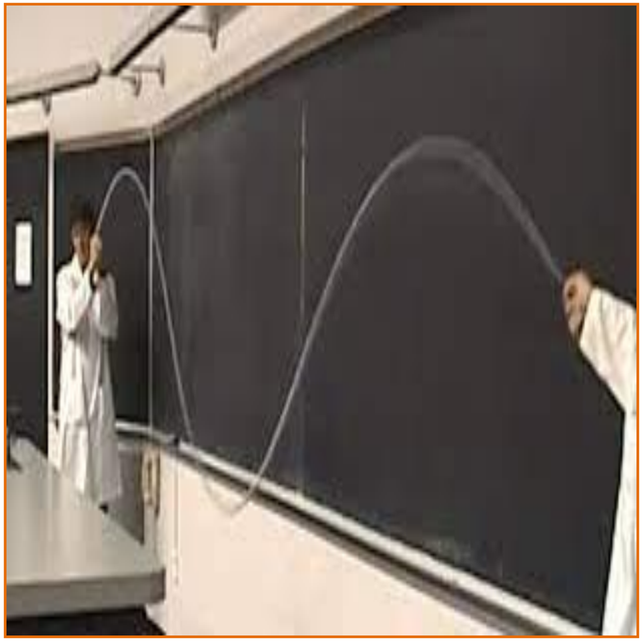
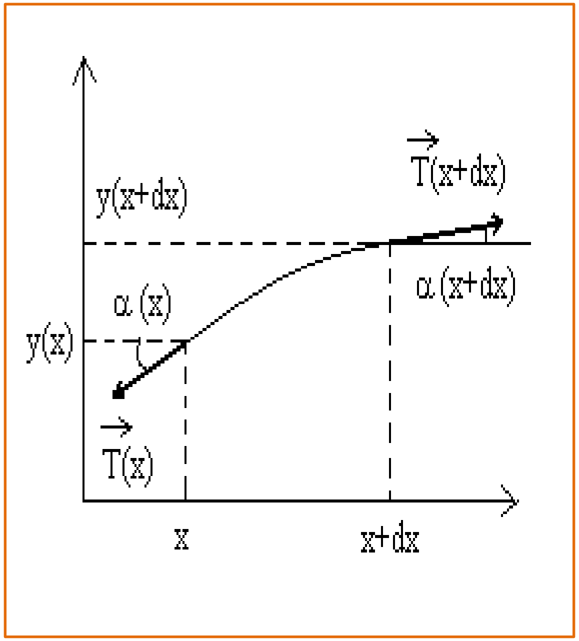
TD physique4





Chapter 1: Generalities of propagation phenomena

Set 02

1- Exercise 1

2- Exercise 2

Exercise 1

1-A string of length $4.35\ m$ and mass $137\ g$ is under a tension of $125\ N$. A standing wave has formed which has seven nodes including the endpoints. What is the frequency of this wave? Which harmonic is it? What is the fundamental frequency?

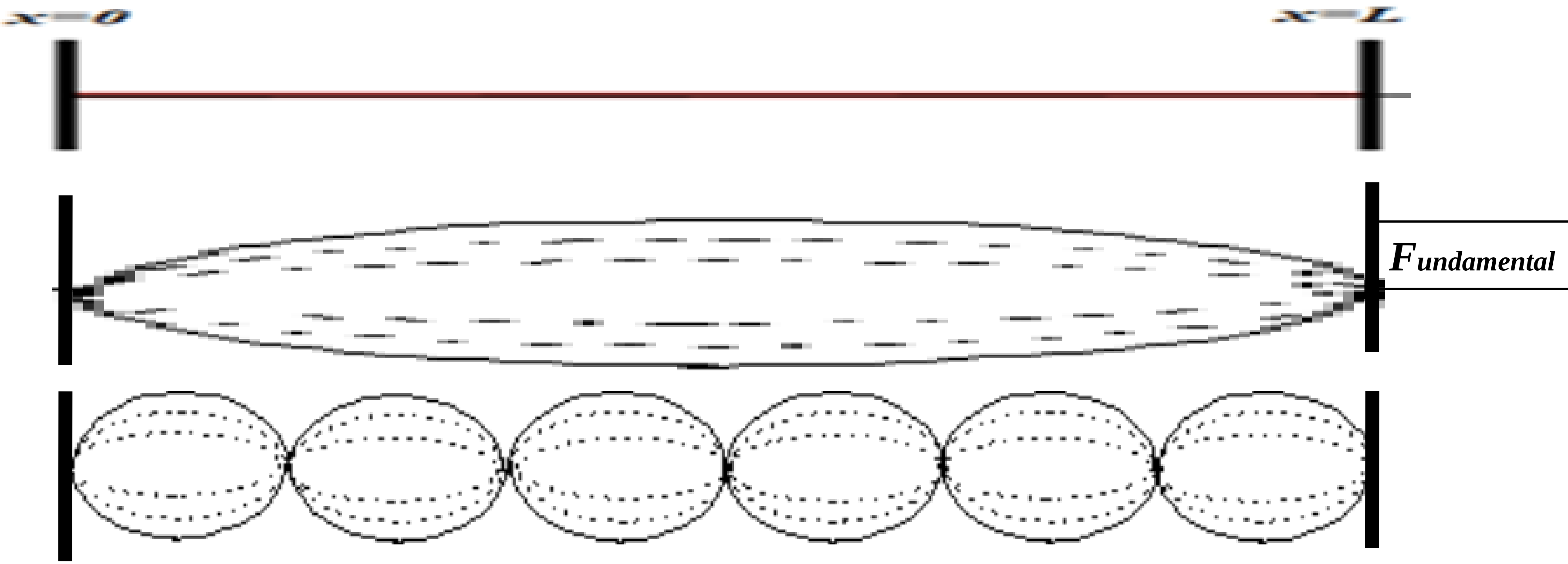
-Represent the fundamental mode and the first three overtones

2-A string fixed at one end only, is vibrating in its ninth harmonic mode. The speed of a wave on the string is $v = 25.8\ m/s$ and the string has a length of $8.25\ m$. What is the frequency of this wave? What is the wavelength of the wave? What is the fundamental frequency?

-Represent the fundamental mode and the first three overtones

Exercise 1

1- Fixed ends



$$u(0,t) = 0$$

$$u(L,t) = 0$$

Exercise 1

1- Fixed ends

$$u(x,t) = u_i(x,t) + u_r(x,t)$$
$$= U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)}$$

- Boundary Conditions:

$$\begin{cases} u(o,t) = 0 \ \ldots\ldots\ldots\ldots (1) \\ u(L,t) = 0 \ \ldots\ldots\ldots\ldots (2) \end{cases}$$

$$(1) \Rightarrow U_r = -U_i \Rightarrow u(x,t) = U_i e^{jwt}\left(e^{-jkx} - e^{+jkx}\right)$$

Using: $\sin a = \frac{e^{ja} - e^{-ja}}{2j}$

The displacement becomes: $u(x,t) = -2jU_i e^{jwt} \sin kx$

Exercise 1

1- Fixed ends

- The real part of the function u(x,t):

Using: $e^{jwt} = coswt + j\, sinwt$

$$\Rightarrow \; u(x,t) = 2U_i\, sinkx\ .\ \sin wt$$

- Using boundary Conditions:(2) $\Rightarrow \; u(L,t) = 0$

$$u(x,t) = 2U_i\, sinkx\ .\ \sin wt \;\Rightarrow\; u(L,t) = 2U_i sinkL\ .sinwt = 0$$

$$sinkL = 0 \;\Rightarrow\; kL = n\pi \;\Rightarrow\; \frac{w}{v}L = n\pi \;\Rightarrow\; w_n = n\frac{\pi v}{L}$$

$$w_n = n\frac{\pi v}{L} \;\Rightarrow\; w_n = n\, w_0 \;\Rightarrow\; f_n = n\frac{v}{2L}$$

Exercise 1

1- Fixed ends

$$v = \sqrt{\frac{T}{\mu}} \quad , \quad \mu = \frac{m}{L} \quad \Rightarrow \quad v = 360\,m/s$$

$$f_n = n\frac{v}{2L} \quad \Rightarrow \quad f_1 = 7.42Hz \quad and \quad f_6 = 43.4Hz$$

-Represent the fundamental mode and the first three overtones

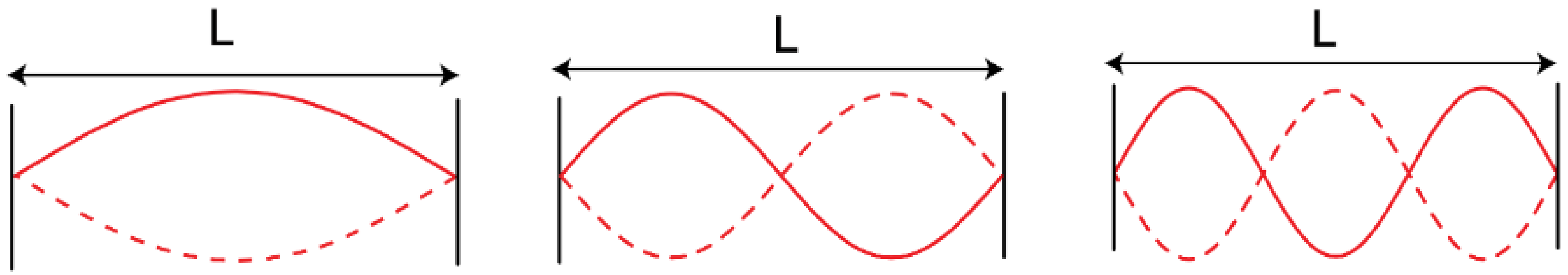
1- Fixed ends

1st harmonic = ***Fundamental***

Standing wave

$$f_n = \frac{v}{\lambda} = \left(\frac{v}{2L}\right) n = f_1 n;\ n = 1, 2, 3, \ldots$$

➢ **Harmonics for Two Fixed Ends**



1st harmonic

$$\lambda = 2L \rightarrow f = \left(\frac{v}{2L}\right)$$

2nd harmonic

$$\lambda = L \rightarrow f = 2\left(\frac{v}{2L}\right)$$

3rd harmonic

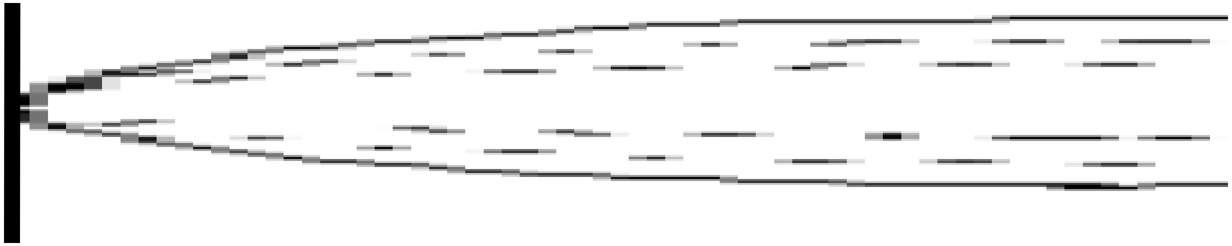
$$\lambda = \frac{2L}{3} \rightarrow f = 3\left(\frac{v}{2L}\right)$$

$$\lambda_n = \frac{2L}{n};\ n = 1, 2, 3, \ldots$$

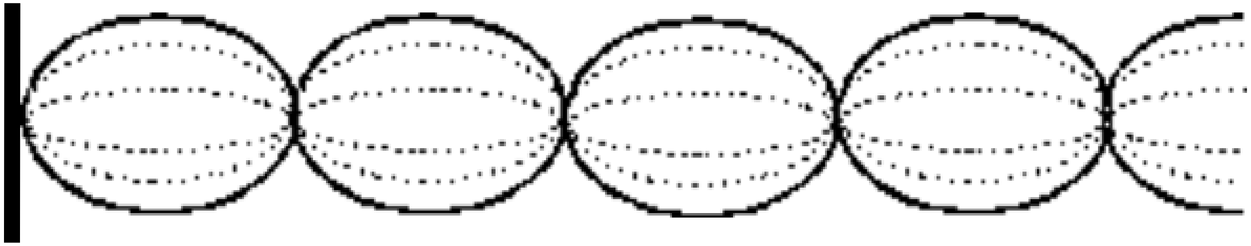
Where: n is a positive integer

Exercise 1

2-One Fixed One Free End



Fundamental



$$u(0,t) = 0$$

$$\left.\frac{\partial U(x,t)}{\partial x}\right|_{x=L} = 0$$

Exercise 1

2-One Fixed One Free End

$$u(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt-kx)} + U_r e^{j(wt+kx)}$$

- Boundary Conditions:

At x=0 $\Rightarrow$ $u(o,t) = 0 \dots \dots \dots .. (1)$

PFD At x=L $\Rightarrow$ $-T \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$ $\Rightarrow$ $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \quad \dots\dots(2$

$$\begin{cases} u(o,t) = 0 \dots \dots \dots .. (1) \\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \quad \dots \dots \dots .. \quad (2) \end{cases}$$

Exercise 1

2-One Fixed One Free End

$$(1) \quad \Rightarrow \quad U_r = -U_i \quad \Rightarrow \quad u(x,t) = U_i e^{jwt}\left(e^{-jkx} - e^{+jkx}\right)$$

Using: $\sin a = \frac{e^{ja} - e^{-ja}}{2j}$

The displacement becomes: $u(x,t) = -2jU_i e^{jwt}\, sinkx$

- The real part of the function u(x,t):

  Using: $e^{jwt} = coswt + j\, sinwt$

$$\Rightarrow \quad u(x,t) = 2U_i\, sinkx\ .\ \sin wt$$

Exercise 1

2-One Fixed One Free End

$$\Rightarrow \; u(x,t) = 2U_i \, sinkx \; . \; \sin wt$$

- Using boundary Conditions:(2) $\Rightarrow \; \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

$$2U_i \, coskx \; . \; \sin wt \,|_{x=L} \;\; \Rightarrow \;\; 2U_i coskL \, .sinwt = 0$$

$$\Rightarrow coskL = 0 \quad \Rightarrow \quad kL = (2n+1)\frac{\pi}{2} \quad \Rightarrow \quad \frac{w}{v}L = (2n+1)\frac{\pi}{2}$$

$$w_n = (2n+1)\frac{\pi v}{L} \quad \Rightarrow \quad f_n = (2n+1)\frac{v}{4L}$$

Exercise 1

2-One Fixed One Free End

$$w_n = (2n+1)\frac{\pi v}{L} \quad \Rightarrow \quad f_n = (2n+1)\frac{v}{4L}$$

$f_n = n' \frac{v}{4L}$, $n' = (2n+1)$ n' is a positive odd integer

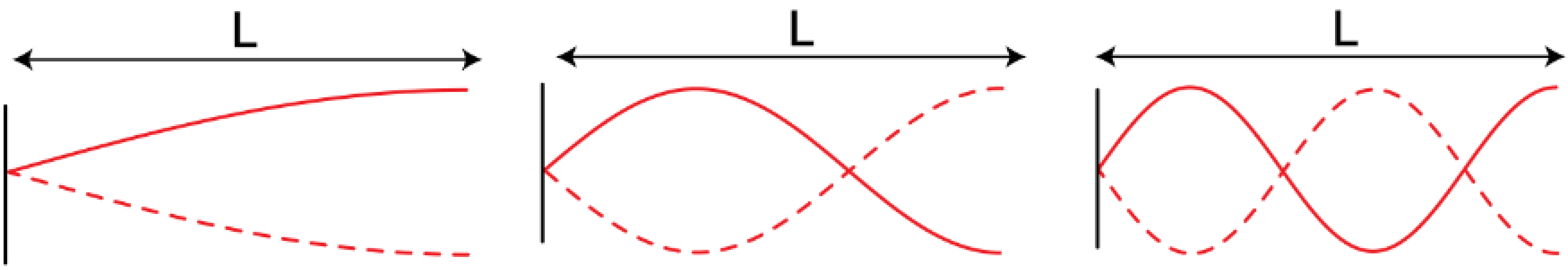
$f_1 = 0.78 \text{ Hz}$ *and* $f_9 = 7.04 \text{ Hz}$

2-One Fixed One Free End

-Represent the fundamental mode and the first three overtones

$$f_n = \frac{v}{\lambda} = \left(\frac{v}{4L}\right) n = f_1 n;\ n = 1, 3, 5, \ldots$$

➢Harmonics for One Fixed One Free End



1st harmonic — $\lambda = 4L \rightarrow f = \left(\frac{v}{4L}\right)$

3rd harmonic — $\lambda = \frac{4L}{3} \rightarrow f = 3\left(\frac{v}{4L}\right)$

5th harmonic — $\lambda = \frac{4L}{5} \rightarrow f = 5\left(\frac{v}{4L}\right)$

$$\lambda_n = \frac{4L}{n};\ n = 1, 3, 5, \ldots$$

where : n is a positive odd integer

Exercise 2

Two semi-infinite strings positioned along an $x'Ox$ axis are connected at $x = 0$. The string in the region $x < 0$ has a linear mass density μ_1 . The string extending from 0 *to* $+\infty$ has a linear mass density $\mu_2 = 0.25\mu_1$.

An incident wave of amplitude U_0 and angular frequency ω arrives from -∞ and propagates in the direction of increasing x. At $x = 0$, the wave undergoes reflection.

1. Calculate the reflection coefficient at $x = 0$.
2. Show that the resulting wave in the region $x < 0$ varies between two values U_{max} and U_{min}. Determine U_{max} and U_{min} , as well as the positions of the vibration maxima and minima. Calculate the standing wave ratio (SWR).
3. Demonstrate that this system is equivalent to a string terminated at $x = 0$ by a damper, and specify the value of the damping coefficient.

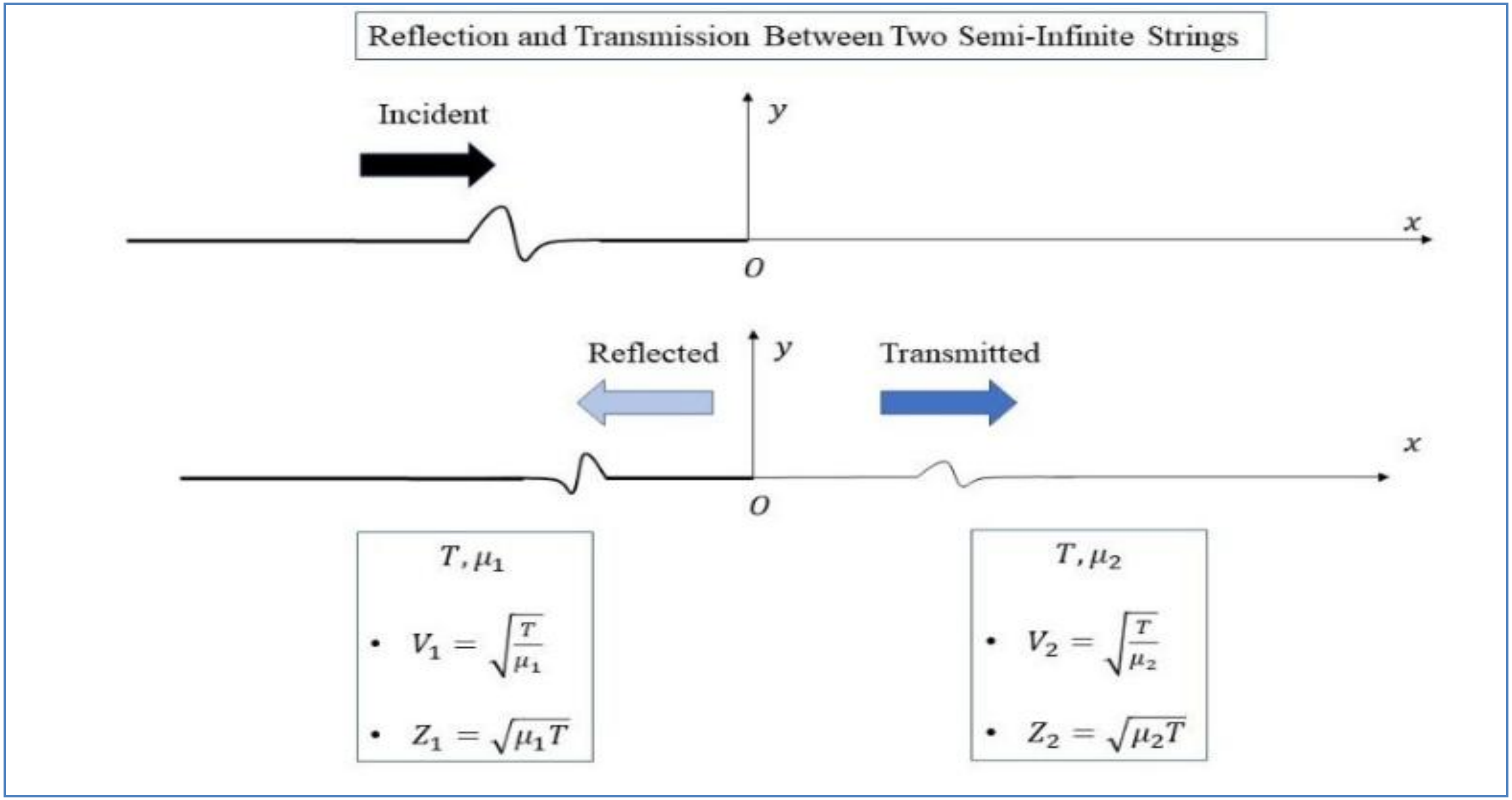
Exercise 2

4. In the case where $\mu_1 = \mu_2$:

- What happens to the reflection coefficient R (in amplitude) at $x = 0$?
- Calculate the linear densities of kinetic and potential energy, e_c and e_p , at any point x along the $x'Ox$ axis.
- Deduce the total energy density and calculate its average value over a wavelength λ.

Exercise 2

1- Calculate the reflection coefficient at x=0



Exercise 2

1- Calculate the reflection coefficient at x=0

$$R = \frac{U_r}{U_i} \quad \text{and} \quad T = \frac{U_t}{U_i}$$

The displacements of the waves in each region are expressed as:

$$x < 0 \qquad u_1(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i e^{j(wt - k_1 x)} + U_r e^{j(wt + k_1 x)}$$

$$x > 0 \qquad u_2(x,t) = U_t e^{j(wt - k_2 x)}$$

Exercise 2

1- Calculate the reflection coefficient at x=0

Using Boundary Conditions

To determine the reflection and transmission coefficients, two conditions are applied at x=0:

- ✓ **Continuity of displacement:**

  $u_1(0,t) = u_2(0,t)$, This ensures the string remains connected.

  $$\Rightarrow \quad U_i + U_r = U_t \ldots\ldots\ldots(1) \qquad \Rightarrow \quad 1 + R = T$$

- ✓ **Continuity of force (PFD at x=0):**

  This represents the continuity of mechanical tension at the junction.

  Oy: $$-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0} + T\left.\frac{\partial u_2}{\partial x}\right|_{x=0} = m.\frac{\partial^2 u}{\partial t^2}$$

.

Exercise 2

1- Calculate the reflection coefficient at x=0

- m is very small

$$\Rightarrow \quad -T\frac{\partial u_1}{\partial x}\bigg|_{x=0} + T\frac{\partial u_2}{\partial x}\bigg|_{x=0} = 0$$

$$-T\left(-jk_1U_ie^{j(wt-k_1.0)} + jk_1U_re^{j(wt+k_1.0)}\right) - Tjk_2U_te^{j(wt+k_2.0)} = 0$$

$$k_1(U_i - U_r) = k_2U_t \quad \Rightarrow \quad U_i - U_r = \frac{k_2}{k_1}U_t \;\ldots\ldots\ldots.(2)$$

Exercise 2

1- Calculate the reflection coefficient at x=0

$$\Rightarrow \qquad \begin{cases} U_i + U_r = U_t \dots\dots\dots (1) \\ U_i - U_r = \frac{k_2}{k_1} U_t \dots\dots .(2) \end{cases}$$

$$(1) + (2) \quad and\ (1) - (2) \ \Rightarrow \qquad \begin{cases} 2U_i = U_t(1 + \frac{k_2}{k_1}) \\ 2U_r = U_t(1 - \frac{k_2}{k_1}) \end{cases}$$

$$R = \frac{U_r}{U_i} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \text{or} \qquad R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$T = \frac{U_t}{U_i} = \frac{2k_1}{k_1 + k_2} \qquad \text{or} \qquad T = \frac{2Z_1}{Z_1 + Z_2}$$

Exercise 2

1- Calculate the reflection coefficient at x=0

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad \Rightarrow \quad R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

$$T = \frac{2 Z_1}{Z_1 + Z_2} \quad \Rightarrow \quad T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

$$T = \frac{2 Z_1}{Z_1 + Z_2}$$

- In the ge $R = |R|.\,e^{j\,\theta}$

Exercise 2

2-Show that the resulting wave in the region $x < 0$ varies between two values U_{max} and U_{min}. Determine U_{max} and U_{min} , as well as the positions of the vibration maxima and minima. Calculate the standing wave ratio (SWR).

$$R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \quad \Rightarrow \quad R = \frac{\sqrt{\mu_1} - \sqrt{0.25\mu_1}}{\sqrt{\mu_1} + \sqrt{0.25\mu_1}}$$

$$R = \frac{1}{3} \quad \Rightarrow \quad \begin{cases} |R| = \frac{1}{3} \\ \theta = 0 \end{cases}$$

Exercise 2

2-Show that the resulting wave in the region $x < 0$ varies between two values U_{max} and U_{min}. Determine U_{max} and U_{min} , as well as the positions of the vibration maxima and minima. Calculate the standing wave ratio (SWR).

$|R| < 1 \quad \Rightarrow \quad$ Standing wave ratio

$$U_r = R\ U_i$$

$$u(x,t) = u_i(x,t) + u_r(x,t)$$

$$= U_i\, e^{j(wt-k_1x)} + U_r\, e^{j(wt+k_1x)}$$

$$= U_i\, e^{j(wt-k_1x)} + U_i\ R\ e^{j(wt+k_1x)}$$

$$u(x,t) = U_i(1 + |R|e^{j(2kx)})\ e^{j(wt-kx)}$$

$$|u(x,t)| = U_i\sqrt{(1 + |R|\cos(2kx))^2 + (|R|(\sin(2kx))^2}$$

$$|u(x,t)| = U_i\sqrt{1 + R^2 + 2|R|(\cos(2kx)}$$

$$U_{min} \leq |u(x,t)| \leq U_{max}$$

Exercise 2

2-Show that the resulting wave in the region $x < 0$ varies between two values U_{max} and U_{min}. Determine U_{max} and U_{min}, as well as the positions of the vibration maxima and minima. Calculate the standing wave ratio (SWR).

$$U_{min} \Rightarrow \cos(2kx) = -1 \Rightarrow U_{min} = (1 - |R|)U_i$$

$$\Rightarrow U_{min} = (1 - \tfrac{1}{3})U_i \Rightarrow U_{min} = \frac{2}{3}$$

$$U_{max} \Rightarrow \cos(2kx) = +1 \Rightarrow U_{max} = (1 + |R|)U_i$$

$$\Rightarrow U_{max} = (1 + \tfrac{1}{3})U_i \Rightarrow U_{max} = \frac{4}{3}$$

- Standing wave ratio : $SWR = \frac{U_{max}}{U_{min}} \Rightarrow SWR = \frac{1+|R|}{1-|R|}$

$$SWR = \frac{\frac{4}{3}}{\frac{2}{3}} \Rightarrow SWR = 2$$

Exercise 2

Positions of the vibration maxima and minima

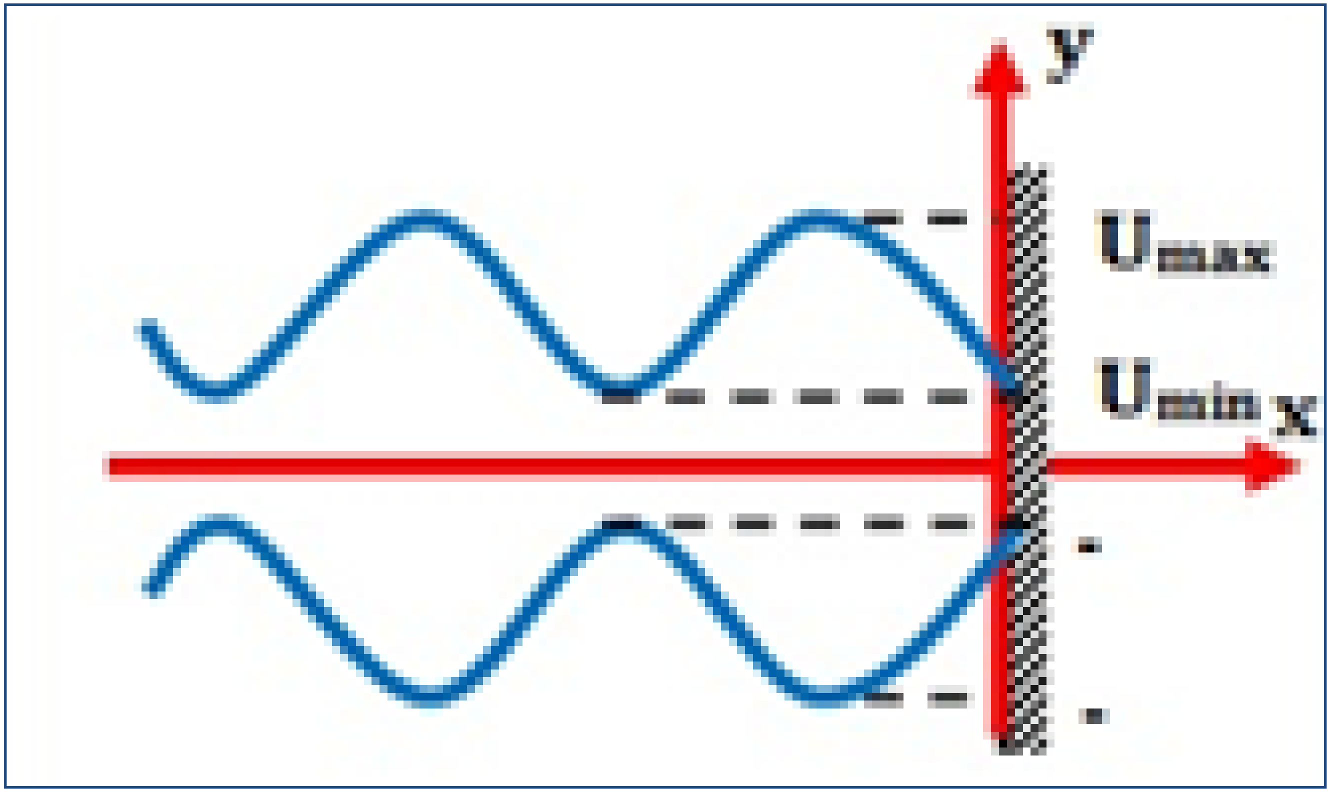
$$U_{max} \quad \Rightarrow \quad \cos(2kx) = +1 \qquad 2kx = 2n\,\pi$$

$$k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad x_{max} = \frac{n\lambda}{2}$$

$$U_{min} \quad \Rightarrow \quad \cos(2kx) = -1 \qquad 2kx = (2n+1)\,\pi$$

$$k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad x_{min} = \frac{(2n+1)\lambda}{4}$$

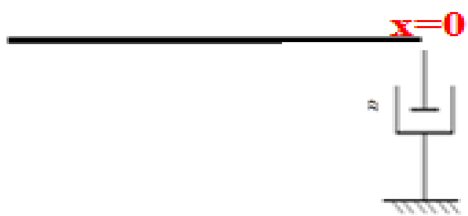
Exercise 2



Exercise 2

3-Demonstrate that this system is equivalent to a string terminated at $x=$ 0 by a damper, and specify the value of the damping coefficient.

➢ **(PFD at x=0):** $\Rightarrow$ Oy: $$-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}+T\left.\frac{\partial u_2}{\partial x}\right|_{x=0}=0$$



➢ **(PFD at x=0):** $\Rightarrow$ Oy: $$-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}-\alpha\dot{u}_2=0$$

$$-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}-\alpha\left.\frac{\partial u_2}{\partial t}\right|_{x=0}=0$$

Exercise 2

3–Demonstrate that this system is equivalent to a string terminated at $x =$ 0 by a damper, and specify the value of the damping coefficient.

➤ $At\ x = 0$ $we\ have\ continuity\ of$ displacement and speed :

$$u_1(0,t) = u_2(0,t) \quad \Rightarrow \quad \dot{u}_1(0,t) = \dot{u}_2(0,t)$$

$$\begin{cases} -T\left.\frac{\partial u_1}{\partial x}\right|_{x=0} + T\left.\frac{\partial u_2}{\partial x}\right|_{x=0} = 0 \\ -T\left.\frac{\partial u_1}{\partial x}\right|_{x=0} - \alpha\left.\frac{\partial u_2}{\partial t}\right|_{x=0} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}}{\dot{u}_1(0,t)} + \dfrac{+T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}}{\dot{u}_2(0,t)} = 0 \\ \dfrac{-T\left.\frac{\partial u_1}{\partial x}\right|_{x=0}}{\dot{u}_1(0,t)} - \dfrac{\alpha\left.\frac{\partial u_2}{\partial t}\right|_{x=0}}{\dot{u}_2(0,t)} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} Z_1(0) - Z_2(0) = 0 \\ Z_1(0) - \alpha = 0 \end{cases} \Rightarrow \quad Z_2(0) = \alpha$$

Exercise 2

4. In the case where $\mu_1 = \mu_2$:

- **What happens to the reflection coefficient R (in amplitude) at $x = 0$?**

$$R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \quad \Rightarrow \quad R = 0$$

Exercise 2

4-In the case where $\mu_1 = \mu_2$:

- **Calculate the linear densities of kinetic and potential energy, e_c and e_p, at any point x along the $x'Ox$ axis.**

$$e_k = \frac{dE_k}{dx} \quad \Rightarrow \quad e_k = \frac{1}{2}\mu\left(\frac{\partial u}{\partial t}\right)^2$$

$$u(x,t) = u_i(x,t) \quad ; \quad R = 0$$

$$= U_i e^{j(wt - k_1 x)} \quad ; \quad U_i = u \; ; \quad k_1 = k$$

➢ The real part of the function u(x,t):

$$u(x,t) = u \cos(wt - kx)$$

$$e_k = \frac{1}{2}\mu\left(\frac{\partial u}{\partial t}\right)^2 \quad \Rightarrow \quad e_k = \frac{1}{2}\mu\, u^2 w^2 \sin^2(wt - kx)$$

$$e_p = \frac{1}{2}T\left(\frac{du}{dx}\right)^2 \quad \Rightarrow \quad e_p = \frac{1}{2}T\, u^2 k^2 \sin^2(wt - kx)$$

$$k^2 T = \mu w^2 \quad \Rightarrow \quad e_p = e_k$$

$$e_T = 2e_p = 2e_k$$

Exercise 2

4-In the case where $\mu_1 = \mu_2$:

- **Calculate the linear densities of kinetic and potential energy, e_c and e_p , at any point x along the $x'Ox$ axis.**

➤ The average values
$$\begin{cases} \langle e_p \rangle = \frac{1}{\lambda} \int \frac{1}{2} T\, u^2 k^2 \sin^2(wt - kx)\, dx \\ \langle e_k \rangle = \frac{1}{\lambda} \int \frac{1}{2}\, u^2 \mu w^2 \sin^2(wt - kx)\, dx \end{cases}$$

$$\begin{cases} \langle e_p \rangle = \frac{1}{4}\, u^2 k^2 T \\ \langle e_k \rangle = \frac{1}{4}\, u^2 \mu w^2 \end{cases} \quad \Rightarrow \quad \langle e_p \rangle = \langle e_k \rangle$$

➤ $$\Rightarrow e_T = \frac{1}{2}\, u^2 k^2 T = \frac{1}{2}\, u^2 \mu w^2$$

Thank You for Your Attention