Introduction To Integrals.

Exercise 1

Let f be a continuous function on $I=[a,b]$ and let F be defined as $F(x)=\displaystyle\int_a^x f(y)\ dy, \forall x\in I$.

1. Prove that $F'(x)=f(x)$.
2. Let G be an arbitrary antiderivative of f in I. Show that $F(x)=G(x)-G(a)$.
3. Show that: F is even $\implies$ f is odd, and f is odd $\implies$ f is even.
4. Prove that if f is k-periodic, then $F(x)=\displaystyle\int_{a+k}^{x+k} f(y)\ dy$, and $\forall N\in\mathbb{N}$ we have $\displaystyle\int_0^{Nk} f(x)\ dx = N\int_0^k f(x)\ dx$
5. Prove that $\displaystyle\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\ dx=\frac{b-a}{2}$.
6. Consider the following function $\Gamma(x)=\displaystyle\int_0^\infty e^{-y}\ y^{x-1}\ dy$. Show that $\Gamma(x)=(x-1)\ \Gamma(x-1), \forall x\in\mathbb{R}^+$.
7. Let $x,y\in\mathbb{R}^+$. Prove that $\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\ dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1\frac{t^{x-1}}{(1+t)^{x+y}}\ dt=2\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\ d\theta$.

Exercise 2

Calculate the following limits:

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{2}{n}\left[\left(1+\frac{4i}{n}\right)^2-1\right] \qquad \lim_{n\to\infty}\sum_{k=1}^n\sqrt{\frac{k}{n^3}} \qquad \lim_{n\to\infty}\frac{\pi}{2}\sum_{k=1}^n\frac{\ln(\sin(\frac{k}{n}))}{n} \qquad \lim_{n\to\infty}\sum_{k=1}^n\frac{\log(n/k)}{n}$$

Exercise 3

Calculate the following integrals:

1) $\displaystyle\int\frac{1}{1+e^x}\ dx$ 2) $\displaystyle\int_0^1\frac{x}{x^3+x^2+x+1}\ dx$ 3) $\displaystyle\int\frac{\sin^2(x)+\sin(x)\cos(x)}{\cos^2(x)}\ dx$ 4) $\displaystyle\int\frac{\sin(x)}{\cos(x)+\sin(x)}\ dx$

5) $\displaystyle\int_{\sqrt[3]{\log 3}}^{\sqrt[3]{\log 4}}\frac{x^2\sin(x^3)}{\sin(x^3)+\sin(\log(12)-x^3)}\ dx$ 6) $\displaystyle\int\frac{\sin(x)+\cos(x)}{\sqrt{\sin(2x)}}\ dx$ 7) $\displaystyle\int_0^{+\infty}\frac{dx}{x^4+x^3+x^2+x+1}$ 8) $\displaystyle\int_0^\infty\frac{\log(2e^x-1)}{e^x-1}dx$

9) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}\ dx$ 10) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin(x)}\ dx$ 11) $\displaystyle\int_0^1\frac{\ln(1-x)}{x}\ dx$ 12) $\displaystyle\int\sqrt{\tan(x)}\ dx$

13) $\displaystyle\int_0^1\frac{\ln(x+1)}{x^2+1}\ dx$ 14) $\displaystyle\int\frac{x^2}{\sqrt{1-x^2}}\ dx$ 15) $\displaystyle\int\frac{x^2}{(x\sin(x)+\cos(x))^2}\ dx$ 16) $\displaystyle\int_0^{\frac{\pi}{2}}\ln(2\cos(x))\ dx$

Hard Integrals:

$$\int_o^{\pi/2}\cos^4 x\ \sin^5 x\ dx \qquad \int_{-\infty}^{+\infty}\frac{\sin(x)}{x}\ dx \qquad \int_0^1 x\left[\frac{1}{x}\right]\ dx \qquad \int_0^1 x^x\ dx \qquad \int_0^{+\infty}e^{-x^2}\ dx$$

The Correction Of Epsilite's Serie:

Exercise 1:

1) We have: $F'(x) = \lim_{h\to 0}\left[\dfrac{\int_a^{x+h} f(t)\ dt - \int_a^{x} f(t)\ dt}{h}\right] = \lim_{h\to 0}\left[\int_x^{x+h}\dfrac{f(t)}{h}\ dt\right]$.

Mean Value Theorem states that there exists $c \in [x; x+h]$ such that $f(c) = \int_x^{x+h}\dfrac{f(t)}{h}\ dt$. So, taking the limit, we find that $c \in [x; x]$, which means $c = x$ and thus $F'(x) = f(x)$.

2) Let $H(x) = G(x) - F(x)$, knowing that $F'(x) = G'(x) = f(x)$, we get $H'(x) = 0$ and $H(a) = G(a)$, which results in $F(x) = G(x) - G(a)$, thus proving Fundamental Theorem of Calculus.

3) We need to prove two implications. First, let F be an even function. We have F is even means $F(x) = F(-x)$, which implies that $F'(x) = -F'(-x)$, and that's the same as $f(x) = -f(-x)$. Next, let F be an odd function. Thus, $F(x) = -F(-x) \implies f(x) = f(-x)$, as desired.

4) If f is k-periodic, then, first using $t = y + k$ and then sequences,

$$\int_{a+k}^{x+k} f(y)\ dy = \int_{a+k}^{x+k} f(y+k)\ dy = \int_a^x f(t)\ dt = F(x)$$

and,

$$\int_0^{Nk} f(x)\ dx = \int_0^k f(x)\ dx + \int_k^{2k} f(x)\ dx + \dots + \int_{k(N-1)}^{Nk} f(x)\ dx = \sum_{Q=0}^{N-1}\int_{kQ}^{k(Q+1)} f(x)\ dx$$

using the previous result,

$$\sum_{Q=0}^{N-1}\int_{kQ}^{k(Q+1)} f(x)\ dx = \sum_{Q=0}^{N-1}\int_0^k f(x)\ dx = N\int_0^k f(x)\ dx = NF(x)$$

5) Using the King's rule (let $y = a + b - x$),

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\ dx = \int_a^b \frac{f(a+b-y)}{f(a+b-y)+f(y)}\ dy$$

We know that x and y are dummy variables, meaning

$$\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\ dx = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\ dx.$$

Then, if we add the two, we get

$$2\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\ dx = \int_a^b 1\ dx = b-a, \text{ which means } \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\ dx = \frac{b-a}{2}.$$

6) Let's try integration by parts! (Taking $U = e^{-y}$ and $V' = y^{x-2}$).

$$\Gamma(x-1) = \int_0^\infty e^{-y}\ y^{x-2}\ dy = \left[\frac{y^{x-1}}{x-1}e^{-y}\right]_0^\infty + \int_0^\infty \frac{e^{-y}y^{x-1}}{x-1}\ dy = \frac{1}{x-1}\int_0^\infty e^{-y}y^{x-1}\ dy$$

Which implies $\Gamma(x-1)(x-1) = \int_0^\infty e^{-y}\ y^{x-2}\ dy = \Gamma(x)$.

Exercise 3:

Calculating the integrals:

1) Putting $y = e^x \implies dx = \dfrac{dy}{y}$

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{y(1+y)}dy = \int \left(\frac{1}{y} - \frac{1}{1+y}\right) dy = \ln(y) - \ln(1+y) + \lambda = x - \ln(e^x+1) + \lambda$$

2) Notice that $x = -1$ is a root for $x^3 + x^2 + x + 1$

$$\int_0^1 \frac{x}{x^3+x^2+x+1}dx = \int_0^1 \frac{x}{(x+1)(x^2+1)}dx = \int_0^1 \left(\frac{1}{x^2+1} - \frac{1}{(x+1)(x^2+1)}\right) dx = \frac{\pi}{4} - \int_0^1 \left(\frac{1}{2(x+1)} + \frac{1-x}{2(x^2+1)}\right) dx$$

So we get:

$$\int_0^1 \frac{x}{x^3+x^2+x+1}dx = \frac{\pi}{4} - \frac{\ln 2}{4}$$

3)

$$\int \frac{\sin^2(x) + \cos(x)\sin(x)}{\cos^2(x)}dx = \int \left(\tan^2(x) + \tan(x)\right) dx = \int \left(\tan(x) - 1\right) dx + \tan(x) + \lambda = \tan(x) - \ln(\cos(x)) - x + \lambda$$

4)

$$\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx = x - \int \frac{\cos(x)}{\sin(x)+\cos(x)}dx = x - \int \frac{1}{\tan(x)+1}dx$$

$$t = \tan(x) \implies dx = \frac{1}{1+t^2}dt$$

$$\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx = x - \int \frac{1}{(t+1)(t^2+1)}dt = x - \frac{1}{2}\ln(t+1) + \frac{1}{4}\ln(t^2+1) - \frac{1}{2}\arctan(t) + \lambda$$

$$\int \frac{\sin(x)}{\cos(x)+\sin(x)}dx = \frac{1}{2}x - \frac{1}{2}\ln(\tan x - 1) - \frac{1}{2}\ln(\cos x) + \lambda$$

5)

$$\int_{\sqrt[3]{\log 3}}^{\sqrt[3]{\log 4}} \frac{x^2 \sin(x^3)}{\sin(x^3) + \sin(\log 12 - x^3)}dx = \frac{1}{3}\log 12$$

6)

$$\int \frac{\sin(x)+\cos(x)}{\sqrt{\sin(2x)}}dx = \int \frac{\sin(x)+\cos(x)}{\sqrt{1-(\sin(x)-\cos(x))^2}}dx$$

$$z = \sin(x) - \cos(x) \implies dx = \frac{dz}{\sin(x)+\cos(x)}$$

$$\int \frac{\sin(x)+\cos(x)}{\sqrt{\sin(2x)}}dx = \int \frac{1}{\sqrt{1-z^2}}dz = \arcsin(z) + \lambda = \arcsin(\sin(x)+\cos(x)) + \lambda$$

7)

$$\int_0^\infty \frac{1}{x^4+x^3+x^2+x+1}\,dx = \int_0^\infty \frac{1-x}{1-x^5}\,dx = \int_0^\infty \frac{1}{1-x^5}\,dx - \int_0^\infty \frac{x}{1-x^5}\,dx$$

Using PV and Euler's Reflection Formula

$$\int_0^\infty \frac{1}{x^4+x^3+x^2+x+1}\ dx = \frac{\pi}{5}\left(\cot\left(\frac{\pi}{5}\right)-\cot\left(\frac{2\pi}{5}\right)\right)$$

8) Feymen Technique: Let $F(z)=\displaystyle\int_0^\infty \frac{\log(1+z(e^x-1))}{e^x-1}\ dx$. Then $F(2)=\displaystyle\int_0^\infty \frac{\log(2e^x-1)}{e^x-1}\ dx$. Note that $F(0)=0$.

According to Leibniz's rule we have:

$$F'(z)=f(z)=\int_0^\infty \frac{\partial}{\partial z}\frac{\log(1+z(e^x-1))}{e^x-1}\ dx=\int_0^\infty \frac{1}{1+z(e^x-1)}\ dx=\int_0^\infty \frac{e^{-x}}{e^{-x}+z(1-e^{-x})}\ dx=\frac{1}{z}\int_0^\infty \frac{e^{-x}}{1+(\frac{1-z}{z})e^{-x}}\ dx$$

Putting $y=\dfrac{1-z}{z}e^{-x} \implies dx=-\dfrac{1}{y}dy$

$$f(z)=\frac{1}{z}\int_{\frac{1-z}{z}}^0 \frac{\frac{z}{z-1}\cdot y}{y(1+y)}\ dy=\frac{1}{z-1}\int_{\frac{1-z}{z}}^0 \frac{1}{1+y}\ dy=\frac{\ln z}{z-1}$$

So:

$$\int_0^2 f(x)\ dx=F(2)-F(0)=\int_0^2 \frac{\ln x}{x-1}\ dx=\int_0^1 \frac{\ln x}{x-1}\ dx+\int_1^2 \frac{\ln x}{x-1}=\sum_{k=0}^\infty\int_1^0 x^k\ln x\ dx+\int_0^1 \frac{\ln x+1}{x}\ dx$$

$$\int_0^2 f(x)\ dx=\sum_{k=0}^\infty\int_0^1 \frac{x^k}{k+1}\ dx+\sum_{k=0}^\infty\int_0^1(-1)^{k+1}\frac{x^k}{k+1}\ dx=\sum_{k=0}^\infty\frac{1}{(k+1)^2}+\sum_0^\infty(-1)^{k+1}\frac{1}{(k+1)^2}=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}$$

9)

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)+\sqrt{\cos(x)}}}dx=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\sin(\frac{\pi}{2}-x)}}\ dx=\frac{\pi}{4}$$

10)

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin(x)}dx=\int_0^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^2}dx=\int_0^{\frac{\pi}{2}} \frac{1+\tan\left(\frac{x}{2}\right)^2}{\left(1+\tan\left(\frac{x}{2}\right)\right)^2}dx$$

$$s=1+\tan(x/2) \implies dx=\frac{2}{1+\tan(x/2)^2}ds$$

So:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin(x)}dx=2\int_1^2 \frac{1}{s^2}ds=1$$

11)

$$\int_0^1 \frac{\ln(1-x)}{x}=-\int_0^1\sum_{k=1}^\infty \frac{x^{k-1}}{k}dx=-\sum_{k=1}^\infty\int_0^1 \frac{x^{k-1}}{k}dx=-\sum_{k=1}^\infty \frac{1}{k^2}=-\frac{\pi^2}{6}$$

12) Let's put $y=\sqrt{\tan(x)}$. We get $dx=\dfrac{2y}{1+y^4}dy$

$$\int\sqrt{\tan(x)}dx=\int\frac{2y^2}{1+y^4}dy=2\int\frac{1}{y^2+\dfrac{1}{y^2}}dy=\int\frac{1-\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy+\int\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy=\int\frac{1-\frac{1}{y^2}}{(y+\frac{1}{y})^2-2}dy+\int\frac{1+\frac{1}{y^2}}{(y-\frac{1}{y})^2+2}dy$$

$$\kappa=y-\frac{1}{y} \implies dx=\frac{1}{1+\frac{1}{y^2}}d\kappa$$

$$\vartheta=y+\frac{1}{y} \implies dx=\frac{1}{1-\frac{1}{y^2}}d\vartheta$$

$$\int\sqrt{\tan(x)}dx=\int\frac{1}{\kappa^2+2}d\kappa+\int\frac{1}{\vartheta^2-2}d\vartheta=\frac{1}{\sqrt{2}}\arctan\left(\frac{\kappa}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{\vartheta}{\sqrt{2}}\right)$$

Therefore:

$$\int \sqrt{\tan(x)}dx = \frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{\tan(x)}-\sqrt{\cot(x)}}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{\sqrt{\tan(x)}+\sqrt{\cot(x)}}{\sqrt{2}}\right) + \lambda$$

13) Putting $u = \arctan(x) \implies dx = (1+\tan^2 t)dt = \dfrac{1}{\cos^2 t}dt$ and knowing that $\cos x + \sin x = \sqrt{2}\cos(x - \frac{\pi}{4})$

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \int_0^{\pi/4} \ln(\tan t+1)dt = \int_0^{\pi/4} (\ln(\cos t + \sin t) - \ln(\cos t))\, dt = \int_0^{\pi/4} \left(\ln(\sqrt{2}\cos(t - \frac{\pi}{4})) - \ln(\cos t)\right) dt = \frac{\pi}{8}\ln 2$$

14) Let $x = \sin y$ and $dx = \cos x$

$$\int \sin^2 y\ dy = \int \frac{1-\cos(2y)}{2}dy = \frac{y}{2} - \frac{\sin(2y)}{4} + \lambda = \frac{\arcsin x}{2} - \frac{\arcsin(2\sin(x))}{4} + \lambda$$

15) The trick here is integration by parts.

$$\int \frac{x^2}{(x\sin x + \cos x)^2}\ dx = \int \frac{-x}{\cos x} \times \frac{-x\cos x}{(x\sin x + \cos x)^2}\ dx = \frac{-x}{\cos x} \times \frac{1}{(x\sin x + \cos x)} + \int \frac{1}{\cos^2(x)}dx + \lambda$$

So we get:

$$\int \frac{x^2}{(x\sin x + \cos x)^2}dx = -\frac{x}{\cos x}\frac{1}{(x\sin x + \cos x)} + \tan x + \lambda = \frac{-x\cos x + \sin x}{x\sin x + \cos x} + \lambda$$

16)

$$\int_0^{\pi/2} \ln(2\cos(x))\ dx = \frac{\pi \ln 2}{2} + \int_0^{\pi/2} \ln(\cos x)\ dx$$

Let's set $A = \int_0^{\pi/2} \ln(\cos x)\ dx$. We know that $\int_0^{\pi/2} \ln(\cos x)\ dx = \int_0^{\pi/2} \ln(\sin x)\ dx$ using change of variable. Therefore:

$$2A = \int_0^{\pi/2} \ln(\sin x \cos x)\ dx = \int_0^{\pi/2} \ln\left(\frac{\sin(2x)}{2}\right)\ dx = -\frac{\pi}{2}\ln 2 + \int_0^{\pi/2} \ln(\sin(2x))\ dx$$

Putting $\beta = 2x$ implies $dx = \dfrac{1}{2}d\beta$. So we get:

$$2A = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi} \ln(\sin(\beta))\ d\beta = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi/2} \ln(\sin(\beta))\ d\beta + \frac{1}{2}\int_{\pi/2}^{\pi} \ln(\sin(\beta))\ d\beta = -\frac{\pi}{2}\ln 2 + \frac{A}{2} + \frac{A}{2}$$

$$\implies A = -\frac{\pi}{2}\ln 2 \implies \frac{\pi}{2}\ln 2 + \int_0^{\pi/2} \ln(\cos x)\ dx = 0$$

So:

$$\int_0^{\pi/2} \ln(2\cos(x))\ dx = 0$$

Bonus part

We know that the function $(\sin x)$ admits a polynomial of the form $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!}$ with infinite roots of the form $2k\pi, \forall k \in \mathbb{Z}$. Then we can write the function as the following product:

$$\sin x = \prod_{k=0}^{\infty}(x-k\pi)\prod_{k=1}^{\infty}(x+k\pi) = (x-0)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots = x(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)...$$

Which is equal to:

$$\sin x = x \cdot \prod_{k=0}^{\infty}(x^2-k^2\pi^2) = x(x^2-\pi^2)(x^2-2^2\pi^2)(x^2-3^2\pi^2)... \tag{1}$$

Deviding by x we get:

$$\frac{\sin x}{x} = (x^2-\pi^2)(x^2-2^2\pi^2)(x^2-3^2\pi^2)...$$

Passing to the limit:

$$\left(\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}(x^2-\pi^2)(x^2-2^2\pi^2)(x^2-3^2\pi^2)...\right)$$

$\iff$

$$\left(1 = (-\pi^2)(-2^2\pi^2)(-3^2\pi^2)(-4^2\pi^2)... = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)(-4^2\pi^2)...}\right)$$

Replacing that into our (1) equation gives:

$$\sin x = x\frac{(x^2-\pi^2)(x^2-2^2\pi^2)(x^2-3^2\pi^2)...}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)(-4^2\pi^2)...} = x\left(1-\left(\frac{x}{\pi}\right)^2\right)\left(1-\left(\frac{x}{2\pi}\right)^2\right)\left(1-\left(\frac{x}{3\pi}\right)^2\right)...$$

Now, if we wanted to know the product of the x^3 term, we will find that they will all be the results of $x \times \left(\frac{x}{k\pi}\right)^2$ multiplied by a series of 1. We are not interested in any other terms, which will mean that:

$$\sin x = x - \left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\frac{1}{4^2\pi^2}...\right)x^3 + Ax^5... = x - \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}...\right)\frac{1}{\pi^2}x^3 + Ax^5... \quad \text{For non-zero } A \in \mathbb{R}.$$

We also know that sin has a DL as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}...$$

That would imply:

$$\sum_{1}^{\infty}\frac{1}{k^2\pi^2} = \frac{1}{6}$$

which would result in the famous equality:

$$\sum_{1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

THE END.