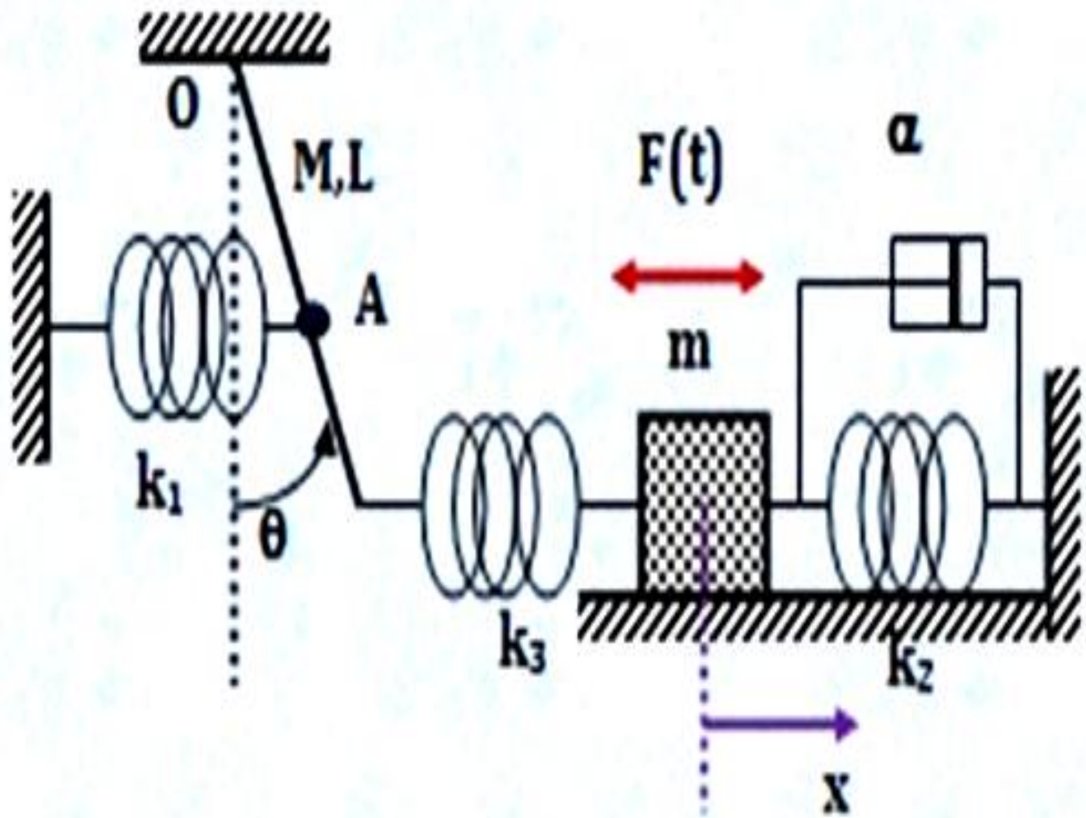



Figure 2