Eigenvalues and Eigenvectors

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Definiton

Definition

Let A be a square matrix of order n. $\lambda \in \mathbb{R}$ is called an eigenvalue of A, if there exists a nonzero vector X in $\mathbf{R}^n$ such that: $AX = \lambda X$.
The vector X is then called an eigenvector corresponding to λ.

Remark

An eigenvalue can have *many* eigenvectors. But every eigenvector has **only** one eigenvalue.

Matrix and Vector Multiplication

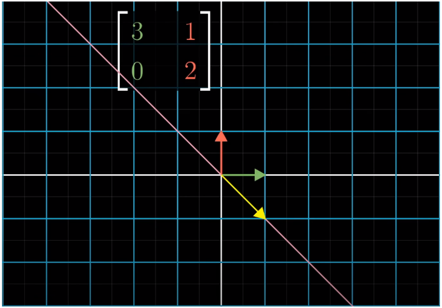
Example

Let $\mathrm{M} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and let $\mathrm{A} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

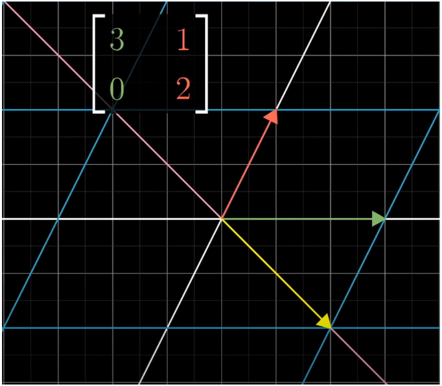
We have:

$$\mathrm{MA} = 1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\mathrm{A}$$

Therefore, 2 is an eigenvalue of M and the vector A is an eigenvector corresponding to 2.



(a) Before Multiplying



(b) After Multiplying

Notice that the unit vectors i and j, after multiplying by the Matrix M, are exactly where the columns of M tell us; $\mathrm{M} \cdot \mathrm{i} = (3, 0), \mathrm{M} \cdot \mathrm{j} = (1, 2)$. So we get the following result:

Multiplication

Let $\mathrm{M} = \begin{bmatrix} \mathrm{a} & \alpha \\ \mathrm{b} & \beta \end{bmatrix}$ and $\mathrm{V} = \begin{bmatrix} x \\ y \end{bmatrix}$ Then we have:

$$\mathrm{MV} = \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix} x + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} y = \begin{bmatrix} \mathrm{a}x + \alpha y \\ \mathrm{b}x + \beta y \end{bmatrix}$$

This can be proven using the fact that the 1st column of M is the cordinates of the unit vector i after the transformation, and the 2nd column is the cordinates of the unit vector j after the transformation. Therefore, we get:

$$\mathrm{MV} = \mathrm{M}\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x \cdot \mathrm{M} \cdot \mathrm{i} + y \cdot \mathrm{M} \cdot \mathrm{j}$$

Linear Maps

Property

If f is a linear map, then there exists a Matrix M such that $\forall \mathrm{X} \in \mathbb{D}_f,\ \mathrm{f}(\mathrm{X}) = \mathrm{MX}$.

With this new knowledge, we can look at linear maps a bit differently. In essence, Linear maps use Matrices to describe a linear transformation i.e. the origin doesn't move and stright, parallel lines remain stright and parallel.

It's worth mentioning that every linear map is associated with a Matrix, and every Matrix can be associated to a linear transformation.

Example

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathrm{B} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

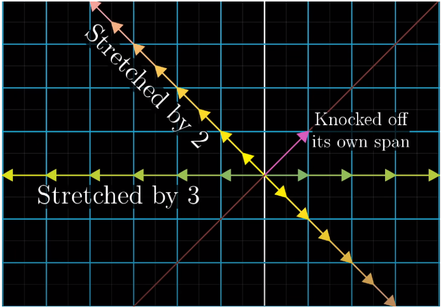
The Matrix A represents projection on the x-y plane, while the Matrix B preforms a rotation by θ degrees.

One can easily verify that A and B meet the conditions to preform Linear Maps.

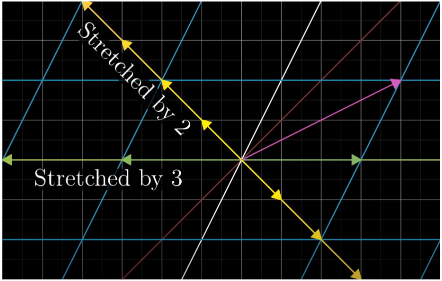
Eigenvectors' and Eigenvalues' visual Charactiristics.

If we knew that multiplying a Matrix with a vector is but a linear transformation, the meaning behind eigenvectors' definition becomes clear: they are the answer to the question 'What are the vectors that do not leave their span?'

And eigenvalues is just the value by which they are multiplied.



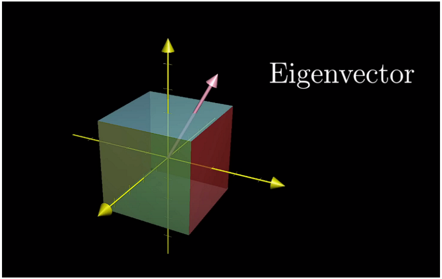
(a) Before Multiplying



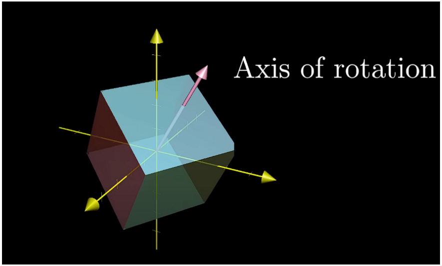
(b) After Multiplying

We can observe that

- The yellow vector and the green vector stayed on line, and they both got stretched by 2 and 3 respectively.
- The violet vector left its span.



(a) Before Multiplying



(b) After Multiplying

In 3D, finding an eigenvector has the same sense of finding the axis of rotation.

Characteristic Equation

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector X. Thus $\mathrm{AX} = \lambda\mathrm{X}$. This equation may be written as:

$$\begin{aligned}(\mathrm{AX} = \lambda\mathrm{X}) &\Leftrightarrow (\mathrm{AX} - \lambda\mathrm{X} = 0_{\mathrm{X}}) \\ &\Leftrightarrow ([\mathrm{A} - \lambda\mathrm{I_n}]\mathrm{X} = 0_{\mathrm{X}})\end{aligned}$$

If $[\mathrm{A} - \lambda\mathrm{I_n}]$ is inversible, then $\mathrm{X} = 0_{\mathrm{X}}$. But we are not interested in this case, which means that $[\mathrm{A} - \lambda\mathrm{I_n}]$ is not inversible. So, $\mathbf{det}(\mathrm{A} - \lambda\mathrm{I_n}) = |\mathrm{A} - \lambda\mathrm{I_n}| = 0$.

Therefore, we arrive at the following result:

Remark

Expending $|\mathrm{A} - \lambda \mathrm{I_n}|$, we get the characteristic polynomial of A in λ. Solving the characteristic equation (polynomial $= 0$) will give us the values of eigenvalues.

Find the eigenvalues and eigenvectors of $\mathrm{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

Solution:

$$(\mathrm{A}-\lambda \mathrm{I}_3) \Leftrightarrow \begin{bmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{bmatrix}$$

$|\mathrm{A}-\lambda \mathrm{I}_3| = (10-\lambda)(\lambda-1)^2 \implies \lambda \in \{1, 10\}$ The corresponding eigenvectors are found by using three values (1 is a double solution) of λ in the equation $[\mathrm{A}-\lambda \mathrm{I}_3]\mathrm{X} = 0_{\mathrm{X}}$

Definition

The set of all eigenvectors corresponding to λ, together with the zero vector of A, is a subspace of $\mathbf{R}^n$.
This subspace is called the eigenspace of λ.

Let M be a square Matrix of order n.

- The product of $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots = |\mathrm{M}|$.
- λ is an eigenvalue of $\mathrm{M} \implies \lambda^{-1}$ is an eigenvalue of M^{-1}.
- λ is an eigenvalue of $\mathrm{M} \implies \lambda^n$ is an eigenvalue of $\mathrm{M}^{\mathrm{n}}, n \in \mathbb{N}$.

Diagonalization

Definition

Let A and B be square matrices of the same size. B is said to be similar to A if there exists an invertible matrix C such that $\mathrm{B} = \mathrm{C}^{-1}\mathrm{AC}$. The transformation of the matrix A into the matrix B in this manner is called a similarity transformation.

Let $\mathrm{A}, \mathrm{B}, \mathrm{C}$ be square matrices of the same size with C invertible. Calculate $\mathrm{A} = \mathrm{C}^{-1}\mathrm{BC}$.

$$\mathrm{B} = \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix}, \ \mathrm{C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$
\begin{aligned}
\mathrm{C}^{-1}\mathrm{BC} &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -10 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Remark

Similar matrices have the same eigenvalues.

Definition

A square matrix A is said to be diagonalizable if there exists an invertible matrix C such that $\mathrm{D} = \mathrm{C}^{-1}\mathrm{AC}$ is a diagonal matrix.

Theorem

Let A be an $n \times n$ Matrix.
A is diagonalizable if and only if it has n linearly independent eigenvectors. Furthermor, The Matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $\mathrm{C}^{-1}\mathrm{AC}$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A in the **same order**.

Example

Show that $\mathrm{A} = \begin{bmatrix} -4 & 6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable.

Find a diagonal matrix B that is similar to A.
Determine the similarity transformation that diagonalizes the Matrix A.

The eigenvalues and corresponding eigenvectors of this matrix can be easily calculated using the characteristic polynomial. They are:

$$\lambda_1 = 2 \ \text{ with } \mathrm{V}_1 = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \forall a \in \mathbb{R}$$

$$\lambda_2 = -1 \ \text{ with } \mathrm{V}_2 = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ \forall b \in \mathbb{R}$$

Solution:

1. The vectors V_1 and V_2 are linearly independent. So A is diagonalizable.
2. A similar to the diagonal Matrix B, which has emelents $\lambda_1 = 2, \ \lambda_2 = -1$. Thus,

$$\mathrm{A} = \begin{bmatrix} -4 & 6 \\ 3 & 5 \end{bmatrix} \text{ is similar to } \mathrm{B} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

3. Let $\mathrm{C} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, we have:

$$\mathrm{C}^{-1}\mathrm{AC} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathrm{B}$$

Theorem

If a $n \times n$ Matrix A has n eigenvalues, it's diagonalizable. If A has less that n eigenvalues i.e. the roots of the characteristic polynomial are not simple, we have to check if the set of all eigenvectors of A form a basis i.e. the vectors are independent.

Let's take our previous example. $\mathrm{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

With eigenvalues $\lambda \in \{1, 10\}$.

- Let $\lambda = 10$. We get:

$$(A - 10I_3)V = 0_V \Longrightarrow \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0_V$$

  We find $V = \{\alpha\ (2, 2, 1), \forall\alpha \in \mathbb{R}.\}$

- Let $\lambda = 1$. We have:

$$(A - I_3)V = 0_V \Longrightarrow \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0_V$$

  We find:
  $V = \{\alpha\ (-1, 1, 0) + \beta\ (-1, 0, 0), \forall\alpha, \beta \in \mathbb{R}.\}$

Proposition

$\mathrm{A} = \mathrm{C}^{-1}\mathrm{BC}$, if and only if, $\mathrm{CAC}^{-1} = \mathrm{B}$.
Meaning (A is similar to B) $\iff$ (B is similar to A).

Let $\mathrm{A} = \mathrm{C}^{-1}\mathrm{BC}$

$$\begin{aligned}(\mathrm{A} = \mathrm{C}^{-1}\mathrm{BC}) &\Leftrightarrow (\mathrm{CA} = \mathrm{BC}) \\ &\Leftrightarrow (\mathrm{CAC}^{-1} = \mathrm{B})\end{aligned}$$

We use this method when calculating the Matrix CAC^{-1} is relatively easy, even thought this is the main goal of this Chapter.

The Importance of Diagonal Matrices

$\mathrm{A} = \mathrm{C}^{-1}\mathrm{BC} \implies \mathrm{A}^{\mathrm{k}} = \mathrm{C}^{-1}\mathrm{B}^{\mathrm{k}}\mathrm{C}, \ \forall k \in \mathbb{N}$

If A is similar to a diagonal matrix D under the transformation $\mathrm{C^{-1}AC}$, then it can be shown that

$$\mathrm{A^k = CD^kC^{-1}}$$

This result can be used to compute $\mathrm{A^k}$. Let us derive this result. We have:

$$\mathrm{A^k = CDC^{-1} \times CDC^{-1} \times \cdots \times CDC^{-1} = CD^kC^{-1}}$$

Proposition

Let A be an $n \times n$ symmetric matrix.
(a) All the eigenvalues of A are real numbers.
(b) The dimension of an eigenspace of A is the multiplicity of the eigenvalues as a root of the characteristic equation.
(c) A has n linearly independent eigenvectors.