National Higher School of Autonomous Systems Technology

Academic year : 2024/2025

Vibrations and waves

By Dr. Malek ZENAD and Dr. Intissar DJOUADA

Chapter 1 :   Introduction to Lagrangian Formalism

contents

1-Equilibrium Points

2- Generalized coordinates and degrees of freedom DOF

3-Kinetic energy and potential energy

4- Lagrange's Equation

Joseph-Louis
Lagrange

1736-1813

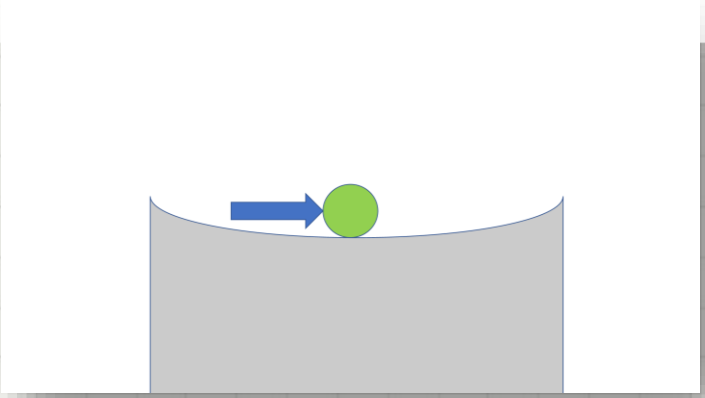
1-Equilibrium Points

By definition, an equilibrium point of a system is a point in physical space where the system remains stationary when it is not subject to any perturbation.

We distinguish two types of equilibrium:

1- **Stable equilibrium**: When a particle is slightly displaced from its position, it returns to the initial position.

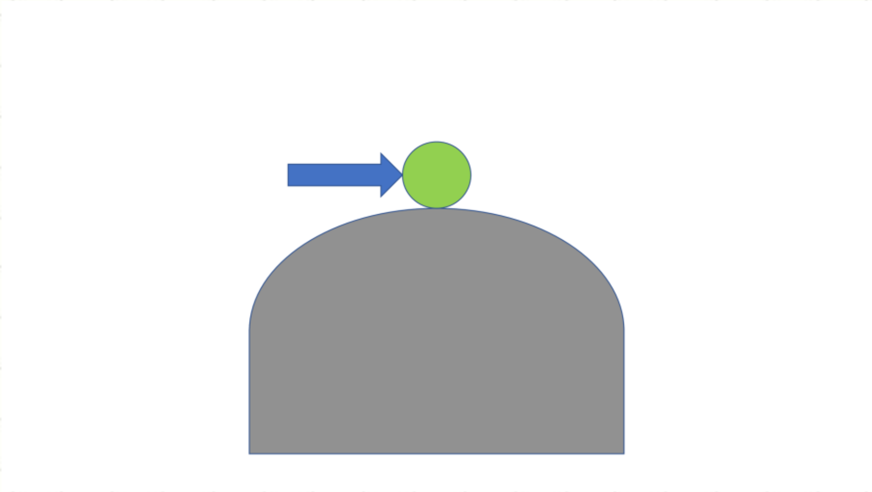
•Stable equilibrium points are located at a **minimum potential energy**.



1-Equilibrium Points

2- **Unstable equilibrium**: When a particle is displaced slightly from a position, the particle **rolls away** from the equilibrium position.

•Unstable equilibrium points are located at a **maximum potential energy**.



1-Equilibrium Points

In one dimension, equilibrium points are defined by

If $\left.\frac{dU(x)}{dx}\right|_{x=x_{eq}} = 0$, (F=0)

Consider a 1D potential $U = U(x)$ and Taylor expands about the equilibrium point $x = x_{eq}$:

$$U(x) = U(x_{eq}) + \left.\frac{\partial U}{\partial x}\right|_{x_{eq}} (x - x_{eq}) + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_{eq}} (x - x_{eq})^2 + \cdots$$

1-Equilibrium Points

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1-Equilibrium Points

the second term is zero $$U(x) - U(x_{eq}) = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_{eq}} (x - x_{eq})^2$$

The sign of this difference is the same as that of the quantity $\frac{1}{2}\frac{\partial^2 U}{\partial x^2}\Big|_{x_{eq}}$

- **If** $\frac{\partial^2 U}{\partial x^2}\Big|_{x_{eq}} > 0 \quad \Rightarrow \quad U(x) - U(x_{eq}) > 0 \quad \Rightarrow$ **Stable equilibrium**

  The system returns to its equilibrium position in order to minimize its potential energy

- **If** $\frac{\partial^2 U}{\partial x^2}\Big|_{x_{eq}} < 0 \quad \Rightarrow \quad U(x) - U(x_{eq}) < 0 \quad \Rightarrow$ **Unstable equilibrium**

- The system moves away to occupy a position corresponding to a lower potential energy

1-Equilibrium Points

the second term is zero $$U(x) - U(x_{eq}) = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_{eq}}(x - x_{eq})^2$$

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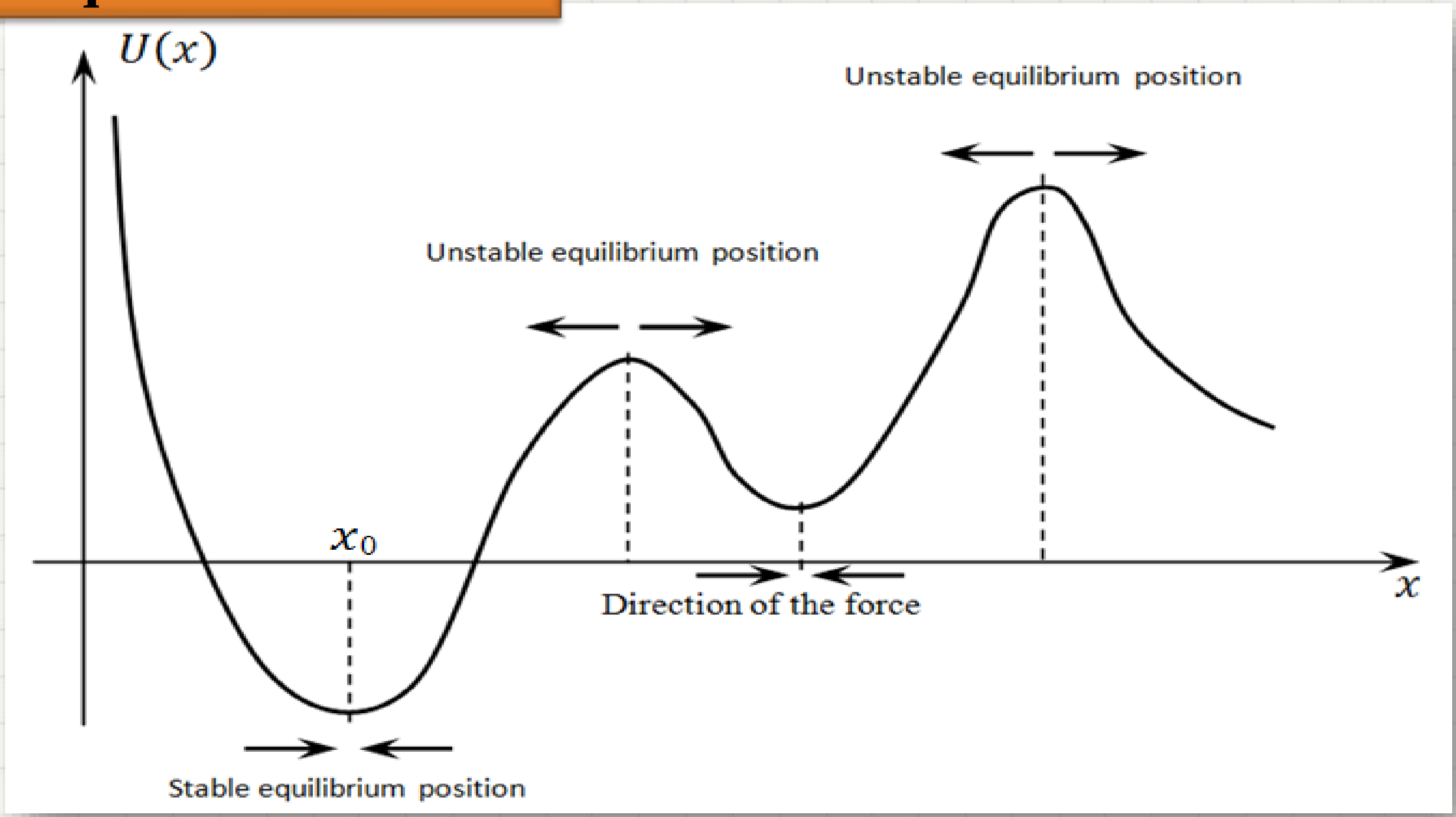
1-Equilibrium Points



Free oscillations of a system can only occur around a stable equilibrium position.

2- Generalized coordinates and degrees of freedom DOF

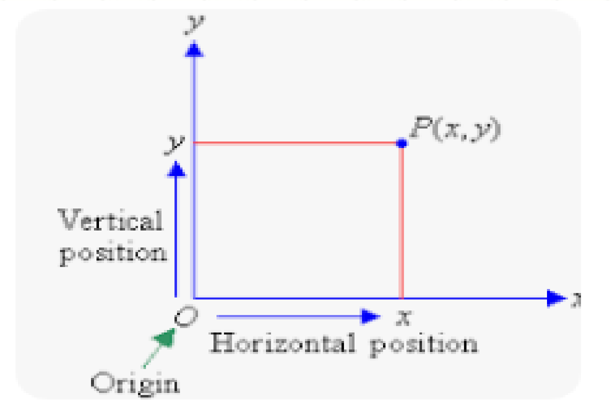
Introduction:

Coordinates: A set of values that **indicate** the exact position.

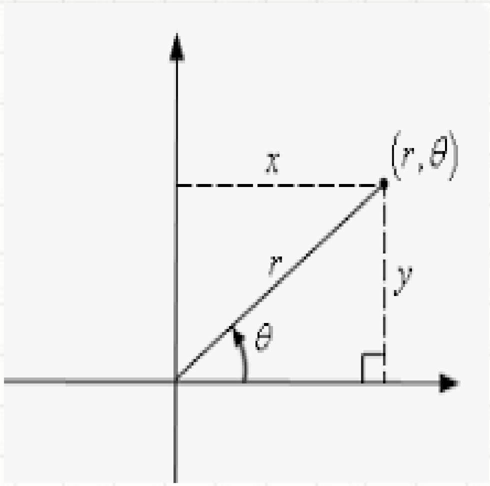
The position of a particle can be expressed using various coordinate systems:

In plan

Cartisian coordinates (x,y)



Polar coordinates



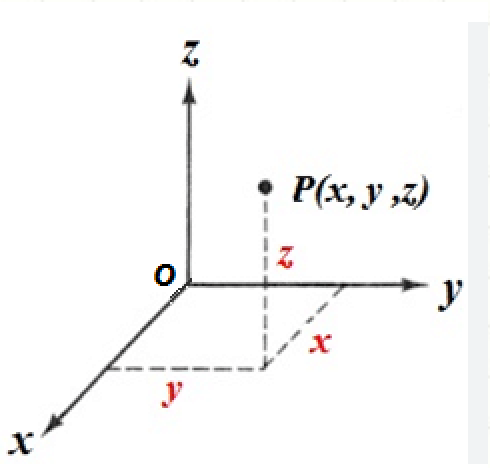
2- Generalized coordinates and degrees of freedom DOF

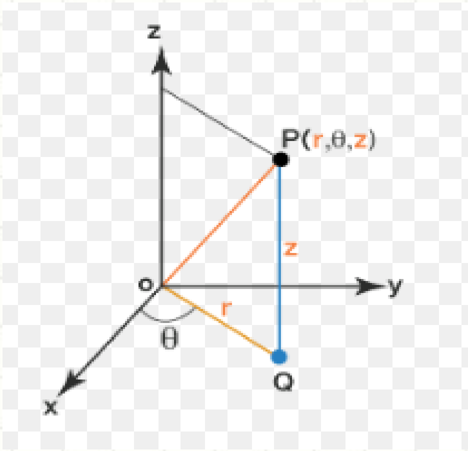
In space

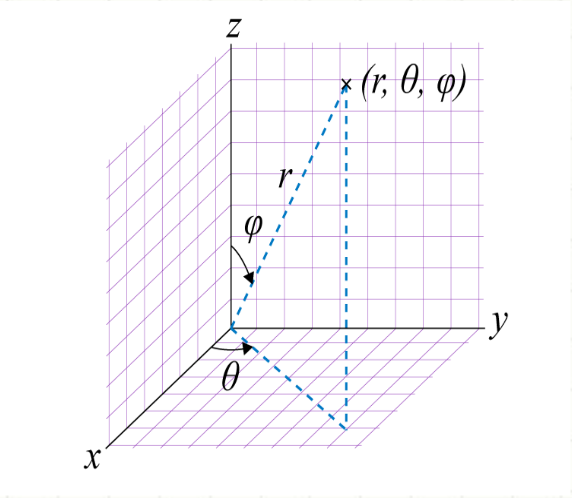
Cartisian coordinates (x, y, z)

Cylindrical coordinates

Spherical coordinates







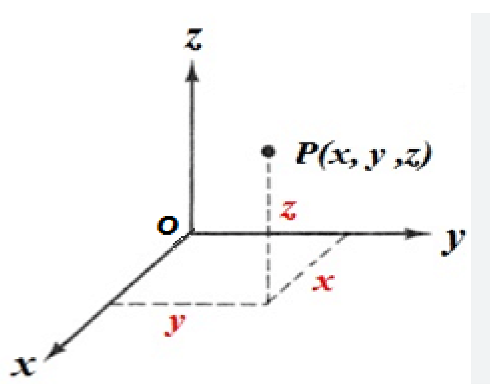
2- Generalized coordinates and degrees of freedom DOF

1- Generalized coordinates: It is the minimum number of independent coordinates necessary to describe the motion of a dynamical system.

Number of generalized coordinates = Number of degrees of freedom DOF

= *Number of differential equations of the system*

- A free particle is positioned by three coordinates (3 translations)



DOF= 3n-C

n: number of particles
C: Number of constraints

2- Generalized coordinates and degrees of freedom DOF

- A free solid is positioned by
Six coordinates (3 translations (x, y, z)
and 3 rotations ((θ, φ, ψ) Euler angles)

DOF= 6N-C N : number of solid

- ***For a system containing particles and solid bodies***

DOF= 6N+3n-C

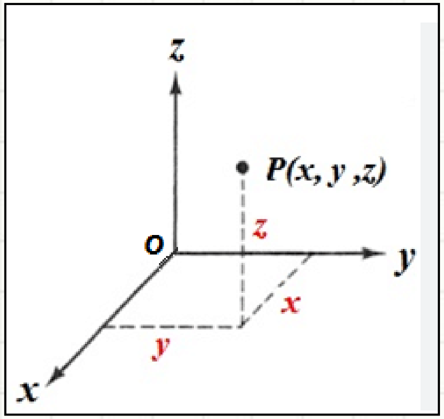
DOF = Total number of variables−Number of constraints

2- Generalized coordinates and degrees of freedom DOF

2-Constraints

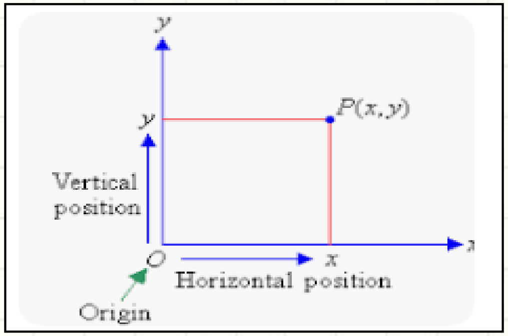
1- Particle in space

$$m(x \neq 0, y \neq 0, z \neq 0)$$
$$DOF = 3n - c \Longrightarrow DOF = 3 * 1 - 0$$
$$DOF = 3$$



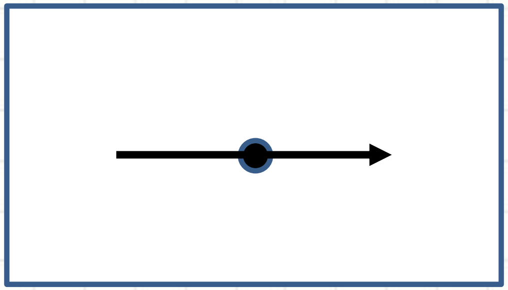
2- Particle in plan

$$m(x \neq 0, y \neq 0, z = 0)$$
$$DOF = 3n - c \Longrightarrow DOF = 3 * 1 - 1$$
$$DOF = 2$$



3- Particle along a line

$$m(x \neq 0, y = 0, z = 0)$$
$$DOF = 3n - c \Longrightarrow DOF = 3 * 1 - 2$$
$$DOF = 1$$



2- Generalized coordinates and degrees of freedom DOF

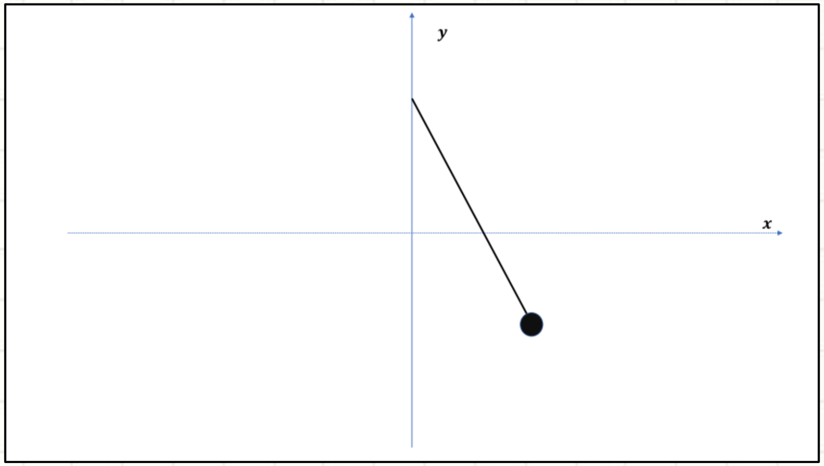
2-Constraints

For the simple pendulum shown in the figure, the motion can be described either in terms of θ or the coordinates x and y. If the coordinates x and y are used to describe the motion, it must be noted that these coordinates are not independent; they are related to each other by the following relation

$$x^2 + y^2 = l^2 \quad \boxed{\Rightarrow \text{ Constraint}}$$

where l is the constant length of the pendulum

Thus, any single coordinate can describe the motion of the pendulum. The choice of θ as the independent coordinate is more convenient than the choice of x and y

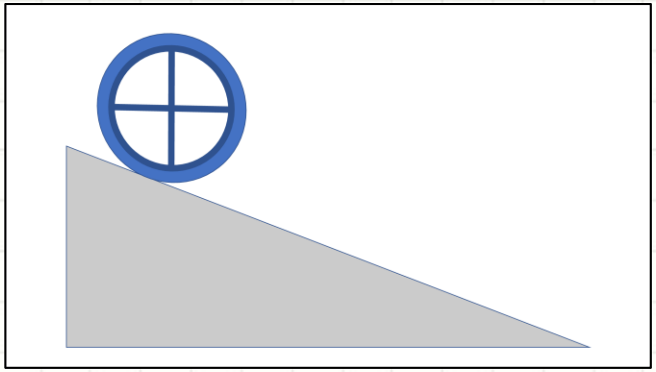


2- Generalized coordinates and degrees of freedom DOF

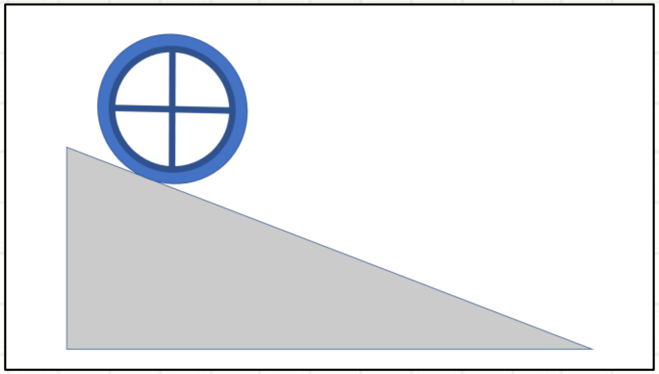
Application exercise

A cylinder rolls down an inclined plane making an angle α with the horizontal.
Determine the different constraints of this system and the number of degrees of freedom in the following cases:

a)The cylinder rolls without slipping.



b)The cylinder rolls with slipping



2- Generalized coordinates and degrees of freedom DOF

Application exercise

a- The cylinder rolls without slipping

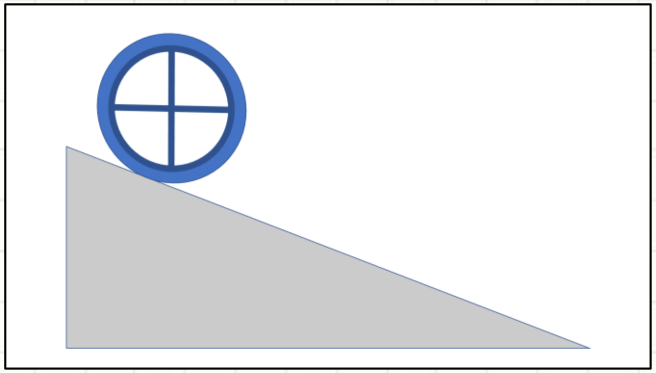
DOF= 6N+3n-C

number of solid N=1
number of particle n= 0

$$(R_x = 0, R_y = 0, T_y = cte, T_z = cst)$$

$$R_z \neq 0 \;\; and \;\; T_x \neq 0$$

- The cylinder makes a single rotation around the z-axis and a single translation along the x-axis



2- Generalized coordinates and degrees of freedom DOF

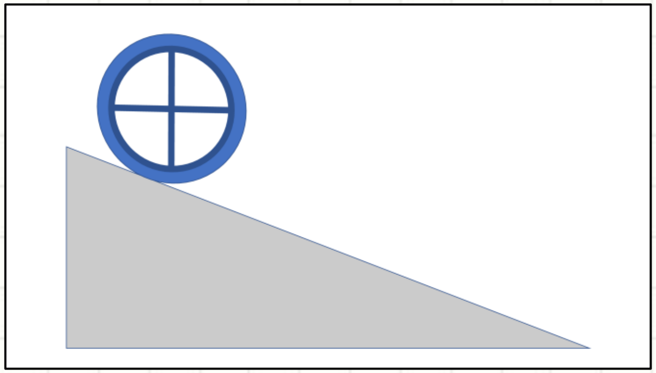
Application exercise

a- The cylinder rolls without slipping

$x = R\theta$ → **condition for rotation without slipping**

C=5 → ***DOF= 6-C =6-5*** → $\boxed{\textit{DOF= 1}}$

$\boxed{\textit{Generalized coordinate is} \quad \theta}$



2- Generalized coordinates and degrees of freedom DOF

Application exercise

b- The cylinder rolls with slipping

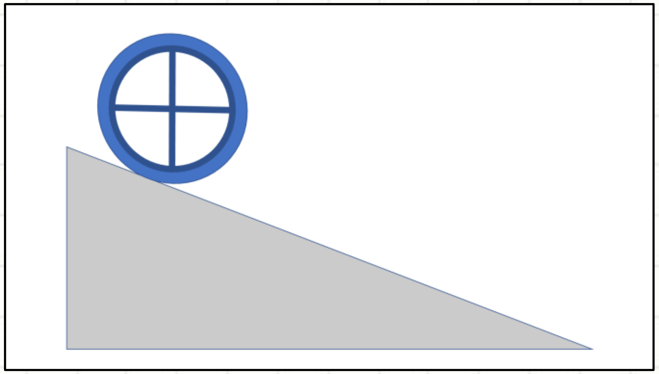
DOF= 6N+3n-C

number of solid N=1
number of particle n= 0

$$(R_x = 0, R_y = 0, T_y = cte, T_z = cst)$$

$$R_z \neq 0 \;\; and \;\; T_x \neq 0$$

➢The cylinder makes a single rotation around the z-axis and a single translation along the x-axis



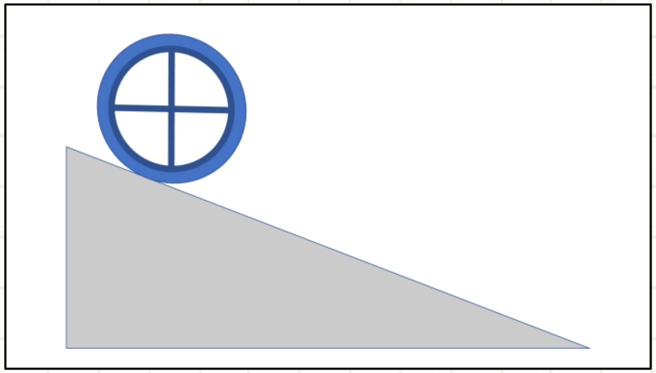
2- Generalized coordinates and degrees of freedom DOF

Application exercise

b- The cylinder rolls with slipping

$x \neq R\theta$ ⟹ **The cylinder rolls with slipping**

C=4 ⟹ ***DOF= 6-C =6-4*** ⟹ $\boxed{DOF = 2}$



$\boxed{\text{Generalized coordinates are } \theta \text{ and } x}$

3-Kinetic energy and potential energy

1- Kinetic energy

Kinetic energy is the energy associated with the movement of an object. It depends on the object's mass and its speed.

- The kinetic energy of a particle with mass m and linear velocity $\vec{v}$ is given by:

$$T = \frac{1}{2}mv^2 \quad , \quad \vec{v}^2 = v^2$$

3-Kinetic energy and potential energy

1-Kinetic energy

- The kinetic energy of a solid with mass M is expressed as:
- $T = \frac{1}{2} M {v_c}^2 + \frac{1}{2} I_{/(\Delta)} \dot{\theta}^2$, $T = T_{Translation} + T_{Rotation}$
- v_c is the linear velocity of the center of mass of the solid.
- $\dot{\theta}$ The angular velocity of the solid body.
- (Δ)Axis of rotation passing through the center of mass of the solid.
- I is the moment of inertia of the solid relative to the axis (Δ)

Moment of inertia about an axis:

1-Kinetic energy

5- Moment of inertia

- By definition, the moment of inertia I_Δ about an axis (Δ), of a material point with mass m located at a distance r from (Δ) is : $I_\Delta = mr^2$

- For a system of N material points with masses m_i, located at distances r_i from (Δ):

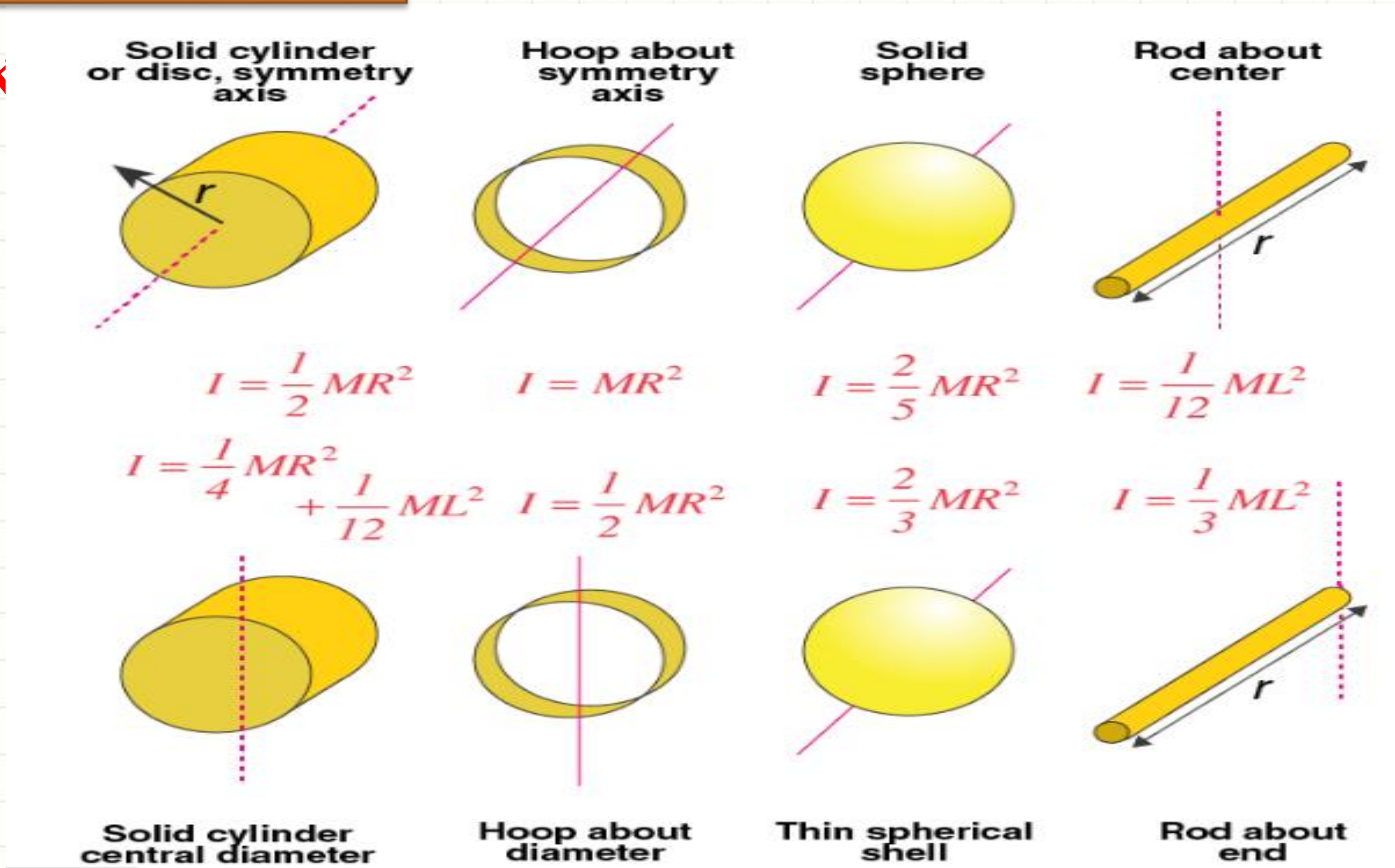
$$I_\Delta = \sum_i^N m_i {r_i}^2$$

- In the case of a solid body composed of an infinite number of material points, we proceed to the following limit: $I_\Delta = \int r^2 dm$

Moment of Inertia for Different Objects

5- Moment of inertia

1-K



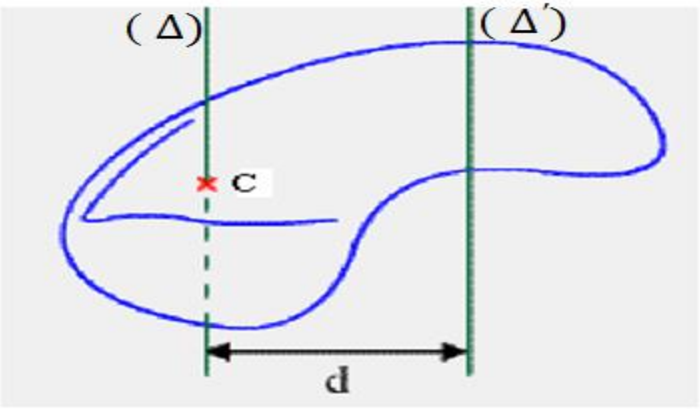
3-Kinetic energy and potential energy

5- Moment of inertia

Moment of inertia about an axis:

Huygens' theorem:

The moment of inertia of a solid, with respect to an axis (Δ') is equal to the moment of inertia of this solid with respect to an axis (Δ), parallel to (Δ'), passing through the center of gravity plus the product Md^2 (where M is the mass of the solid and d is the distance between the two axes: $I_{\Delta'} = I_\Delta + dM^2$



3-Kinetic energy and potential energy

2- Potential energy

Potential energy is the energy stored by an object due to its position or shape

Let a body be moving under the action of a conservative force $\vec{F}$. The potential energy U associated with this force is given by: $U = -\int \vec{F}.d\vec{r_c}$

Integration is generally performed between the equilibrium position and the position of the system at time t.

$d\vec{r_c}$: Position vector of the center of mass of the body.

2- Potential energy of spring:

1-Linear spring (simple spring) : It stores elastic potential energy when it compressed or extended. So the spring stores energy in its longitudinal deformation. The elastic potential energy in a linear spring is given by:

$$U_k = \frac{1}{2}k(x + \Delta l)^2 \quad \text{Where:}$$

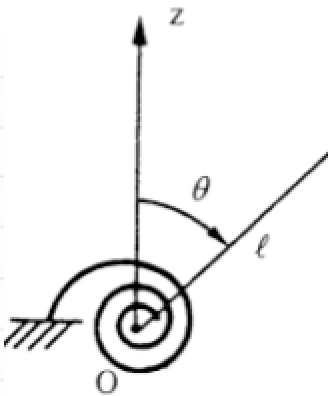
- U_k is the potential energy.
- k is the stiffness of spring (N/m).
- Δl the elongation of the spring at equilibrium
- x is the displacement (compression or extension) of the spring.

2- Potential energy of spring:

2-Torsion Spring: A torsion spring works by rotating around its axis. It stores energy in its rotational deformation. The potential energy in a torsion spring is given by:

$$U_{k_t} = \frac{1}{2} k_t \, \theta^2$$ Where:

- U_{k_t} is the potential energy.
- k_t is the torsional stiffness of spring constant (depending on the geometry and material of the spring). (Nm/rad)
- θ is the angle of rotation (angular displacement of the spring).



2- Potential energy

There are several types of potential energy, each associated with different physical situations:

- Gravitational potential energy.
- Elastic potential energy, also called spring energy
- Chemical energy.
- Nuclear energy.
- Electrical potential energy especially in a capacitor

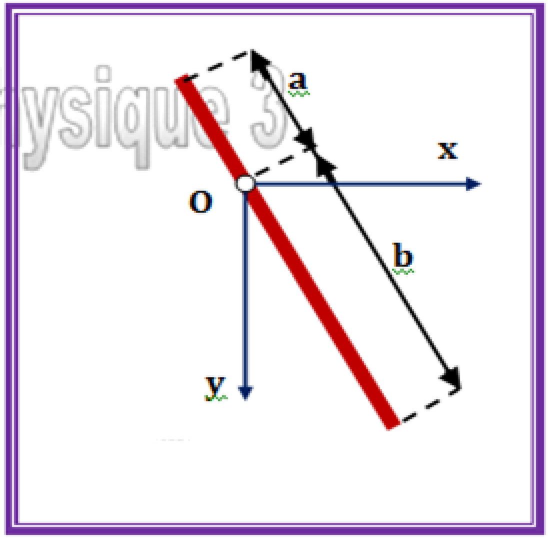
3-Kinetic energy and potential energy

Application exercise

A thin homogeneous rod of mass **m** and length **l**, which can oscillate without friction in the vertical plane around the point O located at a distance "a" from one of its ends and a distance "b" from the other end **(a + b = l)**.

- Determine the components of the position vector of the rod's center of mass, then deduce its kinetic energy.

- Calculate the potential energy of the rod.



Application exercise

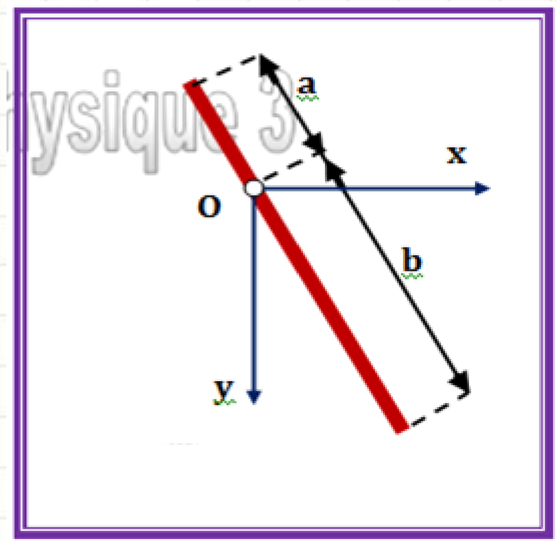
➢ **the components of the position vector of the center of mass of the rod**

$$\overrightarrow{r_C} = \overrightarrow{OC}$$

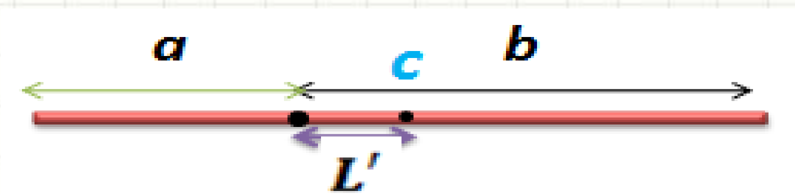
$$L' = b - \frac{a+b}{2} = \frac{b-a}{2}$$

$$Ou: L' = \frac{a+b}{2} - a = \frac{b-a}{2}$$

$$L' = \frac{b-a}{2}$$



$$\boxed{\overrightarrow{OC} = \frac{b-a}{2}\sin\theta\,\vec{i} + \frac{b-a}{2}\cos\theta\,\vec{j}}$$

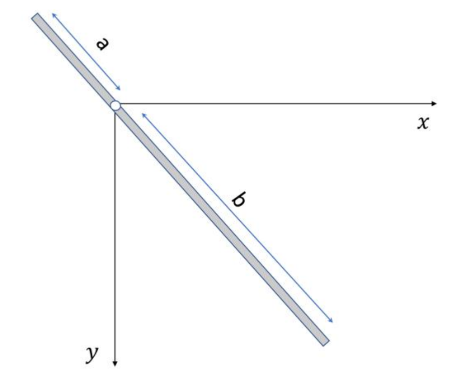


Application exercise

➢*Kinetic energy ?*

1- First method

$$\vec{r}_{cm}\begin{cases}\dfrac{b-a}{2}\sin\theta\\ \dfrac{b-a}{2}\cos\theta\\ 0\end{cases}\qquad \overrightarrow{dr}_{cm}\begin{cases}\dfrac{b-a}{2}\cos\theta\ d\theta\\ -\dfrac{b-a}{2}\sin\theta\ d\theta\\ 0\end{cases}\qquad \vec{V}_{cm}\begin{cases}\dfrac{b-a}{2}\dot{\theta}\cos\theta\\ -\dfrac{b-a}{2}\dot{\theta}\sin\theta\\ 0\end{cases}$$



$$\mathbf{T} = \frac{1}{2}mV_{cm}{}^{2} + \frac{1}{2}I_{/\Delta}\dot{\theta}^{2}$$

$I_{/(\Delta)}$ the moment of inertia of the rod relative to the axis (Δ) passing through its center of mass

3-Kinetic energy and potential energy

Application exercise

$$T = \frac{1}{2}m(L'\dot{\theta})^2 + \frac{1}{2}(m\frac{L^2}{12})\dot{\theta}^2 \Rightarrow \boxed{T = \frac{1}{6}m(a^2 + b^2 - ab)\dot{\theta}^2}$$

2-Second method (Huygens' theorem)

$$T = \frac{1}{2} I_{/\Delta'} \dot{\theta}^2$$

Where

$$I_{/\Delta'} = I_{/\Delta} + md^2$$

d is distance between (Δ) *and* (Δ')

d=L

$$T = \frac{1}{2}\left(m\frac{(a+b)^2}{12} + m(\frac{b-a}{2})^2\right)\dot{\theta}^2 \Rightarrow \boxed{T = \frac{1}{6}m(a^2 + b^2 - ab)\dot{\theta}^2}$$

➢Potential energy of the rod?

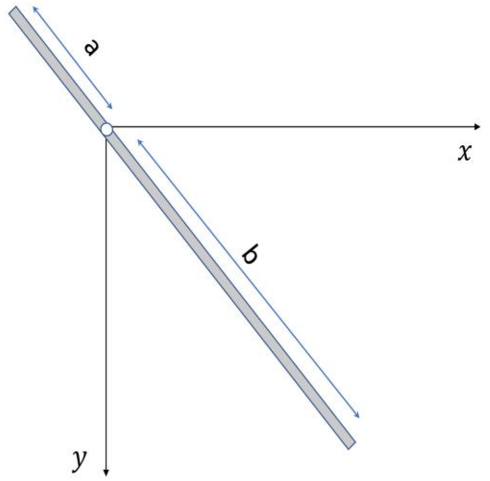
$$\vec{P}_m \begin{cases} 0 \\ +mg \\ o \end{cases}$$

$$U = -\int \vec{P}_m . \vec{dr}_{cm} = -\int_0^{\theta} mg\left(-\frac{b-a}{2}\sin\theta\right) d\theta =$$

$$= \int_0^{\theta} mg\left(\frac{b-a}{2}\sin\theta\right) d\theta$$

$$= mg\frac{b-a}{2}\int_0^{\theta} (\sin\theta)\, d\theta$$

$$= mg\frac{b-a}{2}[-\cos\theta]_0^{\theta}$$



3-Kinetic energy and potential energy

$$\boxed{U = mg\frac{b-a}{2}(1-\cos\theta)}$$

For small oscillations $\Rightarrow$ $$\cos\theta \cong 1 - \frac{\theta^2}{2}$$

$$U = mg\frac{b-a}{2}\left(\frac{\theta^2}{2}\right) \Rightarrow U = mg\frac{b-a}{4}\theta^2$$

Chapter 1 :   Introduction to Lagrangian Formalism

Sir
Isaac
Newton

1642-1727

$$\sum \vec{F} = m\,\vec{a}$$

Joseph-Louis
Lagrange

1736-1813

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} = 0$$

✓Physics without vector

4-Lagrange's Equation for a Particle:

Lagrange's Equation for a Particle:

Consider a particle of mass m and position vector $\vec{r}$. The motion of m is described by the generalized coordinate q, under the action of holonomic constraints.

$$\vec{r} = f(q, t)$$

- The fundamental relation of dynamics is written as:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} \qquad \Rightarrow \qquad m\vec{\ddot{r}} = \vec{F} \ \ldots\ldots\ldots\text{(I)}$$

- The displacement $\partial\vec{r}$ can be expressed as a function of the variation ∂q as:

$$\partial\vec{r} = \frac{\partial\vec{r}}{\partial q}\partial q$$

Chapter 1 : Introduction to Lagrangian Formalism

4-Lagrange's Equation for a Particle:

Lagrange's Equation for a Particle:

- $\frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{r}}{\partial q}\frac{\partial q}{\partial t} \quad \Rightarrow \quad \partial \dot{\vec{r}} = \frac{\partial \vec{r}}{\partial q}\,\partial \dot{q} \quad \Rightarrow \quad \frac{\partial \dot{\vec{r}}}{\partial \dot{q}} = \frac{\partial \vec{r}}{\partial q} \;\ldots\ldots\ldots\ldots (1)$

- $\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{d\vec{r}}{dt}\right) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q}\right) = \frac{\partial \dot{\vec{r}}}{\partial q} \;\ldots\ldots\ldots (2)$

$$(\mathrm{I})\times \frac{\partial \vec{r}}{\partial q} \quad \Rightarrow \quad m\frac{d^2\vec{r}}{dt^2}\cdot\frac{\partial \vec{r}}{\partial q} = \vec{F}\cdot\frac{\partial \vec{r}}{\partial q}$$

4-Lagrange's Equation for a Particle:

Lagrange's Equation for a Particle:

The quantity F_q defined by is called the generalized force: $\vec{F}\,.\dfrac{\partial\vec{r}}{\partial q} = F_q$

On the other hand: $\dfrac{d}{dt}\left(\dot{\vec{r}}.\dfrac{\partial\vec{r}}{\partial q}\right) = \ddot{\vec{r}}\,.\dfrac{\partial\vec{r}}{\partial q} + \dot{\vec{r}}.\dfrac{d}{dt}\left(\dfrac{\partial\vec{r}}{\partial q}\right) \quad\Rightarrow\quad \ddot{\vec{r}}\,.\dfrac{\partial\vec{r}}{\partial q} = \dfrac{d}{dt}\left(\dot{\vec{r}}.\dfrac{\partial\vec{r}}{\partial q}\right) - \dot{\vec{r}}.\dfrac{d}{dt}\left(\dfrac{\partial\vec{r}}{\partial q}\right)$

We introduce equation (2) $\quad\Rightarrow\quad \ddot{\vec{r}}\,.\dfrac{\partial\vec{r}}{\partial q} = \dfrac{d}{dt}\left(\dot{\vec{r}}.\dfrac{\partial\vec{r}}{\partial q}\right) - \dot{\vec{r}}.\dfrac{\partial\dot{\vec{r}}}{\partial q}\ldots\ldots(3)$

4-Lagrange's Equation for a Particle:

Lagrange's Equation for a Particle:

$(3) \times m \qquad \Rightarrow \qquad m.\vec{\ddot{r}}.\frac{\partial \vec{r}}{\partial q} = \frac{d}{dt}\left(m.\vec{\dot{r}}.\frac{\partial \vec{r}}{\partial q}\right) - m.\vec{\dot{r}}.\frac{\partial \vec{\dot{r}}}{\partial q}$

$\vec{F}.\frac{\partial \vec{r}}{\partial q} = \frac{d}{dt}\left(m.\vec{\dot{r}}.\frac{\partial \vec{r}}{\partial q}\right) - m.\vec{\dot{r}}.\frac{\partial \vec{\dot{r}}}{\partial q} \qquad \Rightarrow \qquad F_q = \frac{d}{dt}\left(m.\vec{\dot{r}}.\frac{\partial \vec{r}}{\partial q}\right) - m.\vec{\dot{r}}.\frac{\partial \vec{\dot{r}}}{\partial q} \ \ldots\ldots(4)$

Kinetic energy $T = \frac{1}{2}m\vec{\dot{r}}^2 \Rightarrow \begin{cases} \frac{\partial T}{\partial \dot{q}} = m.\vec{\dot{r}}.\frac{\partial \vec{\dot{r}}}{\partial \dot{q}} \quad \Rightarrow \quad \frac{\partial T}{\partial \dot{q}} = m.\vec{\dot{r}}.\frac{\partial \vec{r}}{\partial q} \\ \frac{\partial T}{\partial q} = m.\vec{\dot{r}}.\frac{\partial \vec{\dot{r}}}{\partial q} \end{cases}$

Chapter 1 : Introduction to Lagrangian Formalism

4-Lagrange's Equation for a Particle:

Lagrange's Equation for a Particle:

$$(4) \qquad \Rightarrow \qquad F_q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = F_q \qquad \text{............}(II)$$

(II) Represents the Lagrange equation for a system with one degree of freedom

4-Lagrange's Equation for a Particle:

a) Case of Conservative Systems:

In conservative systems, the force applied to the system is derived from a potential U and is written as: $F_q = -\frac{\partial U}{\partial q}$

(II) Then becomes: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = F_q \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial U}{\partial q}$

Since the potential energy does not depend on the velocity $\frac{\partial U}{\partial \dot{q}} = 0$

4-Lagrange's Equation for a Particle:

a) Case of Conservative Systems:

$$(II) \Rightarrow \frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{q}}\right)-\left(\frac{\partial T}{\partial q}-\frac{\partial U}{\partial q}\right)=0 \qquad \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{q}}\right)-\frac{\partial(T-U)}{\partial q}=0$$

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}}\right)-\frac{\partial(L)}{\partial q}=0$$ Lagrange's equation in the case of a conservative system

4-Lagrange's Equation for a Particle:

b) Case of viscous friction forces:

In the case of viscous friction, the friction force is proportional to the velocity but acts in the opposite direction.

$$\vec{F}_\alpha = -\alpha\,\vec{v}$$

Where α is the coefficient of viscous friction and $\vec{v}$ is the velocity of the particle

4-Lagrange's Equation for a Particle:

b) Case of viscous friction forces:

let $F_{q,\alpha}$ the generalized force for viscous friction forces

$$F_{q,\alpha} = \vec{F}\,.\frac{\partial\vec{r}}{\partial q} = -\,\alpha\,\vec{v}.\frac{\partial\vec{r}}{\partial q} = -\,\alpha.\frac{\partial\vec{r}}{\partial t}.\frac{\partial\vec{r}}{\partial q} \quad\Rightarrow\quad F_{q,\alpha} = -\,\alpha\,\frac{\partial\vec{r}}{\partial q}\frac{\partial q}{\partial t}\frac{\partial\vec{r}}{\partial q} = -\,\alpha\,(\frac{\partial\vec{r}}{\partial q})^2\,\frac{\partial q}{\partial t}$$

- The work done of $\vec{F}_\alpha$: $dw = \vec{F}_\alpha.d\vec{r} = -\,\alpha\,\vec{v}.\vec{v}\,dt = -\,\alpha\,v^2\,dt$

- The power dissipated by the viscous friction forces is: $P_d = \alpha\,v^2$

$$P_d = \alpha(\frac{\partial\vec{r}}{\partial t})^2 = \alpha.\,(\frac{\partial q}{\partial t}.\frac{\partial\vec{r}}{\partial q})^2 = \alpha\,(\frac{\partial\vec{r}}{\partial q})^2.\,(\frac{\partial q}{\partial t})^2 \quad\Rightarrow\quad P_d = \beta\dot{q}^2$$

4-Lagrange's Equation for a Particle:

b) Case of viscous friction forces:

- By definition, the dissipation function is equal to half of the dissipated power:

$$D = \frac{1}{2} P_d \quad \Rightarrow \quad D = \frac{1}{2} \beta \dot{\boldsymbol{q}}^2$$

- $F_{q,\alpha} = -\beta \dot{\boldsymbol{q}} \quad \Rightarrow \quad F_{q,\alpha} = -\frac{\partial D}{\partial \dot{q}}$

- The Lagrange's Equation for conservative forces and viscous friction forces becomes :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = \boldsymbol{F_q} = \boldsymbol{F_{q,U}} + F_{q,\alpha} \quad \textbf{where} \quad \boldsymbol{F_q} = \boldsymbol{F_{q,U}} + F_{q,\alpha}$$

4-Lagrange's Equation for a Particle:

b) Case of viscous friction forces:

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}}\right)-\frac{\partial(L)}{\partial q}=F_{q,\alpha}\quad\Rightarrow\quad\frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}}\right)-\frac{\partial(L)}{\partial q}+\frac{\partial D}{\partial\dot{q}}=0$$

4-Lagrange's Equation for a Particle:

c) Case of an external force depending on time:

c) Case of an external force depending on time:

In the general case, of a time-dependent external force $\vec{F}_{ext,}$ acting on a system that is subject to viscous friction forces, the Lagrange equation can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial\dot{q}}\right) - \frac{\partial(L)}{\partial q} + \frac{\partial D}{\partial\dot{q}} = F_{eq}$$

where

$$F_{eq} = \vec{F}_{ext}.\frac{\partial\vec{r}}{\partial q}$$