TD 2:

PROCEDURE FOR ANALYSIS

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For areas the differential element is generally a *rectangle* of area dA, having a finite length and differential width.
- For volumes the differential element can be a circular *disk* of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

- Express the area dA or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

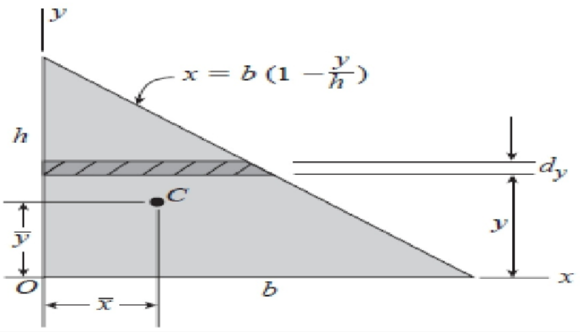
Integrations.

- Substitute the formulations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and dA or dV into the appropriate equations (Eqs. 6–1 through 6–3).
- Express the function in the integrand in terms of the same variable as the differential thickness of the element.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.*

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

<u>Problem 1 :</u> (triangular area)

By using an horizontal rectangular differential element (strip), determine the distances ($\bar{x}$, $\bar{y}$)to thecentroid C of a right triangle having base b andaltitude h.



$$A = \int dA = \int_0^h x\, dy = \int_0^h b\left(1 - \frac{y}{h}\right) dy$$

$$= \left[by - \frac{by^2}{2h}\right]_0^h = bh - \frac{bh}{2}$$

$$\boxed{A = \frac{bh}{2}}$$

$$Q_y = \int_0^h \underset{\sim}{x}\, dA$$

$\downarrow$ distance du centroïde de l'element de surface et l'axe y

$$Q_y = \int_0^h \underbrace{\frac{b\left(1-\frac{y}{h}\right)}{2}}_{x} \times b\left(1-\frac{y}{h}\right) dy$$

$$= \frac{b^2}{2h^2}\int_0^h (h-y)^2\, dy = \frac{b^2}{2h^2}\int \left[y^2 - 2hy + h^2\right] dy$$

$$= \frac{b^2}{2h^2}\left[\frac{y^3}{3} - hy^2 + h^2 y\right]_0^h$$

$$= \frac{b^2}{2h^2}\left(\frac{h^3}{3} - h^3 + h^3\right)$$

$$\boxed{Q_y = \frac{b^2 h}{6}}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{\frac{b^2 h}{6}}{\frac{bh}{2}} = \boxed{\frac{bh}{3}}$$

$$Q_x = \int y\, dA = \int_0^h by\left(1-\frac{y}{h}\right) dy$$

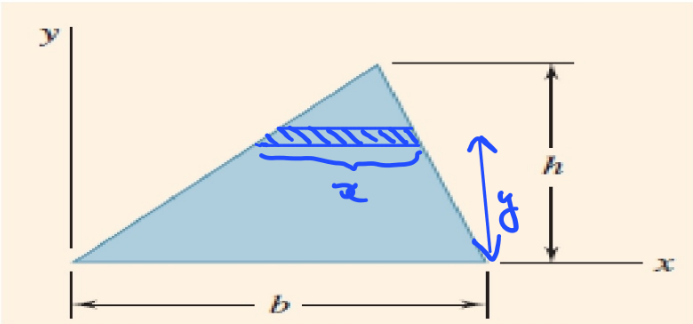
$$= \frac{b}{h}\int_0^h (hy - y^2)\, dy$$

$$= \frac{b}{h}\left[h\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{bh}{6}$$

$$\bar{y} = \frac{Q_x}{A} = \boxed{\frac{bh}{3}}$$

<u>Problem 2 :</u> (triangular area)

By using an horizontal rectangular differential element (strip), determine :
1) The triangle surface (A).
2) The first moment with repect to x axe.
3) The coordinate ($\bar{y}$) of the centroid C.



① - Talès : $\frac{x}{b} = \frac{h-y}{h}$

$$x = b\left(1 - \frac{y}{h}\right)$$

$$A = \int dA = \int_0^h b\left(1 - \frac{y}{h}\right) dy = \frac{bh}{2}$$

② - $Q_x = \int_0^h y\, dA = \int_0^h by\left(1 - \frac{y}{h}\right) dy$

$$= \frac{b}{h}\int_0^h \left(-y^2 + hy\right) dy$$

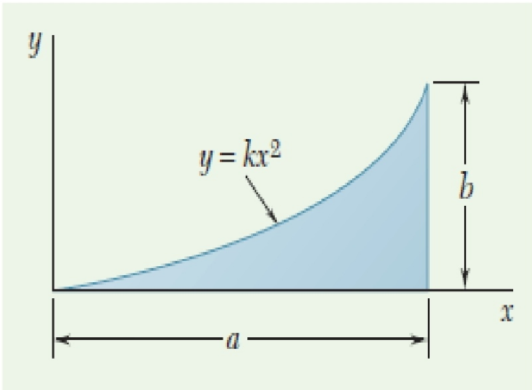
$$= \frac{b}{h}\left[\frac{-y^3}{3} + h\frac{y^2}{2}\right]_0^h$$

$$= \frac{b}{h}\left[\frac{-h^3}{3} + \frac{h^3}{2}\right] = \boxed{\frac{bh^2}{6}}$$

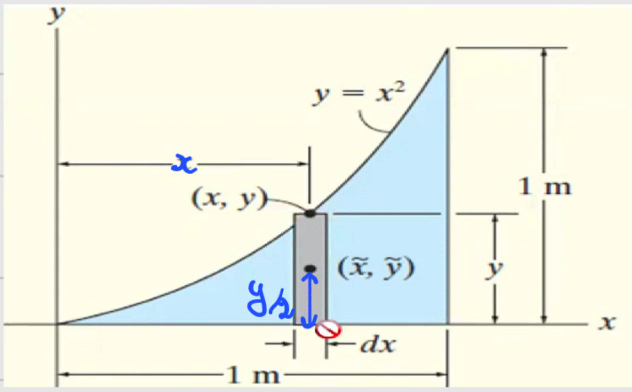
$$\bar{y} = \frac{Q_x}{A} = \frac{\frac{bh^2}{6}}{\frac{bh}{2}} = \frac{h}{3}$$

Problem 3 : **(Parabolic Spandrel)**

Locate the centroid of the area shown in Fig. ($k = 1$, $a = b = 1$ m). Use both horizontal and vertical strip and compare.



① Vertical differential element:



$$dA = y\, dx$$

and its centroid is located at x, $y/2$

$$\bar{x} = \frac{Q_y}{A} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} = \frac{\int_0^1 x\, y\, dx}{\int_0^1 y\, dx}$$

$$= \frac{\int_0^1 x^3\, dx}{\int_0^1 x^2\, dx} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4} = 0,75\ m$$

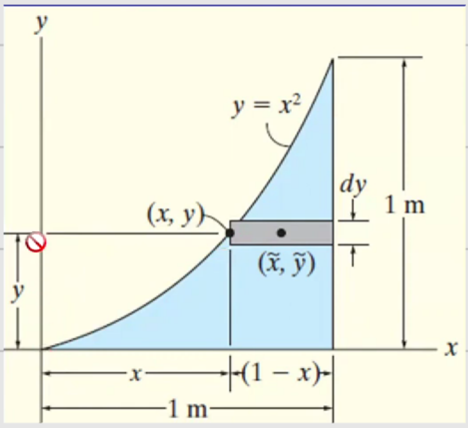
$$\boxed{\bar{x} = 0,75\ m}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{\int_0^1 (y/2)\, y\, dx}{\int_0^1 y\, dx}$$

$$= \frac{\frac{1}{2}\int_0^1 x^4\, dx}{\int_0^1 x^2\, dx} = \frac{\frac{1}{2} \times \frac{1}{5}}{\frac{1}{3}} = \frac{3}{10}$$

$$\boxed{\bar{y} = 0,3\ m}$$

(2) Horizontal differential element:



$$dA = (1 - x)\, dy$$

$$\bar{x} = \frac{Q_y}{A} = \frac{\int_A \tilde{x}\, dA}{\int_A dA}$$

$$= \frac{\int_0^1 \left[x + \frac{(1-x)}{2} \right] (1-x)\, dy}{\int_0^1 (1-x)\, dy}$$

$$= \frac{\frac{1}{2}\int_0^1 (1+x)(1-x)\, dy}{\int_0^1 (1-x)\, dy} = \frac{\frac{1}{2}\int_0^1 (1-x^2)\, dy}{\int_0^1 (1-x)\, dy}$$

$$= \frac{\frac{1}{2}\int_0^1 (1-y)\, dy}{\int_0^1 (1-\sqrt{y})\, dy} = \frac{\frac{1}{2}\left[1 - \frac{1}{2}\right]}{1 - \frac{2}{3}}$$

$$\boxed{\bar{x} = \frac{3}{4}}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{\int_0^1 y\,(1-x)\, dy}{\int_0^1 (1-x)\, dy}$$

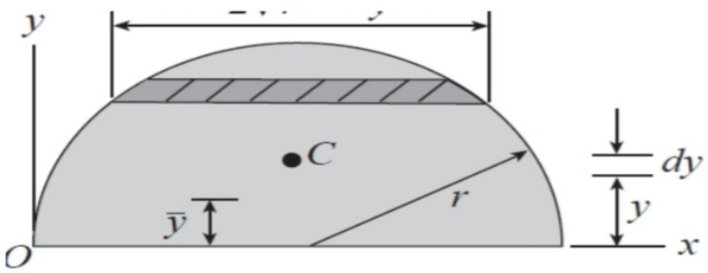
$$= \frac{\int_0^1 y\,(1-\sqrt{y})\, dy}{\int_0^1 (1-\sqrt{y})\, dy} = \frac{\frac{1}{2} - \frac{2}{5}}{1 - \frac{2}{3}}$$

$$= \frac{3}{10}$$

$$\boxed{\bar{y} = \frac{3}{10}}$$

Problem 4 : **(circular area)**

By using an horizontal rectangular differential element (strip), determine the distance($\bar{y}$)to thecentroid *C* of a semicircle (radius r).



$$\bar{y} = \frac{Q_x}{A} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

$$(x-r)^2 + y^2 = r^2 \Rightarrow |x-r| = \sqrt{r^2-y^2}$$

$$\Rightarrow \begin{cases} x-r = \sqrt{r^2-y^2} & \text{if } x > r \\ x-r = -\sqrt{r^2-y^2} & \text{if } x < r \end{cases}$$

$$A = \int_0^r \int_{r-\sqrt{r^2-y^2}}^{r+\sqrt{r^2-y^2}} dx\, dy$$

$$= \int_0^r \left(\sqrt{r^2-y^2} + r + \sqrt{r^2-y} - r \right) dy$$

$$dA = 2\sqrt{r^2-y^2}\, dy$$

$$\bar{y} = \frac{-\int_0^r -2y\sqrt{r^2-y^2}\;dy}{\int_0^r 2\sqrt{r^2-y^2}\;dy}$$

$$= \frac{-\frac{2}{3}\left[(r^2-y^2)^{3/2}\right]_0^r}{\int_0^r 2\sqrt{r^2-y^2}\,dy} = \frac{\frac{2r^3}{3}}{A}$$

$$A = 2\int_0^r \sqrt{r^2-y^2}\;dy$$

$$y = r\sin u \Rightarrow dy = r\cos u\,du$$

$$y = 0 \Rightarrow r\sin(u) = 0 \Rightarrow u = 0$$

$$y = r \Rightarrow r\sin(u) = r \Rightarrow r = \frac{\pi}{2}$$

$$A = 2\int_0^{\frac{\pi}{2}} r\cos^2(u)\,du$$

$$= 2r\int_0^{\frac{\pi}{2}} \frac{\cos(2u)+1}{2}\,du$$

$$= \cancel{r\left[\sin 2u\right]_0^{\pi/2}} + r\left[u\right]_0^{\frac{\pi}{2}}$$

$$\boxed{A = \frac{r\pi}{2}}$$

$$\Rightarrow \bar{y} = \frac{\frac{2r^3}{3}}{\frac{r\pi}{2}} = \boxed{\frac{4r^2}{3\pi}}$$