Introduct$\imath$on to Complex Numbers

1 Historical Contex

We begin this document with a brief overview of the history of complex numbers. The concept emerged in the 16th century as mathematicians sought solutions to cubic equations, leading to encounters with the square roots of negative numbers. Notable figures such as Gerolamo Cardano and Rafael Bombelli wrestled with these perplexing values, revealing the limitations of real numbers. Their work laid the foundation for the eventual acceptance of what were initially dubbed 'imaginary' numbers—a term that Leonhard Euler later deemed a misnomer.

As time progressed, the understanding and formalization of complex numbers evolved significantly. In the following pages, we will establish a mathematical framework that clarifies the existence of imaginary numbers. Very importantly, this discussion requires no more than a high school level of mathematical knowledge. First year students, worry not. This course was created with your circumstances in mind—assuming you have some degrees of comprehension about vectors, Set and Group Theory, that is. Through this exploration, Epsilite readers will come to understand that complex numbers are just as real as any other number. Let's dive in!

2 Basic Definitions

You are likely familiar with $\mathbb{R}^2$, the Cartesian product of $\mathbb{R}$ with itself. This set, under the usual operations of addition and scalar multiplication, forms a vector space. If this terminology perplexed you, we are just saying that the ordered pairs (x, y) where x and y are real numbers, meet the requirements to be considered vectors. In this document, we will utilize this set of vectors, but instead of just the standard operations, we will define new ones that will help us formally introduce the idea of complex numbers, not as a made-up concept to solve unsolvable equations, but as vectors in their own right.

Definition

Let x, y, x', y' and c be some real numbers. We define the following operations:

$$\begin{aligned} c \cdot (x, y) &= (cx, cy), \\ (x, y) + (x', y') &= (x + x', y + y'), \\ (x, y) \times (x', y') &= (xx' - yy', xy' + x'y). \end{aligned}$$

While the definition of '+' and '$\cdot$' may seem intuitive, the internal law '$\times$' may not be as straightforward. Let's set aside the "why" questions for now; by the end of this document, the reasoning will become clear. Using our definitions, we can perform some basic manipulations. For instance, we can express (x, y) as $(x, 0) + (0, y)$, which is equivalent to $x \cdot (1, 0) + y \cdot (0, 1)$. This representation shows that any pair can be expressed in this manner.

Next, observe that both $(1,0) \times (x,y)$ and $(x,y) \times (1,0)$ yield (x,y), meaning $(1,0)$ acts as the identity element for multiplication. In simpler terms, $(1,0)$ is analogous to 1 in real multiplication, since any real number multiplied by 1 equals itself. We are basically saying that $1 \times x$ and $x \times 1$ both equal x. Thus, we will set $(1,0)$ equal to $\underline{\mathbf{1}}$. But let us make it clear that $1 \neq \underline{\mathbf{1}}$.

Therefore, any pair (x,y) can be expressed as $x \cdot \underline{\mathbf{1}} + y(0,1)$. Consequently, just as we typically write x instead of $1 \times x$ or $x \times 1$, we will adopt this convention and simply write x instead of $x \cdot \underline{\mathbf{1}}$, which is ultimately $(x,0)$. Thus, we get $x + y(0,1)$ as a result. This prompts the question: can we apply a similar notation to $(0,1)$? To explore this, we need to perform some basic operations. Notice that

$$\begin{aligned}(0,1) \times (0,1) &= (-1,0) \\ &= -1 \cdot (1,0) \\ &= -1 \cdot \underline{\mathbf{1}} \\ &= -1.\end{aligned}$$

Following the previous reasoning, we can see why -1 equals $(-1,0)$. However, just as we said before, this does not imply that $(0,1) \times (0,1)$ belongs to $\mathbb{R}$. The result remains a pair in $\mathbb{R}^2$. Now, did you notice a similarity? If not, consider writing it as $(0,1)^2 = -1$ instead. Does that form refresh your memory? Infact, if we denote $(0,1)$ with a specific symbol similarly to what we did before—say i, for example—then the equation above simplifies to the very famous

$$i^2 = -1.$$

So, we get $(x,y) = x + yi$, whereas $i^2 = -1$. But neither $i^2 \in \mathbb{R}$, nor x being multiplied by 1 are true. It is just a convention, mainly because when $y = 0$, operating the pair $(x,0)$—be it addition, multiplication, exponent, or even functions—is almost exactly identical to operating with x. So, in strict terms, $\mathbb{R} \not\subset \mathbb{C}$. This notation encapsulates the very reason why complex numbers were initially met with skepticism.