

– Course 6 –

Chapter 6 : Linear Programming

Linear programming? (1/3)

- **Linear programming** is at the heart of operational research (OR) or decision support: it offers **conceptual models** to analyze complex situations and **allows decision-makers to make the most effective choices**.
- Many economic and industrial phenomena can be modeled by mathematical systems of **linear inequalities** and **equalities** leading to **linear optimization problems**.

Linear programming? (2/3)

- In these linear optimization problems, we seek to **minimize** or **maximize** a **linear function** under **linear constraints** on the **variables** of the problem.
- Linear programming (or linear program), the term **programming** referring to the idea of **organization** and **planning** linked to the nature of the phenomena modeled.

=> Programming = Planning or Organizing

Linear programming? (3/3)

- **Definition 1 (William J. BAUMAUL)** Linear programming is a mathematical technique for **optimizing** (**maximizing** or **minimizing**) a function with a **linear objective** under **constraints** in the form of **linear inequations**. It aims to select among different actions the one that will most likely achieve the desired objective.
- **Definition 2 (Robert DORFMAN and Paul SAMUELSON)** Linear programming is a method of determining the **best plan of action** to achieve **given objectives** in a situation where **resources are limited**. It is therefore a method of solving the economic problem, either in the context of a global economy, or in that of the public sector, or in a particular company.

OR process (Reminder)



Find a problem



Formalize the problem



Construct a model



Implement a solution

Refine model

Carry out decisions

Solve the model

$$\begin{aligned} \min & x + 3y + 2z - a \\ 10x + 5y & \leq 10 \\ 42y - 1.2z + a & \geq x \\ x, y, z & \geq 0 \end{aligned}$$

$$\begin{aligned} \min & x + 3y + 2z \\ 2x + 5y & \leq 10 \\ 4y - 1.2z & \geq x \\ x, y, z & \geq 0 \end{aligned}$$



Formulation of a linear program (LP) (Modeling) (1/3)

The conditions for formulating a linear program (1/2)

- Linear programming as a model admits hypotheses (conditions) that the decision maker must validate before being able to use them to model his problem. These hypotheses are:
 1. The **decision variables of the problem are positive**.
 2. The **selection criterion** of the best decision is described by a **linear function of these variables**, that is, the function cannot contain, for example, a cross product of two of these variables. The function that represents the selection criterion is called the **objective function** (or economic function).

Formulation of a linear program (LP) (Modeling) (2/3)

The conditions for formulating a linear program (2/2)

- 3) The restrictions on the decision variables (e.g. resource limitations) can be expressed by a set of linear equations. **These equations form the set of constraints.**
- 4) The parameters of the problem outside the decision variables have a value known with certainty.

Formulation of a linear program (LP) (Modeling) (3/3)

The steps of formulating a linear program

1. Identify the variables of the problem with unknown values (**decision variable**) and represent them in symbolic form (example: x , y).
2. Identify the restrictions (**constraints**) of the problem and express them by a **system of linear equations**.
3. Identify the objective or selection criterion and represent it in linear form as a **function of the decision variables**. Specify whether the selection criterion is to be **maximized** or **minimized**.

Illustrative example 1: Agricultural problem (1/6)

A farmer wants to allocate 150 hectares of irrigable area between tomato and pepper cultivation. He has 480 hours of labor and 440 m³ of water. One hectare of tomatoes requires 1 hour of labor, 4 m³ of water and gives a net profit of 100 dinars. One hectare of peppers requires 4 hours of labor, 2 m³ of water and gives a net profit of 200 dinars. The irrigated area office wants to protect the price of tomatoes and does not allow him to cultivate more than 90 hectares of tomatoes.

What is the best allocation of his resources?

Illustrative example 1: Agricultural problem (2/6)

Step 1: Identification of **decision variables**. The two activities that the farmer must determine are the areas to allocate for growing tomatoes and peppers:

- x_1 : the area allocated to growing tomatoes.
- x_2 : the area allocated to growing peppers.

We check that the decision variables x_1 and x_2 are positive:

$$x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Illustrative example 1: Agricultural problem (3/6)

Step 2: Identification of **constraints**. In this problem, the constraints represent the availability of production factors:

- **Agricultural land**: the farmer has 150 hectares of land, so the constraint related to the limitation of the land area is $x_1 + x_2 \leq 150$.
- **Water**: growing one hectare of tomatoes requires 4 m³ of water and growing one hectare of peppers requires 2 m³, but the farmer only has 440 m³. The constraint that expresses the limitations of water resources is $4x_1 + 2x_2 \leq 440$.

Illustrative example 1: Agricultural problem (4/6)

- **Labor:** The 480 hours of labor will be divided (not necessarily in full) between the cultivation of tomatoes and that of peppers. Knowing that **one hectare of tomatoes requires one hour of labor** and **one hectare of peppers requires 4 hours of labor** then the constraint representing the limitations of human resources is **$x_1 + 4x_2 \leq 480$** .
- **Irrigated Perimeter Office Limitations:** These limitations require that the farmer does not grow more than **90 hectares** of tomatoes. The constraint that represents this restriction is **$x_1 \leq 90$** .

Illustrative example 1: Agricultural problem (5/6)

- **Step 3:** Identification of the **objective function**. The **objective function** consists of maximizing the profit brought by the cultivation of tomatoes and peppers. The respective contributions 100 and 200, of the two decision variables x_1 and x_2 are proportional to their value.

=> The **objective function** is therefore **$100 x_1 + 200x_2$**

Illustrative Example 2: Agricultural problem (6/6)

The linear program that models the agricultural problem is:

$$\text{Max } z = 100x_1 + 200x_2$$

$$\left\{ \begin{array}{l} x_1 + x_2 \leq 150 \\ 4x_1 + 2x_2 \leq 440 \\ x_1 + 4x_2 \leq 480 \\ x_1 \leq 90 \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right.$$

Illustrative Example 2: Medicine problem (1/5)

A **medical specialist** has made a medicine (pills) to cure people with a cold. These pills are made in two sizes:

- **Small size**: it contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine.
- **Large size**: it contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine.

To cure the disease, the subject needs 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine.

Determine the minimum number of pills to prescribe to the subject so that he is cured.

Illustrative Example 2: Medicine problem (2/5)

Step 1: The decision variables that represent values unknown to the decision maker (medical specialist) are:

- x_1 : the number of small pills to prescribe.
- x_2 : the number of large pills to prescribe.

We check that the decision variables x_1 and x_2 are positive:

$x_1 \geq 0$ and $x_2 \geq 0$.

Illustrative Example 2: Medicine problem (3/5)

Step 2: The constraints imposed by the problem on the possible values of x_1 and x_2 are:

- The prescription must contain pills with at least 12 grains of aspirin. Knowing that a small pill contains 2 grains of aspirin and a large pill contains a single grain of aspirin, we obtain the following constraint: $2x_1 + x_2 \geq 12$.

- In the same way as for aspirin, the prescription of the medical specialist must contain at least 74 grains of bicarbonate. Thus the following constraint must be satisfied: $5x_1 + 8x_2 \geq 74$.

Illustrative Example 2: Medicine problem (4/5)

Finally the constraint imposed by the fact that the prescription must contain at least **24 grains of codeine** is:

$$x_1 + 6x_2 \geq 24.$$

Step 3: We notice that there are several pairs of solutions that can satisfy the constraints specified in **step 2**. The prescription must contain the minimum possible number of pills. Therefore the selection criterion for the quantity of pills to be prescribed is the one that **minimizes the total number of pills** $z = x_1 + x_2$.

Illustrative Example 2: Medicine problem (5/5)

The linear program that models the medicine problem is:

$$\text{Min } z = x_1 + x_2$$

$$\left\{ \begin{array}{l} 2x_1 + x_2 \geq 12 \\ 5x_1 + 8x_2 \geq 74 \\ x_1 + 6x_2 \geq 24 \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right.$$

Illustrative Example 3: Production problem (1/3)

To manufacture two products **P1** and **P2**, operations must be performed on three machines **M1**, **M2** and **M3**, successively but in any order. The unit execution times are given by the following table:

	M1	M2	M3
P1	11mn	7mn	6mn
P2	9mn	12mn	16mn

We assume that the machines have no idle time. The availability for each machine is:

- 165 hours (9900 minutes) for machine M1.
- 140 hours (8400 minutes) for machine M2.
- 160 hours (9600 minutes) for machine M3.

Product P1 gives a unit profit of 900 dinars and product P2 a unit profit of 1000 dinars.

In these conditions, how many products P1 and P2 must be produced monthly to have a maximum total profit?

Illustrative Example 3: Production problem (2/3)

Step 1: The decision variables are:

- x_1 : the number of units of product P1 to be manufactured.
- x_2 : the number of units of product P2 to be manufactured.

Step 2: The constraints besides the non-negativity constraints are:

- $11x_1 + 9x_2 \leq 9900$: for machine M1.
- $7x_1 + 12x_2 \leq 8400$: for machine M2.
- $6x_1 + 16x_2 \leq 9600$: for machine M3.

Step 3: The profit to be maximized is : $\text{Max } z = 900x_1 + 1000x_2$

Illustrative Example 3: Production problem (3/3)

The linear program that models the production problem is:

$$\text{Max } z = 900x_1 + 1000x_2$$

$$\left\{ \begin{array}{l} 11x_1 + 9x_2 \leq 9900 \\ 9x_1 + 12x_2 \leq 8400 \\ 6x_1 + 16x_2 \leq 9600 \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right.$$

Forms of a Linear Program (1/8)

There are three formulations of the linear program with the condition of non-negativity (or positivity or realizability) of the set of variables $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$

Standard Form	Canonical Form	Mixed Form
$\begin{aligned} \text{Max } z = & C.X \\ \begin{cases} A.X = 0 \\ X \geq 0 \end{cases} \end{aligned}$	$\begin{aligned} \text{Max } z = & C.X \\ \begin{cases} A.X \leq b \\ X \geq 0 \end{cases} \end{aligned}$	$\begin{aligned} \text{Max } z = & C.X \\ \begin{cases} \alpha_i.X \leq b_i \quad i \in M_1 \\ \alpha_i.X = b_i \quad i \in M - M_1 \\ X \geq 0 \end{cases} \end{aligned}$

Forms of a Linear Program (2/8)

Property: The mixed form can be reduced to the standard form or to the canonical form. Also, we can pass from the standard form to the canonical form and vice versa by elementary operations.

Operation 1: $\text{Min } f(x) = -\text{Max}(-f(x))$.

Operation 2: we can replace each variable by a difference of positive variables $x_j = x_j' - x_j'' \geq 0$ where $x_j' \geq 0$ and $x_j'' \geq 0$.

Forms of a Linear Program (3/8)

a) If a variable has no sign constraint, we replace it by two positive variables x_j' and x_j'' such that $x_j = x_j' - x_j''$.

For example:

$$\begin{array}{ll} \text{Max } z = 3x_1 - 2x_2 + 8x_3 & \\ \text{s.t. } \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3 \leq 8 \\ x_1 + 3x_2 + 8x_3 \leq 25 \\ 9x_1 + 6x_2 - 3x_3 \leq 17 \\ x_1 \geq 0, x_2 \geq 0, x_3 \in \mathbb{R} \end{array} \right. & \Rightarrow \end{array}$$
$$\begin{array}{ll} \text{Max } z = 3x_1 - 2x_2 + 8x_3' - 8x_3'' & \\ \text{s.t. } \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3' - 4x_3'' \leq 8 \\ x_1 + 3x_2 + 8x_3' - 8x_3'' \leq 25 \\ 9x_1 + 6x_2 - 3x_3' + 3x_3'' \leq 17 \\ x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0 \end{array} \right. & \end{array}$$

Forms of a Linear Program (4/8)

b) If a variable x_j is negative, we replace it with a positive variable $x_j' = -x_j$.

For example:

$$\begin{array}{ll} \text{Max } z = & 3x_1 - 2x_2 + 8x_3 \\ \text{S.C} & \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3 \leq 8 \\ x_1 + 3x_2 + 8x_3 \leq 25 \\ 9x_1 + 6x_2 - 3x_3 \leq 17 \\ x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \end{array} \right. \end{array}$$

\Rightarrow

$$\begin{array}{ll} \text{Max } z = & 3x_1 - 2x_2 - 8x_3' \\ \text{S.C} & \left\{ \begin{array}{l} 5x_1 - 2x_2 - 4x_3' \leq 8 \\ x_1 + 3x_2 - 8x_3' \leq 25 \\ 9x_1 + 6x_2 + 3x_3' \leq 17 \\ x_1 \geq 0, x_2 \geq 0, x_3' \geq 0 \end{array} \right. \end{array}$$

Forms of a Linear Program (5/8)

Operation 3: each equation $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = d_i$ can be replaced by the inequations:

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq d_i \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq d_i \end{cases}$$

or by the following equivalent inequations:

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq d_i \\ -a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -d_i \end{cases}$$

Forms of a Linear Program (6/8)

If the linear program has an equality constraint, we replace it with two equivalent constraints, one of less than, the other of greater than. The program variables must satisfy these two constraints, which then amounts to the initial equality.

For example:

$$\begin{array}{lcl} \text{Max } z = 3x_1 - 2x_2 + 8x_3 & & \text{Max } z = 3x_1 - 2x_2 + 8x_3 \\ \text{S.C. } \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3 \leq 8 \\ x_1 + 3x_2 + 8x_3 \leq 25 \\ 9x_1 + 6x_2 - 3x_3 = 17 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \right. & \Rightarrow & \text{S.C. } \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3 \leq 8 \\ x_1 + 3x_2 + 8x_3 \leq 25 \\ 9x_1 + 6x_2 - 3x_3 \leq 17 \\ 9x_1 + 6x_2 - 3x_3 \geq 17 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \right. \\ & & \Rightarrow \text{S.C. } \left\{ \begin{array}{l} 5x_1 - 2x_2 + 4x_3 \leq 8 \\ x_1 + 3x_2 + 8x_3 \leq 25 \\ 9x_1 + 6x_2 - 3x_3 \leq 17 \\ -9x_1 - 6x_2 + 3x_3 \leq -17 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \right. \end{array}$$

Forms of a Linear Program (7/8)

Operation 4: Any inequality $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq d_i$ (**or** $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq d_i$) can be replaced by the equations:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + E_i = d_i \text{ avec } x_{n+i} \geq 0$$

$$(\text{or } a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - x_{n+i} = d_i \text{ avec } x_{n+i} \geq 0)$$

x_{n+i} is called a **deviation variable** that can be introduced into the problem and which is assigned a **zero coefficient** in the function to be optimized. The x_i are called “**structural variables**”.

Forms of a Linear Program (8/8)

- From the **mixed** and **standard** form to the **canonical form**, we proceed with **operation 3**.
- From the **mixed** and **canonical** form to the **standard form**, we proceed with **operation 4**.