Serie N°3 Numerical Series

Ex 1:

A)

① $U_n = \dfrac{\cos \alpha n}{2^n}$, $\alpha \in \mathbb{R}$

$= Re\left[\left(\frac{1}{2} e^{i\alpha}\right)^n\right]$

$V_n = \sum_{k=0}^{n} U_k = \sum_{k=0}^{n} Re\,(q^k)$ / $q = \frac{1}{2} e^{i\alpha} \neq 1$ because $|q| = \frac{1}{2}$

$= Re\left(\sum_{k=0}^{n} q^k\right) = Re\left(\dfrac{q^{n+1}-1}{q-1}\right)$

$= Re\left(\dfrac{1-q^{n+1}}{1-q}\right)$

$\lim V_n = Re\left(\dfrac{1}{1-q}\right)$ because $|q^{n+1}| = \dfrac{1}{2^{n+1}}$

$$\sum_{n=0}^{+\infty} \frac{\cos n\alpha}{2^n} = Re\left(\frac{1}{1-\frac{1}{2}\cos\alpha - i\frac{1}{2}\sin\alpha}\right)$$

$$= \frac{1-\frac{1}{2}\cos\alpha}{(1-\frac{1}{2}\cos\alpha)^2+(\frac{1}{2}\sin\alpha)^2}$$

$$\sum_{n=0}^{+\infty} \frac{\cos n\alpha}{2^n} = \frac{1-\frac{1}{2}\cos\alpha}{1+\frac{1}{4}-\cos\alpha}$$

$$= \frac{\frac{1}{2}(2-\cos\alpha)}{\frac{5}{4}-\cos\alpha}$$

$$\sum_{n=0}^{+\infty} \frac{\cos n\alpha}{2^n} = \frac{4-2\cos\alpha}{5-4\cos\alpha}$$

$Re\left(\dfrac{1}{z}\right) = Re\left(\dfrac{\bar{z}}{|z|^2}\right)$

$\dfrac{1}{|z|^2} Re(\bar{z})$

$= \dfrac{Re(\bar{z})}{|z|^2}$

$u_n = (1+n)\, q^n$

(2) $u_n = \frac{n+1}{3^n}$

$$\sum_{n=0}^{+\infty} u_n = \left(\frac{1}{1-q}\right)^2 = \frac{1}{(1-q)^2}$$

$q = \frac{1}{3}$ $\quad \sum_{0}^{+\infty} \frac{n+1}{3^n} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}$

$$\frac{1}{(1-q)^2} = \left(\sum_{n=0}^{+\infty} q^n\right)^2 = \sum_{n=0}^{+\infty} q^n \times q^n$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} q^k . q^{n-k}\right]$$

$$= \sum_{n=0}^{+\infty} q^n (\underbrace{1 + \dots + 1}_{n+1}) = \sum_{n=0}^{+\infty} (n+1) q^n$$

(3) $u_n = \frac{n-(-1)^n}{3^n} = \frac{n+1}{3^n} - \frac{1+(-1)^n}{3^n}$

$$\sum_{n=0}^{+\infty} u_n = \sum_{n=0}^{+\infty} \frac{n+1}{3^n} - \sum_{0}^{+\infty} \frac{1+(-1)^n}{3^n}$$

$$= \frac{9}{4} - \sum_{n=0}^{+\infty} \frac{2}{3^{2n}} = \frac{9}{4} - 2\sum_{n=0} \frac{1}{9^n}$$

$$= \frac{9}{4} - 2\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{9}{4} - 2\times\frac{9}{8} = 0$$

(4) $u_n = \frac{n}{n^4+n^2+1} = \frac{n}{(n^4+2n^2+1)-n^2}$

$$= \frac{n}{(n^2+1)^2 - n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)}$$

$$= \frac{1}{2} \times \frac{(n^2+n+1)-(n^2-n+1)}{(n^2-n+1)(n^2+n+1)}$$

$$= \frac{1}{2}\left[\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right]$$

$$= a_n - a_{n+1} \,/\, a_n = \frac{1}{2} \cdot \frac{1}{n^2-n+1} = \frac{1}{2(1+n(n-1))}$$

$$a_{n+1} = \frac{1}{2(1+n(n+1))}$$

$U_n = (a_0 - a_1) + (a_1 - a_2) + \dots + (a_{n-1} - a_n) + (a_n - a_{n+1})$

$$U_n = a_0 - a_{n+1} = \frac{1}{2} - \frac{1}{2(1+n+n^2)}$$

$$\boxed{\lim U_n = \frac{1}{2} = \sum_{n=0}^{+\infty} \frac{n}{n^4+n^2+1}}$$

(5) $\sum_{1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$

$$U_n = \frac{\sum_1^n k}{(\sum_1^n k)^2} = \frac{1}{\sum_1^n k}$$

$$= \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= a_n - a_{n+1}$$

$$\sum_{n=1}^{+\infty} U_n = a_1 - \lim a_{n+1} = 2 - 0 = 2$$

(6) $U_n = \ln(1-n^{-2}) = \ln\left(1 - \frac{1}{n^2}\right)$

$$= \ln\left(1-\frac{1}{n}\right) + \ln\left(1+\frac{1}{n}\right)$$

$$= \ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left(\frac{n-1}{n}\right) - \ln\left(\frac{n}{n+1}\right) = a_n - a_{n+1}$$

$$U_n = a_{n_0} - a_{n+1} = a_2 - \ln\left(\frac{n}{n+1}\right)$$

$$a_2 = \ln\frac{1}{2} = -\ln 2$$

II) $U_n = \ln n + a\ln(n+2) + b\ln(n+3)$

$\sum U_n$ Cv $\Longrightarrow \lim U_n = 0$

$$U_n = \ln n + a\left[\ln n + \ln\left(1+\frac{2}{n}\right)\right] + b\left[\ln n + \ln\left(1+\frac{3}{n}\right)\right]$$

$$= (1+a+b)\ln n + a\ln\left(1+\frac{2}{n}\right) + b\ln\left(1+\frac{3}{n}\right)$$

$1+a+b = 0$ $\quad$ $1+a+b \neq 0$ $\quad \infty \quad 0$

$u_n = (1+n)q^n$

② $u_n = \frac{n+1}{3^n}$

$\sum_{n=0}^{+\infty} u_n = \left(\frac{1}{1-q}\right)^2 = \frac{1}{(1-q)^2}$

$q = \frac{1}{3}$ $\quad \sum_{0}^{+\infty} \frac{n+1}{3^n} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}$

$\frac{1}{(1-q)^2} = \left(\sum_{n=0}^{+\infty} q^n\right)^2 = \sum_{n=0}^{+\infty} q^n \times q^n$

$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} q^k \cdot q^{n-k}\right]$

$= \sum_{n=0}^{+\infty} q^n (\underbrace{1 + \dots + 1}_{n+1}) = \sum_{n=0}^{+\infty} (n+1) q^n$

③ $u_n = \frac{n-(-1)^n}{3^n} = \frac{n+1}{3^n} - \frac{1+(-1)^n}{3^n}$

$\sum_{n=0}^{+\infty} u_n = \sum_{n=0}^{+\infty} \frac{n+1}{3^n} - \sum_{0}^{+\infty} \frac{1+(-1)^n}{3^n}$

$= \frac{9}{4} - \sum_{n=0}^{+\infty} \frac{2}{3^{2n}} = \frac{9}{4} - 2\sum_{n=0}^{+\infty} \frac{1}{9^n}$

$= \frac{9}{4} - 2\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{9}{4} - 2 \times \frac{9}{8} = 0$

④ $u_n = \frac{n}{n^4+n^2+1} = \frac{n}{(n^4+2n^2+1)-n^2}$

$= \frac{n}{(n^2+1)^2 - n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)}$

$= \frac{1}{2} \times \frac{(n^2+n+1)-(n^2-n+1)}{(n^2-n+1)(n^2+n+1)}$

$= \frac{1}{2}\left[\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right]$

$= a_n - a_{n+1} \,/\, a_n = \frac{1}{2} \cdot \frac{1}{n^2-n+1} = \frac{1}{2(1+n(n-1))}$

$a_{n+1} = \frac{1}{2(1+n(n+1))}$

$U_n = (a_0 - a_1) + (a_1 - a_2) + \dots + (a_{n-1} - a_n) + (a_n - a_{n+1})$

$U_n = a_0 - a_{n+1} = \frac{1}{2} - \frac{1}{2(1+n+n^2)}$

$\boxed{\lim U_n = \frac{1}{2} = \sum_{n=0}^{+\infty} \frac{n}{n^4+n^2+1}}$

⑤ $\sum_{1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$

$u_n = \frac{\sum_1^n k^3}{(\sum_1^n k)^2} = \frac{1}{\sum_1^n k}$

$= \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= a_n - a_{n+1}$

$\sum_{n=1}^{+\infty} u_n = a_1 - \lim a_{n+1} = 2 - 0 = 2$

⑥ $u_n = \ln(1-n^{-2}) = \ln\left(1 - \frac{1}{n^2}\right)$

$= \ln\left(1-\frac{1}{n}\right) + \ln\left(1+\frac{1}{n}\right)$

$= \ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)$

$= \ln\left(\frac{n-1}{n}\right) - \ln\left(\frac{n}{n+1}\right) = a_n - a_{n+1}$

$U_n = a_2 - a_{n+1} = a_2 - \ln\left(\frac{n}{n+1}\right)$

$a_2 = \ln \frac{1}{2} = -\ln 2$

II) $u_n = \ln n + a\ln(n+2) + b\ln(n+3)$

$\sum u_n \text{ Cv} \Longrightarrow \lim u_n = 0$

$u_n = \ln n + a\left[\ln n + \ln\left(1+\frac{2}{n}\right)\right] + b\left[\ln n + \ln\left(1+\frac{3}{n}\right)\right]$

$= (1+a+b)\ln n + a\ln\left(1+\frac{2}{n}\right) + b\ln\left(1+\frac{3}{n}\right)$

$1+a+b=0$ $\quad$ $1+a+b \neq 0$ $\quad \infty$ $\quad 0$

$\sum u_n$ Cv $\Rightarrow a+b+1=0$

$\Leftrightarrow b=-1-a$

so when $b=-1-a$

$u_n = \ln n + a\ln(n+2) - (1+a)\ln(n+3)$

$= \ln n + a\ln(n+2) - \ln(n+3) - a\ln(n+3)$

$= \ln n - \ln(n+3) + a[\ln(n+2) - \ln(n+3)]$

$= \ln n - \ln(n+3) + \ln(n+1) - \ln(n+2) + \ln(n+2)$

$- \ln(n+3) + a\ln(n+2) - a\ln(n+3)$

$n_0 = 1$

$= [\ln n + \ln(n+1) + (a+1)\ln(n+2)]$

$- [\ln(n+1) + \ln(n+2) + (a+1)\ln(n+3)]$

$= a_n - a_{n+1} = a_1 - a_{n+1};\ \forall n \geq 1$

$u_n = a_{n_0} - a_{n+1} = a_1 - a_{n+1}$

$\sum u_n$ Cv $\Leftrightarrow (a_{n+1})_n$ Cv

$a_{n+1} = \ln(n+1) + \ln(n+2) + (a+1)\ln(n+3)$

$= (a+3)\ln n + \underbrace{\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + (a+1)\ln\left(1+\frac{3}{n}\right)\right]}_{0}$

$\sum u_n$ Cv $\Rightarrow a+3=0 \Rightarrow a=-3$

$b=2$

$$\sum_{n=1}^{+\infty} u_n = \ln 2 - 2\ln 3 = \ln\frac{2}{9}$$

Ex2:

I) ① $u_n = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$

$= \frac{1}{1+\sqrt{\frac{n^2+n}{n^2}}} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$

$\lim u_n = \frac{1}{2} \neq 0 \Rightarrow \sum u_n$ div,

② $u_n = \frac{\ln n}{n^2}$ (The Bertrand series)

$\alpha = 2 \quad \beta = -1$

$u_n = \frac{1}{n\sqrt{n}} \times \underbrace{\frac{\ln n}{\sqrt{n}}}_{\to 0} = o\left(\frac{1}{n\sqrt{n}}\right)$ $\alpha > 1$

$\sum_{n\in\mathbb{N}^*} \frac{1}{n\sqrt{n}} = \sum_{n\in\mathbb{N}^*} \frac{1}{n^{3/2}}$ (the Riemann series) $\alpha = \frac{3}{2} > 1$

Cv

$\sum u_n$ cv

③ $u_n = \frac{1+\ln n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\frac{1}{2} < 1 \Rightarrow \sum \frac{1}{\sqrt{n}}$ div $\Rightarrow \sum u_n$ di

(comparaison criterion)

④ $u_n = \frac{n+\sqrt{n}}{2n^3-1} = \frac{n\left(1+\frac{1}{\sqrt{n}}\right)}{2n^3\left(1-\frac{1}{2n^3}\right)}$

$= \frac{1}{2n^2} \times \frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{2n^3}} \sim \frac{1}{2n^2}$

$\sum \frac{1}{2n^2}$ cv $\Rightarrow \sum u_n$ Cv

(equivalence criterion

⑤ $u_n = (\sqrt[n]{2} + \sqrt[n]{3})^{-n^2}$

$\rightarrow \sqrt[n]{u_n} = (\sqrt[n]{3} + \sqrt[n]{2})^{-n}$

$= \frac{1}{(\sqrt[n]{3} + \sqrt[n]{2})^n} = \frac{1}{\left[\sqrt[n]{3}\left(1+\sqrt[n]{\frac{2}{3}}\right)\right]^n}$

$= \frac{1}{3\left(1+\sqrt[n]{\frac{2}{3}}\right)^n}$ $0<1$ by Cauchy's criter

$\sum u_n$ Cv

⑥ $u_n = \frac{(n+1)(n+2)\cdots(2n)}{(2n)^n}$

$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+2)(n+3)\cdots(2n+2)}{(2(n+1))^n}}{(n+1)(n+2)\cdots(2n)}$

$$= \frac{\left[(\cancel{n+2})(\cancel{n+3})\cdots(\cancel{2n})\,(2n+1)\,(\cancel{2n+2})\right]\,(2n)^n}{(n+1)(\cancel{n+2})\cdots(\cancel{2n})\;\;(2n+2)^n\,(\cancel{2n+2})}$$

$$= \frac{2n+1}{n+1} \times \frac{(2n)^n}{(2n+2)^n} = \frac{2n+1}{n+1} \times \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\longrightarrow \frac{2}{e} < 1$$

So $\sum u_n$ cv