# Serie N°2 :

## Exercice 01 :

$T = 2\pi \qquad S(f) = \sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx))$

where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \qquad b_0 = 0$

and $\forall n \in \mathbb{N}^*$ :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

or :

$$S(f) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}^*} (a_n \cos(nx) + b_n \sin(nx))$$

where :

$$\begin{cases} a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \quad \forall n\in\mathbb{N} \\ b_n = \text{---} \end{cases}$$

tbeddl si 0

1/ $f(x) = e^x \; ; \quad x \in ]-\pi, \pi[$

$f(-\pi) = f(\pi) = e^{\pi} = f(-\pi + 2\pi) = f(\pi)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

because $f$ is periodic

$$a_0 = \frac{1}{2\pi}\int_{0-\pi}^{2\pi-\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{\sinh(\pi)}{\pi}$$

$$a_0 = \frac{\sinh(\pi)}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx$$

Let $C_n = a_n + i b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[e^x \cos(nx) + i e^x \sin(nx)\right]dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x(\cos nx + i \sin nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} e^{inx}\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{(1+in)x}\, dx = \frac{1}{\pi(1+in)}\left[e^{(1+in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+in)}\left[e^{(1+in)\pi} - e^{-(1+in)\pi}\right]$$

$$= \frac{1}{\pi(1+in)}\left[e^{\pi} e^{in\pi} - e^{-\pi} e^{-in\pi}\right]$$

we have : $e^{in\pi} = \cos n\pi + i\sin \pi = (-1)^n$ and $e^{-in\pi} = (-1)^n$.

$$C_n = \frac{1}{\pi(1+in)}\left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi(1+in)}\left[e^{\pi} - e^{-\pi}\right] = \frac{2(-1)^n \sinh \pi}{\pi(1+in)}$$

$$= \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}\left[1 - in\right] \Rightarrow \begin{cases} a_n = Re(c_n) = \dfrac{2(-1)^n sh\pi}{\pi(1+n^2)} \\ b_n = Im(C_n) = \dfrac{-2(-1)^n n \sinh\pi}{\pi(1+n^2)} = -n a_n \end{cases}$$

ad rekhes ugr comme facteur
ad yegri $a_n = 1$ daffen $b_n = -n$ tagara ad d-ffag abba.

$$S(f) = \frac{sh\pi}{\pi} + \sum_{n\in\mathbb{N}^*} \frac{2(-1)^n sh\pi}{\pi(1+n^2)}(\cos nx - n\sin nx)$$

$$= \frac{sh\pi}{\pi} + \frac{2sh\pi}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin x)$$

Exercice 02 :

$f(x) = \alpha x^2 + \beta x + \gamma$ — $f$ $2\pi$-periodic even

1)- $\forall x \in [0,\pi]$.
- $f$ is continous on $\mathbb{R}$ of class $C$ ... $(2k+1)\pi$.
- $f$ is developable on $\mathbb{R} - (2k+1)\pi$ on fourrier series and its fourrier series is uniformly convergent.

2)- $f$ is even $\Rightarrow \begin{cases} b_n = 0 \\ a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx \;;\; a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\,dx \quad \forall n\in\mathbb{N}^* \end{cases}$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} (\alpha x^2 + \beta x + \gamma)\, dx = \frac{1}{\pi}\left[\frac{\alpha}{3}x^3 + \frac{\beta}{2}x^2 + \gamma x\right]_0^{\pi}$$

$$a_0 = \frac{1}{6}\left(2\alpha\pi^2 + 3\beta\pi + 6\gamma\right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\alpha x^2 + \beta x + \gamma)\cos nx\, dx$$

integral by part ① :

$$\frac{2}{n\pi}\left[\left[(\alpha x^2 + \beta x + \gamma)\sin nx\right]_0^{\pi} - \int_0^{\pi} (2\alpha x + \beta)\sin nx\, dx\right]$$

$$= \frac{-2}{\pi n}\int_0^{\pi} (2\alpha x + \beta)\sin nx\, dx$$

integral by part ② :

$$= \frac{2}{\pi n^2}\left[\left[(2\alpha x + \beta)\cos nx\right]_0^{\pi} - \underbrace{\int_0^{\pi} 2\alpha \cos nx\, dx}_{0}\right]$$

$$= \frac{2}{\pi n^2}\left[(2\alpha\pi + \beta)\cos n\pi - \beta\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n(2\alpha\pi + \beta) - \beta\right]$$

$$a_n = \frac{2(2\alpha\pi + \beta)}{\pi}\,\frac{(-1)^n}{n^2} - \frac{2\beta}{\pi n^2}$$
$\forall n \geq 1$

$$S(f) = \sum_{n=1}^{+\infty} \frac{1}{n^2}\cos nx.$$

$$\Rightarrow \begin{cases} a_0 = 0 \\ a_n = \dfrac{1}{n^2}, \ \forall n \in \mathbb{N} \end{cases}$$

$$\Rightarrow \begin{cases} 2\alpha\pi^2 + 3\beta\pi + 6\gamma = 0 \\ 2\alpha\pi + \beta = 0 \\ \dfrac{-2\beta}{\pi} = 1 \end{cases} \quad . \quad \beta = \frac{-\pi}{2} \quad \alpha = \frac{1}{4} \quad \gamma = \frac{\pi^2}{6}$$

$$f(x) = \frac{x^2}{4} - \frac{\pi}{2}x + \frac{\pi^2}{6} \quad \forall x \in [0, \pi]$$

3)- since $f$ satisfies the Dirichlet's conditions:

at $x = 0$ ; $f(0^+) = f(0^-) = f(0) = \gamma = \frac{\pi^2}{6}$

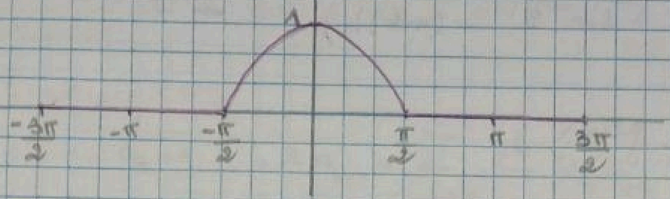
$f'(0^+) = \beta = -f'(0^-)$

fourier series:

- Exercice 03:

$$f : x \longmapsto \begin{cases} \cos x & \text{if } \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \end{cases}$$

①-




② - The sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n^2-1}$$

we compute the fourier series of $f$,

if $f$ is even, then $b_n = 0$, $\forall n \in \mathbb{N}$.

and

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_0^{\pi/2} 0 . dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{1}{\pi}\left[\sin x\right]_0^{\pi/2} = \frac{1}{\pi}$$

$\forall n \in \mathbb{N}$:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\,dx + \int_{-\frac{\pi}{2}}^{\pi} 0 . \cos nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \left[\cos(n+1)x + \cos(n-1)x\right] dx$$

Since $\int \cos \alpha x\,dx = \frac{1}{\alpha}\sin \alpha x + C \quad \forall \alpha \neq 0$

if $n = 1$ :

$$a_1 = \frac{1}{\pi}\int_0^{\pi/2} (\cos 2x + 1)\, dx = \frac{1}{\pi}\left[\frac{1}{2}\sin 2x + x\right]_0^{\pi/2} = \frac{1}{2}$$

and $\forall n \in \mathbb{N} \setminus \{0, 1\}$, we have :

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1}\sin(n+1)x + \frac{1}{n-1}\sin(n-1)x\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}\sin\left[(n+1)\frac{\pi}{2}\right] + \frac{1}{n-1}\sin\left[(n-1)\frac{\pi}{2}\right]\right]$$

$$\sin p\frac{\pi}{2} = \begin{cases} 0 & \text{if } p = 2k \\ (-1)^k & \text{if } p = 2k+1 \end{cases}$$

So $a_{2n+1} = 0$ and
$$a_{2n} = \frac{1}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n-1}}{2n-1}\right)$$
$$= \frac{1}{\pi}\left(\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right)$$
$$= \frac{(-1)^n}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$
$$a_{2n} = \frac{-2(-1)^n}{(4n^2-1)} \quad \forall n \in \mathbb{N}^*$$

the fourier series of $f$ is :

$$S(f) : \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{n\in\mathbb{N}^*} \frac{(-1)^n}{4n^2-1}\cos 2nx$$

using Diriclet's theorem :

since $f$ is derivable at $x = 0$ then :

$$f(0) = \cos(0) = \frac{1}{\pi} + \frac{1}{2}\cos 0 - \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Longleftrightarrow 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n}{4n^2-1} = \frac{-2}{\pi}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{4n^2-1} - \frac{(-1)^0}{4\cdot 0^2-1}\right)$$

$$\frac{1}{2} - \frac{1}{\pi} = \frac{-2}{\pi}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{4n^2-1} + 1\right)$$

$$\frac{1}{2} - \frac{1}{\pi} = \frac{-2}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{4n^2-1} - \frac{2}{\pi}$$

$$\frac{1}{2} + \frac{1}{\pi} = \frac{-2}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{4n^2-1} \implies \sum_{n=0}^{+\infty}\frac{(-1)^n}{4n^2-1} = -\frac{\pi}{4} - \frac{1}{2}$$

## Exercice 04:

$1 - \frac{4x}{\pi}$

1) $f : x \longmapsto 1 - \frac{2x}{\pi}$ , $\forall x \in [0, \pi[$

$f$ even and $2\pi$ - periodic.

$b_n = 0$ , $\forall n \in \mathbb{N}$ ; $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{1}{\pi} \left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$

$\forall n \in \mathbb{N}^*$ :

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx \, dx = \frac{2}{\pi} \left[ \left[\left(1 - \frac{2x}{\pi}\right) \times \frac{1}{n} \sin nx\right]_0^{\pi} - \frac{(-2)}{\pi n} \int_0^{\pi} \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[ 0 + \frac{2}{n\pi} \int_0^{\pi} \sin nx \, dx \right] = \frac{4}{\pi^2 n} \int_0^{\pi} \sin x \, dx.$$

$$a_n = \frac{-4}{\pi^2 n^2} \left[\cos nx\right]_0^{\pi} = \frac{-4}{\pi^2 n^2} \left[(-1)^n - 1\right]$$

$$a_n = \frac{4\left(1 - (-1)^n\right)}{\pi^2 n^2} \quad \forall n \in \mathbb{N}^*$$

$$a_{2n} = 0 \quad , \forall n \in \mathbb{N}^*$$

$$a_{2n+1} = \frac{8}{\pi^2 (2n+1)^2} \quad \forall n \in \mathbb{N}$$

conclusion: The fourier series of $f$ is given by:

$$S(f) : \sum_{n \in \mathbb{N}} a_{2n+1} \cos(2n+1)x$$

$$: \sum_{n \in \mathbb{N}} \frac{8}{\pi^2 (2n+1)^2} \cos(2n+1)x.$$

using the Diriclet's theorem:

since: $f_L(0) = \lim_{x \overset{<}{\to} 0} f(x) = f_R(0) = \lim_{x \overset{>}{\to} 0} f(x) = f(0) = 1.$

$f'_L(0) = - f'_R(0) = \frac{2}{\pi}$

then: $f(0) = 1 = \sum_{n=0}^{n=+\infty} \frac{8}{\pi^2 (2n+1)^2} \cos\left[(2n+1) \cdot 0\right]$

$$\Longrightarrow \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$a_n = \frac{2(-1)^n \sinh(\alpha\pi)}{\alpha\pi} - \frac{n^2}{\alpha^2} a_n \Rightarrow a_n + \frac{n^2}{\alpha^2} a_n = \frac{2(-1)^n \sinh(\alpha\pi)}{\alpha\pi}.$$

$$\frac{(\alpha^2+n^2)}{\alpha^2} a_n = \frac{2(-1)^n \sinh(\alpha\pi)}{\alpha\pi}$$

$$a_n = \frac{2\alpha^2(-1)^n \sinh(\alpha\pi)}{\alpha\pi(\alpha^2+n^2)}$$

The fourier series of $f$ is :

$$Sf : \frac{\sinh(\alpha\pi)}{\alpha\pi}\left[1 + 2\alpha^2 \sum \frac{(-1)^n}{n^2+\alpha^2} \cos nx\right]$$

using Drichlet's theorem :

at $x = \pi$ : $$f(\pi) = \cosh(\alpha\pi) = \frac{\sinh(\alpha\pi)}{\alpha\pi}\left[1 + 2\alpha^2 \sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^n}{n^2+\alpha^2}\right]$$

$$\Rightarrow \alpha\pi \cosh(\alpha\pi) = 1 + 2\alpha^2 \sum_{n=1}^{+\infty} \frac{1}{n^2+\alpha^2}$$

$$= 1 + 2\alpha^2 \left[\sum_{n=0}^{+\infty} \frac{1}{n^2+\alpha^2} - \frac{1}{\alpha^2}\right]$$

$$= 1 + 2\alpha^2 \sum_{n=0}^{+\infty} \frac{1}{n^2+1} - 2 = -1 + 2\alpha^2 \sum_{n=0}^{+\infty} \frac{1}{n^2+\alpha^2}$$

$$\frac{1+\alpha\pi\cosh(\alpha\pi)}{2\alpha^2} = \sum_{n=0}^{+\infty} \frac{1}{n^2+\alpha^2}$$

at $x = 0$ :

$$f(0) = 1 = \frac{\sinh(\alpha\pi)}{\alpha\pi}\left[1 + 2\alpha^2 \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+\alpha^2}\right]$$

$$\Leftrightarrow \frac{\alpha\pi}{\sinh(\alpha\pi)} = 1 + 2\alpha^2 \left[\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+\alpha^2} - \frac{1}{\alpha^2}\right]$$

$$= -1 + 2\alpha^2 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+\alpha^2}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+\alpha^2} = \frac{\alpha\pi + \sinh(\alpha\pi)}{2\alpha^2 \sinh(\alpha\pi)}$$

Using Parseval's identity:

$$\sum_{n=0}^{+\infty} a_{2n+1}^2 = \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx$$

$$\iff \sum_{n=0}^{\infty}\frac{64}{\pi^4}\cdot\frac{1}{(2n+1)^4} = \frac{2}{\pi}\int_0^\pi\left(1-\frac{2x}{\pi}\right)^2 dx = \frac{2}{\pi}\int_0^\pi\left(1+\frac{4x^2}{\pi^2}-\frac{4x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x+\frac{4x^2}{3\pi}-\frac{2x^2}{\pi}\right]_0^\pi = 2\left(1+\frac{4}{3}-2\right) = \frac{2}{3}$$

• $$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} = \frac{2}{3}\times\frac{\pi^4}{64} = \frac{\pi^4}{96} \implies \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{+\infty}\frac{1}{n^4} = \sum_{n=1}^{+\infty}\frac{1}{(2n)^4} + \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} = \frac{1}{2^4}\sum_{n=1}^{+\infty}\frac{1}{n^4} + \frac{\pi^4}{96}$$

$$\implies \left(1-\frac{1}{16}\right)\sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{96} \implies \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{16}{15}\times\frac{\pi^4}{96} = \frac{\pi^4}{90}$$

$$\implies \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

(2) - $f$ is even so $b_n = 0$ $\forall n\in\mathbb{N}$.

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi \cosh(\alpha x)\,dx = \frac{\sinh(\alpha x)}{\alpha\pi}$$

$\forall n\in\mathbb{N}^*$

$$a_n = \frac{2}{\pi}\int_0^\pi \cosh(\alpha x)\cos nx\,dx$$ integral by part 2 fois:

$$= \frac{2}{\pi}\left[\left[\frac{1}{\alpha}\sinh(\alpha x)\cos nx\right]_0^\pi - \frac{1}{\alpha}(-n)\int_0^\pi \sinh(\alpha x)\sin nx\,dx\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{\sinh(\alpha\pi)}{\alpha}\cos n\pi + \frac{n}{\alpha}\int_0^\pi \sinh(\alpha x)\sin nx\,dx\right]$$

$$= \frac{2(-1)^n}{\pi\alpha}\sinh(\alpha\pi) + \frac{2n}{\alpha\pi}\int_0^\pi \sinh(\alpha x)\sin\cancel{\alpha}x\,dx$$

$$= \frac{2(-1)^n\sinh(\alpha\pi)}{\alpha\pi} + \frac{2n}{\pi\alpha^2}\left[\left[\cosh(\alpha x)\sin nx\right]_0^\pi - \int_0^\pi n\cos nx\sinh(\alpha x)\right]$$

Exercice 05 :

$$f : x \mapsto \sum_{n=1}^{+\infty} \frac{\sin nx}{\sqrt{n}} \quad \forall x \in \mathbb{R}.$$

$f(\pi k) = 0 \quad , \forall k \in \mathbb{Z}.$

so $f$

By the Abel's criterion for numrical series; $f$ is well defined on $\mathbb{R} - \pi\mathbb{Z}$ so $f$ is well defined on $\mathbb{R}$.

since $f(x+2\pi) = f(x)$, we can prove that $f$ is continous on $]-\pi, \pi[$.

But $f$ is odd, so we can proof that $f$ is continous on $[0; \pi[$

$f(\pi - x) = f(x)$ so it is sufficiant to prove that $f$ is continous on $[0; \frac{\pi}{2}[$

. By Abel's criterian for the uniform convergence of function series.

$\sum \frac{\sin(nx)}{\sqrt{n}}$ is uniformly convergent on $[0, \pi/2[$

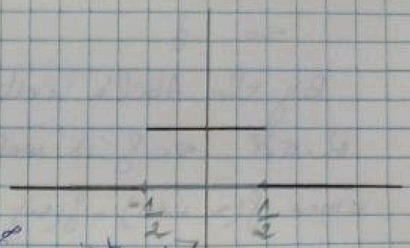
. If $\sum \frac{1}{\sqrt{n}} \sin nx$ is the fourier series of a function $g \Rightarrow \sum_{n \in \mathbb{N}^*} \left(\frac{1}{\sqrt{n}}\right)^2$ cv

so if $f$ is convergent.

Serie 03 :

Exercice 02 :

$$\pi(t) = \begin{cases} 1 & \text{if} \quad 2|t| \leq 1 \\ 0 & \text{if} \quad 2|t| > 1 \end{cases}$$



$$\mathcal{F}_1(\pi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \pi(t) e^{-ixt} \, dt$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \pi(t) e^{-ixt} \, dt = \frac{1}{\sqrt{2\pi}} \left[ \int_{+\infty}^{-1/2} \pi(t) e^{ixt} \, dt + \int_{-1/2}^{1/2} \pi(t) e^{ixt} \, dt + \int_{1/2}^{\infty} \pi(t) e^{ixt} \, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{+\infty}^{-1/2} 0 \, e^{ixt} \, dt + \int_{-1/2}^{1/2} e^{ixt} \, dt + \int_{1/2}^{\infty} 0 \, e^{ixt} \, dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{ixt} \, dt$$

$$\mathcal{F}_1(\pi)(0) = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{0} \, dt = \frac{1}{\sqrt{2\pi}}$$

$\forall x \in \mathbb{R} - \{0\} = \mathbb{R}^*$

$$\mathcal{F}_1(\pi)(x) = \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{-ix} e^{-ixt} \right]_{-1/2}^{1/2} = \frac{1}{-ix\sqrt{2\pi}} \left[ e^{-i\frac{x}{2}} - e^{i\frac{x}{2}} \right]$$

$$= \frac{-\cancel{i} 2 \sin \frac{x}{2}}{-\cancel{i} x \sqrt{2\pi}} = \frac{2 \sin \frac{x}{2}}{x \sqrt{2\pi}}$$

$$\mathcal{F}_1(\pi)(x) = \begin{cases} \dfrac{2 \sin x/2}{x \sqrt{2\pi}} & \text{if } x \neq 0 \\ \dfrac{1}{\sqrt{2\pi}} & \text{if } x = 0 \end{cases}$$

2- 1)

$$\mathcal{F}_1(t \mapsto f(at+b))(x) = \frac{1}{|a|} e^{\frac{ixb}{a}} \mathcal{F}_1(f)\left(\frac{x}{a}\right)$$

$$\mathcal{F}_1\left(t \to \pi\left(\frac{t-1}{2}\right)\right)(x) = \frac{1}{|\frac{1}{2}|} e^{ix\frac{-1/2}{1/2}} \mathcal{F}_1(\pi)\left(\frac{x}{\frac{1}{2}}\right) = 2 e^{-ix} \mathcal{F}_1(\pi)(2x)$$

$a = \frac{1}{2}$ , $b = \frac{-1}{2}$ .

$$\mathcal{F}\left(t \mapsto \pi\left(\frac{t-1}{2}\right)\right)(x) = \begin{cases} \dfrac{2\, e^{-ix} \cdot \sin x}{x\sqrt{2\pi}} & x \neq 0 \\ \dfrac{2}{\sqrt{2\pi}} & x = 0 \end{cases}$$

dil cours abba

$$\mathcal{F}\left(t \mapsto f\left(a\left(t+\frac{b}{a}\right)\right)\right)(x) = \frac{1}{|a|}\,\mathcal{F}\left(t \mapsto f\left(t+\frac{b}{a}\right)\right)\left(\frac{x}{a}\right)$$

$a = 1 \rightarrow$ translation $= e^{\frac{ixb}{a}} \cdot \frac{1}{|a|}\,\mathcal{F}(f)\left(\frac{x}{a}\right)$

$b = 0 \Rightarrow$ تكبي

ii)

maydllo [illegible] + تلقى translation

$$\mathcal{F}(t \mapsto t f(t))(x) = i\,\mathcal{F}(f)'(x).$$

$$\mathcal{F}(t \mapsto t f(t))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t f(t)\, e^{-ixt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \left[\frac{\delta}{\delta x}\left(\frac{1}{-i}\, e^{-ixt}\right)\right] dt$$

$$= \frac{1}{-i}\,\frac{\delta}{\delta x}\left[\int_{-\infty}^{+\infty} f(t)\, e^{-ixt}\, dt\right]$$

$$= i\,\mathcal{F}'(f)(x)$$

$$\left[\frac{\sin x/2}{x}\right]' = \frac{\frac{x}{2}\cos x/2 - \sin x/2}{x^2}$$

$$\mathcal{F}(t \mapsto t\,\pi(t))(x) = \begin{cases} i\,\dfrac{x\cos x/2 - 2\sin x/2}{\sqrt{2\pi}\; x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

iii)

$$\mathcal{F}(t \mapsto t^2\, \pi(t))(x)$$

$$= i\,\mathcal{F}(t \mapsto t\,\pi(t))'(x)$$

$$\left(\frac{x\cos x/2 - 2\sin x/2}{x^2}\right)' = \frac{x\left(-\frac{x}{2}\sin\frac{x}{2}\right) - 2x\left(\cos\frac{x}{2} - 2\sin\frac{x}{2}\right)}{x^4}$$

$$= \frac{-x^2 \; x/2 - 4x\cos x/2 + 8\sin x/2}{2x^3}$$

$$= \frac{(8-x^2)\sin x/2 - 4x\cos x/2}{2x^3}$$

$$\mathcal{F}\left(t \mapsto t^2 \pi(t)\right)(x) = \begin{cases} \dfrac{(x^2-8)\sin\frac{x}{2} + 4x\cos x/2}{2x^3\sqrt{2\pi}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$= -\mathcal{F}(\pi)''(x).$$

Exercice 05 :

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\tilde{\mathcal{F}}(f)\,\mathcal{F}(g)$$

$$\int_{-\infty}^{+\infty} y(u)\, e^{-\beta(t-u)^2}\, du = e^{-t^2} \qquad \forall t \in \mathbb{R}$$

les formules si libero 1

$\forall \beta > 0$, we put $\quad f_\beta(t) = e^{-\beta t^2} = e^{\frac{-t^2}{2\frac{1}{2\beta}}} = \frac{1}{\sqrt{2\beta}}\, \Gamma_{\frac{1}{2\beta}}(t) \qquad s = \frac{1}{2\beta}$

$$y * f_\beta = f_1$$

$$\mathcal{F}(f_1) = \tilde{\mathcal{F}}(y * f_\beta) = \sqrt{2\pi}\,\mathcal{F}(y)\,\mathcal{F}(f_\beta)$$

$$\mathcal{F}(y) = \frac{1}{\sqrt{2\pi}} \frac{\mathcal{F}(f_1)(x)}{\mathcal{F}(f_\beta)(x)} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\mathcal{F}\left(\frac{1}{\sqrt{2}}\Gamma_{1/2}\right)}{\mathcal{F}\left(\frac{1}{\sqrt{2\beta}}\Gamma_{1/2\beta}\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2\beta}}} \cdot \frac{\mathcal{F}(\Gamma_{1/2})}{\mathcal{F}(\Gamma_{1/2\beta})} = \sqrt{\frac{\beta}{2\pi}} \cdot \frac{\frac{1}{\sqrt{1/2}}\Gamma_2}{\frac{1}{\sqrt{1/2\beta}}\Gamma_{2\beta}}$$

$$\frac{1}{\sqrt{2\beta}}\left(\frac{1}{\sqrt{1/2\beta}}\, e^{-\beta t^2}\right) \Rightarrow t \mapsto \sqrt{\frac{\beta}{2\pi}}\, \frac{e^{-t^2/4}}{e^{-t^2/4\beta}}$$

$$t \mapsto \sqrt{\frac{\beta}{2\pi}}\, e^{-t^2/2\gamma}$$

where $\frac{1}{2\gamma} = \frac{1}{4} - \frac{1}{4\beta} \Rightarrow \frac{1}{\gamma} = \frac{1}{2}\left(1 - \frac{1}{\beta}\right) \qquad \gamma = \frac{2\beta}{\beta - 1} > 0$

$$\mathcal{F}(y)(x) = \sqrt{\frac{\beta}{2\pi}}\, e^{-x^2/2\gamma} = \sqrt{\frac{\beta}{2\pi}} \cdot \underbrace{\frac{1}{\sqrt{\gamma}}\, \Gamma_\gamma(x)}$$

$$= \sqrt{\frac{\beta}{2\pi}}\, \mathcal{F}\left(\sqrt{\gamma}\, \Gamma_{1/\gamma}\right)(x)$$

$$= \mathcal{F}\left(\sqrt{\frac{\beta\gamma}{2\pi}}\, \Gamma_{1/\gamma}\right)$$

$$y(t) = \sqrt{\frac{\beta\gamma}{2\pi}}\ \Gamma_{1/\gamma}(t)$$

$$= \sqrt{\frac{2\beta^2}{2\pi(\beta-1)}}\ \sqrt{\gamma}\ e^{-\gamma t^2/2} = \frac{\beta}{\sqrt{\pi(\beta-1)}}\sqrt{\frac{2\beta}{\beta-1}}\ e^{-\beta t^2/\beta-1}.$$

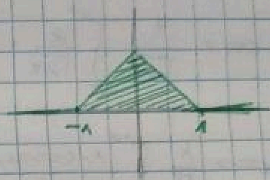
Exercice 03 :

$$f(t) = \begin{cases} 1-|t| & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

①- $\mathcal{L}^1(\mathbb{R},\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \ \middle|\ \int_{-\infty}^{+\infty} |f(t)|\,dt \in \mathbb{R} \right\}.$

Since

$$\int_{-\infty}^{+\infty} |f(t)|\,dt = 1 \in \mathbb{R} \quad \text{then } f \in \mathcal{L}^1(\mathbb{R},\mathbb{R})$$




$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-ixt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{-1} f(t) e^{-ixt}\,dt + \int_{-1}^{1} f(t) e^{-ixt}\,dt + \int_{1}^{+\infty} f(t) e^{-ixt}\,dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|t|)\, e^{-ixt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|t|)(\cos xt - i \sin xt)\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underbrace{(1-|t|)\cos xt}_{\text{even}}\,dt - \frac{i}{\sqrt{2\pi}} \underbrace{\int_{-1}^{1} \underbrace{(1-|t|)\sin xt}_{\text{odd}}\,dt}_{\to 0}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-t)\cos xt\,dt$$

$$\mathcal{F}(f)(0) = \frac{2}{\sqrt{2\pi}} \int_0^1 (1-t)\,dt = \frac{1}{\sqrt{2\pi}}$$

$\forall x \in \mathbb{R}-\{0\}$ : intégration par parties.

$$\mathcal{F}(f)(x) = \frac{2}{x\sqrt{2\pi}} \int_0^1 1-t\,(\sin xt)'\,dt$$

$$= \frac{2}{x\sqrt{2\pi}} \left[ \left[(1-t)\sin xt\right]_0^1 - \int_0^1 (1-t)' \sin xt\,dt \right]$$

$$= \frac{2}{x\sqrt{2\pi}} \int_0^1 \sin xt\,dt = \frac{2}{x\sqrt{2\pi}} \left[\frac{-\cos xt}{x}\right]_0^1 = \frac{2}{x^2\sqrt{2\pi}}(1-\cos x)$$

$$\mathcal{F}(f)(x) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} \, \dfrac{1-\cos x}{x^2} & x \neq 0 \\[2mm] \dfrac{1}{\sqrt{2\pi}} & x = 0 \end{cases}$$

2)- formule d'inversion : $\mathcal{F}^2(f)(x) = f(-x)$ , $\forall x \in \mathbb{R}$.

Since $f$ is even, then $\mathcal{F}^2(f)(x) = f(x)$.

$$\mathcal{F}(\mathcal{F}(f))(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(f)(x)\, e^{-ixt}\, dx = f(x)$$

$$\Longleftrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{2}{\sqrt{2\pi}} \cdot \frac{1-\cos x}{x^2}\, e^{-ixt}\, dx = \begin{cases} 1-|t| & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

$$\Longleftrightarrow \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1-\cos x}{x^2}\, e^{-ixt}\, dx = \begin{cases} 1-|t| & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

3)-

$$\frac{1}{\pi}\Big[\int_{-\infty}^{+\infty} \frac{1-\cos x}{x^2} \cos xt\, dt - i \int_{-\infty}^{+\infty} \underbrace{\frac{1-\cos xt}{x^2} \sin xt}_{\text{odd}}\, dx\Big] = f(t)$$

$$\frac{2}{\pi} \int_0^{+\infty} \frac{1-\cos x}{x^2} \cos xt = f(t)$$

$$\Longleftrightarrow \int_0^{+\infty} \frac{1-\cos x}{x^2} \cos(\alpha x)\, dx = \begin{cases} \frac{\pi}{2}(1-\alpha) & 0 \leq \alpha \leq 1 \\ 0 & \alpha > 1 \end{cases}$$

4)-

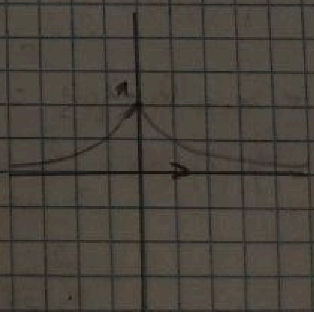
$\alpha = 0$ :

$$\int_0^{+\infty} \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2} \Longleftrightarrow \int_0^{+\infty} \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}\, dx = \pi/2 . \qquad \cancel{x} \to \frac{x}{2}$$

$$\int_0^{+\infty} \frac{2\sin^2 x}{4x^2}\, 2\, dx = \frac{\pi}{2} \Longleftrightarrow \int_0^{+\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$$

Exercice 04 :

Let $\alpha > 0$ $\quad f_\alpha(t) = e^{-\alpha|t|}$



1)- $\mathcal{F}(f)(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} e^{-\alpha|t|}\, e^{-ixt}\, dt$

$$= \frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{0} e^{\alpha t} e^{-ixt}\, dt + \int_0^{+\infty} e^{-\alpha t} e^{-ixt}\, dt\Big] = \frac{1}{\sqrt{2\pi}}\Big[\frac{1}{\alpha - ix} e^{\alpha t} e^{-ixt}\Big|_{-\infty}^{0} - \frac{1}{\alpha+ix} e^{-\alpha t} e^{-ixt}\Big|_0^{+\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\Big[\frac{1}{\alpha - ix}(1-0) - \frac{1}{\alpha+ix}[0-1]\Big]$$

$$\mathcal{F}(f_\alpha)(x) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}\right] = \frac{2\alpha}{\sqrt{2\pi}\,(x^2+\alpha^2)} \qquad \forall x \in \mathbb{R}.$$

2).-

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathcal{F}(f)(x)\, e^{-ixt}\, dx = f(-t) = f(t) \qquad \forall t \in \mathbb{R}.$$

$$\frac{1}{\sqrt{2\pi}} \times \frac{2\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ixt}}{x^2+\alpha^2}\, dx = e^{-\alpha|t|}.$$

$$\int_{-\infty}^{+\infty} \frac{\cos xt - i\sin xt}{x^2+\alpha^2}\, dx = \frac{\pi}{\alpha}\, e^{-\alpha|t|}$$

$$\int_{0}^{+\infty} \frac{\cos xt}{x^2+\alpha^2}\, dx = \frac{\pi}{2\alpha}\, e^{-\alpha|t|} \quad , \forall\ t \in \mathbb{R}.$$

3).-

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)$$

4).-

$$\mathcal{F}(-y'' + y)(x) = \mathcal{F}(f_2)(x) \qquad \forall x \in \mathbb{R}.$$

$$\Longleftrightarrow -\mathcal{F}(y'')(x) + \mathcal{F}(y)(x) = \frac{4}{\sqrt{2\pi}\,(x^2+4)}$$

$$\Longleftrightarrow -\left[(ix)^2\, \mathcal{F}(y)(x)\right] + \mathcal{F}(y)(x) = \frac{4}{\sqrt{2\pi}\,(x^2+4)}$$

$$\Longleftrightarrow -(-x^2)\, \mathcal{F}(y)(x) + \mathcal{F}(y)(x) = \frac{4}{\sqrt{2\pi}\,(x^2+4)}$$

$$\Longleftrightarrow (1+x^2)\, \mathcal{F}(y)(x) = \frac{4}{\sqrt{2\pi}\,(x^2+4)}$$

$$\Longleftrightarrow \mathcal{F}(y)(x) = \frac{4}{\sqrt{2\pi}} \times \frac{1}{(x^2+4)(x^2+1)} = \frac{4}{3\sqrt{2\pi}}\left(\frac{1}{1+x^2} - \frac{1}{x^2+4}\right)$$

$$\mathcal{F}(y)(x) = \frac{4}{3\sqrt{2\pi}}\left[\frac{1}{2}\,\frac{2\times 1}{x^2+1^2} - \frac{1}{4}\,\frac{2\times 2}{x^2+2^2}\right]$$

$$= \frac{2}{3}\left[\frac{1}{\sqrt{2\pi}}\,\frac{2}{1+x^2}\right] - \frac{1}{3}\left(\frac{4}{\sqrt{2\pi}\,(x^2+4)}\right)$$

$$= \frac{2}{3}\,\mathcal{F}(f_1)(x) - \frac{1}{3}\,\mathcal{F}(f_2)(x)$$

$$= \mathcal{F}\left(\frac{2}{3} f_1 - \frac{1}{3} f_2\right)(x)$$

$$y(x) = \frac{2}{3} f_1(x) - \frac{1}{3} f_2(x)$$

$$y_f(x) = \frac{2}{3} e^{-|x|} - \frac{1}{3} e^{-2|x|} \qquad \forall x \in \mathbb{R}.$$

general solution : $y(t) = c_1 e^{x} + c_2 e^{-x} + \frac{2}{3} e^{-|x|} - \frac{1}{3} e^{-2|x|}$

II Exercice 01 :

2) . $\Gamma(t) = e^{-t^2/2}$ , Let $f = \mathcal{F}_1(\Gamma)$

$$f'(x) = \mathcal{F}_1(\Gamma)'(x) = -i\,\mathcal{F}_1(t \longmapsto t\,\Gamma(t))(x).$$

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t e^{-t^2/2} e^{-ixt}\, dt = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2 - ixt}\, dt$$

$$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[(-t - ix) + ix\right] e^{-t^2/2 - ixt}\, dt$$

$$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(e^{-t^2/2 - ixt}\right)' dt - \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2 - ixt}\, dt$$

$$= \frac{i}{\sqrt{2\pi}} \left[\lim_{t \to +\infty} e^{-t^2/2 - ixt} - \lim_{t \to -\infty} e^{-t^2/2 - ixt}\right] - x f(x)$$

$$f'(x) = -x f(x).$$

$$\frac{f'(x)}{f(x)} = -x \Rightarrow \ln|f(x)| = \frac{-x^2}{2} + \lambda$$

$$f(x) = c\, e^{-x^2/2}$$

$$c = f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2}\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

$$f = \Gamma \implies \mathcal{F}_1(\Gamma) = \Gamma$$

3)

$$f_s(t) = \frac{1}{\sqrt{s}} e^{-t^2/2s} = \frac{1}{\sqrt{s}} e^{-\frac{\left(\frac{t}{\sqrt{s}}\right)^2}{2}} = \frac{1}{\sqrt{s}} \Gamma\left(\frac{t}{\sqrt{s}}\right)$$

$$\mathcal{F}_1(\Gamma_s)(x) = \mathcal{F}_1\left(t \longmapsto \frac{1}{\sqrt{s}} \Gamma\left(\frac{t}{\sqrt{s}}\right)\right)(x).$$

$$= \frac{1}{\sqrt{s}} \left(\frac{1}{\frac{1}{\sqrt{s}}} \mathcal{F}_1(\Gamma)\left(\frac{x}{\frac{1}{\sqrt{s}}}\right)\right) = \mathcal{F}_1(\Gamma)(\sqrt{s}\, x)$$

$$= \Gamma\left(\frac{x}{\sqrt{\frac{1}{s}}}\right) = \sqrt{\frac{1}{s}}\, \Gamma_{1/s}(x) = \frac{1}{\sqrt{s}} \Gamma_{1/s}(x).$$

$$\Gamma\left(\frac{t}{\sqrt{s}}\right) = \sqrt{s}\,\Gamma_s(t).$$

ii).

$$\mathcal{F}^2(\Gamma_s) = \mathcal{F}(\mathcal{F}(\Gamma_s)) = \mathcal{F}\left(\frac{1}{\sqrt{s}}\Gamma_{1/s}\right)$$

$$= \frac{1}{\sqrt{s}}\,\mathcal{F}(\Gamma_{1/s}) = \frac{1}{\sqrt{s}}\left[\frac{1}{\sqrt{1/s}}\;\Gamma_{1/s}\right]$$

$$\mathcal{F}^2(\Gamma_s) = \Gamma_s$$

iii).

$$\mathcal{F}^2(f*\Gamma_s)(x) = \mathcal{F}\left(\mathcal{F}(f*\Gamma_s)\right)(x).$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathcal{F}(f*\Gamma_s)(t)\,e^{-ixt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{2\pi}\,\mathcal{F}(f)(t)\,\mathcal{F}(\Gamma_s)(t)\,e^{-ixt}\,dt$$

$$= \int_{-\infty}^{+\infty}\mathcal{F}(f)(t)\,\mathcal{F}(\Gamma_s)\,(t+x)\,dt$$

$$= \int_{-\infty}^{+\infty}f(t)\,\mathcal{F}^2(\Gamma_s)\,(t+x)\,dt$$

$$\mathcal{F}^2(f*\Gamma_s)(x) = \int_{-\infty}^{+\infty}f(t)\,\Gamma_s\,(t+x)\,dt$$

pique du ~~D~~ iraque

$$= \int_{-\infty}^{+\infty}f(t)\,\Gamma_s\,(-x-t)\,dt$$

$$= (f*\Gamma_s)(-x).$$

4)

$$\mathcal{F}^2(f*\gamma_n)\,(x) = (f*\gamma_n)(-x)$$

$$f*\gamma_n \xrightarrow{CV} f$$

$$\lim_{n\to+\infty}\mathcal{F}^2(f*\gamma_{\varphi(n)})(x) = \lim_{n\to+\infty}(f*\gamma_{\varphi(n)})(-x)$$

$$\exists\,\gamma_{\varphi(n)} \qquad f*\gamma_{\varphi(n)} \xrightarrow{CV} f. \quad \text{th R.F}$$

$$\mathcal{F}^2(f)(x) = f(-x) \qquad \forall x\in\mathbb{R}.$$

Serie 01 :

Exercice 01 :

① $\sum_{n\geq n_0} a_n(x-x_0)^n = \sum_{n\geq n_1} a_{n+(n_0-n_1)}(x-x_0)^{n+(n_0-n_1)}$

$n \longleftrightarrow n+p$ such that if $n=n_1$ then $n+p=n_0$

$p = n_0 - n_1$

. In particular: $n_0=0$ and $n_1=1$.

we get $\sum_{n\in\mathbb{N}} a_n(x-x_0)^n = \sum_{n\in\mathbb{N}^*} a_{n-1}(x-x_0)^{n-1}$

• $\sum_{n\in\mathbb{N}} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n\in\mathbb{N}^*} \frac{x^{2(n-1)+1}}{(2(n-1)+1)!} = \sum_{n\in\mathbb{N}^*} \frac{x^{2n-1}}{(2n-1)!}$

• $\sum_{n\in\mathbb{N}} (-1)^n(n+1)x^n = \sum_{n\in\mathbb{N}^*} (-1)^{n-1}((n-1)+1)x^{n-1} = \sum_{n\in\mathbb{N}^*} (-1)^{n+1} n x^{n-1}$

non n fulfills
even non odd

• $\sum_{n\in\mathbb{N}} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n\in\mathbb{N}^*} \frac{(-1)^{n+1}}{2n-1} x^{2n-1}$

②- $R = \sup\{r\geq 0, \sum |a_n| r^n \text{ converge}\}$.

R is independant of $x_0$

: $x_0 = 0$.

* $\sum (\sin n) x^n$ $\qquad a_n = \sin n$

$A = \{r \geq 0, \sum_{n\in\mathbb{N}} |\sin n| r^n \text{ cv}\} \Rightarrow R = \sup A$.

$\forall r<1$ : $\quad |\sin n| r^n \leq r^n$ and $\sum_{n\in\mathbb{N}} r^n$ cv $\Rightarrow \sum |\sin n| r^n$ cv

so $[0,1[ \subset A \Rightarrow \sup([0,1[) = 1 \leq \sup A = R$

$\Rightarrow R \geq 1$.

$R \geq 1$ : $\quad 1 \in A$ but $\sum |\sin n| 1^n = \sum |\sin n|$ div

$$f'(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_n^k \, a_k \, a_{n-k}}{n!} x^n = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\frac{n!}{k!\,(n-k)!}}{n!} a_k \, a_{n-k} \, x^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{a_k}{k!} \cdot \frac{a_{n-k}}{(n-k)!} \right) x^n = \frac{1}{2} f^2(x)$$

$$f'(x) = \frac{1}{2} f^2(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2} \Rightarrow \left[ -\frac{1}{f(x)} \right]' = \frac{1}{2}$$

$$\frac{-1}{f(x)} = \frac{x}{2} + c \Rightarrow c = \frac{-1}{f(0)} = \frac{-1}{a_0} = -1.$$

$$f(x) = \frac{-1}{\frac{x}{2} - 1} = \frac{1}{1 - \frac{x}{2}} = \sum_{n=0}^{+\infty} \left( \frac{x}{2} \right)^n$$

$$\frac{a_n}{n!} = \frac{1}{2^n} \Rightarrow a_n = \frac{n!}{2^n} = \sum_{n=0}^{+\infty} \frac{1}{2^n} x^n \qquad \forall n \in \mathbb{N}$$

$$R = 2$$

(3) -

* $\sum \frac{(-1)^n}{2^n} (x+1)^n$  $x_0 = -1$
$R = 2$

so, this serie cv on $]-3; 1[$.
and div on $]-\infty, -3[ \cup ]1, +\infty[$
If:
$x = -3$ : $\sum_{n\in\mathbb{N}} \frac{(-1)^n}{2^n} (-3+1)^n = \sum_{n\in\mathbb{N}} 1 \Rightarrow$ div.

$x = 1$ : $\sum \frac{(-1)^n}{2^n} \cancel{(1+1)^n} = \sum_{n\in\mathbb{N}} (-1)^n$ div

the interval of convergence is $]-3; 1[$.

* $\sum_{n\in\mathbb{N}} \frac{x^n}{n}$, $x_0 = 0$  $R = 1$

If:
$x = -1$ : $\sum_{n\in\mathbb{N}} \frac{(-1)^n}{n}$ cv by Leibniz criterion.

$x = 1$ : $\sum \frac{1}{n}$ div Reiman serie $(\alpha = 1)$.
so the interval of convergence is $[-1, 1[$.

* $\sum \frac{(-1)^n n!}{3^n} (x-5)^n$  $x_0 = 5$
$R = 0$

the interval of convergence is $\{5\} = [5, 5]$.

Exercice 02:

$$\sum_{n\in\mathbb{N}} \cosh(na)\, x^n = \sum_{n\in\mathbb{N}} \left(\frac{1}{2}\left(e^{na} + e^{-na}\right)\right) x^n$$

$$= \frac{1}{2} \underbrace{\sum_{n\in\mathbb{N}} (x e^{a})^n}_{R_2 = e^{-a}} + \frac{1}{2} \underbrace{\sum_{n\in\mathbb{N}} (x e^{-a})^n}_{R_2 = e^{a}}$$

so $1 \notin A \Rightarrow R \leq 1$ $\Rightarrow$ $R = 1$

* $\sum_{n\in\mathbb{N}} \frac{x^n}{(n+1)^{\alpha+1}3^n}$ we use the Caushy's rule.

$$a_n = \frac{1}{(n+1)^{\alpha+1}3^n} \Rightarrow \sqrt[n]{|a_n|} = \frac{1}{3} \times \frac{1}{(n+1)^{\alpha+1/n}}$$

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \frac{1}{3} \lim e^{-\frac{\alpha+1}{n}\ln(n+1)} = \frac{1}{3}$$

so $R = \frac{1}{\frac{1}{3}} = 3$ $R = 3$

* $\sum_{n\in\mathbb{N}} \arccos\left(1+\frac{1}{n^2}\right)x^n$ arccos : $[-1,1] \longmapsto [0,\pi]$.

$$R = \sup\left\{r \geq 0,\ \sum \arccos\left(1+\frac{1}{n^2}\right) r^n \text{ cv}\right\}.$$

$$\arccos(1-t) \underset{0}{\sim} \sqrt{2}\,t$$

since :

$x = \arccos(1-t)$

$1-t = \cos x \iff t = 1-\cos x \Rightarrow \frac{x^2}{2} = \frac{1}{2}(\arccos(1-t))$

$\sim$ equivalent $\lim_{x\to 0} \frac{1-\cos x}{x^2/2} = 1$

so $\frac{1}{2}(\arccos(1-t))^2 \underset{0}{\sim} t$

then $\arccos(1-t) \underset{0}{\sim} \sqrt{2t}$

$$R = \sup\left\{r \geq 0,\ \sum \sqrt{\frac{2}{n^2}}\, r^n \text{ cv}\right\}$$

$$= \sup\left\{r \geq 0,\ \sqrt{2} \sum \frac{1}{n} r^n \text{ cv}\right\}$$

$R = 1$

* $\sum \frac{n^n}{n!}x^n$ $a_n = \frac{n^n}{n!} > 0 \quad \forall n$

$$R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \left[\frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}}\right]$$

$R = \frac{1}{e}$

$$= \lim_{n\to+\infty} \left[\frac{n^n}{\cancel{n!}} \frac{\cancel{(n+1)}\,\cancel{n!}}{(n+1)^n \cancel{(n+1)}}\right] = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$$I = e - 1 - \sum_{n=0}^{\infty} \int_0^1\int_0^1 \frac{(xy)^n}{n!}\, dx\, dy$$

$$= e - 1 - \sum_{n=0}^{\infty} \int_0^1 x^n\, dx \int_0^1 y^n\, dy = e - 1 - \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{(n+1)} \cdot \frac{1}{(n+1)}$$

$$= e - 1 - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \cdot \frac{1}{(n+1)} = e - 1 - \sum_{n=1}^{+\infty} \frac{1}{n!} \cdot \frac{1}{n}$$

Exercice 06:

$(a_n)_{n\in\mathbb{N}}$ $\quad a_0 = 1 \quad , 2a_{n+1} = \sum_{k=0}^{n} C_n^k\, a_k\, a_{n-k}$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

1) - showing $0 \leq a_n \leq n!$ by induction.

$0 < a_0 = 1 \leq 0! = 1$.

if $0 < a_k < k!$ , $\forall k \in \{0, 1, \ldots, n\}$ then $a_{n+1} \leq (n+1)!$ ?

we have $2a_{n+1} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k a_{n-k}$

$$2a_{n+1} = n! \sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{a_{n-k}}{(n-k)!}$$

we have: $\quad a_k \leq k! \Rightarrow \frac{a_k}{k!} \leq 1$.

$\quad a_{n-k} \leq (n-k)! \Rightarrow \frac{a_{n-k}}{(n-k)!} \leq 1$

$$\sum_{k=0}^{n} \frac{a_k}{k!} \leq \sum_{k=0}^{n} 1 = (n+1)$$

$$2a_{n+1} \leq n!(n+1) = (n+1)! \Rightarrow a_{n+1} \leq \frac{(n+1)!}{2} \leq (n+1)!$$

$$\frac{a_n}{n!} \leq 1 \Rightarrow \left|\frac{a_n}{n!} x^n\right| \leq |x|^n$$

if $\sum |x|^n$ CV $\Rightarrow \sum \left|\frac{a_n}{n!} x^n\right|$ CV $\Rightarrow R \geq 1$.

2) - $f'(x) = \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right)' = \sum_{n=1}^{\infty} \frac{a_n}{n!} n x^{n-1}$

If $R_1 \neq R_2$ $(a \neq 0)$

then $R = \text{Min}(R_1, R_2) = \text{Min}(e^{a}, e^{-a}) = \begin{cases} e^{-a} & \text{if } a \geq 0 \\ e^{a} & \text{if } a \leq 0 \end{cases}$

when $a = 0$ $\cosh(0) = 1 \Rightarrow R = 1$.

. Let $f : x \longmapsto \sum_{n=0}^{+\infty} \cosh(na) x^n$

$$f(x) = \frac{1}{2} \sum_{n=0}^{+\infty} (x e^{a})^n + \frac{1}{2} \sum_{n=0}^{+\infty} (x e^{-a})^n$$

$$= \frac{1}{2} \frac{1}{1 - x e^{a}} + \frac{1}{2} \cdot \frac{1}{1 - x e^{-a}}$$

$$= \frac{1}{2} \left( \frac{1}{1 - x e^{a}} + \frac{1}{1 - x e^{-a}} \right) = \frac{2 - x(e^{a} + e^{-a})}{2(1 + x^2 - x(e^{a} + e^{-a}))}$$

$$= \frac{2 - 2x \operatorname{ch} a}{2(1 + x^2 - 2x \operatorname{ch} a)} = \frac{1 - x \operatorname{ch} a}{1 + x^2 - 2x \operatorname{ch} a}$$

. In particular, if $a = 0$:

$$f(x) = \frac{1 - x}{x^2 + 1 - 2x} = \frac{1 - x}{(1 - x)^2} = \frac{1}{1 - x} = \sum_{n=0}^{+\infty} x^n$$

$$f_3 : x \longmapsto \arctan\left(\frac{1-x^2}{1+x^2}\right)$$

$$t = x^2 \quad \Rightarrow \quad f_3(x) = \arctan\left(\frac{1-t}{1+t}\right) = h(t)$$

$$h'(t) = \frac{\left(\frac{1-t}{1+t}\right)'}{1+\left(\frac{1-t}{1+t}\right)^2} = \frac{\frac{(-1)(1+t)-(1-t)}{(1+t)^2}}{\frac{(1+t)^2+(1-t)^2}{(1+t)^2}} = \frac{-1-\cancel{t}-1+\cancel{t}}{1+\cancel{2t}+t^2+1-\cancel{2t}+t^2} = \frac{-2}{2+2t^2}$$

$$h'(t) = \frac{1}{1-(-t^2)} = -\sum_{n=0}^{\infty} (-t)^{2n} = \sum_{n=0}^{\infty} (-1)^{n+1} t^{2n}$$

$$h(t) = h(0) + \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{2n+1} t^{2n+1}$$

$$h(t) = \frac{\pi}{4} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} t^{2n+1}$$

$$f_3(x) = \frac{\pi}{4} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{2(2n+1)} = h(x^2)$$

$$f_3(x) = \frac{\pi}{4} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{4n+2}$$

Exercice 04:

$$f_1(x) = \cos x \qquad x_0 = \frac{\pi}{4}$$

$$t = x - \frac{\pi}{4} \qquad x = t + \frac{\pi}{4}$$

$$f_1(x) = h\left(t+\frac{\pi}{4}\right) = \cos t \cos\left(\frac{\pi}{4}\right) - \sin(t)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}(\cos t - \sin t) = \frac{\sqrt{2}}{2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} t^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}\right)$$

$$= \frac{\sqrt{2}}{2}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} t^{2n} - \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}\right)$$

$$\sum_{n=0}^{+\infty} a_n t^n \quad \text{where} \quad \begin{cases} a_{2n} = \dfrac{\sqrt{2}}{2} \dfrac{(-1)^n}{(2n)!} = \dfrac{\sqrt{2}}{2} \dfrac{(-1)^{\frac{2n}{2}}}{(2n)!} \quad (-1)^{2n} \\[2ex] a_{2n+1} = \dfrac{\sqrt{2}}{2} (-1) \dfrac{(-1)(-1)^n}{(2n+1)!} = \dfrac{\sqrt{2}}{2} \dfrac{(-1)^{2n+1} (-1)^{E\left(\frac{2n+1}{2}\right)}}{(2n+1)!} \end{cases}$$

$$a_n = \frac{\sqrt{2}}{2} \frac{(-1)^{n+E\left(\frac{n}{2}\right)}}{n!}$$

$$f(x) = \sum_{n=0}^{+\infty} \frac{\sqrt{2}}{2} \frac{(-1)^{n+E\left(\frac{n}{2}\right)}}{n!} \left(x - \frac{\pi}{4}\right)^n$$

$f_2(x) = e^x$ $\quad x_0 = 1$ $\qquad f^n(x) = e^x$

$\qquad f^n(1) = e$

$$= \sum_{n=0}^{+\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \sum_{n=0}^{+\infty} \frac{e}{n!}(x-1)^n$$

$f_3(x) = \sqrt{x}$ $\quad x_0 = 4.$

$t = x-4 \Rightarrow x = t+4.$

$$f_3(x) = f_3(t+4) = \sqrt{t+4} = (t+4)^{1/2} = 2\left(\frac{t}{4}+1\right)^{1/2}$$

$$\alpha = \frac{1}{2} \Rightarrow \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{\frac{1}{2}\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$

$$= \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\cdots\left(\frac{3}{2}-n\right)}{n!} = \frac{(-1)^{n+1}\times 1\times 3\times 5\times 7\cdots(2n-3)}{2^n\, n!} = \frac{(-1)^{n+1}(2n-2)!}{n!\times 2^n\times 4\times 6\times\cdots\times(2n-2)}$$

$$f_3(x) = 2 + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(2n-2)!}{((n-1)!)^2}(x-4)^n$$

Exercice 05 :

1) Solving : $\sum_{n=0}^{\infty}(3n+1)^2 x^n = 0$

We have : $\sum_{n=0}^{+\infty}(3n+1)^2 x^n = \sum_{n=0}^{+\infty}(9n^2+6n+1)x^n$

$$9\sum n^2x^n + \sum x^n + 6\sum n x^n$$

$$9\left(\frac{x(1+x)}{(1-x)^3}\right) + \frac{1}{1-x} + 6\left(\frac{x}{(1-x)^2}\right) = \frac{9(1-x)+(1-x)^2+6x(1-x)}{(1-x)^3}$$

$$= \frac{9-9x+1-2x+x^2+6x-6x^2}{(1-x)^3} = \frac{-5x^2-5x+10}{(1-x)^3} = \frac{-5(x^2+x+10)}{(1-x)^3}$$

$$\sum_{n=0}^{\infty}(3n+1)^2x^n = 0 \Rightarrow x^2+x+10 = 0 \Rightarrow \{\, x = -2,$$

$$9n^2+6n+1 = A(n+2)(n+1)+B(n+1)+C$$

$C = 9n^2+6n+1\big|_{n=-1} = 4$ $\qquad -B+4 = 25 \Rightarrow B = -21$

$\qquad A = 9$

$$\sum_{n=0}^{+\infty} (3n+1)^2 x^n = 0 \iff 9\sum_{n=0}^{+\infty} (n+2)(n+1)x^n - 21\sum_{n=0}^{+\infty}(n+1)x^n$$
$$+4\sum_{n=0}^{+\infty} x^n = 0$$

$$\iff \frac{9x^2}{(\cancel{1}-x)^3} - \frac{21}{(1-x)^2} + \frac{4}{1-x} = 0$$

$$\iff \frac{18 - 21(1-x) + 4(1-x)^2}{(1-x)^3} = 0$$

$$\iff \begin{cases} 18 - 21 + 21x + 4 + 4x^2 - 8x = 0 \\ -1 < x < 1 \text{ because } \underline{\underline{R=1}} \end{cases}$$

$$\iff \begin{cases} 1 + 13x + 4x^2 = 0 \\ -1 < x < 1 \end{cases}$$

$\Delta = 169 - 16 = 153$

$x = \frac{-13 - \sqrt{153}}{8}$ X

$x = \frac{-13 + \sqrt{153}}{8}$ ✓

the unique root of this equation is $\left\{\frac{-13+\sqrt{153}}{8}\right\}$.

2)- ~~Exercice~~

$$I = \int_0^1\int_0^1 xye^{xy}\,dx\,dy = \int_0^1 x\underbrace{\left[\int_0^1 ye^{xy}\,dy\right]}_{u(x)} dx.$$

$$= \int_0^1 x\,h(x)\,dx, \text{ where } h(x) = \int_0^1 ye^{xy}\,dy = H'(x).$$

$$h(x) = \int_0^1 \left[e^{xy}\right]' dy =$$

$$H(x) = \int_0^1 e^{xy}\,dy$$

$$I = \int_0^1 x\,H'(x)\,dx = \left[xH(x)\right]_0^1 - \int_0^1 H(x)$$

$$= H(1) - \int_0^1 H(x)\,dx = \int_0^1 e^y\,dy - \int_0^1\int_0^1 e^{xy}\,dx\,dy$$

$$= e - 1 - \int_0^1\int_0^1\left[\sum_{n=0}^{+\infty}\frac{(xy)^n}{n!}\right]dx\,dy$$

then $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1} \frac{\cos(n.0)}{n^2} = f(0) = \frac{\pi}{6}$

$\ln(1-x) = -\sum \frac{x^n}{n}$

Power series:

$\ln(1+x) = \sum \frac{(-1)^{n+1} x^n}{n}$

Exercice 03:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-x} = \sum_{n=0} x^n$$

$$\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0} (n+1)\, x^n$$

$$e^x = \sum_{n=0} \frac{x^n}{n!} \;/\; \cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$(1+x)^\alpha = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$$

$$f_1 : x \longmapsto \frac{1}{(3+x^2)^2}$$

$$f_1(x) = \frac{1}{9\left(1+\frac{x}{3}\right)^2} = \frac{1}{9} \frac{1}{\left(1-\left(\frac{-x}{3}\right)\right)^2}$$

$$f_1(x) = \frac{1}{9} \sum_{n=0}^{\infty} (n+1)\left(\frac{-x}{3}\right)^n = \frac{1}{9} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{3^n} x^n$$

$$f_2 : x \longmapsto x \ln\left(x + \sqrt{x^2+1}\right)$$

$f_2(x) = x\, g(x)$ such that $g(x) = \ln\left(x+\sqrt{x^2+1}\right)$

$$g'(x) = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{x+\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)} = \frac{1}{\sqrt{1+x^2}}$$

$$g'(x) = \frac{1}{(1+x^2)^{1/2}} = (1+x^2)^{-1/2},$$

$\alpha = -\frac{1}{2}$:

$$\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} = \frac{-\frac{1}{2}\left(\frac{-1}{2}-1\right)\ldots\left(\frac{-1}{2}-n+1\right)}{n!}$$

$$= \frac{-\frac{1}{2}\left(\frac{-3}{2}\right)\ldots\left(\frac{1-2n}{2}\right)}{n!} = \frac{(-1)^n . 1.3.5.7 \ldots (2n-1)}{n!} \times \frac{1}{2^n}$$

$$= \frac{(-1)^n \times 1\times3\times5\times\ldots\times(2n-1)}{2^n\, n!} \times \frac{2\times4\times6\times\ldots\times(2n)}{2\times4\times6\times\ldots\times(2n)}$$

$$= \frac{(-1)^n (2n)!}{4^n . (n!)^2}$$

$$g(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} . \frac{x^{2n+1}}{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+2}$$