# Chapter 2. Kinematics

**Definition:** Kinematics is the study of motion in space and time, independent of the causes that produce it and the phenomena that influence it. The position of the material point $P$ is determined in space at each moment of the motion.

# II. Motion of one reference frame relative to another

Let's consider $R_i(O_i, x_i, y_i, z_i)$ a fixed reference frame linked to the observer, and $R_k(O_k, x_k, y_k, z_k)$ a moving reference frame relative to $R_i$. The motion of $R_k$ is fully known if:

- The position of $O_k$ is completely known in $R_i$ ;
- And the orientation of the axes of $R_k$ is known relative to those of $R_i$

## II.1. Positioning of $\boldsymbol{O_k}$

The positioning of $\boldsymbol{O_k}$ is determined by the components of the vector $\overrightarrow{\boldsymbol{O_iO_k}}$ in $R_i$ or $R_k$ :

$$\overrightarrow{\boldsymbol{O_iO_k}} = \begin{Bmatrix} \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{x}}_{\boldsymbol{i}} \\ \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{y}}_{\boldsymbol{i}} \\ \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{z}}_{\boldsymbol{i}} \end{Bmatrix}_{/R_i} = \begin{Bmatrix} \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{x}}_{\boldsymbol{k}} \\ \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{y}}_{\boldsymbol{k}} \\ \overrightarrow{\boldsymbol{O_iO_k}}.\vec{\boldsymbol{z}}_{\boldsymbol{k}} \end{Bmatrix}_{/R_k}$$

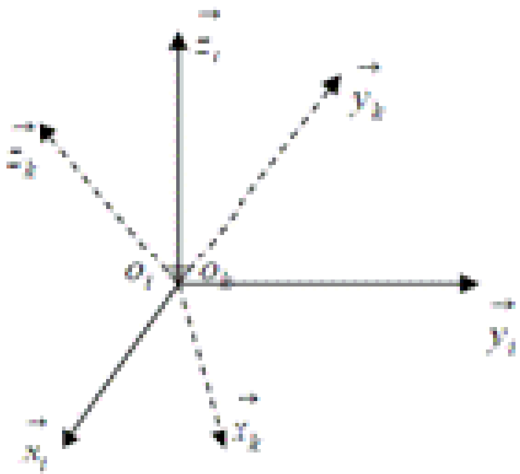
## II.2. Positioning of the orientation of the axes of $\boldsymbol{R_k}$

To determine the orientation of the axes of $R_k$, we bring$R_k$ en $O_i$ in such a way that the centers $O_i$ and $O_k$ coincide. The axes of the reference frame $R_k$ are given as follows:

$$\begin{cases} \vec{x}_k = a_{11}\,\vec{x}_i + a_{12}\,\vec{y}_i + a_{13}\,\vec{z}_i \\ \vec{y}_k = a_{21}\,\vec{x}_i + a_{22}\,\vec{y}_i + a_{23}\,\vec{z}_i \\ \vec{z}_k = a_{31}\,\vec{x}_i + a_{32}\,\vec{y}_i + a_{33}\,\vec{z}_i \end{cases}$$

or in matrix form

$$\begin{Bmatrix} \vec{x}_k \\ \vec{y}_k \\ \vec{z}_k \end{Bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\substack{Transiti\ on\ matrix\ \ from\ R_k\ to\ R_i \\ P_{k\rightarrow i}}} \begin{Bmatrix} \vec{x}_i \\ \vec{y}_i \\ \vec{z}_i \end{Bmatrix}$$



# II.3. Notation

Let $R_i(O_i, x_i, y_i, z_i)$ be a reference frame and $M$ a point in motion. We denote the velocity and acceleration of point M relative to the reference frame $R_i$ as follows:

$$\vec{V}^i(M) = \frac{d^i\overrightarrow{O_iM}}{dt}$$

$$\vec{\gamma}^i(M) = \frac{d^i\vec{V}^i(M)}{dt}$$

such that the index « i » is associated with the reference frame relative to which the motion is considered.

# II.4. Mobile basis formula

Let $R_i(O_i, x_i, y_i, z_i)$ be a fixed reference frame linked to the observer, and $R_k(O_k, x_k, y_k, z_k)$ be a reference frame obtained by rotating $R_i$ around an axis $\vec{U}$ by an angle $\theta$ such that:

$$\vec{U}\begin{Bmatrix} a \\ b \\ c \end{Bmatrix}_{R_k}$$

We then define the instantaneous rotation velocity vector as follows:

$$\vec{\Omega}_k^i = \dot{\theta}\,\vec{U} = \dot{\theta}\,(a\,\vec{x}_k + b\,\vec{y}_k + c\,\vec{z}_k)$$

## Properties of the vector $\vec{\boldsymbol{\Omega}}_{\boldsymbol{k}}^{\boldsymbol{i}}$

*a)* The vector $\vec{\Omega}_k^i$ is antisymmetric with respect to the indices.

$$\vec{\Omega}_k^i = -\vec{\Omega}_i^k$$

*b)* Principle of composition

$$\vec{\Omega}_k^i = \vec{\Omega}_j^i + \vec{\Omega}_k^j$$

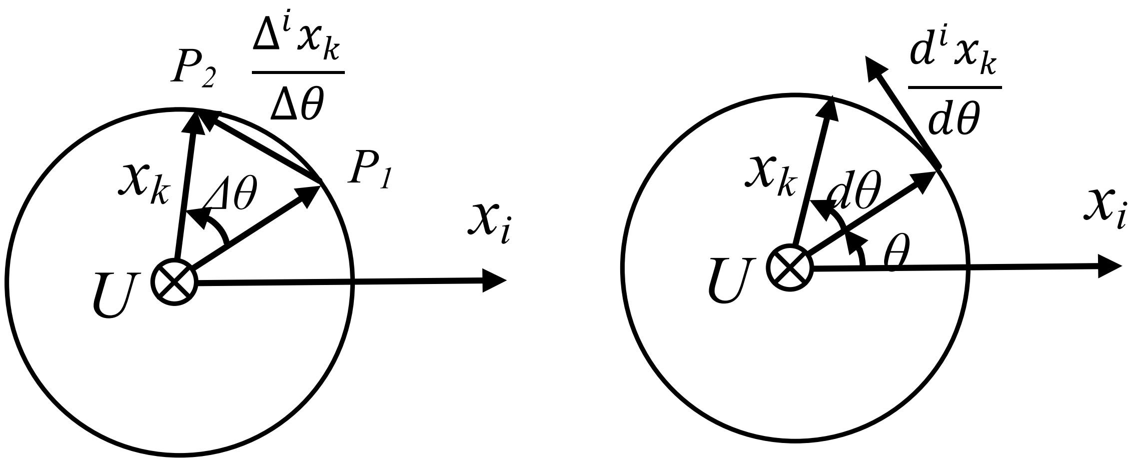
*c)* Equality of derivatives with respect to the indices

$$\frac{d^i \vec{\Omega}_k^i}{dt} = \frac{d^k \vec{\Omega}_k^i}{dt}$$

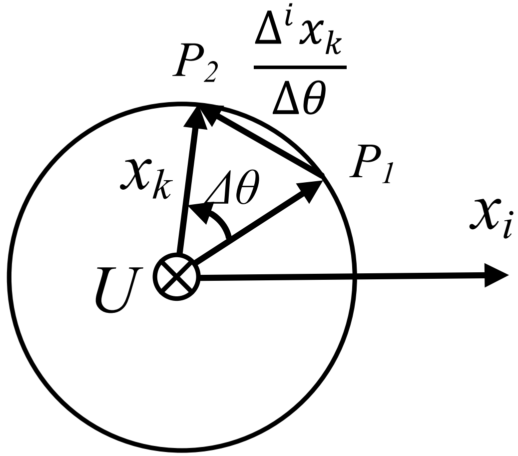
**Calculation of the derivative of $\vec{\boldsymbol{x}}_{\boldsymbol{k}}$ with respect to the reference frame $\boldsymbol{R_i}$**

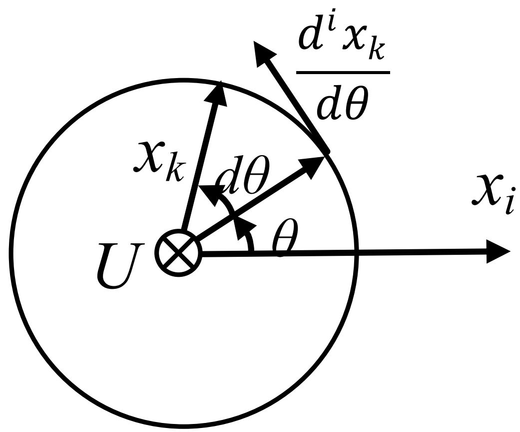
*If* $x_k$ rotates around the axis $\vec{U}$ by an angle $\Delta\theta$ *then :*

$$\frac{\Delta^i \overrightarrow{x_k}}{\Delta\theta} = \frac{\overrightarrow{OP_2} - \overrightarrow{OP_1}}{\Delta\theta} = \frac{\overrightarrow{P_1P_2}}{\Delta\theta}$$



$$\frac{\Delta^i \overrightarrow{x_k}}{\Delta\theta} = \frac{\overrightarrow{OP_2} - \overrightarrow{OP_1}}{\Delta\theta} = \frac{\overrightarrow{P_1P_2}}{\Delta\theta}$$




For an instantaneous rotation, the vector $\overrightarrow{P_1P_2}$ becomes tangent to the circle described by the rotation of $\overrightarrow{x_k}$. Thus $\overrightarrow{P_1P_2}$ becomes perpendicular to $\vec{U}$ and $\overrightarrow{x_k}$

$$\frac{d^i \overrightarrow{x_k}}{d\theta} \perp \overrightarrow{x_k} \text{ et } \frac{d^i \overrightarrow{x_k}}{d\theta} \perp \vec{U}$$

Then : $\frac{d^i \overrightarrow{x_k}}{d\theta} = \vec{U} \wedge \overrightarrow{x_k}$

$$\frac{d^i\overrightarrow{x_k}}{d\theta} = \vec{U} \wedge \overrightarrow{x_k}$$

$$\frac{d^i\overrightarrow{x_k}}{dt} = \frac{d^i\overrightarrow{x_k}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\vec{U} \wedge \overrightarrow{x_k} = \vec{\Omega}_k^i \wedge \overrightarrow{x_k}$$

In the same way, it can be shown that:

$$\frac{d^i\overrightarrow{y_k}}{dt} = \vec{\Omega}_k^i \wedge \overrightarrow{y_k}$$

*and*

$$\frac{d^i\overrightarrow{z_k}}{dt} = \vec{\Omega}_k^i \wedge \overrightarrow{z_k}$$

## *II.5.* Derivative in $R_i$ of a vector expressed in $R_k$

Let $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a fixed reference frame linked to the observer, and $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ be a moving reference frame relative to $R_i$.

Let a vector $V$ be expressed in the reference frame $R_k$ as follows:

$$\vec{V}(t) = a_k(t).\vec{x}_k + b_k(t).\vec{y}_k + c_k(t).\vec{z}_k$$

Its derivative in the reference frame $R_k$:

$$\frac{d^k\vec{V}}{dt} = \dot{a}_k(t).\vec{x}_k + \dot{b}_k(t).\vec{y}_k + \dot{c}_k(t).\vec{z}_k \Rightarrow \frac{d^k\vec{V}}{dt} = \begin{Bmatrix} \dot{a}_k(t) \\ \dot{b}_k(t) \\ \dot{c}_k(t) \end{Bmatrix}_{R_k}$$

Its derivative in the reference frame $R_i$:

$$\frac{d^i\vec{V}}{dt} = \dot{a}_k(t).\vec{x}_k + a_k(t).\frac{d^i\vec{x}_k}{dt} + \dot{b}_k(t).\vec{y}_k + b_k(t).\frac{d^i\vec{y}_k}{dt} + \dot{c}_k(t).\vec{z}_k + c_k(t).\frac{d^i\vec{z}_k}{dt}$$

$$\frac{d^i\vec{V}}{dt} = \frac{d^k\vec{V}}{dt} + \vec{\Omega}_k^i \wedge (a_k(t).\vec{x}_k + b_k(t).\vec{y}_k + c_k(t).\vec{z}_k)$$

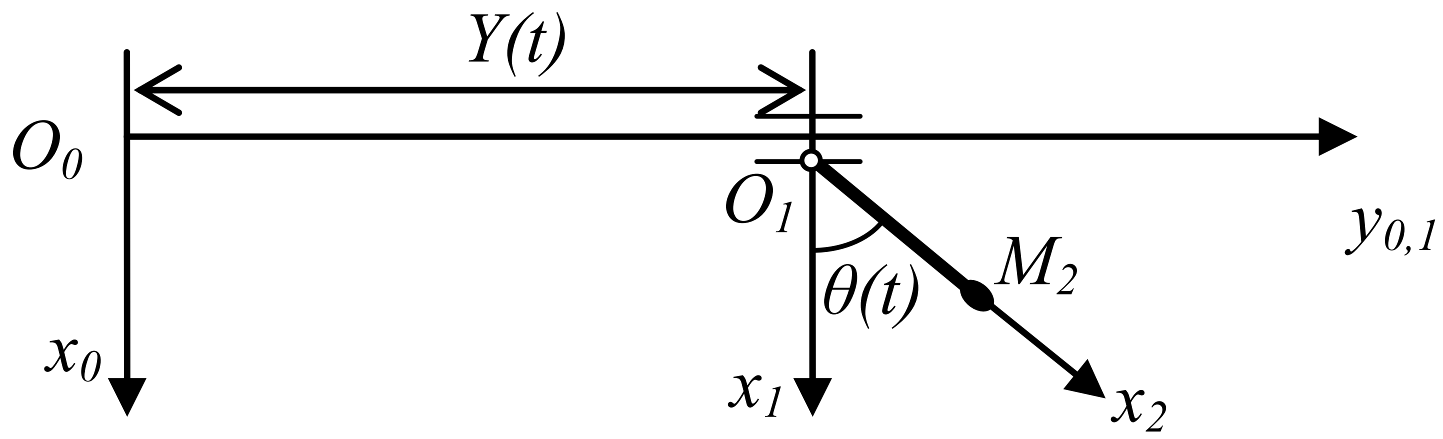
$$\frac{d^i\vec{V}}{dt} = \frac{d^k\vec{V}}{dt} + \vec{\Omega}_k^i \wedge \vec{V}$$

## *Application* Euler's Pendulum

Determine the velocity and acceleration vectors of point $M_2$ relative to $R_0$.

Projection reference frame is $R_1$.

$O_1M_2 = L$



## *III.* Fields of velocities and accelerations of a rigid body

Let $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a fixed reference frame attached to the observer, $(S_k)$ be a mobile solid relative to the reference frame $R_i$ and $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ be a reference frame attached to $(S_k)$.

Let $A_k$ and $B_k$ be two points on $(S_k)$. The velocities of these two points relative to the reference frame $R_i$ are given by:

$$\vec{V}^i(A_k) = \frac{d^i\overrightarrow{O_iA_k}}{dt} = \frac{d^k\overrightarrow{O_iA_k}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_iA_k}$$

$$\vec{V}^i(B_k) = \frac{d^i\overrightarrow{O_iB_k}}{dt} = \frac{d^k\overrightarrow{O_iB_k}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_iB_k}$$

The subtraction of the two previous relations gives:

$$\vec{V}^i(B_k) - \vec{V}^i(A_k) = \frac{d^k\overrightarrow{O_iB_k}}{dt} - \frac{d^k\overrightarrow{O_iA_k}}{dt} + \vec{\Omega}_k^i \wedge \left(\overrightarrow{O_iB_k} - \overrightarrow{O_iA_k}\right)$$

$$= \underbrace{\frac{d^k\overrightarrow{A_kB_k}}{dt}}_{0} + \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}$$

Rigid body $\Rightarrow$ $\qquad \vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}$

**Remarks *:***

*If* $\vec{\Omega}_k^i = 0$ then the solid is in pure translational motion, and all points of the solid have the same velocity $\vec{V}^i(B_k) = \vec{V}^i(A_k)$

*If* $\vec{V}^i(A_k) = 0$ *then* $\vec{V}^i(B_k) = \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}$ and the solid is in pure rotation around $A_k$ which belongs to the solid..

**Acceleration:**

The acceleration of $A_k$ is given as follows:

$$\vec{\gamma}^i(A_k) = \frac{d^i\vec{V}^i(A_k)}{dt}$$

We calculate the acceleration of $B_k$ in terms of the acceleration of $A_k$:

$$\vec{\gamma}^i(B_k) = \frac{d^i\vec{V}^i(B_k)}{dt} = \frac{d^i}{dt}\left(\vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}\right) = \frac{d^i\vec{V}^i(A_k)}{dt} + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_kB_k} + \vec{\Omega}_k^i \wedge \frac{d^i\overrightarrow{A_kB_k}}{dt}$$

*Since :* $\frac{d^i\overrightarrow{A_kB_k}}{dt} = \frac{d^k\overrightarrow{A_kB_k}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k} = \vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}$

*Then :*

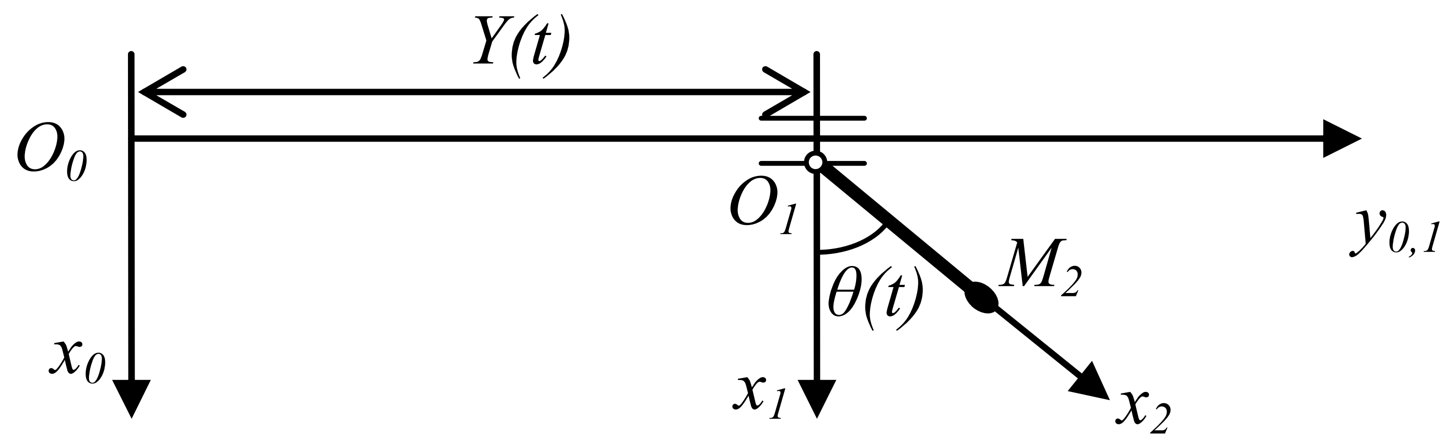
$$\vec{\gamma}^i(B_k) = \vec{\gamma}^i(A_k) + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_kB_k} + \vec{\Omega}_k^i \wedge \left(\vec{\Omega}_k^i \wedge \overrightarrow{A_kB_k}\right)$$

***Application* Euler's Pendulum**

Determine the velocity and acceleration vectors of point $M_2$ relative to $R_0$.

Projection reference frame is $R_1$.

$O_1M_2 = L$



# Composition of motion

Let $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ and $R_j\left(O_j, \vec{x}_j, \vec{y}_j, \vec{z}_j\right)$ be two reference frames.

The motion of a solid $(S)$ relative to $R_j$ $\left(\vec{V}^j(M)\right)$ is known,
and the motion of $R_j$ is known relative to $R_i$.

Let $M$ be a point belonging to $(S)$.

# Composition of velocities :

We aim to determine $\vec{V}^i(M)$ :

*We have* $\overrightarrow{O_iM} = \overrightarrow{O_iO_j} + \overrightarrow{O_jM}$

$$\vec{V}^i(M) = \frac{d^i\overrightarrow{O_iM}}{dt} = \underbrace{\frac{d^i\overrightarrow{O_iO_j}}{dt}}_{\vec{V}^i(O_j)} + \frac{d^i\overrightarrow{O_jM}}{dt}$$

By changing the reference frame, we find :

$$\frac{d^i\overrightarrow{O_jM}}{dt} = \frac{d^j\overrightarrow{O_jM}}{dt} + \vec{\Omega}_\mathrm{j}^\mathrm{i} \wedge \overrightarrow{O_jM} = \vec{V}^j(M) + \vec{\Omega}_\mathrm{j}^\mathrm{i} \wedge \overrightarrow{O_jM}$$

Then :

$$\vec{V}^i(M) = \vec{V}^i(O_j) + \vec{V}^j(M) + \vec{\Omega}_\mathrm{j}^\mathrm{i} \wedge \overrightarrow{O_jM} = \vec{V}^j(M) + \underbrace{\left(\vec{V}^i(O_j) + \vec{\Omega}_\mathrm{j}^\mathrm{i} \wedge \overrightarrow{O_jM}\right)}_{\vec{V}_j^i(M)}$$

Donc *:* $\vec{V}^i(M) = \vec{V}^j(M) + \vec{V}_j^i(M)$

$\vec{V}^i(M)$ is the absolute velocity. It is the velocity of point $M$ for an observer attached to $R_i$.

$\vec{V}^j(M)$ is the relative velocity. It is the velocity of point $M$ for an observer attached to $R_j$.

$\vec{V}_j^i(M)$ is the driving (entraining) velocity. It is the velocity of point $M$ belonging to $R_j$ that coincides at time $t$ with point $M$.

Properties of the vector $\boldsymbol{\vec{V}_j^i(M)}$ **:**

- Antisymmetry with respect to the indices $\vec{V}_j^i(M) = -\vec{V}_i^j(M)$
- $\vec{V}_j^i(M) = \vec{V}_k^i(M) + \vec{V}_j^k(M)$

# Composition of accelerations

We seek to determine $\vec{\gamma}^{\,i}(M)$ :

*We have :* $\vec{V}^{i}(M) = \vec{V}^{j}(M) + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M} + \vec{V}^{i}(O_j)$

*Then :*

$$\vec{\gamma}^{\,i}(M) = \frac{d^i \vec{V}^{i}(M)}{dt} = \frac{d^i \vec{V}^{j}(M)}{dt} + \frac{d^i}{dt}\left(\vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M}\right) + \frac{d^i \vec{V}^{i}(O_j)}{dt}$$

- $\frac{d^i \vec{V}^{j}(M)}{dt} = \frac{d^j \vec{V}^{j}(M)}{dt} + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \vec{V}^{j}(M) = \vec{\gamma}^{\,j}(M) + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \vec{V}^{j}(M)$

- $\frac{d^i}{dt}\left(\vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M}\right) = \frac{d^i \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}}}{dt} \wedge \overrightarrow{O_j M} + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \frac{d^i \overrightarrow{O_j M}}{dt} = \frac{d^i \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}}}{dt} \wedge \overrightarrow{O_j M} + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \left(\frac{d^j \overrightarrow{O_j M}}{dt} + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M}\right) = \frac{d^i \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}}}{dt} \wedge \overrightarrow{O_j M} + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \left(\vec{V}^{j}(M) + \vec{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M}\right)$

- $\frac{d^i \vec{V}^{i}(O_j)}{dt} = \vec{\gamma}^{\,i}(O_j)$

Then :

$$\vec{\gamma}^{i}(M) = \vec{\gamma}^{j}(M) + \underbrace{\left[\vec{\gamma}^{i}(O_j) + \frac{d^{i}\overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}}}{dt} \wedge \overrightarrow{O_j M} + \overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \left(\overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \overrightarrow{O_j M}\right)\right]}_{\vec{\gamma}_{j}^{i}(M)} + 2\overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \vec{V}^{j}(M)$$

Therefore :

$$\vec{\gamma}^{i}(M) = \vec{\gamma}^{j}(M) + \vec{\gamma}_{j}^{i}(M) + 2\overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \vec{V}^{j}(M)$$

$\vec{\gamma}^{i}(M)$ the absolute acceleration

$\vec{\gamma}^{j}(M)$ the relative acceleration

$\vec{\gamma}_{j}^{i}(M)$ the driving (entraining) acceleration

$2\overrightarrow{\Omega}_{\mathrm{j}}^{\mathrm{i}} \wedge \vec{V}^{j}(M)$ the complementary or Coriolis acceleration.