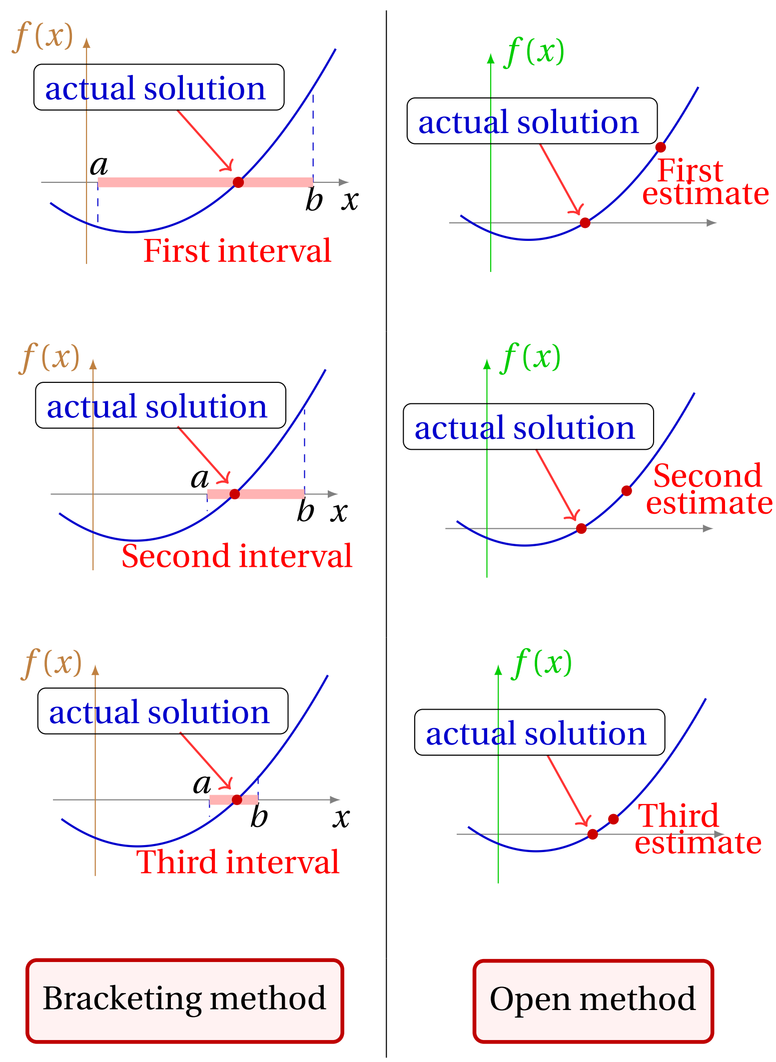
Sect.1 Solution of nonlinear equations

In this chapter we consider one of the most basic problems of numerical approximation, the **root-finding problem**. This process involves finding a **root** (or solution) of an equation of the form $f(x) = 0$, for a given function f. A root of this equation is also called a **zero** of the function f.

The process of solving an equation numerically is different from the procedure used to find an analytical solution. The latter is obtained by deriving an expression that has an exact numerical value. While the former is obtained in a process that starts by finding an approximate solution followed by a numerical procedure in which a more accurate solution is determined.

The methods used for solving equations numerically can be divided into two groups: bracketing methods and open methods. In bracketing methods, an interval that includes the solution is identified. Therefore, the endpoints of the interval are the upper bound and lower bound of the solution. Then, by using a numerical scheme, the size of the interval is successively reduced until the distance between the endpoints is less than the desired accuracy of the solution. In open methods, an initial estimate (one point) for the solution is assumed. The value of this initial guess for the solution should be close to the actual solution. Then, by using a numerical scheme, better values for the solution are calculated. It is worthy to mention that Bracketing methods always converge to the solution while open methods are usually more efficient but sometimes might not yield the solution.



We shall consider two bracketing methods: the bisection method and the false position method; and three open methods: Newton's method, secant method, and fixed-point iteration.

A root of an equation is usually computed in two stages. First, we find the location of a root in the form of a crude approximation of the root. Next we use an iterative technique for computing a better value of the root to a desired accuracy in successive approximations/computations. This is done by using an iterative function.

Sect.2 The bisection method

Based on the Intermediate Value Theorem, the **bisection method** is a root finding method which repeatedly bisects an interval and then selects a subinterval in which a root must lie for further processing. It is an extremely simple and robust method, but it is relatively slow.

Thus, the bisection method is the simplest method for finding a root to an equation. It needs two initial estimates x_a and x_b which bracket the root. Let f is a continuous function defined on the interval $[a,b]$, with $f(a)$ and $f(b)$ of opposite sign. Since $f(a)f(b)<0$, the function f changes sign on the interval $[a,b]$ and, therefore, it has at least one zero in the interval, i.e. a number p exists in $]a,b[$ with $f(p)=0$. This is a consequence of the Intermediate-Value Theorem for Continuous Functions, which asserts that if f is continuous on $[a,b]$, and if $f(a)<y<f(b)$, then $f(x)=y$, for some $x\in]a,b[$.

☞ The method calls for a repeated halving of subintervals of $[a,b]$ and, at each step, locating the half containing p.

To begin, set $a_1=a$ and $b_1=b$, and let p_1 be the midpoint of $[a,b]$, that is,

$$p_1 = a_1 + \frac{b_1-a_1}{2} = \frac{a_1+b_1}{2}$$

- ✓ If $f(p_1)=0$, then $p=p_1$, and we are done.
- ✓ If $f(p_1)\neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
  - ✗ If $f(p_1)$ and $f(a_1)$ have the same sign, then $p\in]p_1,b_1[$. Set $a_2=p_1$ and $b_2=b_1$.
  - ✗ If $f(p_1)$ and $f(a_1)$ have opposite signs, then $p\in]a_1,p_1[$. Set $a_2=a_1$ and $b_2=p_1$.
- ✓ Then reapply the process to the interval $[a_2,b_2]$.

An interval $[a_{n+1}, b_{n+1}]$ containing an approximation to a root of $f(x) = 0$ is constructed from an interval $[a_n, b_n]$ containing the root by first letting

$$p_n = a_n + \frac{b_n - a_n}{2}$$

Then set

$$a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = p_n \quad \text{if} \quad f(a_n) f\left(p_n\right) < 0$$

otherwise we set

$$a_{n+1} = p_n \quad \text{and} \quad b_{n+1} = b_n.$$

We repeat this process until the latest interval (which contains the root) is as small as desired, say ϵ. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n$^{\text{th}}$ step, the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$. We then have

$$\frac{|b-a|}{2^n} < \epsilon \qquad \Longrightarrow \qquad \frac{1}{\ln(2)} \ln\left(\frac{|b-a|}{\epsilon}\right) \leqslant n \tag{1}$$

Equation (1) gives the number of iterations required to achieve an accuracy ϵ. For example, if $|b-a| = 1$ and $\epsilon = 0.001$, then it can be seen that $n \geqslant 10$.

Theorem 1.
Suppose that f continuous on $[a, b]$ and $f(a) f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$\left|p_n - p\right| \leqslant \frac{b-a}{2^n}, \qquad n \geqslant 1.$$

Proof. For each $p \geqslant 1$, we have

$$|b_n - a_n| = \frac{1}{2^{n-1}} (b-a), \qquad \text{with} \qquad p \in]a_n, b_n[$$

Since $p_n = (a_n + b_n)/2$ for all $n \geqslant 1$, it follows that

$$\left|p_n - p\right| \leqslant \frac{b_n - a_n}{2} = \frac{b-a}{2^n}.$$

□

It is worthy to notice that the bisection method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods.

Example 1.

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Since $f(1) = -1$ *is negative and* $f(2) = 5$ *positive, a root lies between* 1 *and* 2 *and, therefore, we take* $x_0 = 3/2$. *Then*

$$f(x_0) = \frac{27}{8} - \frac{3}{2} - 1 = \frac{7}{8} > 0$$

Hence the root lies between 1 *and* 1.5 *and we obtain*

$$x_1 = \frac{1+1.5}{2} = 1.25$$

We find $f(1.25) = -19/64$, *which is negative. We, therefore, conclude that the root lies between* 1.25 *and* 1.5. *If follows that*

$$x_2 = \frac{1.25+1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \qquad x_4 = 1.34375, \qquad x_5 = 1.328125, \qquad x_6 = 1.3203125, \qquad \ldots$$

Example 2.

Find a real root of the following equations

a $2x^3 - x^2 + x = 1$, b $x^4 - 2x^3 - 4x^2 + 4x = -4$, c $(x+1)^2 e^{x^2-2} = 1$

Algorithm 1: Given a function f continuous on the $[a_0, b_0]$ and such that $f(a_0)f(b_0) < 0$.

Algorithm 1: Bisection Method

input : f, a, b, ϵ initialization $fplot(f, [a, b])$

while $abs(a-b)/2 > \epsilon$ **do**

    $c = (a+b)/2$

    if $f(a) * f(b) > 0$ **then**

        $b = c$

    else

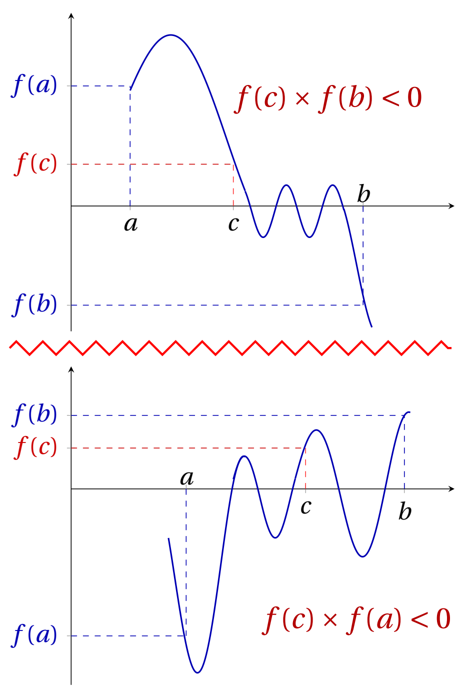
        $a = c$

    end

end

output: display estRoot= $(a+b)/2$,

% (check that f(estRoot) is indeed small)



The bound for the number of iterations for the Bisection method assumes that the calculations are performed using infinite-digit arithmetic. When implementing the method on a computer, we need to consider the effects of round-off error. For example, the computation of the midpoint of the interval $[a_n, b_n]$ should be found from the equation

$$p_n = a_n + \frac{b_n - a_n}{2} \qquad \text{instead of} \qquad \frac{b_n + a_n}{2}$$

in order to adhere to the general stratagem that in numerical calculations it is best to compute a quantity by adding a small correction term to a previous approximation.

Sect.3 Fixed-Point Iteration

A fixed point for a function is a number at which the value of the function does not change when the function is applied. Fixed-point results occur in many areas of mathematics, and are a major tool of economists for proving results concerning equilibria.

Definition 1.
*The number p is a **fixed point** for a given function f if $f(p) = p$.*

Root-finding problems and fixed-point problems are equivalent classes in the following sense:

✓ Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as

$$g(x) = x - f(x), \qquad \text{or} \qquad g(x) = x + af(x)$$

✓ Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.

Example 3.
Determine any fixed points of the function $g(x) = x^2 + 3x - 3$.
A fixed point p for g has the property that

$$p = g(p) \quad \textit{which implies that} \quad p^2 + 3p - 3 = p \quad \Longleftrightarrow \quad (p+3)(p-1) = 0$$

A fixed point for g occurs precisely when the graph of $y = g(x)$ intersects the graph of $y = x$, so g has two fixed points, one at $p = 1$ and the other at $p = -3$.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem 2.
1) *If g is continuous on $[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.*
2) *If, in addition, $g'(x)$ exists on $]a,b[$ and a positive constant $k < 1$ exists with*

$$\left|g'(x)\right| \leqslant k, \qquad \forall x \in]a,b[$$

then there is exactly one fixed point in $[a,b]$.

Proof. **1)** If $g(a) = a$ and $g(b) = b$, then g has a fixed point at an endpoint. If not, then $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a,b]$, with

$$h(a) = g(a) - a > 0 \qquad \text{and} \qquad h(b) = g(b) - b < 0,$$

because $g(x) \in [a,b]$. The Intermediate Value Theorem implies that there exists $p \in]a,b[$ for which $h(p) = 0$. This number p is a fixed point for g because

$$0 = h(p) = g(p) - p \qquad \text{implies that} \qquad g(p) = p$$

2) Suppose, in addition, that $\left|g'(x)\right| \leqslant k < 1$ and that p and q are both fixed points in $[a,b]$. If $p \neq q$, then the Mean Value Theorem implies that a number ξ exists between p and q, and hence in $[a,b]$, with

$$\frac{g(a) - g(b)}{a - b} = g'(\xi)$$

Thus

$$|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leqslant k|p - q| < |p - q|.$$

which is a contradiction. This contradiction must come from the only supposition, $p \neq q$. Hence $p = q$ and the fixed point in $[a,b]$ is unique. □

Example 4.
Consider the function $f(x) = e^x - 2x - 1$ on $[1,2]$. We have $f(1) < 0$ and $f(2) > 0$, then f has a zero in $[1,2]$. Let $g(x) = (e^x - 1)/2$. Therefore, the equation $g(x) = x$ is equivalent to $f(x) = 0$. We see that g is continuous but $g(2) \notin [1,2]$. Set $g(x) = \ln(2x + 1)$, this function is continuous, increasing and $g([1,2]) \subset [1,2]$ and then satisfies the hypothesis of Theorem 2.

Example 5.
Show that $g(x) = (x^2 - 1)/3$ *has a unique fixed point on the interval* $[-1, 1]$.

Since $g'(x) = 2x/3$, *the function* g *is continuous and* $g'(x)$ *exists on* $[-1, 1]$. *The maximum and minimum values of* $g(x)$ *occur at* $x = -1$, $x = 0$, $x = 1$. *But* $g(-1) = 0$, $g(1) = 0$ *and* $g(0) = -1/3$, *so an absolute maximum for* $g(x)$ *on* $[-1, 1]$ *occurs at* $x = -1$ *and* $x = 1$, *and an absolute minimum at* $x = 0$. *Moreover*

$$\left|g'(x)\right| = \left|\frac{2x}{3}\right| \leqslant \left|\frac{2}{3}\right|, \qquad \forall x \in]-1, 1[$$

So g *satisfies all the hypotheses of the Theorem 2 and has a unique fixed point in* $[-1, 1]$.

For the function in latter Example, the unique fixed point p in the interval $[-1, 1]$ can be determined algebraically. If

$$p = g(p) = \frac{p^2 - 1}{3} \quad \text{then} \quad p^2 - 3p - 1 = 0$$

which, by the quadratic formula, implies, that

$$p = \frac{1}{2}\left(3 - \sqrt{13}\right).$$

Note that g also has a unique fixed point $p = \frac{1}{2}\left(3 + \sqrt{13}\right)$ for the interval $[3, 4]$. However, $g(4) = 5$ and $g'(4) = \frac{8}{3} > 1$, so g does not satisfy the hypotheses of above Theorem on $[3, 4]$. This demonstrates that the hypotheses of the latter Theorem 2 are sufficient to guarantee a unique fixed point but are not necessary.

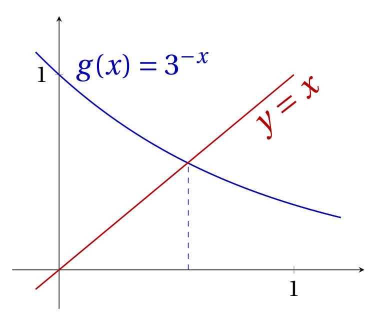
Example 6.
Show that Theorem 2 does not ensure a unique fixed point of $g(x) = 3^{-x}$ *on the interval* $[0, 1]$, *even though a unique fixed point on this interval does exist.*
$g'(x) = -3^{-x}\ln(3) < 0$ *on* $[0, 1]$ *the function* g *is strictly decreasing on* $[0, 1]$. *So*

$$g(1) = \frac{1}{3} \leqslant g(x) \leqslant 1 = g(0), \qquad 0 \leqslant x \leqslant 1$$

Thus, for $x \in [0, 1]$, *we have* $g(x) \in [0, 1]$. *The first part of the Theorem ensures that there is at least one fixed point in* $[0, 1]$. *However,*

$$g'(0) = -\ln(3) < -1$$

$|g'(x)| \nleqslant 1$ *on* $[0, 1]$ *and Theorem 2 cannot be used to determine uniqueness. But* g *is always decreasing, and it is clear from its graph that the fixed point must be unique.*



The convergence

To find a root p of $f(x) = 0$, we must therefore construct a sequence (x_r) which satisfies two criteria:

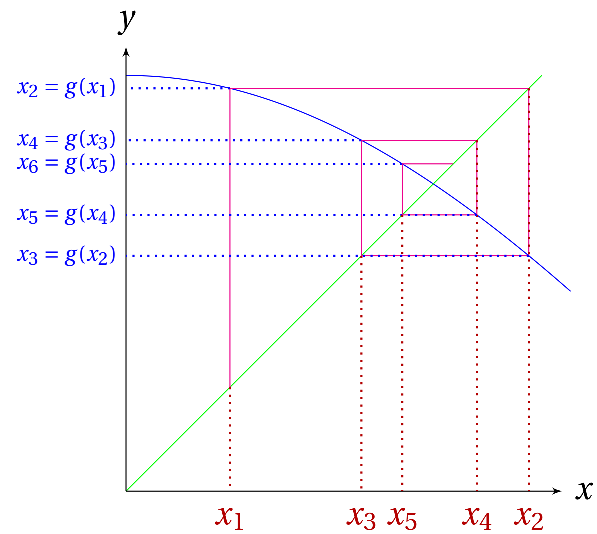
(i) the sequence (x_r) converges to p.

(ii) x_{r+1} depends directly only on its predecessor x_r.

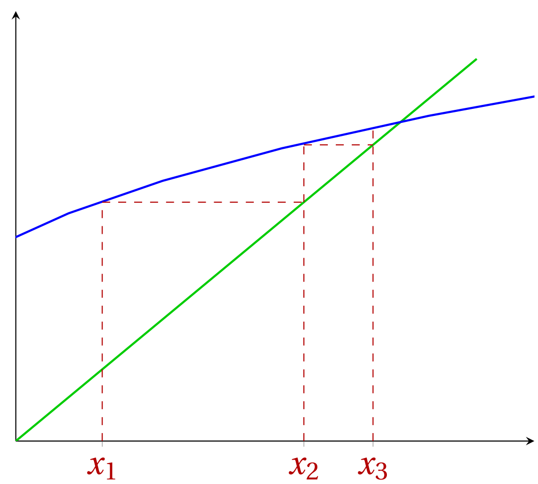
From (ii), we need to find some function g, so that the sequence (x_r) may be computed from

$$x_{r+1} = g(x_r), \qquad r = 0, 1, \dots \tag{2}$$

given an initial value x_0.



Example 7.
Consider the equation $x = e^{-x}$ which is already of the form $x = g(x)$. The graphs of $y = x$ and $y = e^{-x}$ intersect at only one point $x = p \in]0, 1[$, which is therefore the only root of $x = e^{-x}$. Following (2), let us choose $x_0 = 0.5$ and compute x_r, $r = 0, 1, \dots$ Rounding the iterates to three decimal places at each stage, we obtain the numbers displayed in the following table.



r	0	1	2	3	4	5	6	7	8	9
x_r	0.5	0.607	0.545	0.580	0.560	0.571	0.565	0.568	0.567	0.567
$x_r - p$	-0.067	0.040	-0.022	0.013	-0.007	0.004	-0.002	0.001	0.000	0.000

To investigate the behaviour of errors in the general case, we return to (2) and subtract from it the equation $p = g(p)$. Thus

$$x_{r+1} - p = g(x_r) - g(p) = (x_r - p)g'(\xi), \qquad x_r < \xi < p \tag{3}$$

This last step follows from the mean value theorem, assuming that g' exists on an interval containing x_r and p. It is easy to see how to impose conditions on g to ensure convergence of the one-point iterative method. Indeed, the uniqueness condition of Theorem 2, i.e. $|g'(x)| \leqslant k < 1$, shows that

$$|x_r - p| \leqslant k|x_{r-1} - p| \leqslant \ldots \leqslant k^r |x_0 - p|, \qquad r \geqslant 0. \tag{4}$$

Since $0 < k < 1$, then $k^r \to 0$ as $r \to \infty$. Therefore, we deduce from (4) that $|x_r - p| \to 0$ as $r \to \infty$ and the sequence (x_r) converges to the root p. Thus, we have the following

Corollary 1.
If g satisfies the hypotheses of Theorem 2, then

❶ *for any number $p_0 \in [a,b]$, the sequence defined by*

$$p_n = g(p_{n-1}), \qquad n \geqslant 1$$

converges to the unique fixed point p in $[a,b]$.

❷ *The bounds for the error involved in using p_n to approximate p are given by*

$$\text{(i)} \quad |p_n - p| \leqslant k^n \max\{p_0 - a, p_0 - b\}, \qquad \forall n \geqslant 1. \tag{5}$$

$$\text{(ii)} \quad |p_n - p| \leqslant \frac{k^n}{1-k}|p_1 - p_0|, \qquad \forall n \geqslant 1. \tag{6}$$

Proof. ❶ First, theorem 2 shows that a unique fixed point $p \in [a,b]$ exists. Since g maps $[a,b]$ into itself, the sequence $\{p_n\}_n$ satisfies $p_n \in [a,b]$, for all $n \geqslant 0$. Using the fact that $g'(x) \leqslant k$, we obtain for each n

$$\left|p_n - p\right| = \left|g(p_{n-1}) - g(p)\right| = \left|g'(\xi_n)\right|\left|p_{n-1} - p\right| \leqslant k\left|p_{n-1} - p\right| \leqslant \ldots \leqslant k^n\left|p_0 - p\right|, \tag{7}$$

where $\xi_n \in]a,b[$. Next, since $0 < k < 1$, then $k^n \to 0$ as $n \to \infty$. We obtain

$$0 \leqslant \left|p_n - p\right| \leqslant \lim_{n\to\infty} k^n\left|p_0 - p\right| = 0,$$

which shows that $\{p_n\}_n$ converges to p.
❷ Because $p \in [a,b]$, the first bound follows from Inequality (7)

$$\left|p_n - p\right| \leqslant k^n\left|p_0 - p\right| \leqslant k^n \max\{p_0 - a, p_0 - b\}.$$

Now, for $n \geqslant 1$, we have

$$\left|p_{n+1} - p_n\right| = \left|g(p_n) - g(p_{n-1})\right| \leqslant k\left|p_n - p_{n-1}\right| \leqslant \ldots \leqslant k^n\left|p_1 - p_0\right|$$

Thus, for $m > n \geqslant 1$, we have

$$\begin{aligned}
|p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \dots + p_{n+1} - p_n| \\
&\leqslant |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \dots + |p_{n+1} - p_n| \\
&\leqslant k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \dots + k|p_1 - p_0| \\
&= k^n|p_1 - p_0|\left(1 + k + k^2 + \dots + k^{m-n-1}\right)
\end{aligned}$$

By Theorem 2, we have $\lim_{m\to\infty} p_m = p$, then

$$|p - p_n| = \lim_{m\to\infty} |p_m - p_n| \leqslant \lim_{m\to\infty} k^n|p_1 - p_0| \sum_{i=0}^{n-m-1} k^i \leqslant k^n|p_1 - p_0| \sum_{i=0}^{\infty} k^i = \frac{k^n}{1-k}|p_1 - p_0|.$$

The latter sum is a geometric series with ration $0 < k < 1$ which converges to $1/1-k$. □

Example 8.

Show that the equation $x^3+4x^2-10=0$ has a unique solution on $[1,2]$. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. Should verify that the fixed point of each functions in below is actually a solution to the original equation,$x^3+4x^2-10=0$.

(i) $g_1(x) = \left(\dfrac{10-x^3}{4}\right)^{1/2}$

(ii) $g_2(x) = \left(10-4x^2\right)^{1/3}$

(iii) $g_3(x) = x - \dfrac{x^3+4x^2-10}{3x^2+8x}$

(iv) $g_4(x) = \left(\dfrac{10}{4+x}\right)^{1/2}$

n	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	Bisection
0	1.5	1.5		1.5	1.5
1	1.286	1.000		1.348	1.250
2	1.403	1.810		1.364	1.375
3	1.345	-1.43		1.365	1.312
4	1.364	1.22		1.365	1.340
5	1.365	1.592		1.365	1.357
6					
7					
8					
9					
10					

Remark that the function g_2 do not satisfy the hypotheses of theorem 2, indeed, $g(-2) \not\in [1,2]$.

By regarding the condition (3), we can give another criterion for the convergence. First

Definition 2.

*A function f is said to be **contractive** if there exists a number k less than 1 such that*

$$|f(x) - f(t)| \leqslant k|x - t|, \tag{8}$$

for all x and t in the domain of f.

Theorem 3 (Contractive Mapping Theorem).
Let f be a contractive function of a closed interval $[a,b]$ into $[a,b]$. Then f has a unique fixed point. Moreover, this fixed point is the limit of every sequence obtained from Equation (2) with any starting point $x_0 \in [a,b]$ with the estimations (5)-(6).

Proof. We use the contractive property (8) together with Equation (2) to write

$$\left|x_n - x_{n-1}\right| = \left|f(x_{n-1}) - f(x_{n-2})\right| \leqslant k\left|x_{n-1} - x_{n-2}\right| \leqslant \ldots \leqslant k^{n-1}\left|x_1 - x_0\right|.$$

Now the conclusion follows using the same argument in the proof of Corollary 1. □

Example 9.
Prove that the sequence $\{x_n\}$ defined recursively as follows is convergent

$$\begin{cases} x_0 = -4 \\ x_n = 3 - \dfrac{1}{2}|x_{n-1}|, \qquad n \geqslant 1. \end{cases}$$

The function $f(x) = 3 - \frac{1}{2}|x|$ is a contraction because

$$\left|f(x) - f(t)\right| = \left|3 - \frac{1}{2}|x| - 3 + \frac{1}{2}|t|\right| = \frac{1}{2}\left||t| - |x|\right| \leqslant \frac{1}{2}\left|t - x\right|.$$

By the Contractive Mapping Theorem, the sequence described must converge to the unique fixed point of f which is clearly equals 2.

Example 10.
Use the Contractive Mapping Theorem to compute a fixed point of the function

$$f(x) = 7 + \frac{2}{5}\sin(2x)$$

Proposition 1.
If a function g satisfies the hypotheses of Theorem 2, then g is a contraction.

Proof. By the mean value theorem we have

$$\left|g(x) - g(t)\right| = \left|g'(c)\right|\left|x - t\right| \leqslant k\left|x - t\right|, \quad \text{with} \quad k < 1,$$

whence the result. □

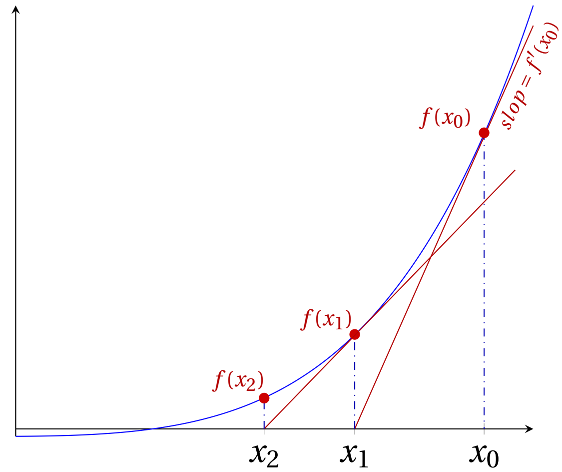
Sect.4 Newton's method

Suppose we wish to solve $f(x) = 0$ and we are given an initial guess x_0 for the solution and that f' exists in an interval containing all the approximations to the exact solution p.

The slope of the tangent line to the graph of f at the point $\big(x_0, f(x_0)\big)$ is $f'(x_0)$, so the equation of this tangent line is

$$y - f(x_0) = f'(x_0)\,(x - x_0)$$

The second estimate x_1 is obtained by taking the tangent line to $f(x)$ at the point $\big(x_0, f(x_0)\big)$ and finding the intersection point of the tangent line with the x-axis. In general, it might give a better approximation to a root than x_0 does, as illustrated in figures.



Since this line crosses the x-axis when the y-coordinate of the point on the line is zero, the next approximation, x_1, to p satisfies

$$0 - f(x_0) = f'(x_0)\big(x_1 - x_0\big)$$

which implies that

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}, \quad f'(x_0) \neq 0.$$

Letting x_2 be the point where the tangent line to $f(x)$ at the point $\big(x_1, f(x_1)\big)$ hits the x-axis, this process can be repeated by constructing the tangent line to f at x_2, determining where it hits the x-axis and calling that point x_3, and so on. This process is known as **Newton's method**. The general iteration formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{9}$$

Newton's method is a functional iteration technique with $x_{n+1} = g(x_n)$, for which

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{10}$$

Example 11.
Consider the function $f(x) = \cos(x) - x$. Approximate a root of f using:
❶ *a fixed-point method.* ❷ *Newton's Method.*
A solution to this root-finding problem is also a solution to the fixed-point problem $x = \cos(x)$ and the graph of $\cos(x)$ shows that a single fixed-point p lies in $[0, \pi/2]$. With $p_0 = \pi/4$, the fixed-point iteration gives a best approximation results of $p_7 \approx 0.7361282565$.

To apply Newton's method to this problem we need $f'(x) = -\sin(x) - 1$. Starting again with $p_0 = \pi/4$, we generate the sequence defined, for $n \geqslant 1$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1} = x_n + \frac{\cos(x_n) - x_n}{\sin(x_n) + 1}$$

An excellent approximation is obtained with $n = 3$, i.e. $p_3 \approx 0.7390851332$

Example 12.
Use $x_0 = 1$ to find an approximate solution to $\sqrt[3]{x} = 0$.
Clearly the solution is $x = 0$. Let us get the general formula for Newton' method.

$$x_{n+1} = x_n - \frac{x^{1/3}}{\frac{1}{3}x^{-2/3}} = x_n - 3x_n = -2x_n.$$

These computations get farther and farther away from the solution $x = 0$, with each iteration. For instance, $x_1 = -2$, $x_2 = 4$, $x_3 = -8$, $x_3 = 16$ ...

The convergence

....

Sect.5 The Secant Method

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation. Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.
The secant method uses two points in the neighborhood of the solution to determine a new estimate for the solution. The two points (marked as x_0 and x_1 in the are used to define a straight line (secant line), and the point where the line intersects the x-axis is the new estimate for the solution, x_3. As shown, the two points can be on one side of the solution or the solution can be between the two points. The slope of the secant line is given by:

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_1) - 0}{x_1 - x_2}$$

which gives

$$x_2 = x_1 - \frac{f(x_1)\big(x_1 - x_0\big)}{f(x_1) - f(x_0)}.$$

Once point x_2 is determined, it is used together with point x_1 to calculate the next estimate of the solution, x_3. The procedure generates an iteration formula in which a new estimate of the solution x_{k+1} is determined from the previous two solutions x_k and x_{k-1}. Therefore, the secant iteration is as follows

$$x_{k+1} = x_k - \frac{f(x_k)\big(x_k - x_{k-1}\big)}{f(x_k) - f(x_{k-1})}. \tag{11}$$

Relationship to Newton's method

Examination of the secant method shows that when the two points that define the secant line are close to each other, the method is actually an approximated form of Newton's method. This can be seen by rewriting (11) in the form

$$x_{k+1} = x_k - \frac{f(x_k)}{\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}} \tag{12}$$