

Serie 1:

Ex 3:

1) Calculo de la superficie

$$S = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy$$

$$\frac{\partial f}{\partial x}(x, y) = 2 \Rightarrow A(S) = \iint \sqrt{1+4} dx dy$$

$$\frac{\partial f}{\partial y}(x, y) = 3 \Rightarrow \iint \sqrt{14} dx dy$$

$$= \sqrt{14} \quad A(D) = \sqrt{14} \cdot 9\pi = 9\sqrt{14}\pi$$

$$2) f(x, y) = 9 - x^2$$

$$D = [0, 2] \times [-2, 2]$$

$$A = \iint_D \sqrt{1 + (-2x)^2} dx dy$$

$$A = 2 \iint_D \sqrt{1 + 4x^2} dx$$

$$A = 2 \int_0^2 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2} dx dt \quad \text{we put } u = \sin t = \frac{e^t - e^{-t}}{2}$$

$$A = 2 \int_0^2 \int_0^{\pi} \sqrt{1 + 4x^2} dt \quad \left\{ \begin{array}{l} du = e^t dt \\ u=0 \rightarrow t=0 \\ u=\pi \rightarrow t=\pi \end{array} \right.$$

$$= \int_0^2 \left(\frac{e^t + e^{-t}}{2} \right)^2 dt$$

$$= \frac{1}{2} \int_0^2 (e^{2t} + e^{-2t} + 2) dt \quad \left\{ \begin{array}{l} u = e^t \\ u=1 \rightarrow t=0 \\ u=e^\pi \rightarrow t=\pi \end{array} \right.$$

$$= \frac{1}{2} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{-2t} + 2t \right]_0^{\pi} \quad \left\{ \begin{array}{l} u^2 - 8u - 1 = 0 \\ (u-4)^2 = 17 \\ u = 4 \pm \sqrt{17} \end{array} \right.$$

$$= \frac{1}{2} \left[\frac{(4+\sqrt{17})^2 - (4-\sqrt{17})^2}{2} + 2 \ln(4+\sqrt{17}) \right]$$

$$\begin{aligned} &= \frac{1}{4} ((4+\sqrt{17})^2 - (4-\sqrt{17})^2) + \ln(4+\sqrt{17}) \\ &= \frac{1}{4} \times 8 \times 2\sqrt{17} + \ln(4+\sqrt{17}) \\ &= \boxed{4\sqrt{17} + \ln(4\sqrt{17})} \end{aligned}$$

$$3) A(S) = \iint_D 1 + \frac{\cos 2x}{\sin^2 x} dx dy$$

$$= \int_0^{\pi/2} \int_0^{\pi} \left[\frac{1}{\sin x} dy \right] dx$$

$$= \int_0^{\pi/2} \frac{\tan x}{\sin x} dx \Rightarrow \int_0^{\pi/2} \frac{1}{\cos x} dx =$$

$$\int_0^{\pi/2} \frac{\cos x}{\cos^2 x} dx = \int_0^{\pi/2} \frac{\cos x}{1 - \sin^2 x} dx$$

$$\begin{aligned} &\text{Let } \sin x = t \\ &dt = \cos x dx \\ &\int_0^{\pi/2} \frac{1}{1-t^2} dt \\ &= \boxed{\int_0^1 \frac{1}{1-t^2} dt} = \frac{1}{2} \int_0^1 \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt \end{aligned}$$

$$= \frac{1}{2} \left[\ln \left(\frac{1+t}{1-t} \right) \right]_0^{\sqrt{2}/2}$$

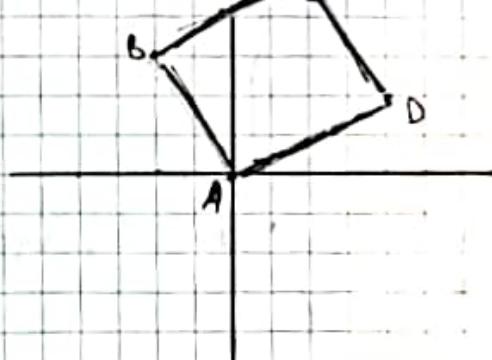
$$= \frac{1}{2} \ln \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{1}{2} \ln \frac{2+\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{1}{2} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln(1+\sqrt{2})$$

Ex 4:

$$① f(x, y) = (3x+2y)^2 \sqrt{2y-x}$$

$$A(0,0), B(-2,3), C(2,5), D(4,2)$$



A(x_A, y_A), B(x_B, y_B)

$$(AB) \cdot y = y_B + \left(\frac{y_B - y_A}{x_B - x_A} \right) (x - x_A)$$

$$AB: y = 2 + \frac{1}{2}(x-4) \Leftrightarrow 2y-x=0$$

$$BC: y = 5 + \frac{1}{2}(x-4) \Leftrightarrow 2y-x=8$$

$$CD: y = 2 + \frac{-3}{2}(x-4) \Leftrightarrow 2y+3x=8$$

$$BA: y = \frac{-3}{2}x \Leftrightarrow 2y+3x=0$$

$$u = 2y - x \quad v = 2y + 3x$$

$$f(u) = (3u+2v)^2 \sqrt{2y-u}$$

$$\left| J_{uv} (u, v) \right| = \frac{1}{\det \begin{pmatrix} u & v \\ 2 & 1 \end{pmatrix}} = \frac{1}{1+u^2}$$

$$\left| J_{uv} (u, v) \right| = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 6+2=8$$

$$\left| J_{uv} (u, v) \right| = \frac{1}{8}$$

$$V = \frac{1}{8} \int_0^4 \int_0^{4u} u^2 \sqrt{v} \, du \, dv$$

$$= \frac{1}{8} \int_0^4 u^2 \, du \int_0^{4u} \sqrt{v} \, dv$$

$$= \frac{1}{8} \left[\frac{u^3}{3} \right]_0^4 \cdot \left[\frac{2}{3} v^{3/2} \right]_0^{4u}$$

$$= \frac{96}{9} \sqrt{2}$$

$$② f(u, v) = \frac{uv}{1+u^2v^2}$$

$$u = xy$$

$$\text{when } u \in [1, 4]$$

$$\text{and } v = \frac{u}{x} \quad v \in [1, 4]$$

$$\left| J_{uv} \right| = \begin{vmatrix} y & u \\ 1 & x \end{vmatrix} = xy$$

$$\left| J_{uv} (u, v) \right| = \frac{1}{xy}$$

$$V = \int_1^4 \int_u^{4u} \frac{u}{1+u^2v^2} \frac{1}{v} \, dv \, du$$

$$= \left(\int_1^4 \frac{u}{1+u^2} \, du \right) \left(\int_1^4 \frac{1}{u^2} \, du \right)$$

$$= \frac{1}{2} \left[\ln(1+u^2) \right]_1^4 \left[\ln(u) \right]_1^4$$

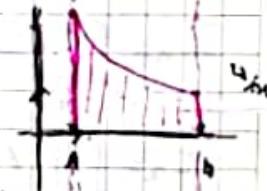
$$= \frac{1}{2} \ln\left(\frac{17}{2}\right) (\ln 4) = \ln(2) \ln\left(\frac{\sqrt{17}}{2}\right)$$

Ex 5:

A) finding the Mass

$$M = \iint_D f(x, y) \, dx \, dy \quad D \begin{cases} 1 \leq x \leq 4 \\ 0 \leq y \leq \frac{4}{x} \end{cases}$$

$$M = \iint_D kx^2 \, dy \, dx$$



$$M = \int_1^4 kx^2 \int_{4/x}^{4/x} 1 \, dy \, dx$$

$$M = \int_1^4 kx^2 [y]_{4/x}^{4/x} \, dx$$

$$= \int_1^4 kx^2 \times \frac{4}{x} \, dx = 4k \int_1^4 x \, dx$$

$$= 30k$$

calculating the centre of mass

$$\bar{x} = \frac{1}{M} \int_0^4 m f(x, y) \, dy \, dx$$

$$\bar{x} = \frac{1}{30k} \int_1^4 \int_{4/x}^{4/x} kx^2 \, dy \, dx$$

$$\bar{x} = \frac{1}{30k} \int_1^4 m \times \frac{4}{x} \, dx$$

$$\bar{x} = \frac{4}{30} \left[\frac{1}{2} x^3 \right]_1^4 = \frac{4}{30} \quad (6)$$

$$= \frac{16}{5}$$

$$\bar{y} = \frac{1}{M} \iiint_D y f(x,y) dm dy$$

$$\bar{y} = \frac{1}{30} \int_0^4 \int_{x=0}^{x=4} K x^2 y dy dm$$

$$\bar{y} = \frac{1}{30} \int_0^4 x^2 \left[\frac{1}{2} y^2 \right]_0^{4/x}$$

$$\bar{y} = \frac{1}{30} \int_0^4 x^2 \left(\frac{1}{2} \times \frac{16}{x^2} \right)$$

$$\bar{y} = \frac{1}{30} \times \frac{16}{2} [3] = \frac{3}{5}$$

the centre of mass $(\bar{x}, \bar{y}) = (14/5, 3/5)$

$$I_m = \iiint_D y^2 f(x,y) dm dy$$

$$I_m = \int_0^4 \int_{x=0}^{x=4} K x^2 y^2 dy dm$$

$$= \int_0^4 K x^2 \left[\frac{1}{3} y^3 \right]_0^{4/x}$$

$$= \int_0^4 4 x^2 \times \frac{1}{3} \frac{64}{x^3} = \frac{64}{3} K \left[\ln x \right]_0^4$$

$$I_m = \frac{64}{3} K \ln 4$$

$$I_y = \iiint_D m^2 f(x,y) dm$$

$$I_y = \iiint_D K m^2 dy dm$$

$$= \int_0^4 K m^2 [y]_0^{4/x}$$

$$= 4 K \left[\frac{1}{4} m^2 \right]_0^4 = K [256 - 1]$$

$$= 255 K$$

Ex7:

$M > 0$

$$1) z = 9 - x^3, y = 2 - x^2, y = 0$$

$$z = 0$$

$$Vol(D) = \iiint_D dm dy dz$$

$$V = \int_0^{\sqrt{2}} \int_0^{2-x^2} \int_0^{9-x^3} dm dy dz$$

$$= \int_0^{\sqrt{2}} \left(\int_0^{2-x^2} (9-x^3) dy \right) dm$$

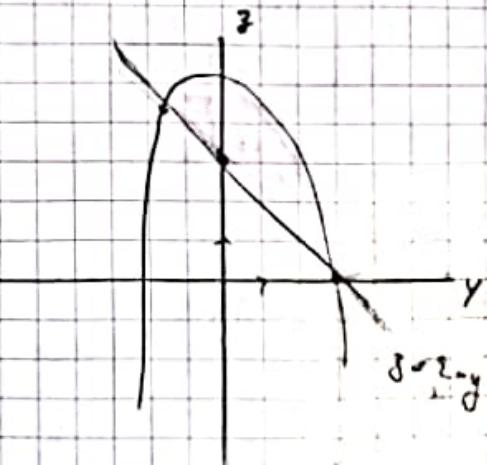
$$= \int_0^{\sqrt{2}} (9-x^3)(2-x^2) dm$$

$$= \int_0^{\sqrt{2}} (18 - 2x^3 + x^5 - 3x^2) dm$$

$$= 18\sqrt{2} - 3 \times \frac{32}{2} - \frac{1}{2}(4) + \frac{8}{6}$$

$$= 18\sqrt{2} - 6\sqrt{2} - 2 + \frac{4}{3}$$

$$2) \begin{cases} z = 2-y \\ z = 4-y^2 \end{cases} \begin{cases} x=0 \\ m=3 \end{cases} \begin{cases} y=0 \\ y=2 \end{cases}$$



$$V = \int_0^3 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dz dy dx$$

$$= \int_0^3 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y^2 - 2xy) dy dx$$

$$= \int_0^3 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (-y^2 + y + 2) dy dx$$

$$= \int_0^3 \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx$$

$$= \int_0^3 \left(\frac{8}{3} + 2 + 4 \right) dx = \int_0^3 \frac{10}{3} dx$$

$$V = 10$$

Ex 8:

$$z \in [x^2 + y^2, \sqrt{4-x^2-y^2}]$$

$$m = r \cos \theta$$

$$dn dy dz$$

$$= r dr d\theta dz$$

$$y = r \sin \theta$$

$$z = \bar{z}$$

$$0 \leq r \leq \sqrt{1+\sqrt{13}}$$

$$0 \leq \theta \leq 2\pi$$

$$r^2 \leq \bar{z} \leq \sqrt{4-r^2}$$

$$M = k \iiint r^3 dz d\theta dr$$

$$= \frac{k}{2} \int r \int \left[\bar{z}^2 \right]_{r^2}^{4-r^2} d\theta dr$$

$$= \frac{k}{2} \iint r (4-r^2-r^4) dr d\theta = \frac{k}{8}$$

$$= \frac{k}{2} \iint (4r-r^3-r^5) dr d\theta$$

$$= \pi k \left[2r^2 - \frac{1}{4}r^4 - \frac{1}{6}r^6 \right]_0^4$$

$$m = \left(2\alpha^2 - \frac{1}{4}\alpha^4 - \frac{1}{6}\alpha^6 \right) \pi k$$

center of mass ($\bar{x}, \bar{y}, \bar{z}$)

$$\bar{z} = \bar{x} = \bar{y} = 0 \text{ because}$$

$$\int \cos \theta = \int \sin \theta = 0$$

$$\bar{z} = \frac{k}{m} \iiint \bar{z}^2 r dz dr d\theta$$

$$= \frac{2k\pi}{3M} \int_0^\alpha r [\bar{z}] \frac{\sqrt{4-r^2}}{r^2} dr$$

$$= \frac{2k\pi}{3M} \int_0^\alpha r (4-r^2)^{1/2} - r^2 dr$$

$$= \frac{2k\pi}{3M} \left[-\frac{1}{2} \times \frac{2}{5} (4-r^2)^{5/2} - \frac{1}{3} r^3 \right]_0^\alpha$$

$$= \frac{2k\pi}{3M} \left[-\frac{1}{5} (4-\alpha^2)^{5/2} - \frac{1}{8} \alpha^8 + \frac{32}{5} \right]$$

$$= \frac{2k\pi}{3M} \left(-\frac{1}{5} \alpha^{10} - \frac{1}{8} \alpha^8 + \frac{32}{5} \right)$$

$$\bar{z} = \frac{2k\pi}{\pi} \times \frac{256 - 8\alpha^{10} - 5\alpha^8}{24\alpha^2 - 3\alpha^4 - 2\alpha^2}$$

$$= 2 \left(\frac{256 - 8\alpha^{10} - 5\alpha^8}{24\alpha^2 - 3\alpha^4 - 2\alpha^2} \right) \alpha = \sqrt{\frac{-1 + \sqrt{13}}{2}}$$

$$I_{xy} = k \iiint y^2 \bar{z} dn dy dz$$

$$I_{yz} = k \iiint m^2 \bar{z} dn dy dz$$

$$I_{xz} = k \iiint \bar{z}^3 dn dy dz$$

$$I_m = I_{xy} + I_{xz}$$

$$I_y = I_{xy} + I_{yz}$$

$$I_z = I_{xz} + I_{yz}$$