National Higher School of Autonomous Systems Technology

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Vibrations and waves

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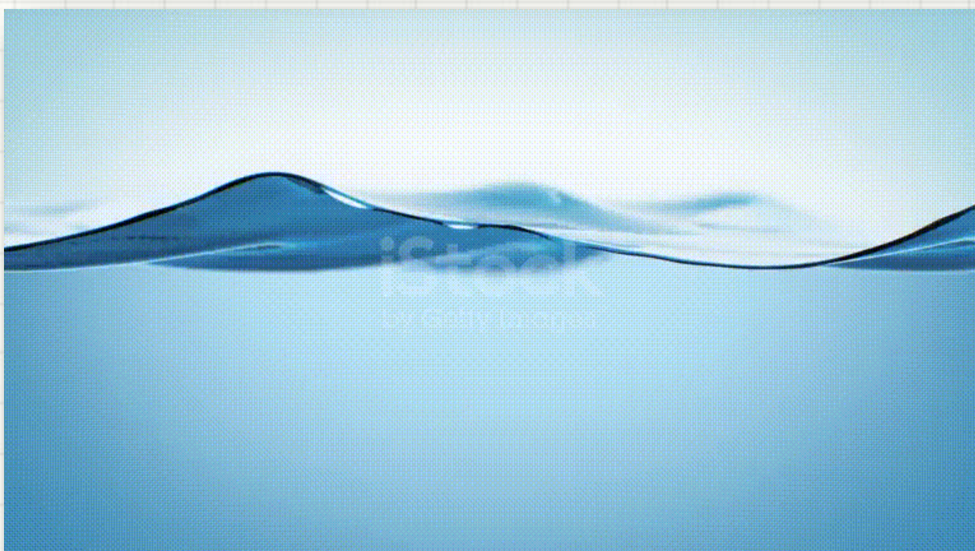
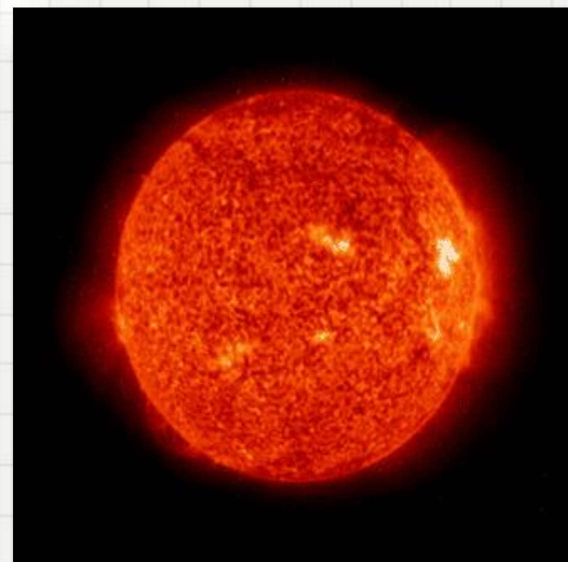
Why vibrations and waves ?

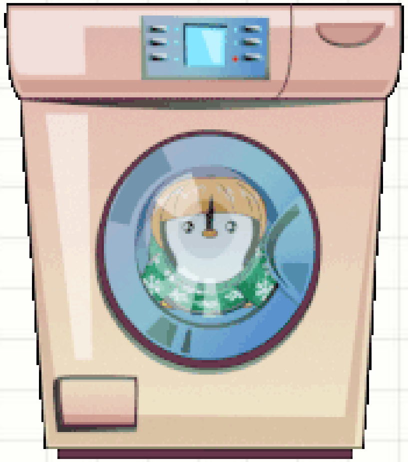
Everything vibrates around us
The stones, the air, the human body, the earth, the sea, the sun





Why vibrations and waves ?

Everything vibrates around us

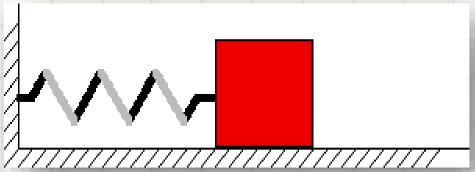


Washing machine

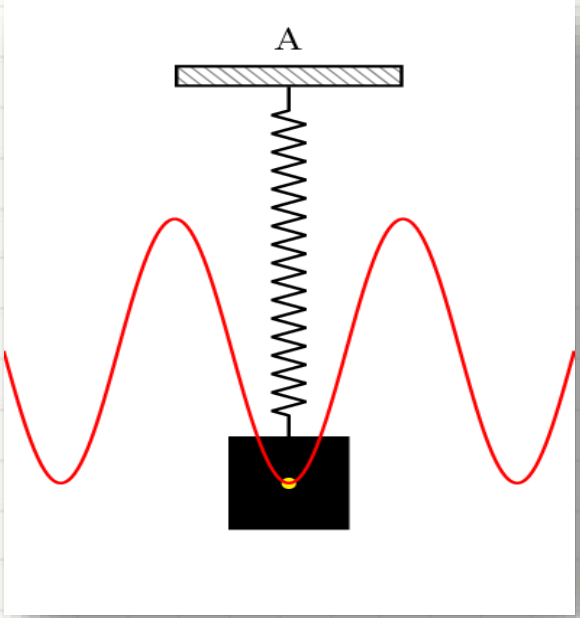
Mixers

Definition of vibration

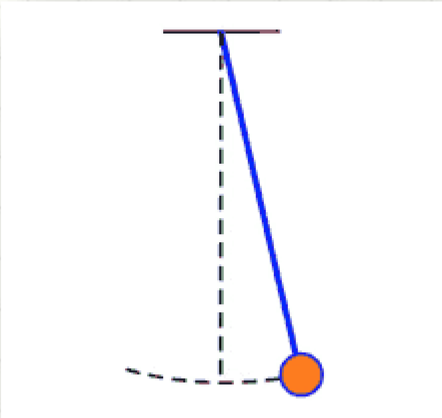
Definition 1 **:** *Any motion that repeat it self after an interval of time colled vibration or oscillation*



Horizontal mass- spring



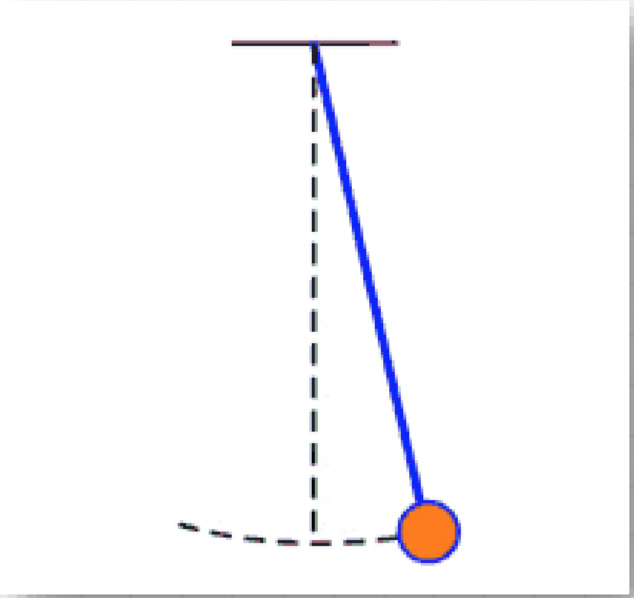
Vertical mass-spring



Simple pendulum

Definition of vibration

Definition 2: It is to and fro motion of whole or part of body about a mean position (Equilibrium position)



Simple pendulum

Classification of Vibrations

1-Free and Forced Vibrations

- **Free Vibration:** When a system, after an initial disturbance, is left to vibrate on its own without the influence of any external force, the resulting vibration is called *free vibration*. No external force acts on the system during this process. The oscillation of a simple pendulum is an example of free vibration.

- **Forced Vibration:** When a system is subjected to an external force, the resulting vibration is known as *forced vibration* (e.g., the oscillation of engines).

Classification of Vibration

2-Undamped and Damped Vibration

- **Undamped Vibration:** If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as *undamped vibration*,.

- **Damped Vibration :** If any energy is lost in this way, however, it is called *damped vibration*.

Summary

Chapter 0:

Review and Generalities

Chapter 0: Review and Generalities

1-Complex numbers

A complex number, z is an expression with two terms

$z = a + jb$, where $j^2 = -1$ and $a, b \in \mathbb{R}$

- Real part: $>$ $\mathrm{Re(z)} = Re(a + jb) = a$
- Imaginary part: $Im(z) = Im(a + jb) = b$
- Modulus or absolut: $|z| = \sqrt{a^2 + b^2}$
- Argument of the complex number : $\tan(\theta) = \frac{Im(z)}{Re(z)}$

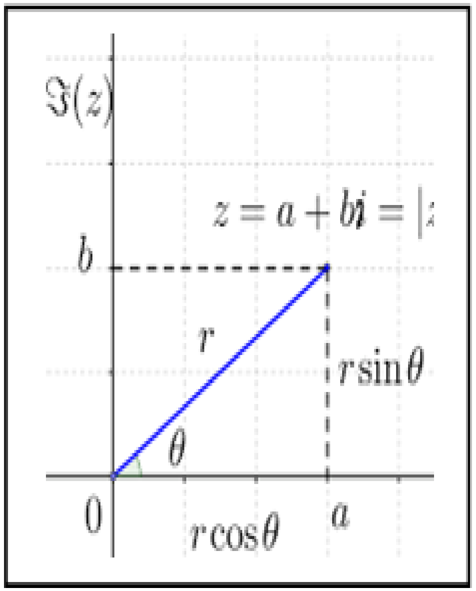
Chapter 0: Review and Generalities

1-Complex numbers

- *Graphical representation of Complex Numbers*

  The complex number is represented by the point

  or by the vector from the origin to the point



1-Complex numbers

- *Euler's formula*

$$z = |z|\,(\cos\theta + j\sin\theta),\quad z = |z|\,e^{j\theta}\ ,\qquad e^{j\theta} = \cos\theta + j\sin\theta.$$

If we have two complex numbers z_1 and z_2:

$$\begin{cases} Z_1 = r_1 e^{j\theta_1} \\ Z_2 = r_2 e^{j\theta_2} \end{cases} \Rightarrow \quad Z_1.Z_2 = r_1.r_2 e^{j(\theta_1+\theta_2)}.$$

$$\begin{cases} Z_1 = r_1 e^{j\theta_1} \\ Z_2 = r_2 e^{j\theta_2} \end{cases} \Rightarrow \quad \frac{Z_1}{Z_2} = \frac{r_1}{r_2}.e^{j(\theta_1-\theta_2)}$$

Chapter 0: Review and Generalities

1-Complex numbers

Properties of the Cosine and Sine functions:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\,cos(b) + \cos(\mathrm{a})\,sin(b)$$

$$\cos(a)+\cos(b)=2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$$

$$\sin(a)+\sin(b)=2\sin(\frac{a+b}{2})cos(\frac{a-b}{2})$$

Chapter 0: Review and Generalities

2-Sinusoidal physical quantity

A physical quantity is considered sinusoidal if its expression as a function of time is of the form:

$x(t) = A\sin(wt + \varphi)$ or $x(t) = A\cos(wt + \varphi)$

In physics $x(t)$ can represent a displacement, an angular deviation, an intensity of an alternating electric current, electric charge …etc.

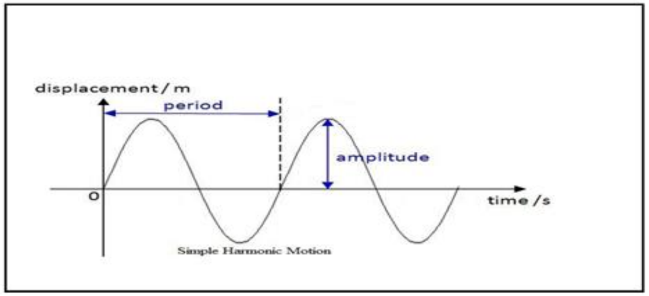
2-Sinusoidal physical quantity

Let $x(t)$ *be a displacement* $x(t) = A\sin(wt + \varphi)$

- A : *is Amplitude represent the maximum value of* $x(t)$ [*same dimension as* $x(t)$]
- w: *angular frequency*, $[w]$ = *radians/second (rad/s)*
- $wt + \varphi$: *Instantaneous phase*, $[wt + \varphi]$ = *radians (rad)*
- φ: *initial phase at* $t = 0$, $[\varphi]$ = *radians (rad)*

$$T = \frac{2\pi}{\omega}\,; \qquad \omega = 2\pi f\,, \quad f = \frac{1}{T}$$

- *Period* (T): *time it takes to complete one oscillation*, $[T]$ = *Second (s)*
- *Frequency* (f): *number of oscillations per unit of time*, $[f]$ = *Hertz (Hz)*



2-Sinusoidal physical quantity

Phase shift between two signals of the same frequency:

The phase shift between two signals is a measure of the offset between two sinusoidal signals of the same frequency. If we have two sinusoidal signals $s_1(t)$ and $s_2(t)$ with the same frequency:

$$s_1(t) = Asin(\omega t + \varphi_1)$$
$$s_2(t) = Asin(\omega t + \varphi_2)$$

The phase shift $\Delta\varphi = \varphi_2 - \varphi_1$

If $\Delta\varphi$ is positive, signal 2 is leading in phase relative to signal 1.
If $\Delta\varphi$ is negative, signal 2 is lagging in phase relative to signal 1.

2-Sinusoidal physical quantity

➢ Special cases of phase shift

1- In-phase signals: *When the phase shift* $\Delta\varphi = 2n\pi$, *the signals are said to be in phase. In this configuration, their maxima and minima coincide.*

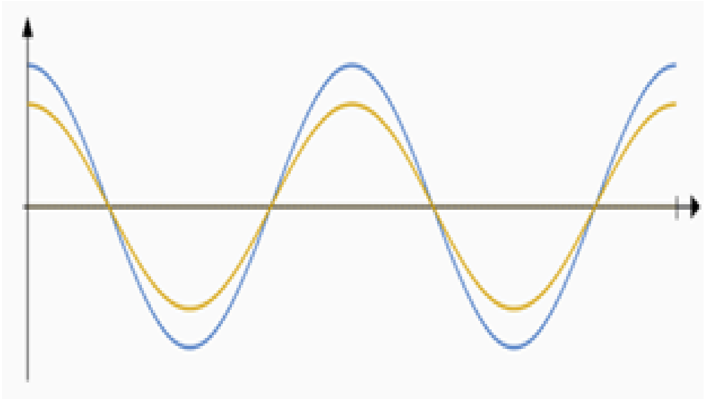
2- Opposite phase signals (Out-of-phase)

When the phase shift $\Delta\varphi = (2n+1)\pi$ *In this configuration, the maxima of one signal coincide with the minima of the other signal.*

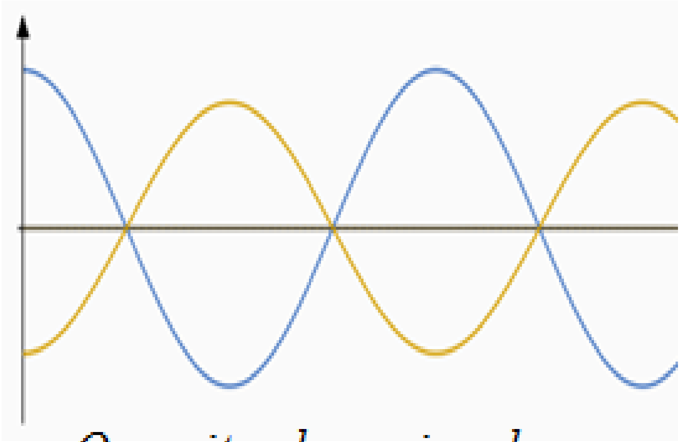
3-Quadrature phase signals

When the phase shift is $\Delta\varphi = (2n+1)\frac{\pi}{2}$, *the signals are said to be in quadrature phase. In this configuration, the maxima of one signal coincide with the zero-crossings of the other signal .*

2-Sinusoidal physical quantity

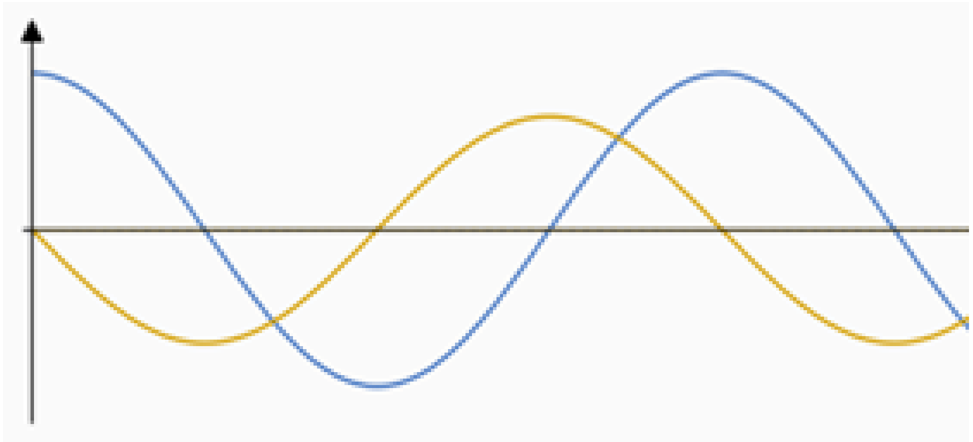


In-phase signals



Opposite phase signals

2-Sinusoidal physical quantity



Quadrature phase signals

3-Average value of a periodic function

If $f(t)$ is periodic with period T, its average value, denoted as $\langle f(t)\rangle$ is defined by:

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$$

3-Average value of a periodic function

- *In practice, it will be rare to calculate the integral directly. It is useful to know the average values of the most commonly encountered functions.*

$$\langle cos\,(wt)\rangle = 0 \qquad \langle sin\,(wt)\rangle = 0$$

$$\langle cos^2\,(wt)\rangle = \frac{1}{2} \qquad \langle sin^2\,(wt)\rangle = \frac{1}{2}$$

$$\langle cos(wt)\,.\,sin\,(wt)\rangle = 0$$

- *The demonstration of these results will be in the tutorial session (TD1)*

4 -Complex representation

A sinusoidal quantity $x(t)$ with amplitude a and angular frequency w can be represented at each instant by a complex number $\overline{X(t)}$ with modulus a, rotating at angular velocity w in the complex plane.

According to Euler's formula, $\overline{X(t)}$ can be expressed as:

$$\overline{X(t)} = a\cos(wt+\varphi) + ja\sin(wt+\varphi)$$

$$\overline{X(t)} = ae^{j(wt+\varphi)} = \bar{a}e^{jwt} \quad , \quad \bar{a} = ae^{j\varphi} \quad , \quad a = |\bar{a}| \; , \quad Tg\varphi = \frac{Im\bar{a}}{Re\bar{a}}$$

4 -Complex representation

Therefore, any real quantity $x_1(t) = a\cos(wt + \varphi)$ can be represented by the complex number $\overline{X(t)}$

$$x_1(t) = Re\overline{X(t)}$$

Similarly for $x_2(t) = a\sin(wt + \varphi)$, $x_2(t) = Im\overline{X(t)}$

4 -Complex representation

Note:

However, the sinusoidal quantities $x_1(t)$ and $x_2(t)$ cannot be represented simultaneously by the same complex number in the same operation. If $x_1(t) = a\cos(wt + \varphi)$ is represented by $\overline{X(t)} = ae^{j(wt+\varphi)}$, then the elongation $x_2(t) = a\sin(wt + \varphi)$ should be written in the form $x_2(t) = a\cos\left(wt + \varphi - \frac{\pi}{2}\right)$, using the same function, either cosine or sine.

4 -Complex representation

Application exercise

Using the complex representation, calculate the amplitude and initial phase of the following sum:

$$x(t) = 2sin\left(wt + \frac{\pi}{4}\right) + 3\cos wt$$

$$x_1(t) = 2sin\left(wt + \frac{\pi}{4}\right) \Rightarrow \overline{x_1(t)} = 2e^{j\left(wt + \frac{\pi}{4}\right)}$$

$$x_2(t) = 3\cos wt = 3sin\left(wt + \frac{\pi}{2}\right) \Rightarrow \overline{x_2(t)} = 3e^{j\left(wt + \frac{\pi}{2}\right)}$$

4 -Complex representation

Application exercise

$$\overline{x(t)} = \overline{x_1(t)} + \overline{x_2(t)} = 2e^{j(\omega t+\frac{\pi}{4})} + 3e^{j(\omega t+\frac{\pi}{2})} = e^{j\omega t}(2e^{j\frac{\pi}{4}} + 3e^{j\frac{\pi}{2}})$$

$$=e^{j\omega t}\left(2\left(cos\left(\frac{\pi}{4}\right) + jsin\left(\frac{\pi}{4}\right)\right) + 3\left(cos\left(\frac{\pi}{2}\right) + jsin\left(\frac{\pi}{2}\right)\right)\right)$$

$$= e^{j\omega t}\left(\sqrt{2} + j\sqrt{2} + 3j\right)$$

$$= e^{j\omega t}\left(\sqrt{2} + j(\sqrt{2} + 3)\right)$$

$$\overline{x(t)} = 4.63\ e^{j(\omega t+1,26)}$$

4 -Complex representation

Application exercise

Using the complex representation, calculate the amplitude and initial phase of the following sum:

$$x(t) = 2sin\left(wt + \frac{\pi}{4}\right) + 3\,cos\,wt$$

$$\overline{x(t)} = 4.63\, e^{j(\omega t + 1,26)} \Longrightarrow x(t) = 4.63\sin(\omega t + 1,26)$$

- **Addition of two harmonic oscillations**

a- Addition of two harmonic oscillations with same frequency

$$\begin{cases} x_1(t) = a_1 cos\,(\omega t + \varphi_1) \\ x_2(t) = a_2\, cos\,(\omega t + \varphi_2) \end{cases} \quad , \qquad x(t) = x_1(t) + x_2(t)$$

$$x(t) = a_1 cos\,(\omega t + \varphi_1) + a_2\, cos\,(\omega t + \varphi_2)$$

$$= a_1 cos\,\omega t\; cos\varphi_1 - a_1 sin\,\omega t\; sin\varphi_1 + a_2 cos\,\omega t\; cos\varphi_2 - a_2 sin\,\omega t\; sin\varphi_2$$

$$= cos\,\omega t\,[a_1 cos\varphi_1 + a_2 cos\varphi_2] - sin\,\omega t[a_1 sin\varphi_1 + a_2 sin\varphi_2]$$

$$= cos\,\omega t\,.\,A cos\varphi - sin\,\omega t\,.\,A sin\varphi$$

➢ Addition of two harmonic oscillations

a- Addition of two harmonic oscillations with same frequency

$$x(t) = A\cos(\omega t + \varphi)$$

We put:
$$\begin{cases} A\cos\varphi = a_1\cos\varphi_1 + a_2\cos\varphi_2 & (1) \\ A\sin\varphi = a_1\sin\varphi_1 + a_2\sin\varphi_2 & (2) \end{cases}$$

Squaring and adding equations (1) and (2) we get:

$$A^2 = a_1{}^2 + a_2{}^2 + 2a_1a_2\cos(\varphi_1 - \varphi_2) \quad \text{or} \quad A = \sqrt{a_1{}^2 + a_2{}^2 + 2a_1a_2\cos(\varphi_1 - \varphi_2)}$$

Dividing equation (2) by equation (1) *we get*

$$\tan(\varphi) = \frac{a_1\sin(\varphi_1) + a_2\sin(\varphi_2)}{a_1\cos(\varphi_1) + a_2\cos(\varphi_2)}$$

➢ Addition of two harmonic oscillations

•Addition of two harmonic oscillations with different frequencies

a- Same Amplitudes

$$\begin{cases} x_1(t) = A\sin\omega_1 t \\ x_2(t) = A\sin\omega_2 t \end{cases} \quad ; \quad x(t) = x_1(t) + x_2(t) \,, \quad \omega_2 > \omega_1$$

According to formula: $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$

$$x(t) = x_1(t) + x_2(t)$$

we get $$x(t) = 2A\cos\frac{\omega_2 - \omega_1}{2}t \;.\; \sin\frac{\omega_2 + \omega_1}{2}t$$

$$x(t) = f_1(t).f_2(t)$$

➢Addition of two harmonic oscillations

•Addition of two harmonic oscillations with different frequencies

a- Same Amplitudes

$$\text{We put}\begin{cases} f_1(t) = 2\ A\cos\frac{\omega_2-\omega_1}{2}t & . \\ f_2(t) = \ \sin\frac{\omega_2+\omega_1}{2}t \end{cases}$$

$$\frac{\omega_2-\omega_1}{2} < \frac{\omega_2+\omega_1}{2} \quad \Rightarrow \quad T_{f_1} > T_{f_2} \quad \Rightarrow \quad \mathrm{f_2(t)\ is\ fast\ then\ f_1(t)}$$

➢ $\frac{\omega_2+\omega_1}{2}$ is called the average frequency

➢ $\frac{\omega_2-\omega_1}{2}$ is called the modulating frequency.

➢ Addition of two harmonic oscillations

•Addition of two harmonic oscillations with different frequencies

a- Same Amplitudes

<u>*Special Case*</u>

➢ *if the two frequencies are slightly different:* $\omega_2 = \omega_1 + \delta$

$$\begin{cases} f_1(t) = 2\ A\cos\frac{\omega_2-\omega_1}{2}t = 2A\ \cos\frac{\delta}{2}t\ . & \text{is the } \textit{slow} \text{ oscillation} \\ f_2(t) = \ \sin\frac{\omega_2+\omega_1}{2}t = \ \sin(\omega_1\, t + \frac{\delta}{2}t) & \text{is the } \textit{fast} \text{ oscillation} \end{cases}$$

$$(I\ ) \Rightarrow \quad x(t) = 2A\ \cos\frac{\delta}{2}t\ .\ \sin(\omega_1\, t + \frac{\delta}{2}t)$$

➢Addition of two harmonic oscillations

•**Addition of two harmonic oscillations with different frequencies**

a- Same Amplitudes

➢ $f_1(t)$ *represent the **envelope** function, The fast oscillation is **modulate** by the slow oscillation.*

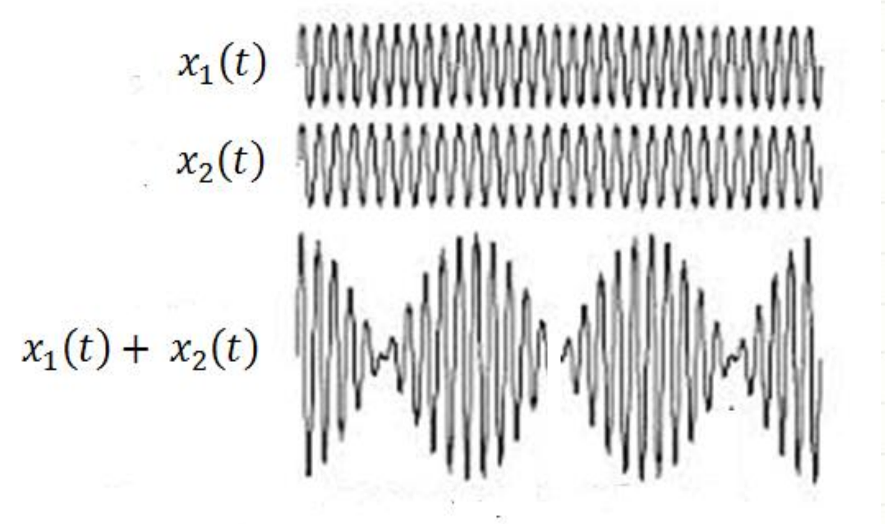
✓ *When tow harmonic oscillations, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as Beat phenomenon*

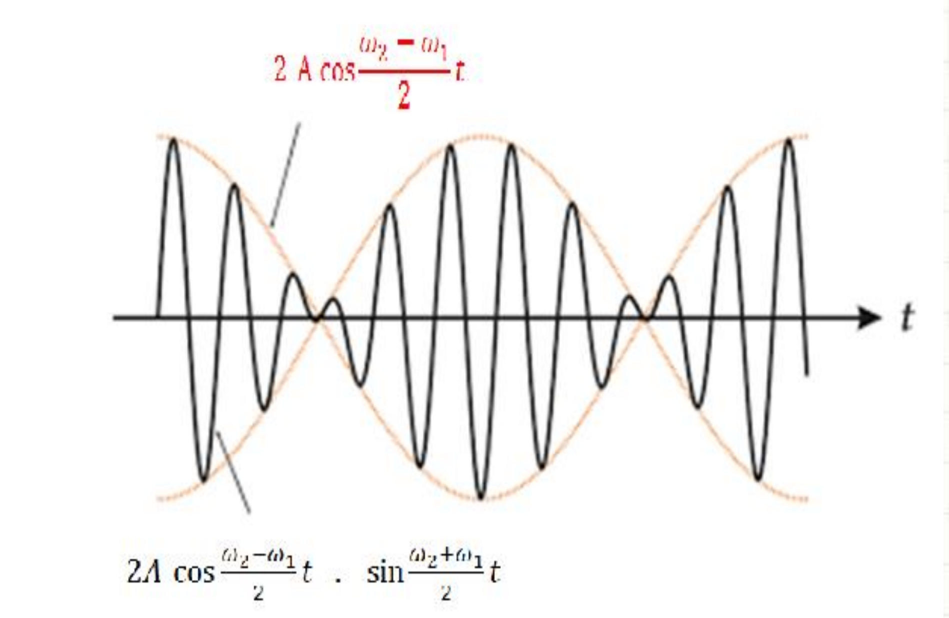
➢ Addition of two harmonic oscillations

•Addition of two harmonic oscillations with different frequencies

a- Same Amplitudes

- ***Graph of equation***





5-Addition of two harmonic oscillations

➢ Addition of two harmonic oscillations

➢ **Addition of two harmonic oscillations with different frequencies**

2- Different Amplitudes

The two frequencies of *slightly different*:

$x_1(t) = A\sin\omega_1 t$ $\qquad\qquad \omega_2 = \omega_1 + \delta$

$x_2(t) = B\sin\omega_2 t$

$\omega_2 = \omega_1 + \delta$ because the two frequencies are *slightly different*

$x(t) = x_1(t) + x_2(t)$ $\qquad x(t) = A\sin\omega_1 t + B\sin(\omega_1 + \delta)t$

➢Addition of two harmonic oscillations

➢ *Addition of two harminic oscilations with different ferquencies*

2- Different amlitudes

$$x_1(t) = A\sin(\omega_1 t) \quad \Rightarrow \quad \overline{x_1(t)} = Ae^{j\omega_1 t}$$
$$x_2(t) = B\sin(\omega_1 t + \delta t) \quad \Rightarrow \quad \overline{x_2(t)} = Be^{j(\omega_1+\delta)t}$$

$$x(t) = x_1(t) + x_2(t) \Rightarrow \overline{x(t)} = \overline{x_1(t)} + \overline{x_2(t)}$$

$$\overline{x(t)} = (\mathrm{A} + Be^{j\delta t})\, e^{j\omega_1 t} = \overline{x_0} e^{j\omega_1 t}$$

$$\text{With} \begin{cases} \overline{x_0} = \left(\mathrm{A} + Be^{j\delta t}\right) = x_0 e^{j(\varphi)} \\ x_0 = |\overline{x_0}| = \left|\mathrm{A} + Be^{j\delta t}\right| \\ \tan(\varphi) = \dfrac{Im(\mathrm{A}+Be^{j\delta t})}{Re(\mathrm{A}+Be^{j\delta t})} \end{cases}$$

5-Addition of two harmonic oscillations

➢ Addition of two harmonic oscillations

➢ **Addition of two harmonic oscillations with different frequencies**

2- Different Amplitudes

➢ *Addition of two harminic oscilations with different ferquencies*

2- Different amlitudes

$$x_0 = |\overline{x_0}| = |(\mathrm{A} + Bcos(\delta t) + jBsin(\delta t)| = \sqrt{(\mathrm{A} + Bcos(\delta t))^2 + (Bsin(\delta t))^2}$$
$$= \sqrt{A^2 + B^2 + 2\mathrm{A}Bcos(\delta t)}$$

x_0 varies between $x_{max} = A + B$ and $x_{min} = A - B$

- If the two frequencies are slightly different we get also beat phenomenon
- The resultant amplitude varies between (A+B) and (A-B)

$$\tan(\varphi) = \frac{Im(\mathrm{A} + Be^{j\delta t})}{Re(\mathrm{A} + Be^{j\delta t})}$$