

Vector space structure

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Definition 1. A vector space V over a field F (or F -vector space) is a set V equipped with a binary operation $(+)$ and the operation multiplication (\cdot) by scalars, i.e. a map

$$\cdot : F \times V \mapsto V,$$

such that :

- (1) $(V, +)$ is a commutative group.
- (2) $\forall (\alpha, \beta) \in F^2, \forall x \in V, (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x.$
- (3) $\forall \alpha \in F, \forall (x, y) \in V^2, \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y.$
- (4) $\forall (\alpha, \beta) \in F^2, \forall x \in V, \alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x.$
- (5) $\forall x \in V, 1 \cdot x = x.$

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Remark

The elements of V are called vectors and the elements of F are called **scalars**.

Examples

(1) Let $F = \mathbb{R}$ and the set $V = \mathbb{R}^2$ equipped by the following binary operation

$$\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and

$$\forall \alpha \in \mathbb{R}, \forall (x_1, x_2) \in \mathbb{R}^2, \alpha (x_1, x_2) = (\alpha x_1, \alpha x_2).$$

Then \mathbb{R}^2 is an \mathbb{R} -vector space.

(2) More generally, let $F = \mathbb{R}$ and let n be a positive integer. The set \mathbb{R}^n of n -tuples of elements of \mathbb{R} is a vector space over \mathbb{R} with

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

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and

$$\alpha (x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n),$$

for all elements (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) of \mathbb{R}^n and for all elements α of F . In particular, \mathbb{R} is a vector space over itself.

(3) Let $F[x]$ be the set of polynomials over F with an indeterminate x . Let $P(x) = a_0 + a_1x + \dots + a_nx^n$ and $Q(x) = b_0 + b_1x + \dots + b_mx^m$ be polynomials in $F[x]$, and let $\alpha \in F$. We define the following operations on $F[x]$:

$$P(x) + Q(x) = \sum_{i=0}^r (a_i + b_i)x^i, \quad \text{where } r = \max(m, n),$$

$$\alpha P(x) = (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n.$$

Equipped with these two operations, $F[x]$ is an F -vector space. In particular, $F_n[x]$, the set of polynomials of degree $\leq n$ over F , equipped with the same operations, is also an F -vector space.

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Denote by $F(E, V)$ the set of functions from E to V . Let $f, g \in F(E, V)$ and $\alpha \in F$. We define the sum of f and g by

$$(f + g)(x) = f(x) + g(x), \forall x \in E$$

and the multiplication by a scalar by

$$(\alpha f)(x) = \alpha f(x), \forall x \in E.$$

Endowed with these two operations, $F(E, V)$ is an F -vector space. In particular, with these operations, the set $F(\mathbb{N}, \mathbb{R})$ of real sequences is an \mathbb{R} -vector space structure.

Properties

Let V be an F -vector space. Then we have: $\forall x \in V, \forall \alpha \in F$,

$$1- 0_F \cdot x = 0_V$$

$$2- \alpha \cdot 0_V = 0_V$$

$$3- \alpha \cdot x = 0_V \Leftrightarrow \alpha = 0_F \text{ or } x = 0_V$$

$$4- (-\alpha) \cdot x = \alpha \cdot (-x) = -(\alpha \cdot x).$$

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1.1. Subspaces.

Definition 2. A subset U of a vector space V over a field F is a **subspace** of V if U itself is a vector space over F under the operations of vector addition and scalar multiplication defined over V .

Proposition 1. Let V be an F -vector space and U be a subset of V . We say that U is a subspace of V if

$$(1) 0_V \in U.$$

$$(2) \forall (x, y) \in U^2, x + y \in U.$$

$$(3) \forall x \in U, \forall \lambda \in F, \lambda \cdot x \in U.$$

Examples

(1) V and $\{0\}$ are subspaces of the vector space V .

(2) The set $U = \{(x, y) \in \mathbb{R}^2, 3x + 5y = 0\}$ is a subspace of \mathbb{R}^2 .

(3) Let the set $F_n[x]$ of polynomials over F of degree $\leq n$ is a subspace of $F[x]$.

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Proof: To verify that U is a subspace of \mathbb{R}^2 , we check the subspace conditions:

- The zero vector $(0, 0)$ belongs to U since $3(0) + 5(0) = 0$.
- Closure under addition: Suppose $(x_1, y_1), (x_2, y_2) \in U$. Then,

$$3x_1 + 5y_1 = 0 \quad \text{and} \quad 3x_2 + 5y_2 = 0.$$

Adding these equations,

$$3(x_1 + x_2) + 5(y_1 + y_2) = (3x_1 + 5y_1) + (3x_2 + 5y_2) = 0 + 0 = 0.$$

Thus, $(x_1 + x_2, y_1 + y_2) \in U$.

- Closure under scalar multiplication: If $(x, y) \in U$ and c is a scalar, then

$$3(cx) + 5(cy) = c(3x + 5y) = c(0) = 0.$$

Thus, $(cx, cy) \in U$.

Since all subspace conditions are satisfied, U is a subspace of \mathbb{R}^2 .

Intersection and union of vector subspaces

Proposition 2. *The intersection of two subspaces of an F -vector space V is a subspace of V .*

Remark

The union of two subspaces of an F -vector space V is not, in general, a subspace of V .

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Sum of subspaces

Definition 3. Let U and W be two subspaces of an F -vector space V . The set

$$U + W = \{u + w : u \in U, w \in W\}$$

is called *sum of U and W* .

Proposition 3. Let U and W be two subspaces of an F -vector space V , then $U + W$ is a subspace of V .

Linear combination of vectors

1.2. Linear combinations.

Definition 4. Let V be an F -vector space and $G = \{v_1, v_2, \dots, v_n\}$ a collection of vectors in V . A vector $v \in V$ is called a **linear combination** of the vectors in G if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$

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Example 1.

- (1) In any F -vector space, the null vector is a linear combination of any collection of vectors, the coefficients are all zero.
- (2) In the F -vector space F^n any vector $u = (x_1, x_2, \dots, x_n)$ is a linear combination of the set of vectors $G = \{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$. In fact, according to the definition of operations in F^n we have

$$u = x_1 e_1 + x_2 e_2 + \dots + x_n e_n.$$

1.3. Spanning set.

Proposition 4. Let V be a vector space, and let G be a set of vectors in V . The set of all linear combinations of the vectors in G , denoted by $\langle G \rangle$ or $\text{span}(G)$, is a subspace of V . Moreover, $\langle G \rangle$ is the smallest subspace of V containing G ; that is, any subspace U of V that contains G also contains $\langle G \rangle$.

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Remark

- We define the span of the empty set to be the zero vector ($\text{span}(\emptyset) = \{0_V\}$).
- $\langle G \rangle$ is called the subspace spanned (or generated) by the set G .
- The elements of G are called generators.
- When G is a finite set of vectors, as $G = \{v_1, v_2, \dots, v_n\}$ then we write $\langle v_1, v_2, \dots, v_n \rangle$.

Example 2.

- (1) In \mathbb{R}^3 , $\text{span} \{(0, 1, 0), (0, 0, 1)\} = \{(0, y, z), y, z \in \mathbb{R}\}$.
- (2) In $\mathbb{R}_2[X]$, $\text{span}\{1, 1 + X\} = \mathbb{R}_1[X]$.

1.4. Linear independence-Linear dependence.

Definition 5. Let V be an F -vector space, and let $G = \{v_1, v_2, \dots, v_n\} \subset V$.

- (1) We say that G is **linearly independent** if for all $\lambda_1, \lambda_2, \dots, \lambda_n \in F$:

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \lambda_2 = \dots = \lambda_n = 0.$$

- (2) We say that G is **linearly dependent** if it is not linearly independent; that is, if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$, not all zero, such that:

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0.$$

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Remark

When G is an infinite set of vectors, we say that G is linearly independent if every finite subset G' of G is linearly independent. Otherwise, if some finite subset G' of G is linearly dependent, we say that G is linearly dependent.

Example 3.

- (1) In F^n , the set of vectors $B = \{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1)\}$ is linearly independent since $\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n = 0$ gives necessarily $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$.
- (2) In $\mathbb{C}[X]$ as \mathbb{C} -vector space $\{1, 1 + X, 1 + X + X^2\}$ is linearly independent since the equation $\lambda_1 + \lambda_2(1 + X) + \lambda_3(1 + X + X^2) = 0$ gives necessarily $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
- (3) Let F be a field. The set $\{1, X, \dots, X^n, \dots\}$ is an infinite linearly independent set of vectors of the space $F[X]$.
- (4) Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of functions from \mathbb{R} to \mathbb{R} , and consider the two functions defined by

$$f(x) = e^x, g(x) = e^{5x} \text{ for all } x \in \mathbb{R}.$$

Then f and g are linearly independent.

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Properties : Let V be a vector space over F .

- Any set in V containing the null vector is linearly dependent.
- Any set in V containing linearly dependent subset is itself linearly dependent.
- A subset G of V is linearly dependent if and only if one of its vectors is a linear combination of other vectors.
- Two vectors v_1 and v_2 of V are linearly dependent if one is a scalar multiple of the other, we say they are colinear.

2. BASES AND DIMENSION

Definition 6. An F -vector space V is said to be **finite dimensional** if it is generated by a finite number of its vectors. In other words, there exists a finite set G of vectors in V such that $V = \text{span}(G)$.

Theorem 1. Let V be an F -vector space spanned by n vectors. Then any set G of vectors in V of cardinality $\geq n + 1$ is linearly dependent.

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Definition 7. Let V be a vector space over F . A **basis** for V is any linearly independent subset B of V that generates V . We also say that the vectors of B form a basis for V .

Example 4.

- (1) In F^n , the set of vectors $B = \{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1)\}$ is a basis for F^n and is called the standard basis.
- (2) In $F_n[X]$, the set $\{1, X, X^2, \dots, X^n\}$ is a basis for $F_n[X]$. We call this basis the standard basis of $F_n[X]$.
- (3) In $F[X]$, the set $\{1, X, X^2, \dots, X^n, \dots\}$ is a basis for $F[X]$.

Theorem 2. Let V be an F -vector space. Then, the vectors v_1, v_2, \dots, v_n of V constitute a basis of V if, and only if, for any vector $u \in V$, there exist uniquely determined scalars

$\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that

$$u = \sum_{i=1}^n \alpha_i v_i.$$

Remark

If $\{v_1, v_2, \dots, v_n\}$ is a basis of V and $u = \sum_{i=1}^n \alpha_i v_i$, then the scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called coordinates or components of u in the basis $\{v_1, v_2, \dots, v_n\}$.

Proposition 5. Any two bases of an F -vector space V have the same number of vectors. This common number for all bases of V is called the dimension of V over F and is denoted $\dim_F V$.

Example 5.

- (1) $\dim_F F^n = n$. Then the dimension of \mathbb{R}^n is n , since $\{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, 0, \dots, 0), \dots\}$ is a basis of cardinality n
- (2) $\dim_F F_n[X] = n + 1$,

Proposition 6. Let V be an F -vector space of dimension n . Then

- (1) Any linearly independent subset of V is of cardinality at most n .
- (2) Any spanning set of V is of cardinality at least equal to n .
- (3) Any linearly independent system of n elements is a basis.
- (4) Any spanning set of V of n elements is a basis.

Theorem 3. *Let V be a finite-dimensional F -vector space. Then:*

- (1) *Any linearly independent subset L of V can be extended to a basis; in other words, any linearly independent set is contained in a basis.*
- (2) *From any spanning set G of V , we can extract a basis; in other words, any spanning set contains a basis.*

Examples

- (1) Let $T = \{(1, 1, 1), (1, 2, 2)\}$. T is linearly independent in \mathbb{R}^3 , we can extend it to a basis by adding $e_3 = (0, 0, 1)$.
- (2) Let $T = \{4, 1 + X\}$. T is linearly independent in $\mathbb{R}_2[X]$, we can extend it to a basis by adding polynomial of degree 2.
- (3) Let $T = \{(1, 1, 1), (1, 2, 2), (1, 1, 3), (1, 0, 0)\}$. T is spanning set of \mathbb{R}^3 , we can extract a basis T by taking $T' = \{(1, 1, 1), (1, 2, 2), (1, 0, 0)\}$.

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Proposition 7. *Let V be a vector space over F and $B = (v_1, v_2, \dots, v_n)$ a subset of V . Then the following are equivalent:*

- (1) *B is a basis for V .*
- (2) *Any $v \in V$, is uniquely written in the form $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ for $\lambda_1, \lambda_2, \dots, \lambda_n \in F$.*

The scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ are called the coordinates of v in the basis B .

Examples

- (1) In \mathbb{R}^3 , the coordinates of the vector $v = (1, 2, 4)$ in the standard basis (e_1, e_2, e_3) are 1, 2, 4. However, in the basis $(v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 2))$, the coordinates are 1, 1, 1, since

$$v = 1v_1 + 1v_2 + 1v_3.$$

- (2) In $\mathbb{R}_3[X]$, the coordinates of the polynomial $P(X) = 1 + 3X + 4X^2 + 2X^3$ in the standard basis $(1, X, X^2, X^3)$ are 1, 3, 4, 2.

Proposition 8. *Let V be an F -vector space, and let U be a subspace of V . Then:*

- (1) $\dim_F U \leq \dim_F V$.
- (2) *If $\dim_F U = \dim_F V$, then $U = V$.*

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2.1. Rank of a family of vectors.

Definition 8. Let V be a vector space over F and $G = \{v_1, v_2, \dots, v_n\} \subset V$. We define the rank of G by $\text{rank}(G) = \dim_F \langle G \rangle$.

Proposition 9. Let V be a vector space over F and $G = \{v_1, v_2, \dots, v_n\}$ a subset of V . Then the rank of G denoted by $\text{rank}(G)$ is the maximum of linearly independent vectors extracted from G .

Example

In $\mathbb{R}_3[X]$,

let $G = \{P_1 = 1 + X, P_2 = 1 + X^2, P_3 = 1 + X^2 + X^3, P_4 = 3 + X + 2X^2 + X^3\}$. $\text{rank} G = 3$ since $P_4 = P_1 + P_2 + P_3$ and $\{P_1, P_2, P_3\}$ linearly independent.

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2.2. Direct sum of subspaces.

Definition 9. Let U and W be two subspaces of an F -vector space V . The sum $U + W$ is called direct if $U \cap W = \{0\}$, and we write $U \oplus W$. If $U \oplus W = V$, we say that U and W are supplementary (or complementary).

Theorem 4. Let U and W be two subspaces of a F -vector space V , then U and W are supplementary (complementary) if and only if

1-The sum $U + W$ is direct

2- $U + W = V$

Theorem 5. Let V_1 and V_2 be finitely generated subspaces of a vector space V . Then

$$\dim_F V_1 + \dim_F V_2 = \dim_F(V_1 \cap V_2) + \dim_F(V_1 + V_2).$$

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