Serie N°2: Vector Analysis

Ex 1:

(1) i) $F(x,y,z) = \frac{yz}{y-z}\vec{i} + \frac{zx}{x-z}\vec{j} + \frac{xy}{x-y}\vec{k}$

If we put $\varphi(t) = \frac{at}{t-a} = a + \frac{a^2}{t-a}$

$$\varphi'(t) = -\frac{a^2}{(t-a)^2}$$

$$\mathrm{Curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{y-z} & \frac{zx}{x-z} & \frac{xy}{x-y} \end{vmatrix}$$

$$= \left[\frac{x^2}{(x-y)^2} - \frac{x^2}{(x-z)^2}\right]\vec{i} + \left[\frac{y^2}{(x-y)^2} + \frac{y^2}{(y-z)^2}\right]\vec{j} + \left[\frac{z^2}{(y-z)^2} - \frac{z^2}{(x-z)^2}\right]\vec{k}$$

ii) $$\mathrm{curl}(F)(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x-y) & \sin(y-z) & \sin(z-x) \end{vmatrix}$$

$$\boxed{\mathrm{Curl}(F)(x,y,z) = \cos(y-z)\vec{i} + \cos(z-x)\vec{j} + \cos(x-y)\vec{k}}$$

(2) i) $F(x,y,z) = (\mathrm{grad}\, f)(x,y,z)$

where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x,y,z) \longrightarrow \frac{1}{2}x^2y^2z^2$

So F is conservative

ii) Since $\mathrm{curl}(F) \neq 0$, then F is not conservative

(3) i) $\mathrm{div}(F)(x,y) = (x+1)e^x + (y+1)e^y$

ii) $\mathrm{div}(F)(x,y,z) = \frac{2x}{x^2+y^2} + x + \frac{2z}{y^2+z^2}$

$D = \mathbb{R}^3 - \{(0,0,0)\}$

Exercise 2.

(1) $F = M_F\vec{i} + N_F\vec{j} + P_F\vec{k}$

$G = M_G\vec{i} + N_G\vec{j} + P_G\vec{k}$

$$F \times G = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ M_F & N_F & P_F \\ M_G & N_G & P_G \end{vmatrix}$$

$$= (N_F P_G - N_G P_F)\vec{i} - (M_F P_G - M_G P_F)\vec{j} + (M_F N_G - M_G N_F)\vec{k}$$

$$\mathrm{div}(F\times G) = \frac{\partial}{\partial x}(N_F P_G - N_G P_F) - \frac{\partial}{\partial y}(M_F P_G - M_G P_F) + \frac{\partial}{\partial z}(M_F N_G - M_G N_F)$$

$$= \left[\frac{\partial N_F}{\partial x}P_G - \frac{\partial P_F}{\partial x}N_G + M_G\frac{\partial P_F}{\partial y} - P_G\frac{\partial M_F}{\partial y} + N_G\frac{\partial M_F}{\partial z} - M_G\frac{\partial N_F}{\partial z}\right]$$
$$- \left[P_F\frac{\partial N_G}{\partial x} - N_F\frac{\partial P_G}{\partial x} + P_F\frac{\partial M_G}{\partial y} - M_F\frac{\partial P_G}{\partial y} - N_F\frac{\partial M_G}{\partial z} + M_F\frac{\partial N_G}{\partial z}\right]$$

$$= M_G\left(\frac{\partial P_F}{\partial y} - \frac{\partial N_F}{\partial z}\right) - N_G\left(\frac{\partial P_F}{\partial x} - \frac{\partial M_F}{\partial z}\right) + P_G\left(\frac{\partial N_F}{\partial x} - \frac{\partial M_F}{\partial y}\right) - \left[\left(\frac{\partial P_G}{\partial y} - \frac{\partial N_G}{\partial z}\right)M_F - \left(\frac{\partial P_G}{\partial x} - \frac{\partial M_G}{\partial z}\right)N_F + \left(\frac{\partial N_G}{\partial x} - \frac{\partial M_G}{\partial y}\right)P_F\right]$$

$$= G.\mathrm{Curl}(F) - F.\mathrm{Curl}(G)$$

(2) $\mathrm{div}(fG) = \mathrm{div}(f M_G\vec{i} + f N_G\vec{j} + f P_G\vec{k})$

$$= \frac{\partial}{\partial x}(f M_G) + \frac{\partial}{\partial y}(f N_G) + \frac{\partial}{\partial z}(f P_G)$$

$$= f\left(\frac{\partial M_G}{\partial x} + \frac{\partial N_G}{\partial y} + \frac{\partial P_G}{\partial z}\right.$$

$(AB): \mathbb{R}^2 \quad t \in \mathbb{R} \begin{cases} x = x_A + t(x_B - x_A) \\ y = y_A + t(y_B - y_A) \end{cases}$ $\quad (AB): \mathbb{R}^3 \begin{cases} x = x_A + t(x_B - x_A) \\ y = y_A + t(y_B - y_A) \\ z = z_A + t(z_B - z_A) \end{cases}$

$$+\frac{\partial f}{\partial x} M_G + \frac{\partial f}{\partial y} N_G + \frac{\partial f}{\partial z} P_G$$

$$= f\,\mathrm{div}(G) + (\mathrm{grad} f) \cdot G$$

③ $\mathrm{div}(\mathrm{Curl} F) = \mathrm{div}\left(\left(\frac{\partial P_F}{\partial y} - \frac{\partial N_F}{\partial z}\right)\vec{i}\right.$

$$\left. - \left(\frac{\partial P_F}{\partial x} - \frac{\partial M_F}{\partial z}\right)\vec{j} + \left(\frac{\partial N_F}{\partial x} - \frac{\partial M_F}{\partial y}\right)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P_F}{\partial y} - \frac{\partial N_F}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial P_F}{\partial x} - \frac{\partial M_F}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial N_F}{\partial x} - \frac{\partial M_F}{\partial y}\right)$$

$$= \left(\frac{\partial^2 P_F}{\partial x \partial y} - \frac{\partial^2 N_F}{\partial x \partial z}\right) + \left(\frac{\partial^2 M_F}{\partial y \partial z} - \frac{\partial^2 P_F}{\partial y \partial x}\right)$$

$$+ \left(\frac{\partial^2 N_F}{\partial z \partial x} - \frac{\partial^2 M_F}{\partial z \partial y}\right)$$

If F is class 2

$$= 0 + 0 + 0 = 0$$

④ $\mathrm{Curl}(fG) = \left(\frac{\partial (f P_G)}{\partial y} - \frac{\partial (f N_G)}{\partial z}\right)\vec{i}$

$$- \left(\frac{\partial (f P_G)}{\partial x} - \frac{\partial (f M_G)}{\partial z}\right)\vec{j} + \left(\frac{\partial (f N_G)}{\partial x} - \frac{\partial (f M_G)}{\partial y}\right)\vec{k}$$

$$= f\,\mathrm{Curl}(G)$$

$$+ \left(\frac{\partial f}{\partial y} P_G - \frac{\partial f}{\partial z} N_G\right)\vec{i} - \left(\frac{\partial f}{\partial x} P_G - \frac{\partial f}{\partial z} M_G\right)\vec{j}$$

$$+ \left(\frac{\partial f}{\partial x} N_G - \frac{\partial}{\partial y} M_G\right)\vec{k}$$

$$= f\,\mathrm{curl}(G) + (\mathrm{grad} f) \times G$$

⑤ $\mathrm{Curl}(\mathrm{grad} f) = \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right)\vec{i}$

$$- \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\vec{k}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

Ex3 :

① $f(x,y) = x^2 + y^2$

i) $C \begin{cases} x = 0 + t(1) = t \\ y = 0 + t(1) = t \end{cases}$

$r(t) = t\vec{i} + t\vec{j} \qquad t \in [0,1]$

$r'(t) = \vec{i} + \vec{j}$

$dr(t) = \|r'(t)\|\,dt = \sqrt{2}\,dt$

$$\int_C f\,dr = \int_0^1 f\,r(t)\,dr = \int_0^1 f(t,t)\sqrt{2}\,dt$$

$$= \int_0^1 2t^2\sqrt{2}\,dt = 2\sqrt{2}\int_0^1 t^2\,dt$$

$$= \frac{2}{3}\sqrt{2}$$

ii) $C : \begin{cases} x = 0 + t(2) = 2t \\ y = 0 + t(4) = 4t \end{cases}$

$r(t) = 2t\vec{i} + 4t\vec{j} \qquad t.$

$r'(t) = 2\vec{i} + 4\vec{j}$

$dr = 2\sqrt{5}\,dt$

$f(r(t)) = f(2t, 4t) = 4t^2 + 16t^2$
$= 20t^2$

$$\int_C f\,dr = \int_0^1 20t^2 \times 2\sqrt{5}\,dt$$

$$= \int_0^1 40\sqrt{5}\,t^2 = \frac{40}{3}\sqrt{5}$$

iii) $C : \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [0, \frac{\pi}{2}]$

$r(t) = \cos t \vec{i} + \sin t \vec{j}$

$r'(t) = -\sin t \vec{i} + \cos t \vec{j}$

$dr = dt$

$f(r(t)) = f(\cos t, \sin t) = 1$

$\int_C f \, dr = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2}$

② $f(x, y, z) = 2x + y^2 - z$

i) $C = C_1 \cup C_2 \cup C_3 \quad \begin{cases} A(0,0,0) \\ B(1,0,0) \\ C(1,0,1) \\ D(1,1,1) \end{cases}$

$C_1 : [AB] \quad \begin{cases} x = t(1) = t \\ y = t(0) = 0 \\ z = t(0) = 0 \end{cases} \quad r(t) = t\vec{i}$

$C_2 : [BC] \quad \begin{cases} x = 1 + t(0) = 1 \\ y = 0 + t(0) = 0 \\ z = 0 + t(1) = t \end{cases} \quad r(t) = \vec{i} + t\vec{k}$

$C_3 : [CD] \quad \begin{cases} x = 1 + t(0) = 1 \\ y = 0 + t(1) = t \\ z = 1 + t(0) = 1 \end{cases} \quad r(t) = \vec{i} + t\vec{j} + \vec{k}$

$C_1 : f(r(t)) = f(t, 0, 0) = 2t \; / dr(t) = dt$

$C_2 : f(r(t)) = f(1, 0, t) = 2 - t \; / dr = dt$

$C_2 : f(r(t)) = f(1, t, 1) = 1 + t^2 \; / dr = dt$

$\int_C f \, dr = \int_{C_1} f \, dr + \int_{C_2} f \, dr + \int_{C_3} f \, dr$

$= \int_0^1 2t \, dt + \int_0^1 (2 - t) dt + \int_0^1 (1 + t^2) dt$

$= \int_0^1 (2t + 2 - t + 1 + t^2) \, dt$

$= \int_0^1 (3 + t + t^2) \, dt = 3 + \frac{1}{2} + \frac{1}{3}$

$= \frac{23}{6}$

ii) $A(0,0,0)$, $B(0,1,0)$, $C(0,1,1)$

$C_1 [AB] \quad \begin{cases} x = t(0) = 0 \\ y = t(1) = t \\ z = t(0) = 0 \end{cases} \quad r(t) = t\vec{j}$

$C_2 [BC] \quad \begin{cases} x = t(0) = 0 \\ y = 1 + t(0) = 1 \\ z = t(1) = t \end{cases} \quad r(t) = \vec{j} + t\vec{k}$

$C_3 [AC] \quad \begin{cases} x = t(0) = 0 \\ y = 1 + t(1) = 1 - t \\ z = 1 + t(-1) = 1 - t \end{cases} \quad r(t) = (1-t)\vec{j} + (1-t)\vec{k}$

$C_1 = f(r(t)) = f(0, t, 0) = t^2 \quad dr = dt$

$C_2 = f(r(t)) = f(0, 1, t) = 1 - t \quad dr = dt$

$C_2 = f(r(t)) = f(0, 1-t, 1-t) = (1-t)^2 - (1-t)$

$= (t^2 - t) \quad dr = \sqrt{2} dt$

$\int_C f dr = \int_0^1 (t^2 + 1 - t + \sqrt{2}(t^2 - t)) dt$

$= \frac{1}{3} + \frac{1}{2} + \sqrt{2} \left(\frac{1}{3} - \frac{1}{2} \right) = \frac{5}{6} - \frac{\sqrt{2}}{6}$

Ex 5 : $= \int \left((x(t)^2 + y(t)^2)\, x'(t) + 2x(t)y(t)\, y'(t) \right) dt$

1/ i) $\int x^2+y^2\, dx + 2xy\, dy$

$r(t) = t^3 \vec{i} + t^2 \vec{j}$ where $t \in [0,2]$

$r'(t) = 3t^2 \vec{i} + 2t \vec{j}$

$dr(t) = r'(t)\, dt$

$F(x,y) = (x^2+y^2)\vec{i} + 2xy\, \vec{j}$

$F(r(t)) = F(t^3, t^2)$

$= ((t^3)^2 + (t^2)^2)\vec{i} + 2(t^3)(t^2)\vec{j}$

$= (t^6 + t^4) i + 2t^5 \vec{j}$

$= \int_0^2 F(r(t)) . r'(t)\, dt$

$= \int_0^2 F(t^3, t^2) . r'(t)\, dt$

$F(r(t)).r'(t) = ((t^6+t^4)\vec{i} + 2t^5 \vec{j}) . (3t^2 \vec{i} + 2t \vec{j})$

$= (t^6+t^4)3t^2 + 2t^5 \times 2t$

$= 3t^8 + 3t^6 + 4t^6$

$\int_0^2 (3t^8 + 3t^6 + 4t^6)\, dt$

$= \left[\frac{3}{9} t^9 + t^7 \right]_0^2$

$= \left[1/3\, (512) + 128 \right] = \frac{896}{3}$

ii) $r(t) = 2\cos t\, \vec{i} + 2 \sin t\, \vec{j}$

$t \in [0, \frac{\pi}{2}]$

$r'(t) = -2\sin t + 2\cos t$

$F(r(t)) = F(2\cos t, 2\sin t)$

$= ((2\cos t)^2 + (2\sin t)^2)\vec{i} + 2(2\cos t \cdot 2\sin t)\vec{j}$

line integral — surface integral

scalar field (or real function): $\int_C f\, dr$

vector field (or vector-valued function): $\int_C F . dr$

$F(r(t)) = 4\vec{i} + 8 \cos t \sin t\, \vec{j}$

$F(r(t)) . r'(t) = -8\sin t + 16 \cos^2 t \sin t$

$\int_0^{\pi/2} (-8\sin t + 16\cos^2 t \sin t)\, dt$

$\left[+8\cos t - \frac{16}{3}\cos^3 t \right]_0^{\pi/2} = \left[-8 + \frac{16}{3} \right]$

$= \frac{-8}{3}$

2/ $F\, dr = df$ / $f(x,y,z) = xyz$

i) $\begin{cases} r(0) = (0,2,0) \Rightarrow f(r(0)) = f(0,2,0) = 0 \\ r(4) = (4,2,4) \Rightarrow f(r(4)) = f(4,2,4) = 32 \end{cases}$

ii) $\begin{cases} r(0) = (0,0,0) \Rightarrow f(r(0)) = 0 \\ r(4) = (4,2,4) \Rightarrow f(r(4)) = f(4,2,4) = 32 \end{cases}$

$\int_C F.dr = 32 - 0 = 32$ In the 2 cases

$df(x,y,z) = yz\, dx + xz\, dy + xy\, dz$

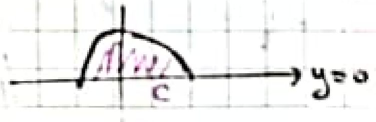
Ex 6 :

$\oint_{C=\partial D} F.dr = \oint_C M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} \ldots \right) dx\, dy$

1/ $\int_C 2xy\, dx + (x+y)\, dy$

$M(x,y) = 2xy$, $N(x,y) = (x+y)$

$\frac{\partial M}{\partial y}(x,y) = 2x$, $\frac{\partial N}{\partial x}(x,y) = 1$



$$D = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \leq y \leq 1 - x^2 \\ -1 \leq x \leq 1 \end{array} \right\}$$

$$\int_{-1}^{1} \left(\int_{0}^{1-x^2} (1-2x)\, dy \right) dx$$

$$= \int_{-1}^{1} ((1-2x)(1-x^2))\, dx$$

$$= \int_{-1}^{1} (1 - x^2 - 2x + 2x^3)\, dx$$

$$= \left[x - \frac{1}{3}x^3 - x^2 + \frac{1}{2}x^4 \right]_{-1}^{1}$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

2/ $\int_C 2\arctan\left(\frac{y}{x}\right) dx + \ln(x^2+y^2)\, dy$

$$\frac{\partial M}{\partial y}(x,y) = 2 \times \frac{1}{x} \times \frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{2}{x + \frac{y^2}{x}} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial N}{\partial x}(x,y) = \frac{2x}{x^2+y^2}$$

since $\frac{\partial N}{\partial x}(x,y) = \frac{\partial M}{\partial y}(x,y)$

$\Rightarrow \oint_C F \cdot dr = 0$ by green's theorem

3/ $\int_C (x^2 - y^2)\, dx + 2xy\, dy$

$$\frac{\partial M}{\partial y}(x,y) = -2y \quad , \frac{\partial N}{\partial x}(x,y) = 2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4y$$

$$D = \begin{cases} x = r\cos\theta & 0 \leq \theta \leq 2\pi \\ y = r\sin\theta & 0 \leq r \leq 4 \end{cases}$$

$$\iint_D 4y\, dx\, dy = 4 \int_0^{2\pi}\int_0^{4} r^2 \sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \sin\theta\, d\theta \times \int_0^{4} r^2\, dr$$

$$= \left[\frac{1}{3} r^3\right]_0^4 \cdot \left[-\cos\theta\right]_0^{2\pi}$$

$$= \left(\frac{4^3}{3}\right)(1-1) = 0$$

Ex 7:

1/ $\mathcal{A}(T) = \iint_T d\sigma$ $\quad d\sigma = \left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}(u,v) \right\| du\, dv$

$$r(u,v) = (2+\cos u)\cos v\, \vec{i} + (2+\cos u)\sin v\, \vec{j} + \sin u\, \vec{k}$$

$$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u\cos v & -\sin u \sin v & \cos u \\ -(2+\cos u)\sin v & (2+\cos u)\cos v & 0 \end{vmatrix} =$$

$$(2+\cos u)\left[-\cos u\cos v\, \vec{i} - \cos u \sin v\, \vec{j} - \sin u\, \vec{k}\right]$$

$$\left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}(u,v) \right\| = 2 + \cos u$$

$$A(T) = \int_0^{2\pi}\int_0^{2\pi} (2+\cos u)\, du\, dv$$

$$= 2\pi \left[2u + \sin u\right]_0^{2\pi} = 8\pi$$

2/ i) $S = \text{graph}\, g = \{ (x,y,z) \in \mathbb{R}^3 : z = g(x,y), (x,y) \in D \}$

$$F(x,y) = x\vec{i} + y\vec{j} + g(x,y)\vec{k}$$

$$g(x,y) = y$$

$$F(x,y) = x\vec{i} + y\vec{j} + y\vec{k} \;/ (x,y) \in \mathbb{R}^2$$

ii) $g(x,y) = \sqrt{4x^2 + 9y^2}$

$$F(x,y) = x\vec{i} + y\vec{j} + \sqrt{4x^2+9y^2}\, \vec{k}$$

$D = \mathbb{R}^2$

iii) $\begin{cases} x = 2\cos u \\ y = 4\sin u \\ z = v \end{cases}$ $\quad (2x)^2 + y^2 = 16$ $\quad \begin{array}{l} \cancel{2}x = \cancel{4}\cos u \\ y = 4\sin u \end{array}$

$F(u,\theta) = 2\cos u\,\vec{i} + 4\sin u\,\vec{j} + v\vec{k}$

$D = [0, 2\pi] \times \mathbb{R}$

iv) $\begin{cases} \frac{x}{3} = \cos u \cos \theta \\ \frac{y}{2} = \cos u \sin \theta \\ z = \sin u \end{cases}$

Exo8 :

1/ i) $S = \{(x,y,z) \in \mathbb{R}^3 : z = x+y = g(x,y)\}$
$x^2+y^2 < 1$

$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$$\iint_S z\,d\sigma = \iint_D f(x,y,g(x,y)) \sqrt{1+\left(\frac{\partial g}{\partial x}(x,y)\right)^2+\left(\frac{\partial g}{\partial y}(x,y)\right)^2}\,dx\,dy$$

$= \sqrt{3} \iint_D (2x^2+2y^2+2xy)\,dx\,dy$

$= 2\sqrt{3} \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos\theta\sin\theta)\,r\,dr\,d\theta$

$= 2\sqrt{3} \left(\int_0^1 r^3\,dr\right)\left(\int_0^{2\pi} (1+\cos\theta\sin\theta)\,d\theta\right)$

$= \frac{2\sqrt{3}}{4} \times (2\pi + 0) = \pi\sqrt{3}$

ii) $S = \{(x,y,z) : z = \sqrt{x^2+y^2}\}$

$(x-1)^2+y^2 \leq 1$

$D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2+y^2 \leq 1\}$

$$\boxed{\iint_S z\,d\sigma = \iint f(x,y,g(x))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy}$$

$\frac{\partial g}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$= \iint \sqrt{x^2+y^2+x^2+y^2}\,\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}$

$= \iint \sqrt{2x^2+2y^2}\,\sqrt{1+1}\,dx\,dy$

$= \sqrt{2} \iint \sqrt{2x^2+2y^2}\,dx\,dy$

$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\,dr\,d\theta$

$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1/3\,(2\cos\theta)^3$

$\int \cos\theta^3\,d\theta = \int (1-\sin^2\theta)(\sin\theta)\,d\theta$

$= \frac{2}{3} \times 2^3 \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{16}{3}\left[\frac{4}{3}\right] = \frac{64}{9}$

2/ $F = M\vec{i} + N\vec{j} + P\vec{k}$

$S = \text{graph } g = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } z = g(x,y)\}$

$\iint_S (F.N)\,d\sigma = \iint_D \left(P - \frac{\partial g}{\partial x}M - \frac{\partial g}{\partial y}N\right) dx\,dy$

i) $\iint_S (F.N)\,d\sigma = \iint_D [y - (-1)3(1-x-y) - (-1)(4)]\,dx\,dy$

$= \iint_D (-2y - 3x - 1)\,dx\,dy$

$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$

$\iint_S (F.N)\,d\sigma = \int_0^1 \left[\int_0^{1-x} (-2y - 3x - 1\right.$

$= \int_0^1 [-y^2 - (3x+1)y]_0^{1-x}\,dx$

$$= -\int_0^1 (1-x)(1-x+3x+1)\,dx$$

$$= -\int_0^1 (1-x)(2x+2)\,dx$$

$$= -2\int_0^1 (1-x^2)\,dx = -2\left(1-\frac{1}{3}\right) = \frac{-4}{3}$$

ii) $F = M\vec{i} + N\vec{j} + P\vec{k}$

$$\iint_S (F.N)\,d6 = \iint_D \left(P - \frac{\partial g}{\partial x} M - \frac{\partial g}{\partial y} N\right) \boxed{dx\,dy}$$

$$= \iint_D (1-x^2-y^2) - \underbrace{(-2x)(x)}_{2x^2} - \underbrace{(-2y)(y)}_{2y^2}\,dx\,dy$$

$$= \iint_D (1+x^2+y^2)\,dx\,dy$$

$z \geq 0$ $\quad x = r\cos\theta$ $\quad 0 \leq \theta \leq 2\pi$

$x^2+y^2 \leq 1$ $\quad y = r\sin\theta$ $\quad 0 \leq r \leq 1$

$$= \int_0^1\int_0^{2\pi} (1+r^2)\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (1+r^2)\,dr\,d\theta = \int_0^{2\pi} \left[r + \frac{1}{3}r^3\right]_0^1$$

$$= \int_0^{2\pi} \frac{4}{3} = \frac{8\pi}{3}$$

Ex 9: divergence theorem:

$$\iint_S (F.N)\,d6 = \iiint_\Omega \operatorname{div}(F)\,dx\,dy\,dz$$

$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = 0,\ x^2+y^2 \leq a^2\}$

$\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq \sqrt{a^2-x^2-y^2}\}$

$\partial\Omega = S \cup \Sigma$ so

$$\iint_{S\cup\Sigma} (f.N)\,d6 = \iiint_\Omega \operatorname{div}(F)\,dx\,dy\,dz$$

$$= \iint_S (f.N)\,d6 + \iint_\Sigma (f.N)\,d6$$

$g(x,y) = z = 0$

$$\iint_\Sigma (F.N)\,d6 = \iint_S f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx$$

$$= \iint_\Sigma (F.N)(x,y,0)\sqrt{1+0}\,dx\,dy$$

$N = -\vec{k}$

$(F.N)(x,y,z) = -2xyz^2$

$(F.N)(x,y,0) = 0$

$= 0$

$$\operatorname{div}(F) = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)$$

$$= 2x - 2x + 2xyz = 2xyz$$

$$2\iiint xyz\,dx\,dy\,dz =$$

$$= \iint_D xy\left[z^2\right]_0^{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \iint_D xy(a^2-x^2-y^2)\,dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq a$

$$= \int_0^{2\pi}\int_0^a r^3\cos\theta\sin\theta\,(a^2-r^2)\,dr\,d\theta$$

$= 0$ / because $\int_0^{2\pi} \cos\theta\sin\theta\,d\theta = \left(\frac{\cos^2\theta}{2}\right)_0^{2\pi} = 0$

Ex 10 :

$$F(x,y,z) = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\text{Curl}(F)(x,y,z) = 0$$

$$\oint_C F.dr = \iint_S (\text{curl}(F).N)\, d\sigma = 0$$

$C = \partial S$

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = y^2 / (x,y) \in [0,a] \times [0,a] = D\}$$

$$C = \partial S = C_1 \cup C_2 \cup C_3 \cup C_4$$

$C_1 : r(t) = t\vec{j} + t^2 \vec{k} \quad t \in [0,a]$

$C_2 : r(t) = t\vec{i} + a\vec{j} + a^2 \vec{k} \quad t \in [0,a]$

$C_3 : r(t) = a\vec{i} + (a-t)\vec{j} + (a-t)^2 \vec{k}$

$C_4 : r(t) = (a-t)\vec{i} \quad t \in [0,a]$

$C_1 : F(r(t)).r'(t) = F(0,t,t^2).(\vec{j}, 2t\vec{k})$
$= t^2 + 2t^5$

$C_2 : F(r(t)).r'(t) = F(t,a,a^2).\vec{i} = t^2$

$C_3 : F(r(t)).r'(t) = F(a,a-t,(a-t)^2).(-\vec{j}, -2(a-t)\vec{k})$
$= -(a-t)^2 - 2(a-t)^5$

$C_4 : F(r(t)).r'(t) = F(a-t,0,0).(-\vec{i}) = -(a-t)^2$

$$\oint_C F.dr = \int_0^a [t^2 + 2t^5 + t^2 - (a-t)^2 - 2(a-t)^5 - (a-t)^2]\,dt$$

$$= 2\int_0^a (t^2 + t^5 - (a-t)^2 - (a-t)^5)\,dt$$

$$= 2\left[\frac{1}{3}t^3 + \frac{1}{6}t^6 + \frac{1}{3}(a-t)^3 + \frac{1}{6}(a-t)^6\right]_0^a$$

$$= 2\left[\left(\frac{1}{3}a^3 + \frac{1}{6}a^6 + 0\right) - \left(0 + \frac{1}{3}a^3 + \frac{1}{6}a^6\right)\right]$$

$= 0$