Epsilite's Summer Puzzles:

Logic and Reasoning.

Pinocchio!

Pinocchio is the hero of Carlo Collodi's children's story, The Adventures of Pinocchio (1883), a wooden puppet who comes to life as a boy and whose nose grows longer whenever he tells a lie.

But for the sake of our proposition, we will use our own definition of Pinocchio, which is the following:

> Definition
>
> Pinocchio is an entity that can speak and has a nose, and whose nose grows longer whenever it tells a lie.

Naturally, we also need to define what we mean by "entity" and "speak." But not to get sidetracked, we will define speaking as the ability to make any understandable sentences (can be either true or false, or neither) in any language, while entity is simply a real existence.

With this nice definition, we can now write the next statement.

> Theorem
>
> **Theorem 1.** (*Pinocchio's Last Paradox*) Pinocchio cannot exist.

Proof. Let's assume for a contradiction that Pinocchio exists. Then, Pinocchio can say that his nose will grow which we will note as proposition A, and Pinocchio lying will be denoted as proposition B. We now have two cases:

Case 1: His nose grows, so he told the truth, but he lied. And if A is true, then B and ¬B is true.

Case 2: His nose does not grow, so he lied, but he told the truth. And if A is true, then B and ¬B is true.

In both cases, we have both B and ¬B, which is always false. We assumed that Pinocchio exists, which implies B and ¬B. Therefore, Pinocchio (as we defined it) can't exist, as desired. □

Note that there is a slight detail we left out. In reality, this proof doesn't prove that Pinocchio doesn't exist, it proves that our Pinocchio, the one we defined, can't exist!

You see, we assumed without mentioning it that Pinocchio can say anything, which actually implied the contradiction. Meaning that the our proof is something like this: If Pinocchio exists, then he can say his nose will grow, and if he can, then he both lies and tell the truth, which is impossible mathematically.

As you see, the proposition that generated the contradiction is not "Pinocchio exists," but "he can say his nose will grow." That shows two things: First that a Pinocchio who can say anything can't exist, second is if Pinocchio existed, then he can't say some phrases. But in our definition, we didn't limit his speaking ability. That's why Pinocchio, as we defined it, can't exist.

The inspiration behind this puzzle was actually something that once caused havoc in the mathematical world. And this something is:

> **Exercise**
>
> Let $\mathbb{A}$ bet a set defines by $\mathbb{A} = \{X : X \notin X\}$. In other words, $\mathbb{A}$ is the set of all the sets that doesn't belong to themselves. Does such a set exist?

Proof. Let's assume for a contradiction that such set exists. Then, the set $\mathbb{A} = \{X : X \notin X\}$ either belongs to itself or it doesn't, there is no third option. Thus we have two cases:

Case 1: $\mathbb{A} \in \mathbb{A}$: In this case we have $\mathbb{A} \in \mathbb{A}$, which means, by the conditions the set has, that $\mathbb{A} \notin \mathbb{A}$.

Case 2: $\mathbb{A} \notin \mathbb{A}$: This time, $\mathbb{A} \notin \mathbb{A}$ implies that the set satisfies the conditions to be in it, meaning $\mathbb{A} \in \mathbb{A}$.

In both cases, we find both $\mathbb{A} \in \mathbb{A}$ and $\mathbb{A} \notin \mathbb{A}$, which is a contradiction.

In the same way as Pinocchio's puzzle a few days ago, we can solve the contradiction in two ways, but we chose that the set $\mathbb{A}$ can't exist, completing the proof. $\square$

Mathematicians, back after establishing the Set Theory and around the 19th century, questioned themselves after the publishing of this seemingly contradiction in Set Theory.

Even though this question was the inspiration behind Pinocchio's one, let's imagine for a moment that it's inversed to better understand the reason of the chaos this one once caused.

Back then, there were no restrictions on sets—meaning that there weren't any rules preventing the set $\mathbb{A}$ from existing. So the contradiction we got was taken as a sign the entire Set Theory is wrong by some individuals, which was without a doubt a thing to worry about. This caused havoc in that era, as Set Theory was seen as a heaven for most mathematicians.

In an analogous way, for people using Pinocchio on a daily bases, finding out that he can't exist must be a shock. Although we can fix that by saying that some versions of Pinocchio can't, which means we are modifying its definition.

In the same way later on, the Set Theory was upgraded to a better version, one which had a few restrictions that avoided any contradiction.

Induction!

Exercise

Let n be a positive integer. Show that the sum of the first n odd numbers is equal to n^2. i.e. for all $n \in \mathrm{N}$:

$$n^2 = 1 + 3 + 5 + \cdots + k_{n-1} + k_n.$$

Where 1, 3, 5... k_{n-1}, and k_n are the first n odd numbers.

Proof. We procced by induction.

Base Case: $n = 2$: We have $1 + 3$ is equal to 2^2, thus it's true.

Inductive Step: Let $n \in \mathrm{N}$, and let's assume that the sum of the first n odd numbers is equal to n^2.

We have $n^2 = (n + 1 - 1)^2$, hence $(n + 1)^2 = n^2 + 2(n + 1) - 1$. Then, using our assumption that n^2 is the sum of the first n odd numbers, and letting 1, 3, 5 .... k_{n-1}, and k_n be these numbers we have:

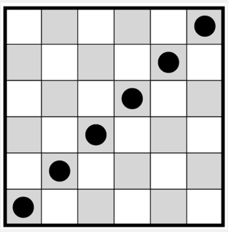
$$\begin{aligned}(n+1)^2 &= 1 + 3 + 5... + k_{n-1} + k_n + 2(n + 1) - 1. \\ &= 1 + 3 + 5... + k_{n-1} + k_n + 2n + 1. \\ &= 1 + 3 + 5... + k_{n-1} + k_n + k_{n+1}.\end{aligned}$$

Because the n^{th} odd number is the number $2n - 1$. So the number $2n + 1$, which is just $n^{\text{th}} + 2$, is the $(n + 1)^{\text{th}}$ odd number.

Conclusion: By induction, for all $n \in \mathrm{N}$, the sum of the first n odd numbers is equal to n^2. □

Exercise

In chess, a rook attacks all the squares in its row and column. Consider the problem of placing n non-attacking rooks on an $n \times n$ chessboard; that is, n rooks such that none attack any other. One way to do this is to place the rooks on a single diagonal, like this:

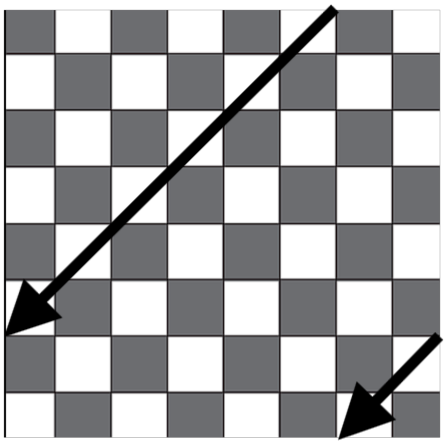


But that's boring. Prove that for every $n \geq 4$, it is possible to place n rooks on the $n \times n$ chessboard so that none of the rooks are on either (both) of the two main diagonals.

Proof Idea. It's painfully obvious that we can place n rooks such that no rook can target another by placing them on one of the two main diagonals. We shall accept this proposition as true, as it's a task for Epsilite's members to use induction and prove it.

Now, after placing the rooks, move each rook by 2 all to the left. Now, because the 2 the last rooks have no squares to go to, move them to the other side of the board.

In an 8×8 this is the result:



This is not a proof, but the idea behind it. $\square$

Proof. We procced by induction.

Base Case: $n = 4$: We can use the *Proof Idea* to show that it holds.

Inductive step: Assume the statement is true for some integer $k \geq 4$, moreover, it's the same way as *Proof Idea*, and we want to prove that the statement holds for $k + 1$.

For that, Consider an $(k+1) \times (k+1)$ chessboard. We place k rooks on the top-right $k \times k$ subgrid, such that none of the rooks are on the two main diagonals (by the hypothesis) as we we showed in the *Proof Idea*, then change the position of the 2 rooks in the $k \times k$ subgrid from $(1; 3)$ and $(2; 2)$ to $(k + 1; 3)$ and $(1; 2)$ respectively, and then place $k + 1$ rook in the position $(2; 1)$. Thus, every rook is in place.

Conclusion: By induction, the statement holds for all $n \geq 4$. In other words, it's possible to place n rooks on an $n \times n$ chessboard such that none can target each other and none of the rooks are on either of the two main diagonals. $\square$

Fake Proofs!

Fake Proposition

Proposition 1 (Fake)**.** Everyone on Earth has the same name.

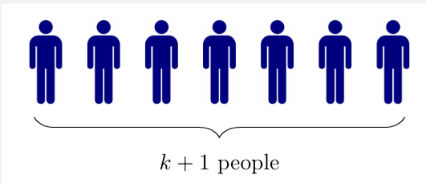
Exercise

WHAT IS WRONG WITH THIS PROOF?

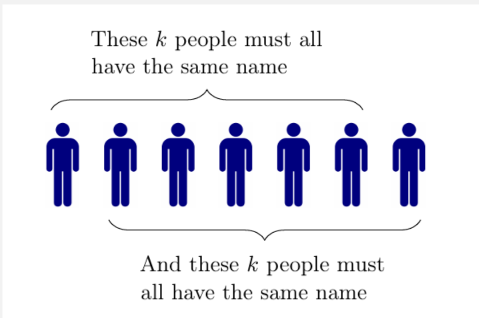
Fake Proof. We will consider groups of n people at a time, and by induction we will "prove" that for every $n \in \mathbb{N}$, every group of n people must have everyone with the same name.

<u>Base Case:</u> If $n = 1$, then of course everyone in the group has the same name, since there's only one person in the group!

<u>Induction Step:</u> Let $k \in N$, and assume that any group of k people all have the same name. Consider a group of $k + 1$ people.



But notice that we can look at the first k of these people and then the last k of these people, and to each of these groups we can apply the inductive hypothesis:



And the only way that this can all happen, is if all $k + 1$ people have the same name.

<u>Conclusion:</u> This "proves" by induction that for every $n \in N$, every group of n people must have the same name. So if you let n be equal to the number of people on Earth, this "proves" that everyone has the same name. □

The answer is simple. There is no relation between our Base Case and Induction Step. Try, in the 2nd step, taking $k = 1$. Our base case needed to be when $k = 2$, not 1, that's the mistake.

Fake Proposition

Proposition 2 (Fake)**.** The harmonic series converges!

Exercise

WHAT IS WRONG WITH THIS PROOF?

Fake Proof. We proceed by induction.

Base Case: If $n = 1$, then of course $1 < \infty$.

Induction Step: Let $k \in N$, and assume that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} < \infty$$

By the hypothesis, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}$ some finite number, which we will call F. Then,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \frac{1}{k+1} = F + \frac{1}{k+1}$$

And since the sum of two finite numbers is finite, then $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \frac{1}{k+1}$ is finite, too.

Conclusion: By induction, this means that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots < \infty$$

completing the "proof." □

You see, in this puzzle, the connection gets cut off in the very last step, in contrast to our previous problem. This induction proves that any finite sum of the form $1 + 1/2...$ is finite, as infinity doesn't belong to the set of natural numbers. The harmonic serie, however, is the limit of such from—meaning it's the limit as n goes to infinity, which is unrelated to our proof. Our proof proves that any finite sum of that form is finite, but the harmonic serie is infinite. That's the error.

Algebra!

Theorem

Theorem 2. (*Bézout's identity*) If a and b are positive integers, then there exists a pair of integers (m, n) such that

$$\gcd(a, b) = am + bn.$$

Proof. Assume that a and b are fixed positive integers. Notice that, for integers x and y, the expression $ax + by$ can take many different values, including positive values, negative values and (if $x = y = 0$) can even be zero. Let E be the set of all the resulting values, and d be the smallest positive value that belongs to E. We now let m and n be the x and y values that give this minimum value of d. That is, for these integers m and n, $d = am + bn$. We defined d to be the smallest positive value that $ax + by$ can take; to prove that this same d is the $\gcd(a, b)$, we must prove that d is a common divisor of a and b, and then that it is the greatest common divisor. We will prove these two parts separately.

Part 1: d is a common divisor of a and b: By Definition, d is a common divisor of a and b if and only if $d \mid a$ and $d \mid b$. To see that $d \mid a$, note that by the division algorithm there exist integers q and r such that $a = dq + r$, with $0 \leq r < d$. Rewriting it, $r = a - dq = a - (am + bn)q = a(1 - mq) + b(-nq)$. And since $(1 - mq)$ and (nq) are both integers (left for Epsilite's members to prove that if α and β are integers, then $\alpha + \beta$ is, too), we have found another expression of the form $ax + by$. But remember, d was chosen to be the smallest positive number that can be written like this. So, since r can be written like this too, and $0 \leq r < d$, this mean $r = 0$. This leaves $a = dq$, which by the definition of divisibility means that $d \mid a$, as desired.

Without a loss of generality, and in the same exact way, one can also show that $d \mid b$. Collectively, these prove that d is a common divisor of a and b.

Part 2: d is the greatest common divisor of a and b: Suppose that d' is some other common divisor of a and b. In order to conclude that d is the greatest among all common divisors, we must show that $d' \mid d$. To do this, observe that since d' is a common divisor, $d' \mid a$ and $d' \mid b$, which by the definition of divisibility means that

$$d' \mid a \text{ and } d' \mid b$$

Which leads to

$$d' \mid am + bn$$

And that means $d' \mid d$, as desired.

We have shown that d is larger than any other divisor of a and b, which means that d is in fact the greatest common divisor of a and b. Thus, $\gcd(a, b) = d = ak + b$ which completes the proof. $\square$

Exercise

Theorem 3. (*Gauss's theorem*) Let a, b and c be positive integers, and $\gcd(a, b) = 1$.

If $a \mid bc$ (we read a divides bc), then $a \mid c$ (we read a divides c). In other words:

$$\begin{cases} a \mid bc \\ \gcd(a,b) = 1 \end{cases} \implies a \mid c$$

Proof. Let a, b and c be positive integers. Assume that $\gcd(a, b) = 1$, and that $a \mid bc$.

We have $\gcd(a, b) = 1$ implies there exists a pair of integers (m, n) such that $am + bn = 1$. We then multiply both sides by c. We get $acm + bcn = c$, and from $a \mid bc$ we get $ak = bc$ for some $k \in \mathrm{N}$. Replacing in the equation, we find $acm + akn = c$. Taking a common factor, $a(cm + kn) = c$, where $(cm + kn)$ is some positive integer, which implies by definition that $a \mid c$, as desired. □

Exercise

Prove that there are infinitely many primes.

Proof. Suppose for a contradiction that there are only finitely many primes, say k in total. Let $\{p_1, p_2, p_3 ... p_k\}$ be the complete list of prime numbers, and consider the number $N = p_1 \cdot p_2 \cdot p_3 ... p_k$, which is the product of every prime. Next, consider the number $N + 1$, which is $p_1 \cdot p_2 \cdot p_3 ... p_k + 1$. Using $N + 1$, we will find a prime not appearing in the list, which will give us our desired contradiction.

First note that, being a natural number, $N + 1$ must either be prime or composite, so consider these two cases.

Case 1: $N + 1$ is prime: Since every prime is an integer at least 2, and $N+1$ is the product of all the primes plus one, $N + 1$ is certainly larger than each p_i in the list. So if $N + 1$ is a prime number, it must be larger than all the primes we had previously considered, and hence is a new prime that doesn't belong to our set.

Case 2: $N + 1$ is not a prime: We begin by showing that no $p_i \in \{p_1, p_2, p_3 ... p_k\}$ can divide $N + 1$. To do so, remember that by the definition, for any integers a and b, we have $a \mid b$ precisely when $b \equiv 0\ [a]$. For instance, because $p_i \mid N$, we know $N \equiv 0\ [p_i]$. Adding 1, $N + 1 \equiv 1\ [p_i]$.

We have shown that $N + 1 \not\equiv 0\ [p_i]$, implying that $p_i \nmid (N + 1)$. And since p_i was arbitrary, this proves that none of our k primes divide $N + 1$.

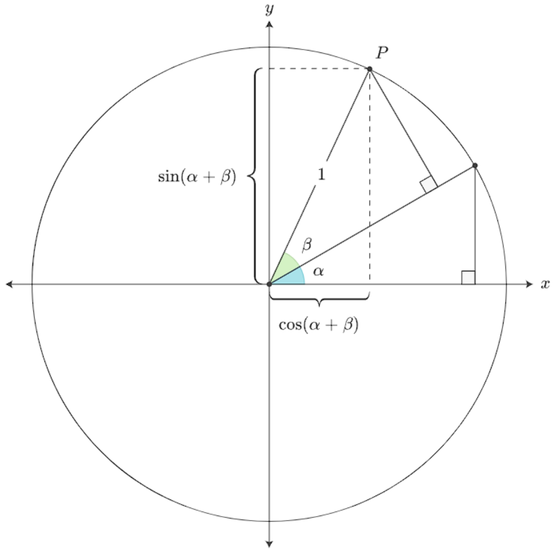
We assumed that $\{p_1, p_2, p_3 ... p_k\}$ was the complete list of prime numbers. And recall that $N + 1$ is assumed to be composite, which means it is a product of primes. But since none of the p_i divide $N + 1$, there must be some other prime number, q, which divides N +1. And hence, we have again found a new prime not in our list.

In either case we have contradicted the claim that $\{p_1, p_2, p_3 ... p_k\}$ was an exhaustive list of the prime numbers. Therefore, there must be infinitely many primes. □

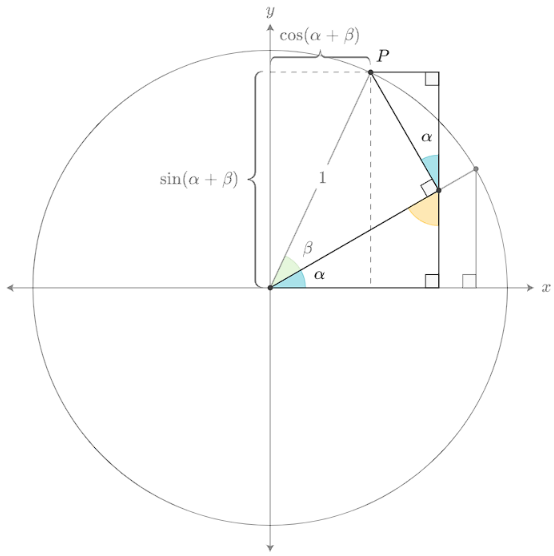
Exercise

Prove that $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and $\sin(x + y) = \cos(x)\sin(y) + \cos(y)\sin(x)$.

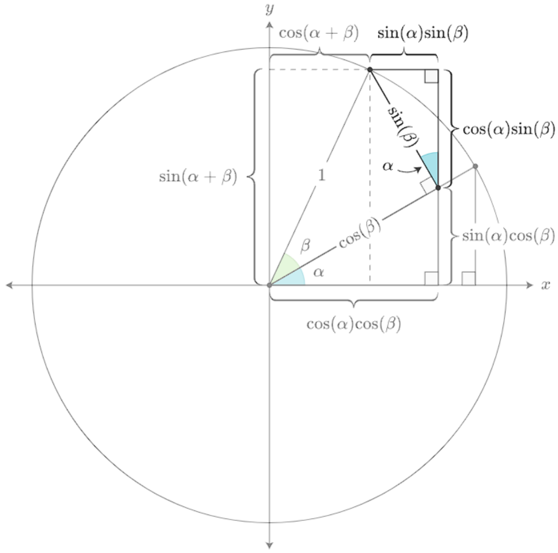
Proof. Start by drawing the an arbitrary point, P, corresponding to sum of two angles α and β on the unit circle. From the circle definitions of the trigonometric functions, we know the horizontal coordinate of this point is equal to $\cos(\alpha+\beta)$ and the vertical coordinate is equal to $\sin(\alpha+\beta)$. This gives us the initial setup to derive the identities, where the goal is to express $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of the trigonometry of the individual angles α and β. Next, draw the trigonometry related to the angles α and β as two right triangles stacked on top of each other. I'll also draw the length $\cos(\alpha+\beta)$ on top of the figure to make room for other expressions.



The key insight here is to draw a vertical line through the right-corner vertex of the right triangle corresponding to the angle β. This forms two right triangles that are similar to the right triangle formed by the angle α. The vertical and horizontal components of the point formed by the sum then can be represented using the side lengths of these two right-triangles.



From the unit circle, we know that the adjacent side of the right triangle formed by β is equal to $\cos(\beta)$ and the opposite side is equal to $\sin(\beta)$. Since we now know the hypotenuse of this first similar right triangle, we can solve for the adjacent and opposite sides using the circle definitions of the trig functions. Note, you can also imagine scaling the adjacent side $\cos(\alpha)$ and opposite side $\sin(\alpha)$ by the value $\cos(\beta)$. Repeat the same process to find the lengths of the second similar right triangle. This time the side lengths are scaled by the value $\sin(\beta)$.



Finally, equate the vertical and horizontal lengths together to derive the identities. This results in the famous sum of two angles identities.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \quad \text{and} \quad \sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x)$$

□

There is a second way to prove this, using some linear algebra—which obviously requires some in-depth knowledge about the purpose of matrices and the graphical representation of linear maps. For interested Epsilite's members, here it is!

Proof. Let $\mathcal{F}$ be a linear map that rotates a point P with coordinates $(\cos y, \sin y)$ on the unit circle with x degrees. Meaning we can define it in the following way:

$\mathcal{F}\ : \{(z, \sqrt{1-z^2}) : \forall\ |z| \leq 1\} \to \{(z, \sqrt{1-z^2}) : \forall\ |z| \leq 1\}$, such that $\mathcal{F}(\cos y, \sin y) = (\cos(x+y), \sin(x+y))$.

Now, because the function is a linear map, it has an associated matrix. To find it, we shall calculate the image of the canonic bases which are $(1,0) = (\cos 0, \sin 0)$ and $(0,1) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$, and then write the images of those vectors in a 2×2 matrix (because $\mathcal{F}$ is bijective, trivial to prove) vertically. We have:

$\mathcal{F}(\cos 0, \sin 0) = (\cos x, \sin x)$ and, $\mathcal{F}(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (\cos(\frac{\pi}{2}+x), \sin(\frac{\pi}{2}+x)) = (-\sin x, \cos x)$.

It's obvious why $\sin(\frac{\pi}{2}+x) = \cos x$, just look at the unit circle. And, if you take the derivative with respect to the variable x, you get $\cos(\frac{\pi}{2}+x) = -\sin x$.

Now that we found the images, we can write

$$\mathcal{F}(\cos y, \sin y) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y \\ \sin y \end{bmatrix} = \begin{bmatrix} \cos(x+y) \\ \sin(x+y) \end{bmatrix}$$

Multiplying,

$$\begin{bmatrix} \cos(x+y) \\ \sin(x+y) \end{bmatrix} = \begin{bmatrix} \cos x\cos y - \sin x\sin y \\ \sin x\cos y + \sin y\cos x \end{bmatrix}$$

which proves the identities above. □

Definition

We call the natural logarithm of a real number x, and we write $\ln(x)$ or $\ln x$, the number $\int_1^x \frac{1}{t}\, dt$.

Theorem

Let a and b be two positive real numbers. Then,

$$\ln(ab) = \ln a + \ln b \qquad (1)$$

And,

$$\ln(a^b) = b \cdot \ln(a) \qquad (2)$$

Proof. We have two objectives this time. Let a and b be two real positive numbers. Then,

We start by proving (1).

$$\begin{aligned}
\ln(ab) &= \int_1^{ab} \frac{1}{t}\, dt \\
&= \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt \\
&= \ln a + \int_a^{ab} \frac{1}{t}\, dt
\end{aligned}$$

Let $s = \frac{t}{a}$, then $ds = \frac{dt}{a}$ and we have this change of integration borders: $t = a \implies s = 1$ and $t = ab \implies s = b$.

$$\begin{aligned}
\ln(ab) &= \ln a + \int_1^{b} \frac{1}{s}\, ds \\
&= \ln a + \ln b.
\end{aligned}$$

Which completes the 1st proof.

Next is proving (2).

$$\ln(a^b) = \int_1^{a^b} \frac{1}{t}\, dt$$

Let $t = s^b$, then $dt = bs^{b-1}ds$ with the following change: $t = 1 \implies s = 1$ and $t = a^b \implies s = a$.

$$\begin{aligned}
\ln(a^b) &= \int_1^{a^b} \frac{1}{t}\, dt \\
&= \int_1^{a} \frac{b}{s}\, ds \\
&= b \cdot \ln(a)
\end{aligned}$$

Which completes the 2nd proof. $\square$

Theorem

Theorem 4. A 0 followed by an infinite serie of 9 is equal to 1. That is to say that

$$0.999... = 1.$$

Proof. Notice that 0.999... can be written in another form,

$$0.999... = 9(0.1^1 + 0.1^2 + 0.1^3 + ...) = 9\left(\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + ...\right)$$

which is a almost like the sum of geometric serie, we only need the to show the 1. Thus,

$$0.999... = 9\left(\frac{1}{10^0} + \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + ...\right) - 9 = \lim_{n\to\infty} \ 9\left[\sum_{k=0}^{n} \frac{1}{10^k} - 1\right] = \lim_{n\to\infty}\left[9\frac{1-\frac{1}{10^k}}{1-\frac{1}{10}} - 9\right]$$

Passing to the limits gives us

$$0.999... = 9\frac{10}{9} - 9 = 1$$

Proving the the result. $\square$

Theorem

Theorem 5. (*Euler's Identity*) $\quad e^{ix} = \cos x + i\sin x$, where $i^2 = -1$.

Proof. We begin by remembering that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ and } \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \text{ and } \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Because they are analytic functions. So,

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} i^k \ \frac{x^k}{k!} = \sum_{k=0}^{\infty} i^{2k} \ \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i^{2k+1} \ \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i \ (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Comparing the two series, we get

$$e^{ix} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + i \ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \cos x + i\sin x$$

which complets the proof. $\square$

Exercise

Calculate the result of this integral; $\int \sqrt{x\sqrt{x\sqrt{x\sqrt{\dots}}}}\, dx$.

Proof. First of all, it is obvious that

$$\int \sqrt{x\sqrt{x\sqrt{x\sqrt{\dots}}}}\, dx = \int x^{\frac{1}{2}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{8}} \cdot x^{\frac{1}{16}} \cdots \ dx$$

Thus, noticing the pattern, we can deduce that

$$\begin{aligned}
\int \sqrt{x\sqrt{x\sqrt{x\sqrt{\dots}}}}\, dx &= \int \prod_{k=1}^{\infty} x^{\frac{1}{2^k}} \ dx \\
&= \int x^{\sum_{k=1}^{\infty} \frac{1}{2^k}} \ dx \\
&= \int \lim_{n\to\infty} x^{\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}-1} \ dx \\
&= \int x \ dx \\
&= \frac{x^2}{2} + \lambda, \quad \lambda \in \mathbb{R}.
\end{aligned}$$

Answering the exercise. □

Physics' Time!

Definition

(Dot Product) Let $n \in \mathbb{N}$ and let $\vec{A}$ and $\vec{B}$ be vectors in the vector-space $\mathbf{R}^n$.

We call the dot product of $\vec{A}$ and $\vec{B}$, noted $\vec{A} \cdot \vec{B}$, the real number $|\vec{A}||\vec{B}|\cos\theta$.

Some may write $||\vec{A}||||\vec{B}||\cos\theta$, which perhaps is acceptable. But I do not like it. Thus, we will use the one-bar notation in this Summer Puzzles file.

Now, the reason behind giving such an opperation (Dot Product) a specific name is because it's important. For instance, we can say that the dot product geometricaly represents the length of the projection of $\vec{A}$ on $\vec{B}$, or vice versa. Experience shows that this property proved to be of significant importance, unlike the cross product which only had meaning in 3D.

Anyway, if you remember, there is 2nd definition of dot product which is equivalent to how we just defined it, which seems counter-intuitive.

Let $\vec{A}$ and $\vec{B}$ be vectors in the space $\mathbf{R}^2$, and let $\vec{A}_x$ and $\vec{B}_x$ be the projections of $\vec{A}$ and $\vec{B}$ on $\vec{\imath}$, respectively, and let $\vec{A}_y$ and $\vec{B}_y$ be the projections of $\vec{A}$ and $\vec{B}$ on $\vec{\jmath}$, respectively.

Exercise

Prove that $\vec{A} \cdot \vec{B} = |\vec{A}_x||\vec{B}_x| + |\vec{A}_y||\vec{B}_y|$

Our task is to prove this. In order to do that, we shall use the "general" formula of Pythagorean Theorem, which few are familiar with.

Theorem

Theorem 6. (The Law of Cosines) Let an arbitrary triangle with sides A, B, and C, and angle θ between A and B in a space. Then we have

$$A^2 + B^2 - 2AB\cos\theta = C^2.$$

(In the case of a right triangle, $\cos\theta = 0$, giving us the famous result.)

As it's not our main objective, we leave it as an exercise for Espilite's members, because it's trivial to prove. Now, using this formula, let's answer the previous exercise!

Proof. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be vectors in $\mathbf{R}^2$ such that $\vec{A} = \vec{B} + \vec{C}$. Using the Law of Cosines above, we have $|\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta = |\vec{C}|^2$ because you can make a triangle with those 3 vector, and θ is the angle between $\vec{A}$ and $\vec{B}$. Now, notice that the equation has the result of the dot product of $\vec{A}$ and $\vec{B}$. So,

$$2|\vec{A}||\vec{B}|\cos\theta = 2(\vec{A}\cdot\vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - |\vec{C}|^2.$$

Thus,

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= \frac{1}{2}\left[|\vec{A}|^2 + |\vec{B}|^2 - |\vec{C}|^2\right] \\
&= \frac{1}{2}\left[|\vec{A}|^2 + |\vec{B}|^2 - |\vec{A}-\vec{B}|^2\right], \quad \left(\text{because } \vec{A} = \vec{B} + \vec{C}\right) \\
&= \frac{1}{2}\left[|\vec{A_x}|^2 + |\vec{A_y}|^2 + |\vec{B_x}|^2 + |\vec{B_y}|^2 - |(\vec{A_x} + \vec{A_y}) - (\vec{B_x} + \vec{B_y})|^2\right], \quad \left(\text{because } |\vec{A}| = \sqrt{|\vec{A_x}|^2 + |\vec{A_y}|^2}\right) \\
&= \frac{1}{2}\left[|\vec{A_x}|^2 + |\vec{A_y}|^2 + |\vec{B_x}|^2 + |\vec{B_y}|^2 - |(\vec{A_x} - \vec{B_x}) + (\vec{A_y} - \vec{B_y})|^2\right] \\
&= \frac{1}{2}\left[|\vec{A_x}|^2 + |\vec{A_y}|^2 + |\vec{B_x}|^2 + |\vec{B_y}|^2 - |(\vec{A_x} - \vec{B_x})|^2 - |(\vec{A_y} - \vec{B_y})|^2\right], \quad (\text{because of Pythagorean Theorem})
\end{aligned}$$

Simplyfing, we get

$$\vec{A}\cdot\vec{B} = |\vec{A_x}||\vec{B_x}| + |\vec{A_y}||\vec{B_y}|,$$

completing the proof. □

Funny Ones!

Exercise

You are investigating murder. The following 5 facts have been established at the scene of the crime:

1. If Colonel Mustard is not guilty, then the crime took place in the library.
2. Either the weapon was the wrench or the crime took place in the billiard room (only one is true.)
3. If the crime took place at midnight, then Colonel Mustard is guilty.
4. Professor Plum is innocent if and only if the weapon was not the wrench.
5. Either Professor Plum or Colonel Mustard is guilty (only one is true.)

Now, as a side hustle, you are also a mathematician. As such, you start wondering what further piece of evidence would conclusively determine the killer.

For each of the following, explain how that piece of evidence—if established—would determine the killer.

(a) The crime took place in the billiard room.

(b) The crime did not take place in the billiard room.

(c) The crime was committed at noon with a knife.

(d) The crime took place at midnight in the kitchen.

Proof. We have 4 cases, each time taking into account a different 6th fact.

Case 1: We have the original five propositions, and proposition (a.)
We have "The crime took place in the billiard room." From proposition 1, "If Colonel Mustard is not guilty, then the crime took place in the library," we can conclude that "If the crime didn't take place in the library, the Colonel Mustard is guilty." So, Colonel Mustard is murderer, because there is only one.

Case 2: We have the original five propositions, and proposition (b.)
We have "The crime did not take place in the billiard room," and from "Either the weapon was the wrench or the crime took place in the billiard room," we deduce that the weapon is the wrench. Proposition 4, "Professor Plum is innocent if and only if the weapon was not the wrench," is equivalent to "Professor Plum is guilty if and only if the weapon was the wrench." Which indicates that Professor Plum is indeed the murderer. Again, because there is only one.

Case 3: We have the original five propositions, and proposition (c.)

We have "The crime was committed at noon with a knife," and with proposition 2, "Either the weapon was the wrench or the crime took place in the billiard room," we can conclude that the crime took place in the billiard room, which in turn makes Colonel Mustard the murderer from proposition 1 and 5.

Case 4: We have the original five propositions, and proposition (d.)
"The crime took place at midnight in the kitchen," implies that Colonel Mustard is guilty from either proposition 1 or 3, and 5. □

Exercise

Every natural number is interesting!

Proof. Assume for a contradiction that not every natural number is interesting. Then, there must be a smallest uninteresting number, which we call n. But being the smallest uninteresting number is a very interesting property for a number to have! So n is both uninteresting and interesting, which gives the contradiction.
Therefore, every natural number must be interesting. □

Sure, this was just a fun example, and "interesting" is impossible to define in the way we mean it. But one of its main ideas was a good one: If we are assuming that there exist uninteresting numbers, let's find a specific one that is in fact interesting. Don't think about all of the cases, focus on a special one; in this case, the special one is the smallest one.