MATRICES

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Introduction of Matrices

1.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as A,B, C etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{m \times n}$$

A is a matrix of order $m \times n$. i^{th} row j^{th} column element of the matrix denoted by a_{ij}

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1.2 Special Types of Matrices:

1. Square matrix:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. Diagonal matrix:

A square matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero. That is $a_{ij} = 0$ if $i \neq j$ and at least one element $a_{ii} \neq 0$.

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Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. Identity Matrix

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

That is $a_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$

That is $a_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$

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Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Upper Triangular matrix:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0 \;\; if \; i > j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

5. Lower Triangular Matrix:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0 \;\; if \; i < j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

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6. Symmetric Matrix:

A square matrix $A = \left(a_{ij}\right)_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all i and j.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

7. Skew- Symmetric Matrix:

A square matrix $A = \left(a_{ij}\right)_{n \times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$$

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8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n \times m}$.

Example:

$$0_{3 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consists a single row is called as a row vector or row matrix.

Example:

$$A = (a_{11} \quad a_{12} \quad a_{13}) \qquad\qquad B = (7 \quad 4 \quad -3)$$

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10. Column Vector

A matrix consists a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

Matrix Algebra

2.1. Equality of two matrices:

Two matrices A and B are said to be equal if

    (i)   They are of same order.
    (ii)   Their corresponding elements are equal.

That is if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then $a_{ij} = b_{ij}$ for all i and j.

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2.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = \left(a_{ij}\right)_{m \times n}$ given denoted by kA and

given by $kA = \left(ka_{ij}\right)_{m \times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

2.3. Addition of two matrices:

Let $A = \left(a_{ij}\right)_{m \times n}$ and $B = \left(b_{ij}\right)_{m \times n}$ are two matrices with same order then sum of the

two matrices are given by

$$A + B = \left(a_{ij}\right)_{m \times n} + \left(b_{ij}\right)_{m \times n} = \left(a_{ij} + b_{ij}\right)_{m \times n}$$

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Example 2.1: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Find (i) 5B   (ii) A + B   (iii) 4A – 2B (iv) 0 A

2.4. Multiplication of two matrices:

Two matrices A and B are said to be confirmable for product AB if number of columns in A equals to the number of rows in matrix B. Let $A = \left(a_{ij}\right)_{m \times n}$ and $B = \left(b_{ij}\right)_{n \times r}$ be two matrices the product matrix $C = AB$, is matrix of order $m \times r$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots\ldots + a_{in}b_{nj}$$

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Example 2.2: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate   (i) AB   (ii) BA

(iii) is AB = BA ?

2.5. Integral power of Matrices:

Let A be a square matrix of order n, and m be positive integer then we define

$A^m = A \times A \times A \dots\dots\times A$ (m times multiplication)

2.6. Properties of the Matrices

Let A, B and C are three matrices and λ and μ are scalars then

(i) $A + (B + C) = (A + B) + C$ Associative Law

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(ii) $\lambda(A + B) = \lambda A + \lambda B$ Distributive law

(iii) $\lambda(\mu A) = (\lambda\mu)A$ Associative Law

(iv) $(\lambda A)B = \lambda(AB)$ Associative Law

(v) $A(BC) = (AB)C$ Associative Law

(vi) $A(B + C) = AB + AC$ Distributive law

2.7. Transpose:

The transpose of matrix $A = \left(a_{ij}\right)_{m \times n}$, written A^t $(A'$ or $A^T)$ is the matrix obtained

by writing the rows of A in order as columns.

That is $A^t = \left(a_{ji}\right)_{n \times m}$.

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Properties of Transpose:

(i) $(A + B)^t = (A^t + B^t)$

(ii) $(A^t)^t = A$

(iii) $(kA)^t = k A^t$ for scalar k.

(iv) $(AB)^t = B^t A^t$

Example 2.3: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix} , B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

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2.8 A square matrix A is said to be symmetric if $A = A^t$.

2.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

Notation

1- We denote by $M_{n,p}(K)$ the set of matrices of n rows and p columns with coefficients in K. If the matrix is square $n = p$, the set is noted $M_n(K)$.

Properties

1- Provided with the following two laws, for $\lambda \in K$ and $(a_{ij}), (b_{ij}) \in M_{n,p}(K)$:

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \text{ and } \lambda.(a_{ij}) = (\lambda a_{ij}), (1 \leq i \leq n, 1 \leq j \leq p)$$

The set $M_{n,p}(K)$ is A K- vector space , where $dim\left(M_{n,p}(K)\right) = np.$

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2- Equipped with the matrix product and the addition of matrices, $M_n(K)$ is a ring. Its neutral element is the matrix identity I_n, which is the diagonal matrix having only 1 on its diagonal.

3- Determinant, Minor and Adjoint Matrices

Definition 3.1:

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n, then the determinant of A of order n being denoted by det(A), $|A|$ or   det $(C_1, C_2,...,C_n)$,($C_1, C_2,...,C_n$ are the columns of A)

is the element of K defined by induction in the following way:
- if $n = 1$, $\det(A) = a_{11}$,
- if $n > 1$, $\det(A) = a_{11}\Delta_{11} - a_{12}\Delta_{12} + a_{13}\Delta_{13} + ... + (-1)^{n+1} a_{1n}\Delta_{1n}$.

where Δ_{1i} is the determinant of the matrix of $M_{n-1}(K)$ obtained by removing the column R_i and the first row from A.

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(i) Determinant of 2 $\times$ 2 matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3 $\times$ 3 matrix

Let B = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$

Exercise 3.1: Calculate the determinants of the following matrices

(i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

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3.1 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

e. det $(C_1,\ldots,\lambda C_j,\ldots,C_n$ $)=\lambda det(C_1,\ldots,C_j,\ldots,C_n)$.

f. det $(C_1,\ldots,C_i,\ldots,C_j,\ldots, C_n$ $)=det(C_1,\ldots,C_j,\ldots,C_i,\ldots,C_n$).

g. det $(C_1,\ldots,C_i,\ldots,C_n$ $)=det(C_1,\ldots,C_i + \lambda C_j,\ldots,C_n)$.

The previous properties are still checked by replacing columns with rows

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3.2 Singular Matrix

If A is square matrix of order n, the A is called singular matrix when $|A| = 0$ and non-singular otherwise.

3.3. Minor and Cofactors:

Let A = $\left(a_{ij}\right)_{n \times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order (n-1) × (n-1) obtained by deleting its ith row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j}|M_{ij}|$.

Proposition We have: $\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} a_{ij}A_{ij}.$

Exercise 3.2: Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

(i)   Compute determinant of A.

(ii)   Find the cofactor matrix.

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3.4. Adjoin Matrix:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj\, A$ is called adjoin of matrix A.

Example 3.3: Find the adjoin matrix of the above example.

Theorem 3.1:

For any square matrix A,

$$A\,(adj\,A) = (adj\,A)A = |A|\,I$$ where I is the identity matrix of same order.

Theorem 3.2: If A and B are two square matrices of order n then

$$|adj\,A| = |A|^{n-1}.$$

$$adj(AB) = (adj\,B)(adj\,A)$$

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4: Inverse of a Matrix and Elementary Row Operations

4.1 Inverse of a Matrix

Definition 4.1:

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}.

Theorem 4.1: (Existence of the inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular) .

Thus $AB = BA = I$ hence B is inverse of A and is given by $A^{-1} = \frac{1}{|A|}\,(adj\,A)$

Example 6: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1}

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Properties

If A and B are two non-singular matrices of order n, then (AB) is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

If A is a non-singular matrix of order n then $(A^t)^{-1} = (A^{-1})^t$.

If A is a non-singular matrix , k is non zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

4.2 Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i)   Interchange of any two rows or columns.

(ii)   Multiplication of the elements of any row (or column) by a non zero number k.

(iii)   Multiplication to elements of any row or column by a scalar k and addition of it to the corresponding elements of any other row or column.

4.2.1 Equivalent Matrix:

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

Definition

A minor of order r is the determinant of any sub square matrix obtained from A by removing m-r rows and n-r columns from A, where the order of A is $m \times n$.

4.3 Rank of a Matrix:

Definition:

A positive integer 'r' is said to be the rank of a non- zero matrix A if

(i)   There exists at least one non-zero minor of order r of A and

(ii)   Every minor of order greater than r of A is zero.

The rank of a matrix A is denoted by r(A) or $\rho(A)$.

4.4 Echelon Matrices:

Definition 4.3:

A matrix $A = (a_{ij})$ is said to be echelon form (echelon matrix) if the number of zeros preceding the first non zero entry of a row increasing by row until zero rows remain.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i)   The only non- zero elements in their respective columns.
(ii)   Each equal to 1.

Remark: The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Example 4.1:

Reduce following matrices to row reduce echelon form

(i) $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$

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4. 6 Solution of System and the Inverse of matrix

Let the linear system $AX= B$, where A *is a square matrix, we know that* if the matrix A is invertible then linear system $AX= B$ have a solution of the form $X= A^{-1} B$. So, it is enough to find X components in terms of components of B.

Example

Let the following I linear system.

$\begin{cases} 2x + 3y = a \\ x + y = b \end{cases} \Leftrightarrow \begin{cases} x + 3y = a \\ 0 + y = -2b + a \quad \leftarrow L_1 - 2L_2 \end{cases}$

$\Leftrightarrow \begin{cases} x + 0 = -a + 3b \qquad \leftarrow \quad L_1 - 3L_2 \\ 0 + y = a + -2b \end{cases}$

So, $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ we get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$.

Using Elementary row operations(Gaussian Elimination) on the matrix writing as (A:I) and reduce it to the matrix of the form (I:B). Then the inverse matrix of A is B (applying only elementary row transformations)

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Example: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1}

Using Elementary row operations

5 Matrices and linear maps

Let V, U and W be vector spaces, equipped respectively by the bases $B = (b_i)_{1 \le i \le p}$, $C = (c_i)_{1 \le i \le n}$ and $D = (d_i)_{1 \le i \le m}$.

Definition 5. 1

Let f $\in L(V, U)$, we call the matrix of f in the bases B and C the matrix of $M_{n,p}(K)$ whose column vectors are the coordinates of the vectors $f(b^1), \dots, f(b_p)$ in the base C=(c_1,...,c_[n]). We denote it $M(f, B, C)$.

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$$f(b_1) \quad \cdots \quad \cdots \quad f(b_p)$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{1p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n1} & & a_{np} \end{pmatrix} \begin{matrix} c_1 \\ \\ \\ c_n \end{matrix}$$

Remark

1.Let $X \in V$ have coordinates $(x^1, \dots, x_p)$ in the basis B and $Y = f(X)$ have coordinates $(y^1, \dots, y_n)$, in the basis C. Then we have: $Y = M(f, B, C)X$.

2.If $f \in L(V, U)$ and $g \in L(U, W)$, then $M(g \circ f, B, D) = M(g, C, D)M(f, B, C)$.

3.If $A = (a_{ij}) \in M_{n,p}(K)$, then A define a linear map $f_A: K^p \to K^n$ defined by $f_A(X) = AX$, where we identify a vector of $K^p(resp. K^n)$ and the column vector formed by the coordinates of this vector in the canonical basis.

4. The kernel, the image, and the rank of A are then by definition the kernel, the rank, the image of the associated linear map.

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5.1 Change of basis

Let V be a finite-dimensional vector space and let B_1, B_2 two bases of V. The passage matrix from the basis B_1 to the basis B_2 is the matrix whose columns are the vector family of B_2 in the basis B_1, denoted as $P(B_1, B_2)$. It is invertible, and its inverse is $P(B_2, B_1)$.

Remark

Let $X \in V$ has coordinates X^1 in the base B_1 and the coordinates X^2 in the base B_2, then $X^1 = P(B_1, B_2)\, X^2$.

5.2 Change of basis for linear maps

Let $f \in L(V, U)$, where V and U are vector spaces, and let B and B' be two bases of V. And C and C' are two bases of U. If we note $A = M(f, B, C)$, $A' = M(f, B', C')$, $P = P(B, B')$ and $Q = P(C, C')$, then we get

$$A' = Q^{-1}AP.$$

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Example

Let the map:

$$f: \mathbb{R}^2 \to \mathbb{R}^3$$

1) Give the associated matrix of f with respect to the canonical bases $\mathbb{R}^2$ and $\mathbb{R}^3$.

2) Give the associated matrix of f with respect to the bases L and H defined by:

$$L = \{l_1 = (2,1), l_2(0,1)\}$$

$$H = \{h_1 = (2,1,1), h_2 = (0,2,1), h_3 = (0,0,1)\}.$$

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Trace of a matrix

* If $A \in M_n(K)$, we call trace of A, notée $Tr(A)$, the sum of the diagonal coefficients of A.

* We have: $tr(AB) = tr(BA)$.

* If $f \in L(V)$, then we call trace of f the trace of the matrix representing f in any basis of V.

Proposition
Let $A, B \in MnK$.
1) $tr(A + B)= tr\, A + tr\, B$.
2) $tr(\alpha A)= \alpha tr (A), \forall \alpha \in K.$
3) $tr(AT)= tr (TA)$.

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