Analysis 4 Summary

Ayad Mohamed

2024–2025

I. Power Series

Definitions

- Let $x_0 \in \mathbb{R}$, and $(a_n)_{n\in\mathbb{N}}$ be a real sequence. The expression

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is called a **power series centered at** x_0 with coefficients a_n.

- The **sum** of the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is the function $f:\mathbb{R}\to\mathbb{R}$ defined by:

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

for values of x for which the series converges.

Properties

Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

be two power series centered at x_0, and let $f(x)$ and $g(x)$ be their respective sums. Then for all $\lambda,\mu\in\mathbb{R}$:

- **Linearity:**

$$\lambda\sum_{n=0}^{\infty} a_n(x-x_0)^n+\mu\sum_{n=0}^{\infty} b_n(x-x_0)^n=\sum_{n=0}^{\infty}(\lambda a_n+\mu b_n)(x-x_0)^n$$

and hence,

$$\lambda f(x)+\mu g(x)=\sum_{n=0}^{\infty}(\lambda a_n+\mu b_n)(x-x_0)^n.$$

- **Cauchy Product (Convolution):** If both series converge absolutely at x, then their product is given by:

$$\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-x_0)^n.$$

- **Derivative and primitive:** If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, then:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n,$$

and a primitive is given by:

$$\int_{x_0}^{x} f(t)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-x_0)^n.$$

If F is any primitive of f (so $F' = f$), then:

$$F(x) - F(x_0) = \int_{x_0}^{x} f(t)\mathrm{d}t = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-x_0)^n,$$

which implies

$$F(x) = F(x_0) + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-x_0)^n.$$

- **Domain and radius of convergence:** The domain of f is

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : \text{the series converges at } x\}.$$

The radius of convergence R is given by:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

or if the limit exists,

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

The series converges absolutely for $|x-x_0| < R$ (i.e., on (x_0-R, x_0+R)) and diverges for $|x-x_0| > R$ (i.e., on $(-\infty, x_0-R) \cup (x_0+R, \infty)$). At the points $x = x_0 \pm R$, convergence must be checked separately.

Example

The geometric series:

$$\begin{cases} x_0 = 0 \text{ and } a_n = 1, \forall n \in \mathbb{N} \\ R = 1, f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \mathrm{Dom}(f) = (-1,1) \end{cases}$$

$$\implies f'(x) = \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\int_0^x \frac{1}{1-t}\mathrm{d}t = \sum_{n=0}^{\infty}\int_0^x t^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\implies -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Cauchy product:

$$[f(x)]^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\cdot 1\right)x^n = \sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$$

*You can check this result by differentiating the geometric series.

Definition

$$\text{If } f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \text{ then: } a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$\text{So, } \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \text{ is the Taylor series expansion of } f.$$

$$\text{In particular where } x_0 = 0 : \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \text{ is the Maclaurin series.}$$

Example:

$$f(x) = \frac{e^x}{1-x} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\right) x^n$$

II. Fourier Series

Trigonometric Series:

- Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}^*}$ be two real sequences and $T > 0$. The trigonometric series of period T with coefficients $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 1}$ is:

$$\sum_{n=0}^{\infty}\left(a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right)\right)$$

- The **sum** of this series is a T-periodic function.
- The **Fourier series** of a T-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ $(T > 0)$ is defined by coefficients:

$$a_0 = \frac{1}{T}\int_0^T f(x)\mathrm{d}x, \quad b_0 = 0$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2n\pi}{T}x\right)\mathrm{d}x, \quad \forall n \in \mathbb{N}^*$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2n\pi}{T}x\right)\mathrm{d}x, \quad \forall n \in \mathbb{N}^*$$

The series $\sum_{n=0}^{\infty}\left(a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right)\right)$ is called the **Fourier series** of f.

In particular:

  - If f is even:

$$b_n = 0, \quad a_0 = \frac{2}{T}\int_0^{T/2} f(x)\mathrm{d}x, \quad a_n = \frac{4}{T}\int_0^{T/2} f(x)\cos\left(\frac{2n\pi}{T}x\right)\mathrm{d}x \quad (n \geq 1)$$

  - If f is odd:

$$a_n = 0 \quad (\forall n \geq 0), \quad b_n = \frac{4}{T}\int_0^{T/2} f(x)\sin\left(\frac{2n\pi}{T}x\right)\mathrm{d}x \quad (n \geq 1)$$

Example:

$$T = 2\pi, \quad f(x) = x^3 \quad \text{on} \quad (-\pi, \pi)$$
$$f \text{ is odd on } (-\pi, \pi), \quad \text{so}$$
$$a_n = 0 \quad \forall n \in \mathbb{N}, \quad b_n = \frac{(2n^2\pi^2 - 12)(-1)^{n+1}}{n^3}$$

Dirichlet's Theorem:

Let f be periodic with period T and piecewise smooth. For any $x_0 \in \mathbb{R}$, the Fourier series converges to:

$$\frac{f(x_0^+) + f(x_0^-)}{2}$$

where the one-sided limits are defined as:

- Left limit: $f(x_0^-) = \lim_{t \to 0^-} f(x_0 + t)$
- Right limit: $f(x_0^+) = \lim_{t \to 0^+} f(x_0 + t)$

$$\sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{T} x_0\right) + b_n \sin\left(\frac{2n\pi}{T} x_0\right) \right] = \frac{f(x_0^+) + f(x_0^-)}{2}$$

In particular, if f is differentiable at x_0 then:

$$\sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{T} x_0\right) + b_n \sin\left(\frac{2n\pi}{T} x_0\right) \right] = f(x_0)$$

Parseval's Identity

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{2}{T} \int_0^T f^2(x) \mathrm{d}x$$

Example:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x < 0, \\ 0 & \text{if } x \in \{-\pi, 0, \pi\}. \end{cases}$$

The Fourier series:

$$\sum_{n=1}^{\infty} \frac{4}{\pi(2n+1)} \sin((2n+1)x) \quad \text{(simplified example)}$$

III. Fourier Transform

$$f \colon \mathbb{R} \to \mathbb{C}, \quad |f(t)| = \sqrt{\Re^2(f(t)) + \Im^2(f(t))}$$
$$f \colon t \mapsto f(t)$$

$$\mathcal{L}^1(\mathbb{R}, \mathbb{C}) = \left\{ f \colon \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t < \infty \right\}$$

$$\mathcal{L}^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R}, \mathbb{R}) = \left\{ f \colon \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t < \infty \right\}$$

For all $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$:

$$\mathcal{F}\{f\} \colon \mathbb{R} \to \mathbb{C},$$
$$\mathcal{F}\{f\}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-ixt} \mathrm{d}t$$

$\mathcal{F}\{f\}$ or $\hat{f}$ is called the **Fourier transform** of f.

Properties:

i) If $f \in \mathcal{L}^1(\mathbb{R})$, $f \colon \mathbb{R} \to \mathbb{R}$:

$$\mathcal{F}\{f\}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cos(xt) \mathrm{d}t - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \sin(xt) \mathrm{d}t$$

In particular:

- If f is even: $\mathcal{F}\{f\}(x) \in \mathbb{R}$ and even,

$$\mathcal{F}\{f\}(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \cos(xt) \mathrm{d}t.$$

- If f is odd: $\mathcal{F}\{f\}(x)$ is purely imaginary and odd,

$$\mathcal{F}\{f\}(x) = -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \sin(xt) \mathrm{d}t.$$

For real-valued f:

$$\mathcal{F}\{f\}(-x) = \overline{\mathcal{F}\{f\}(x)}.$$

In general:

$$\mathcal{F}\{f\}(x) = \mathcal{F}\{\Re(f)\}(x) + i\mathcal{F}\{\Im(f)\}(x).$$

ii) $\mathcal{F}\{\lambda f + \mu g\} = \lambda \mathcal{F}\{f\} + \mu \mathcal{F}\{g\}$.

iii) $\mathcal{F}\{f * g\} = \sqrt{2\pi}\, \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$, where the convolution is:

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(s) g(t-s) \mathrm{d}s.$$

In particular, if $f = g$:

$$\mathcal{F}\{f * f\} = \sqrt{2\pi}\, [\mathcal{F}\{f\}]^2.$$

Example 1:

$$f(t) = e^{-\alpha|t|}, \quad \alpha > 0 \implies \mathcal{F}\{f\}(x) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{x^2 + \alpha^2}.$$

Example 2:

$$f(t) = e^{-t^2/2} \implies \mathcal{F}\{f\}(x) = e^{-x^2/2}.$$

iv) **Derivative:**

$$\mathcal{F}\{f^{(n)}\}(x) = (ix)^n \mathcal{F}\{f\}(x)$$
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathcal{F}\{f\}(x) = (-i)^n \mathcal{F}\{t^n f(t)\}(x).$$

Fourier Inversion Formula

If $f \in \mathcal{L}^1(\mathbb{R})$ and $\mathcal{F}\{f\} \in \mathcal{L}^1(\mathbb{R})$, then:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(x) e^{ixt} \mathrm{d}x = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(t).$$

Equivalently:

$$\mathcal{F}^2\{f\}(t) = f(-t).$$

If f is even, $\mathcal{F}^2\{f\} = f$.

Definition: The inverse Fourier transform is:

$$\mathcal{F}^{-1}\{f\}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{ixt} \mathrm{d}x.$$

Plancherel Identity

If $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |\mathcal{F}\{f\}(x)|^2 \mathrm{d}x.$$

Example:

$$f(t) = e^{-\alpha|t|} \implies \int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \frac{1}{\alpha}, \quad \int_{-\infty}^{\infty} |\mathcal{F}\{f\}(x)|^2 \mathrm{d}x = \frac{1}{\alpha}.$$

IV. Partial Differential Equations

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (*)$$

where $A, B, C, D, E, F \in \mathbb{R}$ and $G\colon \mathbb{R}^2 \to \mathbb{R}$. If $G = 0$, $(*)$ is homogeneous.

- If $A = B = C = 0$, $(*)$ is first-order.
- The discriminant is $\Delta = B^2 - 4AC$:
  - $\Delta > 0$: hyperbolic,
  - $\Delta = 0$: parabolic,
  - $\Delta < 0$: elliptic.

Examples:

- Heat equation: $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ (parabolic, $\Delta = 0$)
- Wave equation: $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ (hyperbolic, $\Delta > 0$)
- Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (elliptic, $\Delta < 0$)

Method 1: Separation of Variables

Assume $u(t,x) = F(x)\Phi(t)$ or $u(x,y) = X(x)Y(y)$.

$$\textbf{1. Heat Equation:} \begin{cases} k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \\ u(t,0) = u(t,L) = 0 \\ u(0,x) = f(x) \end{cases} \implies \begin{cases} F'' + \lambda F = 0, \quad F(0) = F(L) = 0 \\ \Phi' + \lambda k \Phi = 0 \end{cases}$$

Solution:

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi}{L}x\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi}{L}s\right)\mathrm{d}s.$$

$$\textbf{2. Wave Equation:} \begin{cases} a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \\ u(t,0) = u(t,L) = 0 \\ u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x) \end{cases} \implies \begin{cases} F'' + \lambda F = 0, \quad F(0) = F(L) = 0 \\ \Phi'' + a^2\lambda \Phi = 0 \end{cases}$$

Solution:

$$u(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(A_n\cos\left(\frac{an\pi}{L}t\right) + B_n\sin\left(\frac{an\pi}{L}t\right)\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \quad B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x.$$

$$\textbf{3. Laplace Equation:} \begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) = 0 \\ u(x,0) = 0, \quad u(x,b) = f(x) \end{cases}$$

Solution:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}y\right)\cos\left(\frac{n\pi}{a}x\right),$$

where

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)\mathrm{d}x.$$

Method 2: Fourier Transform

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t,x)e^{-i\xi x}\mathrm{d}x$$

$$\textbf{Heat Equation:} \begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ u(0,x) = f(x), \quad f \in \mathcal{L}^1(\mathbb{R}) \end{cases} \quad \text{Solution for } t > 0:$$

$$u(t,x) = \int_{-\infty}^{\infty} f(s)\cdot\frac{1}{\sqrt{4\pi kt}}e^{-\frac{(x-s)^2}{4kt}}\mathrm{d}s.$$

$$\textbf{Wave Equation:} \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \\ u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x) \end{cases} \quad \text{Solution (d'Alembert):}$$

$$u(t,x) = \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\mathrm{d}s.$$

Laplace Equation (Half-Plane): $\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & y > 0 \\ u(x,0) = f(x) \end{cases}$ Solution:

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (x-s)^2} \mathrm{d}s.$$