National Higher School of Autonomous Systems Technology

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Vibrations and waves

By Dr. Malek ZENAD and Dr. Intissar DJOUADA

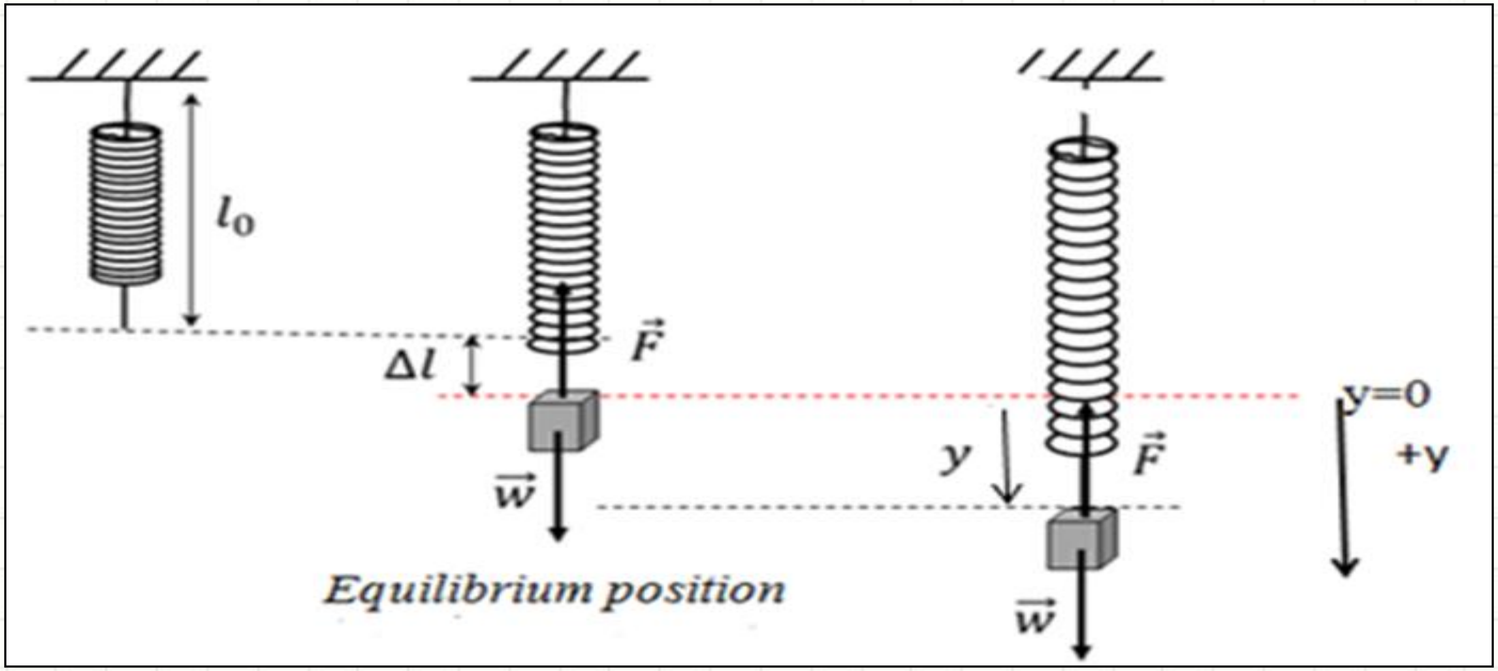
Chapter 2 : Free vibration of undamped single degree of freedom systems

$$\boxed{\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} = 0}$$

Contents

1- Vertical mass- spring

Consider the single-degree-of-freedom (SDOF) system shown in the figure below. The spring is initially in the unstretched position. It is assumed that the spring obeys Hooke's law.



1- Vertical mass- spring

- **Hooke's law** describes the relationship between the force exerted by a spring and its extension or compression. The law is expressed as:

$$F = -k \cdot x \quad or \quad F = -ky$$ Where:

- F is the force exerted by the spring (N),
- k is the spring constant or stiffness (N/m),
- x is the extension or compression of the spring from its equilibrium position (m).

- The negative sign indicates that the force exerted by the spring is always in the opposite direction of the extension or compression. This force is known as the *restoring force* because it works to return the spring to its equilibrium position.

1- Vertical mass- spring

a) Using Newton's Second Law

When the mass m ((weight W)) is applied, the spring will deflect to a static equilibrium position

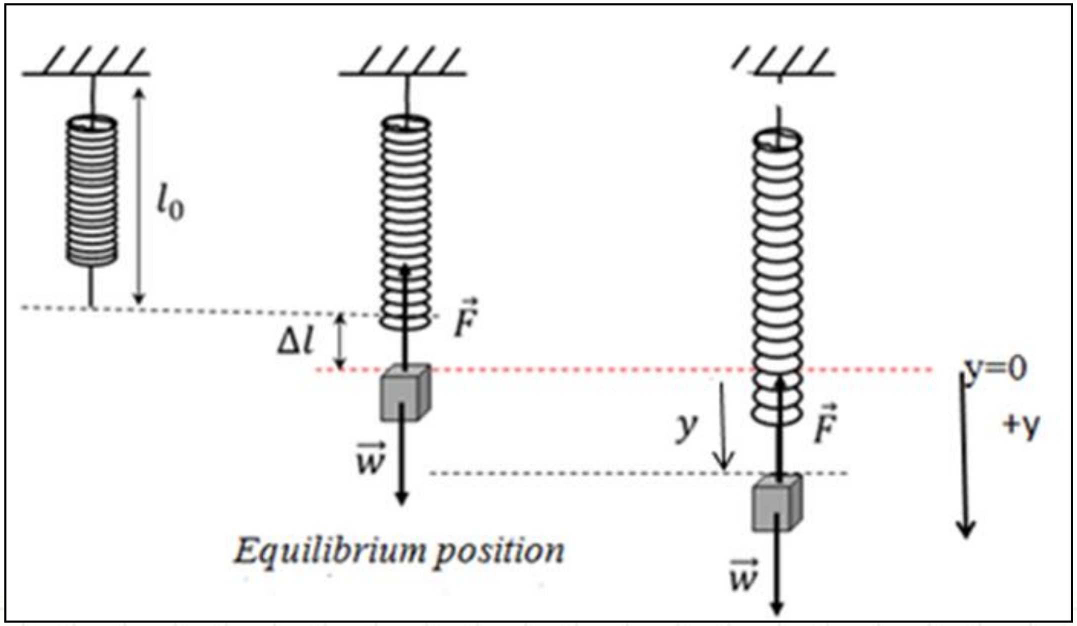
l_0: The spring's natural length

- At equilibrium: y=0

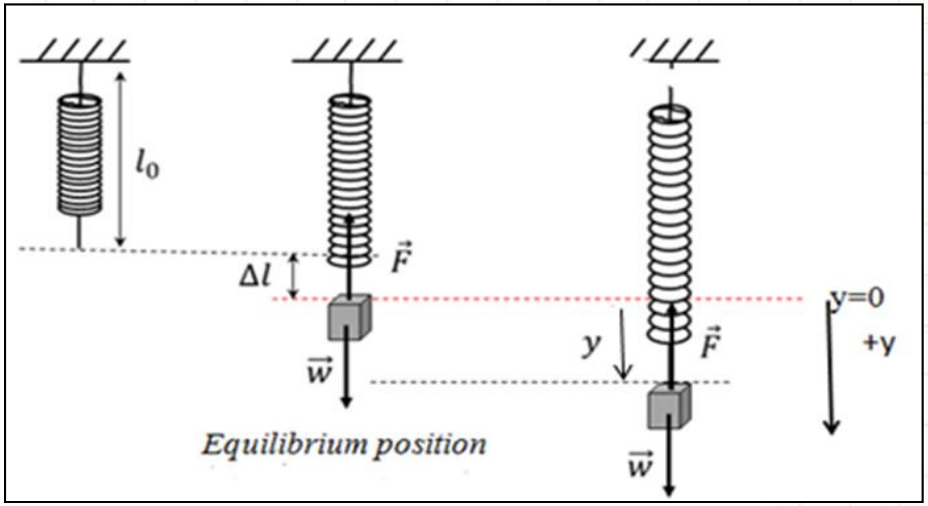
$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{W} + \vec{T} = \vec{0}$

Oy: $mg - k\Delta l = 0 \dots\dots\dots\dots\dots (1)$

$$\Rightarrow \Delta l = \frac{mg}{k}$$



1- Vertical mass- spring



- When the system is in the motion:

$$\sum \vec{F} = m\vec{a} \quad \Rightarrow \quad \vec{W} + \vec{T} = m\vec{a}$$

Oy: $mg - k(y + \Delta l) = ma$

$$mg - k\Delta l - ky = ma \ldots\ldots\ldots\ldots(2)$$

According to equation (1) $\quad mg - k\Delta l = 0$

(2) $\quad ma + ky = 0 \quad \Rightarrow \quad m\ddot{y} + ky = 0 \quad \Rightarrow \quad \ddot{y} + \frac{k}{m}y = 0$

$\Rightarrow \quad \ddot{y} + {\omega_n}^2 y = 0 \qquad$ Where $\quad \omega_n = \sqrt{\frac{k}{m}}$

ω_n is called natural frequency

1- Vertical mass- spring

b) Using Lagrange's equation

The *Lagrangian* L = T-U

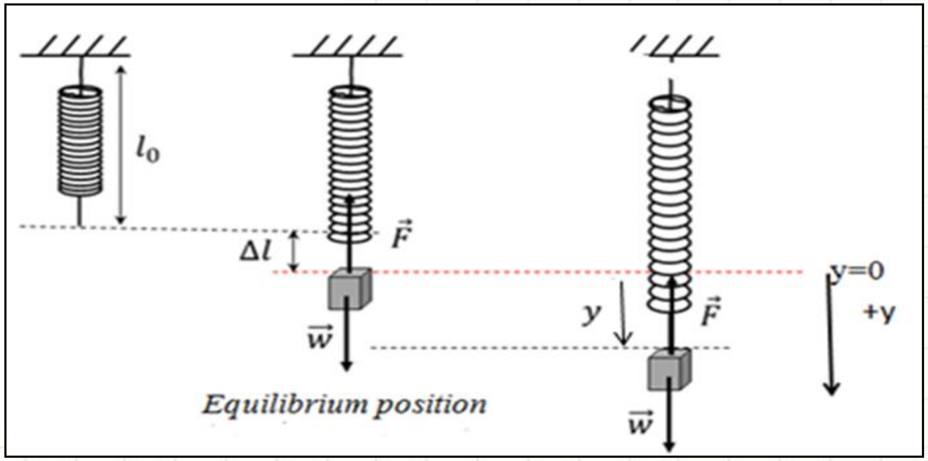
$$\vec{r}_m \begin{pmatrix} 0 \\ y \end{pmatrix} \Rightarrow d\vec{r}_m \begin{pmatrix} 0 \\ dy \end{pmatrix} \Rightarrow \vec{V}\begin{pmatrix} 0 \\ \dot{y} \end{pmatrix} \quad ; \vec{W}\begin{pmatrix} 0 \\ mg \end{pmatrix}$$

- Kinetic energy

$$T = T_m \Rightarrow T = \frac{1}{2} m\dot{y}^2$$

- Potential energy

$$U = U_m + U_k$$



1- Vertical mass- spring

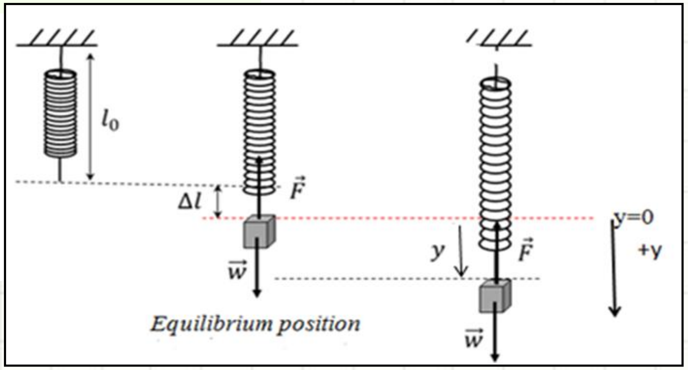
$$U = -\int_0^y \vec{W} \cdot d\vec{r}_m = -\int_0^y mg\, dy = -mgy$$

$$U_K = \frac{1}{2} k(y + \Delta l)^2$$

$$U = \frac{1}{2} k(y + \Delta l)^2 - mgy \;\ldots\ldots\ldots\ldots(3)$$

At equilibrium $\left.\frac{\partial U(y)}{\partial y}\right|_{y_{eq}=0} = 0$ $\quad k(y + \Delta l) - mg|_{y=o} = 0$

$\Rightarrow$ $\boldsymbol{\Delta l = \frac{mg}{k}}$ $\ldots\ldots\ldots(4)$



By substituting equation (4) into (3)

1- Vertical mass- spring

By substituting equation (4) into (3)

$$U = \frac{1}{2}k(y+\Delta l)^2 - mgy \quad \Rightarrow \quad U = \frac{1}{2}ky^2 + \frac{1}{2}.2ky.\Delta l + \frac{1}{2}k\Delta l^2 - mgy$$

$$U = \frac{1}{2}ky^2 + ky\frac{mg}{k} - mgy + cte \quad \Rightarrow \quad U = \frac{1}{2}ky^2 + mgy - mgy + cte \quad , cte = \frac{1}{2}k\Delta l^2$$

$$\Rightarrow U = \frac{1}{2}ky^2 + cte$$

$$L = T - U \quad \Rightarrow \quad L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 + cte$$

1- Vertical mass- spring

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$ Lagrange's equation in the case of a conservative system

$$\frac{\partial L}{\partial \dot{q}} = m\dot{y} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

$$\frac{\partial L}{\partial q} = -ky$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \Rightarrow \quad m\ddot{y} + ky = 0 \quad \Rightarrow \quad \ddot{y} + \frac{k}{m}y = 0$$

$$\Rightarrow \quad \ddot{y} + {\omega_n}^2 y = 0 \qquad \text{Where} \quad \omega_n = \sqrt{\frac{k}{m}}$$

ω_n *is* **called natural frequency**

Chapter 2 : Free vibration of undamped single degree of freedom systems

1- Vertical mass- spring

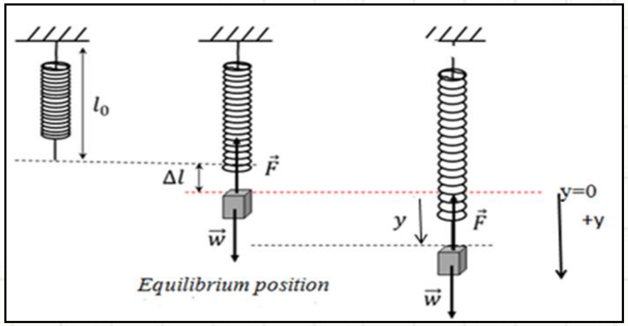
- $\ddot{y} + \frac{k}{m} y = 0$
- This equation is a second order linear homogeneous ordinary differential equation with constant coefficients.

The general solution of this differential equation takes a form of

$$y(t) = y_0 \cos(wt + \varphi) \quad or \quad y(t) = y_0 \sin(wt + \varphi')$$

We can find the values of y_0 and φ using initial conditions

- The solution is based on the initial conditions



1- Vertical mass- spring

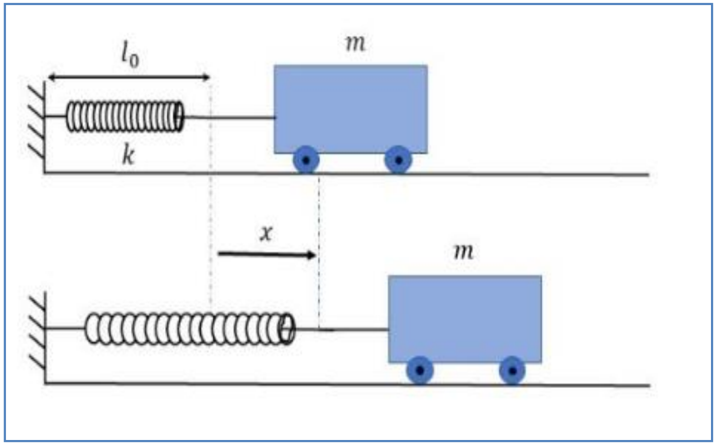
- The motion of an object can be describe in terms of its
  - Displacement $y(t) = y_0 \cos(w_n t + \varphi)$
  - Velocity $v(t) = \frac{dy}{dt} = \dot{y}$ $\Rightarrow$ $\dot{y}(t) = -w_n\, y_0 \sin(w_n t + \varphi)$
  - Acceleration $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2} = \ddot{y}$ $\Rightarrow$ $\ddot{y}(t) = -w_n{}^2\, y_0 \cos(w_n t + \varphi)$

$\ddot{y}(t) = -w_n{}^2 y(t)$ $\Rightarrow$ Simple Harmonic motion

2- Horizontal mass- spring

2- Horizontal mass spring

Consider a mass attached to a massless spring on a frictionless surface, as shown below.



- **Using Lagrange's equation:**

$$T = \frac{1}{2} m\dot{x}^2 \quad , \quad U = \frac{1}{2} kx^2$$

- *Lagrangian of the system*:

$$L = T - U \quad \Rightarrow \quad L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

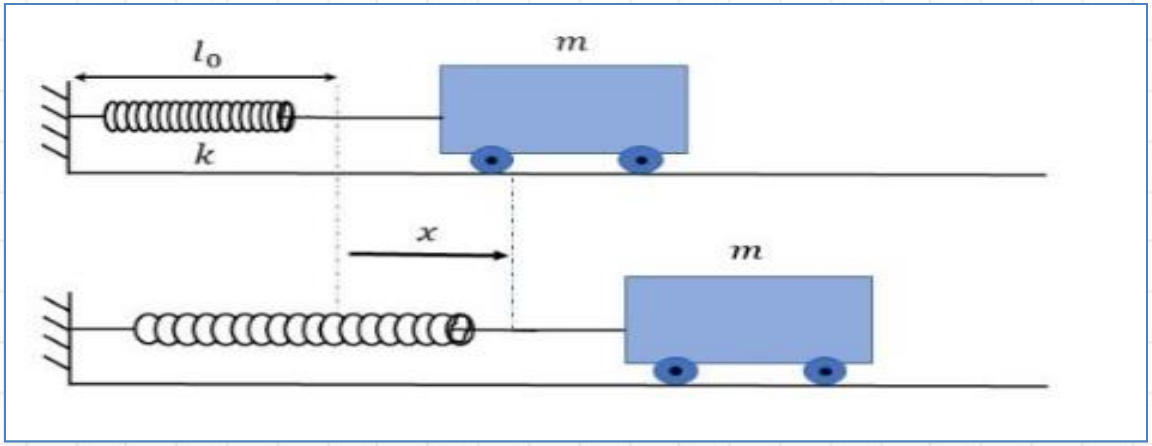
- $$\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} = 0 \quad \Rightarrow \quad \ddot{x} + \frac{k}{m} x = 0 \quad \Rightarrow \quad \ddot{x} + {w_n}^2 x = 0$$

2- Horizontal mass- spring

- This equation is a second order linear homogeneous ordinary differential equation with constant coefficients.

  Where $\omega_n = \sqrt{\frac{k}{m}}$ The natural frequency of the system.

- It is important to note that this frequency is independent of the initial conditions



3- Simple Pendulum

- **Positon vector**

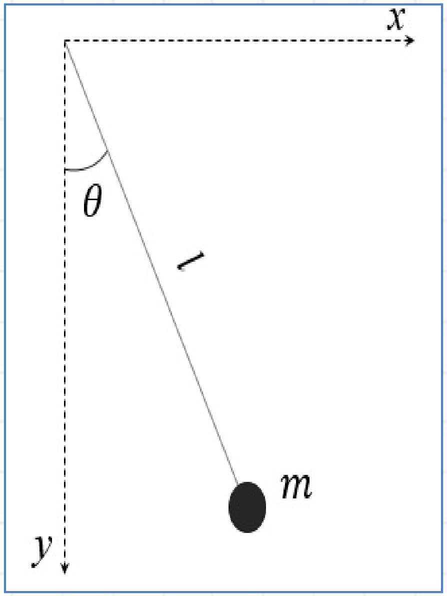
$$\vec{r}_m \begin{pmatrix} l\,sin\theta \\ l\,cos\theta \end{pmatrix} \Rightarrow \quad \overrightarrow{dr}_m \begin{pmatrix} l\,cos\theta\ \ d\theta \\ -l\,sin\theta\ \ d\theta \end{pmatrix} \quad \Rightarrow \quad \vec{v}_m \begin{pmatrix} lcos\theta\ \dot{\theta} \\ -lsin\theta\ \dot{\theta} \end{pmatrix}$$

- **Weight**

$$\vec{w}\begin{pmatrix} 0 \\ mg \end{pmatrix}$$

- **Potential energy**

$$U = -\int_0^{\theta} \vec{w}.\ \overrightarrow{dr}_m = -\int_0^{\theta} mg.\ (-l\ sin\theta\ \ d\theta) = +lmg\int_0^{\theta} sin\theta\ \ d\theta$$

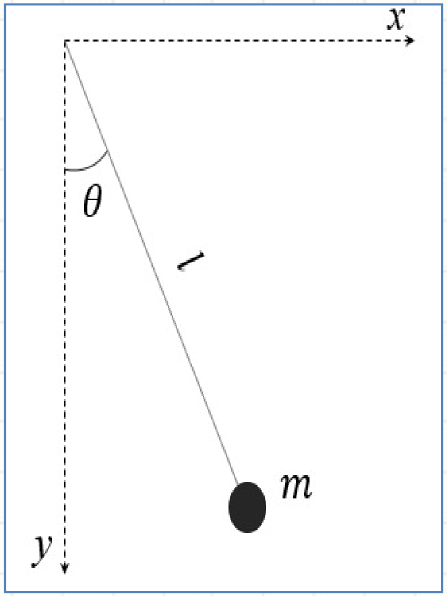


3- Simple Pendulum

✓ **For small oscillations** $\Rightarrow \quad \theta \ll 1 \quad \Rightarrow \quad cos\theta = 1 - \frac{\theta^2}{2}$

$$\Rightarrow \quad U = lmg\frac{\theta^2}{2}$$

$$T = \frac{1}{2}mv^2 \qquad \Rightarrow \qquad T = \frac{1}{2}m\,l^2\dot{\theta}^2$$



➢ The Lagrangian of the system

L= T-U $\Rightarrow \quad L = \frac{1}{2}m\,l^2\dot{\theta}^2 - lmg\frac{\theta^2}{2}$

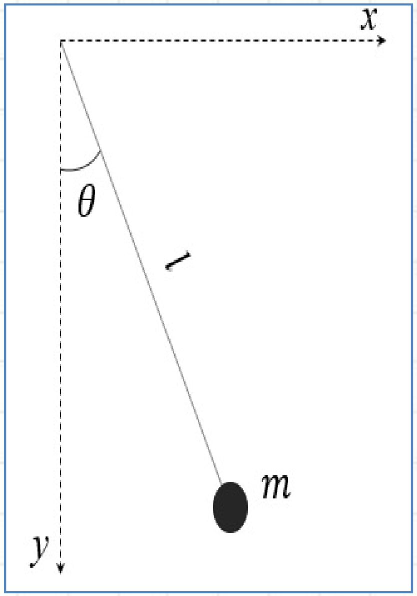
3- Simple Pendulum

Lagrange equation $\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{\theta}}\right) - \frac{\partial(L)}{\partial \theta} = 0$

- $\frac{\partial L}{\partial \dot{\theta}} = m\, l^2 \dot{\theta}$

- $\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m\, l^2 \dot{\theta}\right) = m\, l^2 \ddot{\theta}$

- $\frac{\partial(L)}{\partial q} = lmg\theta$

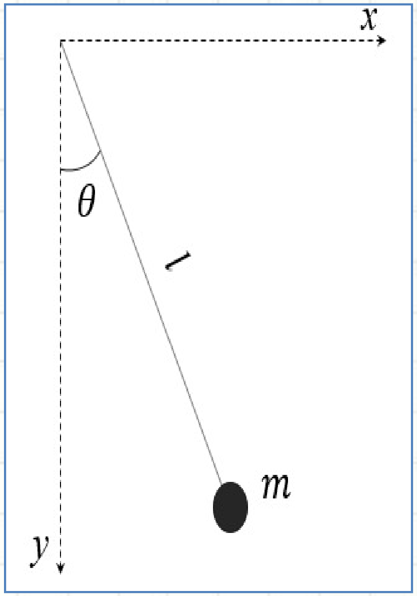


3- Simple Pendulum

$$\frac{d}{dt}\left(\frac{\partial(L)}{\partial \dot{q}}\right) - \frac{\partial(L)}{\partial q} = 0 \quad \Rightarrow \quad m\,l^2\ddot{\theta} + lmg\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \omega_n{}^2\,\theta = 0$$

Where $\omega_n = \sqrt{\frac{g}{l}}$ is the natural frequency of the system



3- Simple Pendulum

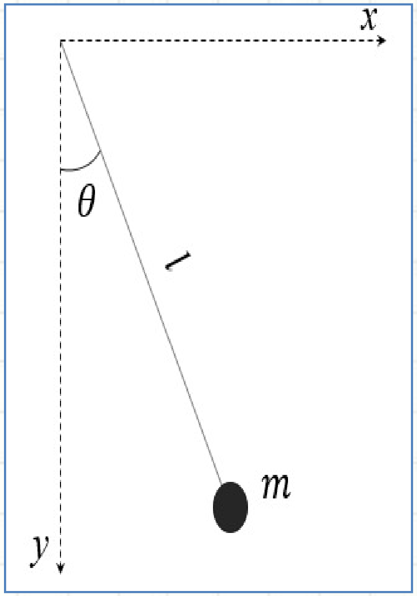
- $\ddot{\theta} + \omega_0^{\ 2}\,\theta = 0$

The general solution of this differential equation takes a form of

$$\theta(t) = \theta_0 \cos(wt + \varphi) \quad or \quad \theta(t) = \theta_0 \sin(wt + \varphi')$$

We can find the values of θ_0 and φ using initial conditions

- The solution is based on the initial conditions



4- LC circuit

4- LC circuit

Considérant un circuit LC

By applying Kirchhoff's loop rule:

$$V_c + V_L = 0 \quad \Rightarrow \quad \frac{q}{C} + L\frac{di}{dt} = 0 \quad \Rightarrow \quad \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\ddot{q} + \frac{q}{LC} = 0 \quad \Rightarrow \quad \ddot{q} + {\omega_n}^2 q = 0 \quad \Rightarrow \quad \omega_n = \sqrt{\frac{1}{LC}}$$

ω_n: The natural frequency of the circuit

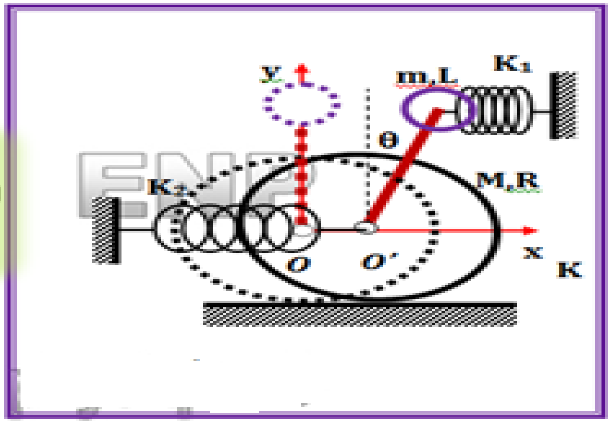
Electro-mechanical analogy for free oscillation

Mechanical system	Electrical system
$m\ddot{x} + kx = 0$	$L\ddot{q} + \frac{q}{c} = 0$
Elongation $\boldsymbol{x}$	Charge $\boldsymbol{q}$
Velocity $\dot{\boldsymbol{x}}$	Current $\boldsymbol{i}$
Mass $\boldsymbol{m}$	Inductance $\boldsymbol{L}$
Spring $\boldsymbol{k}$	Inverse of capacitance $\frac{1}{c}$
Kinetic energy $\frac{1}{2}\boldsymbol{m\dot{x}^2}$	Energy of the inductor $\frac{1}{2}L\dot{q}^2$
Potential energy $\frac{1}{2}kx^2$	Energy of the capacitor $\frac{1}{2c}q^2$

Application exercise

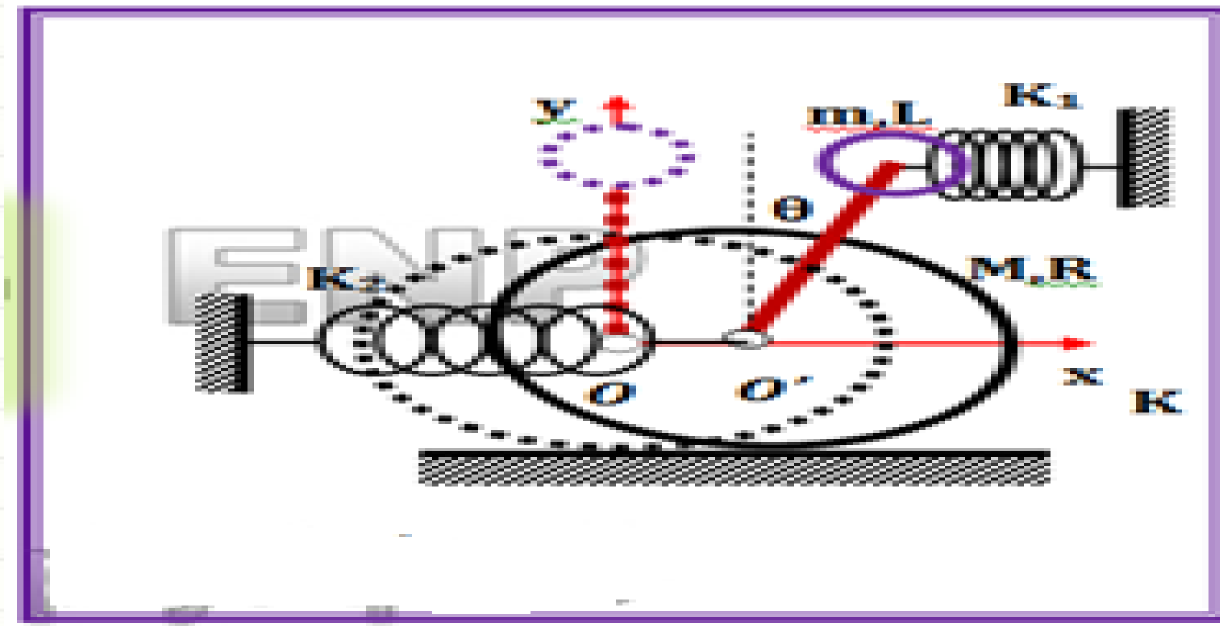
In the system shown in the figure, the cylinder (M, R) can roll freely on a horizontal support. Along a radius, a rod of negligible mass and length $L = 2R$ is welded, at the end of which a point mass m is attached. The cylinder is connected at its center of mass to a fixed wall by a spring with stiffness K_2, and the point mass is connected to another fixed wall using a spring with stiffness K_1. At equilibrium, both springs are neither stretched nor compressed, and the system's position is represented by dashed lines. For small oscillations, determine:

- The potential energy of the system and the condition for oscillation, if it exists
- The kinetic energy of the system
- The natural angular frequency of the oscillations



Application exercise

$$\vec{r}_m \begin{cases} R\theta + L\sin\theta \\ L\cos\theta \\ 0 \end{cases} \Rightarrow \overrightarrow{dr_m} \begin{cases} (R + L\cos\theta)\, d\theta \\ -L\sin\theta\, d\theta \\ 0 \end{cases} \Rightarrow \vec{V}_m \begin{cases} (R + L\cos\theta)\dot{\theta} \\ -L\sin\theta\, \dot{\theta} \\ 0 \end{cases}$$



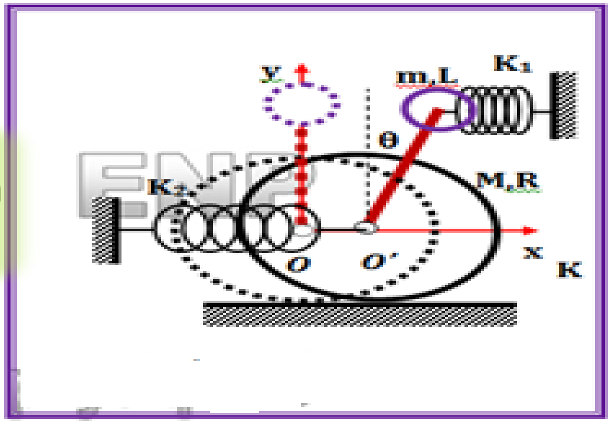
Application exercise

$$\mathbf{U} = U_m + U_{K_1} + U_{K_2}$$

$$\overrightarrow{W_m}\begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

$$U = -\int_0^\theta \overrightarrow{w_m}.\ \overrightarrow{dr_m} = -\int_0^\theta -mg.\ (-L\,sin\theta\ \ d\theta)$$

$$= \int_0^\theta mg\mathrm{L}(-\sin\theta)\,d\theta = mg\mathrm{L}[\cos\theta]_0^\theta = mg\mathrm{L}(\cos\theta - 1)$$



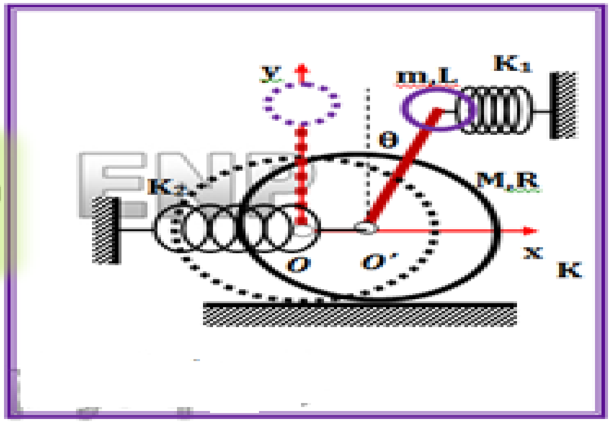
Application exercise

$$\cos\theta \cong 1 - \frac{\theta^2}{2} \Longrightarrow U_m = -mgL\frac{\theta^2}{2}$$

$$U_{K_1} = \frac{1}{2}K_1(R\theta + L\theta)^2$$

$$U_{K_1} = \frac{1}{2}K_1(R+L)^2\theta^2$$

$$U_{K_2} = \frac{1}{2}K_2R^2\theta^2$$

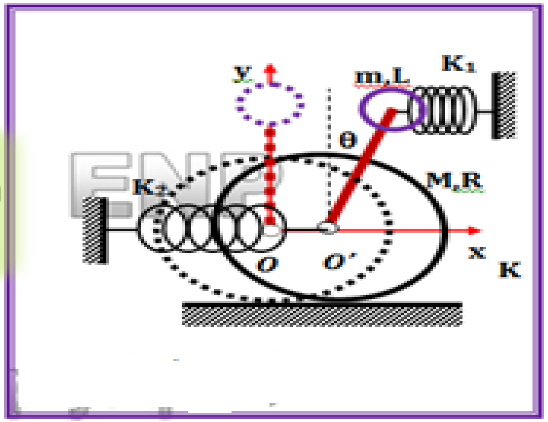


Application exercise

$$U = -mgL\frac{\theta^2}{2} + \frac{1}{2}(K_1(R+L)^2 + K_2R^2)\theta^2$$

- **Condition of oscillation**

$$\left.\frac{\partial^2 U}{\partial \theta}\right|_{\theta=0} > 0 \Rightarrow \boxed{K_1(R+L)^2 + K_2R^2 > mgL}$$



Application exercise

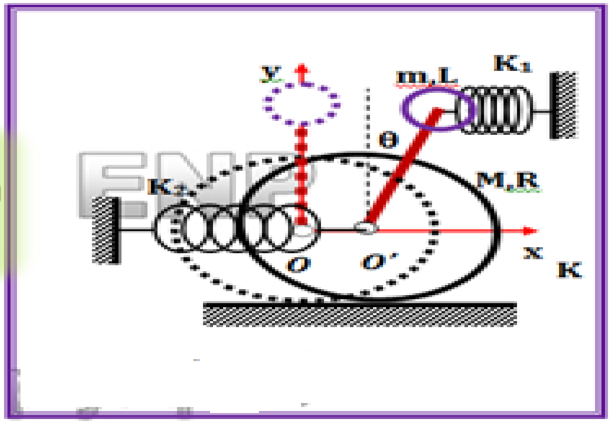
$$T = T_m + T_M$$

$$T = \frac{1}{2} m V_m^2 + \frac{1}{2} M V_M^2 + \frac{1}{2} I_{M/\Delta} \dot{\theta}^2$$

$$\vec{V}_m \begin{cases} (R + L\cos\theta)\dot{\theta} \\ -L\sin\theta\,\dot{\theta} \\ 0 \end{cases}$$

$$V_m^2 = ((R + L\cos\theta)\dot{\theta})^2 + (-L\sin\theta\,\dot{\theta})^2$$

$$V_m^2 = (R\dot{\theta})^2 + (2RL\cos\theta)(\dot{\theta})^2 + (L\cos\theta\,\dot{\theta})^2 + (L\sin\theta\,\dot{\theta})^2$$



$\cos\theta \cong 1$

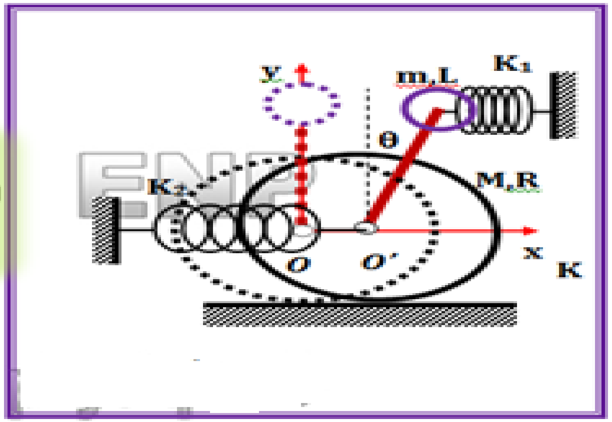
$$V_m^2 = (R^2 + 2RL + L^2)\dot{\theta}^2$$

Application exercise

$$V_m^2 = (R + L)^2 \dot{\theta}^2$$

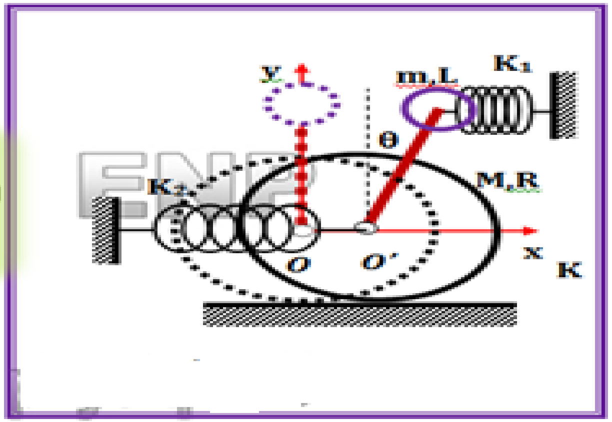
$$T = \frac{1}{2} m V_m^2 + \frac{1}{2} M V_M^2 + \frac{1}{2} I_{M/\Delta} \dot{\theta}^2$$

$$T = \frac{1}{2} m (R + L)^2 \dot{\theta}^2 + \frac{1}{2} M (R\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right) \dot{\theta}^2$$



Application exercise

$$T = \frac{1}{2}\left(m(R+L)^2 + \frac{3}{2}MR^2\right)\dot{\theta}^2$$



$$L = T - U$$

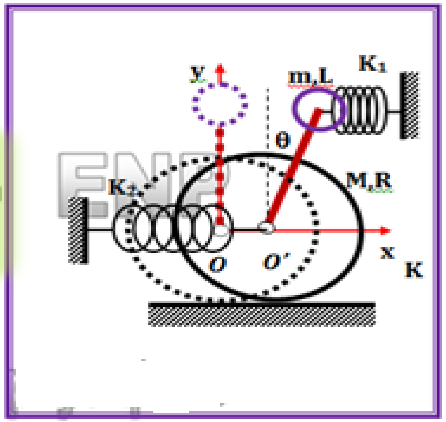
$$L = \frac{1}{2}\left(m(R+L)^2 + \frac{3}{2}MR^2\right)\dot{\theta}^2 - \frac{1}{2}\left(K_1(R+L)^2 + K_2R^2 - mgL\right)\theta^2$$

Application exercise

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

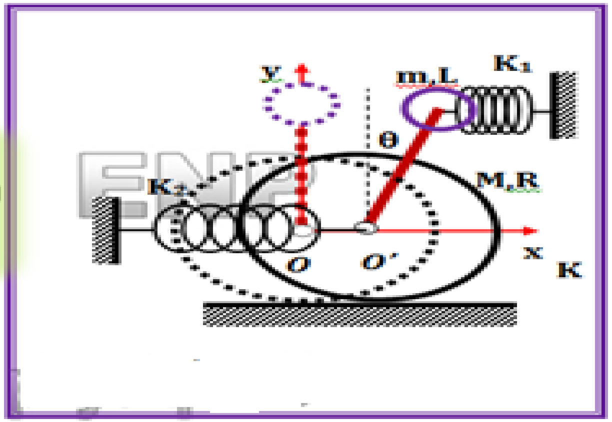
$$\left(m(R+L)^2 + \frac{3}{2}MR^2\right)\ddot{\theta} + (K_1(R+L)^2 + K_2R^2 - mgL)\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{K_1(R+L)^2 + K_2R^2 - mgL}{m(R+L)^2 + \frac{3}{2}MR^2}\theta = 0}$$



Application exercise

$$\omega = \sqrt{\frac{K_1(\mathrm{R}+\mathrm{L})^2 + K_2R^2 - mgL}{m(R+L)^2 + \frac{3}{2}MR^2}}$$



- **Condition of oscillation**

$$\boxed{K_1(\mathrm{R}+\mathrm{L})^2 + K_2R^2 > mgL}$$

Application exercise

➢ Condition of oscillation

$$\frac{\partial^2 U}{\partial \theta^2}\bigg|_{\theta=0} > 0 \Longrightarrow \boxed{K_1(R+L)^2 + K_2 R^2 > mgL}$$