Power Series.

Exercise 1:

Determine the radius of convergence for the following power series:

$$1)\ \sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}+(-1)^n}\, x^n\ , \qquad 2)\ \sum_{n\geq 1} n^{(-1)^n} x^n\ , \qquad 3)\ \sum_{n\in\mathbb{N}} \frac{n!}{\prod_{k=0}^{n}(2k+1)}\, x^{2n+1}, \qquad (\star)$$

$$4)\ \sum_{n\geq 1} 2^n z^{n!}\ , \qquad 5)\ \sum_{n\in\mathbb{N}} \sqrt[n+1]{n+1}-\sqrt[n]{n}\ , \qquad 6)\ \sum_{n\geq 1} \frac{n^{3n}}{(3n)!}\, x^{3n}, \qquad (\star\star)$$

$$7)\ \sum_{n\geq 1} \mathrm{Argcosh}\left(1+\frac{1}{n}+\frac{1}{n^2}\right) x^n\ , \quad 8)\ \sum_{n\in\mathbb{N}} \sinh\left(\frac{\pi}{3}-\arctan\left(\frac{n\sqrt{3}}{n+1}\right)\right) z^n\ , \quad 9)\ \sum_{n\geq 0} \left(\frac{2n-1}{n+1}\right)^{2n} x^n, \quad (\star\star\star)$$

$$10)\ \sum_{n\geq 0} \frac{(2n)!}{n!n^n} x^n\ , \qquad 11)\ \sum_{n\geq 0} \left(\log\left(1+\frac{1}{2^n}\right)\right) x^n\ , \qquad 12)\ \sum_{n\in\mathbb{N}} \left(\frac{4}{2^{n+1}}-\frac{4}{3^{n+1}}\right) x^n, \qquad (\star\star\star)$$

$$13)\ \sum_{n\in\mathbb{N}} (n^2+n+1)x^n\ , \qquad 14)\ \sum_{n\in\mathbb{N}} (n^2+1)2^{n+1}x^n, \qquad (\star\star).$$

Exercise 2:

I. For the following series, verify its convergence and give the radius of convergence and the expression of the sum in terms of usual functions.

$$\sum_{n\geq 1} \cos(nx)x^n\ , \qquad \sum_{n\in\mathbb{N}} \frac{n(-1)^n}{(2n)!}x^n \qquad \sum_{n\in\mathbb{N}} \frac{\sin \alpha n}{n!}x^n\ , \qquad \sum_{n\in\mathbb{N}} \frac{n^3}{n!}x^n, \qquad (\star\star)$$

$$\sum_{n\in\mathbb{N}} (-1)^{n+1} n x^{2n+1}\ , \qquad \sum_{n\in\mathbb{N}} [3+(-1)^n]^n z^n\ , \qquad \sum_{n\in\mathbb{N}} (n^2-n-3)3^{n-1}z^{2n}, \qquad (\star\star\star)$$

$$\sum_{n\in\mathbb{N}} \frac{x^n}{3^{n+1}}\ , \qquad \sum_{n\in\mathbb{N}} (-1)^n(n+1)x^n\ , \qquad \sum_{n\in\mathbb{N}} (-1)^n(n+1)(n+2)x^n, \qquad (\star\star).$$

II. 1. Compute $\displaystyle\sum_{n\in\mathbb{N}} \frac{x^n}{4n^2-1}$, $\forall |x|<1$, then deduce the sums $\displaystyle\sum_{n\in\mathbb{N}} \frac{1}{4n^2-1}$, and $\displaystyle\sum_{n\in\mathbb{N}} \frac{(-1)^n}{4n^2-1}$. $(\star)$

2. a) Let, for all $x \in \mathbb{R}$, $S(x) = \sum_{n\geq 1} \frac{x^3}{(3n)!}$. Find the radius of convergence R of $S(x)$.

   b) Show that $S''(x) + S'(x) + S(x) = e^x$.

   c) Determine the sum S.

3. We let
$$f(x) = \sum_{n\geq 1} \frac{x^{2n+2}}{n(n+1)(2n+1)}, \ \forall x \in \mathbb{R}.$$
   Find its definition domain and express f using usual functions.

4. Let $S(x) = \sum_{n\geq 1} \frac{(-1)^n x^{2n} e^{-nx}}{n}$.

   a) Show that S is continuous over $[0\ ;+\infty[$.

   b) Show that S is differentiable over $[0\ ;+\infty[$.

   c) Compute $S'(x)$ over $[0\ ;+\infty[$.

   d) Deduce $S(x)$.

5. We define the sequences (a_n) and b_n, with $a_0 = 1$, $2a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k\ a_{n-k}$ and $b_n = \frac{a_n}{n!}$.

   a) Show that $|a_n| \leq \frac{n!}{2^n}$. Deduce that the radius of convergence of $b_n x^n$ is not null.

   b) Compute the derivative of $\sum_{n\in\mathbb{N}} b_n x^n$ and express it using it's antiderivative.

   c) Deduce the sum.

6. We consider the complex sequence $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$, with $a_0 = 1$, $a_1 = 3$, and $\forall n \geq 2\ :\ 3a_{n-1} - 2a_{n-2} = a_n$. (⋆⋆⋆)

   a) Show that the radius of convergence R of this sequence is greater than $\frac{1}{4}$.

   b) Show that $\forall z \in \mathbb{C}$ such that $|z| < R$:
$$f(z) = \frac{1}{2z^2 - 3z + 1}$$

   c) Deduce the value of R and the expression of a_n in function of n.

7. Let the power serie : $S_1(x) = \sum_{n\geq 1} (n-1)x^n$.

   a) Compute the radius of convergence R.

   b) Compute the sum of the serie for all $x < R$.

   c) Deduce the radius of convergence of $S_2(x) = \sum_{n\geq 1} (n^2 - n)x^{n-1}$.

   d) Deduce $\sum_{n\geq 1} \frac{n-1}{2^n}$, $\sum_{n\geq 1} \frac{n^2-n}{3^n}$.

Exercise 3: $(\star\star\star)$

Show the following.

$$\int_0^1 \frac{\ln x}{1+x^2}\,dx = \sum_{n\in\mathbb{N}} \frac{(-1)^{n-1}}{(2n+1)^2}.$$

$$\forall |x|<1, \int_0^{\frac{\pi}{2}} \frac{\ln(1+x\sin^2 t)}{\sin^2 t}\,dt = \pi(\sqrt{1+x}-1).$$

$$\ln(5-x) = \ln 5 - \sum_{n\geq 1} \frac{x^n}{n5^n}\ , \quad R=5.$$

$$\frac{1}{(1+6x)^2} = \sum_{n\in\mathbb{N}} (n+1)6^n x^n.$$

Exercise 4:

Give the development of the series bellow:

$$\frac{2x+4}{(x-2)(x-3)}\ , \quad (\sin x \sinh x)^2\ , \quad (4+x^2)^{-3/2}\ , \quad \ln(x^2-5x+6)\ , \quad \ln(1-4x^2)\ , \quad \int_0^x e^{-t^2}\,dt$$

$$\boldsymbol{e^{ax}} = \sum_{n=0}^{\infty} \frac{a^n}{n!}\ x^n \qquad a \in \mathbb{C}\,,\ x \in \mathbb{R}$$

$$\mathbf{sh}\ \boldsymbol{x} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\ x^{2n+1} \qquad x \in \mathbb{R}$$

$$\mathbf{ch}\ \boldsymbol{x} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\ x^{2n} \qquad x \in \mathbb{R}$$

$$\mathbf{sin}\,\boldsymbol{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\ x^{2n+1} \qquad x \in \mathbb{R}$$

$$\mathbf{cos}\,\boldsymbol{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\ x^{2n} \qquad x \in \mathbb{R}$$

$$\boldsymbol{(1+x)^{\alpha}} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\ x^n \qquad (\alpha \in \mathbb{R}) \qquad x \in \,]-1\,;1\,[$$

$$\boldsymbol{\frac{1}{a-x}} = \sum_{n=0}^{\infty} \frac{1}{a^{n+1}}\ x^n \qquad (a \in \mathbb{C}^*) \qquad x \in \,]-|a|\,;|a|\,[$$

$$\boldsymbol{\frac{1}{(a-x)^2}} = \sum_{n=0}^{\infty} \frac{n+1}{a^{n+2}}\ x^n \qquad (a \in \mathbb{C}^*) \qquad x \in \,]-|a|\,;|a|\,[$$

$$\boldsymbol{\frac{1}{(a-x)^k}} = \sum_{n=0}^{\infty} \frac{\mathrm{C}_{n+k-1}^{k-1}}{a^{n+k}}\ x^n \qquad (a \in \mathbb{C}^*) \qquad x \in \,]-|a|\,;|a|\,[$$

$$\mathbf{ln}\boldsymbol{(1-x)} = -\sum_{n=1}^{\infty} \frac{1}{n}\ x^n \qquad x \in [-1\,;1\,[$$

$$\mathbf{ln}\boldsymbol{(1+x)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\ x^n \qquad x \in \,]-1\,;1\,]$$

$$\boldsymbol{\sqrt{1+x}} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \times 3 \times \cdots \times (2n-3)}{2 \times 4 \times \cdots \times (2n)}\ x^n \qquad x \in \,]-1\,;1\,[$$

$$\boldsymbol{\frac{1}{\sqrt{1+x}}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}\ x^n \qquad x \in \,]-1\,;1\,[$$

$$\mathbf{Arctan}\ \boldsymbol{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\ x^{2n+1} \qquad x \in [-1\,;1\,]$$

$$\mathbf{Argth}\ \boldsymbol{x} = \sum_{n=0}^{\infty} \frac{1}{2n+1}\ x^{2n+1} \qquad x \in \,]-1\,;1\,[$$

$$\mathbf{Arcsin}\ \boldsymbol{x} = x + \sum_{n=1}^{\infty} \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}\ \frac{x^{2n+1}}{2n+1} \qquad x \in \,]-1\,;1\,[$$

$$\mathbf{Argsh}\ \boldsymbol{x} = x + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}\ \frac{x^{2n+1}}{2n+1} \qquad x \in \,]-1\,;1\,[$$