Vector analysis

Chapter 2

2-1- Vector fields and scalar fields

Definition:

① A scalar field over a ~~region~~ plane region $D \subset \mathbb{R}^2$ (resp. a solid region Ω in space) is a ~~function~~ real function $f : D \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ (resp. $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto f(x,y,z)$).

② A vector field over $D \subset \mathbb{R}^2$ is a function F that assigns a vector $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ ~~$= (P(x,y), Q(x,y))$~~ to each pt $(x,y) \in D$

③ A vector field over $\Omega \subset \mathbb{R}^3$ is a function F that assigns a vector $F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ to each pt $(x,y,z) \in \Omega$

Example: Gradient field

A vector field F is a gradient field, if there exists a real function (or scalar field) f, called potential of F, such that $F = \nabla f = \text{grad}(f)$, that is:

• If F is a plane vector field, then $F(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$, and if F is a vector field in space, then $F = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.

Definition: Conservative vector field

A vector field F is called conservative if it is a gradient field.

Examples: ① An inverse square field is $F(x,y,z) = \frac{k}{\|r\|^3} r$ where $r(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector field is conservative, and $f(x,y,z) = \frac{-k}{\sqrt{x^2+y^2+z^2}}$, $\Omega = \mathbb{R}^3 - \{(0,0,0)\}$

theorem:

Let $M, N : R \subset \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^1$ on an open disk $R \subset \mathbb{R}$. The vector field $F = M\vec{i} + N\vec{j}$ is conservative if, and only if, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

Example:

• The vector field $F(x,y) = x^2y\,\vec{i} + xy\,\vec{j}$ is not conservative because $\frac{\partial M}{\partial y}(x,y) = x^2 \neq \frac{\partial N}{\partial x}(x,y) = y$.

• If $F(x,y) = \varphi(x)\vec{i} + \psi(y)\vec{j}$, then F is conservative, a potential f on F is $f(x) = H(x) + G(y)$ / $H' = \varphi$, $G' = \psi$.