Fourier Series

Exercise 1:

We consider the 2π-periodic function given by

$$f_a(x) = \frac{1}{\cosh a - \cos x},$$

where a is a strictly positive real parameter.

1. Verify that for all $x \in \mathbb{R}$,
$$\frac{1}{\cosh a - \cos x} = \frac{-2z}{z^2 - (e^{-a} + e^{a})z + 1}, \quad \text{where } z = e^{ix}.$$
2. Using partial fraction decomposition of this rational function, show that
$$f_a(x) = \frac{1}{\sinh a}\left(\frac{e^{-a}}{e^{ix} - e^{-a}} - \frac{e^{a}}{e^{ix} - e^{a}}\right).$$
3. Use the geometric series to expand f_a in the form
$$f_a(x) = \frac{1}{\sinh a}\left(\sum_{n=0}^{\infty} e^{-na} e^{inx} + \sum_{n=1}^{\infty} e^{-na} e^{-inx}\right).$$
Justify this expansion with respect to the parameter a.
4. Deduce the Fourier series expansion of the function f_a.
5. Deduce the following integrals:
$$I_1 = \int_0^{\pi} \frac{\cos(nt)}{\cosh a - \cos t}\, dt \quad , \qquad I_2 = \int_0^{\pi} \frac{dt}{(\cosh a - \cos t)^2}$$
.

Exercise 2:

We consider the function f defined on the interval $[0, \pi]$ by $f(x) = \sin x$.

1. Expand f into a cosine series.
2. Using Dirichlet's theorem, determine the sum of each of the two following series:
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \quad , \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)}.$$
3. Compute, using Parseval's identity, the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2}.$$

Exercise 3:

Let f be a 2π-periodic function defined on the interval $[-\pi, \pi]$ by $f(x) = \cosh x$.

1. Give the Fourier series expansion of this function.
2. Deduce the sum of the trigonometric series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(1+n^2)} \sin(nx), \quad \text{for } x \in [-\pi, \pi].$$
3. Using Dirichlet's convergence theorem, determine the sums of the following numerical series:
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \quad , \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \quad , \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{1+4n^2}$$
.
4. Compute, using Parseval's identity, the sum of the following series:
$$\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^2} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^2(1+n^2)^2}$$
.
5. From the previous results, verify that:
   (a)
$$\sum_{n=0}^{\infty} \frac{1}{1+4n^2} = \frac{\pi}{4}\left(\frac{C_1}{\tanh \pi} + \frac{C_2}{\sinh \pi} + C_3\right),$$
   (b)
$$\sum_{n=0}^{\infty} \frac{1}{1+2n+2n^2} = \frac{\pi}{2}\left(\frac{C_4}{\tanh \pi} + \frac{C_5}{\sinh \pi} + C_6\right),$$
   (c)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(1+2n)(1+2n+2n^2)} = \frac{\pi}{2}\left[\frac{C_7}{\cosh(\pi/2)} + C_8\right],$$
   where the coefficients C_k, $k = 1, 2, \ldots, 8$, are to be determined.

Exercise 4:

Let α be a strictly positive real number, and let f be a 2π-periodic function defined on the interval $[-\pi, \pi]$ by $f(x) = \cosh(\alpha x)$.

1. Determine the Fourier series expansion of this function.
2. Compute the sums of the following numerical series:
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}.$$
3. For all $t \in \mathbb{R}^*$, show that:
$$\sum_{n=1}^{\infty} (-1)^n \frac{t}{t^2+n^2\pi^2} = \frac{1}{2}\left(\frac{1}{\sinh t} - \frac{1}{t}\right) \quad , \quad \sum_{n=1}^{\infty} \frac{t}{t^2+n^2\pi^2} = \frac{1}{2}\left(\coth t - \frac{1}{t}\right).$$

Exercise 5:

Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function such that $\forall x \in [-\pi, \pi], \quad f(x) = \cos(\alpha x)$.

1. Expand the function f into a Fourier series.
2. Deduce that for all $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{+\infty} (-1)^n \frac{2z}{z^2 - n^2} \quad \text{and} \quad \pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{+\infty} \frac{2z}{z^2 - n^2}.$$

Exercise 6:

Let α be a real number in the interval $(0, \pi)$, and let f be the 2π-periodic function defined on $(-\pi, \pi]$ by:

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

a) Study the Fourier series of f and its convergence.

b) What is the value of this series at $x = 0$? At $x = \alpha$?

c) Compute

$$\sum_{n=1}^{+\infty} \frac{\sin^2(n\alpha)}{n^2}$$

d) Justify and evaluate

$$\int_0^{+\infty} \frac{\sin^2 t}{t^2} \, dt$$