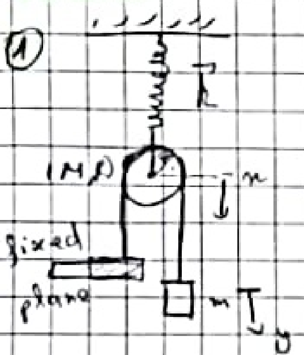
5.96

Set N°1

Degrees of freedom, kinetic & potential energy

Ex 1:

①



$DOF = 6N + 3n - C$

$n = 0$

$N = 2$

$m \begin{cases} T_x = 0 & R_x = 0 \\ T_y \neq 0 & R_y = 0 \\ T_z = 0 & R_z = 0 \end{cases} \quad C = 5$

$M \begin{cases} T_x = 0 & R_x = 0 \\ T_y \neq 0 & R_y = 0 \\ T_z = 0 & R_z \neq 0 \end{cases} \quad C = 4$

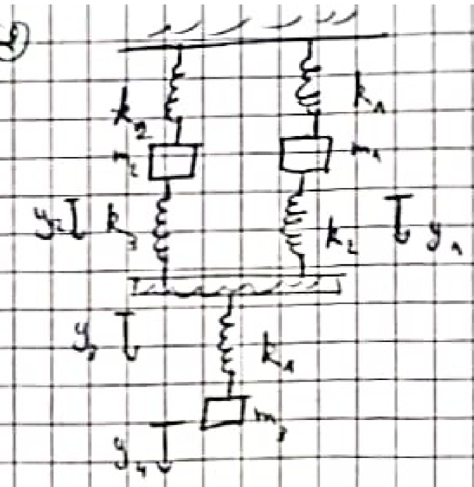
$C = 1 \quad y = 2x$

$C = 1 \quad R\theta = x$

$DOF = 12 - 11 = 1$

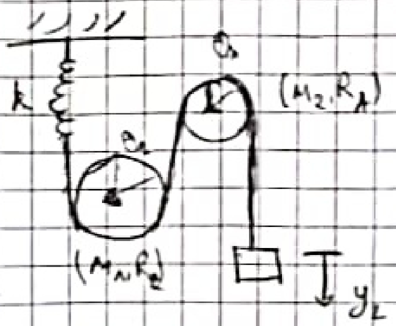
$5_C + 4_C + 1_C + 1_C$

②



$DOF = 4 \qquad C = 0$

③

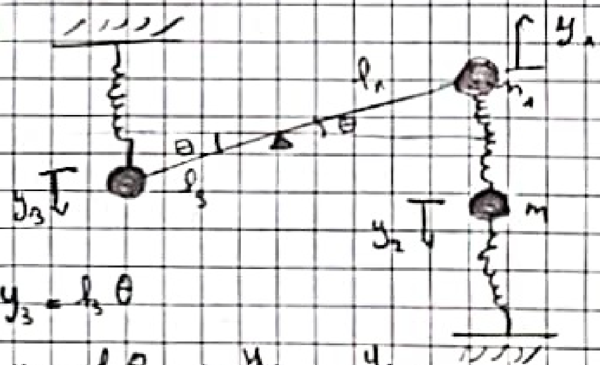


$y_2 = R_1 \theta_1 \text{ ---- } C = 1$

$R_2 \theta_2 = R_1 \theta_1 \text{ ---- } C = 1$

$DOF = 3 - 2 = 1$

④



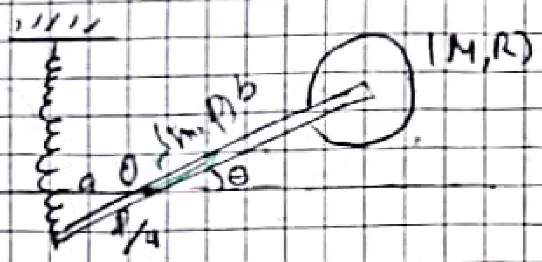
$y_3 = l_3 \theta$

$y_1 = l_1 \theta \Rightarrow \frac{y_3}{l_3} = \frac{y_1}{l_1}$

$y_3 l_1 = y_1 l_3 \longrightarrow C = 1$

$DOF = 3 - 1 = 2 \begin{pmatrix} y_2, y_1 \\ y_2, y_3 \\ y_2, \theta \end{pmatrix}$

Ex 2 :



the disk is rigidly connected

the connection is pinned

DOF = 3 - 2 = 1

DOF = 3 - 1 = 2

$\Delta = l\cos\theta$

Kinetic energy :

$$T_m + T_M$$

$$T_{m_T} + T_{m_R} \qquad T_{M_T} + T_{M_R}$$

$$\vec{r}_{cm}\begin{cases}\frac{b-a}{2}\cos\theta\\ \frac{b-a}{2}\sin\theta\end{cases} \qquad b = \frac{3}{4}l \qquad a = \frac{1}{4}l$$

$$\vec{r}_{cm} = \begin{cases}\frac{1}{4}l\cos\theta\\ \frac{1}{4}l\sin\theta\end{cases}$$

$$d\vec{r}_{cm} = \begin{cases}-\frac{1}{4}l\sin\theta\, d\theta\\ \frac{1}{4}l\cos\theta\, d\theta\end{cases}$$

$$\vec{v}_m = \begin{cases}-\frac{1}{4}l\sin\theta\,\dot\theta\\ \frac{1}{4}l\cos\theta\,\dot\theta\end{cases} \Rightarrow v = \sqrt{\left(\frac{1}{4}l\dot\theta\right)^2}$$

$$T_m = T_{m_T} + T_{m_R} = \frac{1}{2}mv^2 + \frac{1}{2}I_{\Delta}\dot\theta^2 \qquad I_{\Delta} \to \frac{1}{12}ml^2$$

$$= \frac{1}{2}m\left(\frac{1}{4}l\dot\theta\right)^2 + \frac{1}{2}\,\frac{1}{12}ml^2\dot\theta^2$$

$$T_m = \frac{1}{32}ml^2\dot\theta^2 + \frac{1}{24}ml^2\dot\theta^2$$

$$\boxed{T_m = \frac{7}{96}ml^2\dot\theta^2}$$

$$I_{/\Delta'} = I_{/\Delta} + md^2$$

$$I_{/\Delta'} = \frac{1}{12}ml^2 + m\left(\frac{l}{4}\right)^2$$

$$I_{/\Delta'} = \frac{7}{48}ml^2$$

$$\boxed{T_m = \frac{1}{2}I_{\Delta}\dot\theta^2 = \frac{7}{96}ml^2\dot\theta^2}$$

$$T_M = T_{M(R)} + T_{M(T)}$$

$$\vec{OR}_{cM}\begin{cases}\frac{3l}{4}\cos\theta\\ \frac{3l}{4}\sin\theta\end{cases}$$

$$d\vec{OR}_{cM}\begin{cases}-\frac{3l}{4}\sin\theta\\ \frac{3l}{4}\cos\theta\end{cases}$$

$$\vec{v}_{cM}\begin{cases}-\frac{3l}{4}\sin\theta\,\dot\theta\\ \frac{3l}{4}\cos\theta\,\dot\theta\end{cases} \qquad v_M = \frac{3l}{4}\dot\theta$$

$$T_M = \frac{1}{2}Mv_{cM}^2 + \frac{1}{2}I_{/\Delta}\dot\theta^2$$

$$= \frac{1}{2}M\left(\frac{3l}{4}\dot\theta\right)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot\theta^2$$

$$\boxed{T_M = \frac{9}{32}Ml^2\dot\theta^2 + \frac{1}{4}MR^2\dot\theta^2}$$

$$T = T_m + T_M$$

$$= \frac{7}{96}ml^2\dot\theta^2 + \frac{9}{32}Ml^2\dot\theta^2 + \frac{1}{4}MR^2\dot\theta^2$$

$$T = \left[\left(\frac{7}{96}m + \frac{9}{32}M\right)l^2 + \frac{1}{4}MR^2\right]\dot\theta^2$$

Potential energy :

$$U = U_m + U_M + U_k$$

$$U_m = -\int \vec{W}_m \cdot d\vec{OR}_{cm}$$

$$\vec{W}_m\begin{cases}0\\ -mg\end{cases},\quad d\vec{OR}_{cm}\begin{cases}-\frac{l}{4}\sin\theta\, d\theta\\ \frac{l}{4}\cos\theta\, d\theta\end{cases}$$

$\vec{W}_m \cdot d\vec{OX}_{cm} = -mg\frac{l}{4}\cos\theta\, d\theta$

$U_m = \int_0^\theta mg\frac{l}{4}\cos\theta\, d\theta$

$U_m = mg\frac{l}{4}[\sin\theta]_0^\theta$

$U_m = mg\frac{l}{4}\sin\theta$ / $\sin\theta \ll 1$

$\Rightarrow \sin\theta = \theta$

$U_m = mg\frac{l}{4}\theta$

$U_M = \frac{3}{4}Mgl\theta$ same steps

$U_k = \frac{1}{2}kx^2$ $\quad U_k = \frac{1}{2}k(x+\Delta l)^2$

$U_k = \frac{1}{2}k(y+\Delta l)^2$

$U_k = \frac{1}{2}k(\frac{l}{4}\theta+\Delta l)^2$

$U = \frac{1}{4}gl(m+3M)\theta + \frac{1}{2}k(\frac{l}{4}\theta+\Delta l)^2$

$\frac{dU}{d\theta}\Big|_{\theta_0} = 0$

$\frac{d^2U}{d\theta^2}\Big|_{\theta_0} > 0$

$\frac{dU}{d\theta} = \frac{1}{4}gl(m+3M) + k\frac{l}{4}(\frac{l}{4}\theta+\Delta l)$

$\frac{dU}{d\theta}\Big|_{\theta=0} \Rightarrow \frac{1}{4}gl(m+3M) + \frac{kl}{4}\Delta l = 0$

$\Delta l = -\frac{\frac{l}{4}g(m+3M)}{k\frac{l}{4}} = -\frac{g(m+3M)}{k}$

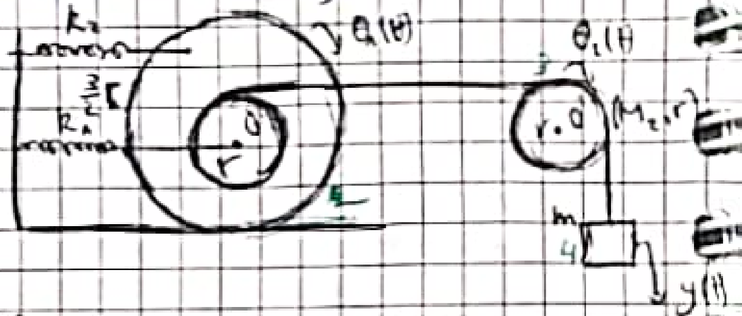
2nd case

$T_M = T_{M(t)} + T_{M(R)}$

$\frac{9}{32}ml^2\dot\theta^2 + \frac{1}{2}\underbrace{I_{/\Delta}}_{\frac{1}{2}MR^2}\dot\varphi^2$

$T_M = \frac{9}{32}ml\dot\theta^2 + \frac{1}{4}MR^2\dot\varphi^2$ potential stays the same

Ex3 : (M_1, R)



$R = 2r$

$k_1 = 2k_2$

$M_1 = 2M_2$

$\boxed{DOF = 4 - 3 = 1}$

$\boxed{y = r\theta_2}$ 1

$r\theta_2 = r\theta_1 + \underbrace{x}_{R\theta_1}$

$r\theta_2 = 3r\theta_1$

$\boxed{\theta_2 = 3\theta_1}$ 2

generalized coordinates θ_1 $\boxed{x = R\theta_1}$ 3

$U_{k_1} = \frac{1}{2}k_1(\underbrace{x}_{R\theta_1}+\Delta l_1)^2$

$U_{k_2} = \frac{1}{2}k_2(\underbrace{x}_{R\theta_1}+\frac{3}{2}r\theta_1)$

$T = T_{M_1} + T_{M_2} + T_m$

$T_{M_1(t)}$ $\quad T_{M_1(R)}$

$T_{M_1} = \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}I_{/\Delta}\dot\theta_1^2$

$= \frac{1}{2}M_1R^2\dot\theta_1^2 + \frac{1}{4}M_1R^2\dot\theta_1^2 = \frac{3}{4}M_1R^2\dot\theta_1^2$

$T_{M_2} = \frac{1}{2}I_{/\Delta}\dot\theta_2^2 = \frac{1}{4}M_2r^2\dot\theta_2^2$

$= \frac{9}{4}M_2r^2\dot\theta_1^2$

$T_m = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}mr^2\dot\theta_2^2 = \frac{9}{2}mr^2\dot\theta_1^2$

$T = \frac{3}{4}M_1R^2\dot\theta_1^2 + \frac{9}{4}M_2r^2\dot\theta_1^2 + \frac{9}{2}mr^2\dot\theta_1^2$

$T = 3M_1r^2\dot\theta_1^2 + \frac{9}{8}M_1r^2\dot\theta_1^2 + \frac{9}{2}mr^2\dot\theta_1^2$

$T = (\frac{33}{8}M_1 + \frac{9}{2}m)r^2\dot\theta_1^2$

$U = U_{k_1} + U_{k_2} + U_m$ $\left.\frac{\partial U}{\partial \theta}\right|_{\theta_{eq}} = 0$

$U_{k_1} = \frac{1}{2} K_1 (x + \Delta l_1)^2 = \frac{1}{2} K_1 (R\theta_1 + \Delta l_1)^2$

$U_{k_2} = \frac{1}{2} K_2 (\frac{3}{2} r\theta_1 + x + \Delta l_2)^2$

$U_{k_2} = \frac{1}{2} K_2 (\frac{3}{2} r\theta_1 + R\theta_1 + \Delta l_2)^2$

$U_{k_2} = \frac{1}{2} K_2 (\frac{7}{2} r\theta_1 + \Delta l_2)^2$

$U_m = -\int_0^y \vec{W} \cdot \vec{dy} = -mgy = -3mgr\theta_1$

$U = \frac{1}{2} K_1 (2r\theta_1 + \Delta l_1)^2 + \frac{1}{2} K_2 (\frac{7}{2} r\theta_1 + \Delta l_2)^2$
$- 3mgr\theta_1$

$$\boxed{U = K_2 \left[(2r\theta_1 + \Delta l_1)^2 + \frac{1}{2}(\frac{7}{2} r\theta_1 + \Delta l_2)^2\right] - 3mgr\theta_1}$$

Ex4: $x = \frac{l}{2}\sin\theta = \frac{l}{2}\theta$ ($\theta \approx \sin\theta$)

$x = R\varphi$ $\boxed{C=2}$

$\frac{l}{2}\theta = R\varphi$

DOF $= 3 - 2 = 1$

$T = T_M + T_m$

$T_M = T_{M(t)} + T_{M(R)}$

$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} I_{/G}\dot{\varphi}^2$

$= \frac{1}{2} M \frac{l^2}{4}\dot{\theta}^2 + \frac{1}{4} MR^2 \left(\frac{l}{2R}\dot{\theta}\right)^2$

$T_M = \frac{1}{8} Ml^2\dot{\theta}^2 + \frac{1}{16} Ml^2\dot{\theta}^2$

$T_M = \frac{3}{16} Ml^2\dot{\theta}^2$

$T_m = \frac{1}{2} I_{/G}\dot{\theta}^2$

$T_m = \frac{1}{24} ml^2\dot{\theta}^2$

$T = \frac{3}{16} Ml^2\dot{\theta}^2 + \frac{1}{24} ml^2\dot{\theta}^2$

$U = U_k = \frac{1}{2} k (x + \Delta l)^2$

$U = \frac{1}{2} k (\frac{l}{2}\theta + \Delta l)^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ $q = \theta$

$\left.\frac{dU}{d\theta}\right|_{\theta=0} = 0$

$\frac{dU}{d\theta} = k\frac{l}{2}\left(\frac{l}{2}\theta + \Delta l\right)$

$\left.\frac{dU}{d\theta}\right|_{\theta=0} = k\frac{l}{2}\Delta l = 0$

$U = \frac{1}{2} k (\frac{l}{2}\theta)^2 = \frac{1}{8} kl^2\theta^2$

$L = T - U$

$L = (3/16\, M + 1/24\, m) l^2\dot{\theta}^2 - \frac{1}{8} kl^2\theta^2$

$\frac{\partial L}{\partial \dot{\theta}} = (\frac{3}{8} M + \frac{1}{12} m) l^2\dot{\theta}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (3/8\, M + \frac{1}{12} m) l^2\ddot{\theta}$

$-\frac{\partial L}{\partial \theta} = +\frac{1}{4} kl^2\theta$

$(\frac{3}{8} M + \frac{1}{12} m) l^2\ddot{\theta} + \frac{1}{4} kl^2\theta = 0$

$\ddot{\theta} + \frac{1/4\, k}{\frac{3}{8} M + \frac{1}{12} m}\theta = 0$ $\ddot{\theta} + \omega_n^2\theta = 0$

$\omega_n = \sqrt{\frac{k}{3/2\, M + 1/3\, m}}$

$\theta(t) = \theta_M \sin(\omega_n t + \gamma)$