Multivariable Functions' Worksheet

Exercise 1

Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}^*$, and let $f_{m,\alpha} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined as: $f_{m,\alpha}(x,y) = \begin{cases} \dfrac{e^{m(x^2+y^2)}-1}{x^2+y^2} & : (x,y) \neq (0,0). \\ \alpha & : (x,y) = (0,0). \end{cases}$

Part 1: Study of $f_{m,\alpha}$ over $\mathbb{R}^2 - \{(0,0)\},\ \forall m \in \mathbb{N}^*$.

1. Show that $f_{m,\alpha}$ is continuous.
2. Compute the gradient of $f_{m,\alpha}$ over it, and show that $f_{m,\alpha}$ is of class C^1 in that domain.
3. What can we deduce?

Part 2: Study of $f_{m,\alpha}$ at $(0,0)$.

When $m = 1$:

1. Compute the limit of $f_{1,\alpha}$ at (0, 0) and deduce that there exists a value, α_0, for which $f_{1,\alpha}$ is continuous at (0, 0).
2. Show that $f_{1,\alpha}$ admits a partial derivative at (0, 0) if and only if $\alpha = \alpha_0$, and calculate the gradient at (0, 0).
3. Using the limit definition, prove that f_{1,α_0} is differentiable at (0, 0).
4. Show that f_{1,α_0} is of a class C^1 at (0, 0).

When $m > 1$:

1. For what values of m is the function $f_{m,\alpha}$ continuos at (0, 0).
2. Compute the gradient of $f_{m,\alpha}$ at that point.
3. For what values of m is the function $f_{m,\alpha}$ differentiable at (0, 0).
4. For what values of m is the function $f_{m,\alpha}$ class C at (0, 0).

---

(1)

Exercise 2

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, such that: $f(x,y) = \begin{cases} 0 & : \text{ if } y = 0. \\ y^2 \sin \dfrac{x}{y} & : \text{ if } y \neq 0. \end{cases}$

1. Study the continuity of f.
2. Study the existence of the value of partial derivative at 1 and 2.
3. For each of the following functions, defined on a subset of $\mathbb{R}^2$, check if they admit an extension by continuity near their domain borders:

$$g_1(x,y) = \frac{y}{x^3} e^{-\dfrac{|y|}{x^3}} \quad , \quad g_2(x,y) = (x-5y)\sin\left(\frac{x^2}{x^2+y^2}\right).$$

---

(2)