Integration Techniques

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Introduction

This is a comprehensive guide to mastering integration techniques, which is packed with integration tips and tricks to simplify and streamline your integration process.

Inside, you will find a curated collection of methods, shortcuts, and strategies to tackle integrals more efficiency. They vary from basic, trivial skills to advanced, hidden methods.

Each trick is proved, and explained concisely with clear examples, ensuring that you do not only understand the concept but also know how to apply it to a variety of problems.

Constants

Example

Calculate the following integral: $\int x \cdot \arctan x \ dx$

In this examplem, the solution to this integral is done by integration by parts.

$$\begin{aligned}\int x \cdot \arctan x \ dx &= \frac{1}{2}x^2 \arctan x - \int \frac{x^2}{2(x^2+1)} \ dx \\ &= \frac{1}{2}x^2 \arctan x - \int \left[\frac{x^2+1}{2(x^2+1)} - \frac{1}{2(x^2+1)}\right] dx \\ &= \frac{1}{2}(x^2 \arctan x - x + \arctan x + \lambda), \quad \lambda \in \mathbb{R}\end{aligned}$$

The answer is obvious, but in many cases, we can skip through many fractions and ecludian divisions if we used the following trick:

Theorem (True Integration By Parts)

$$\int f(x)g'(x)\ dx = f(x)(g(x)+\alpha) - \int f'(x)(g(x)+\alpha)\ dx$$

Applying this rule, we get a simplier integral if we took $\alpha = 1$:

$$\begin{aligned}\int x \cdot \arctan x\ dx &= \frac{1}{2}(x^2+1)\arctan x - \int \frac{x^2+1}{2(x^2+1)}\ dx \\ &= \frac{1}{2}((x^2+1)\arctan x - x + \lambda), \quad \lambda \in \mathbb{R}\end{aligned}$$

Proof.

We can easily show that the Theorem is true just by taking the derivative of the right side. □

Inverse Functions

Theorem (Inverse integration)

Let F be an antiderivative of f. If we assume that f^{-1} is differentiable, then:

$$\int f^{-1}(x)\,dx = x \cdot f^{-1}(x) - F(f^{-1}(x)) + \lambda, \quad \lambda \in \mathbb{R}$$

Although it can seem counterintuitive, finding the integral of inverse functions are easy if we knew the integral of the functions themselves.

Example

Calculate the antiderivatives of
$\arccos x,\ \arcsin x,\ \arctan x,\ \cosh^{-1} x,\ \sinh^{-1} x,\ \tanh^{-1} x.$

Proof.

Let $x = f(y)$. We get $dx = f'(x)dy$

$$\begin{aligned} \int f^{-1}(x)\ dx &= \int y \cdot f'(y)\ dy \\ &= y \cdot f(y) - \int f(y)\ dy \\ &= x \cdot f^{-1}(x) - F(f^{-1}(x)) + \lambda, \quad \lambda \in \mathbb{R} \end{aligned}$$

$\square$

King's Property

Example

Calculate $\int_0^1 \frac{\ln(x+1)}{x^2+1}\ dx$, $\int_0^1 \frac{\ln(\frac{1+x}{2x})}{1-x^2}\ dx$, $\int_{-1}^1 \frac{\arccos x}{x^2+1}\ dx$

As shown in our session, finding these direct integrals without any trigonometry identities is difficult. Instead of trig identities however, we used a method of reverse integration:

Theorem (King's Rule)

$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$$

Remark

We can prove it by a simple change of variable $y = a + b - x$.

Proof.

Putting $t = a + b - x \implies dx = -dt$. With $t : b \to a$.

$$\int_a^b f(x)\ dx = \int_b^a -f(a+b-t)\ dt$$
$$= \int_a^b f(a+b-t)\ dt$$

Because it's a dummy variable, we get:

$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$$

$\square$

Famous Froms

Example

Find $\int \left[\frac{1}{\ln x} - \frac{1}{\ln^2 x}\right] \, dx$

Trying to calculate each integral separately is impossible, more so when they do not have an elementary function. However, aiming for a well-known general integral with similar nature as Integration By Parts can effectively reduce the difficulty.

If we united the denominators, we will get:

$$\int \left[\frac{1}{\ln x} - \frac{1}{\ln^2 x}\right] \, dx = \int \frac{\ln x - \frac{x}{x}}{\ln^2 x} \, dx$$
$$= \int \left(\frac{x}{\ln x}\right)' \, dx$$
$$= \frac{x}{\ln x} + \lambda, \quad \lambda \in \mathbb{R}$$

And this way we had solved a sum of two seemingly unsolvable integrals.

Belgian Way

It may seem obvious, but treating the 0 as a sum of two symetric element is usualy time saving.

Theorem

$$\int f(x)\ dx = \int [f(x) + g(x)]\ dx - \int g(x)\ dx$$

We use this method because finding the integral of the sum of $f + g$ and g is easier than just f.

Example

Using this technique, calculate:

$$\int_{-1}^{1} \frac{\arccos x}{x^2 + 1}\ dx$$

Applying the rule, and taking $g(x) = \arcsin x$ we have:

$$\begin{aligned}\int_{-1}^{1} \frac{\arccos x}{x^2+1}\, dx &= \int_{-1}^{1} \frac{\pi}{2(x^2+1)}\, dx - \int_{-1}^{1} \frac{\arcsin x}{x^2+1}\, dx \\ &= \frac{\pi}{2} \cdot \frac{\pi}{2} + 0 \\ &= \frac{\pi^2}{4}\end{aligned}$$

Notice that $\int_{-1}^{1} \frac{\arcsin x}{x^2+1}\, dx = 0$ because the function is odd, and $\int_{-1}^{1} \frac{\pi}{2(x^2+1)}\, dx = 2\int_{0}^{1} \frac{\pi}{2(x^2+1)}\, dx$ because it's even.