Annotations for Ravi Vakil's The Rising Sea

Foundations of Algebraic Geometry

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Part I

Backgroud and conventions

Chapter 1 Backgroud and conventions

1.1 Some conventions

(1) All rings are assumed to be commutative unless explicity stated otherwise. All rings are assumed to contain a unit, denoted 1.
(2) Maps of rings must send 1 to 1.
(3) We don't require that $0 \neq 1$, in other words, "0-ring" (with one element) is a ring.
There is a ring map from any ring to the 0-ring, the 0-ring only maps to itself. The 0-ring is the final object in the category of rings.
(4) The definition of "integral domain" includes $1 \neq 0$, so the 0-ring is not integral domain.
(5) We accept the Axiom of Choice, usually in guise of Zorn's Lemma.
In particular, any proper ideal in a ring is contained in a maximal ideal.
The Axiom of Choice also arise in the argument that the category of A-modules have enough injectives.

1.2 Notations

(1) We will use the notation $(A, \mathfrak{m})$ or $(A, \mathfrak{m}, k)$ for local rings (rings with a unique maximal ideal), where A is the ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field.
(2) We will use the structure theorem for finitely generated modules over a principle ideal domain A: any such module can be written as the direct sum of principal modules $A/(a)$.

Theorem 1.2.1

Let A be a PID and M be a finitely generated module over A. Then there exists a unique finite sequence (d_i) such that $d_1 \mid d_2 \mid \cdots \mid d_n$ and that:

$$M \cong A/(d_1) \oplus \cdots \oplus A/(d_n).$$

(3) Some expirence with field theorey will be important from time to time.=

Part II

Preliminaries

Chapter 2 Category Theory

Example 2.1 (Product). For example, we will define the notion of **product** of schemes. As a motivation, we revisit the notion of product in a situation we know well: the category of sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u, v) : u \in U, v \in V\}$. But someone from a different mathematical culture might resonably define it as the set of symbols $\{\substack{u \\ v} : u \in U, v \in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a **universal property**.

Definition 2.0.1 (Product)

*Given two sets M and N, a **product** is a set P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for any set P' with maps $\nu‘ : P' \to M$ and $\nu' : P' \to N$, these maps must factor **uniquely** through P:*

$$\begin{tikzcd}
P' \arrow[dr, dashed, "\exists!"] \arrow[drr, "\nu'"] \arrow[ddr, "\mu'"'] & & \\
& P \arrow[r, "\nu"'] \arrow[d, "\mu"] & N \\
& M &
\end{tikzcd}$$

*Thus a **product** is a diagram:*

$$\begin{tikzcd}
P \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \\
M &
\end{tikzcd}$$

and not just a set P, although the maps μ and ν are often left implicit.

This definition agrees with the traditional definition, with one twist: there isn't just a single product, but any two products come with a **unique** isomorphism between them.

2.1 Categories and functors

2.1.1 Categories

Definition 2.1.1 (Category)

*A **category** consists of a collection of **objects**, and for each pair of objects, a set of **morphism** (or arrows) between them. Morphsims are often informally called **maps**. The collection of objects of a category $\mathscr{C}$ is often denoted $\mathrm{obj}(\mathscr{C})$, but we will usually denote the collection also by $\mathscr{C}$. If $A, B \in \mathscr{C}$, then the set of morphisms from A to B is denoted by $\mathrm{Mor}(A, B)$. A morphism is often written $f : A \to B$, and A is said to be the **source** of f, and B the **target** of f.*

Morphisms compose as expected: there is a composition $\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is demoted $g \circ f$.

*(1) (**Composition is associative**) If $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.*

*(2) (**Identity morphism**) For each obeject $A \in \mathscr{C}$, there is always an **identity morphism** $\mathrm{id}_A : A \to A$, for any morphism $f : A \to B$ and $g : B \to C$, $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$.*

Definition 2.1.2 (Isomorphism)

We called morphism f (or g) is an isomorphism, if $f : A \to B$ such that there exists some — necessarily unique — morphism $g : B \to A$, where $f \circ g$ and $g \circ f$ are the identity on B and A respectively.

Example 2.2 (Category of sets). The prototypical example to keep in mind is the category of sets, denoted $\mathbf{Sets}$. The objects are sets, and the morphisms are maps of sets.

Example 2.3 (Category of vector spaces). The category of vector spaces over a given field k, denoted $\mathbf{Vec}_k$. The objects are k-vector spaces, and the morphisms are linear transformations.

✍ **Exercise 2.1** A category in which each morphism is an isomorphism is called a **groupoid**.

(a) A perverse definition of a **group** is: a groupoid with one object.

(b) Describe a groupoid that is not a group.

Solution

(a) Look group G as a category with one obeject G and its morphism is the element of group. Since each element belong to a group is invertible, category G is a groupoid. Then we defined a group.

(b) A groupoid with two distinct object is not a group.

✍ **Exercise 2.2**

(1) If A is an object in category $\mathscr{C}$, show that the invertible elements of $\mathrm{Mor}(A, A)$ form a group, called the **automorphism group of** A, denoted $\mathbf{Aut}(A)$.

(2) What are the automorphism groups of the objects in Example 2.2 and Example 2.3 ?

(3) Show that two isomorphic objects have isomorphic automorphism groups. (Topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of X, and the morphisms $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the pointed fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid, the automorphism groups of the obejects are all isomorphic, but not canonically isomorphic.)

Proof

(1) $\mathbf{Aut}(A) = \{f \in \mathrm{Mor}(A, A) : f \text{ is invertible}\}$. Since each element in $\mathbf{Aut}(A)$ is invertible, and its identity is id_A, $\mathbf{Aut}(A)$ is a group.

(2) Let $A \in \mathbf{Sets}$, $\mathbf{Aut}(A) = \{f \in \mathrm{Mor}(A, A) : f \text{ is bijection}\}$. Let $V \in \mathbf{Vec}$, $\mathbf{Aut}(V) = \mathbf{GL}(V)$.

(3) Suppose that $X, Y \in \mathrm{obj}(\mathscr{C})$, where X is isomorphic to Y. Then exists morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Define $\varphi : \mathbf{Aut}(X) \to \mathbf{Aut}(Y)$ by setting $\varphi(\alpha) = f \circ \alpha \circ g$. It is easy to see that φ is a group homomorphism, and its inverse is $\varphi^{-1}(\beta) = g \circ \beta \circ f$. Hence, $\mathbf{Aut}(X) \cong \mathbf{Aut}(Y)$.

□

Example 2.4 (Category of abelian groups). The abelian groups, along with group homomorphisms, form a category $\mathbf{Ab}$.

Example 2.5 (Category of modules over a ring). If A is a ring, then the A-modules form a category $\mathbf{Mod}_A$. (This category has additional structure; it will be the prototypical example of an **abelian category**.) Taking $A = k$, we obtain Example 2.3 ; taking $A = \mathbb{Z}$, we obtain Example 2.4 .

Example 2.6 (Category of rings). There is a category $\mathbf{Rings}$, where the objects are rings, and the morphisms are maps of rings in the usual sense (maps of sets which respect adition and multiplication, and which send 1 to 1)

Example 2.7 (Category of topological spaces). The topological spaces, along with continuous maps, form a category $\mathbf{Top}$. The isomorphisms are homeomorphisms.

In all of the above examples, the obejects of the categories were in obvious ways sets with additional structure. This needn't be the case, as the next example shows.

Definition 2.1.3 (Partially ordered sets)

*A **partially ordered set**, (or **poset**), is a set S along with a binary relation $\geq$ on S satisfying:*

(i) $x \geq x$ (reflexivity),

(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity),

(iii) if $x \geq y$ and $y \geq x$ then $x = y$ (antisymmetry).

A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S, and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

Example 2.8

(1) A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$.

(2) Another example is

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \tag{2.1}$$

Here there are three objects. The identity morphisms are omitted for convenience, and the two non-identity morphisms are depicted.

(3) A third example is

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \tag{2.2}$$

Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right.

(4)

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \tag{2.3}$$

depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

Example 2.9 (The category of subsets of a set, and the category of open subsets of a topological space.) If X is a set, then the subsets form a partially ordered set, where arrows are given by inclusion. Informally, if $U \subseteq V$, then we have exactly one morphism $U \to V$ in the category (and otherwise none).

Similarly, if X is a topological space, then the open sets form a partially ordered set, where the maps are given by inclusions.

Definition 2.1.4 (Subcategory)

*A **subcategory** $\mathscr{A}$ of a category $\mathscr{B}$ has as its objects some of the objects of $\mathscr{B}$, and some of the morphisms of $\mathscr{B}$, such that the objects of $\mathscr{A}$ include the sources and targets of the morphisms of $\mathscr{A}$, and are preserved by composition.*

Remark Also we have an obvious "inclusion" $i : \mathscr{A} \to \mathscr{B}$, which will soon be an example of a functor.

Example 2.10 (2.1) is in an obvious way a subcategory of (2.2).

2.1.2 Functors

Definition 2.1.5 (Covariant functor)

A ***covariant functor*** *F from a category $\mathscr{A}$ to a category $\mathscr{B}$, denoted $F : \mathscr{A} \to \mathscr{B}$, is the following data.*

(1) *It is a map of obejects $F : \mathrm{obj}(\mathscr{A}) \to \mathrm{obj}(\mathscr{B})$,*
(2) *For each $A_1, A_2 \in \mathscr{A}$ and morphism $m \in \mathrm{Mor}_{\mathscr{A}}(A_1, A_2)$, $F(m) \in \mathrm{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$,*
(3) *F preserves identity morphisms: for $A \in \mathscr{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,*
(4) *F preserves composition: $F(m_2 \circ m_1) = F(m_2) \circ F(m_1)$.*

Example 2.11 (Identity functors). A trivial example is the **identity functors** $\mathrm{id} : \mathscr{A} \to \mathscr{A}$.

Example 2.12 (Forgetful functors). Consider the functor from the category of vector space (over a field k) $\mathbf{Vec}_k$ to $\mathbf{Sets}$, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a **forgetful functor** is $\mathbf{Mod}_A \to \mathbf{Ab}$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module.

Example 2.13 (Topological examples). Examples of covariant functors include the fundamental group functor π_1, which sends a topological space X with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$, and the i-th homology functor $\mathbf{Top} \to \mathbf{Ab}$, which sends a topological space X to its i-th homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a continuous morphism of pointed topological spaces $\varphi : X \to Y$ with $\varphi(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups.

Example 2.14 (Hom functor). Suppose A is an object in a category $\mathscr{C}$. Then there is a functor $h^A : \mathscr{C} \to \mathbf{Sets}$ sending $B \in \mathscr{C}$ to $\mathrm{Mor}(A, B)$, and sending $f : B_1 \to B_2$ to $\mathrm{Mor}(A, B_1) \to \mathrm{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

Definition 2.1.6 (Composition)

If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are covariant functors, then we define a functor $G \circ F : \mathscr{A} \to \mathscr{C}$ (the ***composition*** *of G and F) in the obvious way.*

Definition 2.1.7 (Faithful, full, fully faithful and full subcategory)

A covariant functor $F : \mathscr{A} \to \mathscr{B}$ is ***faithful*** *if for all $A, A' \in \mathscr{A}$, the map $\mathrm{Mor}_{\mathscr{A}}(A, A') \to \mathrm{Mor}_{\mathscr{B}}(F(A), F(A'))$ is injective, and* ***full*** *if it is surjective. A functor that is full and faithful is* ***fully faithful****.*

A subcategory $i : \mathscr{A} \to \mathscr{B}$ is a ***full subcategory*** *if i is full. Thus a subcategory $\mathscr{A}'$ of $\mathscr{A}$ is full if and only if for all $A, B \in \mathrm{obj}(\mathscr{A}')$, $\mathrm{Mor}_{\mathscr{A}'}(A, B) = \mathrm{Mor}_{\mathscr{A}}(A, B)$.*

Remark

(1) For various philosophical reasons, the notion of "full" functor on its own is unimportant; "fully faithful" is the useful notion.
(2) Inclusions are always faithful, so there is no need for the phrase "faithful subcategory".

Example 2.15

(1) The forgetful functor $\mathbf{Vec}_k \to \mathbf{Sets}$ is faithful, but not full.
(2) If A is a ring, the category of f.g. A-modules is a full subcategory of the category $\mathbf{Mod}_A$ of A-modules.

Definition 2.1.8 (Contravariant functor)

A ***contravariant functor*** *is defined in the same way as a covariant functor, except the arrows switch directions: in the above langrage (Definition 2.1.5), $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$, i.e., $F(A_2) \to F(A_1)$.*

Remark Thus $F(m_2 \circ m_1) = F(m_1) \circ F(m_2)$, not $F(m_2) \circ F(m_1)$

Sometimes people describe a contravariant functor $\mathscr{C} \to \mathscr{D}$ as a covariant functor $\mathscr{C}^{opp} \to \mathscr{D}$, where $\mathscr{C}^{opp}$ is said to be the **opposite category** to $\mathscr{C}$.

Example 2.16 (Linear algebra example.) If $\mathbf{Vec}_k$ is the category of k-vector spaces, then taking duals gives a contravariant functor $(\cdot)^\vee : \mathbf{Vec}_k \to \mathbf{Vec}_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

Example 2.17 (Topological example.) The i-th cohomology functor $H^i(\cdot, \mathbb{Z}) : \mathbf{Top} \to \mathbf{Ab}$ is a contravariant functor.

Example 2.18 There is a contravariant functor $\mathbf{Top} \to \mathbf{Rings}$ taking a topological space X to the ring of real-valued continuous functions on X. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on Y to functions on X.

Example 2.19 (The functor of points.) Suppose A is an object of a category $\mathscr{C}$. Then there is a cotravariant functor $h_A : \mathscr{C} \to \mathbf{Sets}$ sending $B \in \mathscr{C}$ to $\mathrm{Mor}(B, A)$, and sending the morphism $f : B_1 \to B_2$ to the morphism $\mathrm{Mor}(B_2, A) \to \mathrm{Mor}(B_1, A)$ via

$$[g : B_2 \to A] \mapsto [g \circ f : B_1 \to B_2 \to A].$$

This functor might reasonably be called the **functor of maps** (to A), but is actually known as the **functor of points**.

Definition 2.1.9 (Natural transformation)

Suppose F and G are two covariant functors from $\mathscr{A}$ to $\mathscr{B}$. A ***natural transformation of covariant functors*** *$F \to G$ is the data of a morphism $m_A : F(A) \to G(A)$ for each $A \in \mathscr{A}$ such that for each $f : A \to A'$ in $\mathscr{A}$, the diagram*

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "m_A"'] & F(A') \arrow[d, "m_{A'}"] \\
G(A) \arrow[r, "G(f)"'] & G(A')
\end{tikzcd}$$

commutes. A ***natural isomorphism*** *of functors is a natural transformation such that each m_A is an isomorphism. (We make analogous definition when F and G are both contravariant.)*

Definition 2.1.10 (Equivalence of categories)

The data of functors $F : \mathscr{A} \to \mathscr{B}$ and $F' : \mathscr{B} \to \mathscr{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $\mathrm{id}_{\mathscr{B}}$ on $\mathscr{B}$ and $F' \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathscr{A}}$ is said to be an ***equivalence of categories****.*

Remark The right notion of when two categories are "essentially the same" is not isomorphism (a functor giving bijections of objects and morphisms) but equivalence.

Two example might make this strangeconcept more comprehensible. The double dual of a finite-dimensional vector space V is not V, but we learn early to say that it is canonically isomorphic to V. We can make that precise as follows. Let $\mathbf{f.d.Vec}_k$ be the category of finite-dimensional vector spaces over k. Note that this category contains oodles of vector spaces of each dimension.

Exercise 2.3 Let $(\cdot)^{\vee\vee} : \mathbf{f.d.Vec}_k \to \mathbf{f.d.Vec}_k$ be the double dual functrion from the category of finite-dimensional vector spaces over k to itself. Show that $(\cdot)^{\vee\vee}$ is natually isomorphic to $\mathrm{id}_{\mathbf{f.d.Vec}_k}$. (Without the finite-dimensionality hypothesis, we only get a natural transformation of functors from id to $(\cdot)^{\vee\vee}$.)

Proof Since $V \cong (V)^{\vee\vee}$, then there is an isomorphism $m_V : V \to (V)^{\vee\vee}$. By the definition of double dual vector space, the diagram

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}(f)=f"] \arrow[d, "m_V"'] & W \arrow[d, "m_W"] \\ (V)^{\vee\vee} \arrow[r, "(f)^{\vee\vee}"'] & (W)^{\vee\vee} \end{tikzcd}$$

commutes. This implies that $(\cdot)^{\vee\vee}$ is natually isomorphic to $\mathrm{id}_{\mathbf{f.d.Vec}_k}$. □

Example 2.20 Let $\mathscr{V}$ be the category whose objects are the k-vector spaces k^n for each $n \geq 0$ (there is one vector space for each n), and whose morphsims are linear transformations. The objects of $\mathscr{V}$ can be thought of as vector spaces with bases, and the morphisms as matrices. There is an obvious functor $\mathscr{V} \to \mathbf{f.d.Vec}_k$, as each k^n is a finite-dimensional vector space.

By linear algebra, $\mathscr{V} \to \mathbf{f.d.Vec}_k$ gives an equivalence of categories.

2.2 Universal properties determine an object up to unique isomorphsim

Products were defined by a universal property.

2.2.1 Product and coproduct

Definition 2.2.1 (Product)

*Let $\mathscr{C}$ be a category, and let $\{A_i\}_{i\in I}$ be a family of objects in $\mathscr{C}$ indexed by a set I. A **product** is an ordered pair $(C, \{p_i : C \to A_i\}_{i\in I})$, consisting of an object C and a family $\{p_i : C \to A_i\}_{i\in I}$ of **projections**, that is a solution to the following universal mapping problem: for every object X equipped with morphisms $f_i : X \to A_i$, there exists a unique morphism $\theta : X \to C$ making the diagram commute for each i.*

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, "f_i"] \arrow[ddr, "f_j"'] & & \\ & C \arrow[r, "p_i"] \arrow[d, "p_j"] & A_i \\ & A_j & \end{tikzcd}$$

Proposition 2.2.1

If $\{A_i\}$ is a family of left R-modules, then the direct product $C = \prod_{i\in I} A_i$ is their product in ${}_R\mathbf{Mod}$.

Proof The statement of the proposition is incomplete, for a product requires projections. For each $j \in I$, define $p_j : C \to A_j$ by $p_j : (a_i) \mapsto a_j \in A_j$. Note that each p_j is an R-map.

Now let X be a module and, for each $i \in I$, let $f_i : X \to A_i$ be an R-map. Define $\theta : X \to C$ by $\theta : x \mapsto (f_i(x))$. First, the diagram commutes: if $x \in X$, then $p_i\theta(x) = f_i(x)$. Second, θ is unique. If

$\psi : X \to C$ makes the diagram commute, then $p_i\psi(x) = f_i(x)$ for all i, that is, for each i, the i-th coordinate of $\psi(x)$ is $f_i(x)$, which is also the i-th coordinate of $\theta(x)$. Hence, $\psi = \theta$. □

Coproducts were defined by a universal property.

Definition 2.2.2 (Coproduct)

*Let $\mathscr{C}$ be a category, and let $\{A_i\}_{i\in I}$ be a family of obejects in $\mathscr{C}$ indexed by a set I. A **coproduct** is an ordered pair $(C, \{\alpha_i : A_i \to C\}_{i\in I})$, consisting of an object C and a family $\{\alpha_i : A_i \to C\}_{i\in I}$ of morphisms, called **injections**, that is a solution to the following universal mapping problem: for every object X equipped with morphisms $\{f_i : A_i \to X\}_{i\in I}$, there exists a unique morphism $\theta : C \to X$ making the diagram commute for each i.*

$$\begin{tikzcd}
& A_i \arrow[d, "\alpha_i"'] \arrow[ddr, "f_i"] & \\
A_j \arrow[r, "\alpha_j"] \arrow[drr, "f_j"'] & C \arrow[dr, dashed, "\exists!\theta"] & \\
& & X
\end{tikzcd}$$

Coproduct is denoted by $\coprod$.

Proposition 2.2.2

If $\{A_i\}_{i\in I}$ is a family of left R-modules, then the direct sum $\bigoplus_{i\in I} A_i$ is their coproduct in ${}_R\mathbf{Mod}$.

Proof The statement of the proposition is incomplete, for a coproduct requires injections α_i. Write $C = \bigoplus_{i\in I} A_i$, and define $\alpha_i : A_i \to C$ by $a_i \mapsto (0, \cdots, 0, a_i, 0, \cdots)$. Note that each α_i is an R-map.

Let X be a module and for each $i \in I$, let $f_i : A_i \to X$ be an R-map. If $(a_i) \in C = \bigoplus_i A_i$, then only finitely many a_i are nonzero, and $(a_i) = \sum_i \alpha_i a_i$. Define $\theta : C \to X$ by $\theta : (a_i) \mapsto \sum_i f_i a_i$. The coproduct diagram does commute: if $a_i \in A_i$, then $\theta\alpha_i a_i = f_i a_i$. We now prove that θ is unique. If $\psi : C \to X$ makes the coproduct diagram commute, then

$$\psi\left((a_i)\right) = \psi\left(\sum_i \alpha_i a_i\right) = \sum_i \psi\alpha_i(a_i) = \sum_i f_i(a_i) = \theta\left((a_i)\right).$$

Therefore, $\psi = \theta$. □

Here are some simple but useful concepts.

Definition 2.2.3 (Initial, final, and zero objects)

*An object A of a category $\mathscr{C}$ is called an **initial object**, if for every object X in $\mathscr{C}$, there exists a unique morphism $A \to X$.*

*An object Ω of a category $\mathscr{C}$ is called a **final object**, if for every object X in $\mathscr{C}$, there exists a unique morphism $X \to \Omega$.*

*An object is called a **zero object** if it is both an initial object and a final object.*

✍ **Exercise 2.4** Show that any two initial objects are uniquely isomorphic . Show that any two final objects are uniquely isomorphic.

Proof Let A, A' be initial objects, then we have arrows $A \to A'$ and $A' \to A$. Note that $\mathrm{id}_A : A \to A$, the arrow $A \to A' \to A$ must be id_A, by the definition of initial object. Similarly, $A' \to A \to A'$ must be $\mathrm{id}_{A'}$. Hence, A is isomorphic to A'.

Similarly, we can prove that any two final objects are isomorphic. □

Remark In other words, if an initial object exists, it is unique up to unique isomorphism, and similarly for final objects.

Remark Convention: we often say "the", not "a", for anything defined up to unique isomorphism.

✍ **Exercise 2.5** What are the initial and final objects in **Sets**, **Rings**, and **Top** (if exist)? How about in Example 2.9 ?

Solution *The initial object in* **Sets** *and* **Top** *is* $\varnothing$. *The final object in* **Sets** *and* **Top** *is the point set* $\{*\}$. *The initial object in* **Rings** *is* $\mathbb{Z}$, *and the final object is* 0*-ring.*

Consider the category of subset of a set X, *the object of this category is* $\mathscr{P}(X)$, *hence, the initial object is* $\varnothing$ *and the final object is* X.

2.2.2 Localization

Another important example of a definition by universal property is the notion of **localization** of a ring.

Definition 2.2.4 (Localization of ring)

A ***multiplicative subset*** S *of a ring* A *is a subset closed under multiplication containing* 1. *We define a ring* $S^{-1}A$ *as follow:*

$$S^{-1}A = \{a/s : a \in A, s \in S\}/\sim,$$

where $\sim$: $a_1/s_1 = a_2/s_2$ *if and only if for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. *We define* $(a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)$, *and* $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$.

We have a canonical ring map

$$A \to S^{-1}A$$

given by $a \mapsto a/1$.

Remark If $0 \in S$, $S^{-1}A$ is the 0-ring. (Since, each element is equivalence by the definiton of $\sim$ if $0 \in S$.)

There are two particularly important flavors of multiplicative subsets.

Example 2.21 The first is $S = \{1, f, f^2, \cdots\}$, where $f \in A$. This localization is denoted A_f. Note that $A[t]/(tf-1) = A[\frac{1}{f}]$, we have an isomorphism $A_f \cong A[t]/(tf-1)$.

Example 2.22 The second is $A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}A$ is denoted $A_{\mathfrak{p}}$.

Remark Sometimes localization is first introduced in the special case where A is an integral domain and $0 \notin S$. In that case, $A \hookrightarrow S^{-1}A$, but this isn't always ture, as shown by the following exercise.

✍ **Exercise 2.6** Show that $A \to S^{-1}A$ is injective if and only if S contains no zero-divisors

Proof Let $a \in A$ and $a/1 = 0$, then $s(a-0) = sa = 0$ for some $s \in S$. Hence, $A \to S^{-1}A$ is an injective iff S contains no zero-divisor. □

If A is an integral domain and $S = A \setminus \{0\}$, then $S^{-1}A$ is called the **fraction field of** A, which we denote $K(A)$. The Exercise 2.6 shows that A is a subring of its fraction field $K(A)$.

We now return to the case where A is a general commutative ring.

Proposition 2.2.3 (Universal property of localization)

Let $g : A \to B$ *be a ring homomorphism such that* $g(s)$ *is a unit in* B *for all* $s \in S$. *Then there exists a unique ring homomorphism* $h : S^{-1}A \to B$ *such that* $g = h \circ f$, *that is, the following diagram is* *commute:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ S^{-1}A, \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

where $f : A \to S^{-1}A$ *defined by* $a \mapsto a/1$

Proof **Uniquness.** If h satisfies the conditions, then $h(a/1) = hf(a) = g(a)$ for all $a \in A$, hence, if $s \in S$,

$$h(1/s) = h((s/1)^{-1}) = h(s/1)^{-1} = g(s)^{-1}$$

and therefore $h(a/s) = h(a/1)h(1/s) = g(a)g(s)^{-1}$, so that h is uniquely determined by g.

Existence. Let $h(a/s) = g(a)g(s)^{-1}$. We shall to show that h is well-defined. If $a_1/s_1 = a_2/s_2$, then $s(a_1s_2 - a_2s_1) = 0$ for all $s \in S$, hence, $g(s)(g(a_1)g(s_2) - g(a_2)g(s_2)) = 0$. Since $g(s)$, $g(s_1)$ and $g(s_2)$ are units, we have $g(a_1)g(s_1)^{-1} = g(a_2)g(s_2)^{-1}$, that is, $h(a_1/s_1) = h(a_2/s_2)$. This implies that h is well-defined. By the definition of h, clearly, h is a ring homomorphism, which proved existence. □

Corollary 2.2.1

$S^{-1}A$ is the initial among A-algebras B where every element of S is sent to an invertible element in B.

Proof In fact, the data of "an A-algebra B" and "a ring map $A \to B$" are the same. Apply Proposition 2.2.3, we done immediately. □

Corollary 2.2.2

Any map $A \to B$ where every element of S is sent to an invertible element must factor uniquely through $A \to S^{-1}A$.

Proof Proposition 2.2.3, obviously! □

Corollary 2.2.3

A ring map out of $S^{-1}A$ is the same thing as a ring map from A that sends every element of S to an invertible element.

Proof Proposition 2.2.3, obviously! □

Corollary 2.2.4

An $S^{-1}A$-module is the same thing as an A-module for which $s \times \cdot : M \to M$ is an A-module isomorphism for all $s \in S$.

Proof $s \times \cdot : M \to M$ is an A-module isomorphism for all $s \in S$ iff each $s \in S$ is a unit iff M is $S^{-1}A$ module. □

In fact, it is cleaner to define $A \to S^{-1}A$ by the universal property to check various properties $S^{-1}A$ has. Now, let's define localizations of modules by universal property.

Definition 2.2.5 (Localization of module)

Suppose M is an A-module. We define the A-module map $\varphi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N ($s \times \cdot : N \to N$ is an isomorphism *for all $s \in S$). More precisely, any such map $\alpha : M \to N$ factors uniquely through φ:*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & N \\ S^{-1}M \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Remark Translation: $M \to S^{-1}M$ is universal initial among A-module maps from M to modules that are actually $S^{-1}A$-modules.

Note

(i) This determines $\varphi : M \to S^{-1}M$ up to unique isomorphism.

(ii) We are defining not only $S^{-1}M$, but also the map φ at the same time.

(iii) Essentially by definition the A-module structure on $S^{-1}M$ extends to an $S^{-1}A$-module structure.

Exercise 2.7 Show that $\varphi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property.

Proof Let $S^{-1}M = \{m/s : m \in M, s \in S\}/\sim$, where $\sim$: $m_1/s_1 = m_2/s_2$ iff for some $s \in S, s(s_2m_1 - s_1m_2) = 0$. Define the additive structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}A$-module structure is given by $(a_1/s_1) \cdot (m_2/s_2) = (a_1m_2)/(s_1s_2)$.

Let $\varphi : M \to S^{-1}M$ defined by $\varphi(m) = m/1$. Let N be any A-module with $s \times \cdot : N \to N$ is an isomorphism for all $s \in S$, and $f : M \to N$ be an A-module morphism, define $h : S^{-1}M \to N$ by setting $h(m/s) = f(m)f^{-1}(s)$. By the same way as Proposition 2.2.3, h unique exists, then we done. □

Exercise 2.8

(a) Show that localization commutes with finite products, or equivalently, with finite direct sums. In other words, if $M_1, \cdots, M_n$ are A-modules, describe an isomorphism of A-modules, and of $S^{-1}A$-modules $S^{-1}(M_1 \times \cdots \times M_n) \to S^{-1}M_1 \times \cdots \times S^{-1}M_n$.

(b) Show that localization commutes with arbitrary direct sums.

(c) Show that "localization does not necessarily commute with infinite products": the obvious map $S^{-1}(\prod_i M_i) \to \prod_i S^{-1}M_i$ induced by the universal property of localization is not always an isomorphism.

Proof

(a) Let $\varphi : S^{-1}(M_1\times\cdots\times M_n) \to S^{-1}M_1\times\cdots\times S^{-1}M_n$ by setting $\varphi\left(\frac{(m_1, \cdots, m_n)}{s}\right) = \left(\frac{m_1}{s}, \cdots, \frac{m_n}{s}\right)$.

The inverse of φ, by setting $\psi\left(\frac{m_1}{s_1}, \cdots, \frac{m_n}{s_n}\right) = \frac{\left(m_1 \prod_{i\neq 1} s_i, \cdots, m_n \prod_{i\neq n} s_i\right)}{\prod_i s_i}$ (reduction of a fraction if possible). φ and ψ are $S^{-1}A$-module homomorphism, note that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$, $S^{-1}(M_1 \times \cdots \times M_n) \cong S^{-1}M_1 \times \cdots \times S^{-1}M_n$.

(b) Let $M = \bigoplus_i M_i$ and $N = \bigoplus_i S^{-1}M_i$, $f : M \to S^{-1}M$ and $f_i : M_i \to S^{-1}M_i$ be the localization map, $\iota_i : M_i \to M$ be embedding. Then we have a map $f \circ \iota_i : M_i \to S^{-1}M$. Appy the universal property of localization to following diagram

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[dr, "f\circ\iota_i"'] & S^{-1}M_i \arrow[d, dashed, "\exists! h_i"] \\ & S^{-1}M \end{tikzcd} \tag{2.4}$$

there exists a unique homomorphism $h_i \to S^{-1}M$ such that (2.4) commute. Consider the following diagram

$$\begin{tikzcd} S^{-1}M_i \arrow[r, "h_i"] \arrow[dr, "e_i"'] & S^{-1}M \arrow[d, dashed, "\psi"] \\ & N \end{tikzcd} \tag{2.5}$$

where $e_i : S^{-1}M_i \to N$ be the embedding. We shall to show (2.5) is commute.

M is a coproduct, by Proposition 2.2.2. Consider the diagram

$$\begin{tikzcd} M_i \arrow[r, "\iota_i"] \arrow[dr, "e_i \circ f_i"'] & M \arrow[d, dashed, "\varphi"] \\ & N \end{tikzcd} \tag{2.6}$$

apply the universal property of coproduct, there exists unique homomorphism $\varphi : M \to N$ such that (2.6) is commute. Consider the diagram

$$\begin{tikzcd} M_i \arrow[r, "h_i \circ f_i"] \arrow[dr, "e_i \circ f_i"'] & S^{-1}M \arrow[d, dashed, "\widetilde{\varphi}"] \\ & N \end{tikzcd} \tag{2.7}$$

by the universal property of localization, there exists unique homomorphism $\widetilde{\varphi} : S^{-1}M \to N$ such that (2.7) is commute, that is, $\widetilde{\varphi} \circ h_i \circ f_i = e_i \circ f_i$. Consider the diagram

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[dr, dashed, "f \circ \iota_i"] \arrow[ddr, "e_i \circ f_i"'] & S^{-1}M_i \arrow[d, "h_i"] \arrow[dd, bend left, "e_i"] \\ & S^{-1}M \arrow[d, "\widetilde{\varphi}"] \\ & N \end{tikzcd} \tag{2.8}$$

by the universal property, $e_i = \widetilde{\varphi} \circ h_i$. Let $\psi = \widetilde{\varphi}$, then (2.5) commute.

By above discussion, $\psi : S^{-1}M \to N$ is uniquely defined by $\psi((m_i)/s) = (m_i/s)$. And it is easy to see ψ is an isomorphism.

(c) Consider $\mathbb{Z}^{-1}\mathbb{Q} \times \cdots$, $(1, 1/2, 1/3, \cdots, 1/n, \cdots)$ has no preimage in $\prod_i \mathbb{Z}^{-1}\mathbb{Q}$. This implies that $\mathbb{Z}^{-1}(\prod_i \mathbb{Q})$ not isomorphic to $\prod_i \mathbb{Z}^{-1}\mathbb{Q}$.

□

Remark Localization does not always commute with Hom.

2.2.3 Tensor product

Another important example of a universal property construction is the notion of **tensor product** of A-modules.

Definition 2.2.6 (Tensor product of A-modules)

*The **tensor product** is often defined as follows:*

$$\begin{aligned} \otimes_A : \mathrm{obj}(\mathbf{Mod}_A) \times \mathrm{obj}(\mathbf{Mod}_A) &\to \mathrm{obj}(\mathbf{Mod}_A) \\ (M, N) &\mapsto M \otimes_A N \end{aligned}$$

Suppose M, N are A-modules. Then elements of the tensor product $M \otimes_A N$ are finite A-linear combinations of symbols $m \otimes n$ ($m \in M$, $n \in N$), subject to relations:

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$
$$a(m \otimes n) = (am) \otimes n = m \otimes (an),$$

where $a \in A$, $m_1, m_2 \in M$, $n_1, n_2 \in N$.
More formally, $M \otimes_A N$ is the free A-module generated by $M \times N$, quotiented by the submodule generated by

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n),$$
$$(m, n_1 + n_2) - (m, n_1) - (m, n_2),$$
$$a(m, n) - (am, n)$$
$$a(m, n) - (m, an),$$

for $a \in A$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$. The image of (m, n) in this quotient is $m \otimes n$.

Remark If A is a field k, we recover the tensor product of vector space.

✍ **Exercise 2.9** Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$.

Proof Note that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12)$ is generated by $\{1 \otimes_{\mathbb{Z}} 1, 1 \otimes_{\mathbb{Z}} 0, 0 \otimes_{\mathbb{Z}} 1, 0 \otimes_{\mathbb{Z}} 0\}$, since $1 \otimes_{\mathbb{Z}} 0 = 0 \otimes_{\mathbb{Z}} 1 = 0 \otimes 0 = 0$, we have $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$. □

Proposition 2.2.4 (Right-exactness of $(\cdot) \otimes_A N$)

$(\cdot) \otimes_A N$ gives a covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. $(\cdot) \otimes_A N$ is a ***right-exact functor****, i.e., if*

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of A-modules, then the induced sequence

$$M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is also exact.

Proof

(1) $(\cdot) \otimes_A N$ gives a covariant functior.
$(\cdot) \otimes_A N$ maps object in $\mathbf{Mod}_A$ to $\mathbf{Mod}_A$. Let $f \in \mathrm{Mor}(M, M')$, where $A, B \in \mathbf{Mod}_A$, define $(f) \otimes_A N = f \otimes_A \mathrm{id}_N : M \otimes_A N \to M' \otimes_A N$, then $f \otimes_A N \in \mathrm{Mor}(M \otimes_A N, M' \otimes_A N)$. By the definition of $f \otimes_A N$, $(\cdot) \otimes_A N$ preserves identity morphism and preserves composition.

(2) Say

$$M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0.$$

is an exact sequence.
We shall to show that the sequence

$$M' \otimes_A N \xrightarrow{i \otimes \mathrm{id}_N} M \otimes_A N \xrightarrow{p \otimes \mathrm{id}_N} M'' \otimes_A N \longrightarrow 0$$

is exact.

(a) $\mathrm{Im}\, i \otimes \mathrm{id}_N = \mathrm{Ker}\, p \otimes \mathrm{id}_N$.
For $\mathrm{Im}\, i \otimes \mathrm{id}_N \subseteq \mathrm{Ker}\, p \otimes \mathrm{id}_N$, it suffices to prove that the composite is 0. Note that

$$(p \otimes \mathrm{id}_N)(i \otimes \mathrm{id}_N) = pi \otimes \mathrm{id}_N = 0 \otimes \mathrm{id}_N = 0,$$

then we done.

Let $E = \mathrm{Im}(i \otimes \mathrm{id}_N)$, then $E \subseteq \mathrm{Ker}\, p \otimes \mathrm{id}_N$, and therefore $i \otimes \mathrm{id}_N$ induces a map $\widehat{p} : (M \otimes N)/E \to M'' \otimes N$ with

$$\widehat{p} : m \otimes n + E \mapsto pm \otimes n,$$

where $m \in M$ and $n \in N$. Now if $\pi : M \otimes N \to (M \otimes N)/E$ is the natural nap, then

$$\widehat{p}\pi = p \otimes \mathrm{id}_N,$$

for both send $a \otimes b \mapsto pa \otimes b$.

$$\begin{tikzcd} M \otimes N \arrow[rr, "\pi"] \arrow[dr, "p\otimes \mathrm{id}_N"'] & & (M \otimes N)/E \arrow[dl, "\widehat{p}"] \\ & M'' \otimes N & \end{tikzcd}$$

Suppose we show that $\widehat{p}$ is an isomorphism. Then

$$\mathrm{Ker}(p \otimes \mathrm{id}_N) = \mathrm{Ker}\, \widehat{p}\pi = \mathrm{Ker}\, \pi = E = \mathrm{Im}(i \otimes \mathrm{id}_N),$$

and we done. To see that $\widehat{P}$ is an isomorphism, we construct its inverse $M'' \otimes N \to (M \otimes N)/E$. Define

$$f : M'' \times N \to (M \otimes N)/E$$

as follows. If $m'' \in M''$, there is $m \in M$ such that $p(m) = m''$, since p is surjective, let

$$f : (m'', n) \mapsto m \otimes n + E.$$

Now f is well-defined: if $pm_1 = m''$, then $p(m - m_1) = 0$ and $m - m_1 \in \mathrm{Ker}\, p = \mathrm{Im}\, i$. Thus there is $m' \in M'$ with $i(m') = m - m_1$, and hence $(m - m_1) \otimes n = i(m') \otimes n \in \mathrm{Im}(i \otimes \mathrm{id}_N) = E$, that is, $m \otimes n + E = m_1 \otimes n + E$. Clearly, f is A-biadditive, and so the definition of tensor product gives a homomorphism $\widehat{f} : M'' \otimes N \to (M \otimes N)/E$ with $\widehat{f}(m'' \otimes n) = m \otimes n + E$. It is easy to check that $\widehat{f}$ is the inverse of $\widehat{p}$, as desired.

Hence, we have $\mathrm{Im}\, i \otimes \mathrm{id}_N = \mathrm{Ker}\, p \otimes \mathrm{id}_N$.

(b) $p \otimes \mathrm{id}_N$ is surjective.

Since, p is surjective, if $\sum m_i'' \otimes n_i \in M'' \otimes N$, then there exist $m_i \in M$ with $pm_i = m_i''$ for all i. Hence, $(p \otimes \mathrm{id}_N)(\sum m_i \otimes n_i) = \sum m_i'' \otimes n_i$, which implies that p is surjective.

□

Remark Tensor product is not left-exact: tensor the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

with $\mathbb{Z}/(2)$.

The constructive definiton of $\otimes$ is a weird definition, and really the "wrong" definition.

Definition 2.2.7 (A-biadditive (or A-bilinear))

*Let A be a ring, and $M, N, P \in \mathbf{Mod}_A$, a map $f : M \times N \to P$ is called A-**bilinear** if*

$$\begin{aligned} f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n), \\ f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2), \\ f(am, n) &= f(m, an) = af(m, n). \end{aligned} \tag{2.9}$$

Any A-bilinear map $M \times N \to P$ factors through the tensor product uniquely: $M \times N \to M \otimes_A N \to P$.

Definition 2.2.8 (Tensor product)

Given a ring R and modules $M, N \in \mathbf{Mod}_A$, then their ***tensor product*** *is an A-module T and an A-bilinear fuction*

$$t : M \times N \to T,$$

such that given any A-bilinear map $t' : M \times N \to T'$, there is a unique A-linear map $f : T \to T'$ such that $t' = f \circ t$.

$$\begin{tikzcd} M \times N \arrow[rr, "t"] \arrow[dr, "t'"'] & & T \arrow[dl, dashed, "\exists! f"] \\ & T' & \end{tikzcd}$$

Proposition 2.2.5

Tensor product $(T, t : M \times N \to T)$ is unique up to unique isomorphism, i.e., if $(T', t' : M \times N \to T)$ another tensor product, $T \cong T'$.

Proof Since T' is a tensor product, for every A-module G and every A-bilinear map $f : M \times N \to G$, there exists a unique A-linear map $f' : T \to G$ making the following diagram commute.

$$\begin{tikzcd} M \times N \arrow[rr, "t'"] \arrow[dr, "f"'] & & T' \arrow[dl, dashed, "f'"] \\ & G & \end{tikzcd}$$

Setting $G = T$ and $f = t$, there is a homomorphism $g : T' \to T$ with $gt' = t$, setting $G = T'$ and $f = t'$ in the diagram defining T, there is a homomorphsim $\widetilde{g} : T \to T'$ with $\widetilde{g}t = t'$.

Consider the following diagram.

$$\begin{tikzcd} & T \arrow[d, "\widetilde{g}"] \arrow[dd, bend left, "\mathrm{id}_T"] \\ M \times N \arrow[ur, "t"] \arrow[r, "t'"] \arrow[dr, "t"'] & T' \arrow[d, "g"] \\ & T \end{tikzcd}$$

Now $g\widetilde{g}$ makes the big triangle with vertices $M \times N$, T, and T commute. But identity id_T also makes this diagram commute. By the definition of tensor product, $g\widetilde{g} = \mathrm{id}_T$. A similar argument shows that $\widetilde{g}g = \mathrm{id}_{T'}$. Hence, $\widetilde{g} : T \to T'$ is an isomorphism. □

Corollary 2.2.5

The construction of Definition 2.2.6 satisfies the universal property of tensor product.

Definition 2.2.9 (Restriction of scalars)

Let $f : A \to B$ be a homomorphism of rings and let N be a B-module. Then N has an A-module structure defined as follows: if $a \in A$ and $x \in N$, then ax is defined to be $f(a)x$. This A-module is said to be obtained from N by ***restriction of scalars****. In particular, f defines in this way an A-module structure on B.*

Definition 2.2.10 (Extension of scalars)

*Let M be an A-module. By Definition 2.2.9, B can be regarded as an A-module, we can form the A-module $B \otimes_A M$. In fact, $B \otimes_A M$ carries a B-module structure such that $b(b' \otimes x)$ for all $b, b' \in B$ and all $x \in M$. The B-module $B \otimes_A M$ is said to be obtained from M by **extension of scalars**.*

Remark

(1) Extension of scalars describes a functor $(\cdot) \otimes_A M : \mathbf{Mod}_A \to \mathbf{Mod}_B$.

(2) Let $A \to B$ and $A \to C$ be morphisms of rings, then $B \otimes_A C$ has a natural structure of a ring. (Multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$.)

Proposition 2.2.6

Let M be an A-module. Then the $S^{-1}A$-modules $S^{-1}M$ and $S^{-1}A \otimes_A M$ are isomorphism; more precisely, there exist a unique isomorphism $f : S^{-1}A \otimes_A M \to S^{-1}M$ for which

$$f((a/s) \otimes m) = am/s$$

for all $a \in A, m \in M, s \in S$

Proof The mapping $S^{-1}A \times M \to S^{-1}M$ defined by

$$(a/s, m) \mapsto am/s$$

is A-bilinear, and therefore by the universal property of the tensor product induces an A-homomorphism

$$f : S^{-1}A \otimes_A M \to S^{-1}M \tag{2.10}$$

satisfying $f((a/s) \otimes m) = am/s$ for all $a \in A$, $m \in M$, $s \in S$.

Clearly, f is surjective, and is A-linear. Also f is uniquely defined by (2.10). We shall to prove f is injective.

Let $\sum_i (a_i/s_i) \otimes m_i$ be any element of $S^{-1}A \otimes M$. If $s = \prod_i s_i \in S$, let $t_i = \prod_{j \neq i} s_j$, we have

$$\sum_i \frac{a_i}{s_i} \otimes m_i = \sum_i \frac{a_i t_i}{s} \otimes m_i = \sum_i \frac{1}{s} \otimes a_i t_i m_i = \frac{1}{s} \otimes \sum_i a_i t_i m_i,$$

so that every element of $S^{-1}A \otimes M$ is of the form $(1/s) \otimes m$. Suppose that $f((1/s) \otimes m) = 0$, thus $m/s = 0$, hence, $tm = 0$ for some $t \in S$, and therefore

$$\frac{1}{s} \otimes m = \frac{t}{st} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0.$$

Hence, f is injective, and therefore an isomorphism. □

Proposition 2.2.7 ($\otimes$ commutes with $\oplus$)

Tensor products commute with arbitrary direct sums: if M and $\{N_i\}$ are all A-modules, there is a A-isomorphism

$$\tau : M \otimes_A (\bigoplus_{i \in I} N_i) \to \bigoplus_{i \in I} (M \otimes_A N_i)$$

with $\tau : m \otimes (n_i) \mapsto (m \otimes n_i)$.

Proof Since the function $f : M \times (\bigoplus_i N_i) \to \bigoplus_i (M \otimes N_i)$ given by $f : (m, (n_i)) \mapsto (m \otimes n_i)$ is A-bilinear, by the universal property of tensor product, there is a A-homomorphism

$$\tau : M \otimes (\bigoplus_i N_i) \to \bigoplus (M \otimes N_i)$$

with $\tau : m \otimes (n_i) \mapsto (m \otimes n_i)$.

To see that τ is an isomorphism, we give its inverse. Denote the injection $N_k \to \bigoplus_i N_i$ by ι_k, where $\iota_k(n_k) \in \bigoplus_i N_i$ has k-th coordinates n_k and all other coordinates 0, so that $\mathrm{id}_M \otimes \iota_k : M \otimes N_k \to M \otimes (\bigoplus_i N_i)$. By Proposition 2.2.2, direct sum is the coproduct in ${}_R\mathbf{Mod}$ gives a homomorphism $\theta : \bigoplus_i (M \otimes N_i) \to M \otimes (\bigoplus_i N_i)$ with $\theta : (m \otimes n_i) \mapsto m \otimes \sum_i \iota_i(n_i)$. It is easy to check that θ is the inverse of τ, and therefore τ is an isomorphism. □

2.2.4 Fibered product and fibered coproduct

Definition 2.2.11 (Fibered product)

Suppose morphsims $\alpha : X \to Z$ and $\beta : Y \to Z$ (in any category). Then the ***fibered product*** *is an object $X \times_Z Y$ along with morphisms $\mathrm{pr}_X : X \times_Z Y \to X$ and $\mathrm{pr}_Y : X \times_Z Y \to Y$, where the two compositions $\alpha \circ \mathrm{pr}_X = \beta \circ \mathrm{pr}_Y$, such that given any object W with maps to X and Y whose compositions to Z agree (commute), these maps factor through some unique $W \to X \times_Z Y$:*

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "\mathrm{pr}_Y"'] \arrow[d, "\mathrm{pr}_X"] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

Remark By the usual universal property argument, if it exists, it is unique up to unique isomorphism.

Definition 2.2.12 (Diagonal morphism)

If $\pi : X \to Y$ is a morphism, and the fibered product $X \times_Y X$ exists, then determines a ***diagonal morphism*** *$\delta_\pi : X \to X \times_Y X$, which satisfy $\delta \circ \mathrm{pr} = \mathrm{id}_{X \times_Y X}$ and $\mathrm{pr} \circ \delta = \mathrm{id}_X$.*

Note *Depending on your religion, the diagram*

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

is called a ***fibered / pullback / Cartesian diagram / square****.*

The right way to interpret the notion of fibered product is first to think about what it means in the category of sets.

Exercise 2.10 Show that in **Sets**,

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}.$$

More precisely, show that the right side, equipped with its evident maps to X and Y, satisfies the universal property of the fibered product.

Proof Let W be an object with $\eta_X : W \to X$ and $\eta_Y : W \to Y$ whose compositions to Z agree $\alpha\eta_X = \beta\eta_Y$, we need to show there exists unique homomorphism maps W to $X \times_Z Y$.

$$\begin{tikzcd} W \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\eta_Y"] \arrow[ddr, bend right, "\eta_X"'] & & \\ & X \times_Z Y \arrow[r, "\mathrm{pr}_Y"'] \arrow[d, "\mathrm{pr}_X"] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

Define the map $\theta : W \to X \times_Z Y$ by $\theta(w) = (\eta_X(w), \eta_Y(w))$. The values of θ do lie in $X \times_Z Y$, for $\alpha\eta_X = \beta\eta_Y$. We need to show that the diagram commutes and that θ is unique.

To show commutative, let $w \in W$, note that

$$\mathrm{pr}_X(\theta(w)) = \mathrm{pr}_X(\eta_X(w), \eta_Y(w)) = \eta_X(w)$$

and

$$\mathrm{pr}_Y(\theta(w)) = \mathrm{pr}_Y(\eta_X(w), \eta_Y(w)) = \eta_Y(w),$$

this implies the diagram commutes.

To show uniqueness, let $\psi : W \to X \times_Z Y$ be another maps which is the solution to the universal mapping problem. Let $w \in W$, say $\psi(w) = (s(w), t(w))$, note that $\mathrm{pr}_X \theta = \eta_X$ and $\mathrm{pr}_Y \theta = \eta_Y$, we have $\mathrm{pr}_X \theta(w) = s(w) = \eta_X(w)$ and $\mathrm{pr}_Y \theta(w) = t(w) = \eta_Y(w)$. Hence, $\psi = \theta$. $\square$

✍ **Exercise 2.11** If X is a topological space, show that fibered products always exists in the category of open sets of X, by describing what a fibered product is.

Proof Let U, V be open subsets in the category of open sets of X, we claim that $U \cap V$ is the fibered product. Let Z be a open subset with inclusion $i : U \to Z$ and $i : V \to Z$. Let W be any open subset of X, which satisfy $W \subseteq U$ and $W \subseteq V$, then $W \subseteq U \cap V$. Hence, $U \cap V$ is the fibered product. $\square$

✍ **Exercise 2.12** If Z is the final object in a category $\mathbb{C}$, and $X, Y \in \mathscr{C}$, show that $X \times_Z Y = X \times Y$: the fibered product over Z is uniquely isomorphic to the product. Assume all relevant fibered products exists.

Proof It suffice to show that product satisfy the universal property of fibered product. Consider the following diagram.

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left, "\beta'"] \arrow[ddr, bend right, "\alpha'"'] & & \\ & X \times Y \arrow[r, "\mathrm{pr}_Y"'] \arrow[d, "\mathrm{pr}_X"] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd} \tag{2.11}$$

By the universal property of product, there exists unique $\theta : W \to X \times Y$ making the diagram

$$\begin{tikzcd} W \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\beta'"] \arrow[ddr, bend right, "\alpha'"'] & & \\ & X \times Y \arrow[r, "\mathrm{pr}_Y"'] \arrow[d, "\mathrm{pr}_X"] & Y \\ & X & \end{tikzcd}$$

commute. Then 2.11 commute, which implies that the fibered product over Z is isomorphic to the product. $\square$

✍ **Exercise 2.13** (Towers of cartesian diagrams are cartesian diagrams.) If the two squares in the following commutative diagram are Cartesian diagrams, show that the "outside rectangle" (involving U, V, Y, and Z) is also a Cartesian diagram.

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Proof Consider the diagram.

$$\begin{tikzcd}
K \arrow[rrd] \arrow[rd, dashed] \arrow[rdd, dashed] \arrow[rddd] & & \\
& U \arrow[r] \arrow[d] & V \arrow[d] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

We need to define a morphism from K to U such that above diagram commute. Since W is the fibered product over Z, we have:

$$\begin{tikzcd}
K \arrow[rrdd] \arrow[rdd, dashed, "\theta"] \arrow[rddd] & & \\
& & V \arrow[d] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}$$

Since U is the fibered product over X, we have:

$$\begin{tikzcd}
K \arrow[rrd] \arrow[rd, dashed, "\psi"] \arrow[rdd, "\theta"] & & \\
& U \arrow[r] \arrow[d] & V \arrow[d] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}$$

Then, we have

$$\begin{tikzcd}
K \arrow[rrd] \arrow[rd, dashed, "\psi"] \arrow[rdd] \arrow[rddd] & & \\
& U \arrow[r] \arrow[d] & V \arrow[d] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z,
\end{tikzcd}$$

as desired. □

✍ **Exercise 2.14** Given morphisms $X_1 \to Y$, $X_2 \to Y$, and $Y \to Z$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming that both fibered products exist.

Proof Since both fibered products exist, we have following commutative diagram immediately.

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& X_1 \times_Z X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \arrow[ddr] & \\
& X_1 \arrow[r] \arrow[drr] & Z & \\
& & & Y \arrow[ul]
\end{tikzcd}$$

□

✍ **Exercise 2.15 (The diagonal-base-change diagram)** Suppose we are given morphisms $X_1 \to Y$, $X_2 \to Y$, and $Y \to Z$. Show that the following diagram is a Cartesian square.

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\
Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

Assume all relevant (fibered) product exist.

Proof By condition, we have following Cartesian diagrams:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\mathrm{pr}^Y_{X_2}"] \arrow[d, "\mathrm{pr}^Y_{X_1}"'] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \times_Z X_2 \arrow[r, "\mathrm{pr}^Z_{X_2}"] \arrow[d, "\mathrm{pr}^Z_{X_1}"'] & X_2 \arrow[d, "g \circ f_2"] \\
X_1 \arrow[r, "g \circ f_1"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times_Z Y \arrow[r, "\mathrm{pr}^Z_Y"] \arrow[d, "\mathrm{pr}^Z_Y"'] & Y \arrow[d, "g"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

By Exercise 2.14, exists a morphism $\theta : X_1 \times_Y X_2 \to X_1 \times_Z X_2$. By the universal property of $Y \times_Z Y$,

$$\begin{tikzcd}
X_1 \times_Z X_2 \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "f_2 \circ \mathrm{pr}^Z_{X_2}"] \arrow[ddr, bend right, "f_1 \circ \mathrm{pr}^Z_{X_1}"'] & & \\
& Y \times_Z Y \arrow[r, "\mathrm{pr}^Z_Y"'] \arrow[d, "\mathrm{pr}^Z_Y"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"] & Z,
\end{tikzcd}$$

there exists a morphism $\eta : X_1 \times_Z X_2 \to Y \times_Z Y$ such that above diagram commute.

Now, we have a diagram,

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\theta"] \arrow[d, "f_1 \circ \mathrm{pr}^Y_{X_1}"'] & X_1 \times_Z X_2 \arrow[d, "\eta"] \\
Y \arrow[r, "\iota"] & Y \times_Z Y,
\end{tikzcd} \tag{2.12}$$

where ι be an inclusion and $\iota = (\mathrm{pr}^Z_Y)^{-1}$.

We need to show two things: (1) (2.12) commute; (2) (2.12) is Cartesian square.

(1) (2.12) is commute.

Note that $\eta = (\mathrm{pr}^Z_Y)^{-1} \circ f_1 \circ \mathrm{pr}^Z_{X_1}$ and $\mathrm{pr}^Z_{X_1} \circ \theta = \mathrm{pr}^Y_{X_1}$, hence

$$\eta \circ \theta = (\mathrm{pr}^Z_Y)^{-1} \circ f_1 \circ \mathrm{pr}^Z_{X_1} \circ \theta = \iota \circ (f_1 \circ \mathrm{pr}^Y_{X_1}),$$

which implies that (2.12) commute.

(2) (2.12) is Cartesian square.

Consider the diagram,

$$
\begin{tikzcd}
W \arrow[drr, "\beta"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, dashed, "\alpha'"'] \arrow[dddr, "\alpha"'] & & & \\
& X_1 \times_Y X_2 \arrow[r, "\theta"'] \arrow[d, "\mathrm{pr}^Y_{X_1}"] & X_1 \times_Z X_2 \arrow[dl, "\mathrm{pr}^Z_{X_1}"'] \arrow[dd, "\eta=(\mathrm{pr}^Z_Y)^{-1}\circ f_1\circ \mathrm{pr}^Z_{X_1}"] \arrow[dddr, bend left, "\mathrm{pr}^Z_{X_2}"] & \\
& X_1 \arrow[d, "f_1"] & & \\
& Y \arrow[r, "\iota=(\mathrm{pr}^Z_Y)^{-1}"] \arrow[dl, "g"] & Y \times_Z Y & \\
Z & & & X_2, \arrow[ull, "f_2"]
\end{tikzcd}
\tag{2.13}
$$

where W any object with $\eta\beta = \iota\alpha$.

Let $\alpha' = \mathrm{pr}^Z_{X_1} \circ\beta$. We shall to show that $f_2 \circ \mathrm{pr}^Z_{X_2} \circ\beta = f_1 \circ \alpha'$, i.e., show the following diagram commute.

$$
\begin{tikzcd}
W \arrow[r, "\mathrm{pr}^Z_{X_2}\circ\beta"] \arrow[d, "\alpha'"'] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & Y
\end{tikzcd}
$$

Since $f_1 \circ \mathrm{pr}^Z_{X_1} = f_2 \circ \mathrm{pr}^Z_{X_2}$, $f_2 \circ \mathrm{pr}^Z_{X_2} \circ\beta = f_1 \circ \alpha'$, hence above diagram commute. By the universal property of $X_1 \times_Y X_2$, there exists unique $\varphi : W \to X_1 \times_Y X_2$ such that (2.13) commute, as we desired. $\square$

✍ **Exercise 2.16** Show that coproduct for **Sets** is disjoint union. This why we use the notation $\coprod$ for disjoint union.

Proof Let $A, B \in$ **Sets**, let $A' = A \times \{1\}$ and $B' = B \times \{2\}$, then $A' \cap B' = \varnothing$. Define the function $\theta : A \coprod B = A' \cup B' \to X$ by

$$
\theta(u) = \begin{cases} f(a) & \text{if } u = (a, 1) \in A', \\ g(b) & \text{if } u = (b, 2) \in B'. \end{cases} \tag{2.14}
$$

The function θ is well-defined because $A' \cap B' = \varnothing$.

We shall to show that the disjoint union $A \coprod B = A' \cup B' \subseteq (A \cup B) \times \{1, 2\}$ is a coproduct in **Sets**. If X is any set and $f : A \to X$ and $g : B \to X$ are functions. θ makes the following diagram commute,

$$
\begin{tikzcd}
& B \arrow[d, "\beta"'] \arrow[ddr, "g"] & \\
A \arrow[r, "\alpha"] \arrow[drr, "f"'] & A \coprod B \arrow[dr, "\theta"] & \\
& & X,
\end{tikzcd}
$$

where $\alpha : A \to A \coprod B$ defined by $\alpha(a) = (a, 1)$ and $\beta : B \to A \coprod B$ defined by $\beta(b) = (b, 2)$.

Now, let us prove uniqueness of θ. If $\psi : A \coprod B \to X$ satisfies $\psi\alpha = f$ and $\psi\beta = g$, then $\psi(a, 1) = f(a) = \theta(a, 1)$ and $\psi(b, 2) = g(b) = \theta(b, 2)$. Hence, $\theta = \psi$. $\square$

Definition 2.2.13 (Fibered coproduct)

*Suppose morphisms $\alpha : Z \to X$ and $\beta : Z \to Y$ (in any category). Then the **fibered coproduct** is an object $X \cup_Z Y$ along with morphisms $\iota_X : X \to X \cup_Z Y$ and $\iota_Y : Y \to X \cup_Z Y$, where the two compositions $\iota_X \circ \alpha = \iota_Y \circ \beta$, such that given any object W with maps from X and Y whos compositions from Z agree, these maps factor through some unique $X \cup_Z Y \to W$:*

$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d, "\iota_Y"'] \arrow[ddr, bend left] & \\
X \arrow[r, "\iota_X"] \arrow[drr, bend right] & X \cup_Z Y \arrow[dr, dashed, "\exists!"] & \\
& & W.
\end{tikzcd}$$

Exercise 2.17 Suppose $A \to B$ and $A \to C$ are two ring homomorphisms, so in particular B and C are A-modules. $B \otimes_A C$ has a natural structure of a ring. (Multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$.) Show that there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. Similarly, there is a natural morphism $C \to B \otimes_A C$. Show that this givs a fibered coproduct on rings, i.e., that

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\iota_Y"'] \\
B \arrow[r, "\iota_X"] & B \otimes_A C
\end{tikzcd}$$

satisfies the universal property of fibered coproduct.

Proof We first show that the diagram

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\iota_Y"'] \\
B \arrow[r, "\iota_X"] & B \otimes_A C
\end{tikzcd}$$

is commute. Let $a \in A$, then

$$\iota_X \circ \alpha(a) = \alpha(a) \otimes_A 1 = a(1 \otimes_A 1)$$

and

$$\iota_Y \circ \beta(a) = 1 \otimes_A \beta(a) = a(1 \otimes_A 1).$$

Hence, the diagram commute.

Let W be an object with $\eta_B : B \to W$ and $\eta_C : C \to W$, which implies that $\eta_B \alpha = \eta_C \beta$.

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C \arrow[d, "\iota_Y"'] \arrow[ddr, bend left, "\eta_C"] & \\
B \arrow[r, "\iota_X"] \arrow[drr, bend right, "\eta_B"'] & B \otimes_A C \arrow[dr, dashed, "\theta"] & \\
& & W
\end{tikzcd}$$

Define $\theta : B \otimes_A C \to W$ by setting $\theta(b \otimes_A c) = \eta_B(b)\eta_C(c)$, then above diagram is commute.

To show the uniqueness of θ, we may suppose there exists $\psi : B \otimes_A C \to W$ such that above diagram commute. Note that $\psi(b \otimes_A c) = \psi(b \otimes 1)\psi(1 \otimes c) = \eta_B(b)\eta_C(c) = \theta(b \otimes_A c)$ for all $b \otimes_A c \in B \otimes_A C$, $\psi = \theta$. $\square$

2.2.5 Monomorphisms and epimorphsims

Definition 2.2.14 (Monomorphism)

*A morphism $\pi : X \to Y$ is a **monomorphism** if any two morphisms $\mu_1 : Z \to X$ and $\mu_2 : Z \to X$ such that $\pi \circ \mu_1 = \pi \circ \mu_2$ must satisfy $\mu_1 = \mu_2$. In other words, there is at most one way of filling in the dotted arrow such that the diagram*

$$\begin{tikzcd} Z \arrow[d, dotted, "\leq 1"'] \arrow[dr] & \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

commutes — for any object Z, the natural map (induced by π) $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$ is an injection.

Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets.

Remark The reason we don't use the word "injective" is that in some contexts, "injective" will have an intuitive meaning which may not agree with "monomorphism".

Example 2.23 In the category of divisible groups, the map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism but not injective.

Proof Clearly, $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not injective.

Let $\mu_1 : G \to \mathbb{Q}$ and $\mu_2 : G \to \mathbb{Q}$ be any two group homomorphism. Say $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Let $\pi \circ \mu_1 = \pi \circ \mu_2$, then $\pi(\mu_1(g)) = \pi(\mu_2(g))$ for all $g \in G$, hence, $(\mu_1 - \mu_2)(g) \in \mathbb{Z}$. We claim that $\mu_1 - \mu_2 = 0$, say $h = \mu_1 - \mu_2$, then $h : G \to \mathbb{Z}$. Since G is divisible, we have $\mathrm{Im}\, h$ is divisible. The only divisible subgroup of $\mathbb{Z}$ is 0, hence $\mathrm{Im}\, h = 0$, i.e., $\mu_1 = \mu_2$. □

Exercise 2.18 Show that the composition of two monomorphisms is a monomorphism.

Proof Let $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ be two monomorphism. Let $\pi_2 \circ \pi_1 \circ \mu_1 = \pi_2 \circ \pi_1 \circ \mu_2$, where $\mu_1 : Z \to X$ and $\mu_2 : Z \to X$. Since π_2 is monomorphism, $\pi_1 \circ \mu_1 = \pi_1 \circ \mu_2$. Since π_1 is monomorphism, $\mu_1 = \mu_2$. □

Exercise 2.19 Prove that a morphism $\pi : X \to Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal morphism $\delta_\pi : X \to X \times_Y X$ is an isomorphism. We may then take this as the definition of monomorphism.

Proof If π is a monomorphism. Consider the following diagram,

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y. \end{tikzcd}$$

Since π is a monomorphism, $\mathrm{pr}_1 = \mathrm{pr}_2 \triangleq \mathrm{pr}$. Consider the diagram,

$$\begin{tikzcd} X \arrow[dr, dashed, "\theta"] \arrow[drr, "\mathrm{id}"] \arrow[ddr, "\mathrm{id}"'] & & \\ & X \times_Y X \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}"'] & X \arrow[d, "\pi"] \\ & X \arrow[r, "\pi"] & Y \end{tikzcd}$$

there exists a unique morphism $\theta : X \to X \times_Y X$ such that $\mathrm{id} = \mathrm{pr}\, \theta$, by the universal property of fibered product. Let $\delta_\pi = \theta$, it is an isomorphism.

Conversely, if the fibered product $X \times_Y X$ exists, and the induced diagonal morphism $\delta_\pi : X \to X \times_Y X$

is an isomorphism. Consider the following diagram,

$$\begin{tikzcd} W \arrow[drr, bend left, "\eta_2"] \arrow[dr, dashed, "\theta"] \arrow[ddr, "\eta_1"'] & & \\ & X \times_Y X \arrow[r, "\delta_\pi"] \arrow[d, "\delta_\pi"'] & X \arrow[d, "\pi"] \\ & X \arrow[r, "\pi"] & Y, \end{tikzcd}$$

where W is any object with $\pi\eta_1 = \pi\eta_2$. We shall to show that $\eta_1 = \eta_2$. By the universal property of fibered product, there exists unique $\theta : W \to X \times_Y X$ such that $\eta_1 = \delta_\pi\theta$ and $\eta_2 = \delta_\pi\theta$, then $\eta_1 = \eta_2$, which implies that $\pi : X \to Y$ is an monomorphism. □

✍ **Exercise 2.20** We use the notation of Exercise 2.14. Show that if $Y \to Z$ is a monomorphism, then the morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ you described in Exercise 2.14 is an isomorphism.

Proof Say $\pi : Y \to Z$. By Exercise 2.19, $\delta_\pi : Y \to Y \times_Z Y$ is an isomorphism. Consider the diagonal-base-change diagram (Exercise 2.15),

$$\begin{tikzcd} X_1 \times_Z X_2 \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "\theta"] \arrow[ddr, "\delta_\pi^{-1}\circ\varphi"'] & & \\ & X_1 \times_Y X_2 \arrow[r, "\eta"'] \arrow[d, "\psi"] & X_1 \times_Z X_2 \arrow[d, "\varphi"] \\ & Y \arrow[r, "\delta_\pi"] & Y \times_Z Y, \end{tikzcd}$$

there exists unique $\theta : X_1 \times_Z X_2 \to X_1 \times_Y X_2$ such that above diagram commute, then we have $\eta\theta = \mathrm{id}$, that is, θ is an isomorphism. □

The notion of an **epimorphism** is "dual" to the definition of monomorphism, where all the arrows are reversed. This concept will not be central for us, although it turns up in the defnition of an abelian category. Intuitively, it is the categorical version of a surjective map.

Definition 2.2.15 (Epimorphism)

A morphism $\pi : X \to Y$ is a ***epimorphism*** *if for any two morphisms $\mu_1 : Y \to Z$ and $\mu_2 : Y \to Z$ such that $\mu_1 \circ \pi = \mu_2 \circ \pi$ we have $\mu_1 = \mu_2$. In other words, there is at most one way of filling in the dotted arrow such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr] & Y \arrow[d, dotted, "\leq 1"] \\ & Z \end{tikzcd}$$

commutes — for any object Z, the natural map (induced by π) $\mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$ is an injection.

Be careful when working with categories of objects that are sets with additional structure, as epimorphisms need not be surjective.

Example 2.24 In the category **Rings**, $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, but not surjective.

Proof Say $\pi : \mathbb{Z} \to \mathbb{Q}$. Let $R \in$ **Rings**, and homomorphisms $\mu_1 : \mathbb{Q} \to R$ and $\mu_2 : \mathbb{Q} \to R$, let $\mu_1 \circ \pi = \mu_2 \circ \pi$, then $\mu_1 \circ \pi(n) = \mu_2 \circ \pi(n)$ for all $n \in \mathbb{Z}$. Since $\pi(n) = n\pi(1) = n$, we have $\mu_1(n) = \mu_2(n)$ for all $n \in \mathbb{Z}$, which implies that π is epimorphism. □

2.2.6 Representable functors and Yoneda's Lemma

Much of our discussion about universal properties can be cleany expressed in termes of representable functors, under the rubric of "Yoneda's Lemma". Yoneda's lemma is an easy fact stated in a complicated way. Informally speaking, you can essentially recover an object in a category by knowing the maps into it. For example, we have seen that the data of maps to $X \times Y$ are naturally (canonically) the data of maps to X and to Y. Indeed, we have now taken this as the definition of $X \times Y$.

Recall Example 2.19 . Suppose A is an object of category $\mathscr{C}$. For any object $C \in \mathscr{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \xrightarrow{\circ f} \mathrm{Mor}(B, A),$$

by composition: given a map from C to A, we get a map from B to A by precomposing with $f : B \to C$. Hence this gives a contravariant functor $h_A = \mathrm{Mor}(\Box, A) : \mathscr{C} \to \mathbf{Sets}$ (or covariant functor $h_A : \mathscr{C}^{op} \to \mathbf{Sets}$). Yoneda's Lemma states that the functor h_A determines A up to unique isomorphism.

Note *If $F, G : A \to B$ are functors of the same variance, then*

$$\mathrm{Nat}(F, G) = \{\textit{natural transformations } F \to G\}.$$

Theorem 2.2.1 (Yoneda's Lemma)

Let $\mathscr{C}$ be a category, let $A \in \mathrm{obj}(\mathscr{C})$. For every functor $G : \mathscr{C}^{op} \to \mathbf{Sets}$, the map

$$y : \mathrm{Nat}(h_A, G) \to G(A), \quad \tau \mapsto \tau_A(\mathrm{id}_A)$$

is a bijection, where $h_A = \mathrm{Mor}(\Box, A)$.

Proof If $\tau : \mathrm{Mor}_{\mathscr{C}}(\Box, A) \to G$ is a natual transformation, then $y(\tau) = \tau_A(\mathrm{id}_A)$ lies in the set $G(A)$, for $\tau_A : \mathrm{Mor}_{\mathscr{C}}(A, A) \to G(A)$. Thus, y is a well-defined function.

Since $\tau \in \mathrm{Nat}(h_A, G)$ is a natural transformation, for each $B \in \mathrm{obj}(\mathscr{C})$ and $\varphi \in \mathrm{Mor}_{\mathscr{C}}(B, A)$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{C}}(A, A) \arrow[r, "\tau_A"] \arrow[d, "\Box\circ\varphi"'] & G(A) \arrow[d, "G(\varphi)"] \\
\mathrm{Mor}_{\mathscr{C}}(B, A) \arrow[r, "\tau_B"] & G(B)
\end{tikzcd}$$

so that

$$(G\varphi)\tau_A(\mathrm{id}_A) = \tau_B(\Box \circ \varphi)(\mathrm{id}_A) = \tau_B(\mathrm{id}_A \circ \varphi) = \tau_B(\varphi).$$

If $\sigma : \mathrm{Mor}_{\mathscr{C}}(\Box, A) \to G$ is another natural transformation, then $\sigma_B(\varphi) = (G\varphi)\sigma_A(\mathrm{id}_A)$. Hence, if $\sigma_A(\mathrm{id}_A) = \tau_A(\mathrm{id}_A)$, then $\sigma_B = \tau_B$ for all $B \in \mathrm{obj}(\mathscr{C})$, and therefore $\sigma = \tau$, which implies y is an injection.

To see that y is a surjection, take $x \in G(A)$. For $B \in \mathrm{obj}(\mathscr{C})$ and $\psi \in \mathrm{Mor}(B, A)$, define

$$\tau_B(\psi) = (G\psi)(x)$$

(note that $G\psi : GA \to GB$, hence $(G\psi)(x) \in GB$). We claim that τ is a natural transformation, that is, if $\theta : C \to B$ is a morphism in $\mathscr{C}$, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{C}}(B, A) \arrow[r, "\tau_B"] \arrow[d, "\Box\circ\theta"'] & GB \arrow[d, "G\theta"] \\
\mathrm{Mor}_{\mathscr{C}}(C, A) \arrow[r, "\tau_C"] & GC.
\end{tikzcd}$$

Going clockwise, we have $(G\theta)\tau_B(\psi) = G\theta G\psi(x)$; going counterclockwise, we have $\tau_C(\Box \circ \theta)\psi = \tau_C(\psi \circ$

$\theta) = G(\psi\theta)(x)$. Since G is a contravariant functor, $G(\psi\theta) = (G\theta)(G\psi)$, hence $(G\theta)\tau_B = \tau_C(\Box \circ \theta)$, thus, τ is a natural transformation. Now $y(\tau) = \tau_A(\mathrm{id}_A) = G(\mathrm{id}_A)(x) = x$, and so y is bijection. $\Box$

Exercise 2.21 (Another form of Yoneda's Lemma)

(a) There are two objects A and A' in a category $\mathscr{C}$, and the commutative diagram.

$$\begin{tikzcd} \mathrm{Mor}(C,A) \arrow[r, "i_C"] \arrow[d, "\mathrm{Mor}(\Box,A)"'] & \mathrm{Mor}(C,A') \arrow[d, "\mathrm{Mor}(\Box,A')"] \\ \mathrm{Mor}(B,A) \arrow[r, "i_B"] & \mathrm{Mor}(B,A') \end{tikzcd}$$

Then i_C are induced from a unique morphism $g : A \to A'$. More precisely, there is a unique morphsim $g : A \to A'$ such that for all $C \in \mathscr{C}$, i_C is $u \mapsto g \circ u$.

(b) If furthermore the i_C are all bijections, then the resulting g is an isomorphism.

Proof

(a) Since $i : \mathrm{Mor}(\Box, A) \to \mathrm{Mor}(\Box, A')$ is a natural transformation, for each $C \in \mathrm{obj}(\mathscr{C})$ and $u \in \mathrm{Mor}(C, A)$, there is a commutative diagram.

$$\begin{tikzcd} \mathrm{Mor}(A,A) \arrow[r, "i_A"] \arrow[d, "\Box\circ u"'] & \mathrm{Mor}(A,A') \arrow[d, "\Box\circ u"] \\ \mathrm{Mor}(C,A) \arrow[r, "i_C"] & \mathrm{Mor}(C,A) \end{tikzcd}$$

Hence, $(\Box \circ u)i_A(\mathrm{id}_A) = (i_A\,\mathrm{id}_A) \circ u = i_C(\Box \circ u)(\mathrm{id}_A) = i_C u$, let $g = i_A(\mathrm{id}_A)$, as we desired.

(b) i is natual isomorphism. By the Yoneda's Lemma 2.2.1, $y : i \mapsto i_A(\mathrm{id}_A) = g$ is bijection, g must be an isomorphism.

$\Box$

Theorem 2.2.2 (Yoneda's Lemma for covariant functors)

*Let $\mathscr{C}$ be a category, let $A \in \mathrm{obj}(\mathscr{C})$. For every covariant functor $G : \mathscr{C} \to$ **Sets**, the map*

$$y : \mathrm{Nat}(h^A, G) \to G(A), \quad \tau \mapsto \tau_A(\mathrm{id}_A)$$

is a bijection, where $h^A = \mathrm{Mor}(A, \Box)$.

Exercise 2.22 Suppose A and B are objects in a category $\mathscr{C}$. Give a bijection between the natural transformations $h^A \to h^B$ of covariant functors $\mathscr{C} \to$ **Sets** and the morphisms $B \to A$.

Proof By Yoneda's Lemma 2.2.2, the map

$$\mathrm{Nat}(h^A, h^B) \to h^B(A) = \mathrm{Mor}(B, A), \quad \tau \mapsto \tau_A(\mathrm{id}_A)$$

is a bijection. $\Box$

Definition 2.2.16 (Representable)

*A covariant functor $F : \mathscr{C} \to$ **Sets** is said to be **representable** if there exists $A \in \mathrm{obj}(\mathscr{C})$ with natural isomorphism $F \cong \mathrm{Mor}_{\mathscr{C}}(A, \Box) = h^A$.*

*A contravariant functors $G : \mathscr{C} \to$ **Sets** is said to be **represantable** if there exists $A \in \mathrm{obj}(\mathscr{C})$ with natural isomorphism $G \cong \mathrm{Mor}_{\mathscr{C}}(\Box, A) = h_A$.*

In fancy terms, Yoneda's lemma states the following. Given a category $\mathscr{C}$, we can produce a new category, called the **functor category of** $\mathscr{C}$ (denote by $\mathbf{Sets}^{\mathscr{C}^{op}}$), where the objects are contravariant functors $\mathscr{C} \to$ **Sets**, and the morphisms are natural transformations of such functors. We have a functor (which we can usefully call h) from $\mathscr{C}$ to its functor category, which sends A to h_A. Yoneda's Lemma states that this is a fully faithful functor, called the Yoneda embedding.

Definition 2.2.17 (Small category)

*A category $\mathscr{C}$ is **small** if the objects form a set and the morphisms form a set.*

Corollary 2.2.6 (Yoneda Imbedding)

If $\mathscr{C}$ is a small category, then there is a functor $h : \mathscr{C} \to \mathbf{Sets}^{\mathscr{C}^{op}}$ that is injective on objects and whose image is a full subcategory of $\mathbf{Sets}^{\mathscr{C}^{op}}$.

Proof Define h on objects by $h(A) = h_A = \mathrm{Mor}_{\mathscr{C}}(\square, A)$. If $A \neq A'$, then pairwise disjointness of Mor sets gives $\mathrm{Mor}(\square, A) \neq \mathrm{Mor}(\square, A')$, that is, $h_A \neq h_{A'}$, and so h is injective on objects. If $\psi : A \to B$ is a morphism in $\mathscr{C}$, then there is a natural transformation $h(\psi) : \mathrm{Mor}(\square, A) \to \mathrm{Mor}(\square, B)$ with $h(\psi)_C = \psi \circ \square$, for all $C \in \mathrm{obj}(\mathscr{C})$, by Yoneda's Lemma 2.2.1. Now, we have $h(\psi\eta) = h(\psi)h(\eta)$, and so h is covariant functor. Surjectivity of the Yoneda's function y in Theorem 2.2.1 shows that every natural transformation $h_A \to h_B$ arises as $h(\psi)$ for some ψ. Therefore, the image of h is a full subcategory of the functor category $\mathbf{Sets}^{\mathscr{C}^{op}}$. $\square$

Corollary 2.2.7 (Useful corollary)

Let $\mathscr{C}$ be a category and let $A, B \in \mathrm{obj}(\mathscr{C})$.

(i) If $\tau : \mathrm{Mor}_{\mathscr{C}}(A, \square) \to \mathrm{Mor}_{\mathscr{C}}(B, \square)$ is a natural transformation, then for all $C \in \mathrm{obj}(\mathscr{C})$, we have $\tau_C = \psi^*$, where $\psi = \tau_A(\mathrm{id}_A) : B \to A$ (and ψ^* is the induced map $\mathrm{Hom}_{\mathscr{C}}(A, C) \to \mathrm{Hom}_{\mathscr{C}}(B, C)$ given by $\varphi \mapsto \varphi \circ \psi$). Moreover, the morphism ψ is unique: if $\tau_C = \theta^*$, then $\theta = \psi$.

(ii) Let

$$\mathrm{Mor}_{\mathscr{C}}(A, \square) \xrightarrow{\tau} \mathrm{Mor}_{\mathscr{C}}(B, \square) \xrightarrow{\sigma} \mathrm{Mor}_{\mathscr{C}}(B', \square)$$

be natural transformations. If $\sigma_C = \eta^*$ and $\tau_C = \psi^*$ for all $C \in \mathrm{obj}(\mathscr{C})$, then

$$(\sigma \circ \tau)_C = (\psi \circ \eta)^*.$$

(iii) If $\mathrm{Mor}_{\mathscr{C}}(A, \square)$ and $\mathrm{Mor}_{\mathscr{C}}(B, \square)$ are naturally isomorphic functor, then $A \cong B$. (The converse is clearly true.)

Proof

(i) Denote $\psi = \tau_A(\mathrm{id}_A) \in \mathrm{Mor}_{\mathscr{C}}(B, A)$. Since τ is a natural transformation, for all $C \in \mathrm{obj}(\mathscr{C})$, for any $\varphi \in \mathrm{Mor}_{\mathscr{C}}(A, C)$, we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{C}}(A, A) \arrow[r, "\mathrm{Mor}_{\mathscr{C}}(A,\varphi)"] \arrow[d, "\tau_A"'] & \mathrm{Mor}_{\mathscr{C}}(A, C) \arrow[d, "\tau_C"] \\
\mathrm{Mor}_{\mathscr{C}}(B, A) \arrow[r, "\mathrm{Mor}_{\mathscr{C}}(B,\varphi)"] & \mathrm{Mor}_{\mathscr{C}}(B, C)
\end{tikzcd}$$

Hence,

$$\begin{aligned}
\varphi_* \circ \tau_A(\mathrm{id}_A) =& \varphi_* \circ \psi = \varphi \circ \psi \\
=& \tau_C \circ \varphi_*(\mathrm{id}_A) = \tau_C(\varphi \circ \mathrm{id}_A) \\
=& \tau_C(\varphi)
\end{aligned}$$

Note that $\varphi \circ \psi = \psi^*(\varphi)$, we have

$$\tau_C = \psi^*.$$

By Yoneda Lemma (Theorem 2.2.2), $\tau \longleftrightarrow \tau_A(\mathrm{id}_A)$ is bijective, the uniqueness assertion follows from injectivity of the Yoneda function y.

(ii) By part (i), there are unique morphisms $\psi \in \mathrm{Mor}_{\mathscr{C}}(B, A)$ and $\eta \in \mathrm{Mor}_{\mathscr{C}}(B', B)$ with

$$\tau_C(\varphi) = \psi^*(\varphi) \quad \text{and} \quad \sigma_C(\varphi') = \eta^*(\varphi')$$

for all $\varphi \in \mathrm{Mor}_{\mathscr{C}}(A, C)$ and $\varphi' \in \mathrm{Mor}_{\mathscr{C}}(B, C)$. By definition, $(\sigma \circ \tau)_C = \sigma_C \circ \tau_C$, and therefore

$$(\sigma \circ \tau)_C(\varphi) = \sigma_C(\psi^*(\varphi)) = \eta^* \circ \psi^*(\varphi) = (\psi \circ \eta)^*(\varphi).$$

(iii) If $\tau : \mathrm{Mor}_{\mathscr{C}}(A, \square) \to \mathrm{Mor}_{\mathscr{C}}(B, \square)$ is a natural isomorphism, then there is a natural isomorphism $\sigma : \mathrm{Mor}_{\mathscr{C}}(B, \square) \to \mathrm{Mor}_{\mathscr{C}}(A, \square)$ with $\sigma \circ \tau = \mathrm{id}_{\mathrm{Mor}_{\mathscr{C}}(A,\square)}$ and $\tau \circ \sigma = \mathrm{id}_{\mathrm{Mor}_{\mathscr{C}}(B,\square)}$. By part (i), there are morphisms $\psi : B \to A$ and $\eta : A \to B$ with $\tau_C = \psi^*$ and $\sigma_C = \eta^*$ for all $C \in \mathrm{obj}(\mathscr{C})$. By part (ii), we have $\tau \circ \sigma = \psi^* \eta^* = (\eta \circ \psi)^* = \mathrm{id}_B^*$ and $\sigma \circ \tau = (\psi\eta)^* = \mathrm{id}_A^*$. The uniqueness in part (i) now gives $\psi\eta = \mathrm{id}_A$ and $\eta\psi = \mathrm{id}_B$, which implies that $A \cong B$.

□

2.3 Limits and colimits

Definition 2.3.1 (Index category)

Suppose $\mathscr{I}$ is any small category, and $\mathscr{C}$ is any category. Then a functor $F : \mathscr{I} \to \mathscr{C}$ (i.e., with an object $A_i \in \mathscr{C}$ for each element $i \in \mathscr{I}$, and appropriate commuting morphisms dictated by $\mathscr{I}$) is said to be a ***diagram indexed by*** $\mathscr{I}$. *We call $\mathscr{I}$ an* ***index category***.

Remark Our index categories will usually be partially ordered sets, in which in particullar there is at most one morphism between any two objects.

Example 2.25 If $\square$ is the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

and $\mathscr{A}$ is a category, then a functor $\square \to \mathscr{A}$ is precisely the data of a commuting square in $\mathscr{A}$.

Definition 2.3.2 (Limit)

The ***limit of the diagram*** *is an object $\lim\limits_{\mathscr{I}} A_i$ (or $\varprojlim\limits_{\mathscr{I}} A_i$) of $\mathscr{C}$ along with morphisms $f_j : \varprojlim\limits_{\mathscr{I}} A_i \to A_j$ for each $j \in \mathscr{I}$, such that if $m : j \to k$ is a morphism in $\mathscr{I}$, then*

$$\begin{tikzcd} \varprojlim\limits_{\mathscr{I}} A_i \arrow[d, "f_j"] \arrow[dr, "f_k"] & \\ A_j \arrow[r, "F(m)"] & A_k \end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property. More precisely, given any other object W along with maps $g_i : W \to A_i$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in $\mathscr{I}$, then $g_k = F(m) \circ g_j$), then there is a unique map

$$g : W \to \varprojlim_{\mathscr{I}} A_i$$

such that $g_i = f_i \circ g$ for all i, i.e,

$$\begin{tikzcd}
\varprojlim_{\mathscr{I}} A_i \arrow[dr, "f_j"] \arrow[ddr, bend right, "f_k"'] & & W \arrow[ll, dashed, "g"'] \arrow[dl, "g_j"'] \arrow[ddl, bend left, "g_k"] \\
& A_j \arrow[d, "F(m)"] & \\
& A_k. &
\end{tikzcd}$$

In some cases, the limit is sometimes called the ***inverse limit*** *or* ***projective limit***.

Remark By the universal property argument, if the limit exists, it is unique up to unique isomorphism.

Example 2.26 (Products.) For example, if $\mathscr{I}$ is the partially ordered set

$$\begin{tikzcd}
& \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

we obtain the fibered product (reveiw the definition of fibered product).

If $\mathscr{I}$ is

$$\bullet \qquad \bullet$$

we obtain the product (reveiw the definition of product).

If $\mathscr{I}$ is a set (i.e., the only morphisms are the identity maps), then the limit is called the **product** of the A_i, and is denoted $\prod_i A_i$.

Exercise 2.23 Suppose that the partially ordered set $\mathscr{I}$ has an initial object e. Show that the limit of any diagram indexed by $\mathscr{I}$ exists.

Proof Say $F : \mathscr{I} \to \mathscr{C}$. We claim that $\varprojlim F(i) = F(e)$. Since e is initial object, set $m : i \to j$, we have the following commutative diagram.

$$\begin{tikzcd}
F(e) \arrow[d, "f_i"'] \arrow[dr, "f_j"] & \\
F(i) \arrow[r, "F(m)"] & F(j)
\end{tikzcd}$$

Let W be an object in $\mathscr{C}$, which makes the following diagram commute.

$$\begin{tikzcd}
W \arrow[d, "g_i"'] \arrow[dr, "g_j"] & \\
F(i) \arrow[r, "F(m)"] & F(j)
\end{tikzcd}$$

Then there is a naturally morphism $g_e : W \to F(e)$, since the commutative diagram

$$\begin{tikzcd}
W \arrow[d, "g_e"'] \arrow[dr, "g_i"] & \\
F(e) \arrow[r] & F(i)
\end{tikzcd}$$

Hence,

$$\begin{tikzcd}
\varprojlim F(i) = F(e) \arrow[dr, "f_i"] \arrow[ddr, bend right, "f_j"'] & & W \arrow[ll, "g_e"'] \arrow[dl, "g_i"'] \arrow[ddl, bend left, "g_j"] \\
& F(i) \arrow[d, "F(m)"] & \\
& F(j), &
\end{tikzcd}$$

as we desired. □

✍ **Exercise 2.24 (The diagonal-base-change diagram)** Solve Exercise 2.15 again by identifiying both $X_1 \times_Y X_2$ and $Y \times_{(Y\times_Z Y)} (X_1 \times_Z X_2)$ as the limit of the diagram.

$$\begin{tikzcd}
X_1 \arrow[dr] & & \\
& Y \arrow[r] & Z \\
X_2 \arrow[ur] & &
\end{tikzcd}$$

Proof $X_1 \times_Y X_2$ is the fibered product over Y, it is the limit of $X_1 \longrightarrow Y \longleftarrow X_2$. Consider the following diagrams.

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[rrrd] \arrow[dr, dashed] \arrow[ddddr] & & & \\
& X_1 \times_Z X_2 \arrow[rr] \arrow[ddd] \arrow[dr, dashed] & & X_2 \arrow[d] \\
& & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& & & \\
& X_1 \arrow[r] & Y \arrow[r] \arrow[ur, "\mathrm{id}"] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr] \arrow[dr, dashed] \arrow[ddr, bend right] & & \\
& Y \times_{(Y\times_Z Y)} (X_1 \times_Z X_2) \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\
& X_1 \times_Z X_2 \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

We see that $X_1 \times_Y X_2$ and $Y \times_{(Y\times_Z Y)} (X_1 \times_Z X_2)$ are both limit of $X_1 \times_Z X_2 \longrightarrow Y \times_Z Y \longleftarrow Y$, by the universal property of limit, we done. □

Example 2.27 (Formal power series) For a ring A, the **(formal) power series)**, $A[[x]]$, are often described informally (and somewhat unnaturally) as being the ring

$$A[[x]] = \{a_0 + a_1 x + a_2 x^2 + \cdots\}$$

where $a_i \in A$, and the ring operations are the "obvious" ones. It is an example of a limit in the category of rings:

$$
\begin{tikzcd}
A[[x]] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] & A[x]/(x^3) \arrow[r] & A[x]/(x^2) \arrow[r] & A[x]/(x). \\
A[x] \arrow[ur] \arrow[urr] \arrow[urrr] \arrow[uu, dashed, bend left] & & &
\end{tikzcd}
$$

The universal property of limits yields a natural ring morphism $A[x] \to A[[x]]$. If $A = \mathbb{R}$ or $\mathbb{C}$, this map factors through the ring of **convergent power series**.

Example 2.28 (The p-adic integers) For a prime number p, the **p-adic integers**, $\mathbb{Z}_p$, are often described informally, and somewhat unnaturally, as being of the form

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

where $0 \le \alpha_i < p$. They are an example of a limit in the category of rings:

$$
\begin{tikzcd}
\mathbb{Z}_p \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] & \mathbb{Z}/(p^3) \arrow[r] & \mathbb{Z}/(p^2) \arrow[r] & \mathbb{Z}/(p).
\end{tikzcd}
$$

Remark Limits do not always exist for any index category $\mathscr{I}$. However, we can often easily check that limits exists if the objects of your category can be interpreted as sets with additional structure, and arbitrary products exist.

Exercise 2.25 Show that in the category **Sets**,

$$\left\{ (a_i)_{i\in\mathscr{I}} \in \prod_i A_i : F(m)(a_j) = a_k \text{ for all } m \in \mathrm{Mor}_{\mathscr{I}}(j,k) \subseteq \mathrm{Mor}(\mathscr{I}) \right\},$$

along with the obvious projection maps to each A_i, is the limit $\varprojlim_{\mathscr{I}} A_i$.

Proof Say $L = \{(a_i)_{i\in\mathscr{I}} \in \prod_i A_i : F(m)(a_j) = a_k \text{ for all } m \in \mathrm{Mor}_{\mathscr{I}}(j,k) \subseteq \mathrm{Mor}(\mathscr{I})\}$. The definition of L gives a commutative diagram.

$$
\begin{tikzcd}
L \arrow[d, "\mathrm{pr}_i"'] \arrow[dr, "\mathrm{pr}_j"] & \\
A_i \arrow[r] & A_j.
\end{tikzcd}
$$

Consider the diagram,

$$
\begin{tikzcd}
L \arrow[dr, "\mathrm{pr}_i"] \arrow[ddr, bend right, "\mathrm{pr}_j"'] & & W \arrow[ll, dashed, "\theta"'] \arrow[dl, "f_i"'] \arrow[ddl, bend left, "f_j"] \\
& A_i \arrow[d, "F(m)"] & \\
& A_j, &
\end{tikzcd}
$$

where W is any object which satisfy $f_j = F(m) \circ f_i$.

Define $\theta : W \to L$ by setting $\theta(w) = (f_i(w))_{i\in\mathscr{I}}$, then θ is well-defined morphism and such that above diagram commute.

To show the uniqueness of θ, let $\psi : W \to L$ be another morphism such that above diagram commute.

Note that

$$\mathrm{pr}_i \circ \psi(w) = f_i(w),$$

we have $\psi(w) = (f_i(w))_{i \in \mathscr{I}}$. Hence, $L = \varprojlim_{\mathscr{I}} A_i$ □

Remark

(1) This clearly also works in the category $\mod_A$ of A-modules (in particular $\mathbf{Vec}_k$ and $\mathbf{Ab}$), as well as $\mathbf{Rings}$.

(2) From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2+3p, 2+3p+2p^2, \cdots)$.

Definition 2.3.3 (Colimits)

*The **colimit of the diagram** is an object $\operatorname{colim}_{\mathscr{I}} A_i$ (or $\varinjlim_{\mathscr{I}} A_i$) of $\mathscr{C}$ along with morphisms $f_j : A_j \to \varinjlim_{\mathscr{I}} A_i$ for each $j \in \mathscr{I}$, such that if $m : j \to k$ is a morphism in $\mathscr{I}$, then*

$$\begin{tikzcd} A_j \arrow[r, "F(m)"] \arrow[d, "f_j"] & A_k \arrow[dl, "f_k"] \\ \varinjlim_{\mathscr{I}} A_i & \end{tikzcd}$$

commutes, and this object and maps from each A_i are universal (initial) with respect to this property. More precisely, given any other object W along with maps $g_i : A_i \to W$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in $\mathscr{I}$, then $g_j = g_k \circ F(m)$), then there is a unique map

$$g : \varinjlim_{\mathscr{I}} A_i \to W$$

such that $g_j = g \circ f_j$ for all i, i.e.,

$$\begin{tikzcd} \varinjlim_{\mathscr{I}} A_i \arrow[rr, dashed, "g"] & & W \\ & A_j \arrow[ul, "f_j"] \arrow[ur, "g_j"'] \arrow[d, "F(m)"] & \\ & A_k. \arrow[uul, bend left, "f_k"] \arrow[uur, bend right, "g_k"'] & \end{tikzcd}$$

*In some cases, the colimit is sometimes called the **direct limit**, **inductive limit**, or **injective limit**.*

Example 2.29 The abelian group $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim_{i \in \mathbb{Z}^+} 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of the diagram

$$\begin{tikzcd} 5^{-\infty}\mathbb{Z} & & & \\ \mathbb{Z} \arrow[u] \arrow[r] & 5^{-1}\mathbb{Z} \arrow[ul] \arrow[r] & 5^{-2}\mathbb{Z} \arrow[ull] \arrow[r] & \cdots \end{tikzcd}$$

in the category of abelian groups.

Example 2.30 The colimit over an index set I is called the **coproduct**, denoted $\coprod_i A_i$, and is the dual (arrow-reversed) notion to the product.

Exercise 2.26

(a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$".

(b) Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.) The objects of the category in question are the subsets of given set.

Proof

(a) Note that $\frac{1}{m}\mathbb{Z} \subseteq \frac{1}{n}\mathbb{Z}$ if $m \mid n$. Then we have the commutative diagram.

$$\begin{tikzcd}
\mathbb{Q} \arrow[rr, dashed, "g"] & & W \\
& \frac{1}{m}\mathbb{Z} \arrow[ul, "\iota"'] \arrow[ur, "\alpha_m"] \arrow[d, "e"] & \\
& \frac{1}{n}\mathbb{Z}, \arrow[uul, bend left, "\iota"] \arrow[uur, bend right, "\alpha_n"'] &
\end{tikzcd}$$

where ι, e be inclusion and W is any object which satisfies $\alpha_m = \alpha_n \circ e$.

Define $g : \mathbb{Q} \to W$ by setting $g(p/q) = \alpha_q$, then above diagram commutes.

To show the uniqueness of g, suppose $f : \mathbb{Q} \to W$ is another morphism which such that above diagram commute. Note that $\alpha_q = f \circ \iota$, we have $f(p/q) = \alpha_q(p/q) = g(p/q)$. Hence, $f = g$, and therefore $\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$.

(b) Morphism in category (the subsets of given set) given by inclusion, i.e., $A_i \to A_j$ if $A_i \subseteq A_j$. Let $\{A_i\}$ be a family of subsets. We consider the following two commutative diagrams.

$$\begin{tikzcd}
\bigcup_i A_i \arrow[rr, dashed, "\theta"] & & W \\
& A_j \arrow[ul, "\iota_j"'] \arrow[ur, "\alpha_j"'] \arrow[d, "i"] & \\
& A_k \arrow[uul, bend left, "\iota_k"] \arrow[uur, bend right, "\alpha_k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigcap_i A_i \arrow[dr, "e_j"] \arrow[ddr, bend right, "e_k"'] & & V \arrow[ll, dashed, "\varphi"'] \arrow[dl, "\beta_j"'] \arrow[ddl, bend left, "\beta_k"] \\
& A_j \arrow[d, "i"] & \\
& A_k &
\end{tikzcd}$$

Define $\theta : \bigcup_i A_i \to W$ by setting $\theta(a) = \alpha_t(a)$, where $a \in A_t$. Since $a \in \bigcup_i A_i$, $a \in A_t$ for some t, hence, θ is well defined and such that left diagram commutes. Define $\varphi : V \to \bigcap_i A_i$ by setting $\varphi(v) = \beta_t$ for all $t \in \mathscr{I}$. Since $\beta_k = i \circ \beta_j$, φ is well-defined and such that right diagram commutes.

To show the uniqueness of θ, suppose there is $\theta' : \bigcup_i A_i \to W$ such that left diagram commutes. Let $a_n \in \bigcup_i A_i$, then exists n such that $a_n \in A_n$. Hence, $\theta'(a_n) = \theta' \circ \iota_n(a_n) = \alpha_n(a_n) = \theta \circ \iota_n(a_n) = \theta(a_n)$, that is, $\theta = \theta'$.

To show the uniqueness of φ, suppose there is $\varphi' : V \to \bigcap_i A_i$ such that right diagram commutes. Let $v \in V$, $\varphi'(v) = e_n \circ \varphi(v) = \beta_n(v) = \varphi(v)$, hence, $\varphi = \varphi'$.

By above discussion, we have

$$\bigcup_i A_i = \varinjlim_{\mathscr{I}} A_i, \qquad \bigcap_i A_i = \varprojlim_{\mathscr{I}} A_i.$$

$\square$

Remark Colimit do not always exist, but there are two useful large classes of examples for which they do.

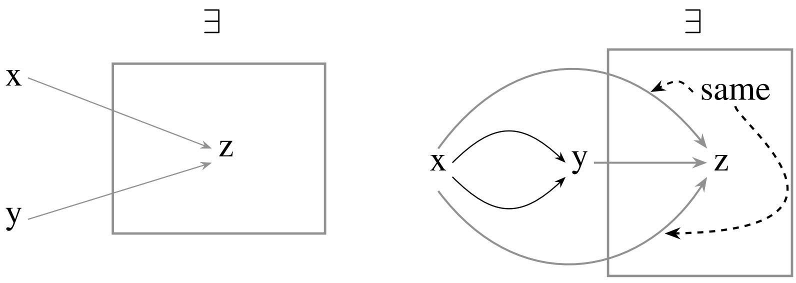
Definition 2.3.4 (Filtered)

*A nonempty partially ordered set $(S, \geq)$ is **filtered** (or is said to be a **filtered set**) if for each $x, y \in S$, there is a z such that $x \geq z$ and $y \geq z$. More generally, a nonempty category $\mathscr{I}$ is filtered if:*

(i) for each $x, y \in \mathscr{I}$, there is a $z \in \mathscr{I}$ and arrows $x \to z$ and $y \to z$,

(ii) for every two arrows $u : x \to y$ and $v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$.

Remark Other terminologies are also commonly used, such as "directed partially ordered set" and "fixed index category", respectively.



(i) (ii)

Figure 2.1: A filtered category (pictorial definition)

✍ **Exercise 2.27** Suppose $\mathscr{I}$ is filtered. Recall the symbol $\coprod$ for disjoint union of sets. Show that any diagram in **Sets** indexed by $\mathscr{I}$ has the following, with the obvious maps to it, as a colimit:

$$\left\{(a_i, i) \in \coprod_{i\in\mathscr{I}} A_i\right\} \Big/ \left(\begin{array}{l}(a_i, i) \sim (a_j, j) \text{ if and only if there are } f : A_i \to A_k \text{ and} \\ g : A_j \to A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k\end{array}\right)$$

Proof Say

$$\mathscr{S} = \left\{(a_i, i) \in \coprod_{i\in\mathscr{I}} A_i\right\} \Big/ \left(\begin{array}{l}(a_i, i) \sim (a_j, j) \text{ if and only if there are } f : A_i \to A_k \text{ and} \\ g : A_j \to A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k\end{array}\right).$$

Then for $A_j \to A_k$, the following diagram commutes,

$$\begin{tikzcd} A_j \arrow[r] \arrow[d, "e_j"'] & A_k \arrow[dl, "e_k"] \\ \mathscr{S}, & \end{tikzcd}$$

where e_j, e_k are embedding.

Cosider ther following diagram.

$$\begin{tikzcd} \mathscr{S} \arrow[rr, dashed, "\theta"] & & W \\ & A_j \arrow[ul, "e_j"'] \arrow[ur, "\alpha_j"] \arrow[d, "g"] & \\ & A_k \arrow[uul, bend left, "e_k"] \arrow[uur, bend right, "\alpha_k"'] & \end{tikzcd}$$

Define $\theta : W \to \mathscr{S}$ by setting $\theta(a_n, n) = \alpha_n(a_n)$. We need to show that θ is well-defined. Let $(a_i, i) \sim (a_j, j) \in \coprod_{i\in\mathscr{I}}$, then there are $f : A_i \to A_k$ and $g : A_j \to A_k$ in the diagram for which $f(a_i) = g(a_j)$. Note that $\alpha_k \circ f = \alpha_i$ and $\alpha_k \circ g = \alpha_j$, $\alpha_i(a_i) = \alpha_k \circ f(a_i) = \alpha_k \circ g(a_j) = \alpha_j(a_j)$, hence $\theta(a_i) = \theta(a_j)$. Hence, θ is well-defined and such that above diagram commutes.

To show the uniqueness of θ, suppose there is ψ such that above diagram is commutes. Let $(a_n, n) \in \coprod_{i\in\mathscr{I}} A_i$, then $\psi(a_n, n) = \psi \circ e_n(a_n) = \alpha_n(a_n) = \theta(a_n, n)$. Hence, $\psi = \theta$. □

For example, in Example 2.29 , each element of the colimit is an element of something upstairs, but you can't say in advance what it is an element of. For instance, $\frac{17}{125}$ is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

This idea applies to many categories whose objects can be interpreted as sets with additional structure (such as abelian groups, A-modules, groups, etc).

✍ **Exercise 2.28** The colimit $\varinjlim M_i$ in the category of A-modules $\mathbf{Mod}_A$ can be described as follows. The set underlying $\varinjlim M_i$ is defined as in Exercise 2.27. To add the elements $m_i \in M_i$ and $m_j \in M_j$, choose an $l \in \mathscr{I}$ with arrows $u : i \to l$ and $v : j \to l$, and then define the sum of m_i and m_j to be $F(u)(m_i) + F(v)(m_j) \in M_l$. The element $m_i \in M_i$ is 0 if and only if there is some arrow $u : i \to k$ for which $F(u)(m_i) = 0$, i.e., if it becomes 0 "later in the diagram". Last, multiplication by an element of A is defined in the obvious way.

Verify that the A-module described above is indeed the colimit.

Proof Let

$$\mathscr{S} = \left\{ (m_i, i) \in \prod_{i \in \mathscr{I}} M_i \right\} \Big/ \left(\begin{array}{l} (m_i, i) \sim (m_j, j) \text{ if and only if there are } f : M_i \to M_k \text{ and} \\ g : M_j \to M_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } M_k \end{array} \right).$$

We need to show that $\mathscr{S}$ indeed an A-module. Let $(m_i, i) \sim (m_{i'}, i')$, then there are $f : M_i \to M_k$, $g : M_{i'} \to M_k$ such that $f(m_i) = g(m_{i'})$.

(1) Addition is well-defined.

Notice that

$$m_i + m_j = f(m_i) + F(j \to k)(m_j) = g(m_{i'}) + F(j \to k)(m_j) = m_{i'} + m_j,$$

addition is independent of the choice of representatives m_i and $m_{i'}$.

Suppose we have another index l' with $u' : i \to l'$ and $v' : j \to l'$. Since $\mathscr{I}$ is filtered, exists $w : l \to k$ and $w' : l' \to k$ such that $w \circ u = w' \circ u'$ and $w \circ v = w' \circ v'$. Then we have

$$m_i + m_j = F(wu)(m_i) + F(wv)(m_j) = F(w'u')(m_i) + F(w'v')(m_j) = m_i + m_j$$

, which implies that addition is independent of the choice of l, and the choice of arrows u and v.

(2) Scalar multiplication is obviously well-defined.

Hence, $\mathscr{S}$, indeed, an A-module. Same as Exercise 2.27, $\varinjlim M_i = \mathscr{S}$. $\square$

Exercise 2.29 (Localization as a colimit) Generalize Exercise 2.26 (a) to interpret localization of an integral domain as a colimit over a filtered set: suppose S is a multipicative set of A, and interpret $S^{-1}A = \varinjlim A_s$ where the colimit is over $s \in S$, and in category of A-modules.

Proof In fact, $A_{s_1} \to A_{s_2}$ exists if $ks_1 = s_2$ for some $k \in A$. Consider the following diagram.

$$\begin{tikzcd}
S^{-1}A \arrow[rr, dashed, "\theta"] & & W \\
& A_{s_i} \arrow[ul, "\iota_i"'] \arrow[ur, "f_i"] \arrow[d, "e"] & \\
& A_{s_j} \arrow[uul, bend left, "\iota_j"] \arrow[uur, bend right, "f_j"'] &
\end{tikzcd}$$

Defined $\theta : S^{-1}A \to W$ by $\theta(\frac{a}{s_i^d}) = f_i(\frac{a}{s_i^d})$, then θ makes above diagram commutes. By the definition of θ, it is easy to see that θ is unique.

Hence, $S^{-1}A = \varinjlim A_s$. $\square$

A variant of this construction works without the filtered condition, if you have another means of "connecting elements in different objects of your diagram". For example:

Exercise 2.30 (Colimits of A-modules without the filtered condition) Suppose you are given a diagram of A-modules indexed by $\mathscr{I}$: $F : \mathscr{I} \to \mathbf{Mod}_A$, where we let $M_i = F(i)$. Show that the colimit is

$$\bigoplus_{i \in \mathscr{I}} M_i \Big/ \{m_i - F(i \to j)(m_i) \text{ for every } i \to j \text{ in } \mathscr{I}\}.$$

Proof Say

$$\mathscr{S} = \bigoplus_{i \in \mathscr{I}} M_i \Big/ \{m_i - F(n)(m_i) \text{ for every } n : i \to j \text{ in } \mathscr{I}\}.$$

Consider the following diagram.

$$\begin{tikzcd}
\mathscr{S} \arrow[rr, dashed, "\theta"] & & W \\
& M_i \arrow[ul, "\iota_i"'] \arrow[ur, "f_i"] \arrow[d, "F(i\to j)"] & \\
& M_j \arrow[uul, bend left, "\iota_j"] \arrow[uur, bend right, "f_j"'] &
\end{tikzcd}$$

Define $\theta : \mathscr{S} \to W$ by setting $\theta((m_k)_{k\in\mathscr{I}}) = f_i(m_i)$. θ is well-defined, since in $\mathscr{S}$, $m_i = F(i \to j)(m_i)$. By the definition of θ, the commmutativity and uniqueness is obvious.

Hence, $\mathscr{S} = \varinjlim M_i$. $\square$

Note *One useful thing to informally keep in mind is the following. In a category where the objects are "set-like", an element of a limit can be thought of as a family of elements of each object in the diagram, that are "compatible" (Exercise 2.25). And an elements of a colimit can be thought of as ("has a representative that is") an element of a single object in the diagram (Exercise 2.27). Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.*

Remark In fact, colimits exist in the category of sets for all reasonable ("smal") index categories, but that won't matter to us.

2.4 Adjoints

Definition 2.4.1 (Adjoint)

*Two covariant functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are **adjoint** if there is a natural bijection for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$*

$$\tau_{AB} : \mathrm{Mor}_{\mathscr{B}}(F(A), B) \to \mathrm{Mor}_{\mathscr{A}}(A, G(B)).$$

*We say that (F, G) form an **adjoint pair**, and that F is **left-adjoint** to G (and G is **right-adjoint** to F). We say F is a **left adjoint** (and G is a **right adjoint**). By "natural" we mean the following. For all $f : A \to A'$ in $\mathscr{A}$, we require*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{B}}(F(A'), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A'B}"'] & \mathrm{Mor}_{\mathscr{B}}(F(A), B) \arrow[d, "\tau_{AB}"] \\
\mathrm{Mor}_{\mathscr{A}}(A', G(B)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathscr{A}}(A, G(B))
\end{tikzcd}$$

to commute, where f^* is the map induced by $f : A \to A'$ and Ff^* is the map induced by $Ff : F(A) \to F(A')$. For all $g : B \to B'$ in $\mathscr{B}$ we want a similar commutative diagram to commute,

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{B}}(F(A), B) \arrow[r, "g_*"] \arrow[d, "\tau_{AB}"'] & \mathrm{Mor}_{\mathscr{B}}(F(A), B') \arrow[d, "\tau_{AB'}"] \\
\mathrm{Mor}_{\mathscr{A}}(A, G(B)) \arrow[r, "Gg_*"] & \mathrm{Mor}_{\mathscr{A}}(A, G(B')),
\end{tikzcd}$$

where g_* is the map induced by $g : B \to B'$, and Gg_* is the map induced by $Gg : G(B) \to G(B')$.

Exercise 2.31 Show that the map τ_{AB} in Definition 2.4.1 has the following properties. For each A there is a map $\eta_A : A \to GF(A)$ so that for any $g : F(A) \to B$, the corresponding $\tau_{AB}(g) : A \to G(B)$ is given by the

composition
$$A \xrightarrow{\eta_A} GF(A) \xrightarrow{Gg} G(B).$$
Similarly, there is a map $\varepsilon_B : FG(B) \to B$ for each B so that for any $f : A \to G(B)$, the corresponding map $\tau_{AB}^{-1}(f) : F(A) \to B$ is given by the composition
$$F(A) \xrightarrow{Ff} FG(B) \xrightarrow{\varepsilon_B} B.$$

Proof

(1) By Definition 2.4.1, we have the following diagram,
$$\begin{tikzcd} \mathrm{Mor}_{\mathscr{B}}(F(A), F(A)) \arrow[r, "g_*"] \arrow[d, "\tau_{A,F(A)}"'] & \mathrm{Mor}_{\mathscr{B}}(F(A), B) \arrow[d, "\tau_{A,B}"] \\ \mathrm{Mor}_{\mathscr{A}}(A, GF(A)) \arrow[r, "Gg_*"] & \mathrm{Mor}_{\mathscr{A}}(A, G(B)) \end{tikzcd}$$
then we have $\tau_{A,B} \circ g_* = Gg_* \circ \tau_{A,F(A)}$. Let $\mathrm{id}_{F(A)} \in \mathrm{Mor}_{\mathscr{B}}(F(A), F(A))$, then
$$\tau_{A,B} \circ g_*(\mathrm{id}_{F(A)}) = \tau_{A,B}(g \circ \mathrm{id}_{F(A)}) = \tau_{A,B}(g) = Gg \circ (\tau_{A,F(A)}(\mathrm{id}_{F(A)})).$$
Say $\eta_A = \tau_{A,F(A)}(\mathrm{id}_{F(A)}) : A \to GF(A)$, then we have $\tau_{A,B}(g) = Gg \circ \eta_A$.

(2) By Definition 2.4.1 again, we have the following diagram,
$$\begin{tikzcd} \mathrm{Mor}_{\mathscr{A}}(G(B), G(B)) \arrow[r, "f^*"] \arrow[d, "\tau^{-1}_{G(B),B}"'] & \mathrm{Mor}_{\mathscr{A}}(A, G(B)) \arrow[d, "\tau^{-1}_{A,B}"] \\ \mathrm{Mor}_{\mathscr{B}}(FG(B), B) \arrow[r, "Ff^*"] & \mathrm{Mor}_{\mathscr{B}}(F(A), B) \end{tikzcd}$$
then we have $\tau^{-1}_{A,B} \circ f^* = Ff^* \circ \tau^{-1}_{G(B),B}$. Let $\mathrm{id}_{G(B)} \in \mathrm{Mor}_{\mathscr{A}}(G(B), G(B))$, then
$$\tau^{-1}_{A,B} \circ f^*(\mathrm{id}_{G(B)}) = \tau^{-1}_{A,B}(f) = Ff^* \circ \tau^{-1}_{G(B),B}(\mathrm{id}_{G(B)}) = Ff(\tau^{-1}_{G(B),B}(\mathrm{id}_{G(B)})).$$
Say $\varepsilon_B = \tau^{-1}_{G(B),B}(\mathrm{id}_{G(B)})$, then we have $\tau^{-1}_{A,B}(f) = Ff \circ \varepsilon_B$.

□

Here is a key example of an adjoint pair.

Lemma 2.4.1

Suppose M, N, and P are A-modules (where A is a ring). There is a bijection $\mathrm{Hom}_A(M \otimes_A N, P) \leftrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$.

Proof Let $f \in \mathrm{Hom}_A(M \otimes_A N, P)$. Define $\tau_{M,N,P} : \mathrm{Hom}_A(M \otimes_A N, P) \to \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ (for simply, say $\tau = \tau_{M,N,P}$) as follow:
$$\tau : f \mapsto \tau(f), \tau(f) : m \mapsto \tau(f)_m, \tau(f)_m : n \mapsto f(m \otimes n).$$
We shall to τ is an A-module isomorphism.

(1) τ is an A-homomorphism.
Let $f, g \in \mathrm{Hom}_A(M \otimes_A N, P)$. The definition of $f + g$ gives, for all $a \in A$,
$$\begin{aligned} \tau(f+g)_m : n \mapsto\ & (f+g)(m \otimes n) \\ &= f(a \otimes b) + g(a \otimes b) \\ &= \tau(f)_a(b) + \tau(g)_a(b). \end{aligned}$$
Therefore, $\tau(f+g) = \tau(f) + \tau(g)$.

(2) τ is injective.
Suppose $\tau(f)_m = 0$ for all $m \in M$, then $0 = \tau(f)_m(n) = f(m \otimes n)$ for all $m \in M$ and $n \in N$. Hence,

$f = 0$, which implies that τ is injective.

(3) τ is surjective.

If $F \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ is an A-map, define $\varphi : M \times N \to P$ by $\varphi(m, n) = F_m(n)$. Now consider the diagram

$$\begin{tikzcd} M \times N \arrow[rr, "h"] \arrow[dr, "\varphi"'] & & M \otimes_A N \arrow[dl, dashed, "\widetilde{\varphi}"] \\ & P. & \end{tikzcd}$$

Since $F_m(n) = m \otimes n$, it is easy to see that φ is A-bilinear. By the universal property of tensor product, there exists unique A-homomorphism $\widetilde{\varphi} : M \otimes_A N \to P$ with $\widetilde{\varphi}(m \otimes n) = \varphi(m, n) = F_m(n)$ for all $m \in M$ and $n \in N$. Hence, $F = \tau(\widetilde{\varphi})$ and τ is surjective.

By above discussion, τ is an A-module isomorphism. $\square$

Theorem 2.4.1 (Tensor-Hom adjunction)

$\square \otimes_A N$ and $\mathrm{Hom}_A(N, \square)$ are adjoint functors.

Proof Let $f : M \to M'$, $g : P \to P'$ in $\mathbf{Mod}_A$. We shall to show that the following diagram commute,

$$\begin{tikzcd} \mathrm{Hom}_A(M' \otimes_A N, P) \arrow[r, "(f \otimes_A N)^*"] \arrow[d, "\tau_{M',N,P}"'] & \mathrm{Hom}_A(M \otimes_A N, P) \arrow[r, "g_*"] \arrow[d, "\tau_{M,N,P}"] & \mathrm{Hom}_A(M \otimes_A N, P') \arrow[d, "\tau_{M,N,P'}"] \\ \mathrm{Hom}_A(M', \mathrm{Hom}(N, P)) \arrow[r, "f^*"] & \mathrm{Hom}_A(M, \mathrm{Hom}(N, P)) \arrow[r, "(g_*)_*"] & \mathrm{Hom}_A(M, \mathrm{Hom}(N, P')). \end{tikzcd}$$

where $(g_*)_* = (\mathrm{Hom}(N, f))_* = \mathrm{Hom}(M, \mathrm{Hom}(N, f))$ and $(f \otimes_A N)^* = \mathrm{Hom}_A(f \otimes \mathrm{id}_N, P)$.

Let $\varphi \in \mathrm{Hom}_A(M' \otimes_A N, P)$, we have

$$f^* \circ \tau_{M',N,P}(\varphi) = f^*(\tau_{M',N,P}(\varphi)) = \tau_{M',N,P}(\varphi) \circ f$$

and

$$\tau_{M,N,P} \circ (f \otimes_A N)^*(\varphi) = \tau_{M,N,P}(\varphi \circ (f \otimes \mathrm{id}_N)).$$

Let $m \in M$ and $n \in N$, by Lemma 2.4.1, we have $\tau_{M',N,P}(\varphi) \circ f(m) = \tau_{M',N,P}(\varphi)_{f(m)}$, and therefore

$$\tau_{M',N,P}(\varphi)_{f(m)}(n) = \varphi(f(m) \otimes n) = (\varphi \circ f \otimes \mathrm{id}_N)(m \otimes n) = \tau_{M,N,P}(\varphi \circ (f \otimes \mathrm{id}_N))_m(n),$$

which implies that $f^* \circ \tau_{M',N,P} = \tau_{M,N,P} \circ (f \otimes_A N)^*$.

Let $\psi \in \mathrm{Hom}_A(M \otimes_A N, P)$, we have

$$(g_*)_* \circ \tau_{M,N,P}(\psi) = g_* \circ \tau_{M,N,P}(\psi)$$

and

$$\tau_{M,N,P'} \circ g_*(\psi) = \tau_{M,N,P'}(g \circ \psi).$$

Let $m \in M$ and $n \in N$, by Lemma 2.4.1, we have

$$g_* \circ \tau_{M,N,P}(\psi)(m) = g_*(\tau_{M,N,P}(\psi)_m) = g \circ \tau_{M,N,P}(\psi)_m,$$

hence

$$g \circ \tau_{M,N,P}(\psi)_m(n) = g(\psi(m \otimes n)) = \tau_{M,N,P'}(g \circ \psi)_m(n),$$

which implies that $(g_*)_* \circ \tau_{M,N,P} = \tau_{M,N,P'} \circ g_*$.

By above dicussion, the diagram commutes, hence, $\square \otimes_A N$ and $\mathrm{Hom}_A(N, \square)$ are adjoint functors. $\square$

Proposition 2.4.1 (This adjoint pair is very important!)

Suppose $B \to A$ is a morphism of rings. If M is an A-module, we can create a B-module M_B by considering it as a B-module. This gives a functor $\square_B : \mathbf{Mod}_A \to \mathbf{Mod}_B$. This functor is right-adjoint to $\square \otimes_B A$. In other words, there is a bijection

$$\mathrm{Hom}_A(N \otimes_B A, M) \cong \mathrm{Hom}_B(N, M_B)$$

functorial in both arguments.

Proof Say $f : B \to A$ is a ring homomorphism.

(1) There is a bijection $\mathrm{Hom}_A(N \otimes_B A, M) \leftrightarrow \mathrm{Hom}_B(N, M_B)$.

Let $\varphi \in \mathrm{Hom}_A(N \otimes_B A, M)$, define $\tau_{N,M} : \mathrm{Hom}_A(N \otimes_B A, M) \to \mathrm{Hom}_B(N, M_B)$ as follow:

$$\tau_{N,M} : \varphi \mapsto \tau_{N,M}(\varphi), \tau_{N,M}(\varphi) : n \mapsto \varphi(n \otimes_B 1_A),$$

define its inverse $\theta_{N,M_B} : \mathrm{Hom}_B(N, M_B) \to \mathrm{Hom}_A(N \otimes_B A, M)$ by setting

$$\theta_{N,M_B} : \psi \mapsto \theta_{N,M_B}(\psi), \theta_{N,M_B}(\psi) : n \otimes a \mapsto a\psi(n).$$

Since

$$\theta_{N,M_B} \circ \tau_{N,M}(\varphi)(n \otimes_B a) = a\tau_{N,M}(\varphi)(n) = a\varphi(n \otimes_B 1_A) = \varphi(n \otimes_B a)$$

and

$$\tau_{N,M} \circ \theta_{N,M_B}(\psi)(n) = \theta_{N,M_B}(\psi)(n \otimes_B 1_A) = \psi(n),$$

θ_{N,M_B} is, indeed, the inverse of $\tau_{N,M}$, and therefore $\tau_{N,M}$ is bijection.

(2) τ is natural isomorphism.

Consider the following diagram, we shall to show that this diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}_A(N' \otimes_B A, M) \arrow[r, "\square\circ(h\otimes_B \mathrm{id}_A)"] \arrow[d, "\tau_{N',M}"'] & \mathrm{Hom}_A(N \otimes_B A, M) \arrow[r, "l\circ\square"] \arrow[d, "\tau_{N,M}"] & \mathrm{Hom}_A(N \otimes_B A, M') \arrow[d, "\tau_{N,M'}"] \\
\mathrm{Hom}_B(N', M_B) \arrow[r, "\square\circ h"] & \mathrm{Hom}_B(N, M_B) \arrow[r, "\widetilde{l}\circ\square"] & \mathrm{Hom}_B(N, M'_B),
\end{tikzcd}$$

where $h : N \to N'$ and $l : M \to M'$, $\widetilde{l} : M_B \to M'_B$ which looks l as B-module homomorphism.

(a) The diagram at the left hand side is commutative.

Let $\varphi \in \mathrm{Hom}_A(N' \otimes_B A, M)$, then $(\square \circ h) \circ \tau_{N',M}(\varphi) = \tau_{N',M}(\varphi) \circ h$. For any $n \in N$, we have

$$\tau_{N',M}(\varphi) \circ h(n) = \tau_{N',M}(\varphi)(h(n)) = \varphi(h(n) \otimes_B 1_A).$$

On the other hand, $\tau_{N,M} \circ (\square \circ (h \otimes_B \mathrm{id}_A))(\varphi) = \tau_{N,M}(\varphi \circ (h \otimes_B \mathrm{id}_A))$, then we have

$$\tau_{N,M}(\varphi \circ (h \otimes_B \mathrm{id}_A))(n) = (\varphi \circ (h \otimes_B \mathrm{id}_A))(n \times_B 1_A) = \varphi(h(n) \otimes_B 1_A).$$

Hence, $(\square \circ h) \circ \tau_{N',M} = \tau_{N,M} \circ (\square \circ (h \otimes_B \mathrm{id}_A))$, this implies that the diagram at left hand side is commutative.

(b) The diagram at the right hand side is commutative.

Let $\psi \in \mathrm{Hom}_A(N \otimes_B A, M)$, then $(\widetilde{l} \circ \square) \circ \tau_{N,M}(\psi)$. For any $n \in N$, we have

$$(\widetilde{l} \circ \square) \circ \tau_{N,M}(\psi)(n) = \widetilde{l} \circ \tau_{N,M}(\psi)(n) = \widetilde{l}(\psi(n \otimes_B 1_A)) = l(\psi(n \otimes_B 1_A)).$$

On the other hand, $\tau_{N,M'} \circ (l \circ \square)(\psi) = \tau_{N,M'}(l \circ \psi)$, for any n, we have

$$\tau_{N,M'}(l \circ \psi)(n) = l \circ \psi(n \otimes_B 1_A) = l(\psi(n \otimes_B 1_A)).$$

Hence, $(\widetilde{l} \circ \square) \circ \tau_{N,M} = \tau_{N,M'} \circ (l \circ \square)$, as we desired.

$\square$

Remark The maps η_A and ε_B of Exercise 2.31 are called the **unit** and **counit** of the adjunction. This leads to a different characterization of adjunction. Suppose functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are given, along with natural transformations $\eta : \mathrm{id}_{\mathscr{A}} \to GF$ and $\varepsilon : FG \to \mathrm{id}_{\mathscr{B}}$ with the property that $G\varepsilon \circ \eta G = \mathrm{id}_G$ (for each $B \in \mathscr{B}$, the composition of $\eta_{G(B)} : G(B) \to GFG(B)$ and $G(\varepsilon_B) : GFG(B) \to G(B)$ is the identity) and $\varepsilon F \circ F\eta = \mathrm{id}_F$. Then F is left-adjoint to G.

Proof Let $f : A \to A'$ in $\mathscr{A}$ and $g : B \to B'$ in $\mathscr{B}$. It is suffice to show that the following diagram is commutative,

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{B}}(F(A'), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A',B}"'] & \mathrm{Mor}_{\mathscr{B}}(F(A), B) \arrow[r, "g_*"] \arrow[d, "\tau_{A,B}"] & \mathrm{Mor}_{\mathscr{B}}(F(A), B') \arrow[d, "\tau_{A,B'}"] \\
\mathrm{Mor}_{\mathscr{A}}(A', G(B)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathscr{A}}(A, G(B)) \arrow[r, "Gg_*"] & \mathrm{Mor}_{\mathscr{A}}(A, G(B')),
\end{tikzcd}$$

where f^* induced by f, Ff^* induced by Ff, g_* induced by g, Gg_* induced by Gg, $\tau_{A,B}$ given by $\tau_{A,B}(\varphi) = G\varphi \circ \eta_A$, we give $\tau_{A'B}$ and $\tau_{A,B'}$.

(1) $\tau_{A,B} : \mathrm{Mor}_{\mathscr{B}}(F(A), B) \to \mathrm{Mor}_{\mathscr{A}}(A, G(B))$ is a bijection.

We give the inverse of $\tau_{A,B}$ directly, $\tau^{-1}_{A,B}(\psi) = \varepsilon_B \circ F\psi$. We shall to show that $\tau^{-1}_{A,B}$, indeed, is the inverse of $\tau_{A,B}$. Let $\varphi \in \mathrm{Mor}_{\mathscr{B}}(F(A), B)$, we have

$$\begin{aligned}
\tau^{-1}_{A,B} \circ \tau_{A,B}(\varphi) =& \tau^{-1}_{A,B}(G\varphi \circ \eta_A) = \varepsilon_B \circ (F(G\varphi \circ \eta_A)) \\
=& \varepsilon_B \circ (F(G\varphi) \circ F(\eta_A)) \\
=& \varepsilon_B \circ F(G\varphi) \circ F(\eta_A),
\end{aligned} \tag{2.15}$$

Since $\varepsilon : FG \to \mathrm{id}_{\mathscr{B}}$ is natural transformation, the following diagram commutes,

$$\begin{tikzcd}
FGF(A) \arrow[r, "FG\varphi"] \arrow[d, "\varepsilon_{F(A)}"'] & FG(B) \arrow[d, "\varepsilon_B"] \\
F(A) \arrow[r, "\varphi"] & B,
\end{tikzcd}$$

that is, $\varphi \circ \varepsilon_{F(A)} = \varepsilon_B \circ FG\varphi$. Hence, note that $\varepsilon F \circ F\eta = \mathrm{id}_F$,

$$\tau^{-1}_{A,B} \circ \tau_{A,B}(\varphi) = \varepsilon_B \circ F(G\varphi) \circ F(\eta_A) = \varphi \circ \varepsilon_{F(A)} \circ F(\eta_A) = \varphi \circ \mathrm{id}_{F(A)} = \varphi.$$

Similarly, for $\psi \in \mathrm{Mor}_{\mathscr{A}}(A, G(B))$, by the commuative diagram,

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\eta_A"'] & G(B) \arrow[d, "\eta_{G(B)}"] \\
GF(A) \arrow[r, "GF\psi"] & GFG(B),
\end{tikzcd}$$

we have $\tau_{A,B} \circ \tau^{-1}_{A,B}(\psi) = \psi$.

(2) Commutativity of diagram.

Let $\varphi \in \mathrm{Mor}_{\mathscr{B}}(F(A'), B)$, then

$$f^* \circ \tau_{A',B}(\varphi) = f^* \circ (G\varphi \circ \eta_{A'}) = G\varphi \circ \eta_{A'} \circ f$$

and

$$\tau_{A,B} \circ Ff^*(\varphi) = \tau_{A,B}(\varphi \circ Ff) = G(\varphi \circ Ff) \circ \eta_A = G\varphi \circ GFf \circ \eta_A = G\varphi \circ \eta_{A'} \circ f.$$

Hence, the diagram at left hand side commutes.

By the same way, we can prove the diagram at right hand side is also commutative.

□

Here are some motivating examples:

Example 2.31 For those familiar with representation theory: Frobenius reciprocity may be understood in terms of adjoints. Suppose V is a finite-dimensional representation of a finite group G, and W is a representation of a subgroup $H < G$. Then induction and restriction are an adjoint pair $(\mathrm{Ind}_H^G, \mathrm{res}_H^G)$ between the category of G-modules and the category of H-modules.

Example 2.32 (Groupification of abelian semigroups) Getting an abelian group from an abelian semigroup. (An **abelian semigroup** is just like an abelian group, except we don't require an identity or an inverse. Morphisms of abelian semigroups are maps of sets preserving the binary operation. One example is the non-negative integers $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \cdots\}$ under addition. Another is the positive integers $\mathbb{Z}^{+} = \{1, 2, 3, \cdots\}$ under addition. Yet another is the positive integers $\mathbb{Z}^{+}$ under multiplication.) From an abelian semigroup, we can create an abelian group. In our examples, from the nonnegative integers under addition $(\mathbb{Z}^{\geq \nvdash}, +)$, we create the integers $(\mathbb{Z}, +)$, and from the positive integers under multiplication $(\mathbb{Z}^{+}, \times)$, we create the positive rationals $(\mathbb{Q}^{+}, \times)$. Here is a formalization of that notion. A **groupification** of an abelian semigroup S is a map of abelian semigroups $\pi : S \to G$ such that G is an abelian group, and any map of abelian semigroups from S to an abelian group G' factors uniquely through G:

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr] & G \arrow[d, dotted, "\exists!"] \\ & G' \end{tikzcd}$$

✍ **Exercise 2.32 (An Abelian group is groupified by itself)** Show that if an abelian semigroup is already a group then the identity morphism is the groupification.

Proof Say G be an abelian group. Note that

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}_G"] \arrow[dr, "\varphi"'] & G \arrow[d, dotted, "\varphi"] \\ & G', \end{tikzcd}$$

then we done. $\square$

✍ **Exercise 2.33** Construct the "groupification functor" H from the category of nonempty abelian semigroups to the category of abelian groups. One possible construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are ordered pairs $(a, b) \in S \times S$, which may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d + e = b + c + e$ for some $e \in S$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Descride the abelian semigroup map $S \to H(S)$. Let F be the forgetful functor from the category of abelian groups $\mathbf{Ab}$ to the category of abelian semigroups. Show that H is left-adjoint to F.

Proof Say $\mathscr{S}$ be the category of nenempty abelian semigroups. We need describe addition in $H(S)$ such that $H(S)$ be a group. Let $(a, b) + (c, d) = (a + c, b + d)$, we need to check this addition is well-defined. Let $(a, b) \sim (a', b')$, then $a + b' + e = a' + b + e$, hence, $a + c + b' + d + e = a' + c + b + d + e$, that is, $(a + c, b + d) \sim (a' + c, b' + d)$, which implies that addition is well-defined. The identity in $H(S)$ is $(0, 0)$. The inverse of (a, b) in $H(S)$ is (b, a). Hence, $H(S)$ is an abelian group. Define the abelian semigroup map $\iota_S : S \to H(S)$ by setting $a \mapsto (a, 0)$.

Now, we shall to show that there is a bijection between $\mathrm{Mor}_{\mathbf{Ab}}(H(S), A)$ and $\mathrm{Mor}_{\mathscr{S}}(S, F(A))$. Define $\tau_{S,A} : \mathrm{Mor}_{\mathbf{Ab}}(H(S), A) \to \mathrm{Mor}_{\mathscr{S}}(S, F(A))$ by setting $\tau_{S,A} : \varphi \mapsto \tau_{S,A}(\varphi), \tau_{S,A}(\varphi) : a \mapsto \varphi(\iota_S(a))$. The inverse of $\tau_{S,A}$ defined by $\tau_{S,A}^{-1} : \psi \mapsto \tau_{S,A}^{-1}(\psi), \tau_{S,A}^{-1}(\psi) : (a, b) \mapsto \psi(a) - \psi(b)$. We shall to show that $\tau_{S,A}^{-1}$

indeed the inverse of $\tau_{S,A}$. Let $\varphi \in \mathrm{Mor}_{\mathbf{Ab}}(H(S), A)$, then

$$\tau_{S,A}^{-1} \circ \tau_{S,A}(\varphi)(a,b) = \tau_{S,A}(\varphi)(a) - \tau_{S,A}(\varphi)(b) = \varphi(a,0) - \varphi(b,0) = \varphi(a,b),$$

hence, $\tau_{S,A}^{-1} \circ \tau_{S,A} = \mathrm{id}_{\mathrm{Mor}_{\mathbf{Ab}}(H(S),A)}$. Let $\psi \in \mathrm{Mor}_{\mathscr{S}}(S, F(A))$, then

$$\tau_{S,A} \circ \tau_{S,A}^{-1}(\psi)(a) = \tau_{S,A}^{-1}(\psi)(a,0) = \psi(a) - \psi(0) = \psi(a),$$

hence, $\tau_{S,A} \circ \tau_{S,A}^{-1} = \mathrm{id}_{\mathrm{Mor}_{\mathscr{S}}(S,F(A))}$. By above discussion, $\tau_{S,A}$ is a bijection.

Let $f : S \to S'$ in $\mathscr{S}$, and $g : A \to A'$ in $\mathbf{Ab}$. Consider the following diagram,

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{Ab}}(H(S'), A) \arrow[r, "Hf^*"] \arrow[d, "\tau_{S',A}"'] & \mathrm{Mor}_{\mathbf{Ab}}(H(S), A) \arrow[r, "g_*"] \arrow[d, "\tau_{S,A}"] & \mathrm{Mor}_{\mathbf{Ab}}(H(S), A') \arrow[d, "\tau_{S,A'}"] \\
\mathrm{Mor}_{\mathscr{S}}(S', F(A)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathscr{S}}(S, F(A)) \arrow[r, "Fg_*"] & \mathrm{Mor}_{\mathscr{S}}(S, F(A')),
\end{tikzcd}$$

where f^* induced by f, Ff^* induced by Ff, g_* induced by g, Fg_* induced by Fg. We shall to show that this diagram is commutative.

Let $\varphi \in \mathrm{Mor}_{\mathbf{Ab}}(H(S'), A)$, then $f^* \circ \tau_{S',A}(\varphi) = \tau_{S',A}(\varphi) \circ f$. For all $s \in S$, we have

$$\tau_{S',A}(\varphi) \circ f(s) = \varphi(f(s), 0).$$

On the other hand, $\tau_{S,A} \circ Hf^*(\varphi) = \tau_{S,A}(\varphi \circ Hf)$. For all $s \in S$, we have

$$\tau_{S,A}(\varphi \circ Hf)(s) = \varphi(Hf(s,0)) = \varphi(f(s),0).$$

Hence, $f^* \circ \tau_{S',A} = \tau_{S,A} \circ Hf^*$.

Similarly, we can shall that $\tau_{S,A'} \circ g_* = Fg_* \circ \tau_{S,A}$. Hence, H is left-adjoint to F. □

Remark We have a full subcategory. We want to "project" from the category to the subcategory. We have

$$\mathrm{Mor}_{\text{category}}(S, H) = \mathrm{Mor}_{\text{subcategory}}(G, H)$$

automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the smaller category, which automatically satisfies the universal property.

✍ **Exercise 2.34** Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a full subcategory of the category of A-modules (via the obvious inclusion $\mathbf{Mod}_{S^{-1}A} \hookrightarrow \mathbf{Mod}_A$). Then $\mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ can be interpreted as an adjoint to the forgetful functor $\mathbf{Mod}_{S^{-1}A} \to \mathbf{Mod}_A$. State and prove the correct statement.

Proof Every $S^{-1}A$-module is an A-module, and this is an injective map, so we have a forgetful functor $F : \mathbf{Mod}_{S^{-1}A} \to \mathbf{Mod}_A$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$-modules as A-modules are just the same when they are considered as $S^{-1}A$-modules. Define the localization functor $S^{-1} : \mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ by setting $S^{-1}(u : M \to N) = S^{-1}u : S^{-1}M \to S^{-1}N$, where $S^{-1}u : m/s \mapsto u(m)/s$. We shall to show that $\mathrm{Mor}(S^{-1}M, N)$ are in natural bijection with $\mathrm{Mor}(M, F(N))$.

Define $\tau_{M,N} : \mathrm{Mor}(S^{-1}M, N) \to \mathrm{Mor}(M, F(N))$ by setting $\tau_{M,N} : \varphi \mapsto \varphi|_M$. Since the morphisms between any two $S^{-1}A$-modules as A-modules are just the same when they are considered as $S^{-1}A$-modules, $\tau_{M,N}$ is well-defined. We give the inverse of $\tau_{M,N}$ directly, $\tau_{M,N}^{-1} : \psi \mapsto \tau_{M,N}^{-1}(\psi), \tau_{M,N}^{-1}(\psi) : m/s \mapsto \psi(m)/s$. We need to check that $\tau_{M,N}^{-1}$ indeed the inverse of $\tau_{M,N}$. Let $\varphi \in \mathrm{Mor}(S^{-1}M, N)$, then

$$\tau_{M,N}^{-1} \circ \tau_{M,N}(\varphi)(m/s) = \tau_{M,N}^{-1}(\varphi|_M)(m/s) = \varphi|_M(m)/s.$$

Let $\psi \in \mathrm{Mor}(M, F(N))$, then

$$\tau_{M,N} \circ \tau_{M,N}^{-1}(\psi)(m) = \tau_{M,N}^{-1}(\psi)|_M(m) = \psi(m).$$

Hence, there is a bijection $\tau_{M,N} : \mathrm{Mor}(S^{-1}M, N) \to \mathrm{Mor}(M, F(N))$.

To show the naturalness, let $f : M \to M'$ and $g : N \to N'$. Now, we consider the following diagram,

$$\begin{tikzcd}
\mathrm{Mor}(S^{-1}M', N) \arrow[r, "S^{-1}f^*"] \arrow[d, "\tau_{M',N}"'] & \mathrm{Mor}(S^{-1}M, N) \arrow[r, "g_*"] \arrow[d, "\tau_{M,N}"] & \mathrm{Mor}(S^{-1}M, N') \arrow[d, "\tau_{M,N'}"] \\
\mathrm{Mor}(M', F(N)) \arrow[r, "f^*"] & \mathrm{Mor}(M, F(N)) \arrow[r, "Fg_*"] & \mathrm{Mor}(M, F(N')),
\end{tikzcd}$$

where f^* induced by f, g_* induced by g, $S^{-1}f^*$ induced by $S^{-1}f$, Fg_* induced by Fg. We shall to show this diagram commutes.

Let $\varphi \in \mathrm{Mor}(S^{-1}M', N)$, then $f^* \circ \tau_{M',N}(\varphi) = \tau_{M',N}(\varphi) \circ f$. For all $m \in M$,

$$\tau_{M',N}(\varphi)(f(m)) = \varphi|_{M'}(f(m)).$$

On the other hand, $\tau_{M,N} \circ S^{-1}f^*(\varphi) = S^{-1}f^*(\varphi)|_M = (\varphi \circ S^{-1}f)|_M$. For all $m \in M$,

$$(\varphi \circ S^{-1}f)|_M(m) = \varphi_{M'}(f(m)).$$

Hence, the diagram at left hand side is commutative.

Let $\psi \in \mathrm{Mor}(S^{-1}M, N)$, then $\tau_{M,N'} \circ g_*(\psi) = \tau_{M,N'}(g \circ \psi)$. For all $m \in M$,

$$\tau_{M,N'}(g \circ \psi)(m) = (g \circ \psi)|_M(m) = g(\psi|_M(m)).$$

On the other hand, $Fg_* \circ \tau_{M,N}(\psi) = Fg \circ \psi|_M$, then we have

$$Fg \circ \psi|_M(m) = g(\psi|_M(m)).$$

Hence, the diagram at right hand side is commutative. $\square$

Table 2.1 gives most of the adjoints that will be come up for us.

Situation	Category $\mathscr{A}$	Category $\mathscr{B}$	Left adjoint $F : \mathscr{A} \to \mathscr{B}$	Right adjoint $G : \mathscr{B} \to \mathscr{A}$
A-modules	$\mathbf{Mod}_A$	$\mathbf{Mod}_A$	$\square \otimes_A N$	$\mathrm{Hom}_A(N, \square)$
Ring maps $B \to A$	$\mathbf{Mod}_B$	$\mathbf{Mod}_A$	$\square \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(Pre)Sheaves on a topological space X	Presheaves on X	Sheaves on X	Sheafification	Forgetful
(Semi)Groups	Semigroups	Groups	Groupification	Forgetful
Sheaves, $\pi : X \to Y$	Sheaves on Y	Sheaves on X	π^{-1}	π_*
Sheaves of abelian groups or $\mathscr{O}$-modules, open embeddings $\pi : U \hookrightarrow Y$	Sheaves on U	Sheaves on Y	$\pi_!$	π^{-1}
Quasicoherent sheaves, $\pi : X \to Y$	$\mathbf{QCoh}_Y$	$\mathbf{QCoh}_X$	π^*	π_*
Ring maps $B \to A$	$\mathbf{Mod}_A$	$\mathbf{Mod}_B$	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto \mathrm{Hom}_B(A, N)$
Quasicoherent sheaves, affine $\pi : X \to Y$	$\mathbf{QCoh}_X$	$\mathbf{QCoh}_Y$	π_*	$\pi^{!?}$

Table 2.1: Some important adjoint pairs

Remark If (F, G) is an adjoint pair of functors, then F commutes with colimits, and G commutes with limits. Also, limits commute with limits and colimits commute with colimits.

2.5 An introduction to abelian categories

2.5.1 Abelian category

We first define the notion of **additive category**. We will use it only as a stepping stone to the notion of an abelian category.

Definition 2.5.1 (Additive category)

*A category $\mathscr{C}$ is said to be **additive** if it satisfies the following properties.*

(1) $\mathrm{Mor}(A, B)$ is an (additive) abelian group for every $A, B \in \mathrm{obj}(\mathscr{C})$,

(2) the distributive laws hold: given morphisms

$$X \xrightarrow{a} A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{b} Y,$$

where X and $Y \in \mathrm{obj}(\mathscr{C})$, then

$$b(f+g) = bf + bg \quad \text{and} \quad (f+g)a = fa + ga,$$

(3) $\mathscr{C}$ has a zero object, denoted 0. (recall that a zero object is an object that is both initial and final object),

(4) $\mathscr{C}$ has finite products and finite coproducts: for all objects A, B in $\mathscr{C}$, both $A \times B$ and $A \coprod B$ exist in $\mathrm{obj}(\mathscr{C})$.

Note *In an additive category, the morphism are often called **homomorphisms**, and Mor is denoted by Hom. In fact, this notation Hom is a good indication that you're working in an additive category.*

Definition 2.5.2 (Additive functor)

*If $\mathscr{C}$ and $\mathscr{D}$ are additive categories, a functor $T : \mathscr{C} \to \mathscr{D}$ (of either variance) is **additive** if, for all A, B and all $f, g \in \mathrm{Hom}(A, B)$, we have*

$$T(f+g) = Tf + Tg,$$

that is, the function $\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(TA, TB)$, given by $f \mapsto Tf$, is a homomorphism of abelian groups.

Remark If T is an additive functor, then $T(0) = 0$, where 0 is either a zero object or a zero morphism.

Proof Note that $T(0) = T(0+0) = T(0) + T(0)$, $T(0) = 0$, where 0 is 0-homomorphsim.

Let Z be a zero object. Since Z is both initial and final object, $\mathrm{id}_Z = 0_Z$, hence, $\mathrm{id}_{T(Z)} = 0_{T(Z)}$, which implies that $T(Z) = 0$. □

Lemma 2.5.1

Let $\mathscr{C}$ be an additive category, and let $M, A, B \in \mathrm{obj}(\mathscr{C})$. Then $M \cong A \times B$ if and only if there are morphisms $i : A \to M$, $j : B \to M$, $p : M \to A$, and $q : M \to B$ such that

$$pi = \mathrm{id}_A, qj = \mathrm{id}_B, pj = 0, qi = 0, \text{ and } ip + jq = \mathrm{id}_M .$$

Moreover, $A \times B$ is also a coproduct with injections i and j, and so

$$A \times B \cong A \coprod B.$$

Proof

"⇒": Since $A \times B$ is product, M is a product, then we have the following commutative diagrams.

$$
\begin{tikzcd}
B \arrow[ddrr, dashed, "j"] \arrow[ddrrr, bend left, "\mathrm{id}_B"] \arrow[dddrr, bend right, "0"'] & & & \\
& A \arrow[dr, dashed, "i"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\mathrm{id}_A"'] & & \\
& & M \arrow[r, "q"] \arrow[d, "p"] & B \\
& & A &
\end{tikzcd}
$$

Hence, $p \circ i = \mathrm{id}_A, q \circ j = \mathrm{id}_B, pj = 0, qi = 0$. Note that both $ip + jq$ and id_M agree with the following commutative diagram.

$$
\begin{tikzcd}
M \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& M \arrow[r, "q"] \arrow[d, "p"] & B \\
& A &
\end{tikzcd}
$$

By the universal property of product, we have $ip + jq = \mathrm{id}_M$.

"⇐": If there are morphisms $i : A \to M, j : B \to M, p : M \to A$, and $q : M \to B$ such that

$$pi = \mathrm{id}_A, qj = \mathrm{id}_B, pj = 0, qi = 0, \text{ and } ip + jq = \mathrm{id}_M .$$

It suffice to show that M is a product. Consider the following diagram.

$$
\begin{tikzcd}
W \arrow[dr, dashed, "\theta"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& M \arrow[r, "q"'] \arrow[d, "p"] & B \\
& A &
\end{tikzcd}
$$

We need to construct $\theta : W \to M$ such that above diagram commutes. Let $\theta = i \circ f + j \circ g$, by condition, above diagram commutes. To show uniqueness, if there is another ψ such that diagram commutes, we have $\psi = \mathrm{id}_M \psi = ip\psi + jq\psi = if + jg = \theta$. Hence, M is product, and therefore $M \cong A \times B$.

"≅": Consider the following diagram.

$$
\begin{tikzcd}
A \times B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "j"] \arrow[ddr, bend left, "f_2"] & \\
A \arrow[r, "i"] \arrow[drr, bend right, "f_1"'] & A \times B \arrow[dr, dashed, "\eta"] & \\
& & W
\end{tikzcd}
$$

We need to construct a morphism $\eta : A \times B \to W$ such that above diagram commutes. Let $\eta = f_1 p + f_2 q$, then

$$\eta \circ i = f_1 pi + f_2 qi = f_1$$

and

$$\eta \circ j = f_1 pj + f_2 qj = f_2,$$

which make above diagram commuts. To show the uniqueness of η, suppose that exists η' such that above diagram commutes. Since $\eta' = \eta' \circ \mathrm{id}_{A\times B} = \eta' \circ (ip+jq) = f_1 p + f_2 q = \eta$, η is unique. Hence, $A \times B$ satisfies the universal property of coproduct, and therefore

$$A \times B \cong A \coprod B.$$

□

Remark It is a consequence of the definition of additive category that finite products are also finite coproducts (i.e. sums). The symbol $\oplus$ is used for this notion.

Proposition 2.5.1

If $\mathscr{C}$ and $\mathscr{D}$ are additive categories and $T : \mathscr{C} \to \mathscr{D}$ is an additive functor of either variance, then $T(A \oplus B) \cong T(A) \oplus T(B)$ for all $A, B \in \mathrm{obj}(\mathscr{C})$.

Proof Let $i : A \to A \oplus B$, $j : B \to A \oplus B$, $p : M \to A$, and $q : M \to B$, then we have

$$pi = \mathrm{id}_A, qj = \mathrm{id}_B, pj = 0, qi = 0, \text{ and } ip + jq = \mathrm{id}_M .$$

Then we have $T(i) : T(A) \to T(A \oplus B)$, $T(j) : T(B) \to T(A \oplus B)$, $T(p) : T(M) \to T(A)$, and $T(q) : T(M) \to T(B)$, since T is an aditive functor, we have

$$T(p)T(i) = \mathrm{id}_{T(A)}, T(q)T(j) = \mathrm{id}_{T(B)}, T(P)T(j) = 0, T(q)T(i) = 0, \text{ and } T(i)T(p)+T(j)T(q) = \mathrm{id}_{T(M)} .$$

By Lemma 2.5.1, we have $T(A \oplus B) \cong T(A) \oplus T(B)$. □

One motivation for the name 0-object is that the 0-morphism in the abelian group $\mathrm{Hom}(A, B)$ is the composition $A \to 0 \to B$. (We also remark that the notion of 0-morphism thus makes sense in any category with a 0-obeject.)

Note *A cleaner axiomatization of additive categories that makes clear that the abelian group structure of $\mathrm{Mor}(A, B)$ is intrinsic to the category itself is the following.*

A0. *$\mathscr{C}$ has a zero object.*

A1. *$\mathscr{C}$ has products of any two objects, and coproducts of any two objects.*

By the universal property of product and coproduct, we have natural morphisms $\varphi_{A,B} : A \coprod B \to A \times B$.

A2. *$\varphi_{A,B}$ is an isomorphism.*

This allows us to define a binary operation on $\mathrm{Mor}(A, B)$, with $f + g$ (for $f, g \in \mathrm{Mor}(A, B)$) defined by the composition

$$A \xrightarrow{(f,g)} B \times B \xrightarrow{\varphi_{B,B}^{-1}} B \coprod B \longrightarrow B$$

*where the last map is the "codiagonal" defined by universal property of coproduct. A little work shows that this endows $\mathrm{Mor}(A, B)$ with the structure of a **commutative monoid**, i.e, an abelian semigroup with identity. The identity is the composition $A \to 0 \to B$.*

A3. *This commutative monoid $\mathrm{Mor}(A, B)$ is an abelian group.*

Example 2.33 The category of A-modules $\mathbf{Mod}_A$ is clearly an additive category, but it has even more structure, which we now formalize as an example of an abelian category.

Definition 2.5.3 (Kernel and cokernel)

*Let $\mathscr{C}$ be a category with a 0-object (and thus 0-morphisms). A **kernel** of a morphism $f : B \to C$ is defined to be a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property.*

Diagramatically:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[dr, "g"] \arrow[drr, "0"] & & \\
A \arrow[r, "i"] \arrow[rr, bend right, "0"'] & B \arrow[r, "f"] & C
\end{tikzcd}$$

Hence it is unique up to unique isomorphism by universal property. The kernel is written $\operatorname{Ker} f \to B$.

A ***cokernel*** *of a morphism* $f : B \to C$ *is defined to be a map* $\pi : C \to D$ *such that* $\pi \circ f = 0$, *and that is universal with respect to this property. Diagramatically:*

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, "0"'] & C \arrow[r, "\pi"] \arrow[dr, "h"] & D \arrow[d, dashed, "\exists!"] \\
& & Y
\end{tikzcd}$$

Hence it is unique up to unique isomorphism by universal property. The cokernel is witten $C \to \operatorname{Coker} f$.

Remark

(1) Note that the kernel is not just an object, it is a morphism of an object to B. In practice, the term is often applied to just the object, and the intended interpretation is clear from the context.

(2) The kernel of $f : B \to C$ is the limit of the diagram

$$\begin{tikzcd}
& 0 \arrow[d] \\
B \arrow[r, "f"] & C.
\end{tikzcd}$$

(3) The cokernel of $f : B \to C$ is the colimit of the diagram

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d] & C \\
0. &
\end{tikzcd}$$

Definition 2.5.4 (Subobject and quotient object)

If $i : A \to B$ *is a monomorphism, then we say that* A *is a* ***subobject*** *of* B, *where the map* i *is implicit.*

If $\pi : A \to B$ *is epimorphism, then we say that* B *is a* ***quotient object*** *of* A, *where the map* π *is implicit.*

Definition 2.5.5 (Abelian category)

An ***abelian category*** *is an additive category satisfying three additional properties.*

(1) Every map has a kernel and cokernel.

(2) Every monomorphism is the kernel of its cokernel.

(3) Every epimorphism is the cokernel of its kernel.

Remark It is a nonobvious (and imprecisely stated) fact that every property you want to be ture about kernals, cokernals, etc. follows from these three.

Definition 2.5.6 (Image)

Let $f : A \to B$ be a morphism in an abelian category, and let $\operatorname{Coker} f$ be $\tau : B \to C$ for some object C. Then its ***image*** *is*

$$\operatorname{Im} f = \operatorname{Ker}(\operatorname{Coker} f) = \operatorname{Ker} \tau.$$

In more suggestive notation,

$$\operatorname{Im}(A \to B) = \operatorname{Ker}(\operatorname{Coker}(A \to B)) = \operatorname{Ker} \tau.$$

Remark $\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) = \operatorname{Ker}(f)$, $\operatorname{Coker}(\operatorname{Ker}(\operatorname{Coker}(f))) = \operatorname{Coker}(f)$.

Proof

(1) $\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) = \operatorname{Ker}(f)$.

Say $k_0 : \operatorname{Ker}(f) \to A$, $k : \operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) \to A$, and $c : A \to \operatorname{Coker}(\operatorname{Ker}(f))$. It suffice to show that there exists unique solution in following universal problem.

$$\begin{tikzcd}
X \arrow[d, dashed] \arrow[r, "h"] \arrow[rr, "0"] & {} & {} \\
\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) \arrow[r, "k"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

By the universal property of $\operatorname{Coker}(\operatorname{Ker}(f))$, there exists unique $\theta : \operatorname{Coker}(\operatorname{Ker}(f)) \to B$ such that the following diagram commute.

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[r, "k_0"] \arrow[rr, bend left, "0"] \arrow[rrd, "0"'] & A \arrow[r, "c"] \arrow[rd, "f"] & \operatorname{Coker}(\operatorname{Ker}(f)) \arrow[d, dashed, "\theta"] \\
& & B
\end{tikzcd}$$

Then we have $f = \theta \circ c$. Note that $\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f)))$ is the kernel of $\operatorname{Coker}(\operatorname{Ker}(f))$, $c \circ k = 0$, and therefore $f \circ k = \theta \circ c \circ k = 0$.

Consider the following diagram.

$$\begin{tikzcd}
X \arrow[d, dashed, "\varphi"'] \arrow[rd, "h"] \arrow[rrd, "0"] \arrow[rrrd, "0"] & & & \\
\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) \arrow[r, "k"] \arrow[rr, bend right, "0"'] \arrow[rrr, bend right=40, "0"'] & A \arrow[r, "c"] & \operatorname{Coker}(\operatorname{Ker}(f)) \arrow[r, "\theta"] & B
\end{tikzcd}$$

By the universal property of $\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f)))$, there exists $\varphi : X \to \operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f)))$ such that above diagram commutes, and note that φ agree with the definition of $\operatorname{Ker}(f)$, hence, $\operatorname{Ker}(\operatorname{Coker}(\operatorname{Ker}(f))) = \operatorname{Ker}(f)$.

(2) $\operatorname{Coker}(\operatorname{Ker}(\operatorname{Coker}(f))) = \operatorname{Coker}(f)$.

Say $c_0 : B \to \operatorname{Coker}(f)$, $k : \operatorname{Ker}(\operatorname{Coker}(f)) \to B$, and $c : B \to \operatorname{Coker}(\operatorname{Ker}(\operatorname{Coker}(f)))$. It suffice to show that there exists unique solution in following universal problem.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, "0"'] & B \arrow[r, "c"] \arrow[dr, "h"] & \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) \arrow[d, dashed] \\
& & Y
\end{tikzcd}$$

By the universal property of $\mathrm{Ker}(\mathrm{Coker}(f))$, there exists unique $\theta : A \to \mathrm{Ker}(\mathrm{Coker}(f))$ such that the following diagram commute.

$$\begin{tikzcd}
A \arrow[d, dashed, "\theta"'] \arrow[dr, "f"] \arrow[drr, "0"] & & \\
\mathrm{Ker}(\mathrm{Coker}(f)) \arrow[r, "k"] \arrow[rr, bend right] & B \arrow[r, "c_0"] & \mathrm{Coker}(f)
\end{tikzcd}$$

Then we have $f = k \circ \theta$. Note that $\mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f)))$ is the cokernel of $\mathrm{Ker}(\mathrm{Coker}(f))$, $c \circ k = 0$, and therefore $c \circ f = c \circ k \circ \theta = 0$.

Consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\theta"'] \arrow[rr, bend left, "0"] \arrow[rrr, bend left, "0"] \arrow[drrr, "0"'] & \mathrm{Ker}(\mathrm{Coker}(f)) \arrow[r, "k"'] \arrow[drr, "0"] & B \arrow[r, "c"'] \arrow[dr, "h"] & \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) \arrow[d, dashed, "\psi"] \\
& & & Y
\end{tikzcd}$$

By the universal property of $\mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f)))$, there exists unique $\psi : \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) \to Y$ such that above diagram commutes, and note that ψ agree with the definition of $\mathrm{Coker}(f)$, hence, $\mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(f))) = \mathrm{Coker}(f)$.

□

Proposition 2.5.2

If $f : A \to B$ is a morphism in an abelian category, then we have a factorization $f = me$, where $m : \mathrm{Im}\, f \to B$ is monomorphism and $e : A \to \mathrm{Im}\, f$ is epimorphism. Also,

$$m = \mathrm{Ker}(B \to \mathrm{Coker}\, f), e = \mathrm{Coker}(\mathrm{Ker}\, f \to A).$$

Diagramatically,

$$\begin{tikzcd}
& \mathrm{Im}\, f \arrow[d, dashed, "m"] \\
A \arrow[ur, dashed, "e"] \arrow[r, "f"] & B.
\end{tikzcd}$$

If also $f = m'e'$, where m' is a kernal, then in the commutative square

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "e'"'] & \bullet \arrow[d, "m"] \arrow[dl, dashed, "t"'] \\
\bullet \arrow[r, "m'"] & \bullet
\end{tikzcd}$$

there is a unique diagonal arrow t with $m = m't$ and $e' = te$.

Proof See, Mac Lane, Categories for the working Mathematician, Chapter VIII, Section 1 and 3. □

Definition 2.5.7 (Quotient)

The cokernal of a monomorphism is called ***quotient****. The quotient of a monomorphism $A \to B$ is often denoted B/A (with the map from B implicit).*

Remark (Small remark on chasing diagrams.) It is useful to prove facts (and solve exercise) about abelian categories by chasing elements. Unfortunately, some commonly used abelian categories, such as the category of complexes, do not have "elements" — they are not naturally "sets with additional structure" in any obvious way. Nonetheless, proof by element-chasing can be justified by the **Freyd-Mitchell Embedding Theorem**:

Theorem 2.5.1 (Freyd-Mitchell Embedding Theorem)

If $\mathscr{C}$ is an abelian category whose object form a set, then there is a ring A and an exact, fully faithful functor from $\mathscr{C}$ into $\mathbf{Mod}_A$, which embeds $\mathscr{C}$ as a full subcategory.

(Unfortunately, the ring A need not be commutative.) The upshot is that to prove something about a diagram in some abelian category, and we may then "diagram-chase" elements. Moreover, any fact about kernels, cokernels, and so on that holds in $\mathbf{Mod}_A$ holds in any abelian category.

In this entire discussion, we assume we are working in an abelian category.

2.5.2 Complexes

Definition 2.5.8 (Complex and exact)

We say a sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is a ***complex at*** *B if $g \circ f = 0$, and is* ***eact at*** *B if $\operatorname{Ker} g = \operatorname{Im} f$.*
A sequence is a ***complex*** *(resp.,* ***exact****) if it is a complex (resp., exact) at each term. A* ***short exact sequence*** *is an exact sequence with five terms, the first and last of which are zeros — in other words, an exact sequence of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Proposition 2.5.3

Some properties of exact sequence (in abelian category):

(1) Sequence

$$0 \longrightarrow A \longrightarrow 0$$

is exact if and only if $A = 0$.

(2) Sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only if f is a monomorphism.

(3) Sequence

$$A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f is a epimorphism.

(4) Sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f is an isomorphism.

(5) Sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if f is kernel of g, and sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if g is cokernel of f.

Remark In additive category, Proposition 2.5.3 also holds.

Proof

(1) If sequence

$$0 \longrightarrow A \longrightarrow 0$$

exact, then $\mathrm{Ker}(A \to 0) = \mathrm{Im}(0 \to A)$. We shall to show that $\mathrm{Ker}(A \to 0) = \mathrm{id}_A : A \to A$ and $\mathrm{Im}(0 \to A) = 0 : 0 \to A$.

Note that $\mathrm{id}_A : A \to A$ is the solution of the following universal problem,

$$\begin{tikzcd} X \arrow[d, dashed, "f"'] \arrow[dr, "f"] \arrow[drr, "0"] & & \\ A \arrow[r, "\mathrm{id}_A"] \arrow[rr, bend right, "0"'] & A \arrow[r, "0"] & 0, \end{tikzcd}$$

hence, $\mathrm{Ker}(A \to 0) = \mathrm{id}_A : A \to A$. Since $\mathrm{Im}(0 \to A) = \mathrm{Ker}(\mathrm{Coker}(0 \to A))$, consider the following diagram,

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[drr, "0"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "g"] & A \arrow[d, dashed, "g"] \\ & & Y \end{tikzcd}$$

$\mathrm{id}_A : A \to A$ is the solution of above universal problem, hence $\mathrm{Coker}(0 \to A) = \mathrm{id}_A : A \to A$. Now, it suffice to show that $\mathrm{Ker}(\mathrm{id} : A \to A) = 0 : 0 \to A$. Consider the following universal problem,

$$\begin{tikzcd} Z \arrow[d, "0"'] \arrow[dr, "0"] \arrow[drr, "0"] & & \\ 0 \arrow[r, "0"'] \arrow[rr, bend right] & A \arrow[r, "\mathrm{id}"'] & A, \end{tikzcd}$$

$0 : 0 \to A$ is the solution, and therefore $\mathrm{Im}(0 \to A) = 0 : 0 \to A$. Since $\mathrm{Ker}(A \to 0) = \mathrm{Im}(0 \to A)$, we have $A = 0$.

Conversely, if $A = 0$, obviously, $\mathrm{Ker}(A \to 0) = \mathrm{Im}(0 \to A)$.

(2) If sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact, then $\mathrm{Im}(0 \to A) = \mathrm{Ker}\, f$. By part (1), $\mathrm{Im}(0 \to A) = 0 : 0 \to A$, hence, $\mathrm{Ker}(f) = 0 : 0 \to A$. Let two morphisms $\mu_1 : X \to A$ and $\mu_2 : X \to A$ with $f \circ \mu_1 = f \circ \mu_2$. Since in Abelian category,

$f \circ (\mu_1 - \mu_2) = 0$ and $\mu_1 - \mu_2 : X \to A$, consider the following diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "0"'] \arrow[dr, "\mu_1 - \mu_2"] \arrow[drr, "0"] & & \\
0 \arrow[r, "0"'] \arrow[rr, bend right] & A \arrow[r, "f"'] & B,
\end{tikzcd}$$

by the universal property of $\operatorname{Ker} f$, $\mu_1 - \mu_2 = 0$, that is, f is monomorphism.
Conversely, if f is monomorphism, we shall to show that $\operatorname{Ker}(f) = 0 : 0 \to A$. It suffice to show that $0 : 0 \to A$ is the solution of the following universal problem.

$$\begin{tikzcd}
X \arrow[d, dashed] \arrow[dr, "g"] \arrow[drr, "0"] & & \\
0 \arrow[r, "0"'] \arrow[rr, bend right] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

Since, f is monomorphism, g is the unique morphism such that $f \circ g = 0$, note that $0 = f \circ 0$, $g = 0$. Hence, $0 : 0 \to A$ is the solution of the universal problem, and therefore $\operatorname{Ker}(f) = 0 : 0 \to A = \operatorname{Im}(0 \to A)$, which implies the sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact.

(3) If sequence

$$A \xrightarrow{f} B \longrightarrow 0$$

is exact, then $\operatorname{Im} f = \operatorname{Ker}(B \to 0) = \mathrm{id}_B : B \to B$. Let two morphisms $\eta_1 : B \to Y$ and $\eta_2 : B \to Y$ with $\eta_1 \circ f = \eta_2 \circ f$, since in Abelian category, we have $(\eta_1 - \eta_2) \circ f = 0$. Note that $\operatorname{Im} f = \operatorname{Ker}(\operatorname{Coker}(f)) = \mathrm{id}_B : B \to B$, $\operatorname{Coker}(f) \circ \mathrm{id}_B = \operatorname{Coker}(f) = 0 : B \to 0$. Now consider the following diagram,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, "0"'] & B \arrow[r, "0"] \arrow[dr] & 0 \arrow[d, dashed] \\
& & Y
\end{tikzcd}$$

$\eta_1 - \eta_2$ agree with above diagram, hence, $\eta_1 = \eta_2$, which implies that f is epimorphism.
Conversely, if f is epimorphism, we shall to show that $\operatorname{Im}(f) = \operatorname{Ker}(\operatorname{Coker}(f)) = \operatorname{Ker}(B \to 0) = \mathrm{id}_B : B \to B$. It suffice to show that $\mathrm{id}_B : B \to B$ is the solution of the following universal problem.

$$\begin{tikzcd}
Z \arrow[d, dashed] \arrow[dr] \arrow[drr, "0"] & & \\
B \arrow[r] \arrow[rr, bend right, "0"'] & B \arrow[r] & \operatorname{Coker}(f)
\end{tikzcd}$$

We need to show that $\operatorname{Coker}(f) \circ \mathrm{id}_B = \operatorname{Coker}(f) = 0 : B \to 0$. Consider the following universal problem,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, "0"'] & B \arrow[r] \arrow[dr] & 0 \arrow[d, dashed] \\
 & & W,
\end{tikzcd}$$

since f is epimorphism, $B \to 0$ and $B \to W$ must be same, hence, $W = 0$, and the unique morphism is $0 : 0 \to 0$. Hence, $\operatorname{Coker}(f) = 0 : B \to 0$. It is clearly $\mathrm{id}_B : B \to B$ is the solution of the universal problem

$$\begin{tikzcd}
Z \arrow[d, dashed, "g"'] \arrow[dr, "g"] \arrow[drr, "0"] & & \\
B \arrow[r] \arrow[rr, bend right, "0"'] & B \arrow[r] & \operatorname{Coker}(f),
\end{tikzcd}$$

hence, $\operatorname{Im}(f) = \mathrm{id}_B : B \to B = \operatorname{Ker}(B \to 0)$.

(4) By part (2) and (3), immediately.

(5) If sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact, then $\operatorname{Ker} g = \operatorname{Im} f$. We shall to show that $g \circ f = 0$. By Proposition 2.5.2, we have $f = me$, where epimorphism $e : A \to \operatorname{Im} f$ and monomorphism $m : \operatorname{Im} f \to B$. Since $\operatorname{Ker} g = \operatorname{Im} f$, we have $e : A \to \operatorname{Ker} g$ and $m : \operatorname{Ker} g \to B$. By the definition of kernal, we have $g \circ m = 0$. Hence, $g \circ f = g \circ m \circ e = 0$.

By Proposition 2.5.2, $f = me$, where $m : \operatorname{Im} f \to B$ monomorphism and $e : A \to \operatorname{Im} f$. We claim that e is an isomorphism. It suffice to show that e is monomorphism. Let $\eta_1 : X \to A$ and $\eta_2 : X \to A$ morphisms with $e \circ \eta_1 = e \circ \eta_2$, then $m \circ e \circ \eta_1 = m \circ e \circ \eta_2$, since $f = me$ is monomorphism, $\eta_1 = \eta_2$. Hence, e is isomorphism. Consider the following universal problem.

$$\begin{tikzcd}
 & X \arrow[dl, dashed] \arrow[d, dashed, "t"] \arrow[dr, "h"] \arrow[drr, "0"] & & \\
A \arrow[r, "e"] \arrow[rrr, bend right, "0"'] & \operatorname{Im} f \arrow[r, "m"] \arrow[rr, bend right, "0"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Since $\operatorname{Im} f = \operatorname{Ker} g$, by the universal property of kernel, exists unique $t : X \to \operatorname{Im} f$. Let $\theta : X \to A$ by setting $\theta = e^{-1} \circ t$, then above diagram commutes. Note that $f = me$ is monomorphism, θ must be unique. Hence, $f = \operatorname{Ker} g$.

Conversely, if $f : A \to B = \operatorname{Ker}(g)$, then $g \circ f = 0$. By Proposition 2.5.2, $f = me$ uniquely, where $e : A \to \operatorname{Im} f$ epimorphism and $m : \operatorname{Im} f \to B$ monomorphism. Consider the following diagram,

$$\begin{tikzcd}
A \arrow[d, dashed, "e"] \arrow[dr, "f"] \arrow[drr, "0"] & & \\
\operatorname{Im} f \arrow[r, "m"'] \arrow[rr, bend right, "0"'] & B \arrow[r, "g"'] & C,
\end{tikzcd}$$

since e is epimorphism, $g \circ f = g \circ m \circ e = 0$ implies $g \circ m = 0$. Hence, e is the solution of above universal problem, and therefore $\operatorname{Im} f = \operatorname{Ker} g$. Let X be any object, $u : X \to A$ and $v : X \to A$ are two morphisms with $f \circ u = f \circ v$, then $f \circ (u - v) = 0$. Consider the following diagram,

$$\begin{tikzcd}
0 \arrow[d, dashed, "\exists!"'] \arrow[dr, "0"] \arrow[drr, "0"] & & \\
A \arrow[r, "f"] \arrow[rr, bend right, "0"'] & B \arrow[r, "g"] & C,
\end{tikzcd}$$

since $f = \operatorname{Ker}(g)$, by the universal property of kernel, $u - v = 0$, that is, $u = v$. Hence, f is monomorphism.

If sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, then $\operatorname{Im} f = \operatorname{Ker} g$ and g is epimorphism. We shall to show that $g = \operatorname{Coker} f$. Note that $\operatorname{Im} f = \operatorname{Ker}(\operatorname{Coker}(f))$, we have $\operatorname{Coker}(\operatorname{Ker}(\operatorname{Coker}(f))) = \operatorname{Coker}(f) = \operatorname{Coker}(\operatorname{Ker} g)$. It suffice to show that $g = \operatorname{Coker}(\operatorname{Ker}(g))$. By Proposition 2.5.2, $g = me$, where $m = \operatorname{Ker}(C \to \operatorname{Coker} g)$ monomorphism and $e = \operatorname{Coker}(\operatorname{Ker}(g) \to B)$ epimorphism. Since g is epimorphism, $\operatorname{Coker}(g) = 0 : C \to 0$, and therefore $m = \operatorname{Ker}(0 : C \to 0) = \mathrm{id}_C : C \to C$. Hence, $g = \operatorname{Coker}(\operatorname{Ker}(g))$.

Conversely, If $g = \operatorname{Coker} f$, then $\operatorname{Ker}(g) = \operatorname{Ker}(\operatorname{Coker}(f)) = \operatorname{Im}(f)$. We shall to show that g is epimorphism. Let morphisms $\mu_1 : C \to X$ and $\mu_2 : C \to X$ with $\mu_1 \circ g = \mu_2 \circ g$, then $(\mu_1 - \mu_2) \circ g = 0$. Since $g = \operatorname{Coker}(f)$, by the universal property of cokernel, $\mu_1 - \mu_2 = 0$, that is, $\mu_1 = \mu_2$. Hence, g is epimorphism. Consequently, sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact.

□

Corollary 2.5.1

Let $u : A \to B$ be a morphism in an additive category $\mathscr{C}$.

(1) If $\operatorname{Ker} u$ exists, then u is monomorphism if and only if $\operatorname{Ker} u = 0$.

(2) If $\operatorname{Coker} u$ exists, then u is epimorphism if and only if $\operatorname{Coker} u = 0$.

Proof See the proof in Proposition 2.5.3. □

Definition 2.5.9 (Homology, cohomology, cycle, and boundary)

If sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

*is a complex at B, then its **homology at** B (often denoted by H) is $\operatorname{Ker} g / \operatorname{Im} f$. More precisely, there is some monomorphism $\operatorname{Im} f \hookrightarrow \operatorname{Ker} g$, and that H is the cokernel of this monomorphism.*

*We say that elements of $\operatorname{Ker} g$ (assuming the objects of the category are sets with some additional structure) are the **cycles**, and elements of $\operatorname{Im} f$ are the **boundaries** (so homology is "cycles mod boundaries").*

*If the complex is indexed in decreasing order, the indices are often written subscripts, and H_i is the homology at $A_{i+1} \to A_i \to A_{i-1}$. If the complex is indexed in increasing order, the indices are often written as superscripts, and the homology H^i at $A^{i-1} \to A^i \to A^{i+1}$ is often called **cohomology**.*

Remark Sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is exact at B if and only if its homology at B is 0.

Theorem 2.5.2

An exact sequence

$$A^\bullet: \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{2.16}$$

can be "factored" into short exact sequences

$$0 \longrightarrow \operatorname{Ker} f^i \longrightarrow A^i \longrightarrow \operatorname{Ker} f^{i+1} \longrightarrow 0$$

which is helpful in proving facts about long exact sequences by reducing them to facts about short exact sequences.

More generally, if (2.16) is assumed only to be a complex, then it can be "factored" into short exact sequences.

$$\begin{gathered} 0 \longrightarrow \operatorname{Ker} f^i \longrightarrow A^i \longrightarrow \operatorname{Im} f^i \longrightarrow 0 \\ 0 \longrightarrow \operatorname{Im} f^{i-1} \longrightarrow \operatorname{Ker} f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0 \end{gathered} \tag{2.17}$$

Proof If $A^\bullet$ is exact, we have

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \\
 & \operatorname{Ker} f^{i-1} \arrow[d, hook] & & \operatorname{Im} f^i = \operatorname{Ker} f^{i+1} \arrow[d, hook] \arrow[ur] & \\
\cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, two heads] & A^i \arrow[r, "f^i"] \arrow[ur, two heads] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d, two heads] & \cdots \\
 & \operatorname{Im} f^{i-1} = \operatorname{Ker} f^i \arrow[d] \arrow[ur, hook] & & \operatorname{Im} f^{i+1} \arrow[d] & \\
0 \arrow[ur] & 0 & & 0. &
\end{tikzcd}$$

If $A^\bullet$ is complex, by Proposition 2.5.2 and Proposition 2.5.3, we have

$$\begin{tikzcd}
 & & 0 & & & \\
 & 0 \arrow[r] & \operatorname{Im} f^i \arrow[u] \arrow[r, hook] \arrow[dr, hook] & \operatorname{Ker} f^{i+1} \arrow[r, two heads] \arrow[d, hook] & H^{i+1}(A^\bullet) \arrow[r] & 0 \\
\cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, two heads] & A^i \arrow[r, "f^i"] \arrow[u, two heads] & A^{i+1} \arrow[r, "f^{i+1}"] & \cdots & \\
0 \arrow[r] & \operatorname{Im} f^{i-1} \arrow[r, hook] \arrow[ur, hook] & \operatorname{Ker} f^i \arrow[r, two heads] \arrow[u] & H^i(A^\bullet) \arrow[r] & 0 & \\
 & & 0. \arrow[u] & & &
\end{tikzcd}$$

$\square$

Exercise 2.35 Describe exact sequence

$$
\begin{gathered}
0 \longrightarrow \operatorname{Im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{Coker} f^i \longrightarrow 0 \\
0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{Coker} f^{i-1} \longrightarrow \operatorname{Im} f^i \longrightarrow 0
\end{gathered}
\tag{2.18}
$$

(These are somehow dual to (2.17). In fact in some mirror universe this might have been given as the standard definition of homology.) Assume the category is that of modules over a fixed ring for convenience, but be aware that the result is true for any abelian category.

Proof Note that

$$
\begin{tikzcd}
 & & & & 0 \\
 & & & \operatorname{Coker} f^{i-1} \arrow[ur] & \\
\cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, two heads] & A^i \arrow[r, "f^i"] \arrow[ur, two heads] & A^{i+1} \arrow[r, "f^{i+1}"] & \cdots \\
 & \operatorname{Im} f^{i-1} \arrow[ur, hook] & & & \\
0, \arrow[ur] & & & &
\end{tikzcd}
$$

then we have the first sequence.

In $\mathbf{Mod}_A$, we have $\operatorname{Coker} f^{i-1} = A^i / \operatorname{Im} f^{i-1}$ and $H^i(A^\bullet) = \operatorname{Ker} f^i / \operatorname{Im} f^{i-1}$, then we have inclusion $H^i(A^\bullet) \hookrightarrow \operatorname{Coker} f^{i-1}$. Since $f^i : A^i \to \operatorname{Im} f^i$ is surjective, $\operatorname{Im} f^i \cong A^i / \operatorname{Ker} f^i$. By the homomorphism fundamental theorem, we have $\operatorname{Coker} f^{i-1} / H^i(A^\bullet) = (A^i / \operatorname{Im} f^{i-1}) / (\operatorname{Ker} f^i / \operatorname{Im} f^{i-1}) \cong A^i / \operatorname{Ker} f^i \cong \operatorname{Im} f^i$. Then sequence

$$
0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{Coker} f^{i-1} \longrightarrow \operatorname{Im} f^i \longrightarrow 0
$$

is exact. $\square$

Proposition 2.5.4

Suppose

$$
0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0
$$

is a complex of finite-dimensional k-vector spaces (often called $A^\bullet$ for short). Define $h^i(A^\bullet) = \dim H^i(A^\bullet)$. Then $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$.

In particular, if $A^\bullet$ is exact, then $\sum (-1)^i \dim A^i = 0$.

Proof By Theorem 2.5.2, spilit up the sequence into short exact sequences

$$
0 \longrightarrow \operatorname{Ker} d^i \longrightarrow A^i \longrightarrow \operatorname{Im} d^i \longrightarrow 0
$$

$$
0 \longrightarrow \operatorname{Im} d^{i-1} \longrightarrow \operatorname{Ker} d^i \longrightarrow H^i(A^\bullet) \longrightarrow 0.
$$

Since dim is additive function (see, Atiyah-MacDonald, Introdution to Commutative Algebra, page 23), we have $\dim A^i = \dim \operatorname{Ker} d^i + \dim \operatorname{Im} d^i$ and $\dim H^i(A^\bullet) = h^i(A^\bullet) = \dim \operatorname{Ker} d^i - \dim \operatorname{Im} d^{i-1}$. Note that $\operatorname{Ker} d^0 = 0$, $\operatorname{Ker} d^n = \mathrm{id}_{A^n}$, $\operatorname{Im} d^0 = 0$, and $\operatorname{Im} d^n = 0$, we have

$$\begin{aligned}\sum (-1)^i \dim A^i &= \sum_{i=0}^{n} (-1)^i (\dim \operatorname{Ker} d^i + \dim \operatorname{Im} d^i) \\ &= \sum_{i=1}^{n} (-1)^i \dim \operatorname{Ker} d^i + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{Im} d^i,\end{aligned}$$

and

$$\begin{aligned}\sum (-1)^i h^i(A^\bullet) &= \sum_{i=1}^{n} (\dim \operatorname{Ker} d^i - \dim \operatorname{Im} d^{i-1}) \\ &= \sum_{i=1}^{n} (-1)^i \dim \operatorname{Ker} d^i + \sum_{i=1}^{n} (-1)^{i+1} \dim \operatorname{Im} d^{i-1} \\ &= \sum_{i=1}^{n} (-1)^i \dim \operatorname{Ker} d^i + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{Im} d^i.\end{aligned}$$

Hence, $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$.

In particular, if $A^\bullet$ is exact, then $h^i(A^\bullet) = \dim \operatorname{Ker} d^i - \dim \operatorname{Im} d^{i-1} = 0$. Hence,

$$\sum (-1)^i \dim A^i = 0.$$

$\square$

$\mathbf{Com}_{\mathscr{C}}$ is an abelian category

Definition 2.5.10 (Category $\mathbf{Com}_{\mathscr{C}}$ of complexes)

*Suppose $\mathscr{C}$ is an abelian category. Define the **category $\mathbf{Com}_{\mathscr{C}}$ of complexes** as follows. The objects are infinite complexes*

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

*in $\mathscr{C}$, each d^i in a complex called **differentials**, and the morphisms $A^\bullet \to B^\bullet$ (called **chain map**) are commuting diagrams.*

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d] & A^i \arrow[r, "d^i"] \arrow[d] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r, "d'^{i-1}"] & B^i \arrow[r, "d'^i"] & B^{i+1} \arrow[r, "d'^{i+1}"] & \cdots . \end{tikzcd}$$

Our goal is to prove that $\mathbf{Com}_{\mathscr{C}}$ is an abelian category.

Lemma 2.5.2

Let $\mathscr{C}$ be an additive category and $\mathscr{S}$ be a subcategory if $\mathscr{S}$ is full, contains a zero object of $\mathscr{C}$, and contains the direct sum $A \oplus B$ (in $\mathscr{C}$) of all $A, B \in \operatorname{obj}(\mathscr{S})$.

Proof Since $\mathscr{S}$ is full,

(1) $\operatorname{Hom}(A, B)$ is an additive abelian group for every $A, B \in \operatorname{obj}(\mathscr{S})$,

(2) distributive laws given by $\mathscr{C}$.

Also, $\mathscr{S}$ contains a zero object of $\mathscr{C}$. By Lemma 2.5.1, product is isomorphic to coproduct, hence, it suffice to check for all $A, B \in \operatorname{obj}(\mathscr{S})$ the direct sum $A \oplus B \in \mathscr{S}$. By the hypothesis, $\mathscr{S}$ is an additive category. $\square$

Lemma 2.5.3

Let $\mathscr{S}$ be a full subcategory of an abelian category $\mathscr{C}$. If for all $A, B \in \mathrm{obj}(\mathscr{S})$ and all $f : A \to B$,

(i) a zero object in $\mathscr{C}$ lies in $\mathscr{S}$,

(ii) the dierect sum $A \otimes B$ in $\mathscr{C}$ lies in $\mathscr{S}$,

(iii) both $\operatorname{Ker} f$ and $\operatorname{Coker} f$ lie in $\mathscr{S}$,

then $\mathscr{S}$ is an abelian category.

Moreover, if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence in $\mathscr{S}$, then it is an exact sequence in $\mathscr{C}$.

Proof The hypothesis gives $\mathscr{S}$ additive, by Lemma 2.5.2, hence, $\mathscr{S}$ is abelian if axiom (iii) in the definition of abelian category holds. If $f : A \to B$ is a monomorphism in $\mathscr{S}$, then $\operatorname{Ker} f = 0$. Since $\operatorname{Ker} f$ is the same in $\mathscr{C}$ as in $\mathscr{S}$, by hypothesis, f is monomorphism in $\mathscr{C}$. By hypothesis, $\operatorname{Coker} f$ is a morphism in $\mathscr{S}$. Since $\mathscr{C}$ is abelian, there is a morphism $g : B \to C$ with $f = \operatorname{Ker} g$, where $g = \operatorname{Coker} f$. Since S contains cokernels, g is a morphism in $\mathscr{S}$. Hence, $f = \operatorname{Ker} g$ in $\mathscr{S}$. The dual argument shows that epimorphisms in $\mathscr{S}$ are cokernels.

Moreover, since kernels and cokernels are the same in $\mathscr{S}$ as in $\mathscr{C}$, images are also the same, and so exactness in $\mathscr{S}$ implies exactness in $\mathscr{C}$. □

Proposition 2.5.5

If $\mathscr{C}$ is an abelian category and $\mathscr{I}$ is a small category, then the functor category $\mathscr{C}^{\mathscr{I}}$ is an abelian category.

Proof We assume that $\mathscr{I}$ is small to gurantee that the morphism set $\mathrm{Mor}(F, G)$, where $F, G : \mathscr{I} \to \mathscr{C}$, are sets, not proper classes. The zero object in $\mathscr{C}^{\mathscr{I}}$ is the constant functor with value 0, where 0 is a zero object in $\mathscr{C}$. If $\tau, \sigma : F \to G$ by $(\tau + \sigma)_i = \tau_i + \sigma_i : F(i) \to G(i)$ for all $i \in \mathrm{obj}(\mathscr{I})$. Finally, define $F \oplus G$ by $(F \oplus G)(i) = F(i) \oplus G(i)$. Since $\mathscr{C}$ is abelian category, $F \oplus G \in \mathscr{C}^{\mathscr{I}}$. Hence, these definitions make $\mathscr{C}^{\mathscr{I}}$ an additive category.

If $\tau : F \to G$, define K by

$$K(i) = \operatorname{Ker}(\tau_i).$$

In the following commutative diagram with exact rows, where $f : i \to j$ in $\mathscr{I}$, there is a unique $Kf : K(i) \to K(j)$ making the augmented diagram commute (by the universal property of $\operatorname{Ker}(\tau_j)$).

$$\begin{tikzcd}
0 \arrow[r] & K(i) \arrow[r, "\iota_i"] \arrow[d, dashed, "Kf"'] & F(i) \arrow[r, "\tau_i"] \arrow[d, "Ff"] & G(i) \arrow[d, "Gf"] \\
0 \arrow[r] & K(j) \arrow[r, "\iota_j"] & F(j) \arrow[r, "\tau_j"] & G(j)
\end{tikzcd}$$

We shall to check that K is a functor, $\iota : K \to F$ is natural transformation, and $\iota = \operatorname{Ker} \tau$.

(1) K is a functor.

Let $i \in \mathscr{I}$, then $K(\mathrm{id}_i) : K(i) \to K(i)$, by the uniqueness of $K(\mathrm{id}_i)$, we have $K(\mathrm{id}_i) = \mathrm{id}_{K(i)}$. Let $i \to j \to k$ in $\mathscr{I}$, also by the uniqueness, we have $K(i \to j \to k) = K(j \to k) \circ K(i \to j)$. Hence, K is a functor.

(2) $\iota : K \to F$ is a natural transformation.

Since, $\tau : F \to G$ is a natural transformation, by the universal property of $\operatorname{Ker}(\tau_i)$, diagram

$$\begin{tikzcd} K(i) \arrow[r, "\iota_i"] \arrow[d, dashed, "Kf"'] & F(i) \arrow[d, "Ff"] \\ K(j) \arrow[r, "\iota_j"] & F(j) \end{tikzcd}$$

commutes, for all $i, j \in \operatorname{Im}(\mathscr{I})$. Hence, $\iota : K \to F$ is a natural transformation.

(3) $\iota = \operatorname{Ker} \tau$.

Note that we have exact sequence,

$$0 \longrightarrow K \longrightarrow F \longrightarrow G,$$

by Proposition 2.5.3, $\iota = \operatorname{Ker} \tau$.

Hence, kernels exists in $\mathscr{C}^{\mathscr{I}}$. Dually, cokernels exists in $\mathscr{C}^{\mathscr{I}}$. By Lemma 2.5.3, $\mathscr{C}^{\mathscr{I}}$ is an abelian category. $\square$

Definition 2.5.11 (Subcomplex)

A complex $(A^\bullet, \delta^\bullet)$ is defined to be a ***subcomplex*** *of complex $(C^\bullet, d^\bullet)$ if there is a chain map $i : A^\bullet \to C^\bullet$ with each i^n is monomorphism.*

Example 2.34 In $\mathbf{Com}_{\mathbf{Mod}_A}$, we have that $(A^\bullet, \delta^\bullet)$ is a subcomplex of $(C^\bullet, d^\bullet)$ if A_n is a submodule of C_n and $\delta_n = d_n|_{A_n}$ for every $n \in \mathbb{Z}$.

Theorem 2.5.3

$\mathbf{Com}_{\mathscr{C}}$ is an abelian category.

Proof We only prove this when $\mathscr{C} = \mathbf{Mod}_A$. View $\mathbb{Z}$ first as a partially ordered set under reverse inequality and then as a small category (with morphisms $m \to n$ if $m \le n$). By Proposition 2.5.5, the functor category $\mathbf{Mod}_A^{\mathscr{I}}$ is an abelian category. Lemma 2.5.3 says that $\mathbf{Com}_{\mathscr{C}}$ is abelian if $\mathbf{Com}_{\mathbf{Mod}_A}$ is a full category of $\mathbf{Mod}_A^{\mathscr{I}}$ containing a zero object, the direct sum $A^\bullet \oplus B^\bullet$ of $A^\bullet, B^\bullet \in \operatorname{obj}(\mathbf{Com}_{\mathbf{Mod}_A})$, and both $\operatorname{Ker} f^\bullet$ and $\operatorname{Coker} f^\bullet$, where $f^\bullet$ is a morphism in $\mathbf{Com}_{\mathbf{Mod}_A}$. All the steps are routine. Note that $\mathbf{Com}_{\mathbf{Mod}_A}$ is, by definition, a full subcategory of $\mathbf{Mod}_A^{\mathscr{I}}$. The **zero complex** is the complex each of whose terms is 0, while $(A^\bullet, d^\bullet) \oplus (B^\bullet, d'^\bullet)$ is, by definition, the complex each of whose n-th term is $A_n \oplus B_n$ and whose n-th differential is $d^n \oplus d'^n$. If $f^\bullet : (A^\bullet, d^\bullet) \to (B^\bullet, d'^\bullet)$ is a chain map, define

$$\operatorname{Ker} f^\bullet = \cdots \longrightarrow \operatorname{Ker} f^{n-1} \xrightarrow{\delta^{n-1}} \operatorname{Ker} f^n \xrightarrow{\delta^n} \operatorname{Ker} f^{n+1} \longrightarrow \cdots,$$

where $\delta^n = d^n|_{\operatorname{Ker} f^n}$, and

$$\operatorname{Im} f^\bullet = \cdots \longrightarrow \operatorname{Im} f^{n-1} \xrightarrow{\Delta^{n-1}} \operatorname{Im} f^n \xrightarrow{\Delta^n} \operatorname{Im} f^{n+1} \longrightarrow \cdots,$$

where $\Delta^n = d'^n|_{\operatorname{Im} f^n}$. Then $\operatorname{Ker} f^\bullet$ is a subcomplex of $A^\bullet$, and $\operatorname{Im} f^\bullet$ is a subcomplex of $B^\bullet$. If $A'^\bullet$ is a subcomplex of $A^\bullet$, define the **quotient complex** to be

$$A^\bullet/A'^\bullet = \cdots \longrightarrow A^n/A'^n \xrightarrow{\overline{d^n}} A^{n+1}/A'^{n+1} \longrightarrow \cdots,$$

where $\overline{d^n} : a_n + A'^n \mapsto d^n a_n + A'^{n+1}$. If $p^n : A^n \to A^n/A'^n$ is the natural map, then $p^\bullet : A^\bullet \to A^\bullet/A'^\bullet$ is a chain map. Finally, define

$$\operatorname{Coker} f^\bullet = \cdots \longrightarrow B^{n-1}/\operatorname{Im} f^{n-1} \xrightarrow{\overline{d'^{n-1}}} B^n/\operatorname{Im} f^n \xrightarrow{\overline{d'^n}} B^{n+1}/\operatorname{Im} f^{n+1} \xrightarrow{\overline{d'^{n+1}}} \cdots.$$

Now, we shall to show that above definitions agree with the categorical definitions of Ker and Coker in $\mathbf{Com}_{\mathbf{Mod}_A}$.

(1) Kernel.

Consider the following universal problem.

$$\begin{tikzcd}
C^\bullet \arrow[d, dashed] \arrow[dr, "g"'] \arrow[drr, "0"] & & \\
\operatorname{Ker} f^\bullet \arrow[r, "i"'] \arrow[rr, bend right, "0"'] & A^\bullet \arrow[r, "f"'] & B^\bullet
\end{tikzcd}$$

Since $f^n \circ i^n : \operatorname{Ker} f^n \to A^n \to B^n$ for all $n \in \mathbb{Z}$, indeed, $f^\bullet \circ i = 0$. By the universal property of $\operatorname{Ker} f^n$, we have the following commutative diagram.

$$\begin{tikzcd}
C^n \arrow[d, dashed, "\exists ! \varphi^n"'] \arrow[dr, "g^n"'] \arrow[drr, "0"] & & \\
\operatorname{Ker} f^n \arrow[r, "i^n"'] \arrow[rr, bend right, "0"'] & A^n \arrow[r, "f^n"'] & B^n
\end{tikzcd}$$

Then we can define a unique map $\varphi : C^\bullet \to \operatorname{Ker} f^\bullet$, since φ^n is unique. We need to show that φ is a chain map, it suffices to check the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & C^n \arrow[r, "d_C^n"] \arrow[d, "\varphi^n"] & C^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{Ker} f^{n-1} \arrow[r, "\delta^{n-1}"] & \operatorname{Ker} f^n \arrow[r, "\delta^n"] & \operatorname{Ker} f^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Consider following diagram.

$$\begin{tikzcd}
& C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[dl, "g^{n-1}"'] \arrow[d, "\varphi^{n-1}"'] & C^n \arrow[d, "\varphi^n"] \arrow[dr, "g^n"] & \\
A^{n-1} \arrow[rrr, bend right, "d^{n-1}"'] & \operatorname{Ker} f^{n-1} \arrow[l, "i^{n-1}"'] \arrow[r, "\delta^{n-1}"] & \operatorname{Ker} f^n \arrow[r, "i^n"] & A^n
\end{tikzcd}$$

Since $d^{n-1} \circ g^{n-1} = g^n \circ d_C^{n-1}$, and note that $g^k = i^k \circ \varphi^k$ and i^k is inclusion for all k,

$$\begin{aligned}
d^{n-1} \circ g^{n-1} =& d^{n-1} \circ i^{n-1} \circ \varphi^{n-1} \\
=& d^{n-1}|_{\operatorname{Ker} f^{n-1}} \circ \varphi^{n-1} \\
=& g^n \circ d_C^{n-1} = i^n \circ \varphi^n \circ d_C^{n-1} \\
=& \varphi^n \circ d_C^{n-1}.
\end{aligned}$$

Note that $\delta^{n-1} = d^{n-1}|_{\operatorname{Ker} f^{n-1}} \circ \varphi^{n-1}$, $\delta^{n-1} \circ \varphi^{n-1} = \varphi^n \circ d_C^{n-1}$. Hence, $\varphi : C^\bullet \to \operatorname{Ker} f^\bullet$ is a chain map, and therefore $\operatorname{Ker} f$ agree with the categorical definition of kernel in $\mathbf{Com}_{\mathbf{Mod}_A}$.

(2) Cokernel.
Similar to the proof of the kernel.

Then both $\operatorname{Ker} f^\bullet$ and $\operatorname{Coker} f^\bullet$ lie in $\mathbf{Com}_{\mathbf{Mod}_A}$, by Lemma 2.5.3, $\mathbf{Com}_{\mathbf{Mod}_A}$ is an abelian category.

□

We introduce some notations and reformulate Definition 2.5.9.

Definition 2.5.12 (Cochains, Cocycles, and Coboundaries)

If $(C^\bullet, d^\bullet)$ is a complex in $\mathbf{Com}_{\mathscr{C}}$, where $\mathscr{C}$ is an abelian category, define

$$\begin{aligned} n\textbf{-cochains} &= C^n, \\ n\textbf{-cocycles} &= Z^n(C^\bullet) = \operatorname{Ker} d^n, \\ n\textbf{-coboundaries} &= B^n(C^\bullet) = \operatorname{Im} d^{n-1}. \end{aligned}$$

Definition 2.5.13 (Cohomology)

*If $C^\bullet$ is a complex in $\mathbf{Com}_{\mathscr{C}}$, where $\mathscr{C}$ is an abelian category, and $n \in \mathbb{Z}$, its n-th **cohomology** is*

$$H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet).$$

*Now $H^n(C^\bullet)$ lies in $\operatorname{obj}(\mathscr{C})$ if quotients are viewed as objects (see Definition 2.5.7). However, if we recognize $\mathscr{C}$ as a full subcategory of $\mathbf{Ab}$ (or $\mathbf{Mod}_A$), then an element of $H^n(C^\bullet)$ is a coset $z + B^n(C^\bullet)$, we call this element a **cohomology class**, and often denote it by $\operatorname{cls}(z)$.*

Proposition 2.5.6

If $\mathscr{C}$ is an abelian category, then $H^n : \mathbf{Com}_{\mathscr{C}} \to \mathscr{C}$ is an additive covariant functor.

Proof By Freyd-Mitchell Embedding Theorem, it suffices to prove this proposition when $\mathscr{C} = \mathbf{Mod}_A$. We have just defined H^n on objects, it remains to define H^n on morphisms. If $f : C^\bullet \to C'^\bullet$ is a chain map, define $H^n(f) : H^n(C^\bullet) \to H^n(C'^\bullet)$ by

$$H^n(f) : \operatorname{cls}(z_n) \mapsto \operatorname{cls}(f_n z_n).$$

We must show that $f_n z_n$ is a cocycle and $H^n(f)$ is independent of the choice of cocycle z_n, both of these follow from f being a chain map, that is, from commutativity of the following diagram:

$$\begin{tikzcd} C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & C^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & C^{n+1} \arrow[d, "f^{n+1}"] \\ C'^{n-1} \arrow[r, "d'^{n-1}"] & C'^n \arrow[r, "d'^n"] & C'^{n+1}. \end{tikzcd}$$

First, let z be an n-cocycle in $Z^n(C^\bullet)$, then $d^n z = 0$. The commutativity of the diagram gives $d'^n f^n z = f^{n+1} d^n z = 0$, hence, $f^n z$ is an n-cocycle.

Next, assume that $z + B^n(C^\bullet) = y + B^n(C^\bullet)$, hence, $z - y \in B^n(C^\bullet)$. We may assume that

$$z - y = d^{n-1} c$$

for some $c \in C^{n-1}$. Applying f^n gives

$$f^n z - f^n y = f^n d^{n-1} c = d'^{n-1} f^{n-1} c \in B^n(C'^\bullet).$$

Thus, $\operatorname{cls}(f^n z) = \operatorname{cls}(f^n y)$, and therefore $H^n(f)$ is well-defined.

Let us now see that H^n is a functor. It is obvious that $H^n(\mathrm{id}_{C^\bullet})$ is the identity. If f and g are chain maps whose composite gf is defined, then for every n-cocycle z, we have

$$\begin{aligned} H^n(gf) : \operatorname{cls}(z) &\mapsto \operatorname{cls}((gf)^n z) \\ &= \operatorname{cls}(g^n f^n z) \\ &= H^n(g) \operatorname{cls}(f^n z) \\ &= H^n(g) H^n(f) \operatorname{cls}(z). \end{aligned}$$

Finally, H^n is additive: if $f, g : C^\bullet \to C'^\bullet$ are chain maps, then

$$\begin{aligned} H^n(f+g) : \mathrm{cls}(z) &\mapsto \mathrm{cls}((f+g)^n z) \\ &= \mathrm{cls}(f^n z + g^n z) \\ &= (H^n(f) + H^n(g))\,\mathrm{cls}(z). \end{aligned}$$

□

Note *We call $H^n(f)$ the **induced map**.*

Homotopic maps induce the same maps on homology

Definition 2.5.14 (Homotopic)

*(1) (**Morphism homotopic.**) A morphism $f : C^\bullet \to D^\bullet$ in $\mathbf{Com}_{\mathscr{C}}$ is **homotopic to zero** if for all n, there exists a morphism $s^n : C^n \to D^{n-1}$, such that*

$$f^n = s^{n+1} \circ d_C^n + d_D^{n-1} \circ s^n.$$

*Two morphisms $f, g : C^\bullet \to D^\bullet$ are **homotopic** if $f - g$ is homotopic to zero.*

(2) An object $C^\bullet \in \mathbf{Com}_{\mathscr{C}}$ is homotopic to zero if $\mathrm{id}_{C^\bullet}$ is homotopic to zero.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd} \cdots \arrow[r] & C^{n-1} \arrow[r] & C^n \arrow[r, "d_C^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[dl, "s^{n+1}"] & \cdots \\ \cdots \arrow[r] & D^{n-1} \arrow[r, "d_D^{n-1}"] & D^n \arrow[r] & D^{n+1} \arrow[r] & \cdots . \end{tikzcd}$$

Proposition 2.5.7

If $f : C^\bullet \to D^\bullet$ is homotopic to zero, then every map $f^n : H^n(C^\bullet) \to H^n(D^\bullet)$ is zero. If f and g are homotopic, then they induce the same maps $H^n(C^\bullet) \to H^n(D^\bullet)$.

Proof It is enough to prove the first assertion, suppose that $f : C^\bullet \to D^\bullet$ defined by $f^n = s^{n+1} \circ d_C^n + d_D^{n-1} \circ s^n$. Every element of $H^n(C^\bullet)$ is represented by an n-cocycle x. Then

$$f^n(x) = s^{n+1} \circ d_C^n(x) + d_D^{n-1} \circ s^n(x) = 0 + d_D^{n-1}(s^n(x)) = d_D^{n-1}(s^n(x)),$$

that is, $f^n(x)$ is an n-coboundary in $B^n(D^\bullet)$, and therefore $f^n(x) = 0$ in $H^n(D^\bullet)$. This implies that f^n is zero.

If f and g are homotopic, then $f - g$ is homotopic to zero, and therefore they induce the same maps $H^n(C^\bullet) \to H^n(D^\bullet)$. □

Long exact sequences

Lemma 2.5.4 (Snake Lemma)

Consider a commutative diagram in abelian category $\mathscr{C}$ with exact rows:

$$\begin{tikzcd} & X' \arrow[r, "f"] \arrow[d, "u'"'] & X \arrow[r, "g"] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y''. & \end{tikzcd}$$

There exists an exact sequence

$$\mathrm{Ker}(u') \longrightarrow \mathrm{Ker}(u) \longrightarrow \mathrm{Ker}(u'') \xrightarrow{\delta} \mathrm{Coker}(u') \longrightarrow \mathrm{Coker}(u) \longrightarrow \mathrm{Coker}(u'').$$

If, moreover, f is monomorphism, then so is $\mathrm{Ker}(u') \to \mathrm{Ker}(u)$*; if g' is epimorphism, then so is* $\mathrm{Coker}(u) \to \mathrm{Coker}(u'')$.

Proof We give a proof by using spectral sequence, Example 2.36 . □

Theorem 2.5.4

A short exact sequence of complexes

$$\begin{tikzcd}
0^\bullet \arrow[d] & : & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet \arrow[d] & : & \cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d] & A^n \arrow[r, "d_A^n"] \arrow[d] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d] & \cdots \\
B^\bullet \arrow[d] & : & \cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] \arrow[d] & B^n \arrow[r, "d_B^n"] \arrow[d] & B^{n+1} \arrow[r, "d_B^{n+1}"] \arrow[d] & \cdots \\
C^\bullet \arrow[d] & : & \cdots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d] & C^n \arrow[r, "d_C^n"] \arrow[d] & C^{n+1} \arrow[r, "d_C^{n+1}"] \arrow[d] & \cdots \\
0^\bullet & : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

induces a ***long exact sequence in cohomology***

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{n-1}(C^\bullet) \arrow[r] & {} \\
H^n(A^\bullet) \arrow[r] & H^n(B^\bullet) \arrow[r] & H^n(C^\bullet) \arrow[r] & {} \\
H^{n+1}(A^\bullet) \arrow[r] & \cdots & &
\end{tikzcd}$$

Proof From the Snake Lemma and the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^{n-1}(A^\bullet) \arrow[r] \arrow[d] & Z^{n-1}(B^\bullet) \arrow[r] \arrow[d] & Z^{n-1}(C^\bullet) \arrow[d] & \\
0 \arrow[r] & A^{n-1} \arrow[r] \arrow[d, "d_A^{n-1}"'] & B^{n-1} \arrow[r] \arrow[d, "d_B^{n-1}"] & C^{n-1} \arrow[r] \arrow[d, "d_C^{n-1}"] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[d] & B^n \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & 0 \\
 & A^n/d_A^{n-1}(A^{n-1}) \arrow[r] \arrow[d] & B^n/d_B^{n-1}(B^{n-1}) \arrow[r] \arrow[d] & C^n/d_C^{n-1}(C^{n-1}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we see that the top and bottom rows are exact. Consider the following commutative diagram

$$\begin{tikzcd} & A^n/d_A^{n-1}(A^{n-1}) \arrow[r] \arrow[d, "\overline{d_A^n}"'] & B^n/d_B^{n-1}(B^{n-1}) \arrow[r] \arrow[d, "\overline{d_B^n}"] & C^n/d_C^{n-1}(C^{n-1}) \arrow[r] \arrow[d, "\overline{d_C^n}"] & 0 \\ 0 \arrow[r] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) & \end{tikzcd}$$

where $\overline{d^n_{(\cdot)}}$ is induced by $d^n_{(\cdot)}$. Note that $H^n(\cdot)$ is the kernel of $\overline{d^n_{(\cdot)}}$ and $H^{n+1}(\cdot)$ is the cokernel of $\overline{d^n_{(\cdot)}}$, apply the Snake lemma again, we obtain

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \xrightarrow{\partial} H^{n+1}(A^\bullet) \longrightarrow H^{n+1}(B^\bullet) \longrightarrow H^{n+1}(C^\bullet).$$

Pasting these sequences together we obtain the result. □

2.5.3 Exactness

Definition 2.5.15 (Right-exact, left-exact, and exact)

*If $F : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor from one abelian category to another, we say that F is **right-exact** if the exactness of*

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

in $\mathscr{A}$ implies that

$$F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$$

*is also exact. Dually, we say that F is **left-exact** if the exactness of*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A''$$

implies

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact.

*An additive contravariant functor is **left-exact** if the exactness of*

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

implies

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A')$$

*is exact. Dually, we say that contravariant functor is **right-exact** if the exactness of*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A''$$

implies

$$F(A'') \longrightarrow F(A) \longrightarrow F(A') \longrightarrow 0$$

*An additive covariant or contravariant functor is **exact** if it is both left-exact and right-exact.*

Exercise 2.36 Suppose F is an exact functor. Show that applying F to an exact sequence preserves exactness. For example, if F is covariant, and $A' \to A \to A''$ is exact, then $FA' \to FA \to FA''$ is exact.

Proof Since $A' \to A \to A''$ is exact, it can be decomposed into some exact sequences, that is, following diagram commutative.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \\
 & \operatorname{Ker} f \arrow[d, hook] & & \operatorname{Im} g \arrow[ur] \arrow[d, hook] & \\
 & A' \arrow[r, "f"] \arrow[d, two heads] & A \arrow[r, "g"] \arrow[ur, two heads] & A'' & \\
 & \operatorname{Im} f = \operatorname{Ker} g \arrow[ur, hook] \arrow[d] & & & \\
 0 \arrow[ur] & 0 & & &
\end{tikzcd}
$$

Since F is an exact sequence, we have exact sequences

$$0 \longrightarrow F(\operatorname{Ker} f) \longrightarrow F(A') \longrightarrow F(\operatorname{Im} f) = F(\operatorname{Ker} g) \longrightarrow 0$$

$$0 \longrightarrow F(\operatorname{Im} f) = F(\operatorname{Ker} g) \longrightarrow F(A) \longrightarrow F(\operatorname{Im} g) \longrightarrow 0$$

and commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \\
 & F(\operatorname{Ker} f) \arrow[d, hook] & & F(\operatorname{Im} g) \arrow[ur] \arrow[d, hook] & \\
 & F(A') \arrow[r, "Ff"] \arrow[d, two heads] & F(A) \arrow[r, "Fg"] \arrow[ur, two heads] & F(A'') & \\
 & F(\operatorname{Im} f) = F(\operatorname{Ker} g) \arrow[ur, hook] \arrow[d] & & & \\
 0 \arrow[ur] & 0. & & &
\end{tikzcd}
$$

Note that

$$
\begin{aligned}
\operatorname{Im} Ff &= \operatorname{Im}(F(A') \to F(\operatorname{Im} f) \to F(A)) \\
&\overset{a}{=} \operatorname{Im}(F(\operatorname{Im} f) \to F(A)) \\
&= \operatorname{Ker}(F(A) \to F(\operatorname{Im} g)) \\
&\overset{b}{=} \operatorname{Ker}(F(A) \to F(\operatorname{Im} g) \to F(A'')) \\
&= \operatorname{Ker} Fg,
\end{aligned}
$$

where "a" holds since $F(A') \to F(\operatorname{Im} f)$ is epimorphism and "b" holds since $F(\operatorname{Im} g) \to F(A'')$ is monomorphism, hence $F(A') \to F(A) \to F(A'')$ is exact. □

Exercise 2.37 Suppose A is a ring, $S \subseteq A$ is a multiplicative subset, and M is an A-module.

(a) Show that localization of A-modules $\mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ is an exact covariant functor.

(b) Show that $\square \otimes_A M$ is a right-exact covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$.

(c) Show that $\operatorname{Hom}(M, \square)$ is a left-exact covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. If $\mathscr{C}$ is any abelian category, and $C \in \mathscr{C}$, show that $\operatorname{Hom}(C, \square)$ is a left-exact covariant functor $\mathscr{C} \to \mathbf{Ab}$.

(d) Show that $\operatorname{Hom}(\square, M)$ is a left-exact contravariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. If $\mathscr{C}$ is any abelian category, and $C \in \mathscr{C}$, show that $\operatorname{Hom}(\square, C)$ is a left-exact contravariant functor $\mathscr{C} \to \mathbf{Ab}$.

Proof

(a) Let $M' \xrightarrow{f} M \xrightarrow{g} M''$, then $\operatorname{Im} f = \operatorname{Ker} g$. Hence, $g \circ f = 0$, and therefore $S^{-1}g \circ S^{-1}f = S^{-1}(0)$. Hence, $\operatorname{Im} S^{-1}f \subseteq \operatorname{Ker} S^{-1}g$. Conversely, let $m/s \in \operatorname{Ker}(S^{-1}g)$, then $g(m)/s = 0$ in $S^{-1}M''$, hence, there exists $t \in S$ such that $tg(m) = 0$ in M''. But $tg(m) = g(tm)$, since g is an A-module homomorphism. Hence, $tm \in \operatorname{Ker} g = \operatorname{Im} f$, there exists $m' \in M'$ such that $tm = f(m')$. Hence, in $S^{-1}M$ we have $m/s = f(m')/st = S^{-1}f(m'/st) \in \operatorname{Im} S^{-1}f$. Hence, $\operatorname{Im} S^{-1}f = \operatorname{Ker} S^{-1}g$, that is sequence

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact. Hence, S^{-1} is an exact covariant functor.

(b) Say

$$M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0.$$

is an exact sequence.
We shall to show that the sequence

$$M' \otimes_A N \xrightarrow{i\otimes \mathrm{id}_N} M \otimes_A N \xrightarrow{p\otimes \mathrm{id}_N} M'' \otimes_A N \longrightarrow 0$$

is exact.

(i) $\operatorname{Im} i \otimes \mathrm{id}_N = \operatorname{Ker} p \otimes \mathrm{id}_N$.
For $\operatorname{Im} i \otimes \mathrm{id}_N \subseteq \operatorname{Ker} p \otimes \mathrm{id}_N$, it suffices to prove that the composite is 0. Note that

$$(p \otimes \mathrm{id}_N)(i \otimes \mathrm{id}_N) = pi \otimes \mathrm{id}_N = 0 \otimes \mathrm{id}_N = 0,$$

then we done.
Let $E = \operatorname{Im}(i \otimes \mathrm{id}_N)$, then $E \subseteq \operatorname{Ker} p \otimes \mathrm{id}_N$, and therefore $i \otimes \mathrm{id}_N$ induces a map $\widehat{p} : (M \otimes N)/E \to M'' \otimes N$ with

$$\widehat{p} : m \otimes n + E \mapsto pm \otimes n,$$

where $m \in M$ and $n \in N$. Now if $\pi : M \otimes N \to (M \otimes N)/E$ is the natural nap, then

$$\widehat{p}\pi = p \otimes \mathrm{id}_N,$$

for both send $a \otimes b \mapsto pa \otimes b$.

$$\begin{tikzcd} M \otimes N \arrow[rr, "\pi"] \arrow[dr, "p\otimes \mathrm{id}_N"'] & & (M \otimes N)/E \arrow[dl, "\widehat{p}"] \\ & M'' \otimes N & \end{tikzcd}$$

Suppose we show that $\widehat{p}$ is an isomorphism. Then

$$\operatorname{Ker}(p \otimes \mathrm{id}_N) = \operatorname{Ker} \widehat{p}\pi = \operatorname{Ker} \pi = E = \operatorname{Im}(i \otimes \mathrm{id}_N),$$

and we done. To see that $\widehat{P}$ is an isomorphism, we construct its inverse $M'' \otimes N \to (M \otimes N)/E$. Define

$$f : M'' \times N \to (M \otimes N)/E$$

as follows. If $m'' \in M''$, there is $m \in M$ such that $p(m) = m''$, since p is surjective, let

$$f : (m'', n) \mapsto m \otimes n + E.$$

Now f is well-defined: if $pm_1 = m''$, then $p(m - m_1) = 0$ and $m - m_1 \in \operatorname{Ker} p = \operatorname{Im} i$. Thus there is $m' \in M'$ with $i(m') = m - m_1$, and hence $(m - m_1) \otimes n = i(m') \otimes n \in \operatorname{Im}(i \otimes \mathrm{id}_N) = E$, that is, $m\otimes n+E=m_1\otimes n+E$. Clearly, f is A-biadditive, and so the definition of tensor product gives a homomorphism $\widehat{f}: M''\otimes N\to (M\otimes N)/E$ with $\widehat{f}(m''\otimes n)=m\otimes n+E$. It is easy to check that $\widehat{f}$ is the inverse of $\widehat{p}$, as desired.

Hence, we have $\operatorname{Im} i\otimes \mathrm{id}_N=\operatorname{Ker} p\otimes \mathrm{id}_N$.

(ii) $p\otimes \mathrm{id}_N$ is surjective.

Since, p is surjective, if $\sum m_i''\otimes n_i\in M''\otimes N$, then there exist $m_i\in M$ with $pm_i=m_i''$ for all i. Hence, $(p\otimes \mathrm{id}_N)(\sum m_i\otimes n_i)=\sum m_i''\otimes n_i$, which implies that p is surjective.

(c) We only prove the case when $\mathscr{C}=\mathbf{Mod}_A$. $\operatorname{Hom}(M,\square)$ is a covariant functor. Let

$$0\longrightarrow N'\xrightarrow{\ i\ } N\xrightarrow{\ p\ } N''$$

be an exact sequence of A-module. Consider the sequence

$$0\longrightarrow \operatorname{Hom}_A(M,N')\xrightarrow{\ i_*\ }\operatorname{Hom}_A(M,N)\xrightarrow{\ p_*\ }\operatorname{Hom}_A(M,N''),$$

we shall to show this sequence is exact.

(i) $\operatorname{Ker} i_*=0$.

If $f\in\operatorname{Ker} i_*$, then $f:M\to N'$ and $i_*(f)=0$, that is, $i(f(m))=0$ for all $m\in M$. Since i is injective, $f(m)=0$ for all $m\in M$, and therefore $f=0$. Hence, $\operatorname{Ker} i_*=0$.

(ii) $\operatorname{Im} i_*\subseteq \operatorname{Ker} p_*$.

If $g\in\operatorname{Im} i_*$, then $g:M\to N$ and there is some $f:M\to N'$ with $g=i_*(f)=if$. Then $p_*(g)=pg=pif=0$, since $\operatorname{Im} i=\operatorname{Ker} p$, and therefore $g\in\operatorname{Ker} p_*$, which implies that $\operatorname{Im} i_*\subseteq\operatorname{Ker} p_*$.

(iii) $\operatorname{Ker} p_*\subseteq\operatorname{Im} i_*$.

If $g\in\operatorname{Ker} p_*$, then $g:M\to N$ and $p_*(g)=pg=0$. Hence, $pg(m)=0$ for all $m\in M$, so that $g(m)\in\operatorname{Ker} p=\operatorname{Im} i$. Thus, $g(m)=i(n')$ for some $n'\in N'$, since i is injective, n' is unique. Hence the map $f:M\to N'$, given by $f(m)=n'$ if $g(m)=i(n')$, is well-defined. We shall to check that $f\in\operatorname{Hom}_A(M,N')$. Note that $g(m_1+m_2)=g(m_1)+g(m_2)=i(n_1')+i(n_2')=i(n_1'+n_2')$, we have $f(m_1+m_2)=n_1'+n_2'=f(m_1)+f(m_2)$. Also note that $g(am)=ag(m)=ai(n')=i(an')$, we have $f(am)=an'=af(m)$. Hence, f is A-map. Since $i_*(f)=if$ and $if(m)=i(n')=g(m)$ for all $m\in M$, that is, $i_*(f)=g$, and so $g\in\operatorname{Im} i_*$.

(d) The proof is same as part (c).

$\square$

Exercise 2.38 Suppose M is a **finitely presented A-module**: M has a finite number of generators, and with these generators it has a finite number of relations; or usefully equivalently, fits in an exact sequence

$$A^{\oplus q}\longrightarrow A^{\oplus p}\longrightarrow M\longrightarrow 0 \tag{2.19}$$

Use (2.19) and the left-exactness of Hom to describe an isomorphism

$$S^{-1}\operatorname{Hom}_A(M,N)\xleftrightarrow{\sim}\operatorname{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N).$$

Proof Since S^{-1} is exact functor, we have following exact sequence.

$$S^{-1}A^{\oplus q}\longrightarrow S^{-1}A^{\oplus p}\longrightarrow S^{-1}M\longrightarrow 0 \tag{2.20}$$

Apply contravariant left-exact fuctor $\operatorname{Hom}_{S^{-1}A}(\square,S^{-1}N)$ to (2.20), sequence

$$0\longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N)\longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}A^{\oplus p},S^{-1}N)\longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}A^{\oplus q},S^{-1}N) \tag{2.21}$$

is exact. Note that localization commutes with direct sum, we have $S^{-1}A^{\oplus k}=(S^{-1}A)^{\oplus k}$ for $k=p,q$. Also, Hom commutes with direct sum (since p, q finite), hence,

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}A^{\oplus k}, N) \cong \mathrm{Hom}_{S^{-1}A}((S^{-1}A)^{\oplus k}, S^{-1}N) \cong \bigoplus_{i=1}^{k} \mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)$$

for $k = p, q$. Note that $\mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)$ can be seen as $S^{-1}A$-module $S^{-1}N$, then we have exact sequence.

$$0 \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \longrightarrow (S^{-1}N)^{\oplus p} \longrightarrow (S^{-1}N)^{\oplus q} \tag{2.22}$$

On the other hand, apply $\mathrm{Hom}_A(\square, N)$ to (2.19), we have following exact sequence.

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(A^{\oplus p}, N) \longrightarrow \mathrm{Hom}_A(A^{\oplus q}, N) \tag{2.23}$$

Apply S^{-1} to (2.23), note that Hom commutes with direct sum and localization commutes with direct sum, we have exact sequence

$$0 \longrightarrow S^{-1}\mathrm{Hom}_A(M, N) \longrightarrow (S^{-1}\mathrm{Hom}_A(A, N))^{\oplus p} \longrightarrow (S^{-1}\mathrm{Hom}_A(A, N))^{\oplus q}. \tag{2.24}$$

Since N is A-module, $\mathrm{Hom}_A(A, N)$ can be seen as A-module N, hence, sequence

$$0 \longrightarrow S^{-1}\mathrm{Hom}_A(M, N) \longrightarrow (S^{-1}N)^{\oplus p} \longrightarrow (S^{-1}N)^{\oplus q} \tag{2.25}$$

is exact.

Hence, (2.22) implies that

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) = \mathrm{Ker}((S^{-1}N)^{\oplus p} \to (S^{-1}N)^{\oplus q}),$$

and (2.25) implies that

$$S^{-1}\mathrm{Hom}_A(M, N) = \mathrm{Ker}((S^{-1}N)^{\oplus p} \to (S^{-1}N)^{\oplus q}).$$

By the universal property of kernel,

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \cong S^{-1}\mathrm{Hom}_A(M, N).$$

$\square$

Example 2.35 (Hom **doesn't always commute with localization.**) In the langrage of Exercise 2.38, take $A = N = \mathbb{Z}$, $M = \mathbb{Q}$, and $S = \mathbb{Z} \setminus \{0\}$.

Proof In fact, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Suppose there is a nonzero morphism $f : \mathbb{Q} \to \mathbb{Z}$. Let $q \in \mathbb{Q}$, and $f(q) = n$ for some $n \in \mathbb{Z}$. For all $k \in \mathbb{Z}$, since f is $\mathbb{Z}$-map, we have

$$f(q) = f(k \cdot \frac{q}{k}) = kf(\frac{q}{k}) = n.$$

Hence, $f(\frac{q}{k}) = \frac{n}{k}$ for all $n \in \mathbb{Z}$. Let n large enough, then $n/k < 1$, a contradiction! Hence, $f = 0$, and therefore $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Thus, $S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Note that $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$, hence,

$$S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \ncong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}).$$

$\square$

Two useful facts in homological algebra.

Note *Ravi Vakil: "We now come to two (sets of) facts I wish I had learned as a child, as they would have saved me lots of grief. They encapsulate what is best and worst of abstract nonsense. The statements are so general as to be nonintuitive. The proofs are very short. They generalize some specific behavior that is easy to prove on an ad hoc basis. Once they are second nature to you, many subtle facts will become obvious to you as special cases. And you will see that they will get used (implicitly or explicitly) repeatedly."*

Interaction of homology and (right/left-)exact functors.

Theorem 2.5.5 (The $FHHF$ Theorem)

Suppose $F : \mathscr{A} \to \mathscr{B}$ is a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathscr{A}$.

*(a) (F **right-exact yields** $FH^\bullet \longrightarrow H^\bullet F$) If F is right-exact, there is a natural morphism $FH^\bullet \longrightarrow H^\bullet F$. More precisely, for each i, the left side is F applied to the cohomology at piece i of $C^\bullet$, while the right side is the cohomology at piece i of $FC^\bullet$.*

*(b) (F **left-exact yields** $FH^\bullet \longleftarrow H^\bullet F$) If F is left-exact, there is a natural morphism $H^\bullet F \longrightarrow FH^\bullet$.*

*(c) (F **exact yields** $FH^\bullet \overset{\sim}{\longleftrightarrow} H^\bullet F$) If F is exact, there is a natural isomorphism between $FH^\bullet$ and $H^\bullet F$.*

Interaction of adjoint, (co)limits, and (left- and right-)exactness.

A surprising number of arguments boil down to the statement:

Limits commute with limits and right adjoints. In particular, in an abelian category, because kernels are limits, both limits and right adjoints are left-exact.

as well as its dual:

Colimits commute with colimits and left adjoints. In particular, because cokernels are colimits, both colimits and left adjoints are right-exact.

The latter has a useful extension:

In Mod_A, colimits over filtered index categories are exact.

Proposition 2.5.8 (Kernels commute with limits)

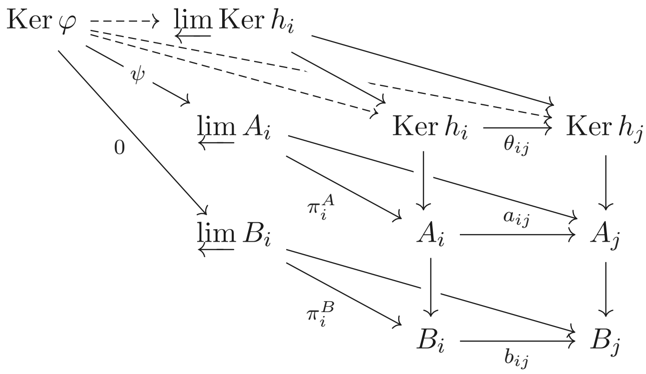
Suppose $\mathscr{C}$ is an abelian category, and $a : \mathscr{I} \to \mathscr{C}$ and $b : \mathscr{I} \to \mathscr{C}$ are two diagrams in $\mathscr{C}$ indexed by $\mathscr{I}$. For convenience, let $A_i = a(i)$ and $B_i = b(i)$ be the objects in those two diagrams. h is a natural transformation of functors $a \to b$. Then there is a cononcial isomorphism

$$\varprojlim \operatorname{Ker} h_i \overset{\sim}{\longleftrightarrow} \operatorname{Ker}(\varprojlim A_i \to \varprojlim B_i).$$

Proof Since $b_{ij} \circ h_i \circ \operatorname{Ker} h_i = 0$, by the universal property of $\operatorname{Ker} h_j$, there exists unique $\theta_{ij} : \operatorname{Ker} h_i \to \operatorname{Ker} h_j$. By the universal property of $\varprojlim B_i$, there exists unique $\varphi : \varprojlim A_i \to \varprojlim B_i$. By the universal property of $\varprojlim A_i$, there exists unique $\iota : \varprojlim \operatorname{Ker} h_i \to \varprojlim A_i$. It suffices to show that $\varprojlim \operatorname{Ker} h_i$ is the kernel of φ. Let W be any object with $\varphi \circ \psi = 0$. Consider the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Ker}\varphi \arrow[dr, "\psi"] \arrow[ddr, "0"'] & \varprojlim \operatorname{Ker} h_i \arrow[l, dashed] \arrow[d, dashed, "\iota"] \arrow[dr, "\pi_i^{\mathrm{Ker}}"] \arrow[drr] & & \\
& \varprojlim A_i \arrow[d, dashed, "\varphi"] \arrow[dr, "\pi_i^A"] \arrow[drr] & \operatorname{Ker} h_i \arrow[r, dashed, "\theta_{ij}"] \arrow[d] & \operatorname{Ker} h_j \arrow[d] \\
& \varprojlim B_i \arrow[dr, "\pi_i^B"] \arrow[drr] & A_i \arrow[r, "a_{ij}"] \arrow[d, "h_i"] & A_j \arrow[d, "h_j"] \\
& & B_i \arrow[r, "b_{ij}"'] & B_j
\end{tikzcd}$$

Note that $h_i \circ \operatorname{Ker} h_i = 0$, $\pi_i^B \circ \varphi \circ \iota = h_i \circ \ker h_i \circ \pi_i^{\mathrm{Ker}} = 0 \circ \pi_i^{\mathrm{Ker}} = 0$ for all i. Hence, $\varphi \circ \iota = 0$. By the universal property of $\operatorname{Ker}\varphi$, there exists unique $\chi : \varprojlim \operatorname{Ker} h_i \to \operatorname{Ker}\varphi$. Now, consider the following diagram,



by the universal property of $\operatorname{Ker} h_i$ and $\operatorname{Ker} h_j$, there are unique morphism $\operatorname{Ker}\varphi \to \operatorname{Ker} h_i$ and $\operatorname{Ker}\varphi \to \operatorname{Ker} h_j$. By the universal property of $\varprojlim \operatorname{Ker} h_i$, there is unique morphism $\kappa : \operatorname{Ker}\varphi \to \varprojlim \operatorname{Ker} h_i$. Then $\chi\circ\kappa : \operatorname{Ker}\varphi \to \operatorname{Ker}\varphi$, by the universal property of $\operatorname{Ker}\varphi$ again, $\chi\circ\kappa = \mathrm{id}_{\operatorname{Ker}\varphi}$. Similarly, $\kappa\circ\chi = \mathrm{id}_{\varprojlim \operatorname{Ker} h_i}$. Hence, there is a canoncial isomorphism

$$\varprojlim \operatorname{Ker} h_i \cong \operatorname{Ker}(\varprojlim A_i \to \varprojlim B_i).$$

□

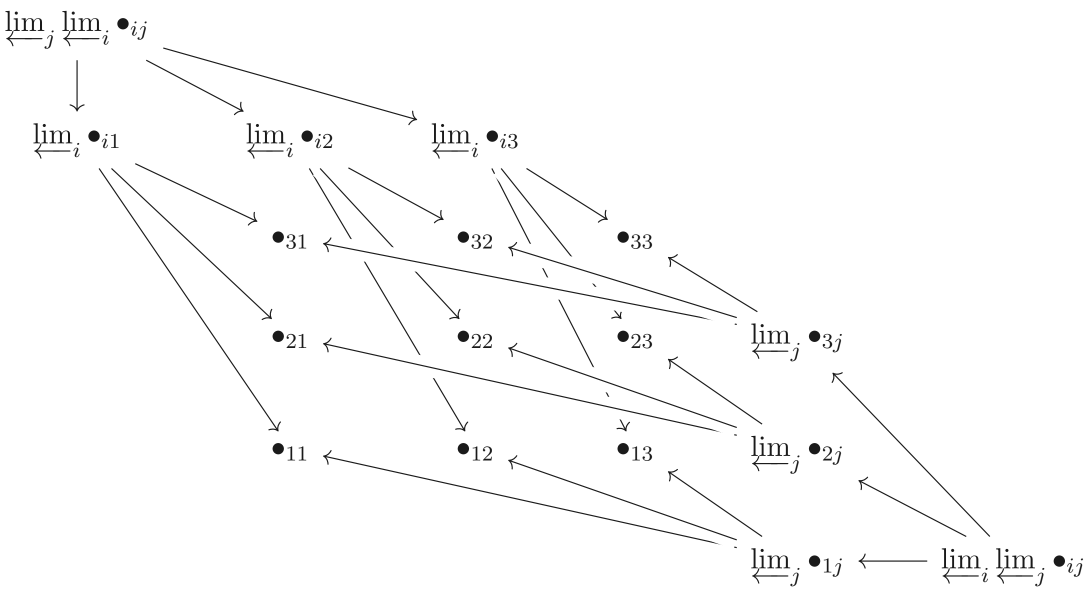
Remark Implicit in the previous exercise is the idea that limits should somehow be understand as functor.

Proposition 2.5.9

In any category, limits commute with limits. More precisely, let $F : I \times J \to \mathscr{C}$ be a functor, and suppose $\varprojlim F(i,j)$ exists for every $i \in I$. Then there is a canoncial isomorphism

$$\varprojlim_i \varprojlim_i F(i,j) \cong \varprojlim_j \varprojlim_i F(i,j).$$

Proof Consider the following diagram.



In the following diagram, all the dashed lines defined by the universal property of limits, and are all isomorphisms.

$$\begin{tikzcd}
\varprojlim_j \varprojlim_i \bullet_{ij} \arrow[rrr, dashed, "\sim"] \arrow[dr] \arrow[ddr] \arrow[dddr] & & & \varprojlim_i \varprojlim_j \bullet_{ij} \arrow[dl] \arrow[ddl] \arrow[dddl] \\
& \varprojlim_i \bullet_{i3} \arrow[r, dashed] \arrow[dr, dashed] \arrow[ddr, dashed] & \varprojlim_j \bullet_{3j} & \\
& \varprojlim_i \bullet_{i2} \arrow[r, dashed] \arrow[ur, dashed] \arrow[dr, dashed] & \varprojlim_j \bullet_{2j} & \\
& \varprojlim_i \bullet_{i1} \arrow[r, dashed] \arrow[ur, dashed] \arrow[uur, dashed] & \varprojlim_j \bullet_{1j} &
\end{tikzcd}$$

Hence,

$$\varprojlim_i \varprojlim_i F(i,j) \cong \varprojlim_j \varprojlim_i F(i,j).$$

□

Proposition 2.5.10

Suppose $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ is a pair of adjoint functors. If $A = \varprojlim A_i$ is a limit in $\mathscr{D}$ of a diagram indexed by $\mathscr{I}$, then $GA = \varprojlim GA_i$ is a limit in $\mathscr{C}$.

Proof We must show that $GA \to GA_i$ satisfies the universal property of limits. Suppose we have maps $W \to GA_i$ commuting with the maps of $\mathscr{I}$. We wish to show that there exists a unique $W \to GA$ extending the $W \to GA_i$. By adjointness of F and G, we can restate this as: Suppose we have maps $FW \to A_i$ commuting with the maps of $\mathscr{I}$. We wish to show that there exists a unique $FW \to A$ extending the $FW \to A_i$. But this is precisely the universal property of the limit. □

Dually, we have the following two propositions.

Proposition 2.5.11

In any category, colimits commute with colimits. More precisely, let $F : I \times J \to \mathscr{C}$ be a functor, and suppose $\varinjlim_j F(i,j)$ exists for ever $i \in I$. Then there is a cononcial isomorphism

$$\varinjlim_i \varinjlim_j F(i,j) \cong \varinjlim_j \varinjlim_i F(i,j).$$

Proposition 2.5.12

Suppose $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ is a pair of adjoint functors. If $A = \varinjlim A_i$ is a colimit in $\mathscr{C}$ of a diagram indexed by $\mathscr{I}$, then $FA = \varinjlim FA_i$.

Proposition 2.5.13

If F and G are additive functors between abelian categories, and (F, G) is an adjoint pair, then (as kernels are limits and cokernels are colimits) G is left-exact and F is right-exact.

Proof Let

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

be an exact sequence. By Proposition 2.5.3, $i = \operatorname{Ker} p$ and $p = \operatorname{Coker} i$.

By Proposition 2.5.10, G commutes with limits, and kernels are limits, hence, $Gi = G(\operatorname{Ker} p) = \operatorname{Ker} Gp$, and therefore sequence

$$0 \longrightarrow GA \xrightarrow{Gi} GB \xrightarrow{Gp} GC$$

is exact, which implies that G is left-exact.

By Proposition 2.5.12, F commutes with colimits, and cokernels are colimits, hence, $Fp = F(\operatorname{Coker} f) = \operatorname{Coker} Fi$, and therefore sequence

$$FA \xrightarrow{Fi} FB \xrightarrow{Fp} FC \longrightarrow 0$$

is exact, which implies that F is right-exact. □

Lemma 2.5.5

In $\mathbf{Mod}_A$, let $\{M_i, \varphi_{ij}\}$ be a direct system of R-modules over a filtered index categories $\mathscr{I}$, and let $\iota_i : M_i \to \bigoplus M_i$ be the i-th injection, then $\varinjlim M_i = (\bigoplus M_i)/S$, where $S = (\iota_j \varphi_{ij} m_i - \iota_i m_i : m_i \in M_i \text{ and } i \to j)$. Also,

(i) Each element of $\varinjlim M_i$ has a representative of the form $\iota_i m_i + S$.

(ii) $\iota_i m_i + S = 0$ if and only if $\varphi_{it}(m_i) = 0$ for some $i \to t$.

Proof

(i) It is easy to check that $(\bigoplus M_i)/S$ satisfies the universal property of colimit, and therefore $\varinjlim M_i = (\bigoplus M_i)/S$. Hence, each element $x \in \varinjlim M_i$ has the form $x = \sum \iota_i m_i + S$. Since, $\mathscr{I}$ is filtered, there exists $j \in \mathscr{I}$, such that $i \to j$ for all i occurring in the sum for x. For each such i, define $b^i = \varphi_{ij} m_i \in M_j$. Let $b = \sum_i b^i$, then $b \in M_j$. It follows that

$$\begin{aligned}\sum \iota_i m_i - \iota_j b &= \sum (\iota_i m_i - \iota_j b^i)\\ &= \sum (\iota_i m_i - \iota_j \varphi_{ij} m_i) \in S.\end{aligned}$$

Therefore, $x = \sum \iota_i m_i + S = \iota_j b + S$, as desired.

(ii) If $\varphi_{it} m_i = 0$ for some $i \to t$, then

$$\iota_i m_i + S = \iota_i m_i + (\iota_t \varphi_{it} m_i - \iota_i m_i) + S = S.$$

Conversely, if $\iota_i m_i + S = 0$, then $\iota_i m_i \in S$, and there is an expression

$$\iota_i m_i = \sum_j a_j (\iota_k \varphi_{jk} m_j - \iota_j m_j) \in S,$$

where $a_j \in A$. We are going to normalize this expression. First, we introduce the following notation for relators: if $j \to k$, define

$$r(j, k, m_j) = \iota_k \varphi_{jk} m_j - \iota_j m_j.$$

Since $a_j r(j, k, m_j) = r(j, k, a_j m_j)$, we may assume that the notion has been adjusted such that

$$\iota_i m_i = \sum_j r(j, k, m_j).$$

As $\mathscr{I}$ is filtered, we may choose an index $t \in \mathscr{I}$ larger than any of the indices i, j, k occuring in the last equation (i.e. $i \to t$, $j \to t$, and $k \to t$). Now

$$\begin{aligned}\iota_t \varphi_{it} m_i &= (\iota_t \varphi_{it} m_i - \iota_i m_i) + \iota_i m_i\\ &= r(i, t, m_i) + \iota_i m_i\\ &= r(i, t, m_i) + \sum_j r(j, k, m_j).\end{aligned}$$

Next,

$$\begin{aligned} r(j,k,m_j) =&\iota_k\varphi_{jk}m_j - \iota_j m_j \\ =&(\iota_t\varphi_{jt}m_j - \iota_j m_j) + [\iota_t\varphi_{kt}(-\varphi_{jk}m_j) - \iota_k(-\varphi_{jk}m_j)] \\ =&r(j,t,m_j) + r(k,t,-\varphi_{jk}m_j), \end{aligned}$$

because $\varphi_{kt}\circ\varphi_{ik} = \varphi_{it}$, by the definition of filtered. Hence,

$$\iota_t\varphi_{it}m_i = \sum_l r(l,t,x_l),$$

where $x_l \in M_l$. For $l \to t$,

$$\begin{aligned} r(l,t,m_l) + r(l,t,m_l') =&\iota_t\varphi_{lt}m_l - \iota_l m_l + \iota_t\varphi_{lt}m_I' - \iota_l m_l' \\ =&\iota_t\varphi_{lt}(m_l + m_l') - \iota_l(m_l + m_l') \\ =&r(l,t,m_l + m_l'). \end{aligned}$$

Therefore, we may amalgamate all relators with the same smaller index l and write

$$\begin{aligned} \iota_t\varphi_{it}m_i =&\sum_l r(l,t,x_l) \\ =&\sum_l(\iota_t\varphi_{lt}x_l - \iota_l x_l) \\ =&\iota_t(\sum_l \varphi_{lt}x_l) - \sum_l \iota_l x_l, \end{aligned}$$

where $x_\iota \in M_\iota$ and all the indices l are distinct. The unique expression of an element in a direct sum allows us to conclude, if $l \neq t$, that $\iota_l x_l = 0$, it follows that $x_l = 0$, for ι_l is an injection. The right side simplifies to $\iota_t\varphi_{tt}m_t - \iota_t m_t = 0$. Since φ_{tt} is the identity, right side is 0 and $\iota_t\varphi_{it}m_i = 0$. Since ι_t is an injection, we have $\varphi_{it}m_i = 0$, as we desired.

□

Proposition 2.5.14

In $\mathbf{Mod}_A$, colimits over filtered index categories are exact.

Proof Suppose sequence

$$0 \longrightarrow A_j \xrightarrow{r} B_j \xrightarrow{p} C_j \longrightarrow 0$$

is exact, for all $j \in \mathscr{I}$, then $p = \operatorname{Coker} r$. Since cokernel is colimit, colimit comutes with colimit, we have $\varinjlim p = \varinjlim \operatorname{Coker} r = \operatorname{Coker} \varinjlim r$, hence, the sequence,

$$\varinjlim_j A_j \xrightarrow{\varinjlim r} \varinjlim_j B_j \xrightarrow{\varinjlim p} \varinjlim_j C_j \longrightarrow 0$$

is exact. This implies that colimit is right-exact.

Now, we need to show that colimit is left adjoint. It suffices to show that $\varinjlim r$ is injection. Suppose that $\varinjlim r(x) = 0$, where $x \in \varinjlim A_j$. Since the index categories is filtered, Lemma 2.5.5 allow us to write $x = \iota_i^A a_i + S_A$. By definition, $\varinjlim r(x) = \iota_i^B r_i a_i + S_B$. Now Lemma 2.5.5 shows that $\iota_i^B r_i a_i + S_B$ in $\varinjlim B_j$ implies that there is an index k with $i \to k$ such that $\varphi_{ik}^B r_i a_i = 0$. Since r is a morphism of direct systems (i.e., r is natural transformation), we have

$$0 = \varphi_{ik}^B r_i a_i = r_k \varphi_{ik}^A a_i.$$

Since r_k is injection, we have $\varphi_{ik}^A a_i = 0$, and therefore,

$$x = \iota_i^A a_i + S_A = \iota_k \circ \varphi_{ik}^A a_i + S_A = S_A = 0.$$

Hence, $\varinjlim r$ is an injection. $\square$

Corollary 2.5.2

Filtered colimits commute with homology in $\mathbf{Mod}_A$.

Proof By Proposition 2.5.14, filtered colimits are exact. By the $FHHF$ Theorem 2.5.5, there is a natural isomorphism between $FH^\bullet$ and $H^\bullet F$, which implies that filtered colimits commute with homology in $\mathbf{Mod}_A$. $\square$

Remark Just as colimits are exact (not just right-exact) in especially good circumstances, limits are exact (not just left-exact) too.

Exercise 2.39 Suppose

$$\begin{tikzcd}
 & \vdots \arrow[d, "\alpha_{n+2}"'] & \vdots \arrow[d, "\beta_{n+2}"'] & \vdots \arrow[d, "\gamma_{n+2}"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "r_{n+1}"] \arrow[d, "\alpha_{n+1}"'] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\beta_{n+1}"'] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "r_n"] \arrow[d, "\alpha_n"'] & B_n \arrow[r, "p_n"] \arrow[d, "\beta_n"'] & C_n \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
 & {} & {} & {} & \\
 & \vdots \arrow[d, "\alpha_1"'] & \vdots \arrow[d, "\beta_1"'] & \vdots \arrow[d, "\gamma_1"] & \\
0 \arrow[r] & A_0 \arrow[r, "r_0"] & B_0 \arrow[r, "p_0"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

is an inverse system of exact sequence of modules over a ring, such that the maps $A_{n+1} \to A_n$ are surjective. (We say: "transition maps of the left term are surjective".) Show that the limit

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$$

is also exact.

Proof By Proposition 2.5.8, left-exactness is obviously. It suffices to show that $\varprojlim B_n \to \varprojlim C_n$ is surjective.

In fact, in $\mathbf{Mod}_R$, it is easy to check $\varprojlim A_i = \Big((a_i) \in \prod A_i : a_i = \psi^A_{ji}(a_j), j \geq i\Big)$ is a submodule of $\prod A_i$.

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim A_i \arrow[r, dashed] \arrow[d, "\pi^A_{n+1}"] \arrow[dd, bend right, "\pi^A_n"'] & \varprojlim B_i \arrow[r, dashed] \arrow[d, "\pi^B_{n+1}"] \arrow[dd, bend right, "\pi^B_n"'] & \varprojlim C_i \arrow[d, "\pi^C_{n+1}"] \arrow[dd, bend right, "\pi^C_n"'] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "r_{n+1}"] \arrow[d, "\alpha_{n+1}"] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\beta_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "r_n"] & B_n \arrow[r, "p_n"] & C_n \arrow[r] & 0
\end{tikzcd}$$

Define $\varprojlim p : \varprojlim B_i \to \varprojlim C_i$ by setting $\varprojlim p((b_i)) = (p_i(b_i))$, then it is easy to see $\varprojlim p$ is well-defined and agree with above diagram. We shall to show that $\varprojlim p$ is surjective. Let $(c_i) \in \varprojlim C_i$. We construct (b_i) by induction. Since p_0 is surjective, there exists b_0 such that $p_0(b_0) = c_0$. Suppose we defined b_i for all $i \leq n$ such that $\psi^B_{ji}(b_j) = b_i$ for all $n \geq j \geq i$. Let $b'_{n+1} \in B_{n+1}$ with $p_{n+1}(b'_{n+1}) = c_{n+1}$. Say $\beta_{n+1}(b'_{n+1}) = b'_n$, since $\gamma_{n+1} \circ p_{n+1} = p_n \circ \beta_{n+1}$ and $\gamma_{n+1}(c_{n+1}) = c_n$, $\gamma_{n+1}(c_{n+1}) = p_n(b'_n) = c_n$. Note that $\operatorname{Ker} p_n = \operatorname{Im} r_n$, we may assume $b'_n = b_n + r_n(a_n)$ for some $a_n \in A_n$. Since α_{n+1} is surjective, there exists a_{n+1} such that

$\alpha_{n+1}(a_{n+1}) = a_n$. Let $b_{n+1} = b'_{n+1} - r_{n+1}(a_{n+1})$, then

$$\begin{aligned}\beta_{n+1}(b_{n+1}) =& \beta_{n+1}(b'_{n+1}) - \beta_{n+1}r_{n+1}(a_{n+1}) \\ =& \beta_{n+1}(b'_{n+1}) - r_n\alpha_{n+1}(a_{n+1}) \\ =& b'_n - r_n(a_n) = b_n.\end{aligned}$$

Hence, we defined (b_i) with $\psi^B_{ji}(b_j) = b_i$, where $j \geq i$ and $\psi^B_{ji} = \beta_{i+1} \circ \cdot \circ \beta_j$, and therefore $\varprojlim p$ is surjective, which implies the right-exactness. □

Remark Based on these ideas, we may suspect that right-exact functors always commute with colimits. The fact that tensor product commutes with infinite direct sums may reinforce this idea. Unfortunately, it is not ture — "double dual" $\square^{\vee\vee} : \mathbf{Vec}_k \to \mathbf{Vec}_k$ is covariant and right exact (in fact, exact), but does not commute with infinite direct sums, as $\bigoplus_{i=1}^{\infty}(k^{\vee\vee})$ is not isomorphic to $(\bigoplus_{i=1}^{\infty} k)^{\vee\vee}$.

Dreaming of derived functors

Remark When we see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in abelian category $\mathscr{A}$, and $F : \mathscr{A} \to \mathscr{B}$ is a left-exact functor, then

$$0 \longrightarrow FM' \longrightarrow FM \longrightarrow FM''$$

is exact, and you should always dream that it should continue in some natural way. For example, the next term should depend only on M', call it R^1FM', and if it is zero, then $FM \to FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In god cases, such a continuation exists, and is incredibly useful.

2.6 ⋆ Spectral sequences

Spectral sequence are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name "spectral" was given because, like specters, spectral sequences are terrifying, evil, and dangerous. I (Ravi Vakil) have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Nonetheless, the goal of this section is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is perhaps different in this presentation is that we will use spectral sequences to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience for how to use them. We will also see them only in the special case of double complexes (the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc).

You should not read this section when you are reading the rest of Chapter 2. Instead, you should read it just before you need it for the first time.

For concreteness, we work in the category $\mathbf{Mod}_A$ of module over a ring A. However, everything we say will apply in any abelian category. (And if it helps you feel secure, work instead in the category $\mathbf{Vec}_k$ of vector spaces over a field k.)

2.6.1 Double complexes

Definition 2.6.1 (Double complex)

*A **double complex** is a collection of A-modules $E^{p,q}$ ($p, q \in \mathbb{Z}$), and "rightward" morphisms $d_{\rightarrow}^{p,q} : E^{p,q} \to E^{p+1,q}$ and "upward" morphisms $d_{\uparrow}^{p,q} : E^{p,q} \to E^{p,q+1}$. In the superscript, the first entry denotes the column number (the "x-coordinate"), and the second entry denotes the row number (the "y-coordinate"). (Warning: this is opposite to the convention for matrices.) The subscript is meant to suggest the direction of the arrows. We will always write these as $d_{\rightarrow}$ and $d_{\uparrow}$ and ignore the superscripts.*

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & E^{p,q+1} \arrow[u] \arrow[r, "d_{\rightarrow}^{p,q+1}"] & E^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & E^{p,q} \arrow[u, "d_{\uparrow}^{p,q}"] \arrow[r, "d_{\rightarrow}^{p,q}"] & E^{p+1,q} \arrow[u, "d_{\uparrow}^{p+1,q}"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

We require that $d_{\rightarrow}$ and $d_{\uparrow}$ satisfy

(a) $d_{\rightarrow}^2 = 0$, $d_{\uparrow}^2 = 0$;

(b) either $d_{\rightarrow} d_{\uparrow} = d_{\uparrow} d_{\rightarrow}$ (all the sequence commute) or $d_{\rightarrow} d_{\uparrow} + d_{\uparrow} d_{\rightarrow} = 0$ (they all anti-commute).

Remark In (b), both come up in nature, and you can switch from one to the other by replacing $d_{\uparrow}^{p,q}$ with $(-1)^p d_{\uparrow}^{p,q}$. So we will assume that all the squares anti-commute, but that you know how to turn the commuting case into this one. (You will see that there is no difference in the recipe, basically because the image and kernel of a homomorphism f equal the image and kernel respectively of $-f$.)

$$\begin{tikzcd}
E^{p,q+1} \arrow[r, "d_{\rightarrow}^{p,q+1}"] \arrow[dr, phantom, "\text{anti-commutes}"] & E^{p+1,q+1} \\
E^{p,q} \arrow[u, "d_{\uparrow}^{p,q}"] \arrow[r, "d_{\rightarrow}^{p,q}"] & E^{p+1,q} \arrow[u, "d_{\uparrow}^{p+1,q}"']
\end{tikzcd}$$

There are variations on this definition, where for example the vertical arrows go downwards, or some subset of the $E^{p,q}$ is required to be zero.

Definition 2.6.2 (Total complex, Cohomology of the double complex)

From the double complex we construct a corresponding (single) complex $E^{\bullet}$ with $E^k = \bigoplus_i E^{i,k-i}$, with $d = d_{\rightarrow} + d_{\uparrow}$. In other words, when there is a single superscript k, we mean a sum of the k-th *anti-diagonal of the double complex. The single complex is sometimes called the* ***total complex****.*

$$\begin{tikzcd}
\vdots \\
E^{0,3} \arrow[u] \arrow[dr, phantom, "\oplus"] \\
E^{0,2} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, phantom, "\oplus"] & E^{1,2} \arrow[dr, phantom, "\oplus"] \\
E^{0,1} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, phantom, "\oplus"] & E^{1,1} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, phantom, "\oplus"] & E^{2,1} \arrow[dr, phantom, "\oplus"] \\
E^{0,0} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, equal] & E^{1,0} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, equal] & E^{2,0} \arrow[u, "d_\uparrow"] \arrow[r, "d_\rightarrow"] \arrow[dr, equal] & E^{3.0} \arrow[r] \arrow[dr, equal] & \cdots \\
E^\bullet : & E^0 \arrow[r, "d^0"] & E^1 \arrow[r, "d^1"] & E^2 \arrow[r, "d^2"] & E^3 \arrow[r] & \cdots
\end{tikzcd}$$

The cohomology of the single complex is sometimes called the ***hypercohomology*** *of the double complex. We will instead use the phrase* ***cohomology of the double complex****.*

Remark Note that $d^2 = (d_\rightarrow + d_\uparrow)^2 = d_\rightarrow^2 + (d_\rightarrow d_\uparrow + d_\uparrow d_\rightarrow) + d_\uparrow^2 = 0$, so $E^\bullet$ is indeed a complex.

Our initial goal will be to find the cohomology of the double complex. You will see later that we secretly also have other goals.

A spectral sequence is a recipe for computing some information about the cohomology of the double complex. I won't yet give the full recipe. Surprisingly, this fragmentary bit of information is sufficient to prove lots of things.

2.6.2 Approximate definition of spectral sequence

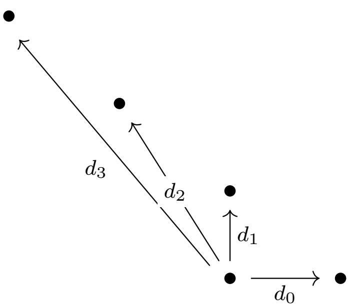
A **spectral sequence** with **rightward orientation** is a sequence of tables or **pages** ${}_\rightarrow E_0^{p,q}, {}_\rightarrow E_1^{p,q}, {}_\rightarrow E_2^{p,q}, \ldots$ $(p, q \in \mathbb{Z})$, where ${}_\rightarrow E_0^{p,q} = E^{p,q}$, along with a differential

$${}_\rightarrow d_r^{p,q} : {}_\rightarrow E_r^{p,q} \longrightarrow {}_\rightarrow E_r^{p-r+1,q+r}$$

$(r \in \mathbb{Z}^{\geq 0})$ with ${}_\rightarrow d_r^{p,q} \circ {}_\rightarrow d_r^{p+r-1,q-r} = 0$, and with an isomorphism of the cohomology of ${}_\rightarrow d_r$ at ${}_\rightarrow E_r^{p,q}$ (i.e., $\operatorname{Ker} {}_\rightarrow d_r^{p,q} / \operatorname{Im} {}_\rightarrow d_r^{p+r-1,q-r}$) with ${}_\rightarrow E_{r+1}^{p,q}$, i.e., ${}_\rightarrow E_{r+1}^{p,q} \cong \operatorname{Ker} {}_\rightarrow d_r^{p,q} / \operatorname{Im} {}_\rightarrow d_r^{p+r-1,q-r}$.

The orientation indicates that our 0-th differential is the rightward one: $d_0 = d_\rightarrow$. The left subscript "$\longrightarrow$" is usually omitted.

The order of the morphisms is best understood visually:



(2.26)

(the morphisms each apply to different pages). Notice that the map always is "degree 1" in terms of the grading of the single complex $E^\bullet$.

Remark Why we draw (2.26) in this way?

Denote $d_r := {}_{\to}d_r$, then we have

$$\begin{aligned} d_0^{p,q} : E_0^{p,q} &\longrightarrow \quad E_0^{p+1,q} \\ d_1^{p,q} : E_1^{p,q} &\longrightarrow \quad E_1^{p,q+1} \\ d_2^{p,q} : E_2^{p,q} &\longrightarrow \quad E_2^{p-1,q+2} \\ d_3^{p,q} : E_3^{p,q} &\longrightarrow \quad E_3^{p-2,q+3}. \end{aligned}$$

Hence the direction of d_0 is along the x-axis, d_1 is along the y-axis, d_2 points in the direction of $(-1,2)$, and d_3 points in the direction of $(-2,3)$.

The actual definition describes what $E_r^{\bullet,\bullet}$ and $d_r^{\bullet,\bullet}$ really are, in terms of $E^{\bullet,\bullet}$. We will describe d_0, d_1, and d_2 below, and you should for now take on faith that this sequence continues in some natural way.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the i-th page $E_i^{p,q}$ for all $i < r$. In particular, if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all r.

Suppose now that $E^{\bullet,\bullet}$ is a first quadrant double complex, i.e., $E^{p,q} = 0$ for $p < 0$ or $q < 0$ (so $E_r^{p,q} = 0$ for all r unless $p, q \in \mathbb{Z}^{\geq 0}$). Then for any fixed p, q, once r is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex

$$\begin{tikzcd} 0 & & \\ & E_r^{p,q} \arrow[ul, "d_r^{p,q}"'] & \\ & & 0 \arrow[ul, "d_r^{p+r-1,q-r}"'] \end{tikzcd}$$

(note that $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r+1,q+r}$ and $E_r^{p-r+1,q+r} = 0$ for sufficiently large r, similarly, $d_r^{p+r-1,q-r} : E_r^{p+r-1,q-r} = 0 \to E_r^{p,q}$ for sufficiently large r) and thus we have canonical isomorphisms

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots.$$

We denote this module $E_\infty^{p,q}$. The same idea works in other circumstances, for example if the double complex is only nonzero in a finite number of rows — $E^{p,q} = 0$ unless $q_0 < q < q_1$. This will come up for example in the mapping cone and long exact sequence discussion.

We now describe the first few pages of the spectral sequence explicitly. As stated above, the differential d_0 on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_{\to}$. The rows are complexes:

$$\text{The 0-th page } E_0 : \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

and so E_1 is just the table of cohomologies of the rows. There are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$ of the row cohomology groups, induced by $d_{\uparrow}$, and that these make the columns into complexes. (This is essentially the fact that a map of complexes induces a map on homology.) We have "used up the horizontal morphisms", but “the vertical differentials live on”.

$$\text{The 1-st page } E_1 : \qquad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \end{tikzcd}$$

We take cohomology of d_1 on E_1, giving us a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is 0. (It is a very worthwhile exercise to work out how this natural morphism d_2 should be defined. Your argument may be reminiscent of the connecting homomorphism in the Snake Lemma 2.5.4 or in the long exact sequence in cohomology arising from a short exact sequence of complexes, Theorem 2.5.4. This is no coincidence.)

$$\text{The 2-nd page } E_2 : \qquad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet \arrow[uul] & \bullet \arrow[uul] \end{tikzcd}$$

This is the beginning of a pattern.

Then it is a theorem that there is a filtration of $H^k(E^\bullet)$ by $E_\infty^{p,q}$ where $p+q=k$. (We can’t yet state it as an official **Theorem** because we haven’t precisely defined the pages and differentials in the spectral sequence.) More precisely, there is a filtration

$$E_\infty^{0,k} \xhookrightarrow{E_\infty^{1,k-1}} F_1 \xhookrightarrow{E_\infty^{2,k-2}} \cdots \xhookrightarrow{E_\infty^{k-1,1}} F_{k-1} \xhookrightarrow{E_\infty^{k,0}} H^k(E^\bullet) \tag{2.27}$$

where the quotients are displayed above each inclusion (i.e., $F_1/E_\infty^{0,k} \cong E_\infty^{1,k-1}$, $\ldots$, $F_i/F_{i-1} \cong E_\infty^{i,k-i}$, $\ldots$, $H^k(E^\bullet)/F_{k-1} \cong E_\infty^{k,0}$). (Here is a tip for remember which way the quotients are supposed to go. The differentials on later and later pages point deeper and deeper into the filtration. Thus the entries in the direction of the later arrowheads are the subobjects, and the entries in the direction of the later “arrowtails” are quotients. This tip has the advantage of being independent of the details of the spectral sequence, e.g, the “quadrant” or the orientation.)

We say that the spectral sequence ${}_{\rightarrow}E_\bullet^{\bullet,\bullet}$ **converges to** $H^\bullet(E^\bullet)$. We often say that ${}_{\rightarrow}E_2^{\bullet,\bullet}$ (or any other page) **abuts to** $H^\bullet(E^\bullet)$.

Although the filtration gives only partial information about $H^\bullet(E^\bullet)$, sometimes one can find $H^\bullet(E^\bullet)$ precisely. One example is if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero (e.g., if $E_r^{\bullet,\bullet}$ has precisely one non-zero row or column, in which case one says that the spectral sequence **collapses** at the r-th step, although we will not use this term). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^\bullet)$. Also, in lucky circumstances, E_2 (or some other small page) already equals E_∞.

Proposition 2.6.1 (Information from the second page)

Let $E^{\bullet,\bullet}$ be a first quadrant double complex, i.e., $E^{p,q}=0$ for $p<0$ or $q<0$. Then we have $H^0(E^\bullet)=E_\infty^{0,0}=E_2^{0,0}$ and

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^\bullet)$$

is exact.

Proof Consider $H^0(E^\bullet)$, by (2.27), we have $H^0(E^\bullet)/E_\infty^{0,0} \cong E_\infty^{1,-1} = 0$, i.e., $H^0(E^\bullet) \cong E_\infty^{0,0}$. We next calculate $E_r^{0,0}$, from complex

$$\begin{tikzcd}
E_r^{-r+1,r} & & \\
& E_r^{0,0} \arrow[ul] & \\
& & E_r^{r-1,-r}, \arrow[ul]
\end{tikzcd}$$

we have $E_\infty^{0,0} = E_2^{0,0}$, i.e., $H^0(E^\bullet) = E_2^{0,0}$.

By filtration (2.27), we have

$$0 \longrightarrow E_\infty^{0,1} \xhookrightarrow{E_\infty^{1,0}} H^1(E^\bullet),$$

where $H^1(E^\bullet)/E_\infty^{0,1} \cong E_\infty^{1,0}$, hence we have exact sequence

$$0 \longrightarrow E_\infty^{0,1} \hookrightarrow H^1(E^\bullet) \twoheadrightarrow E_\infty^{1,0} \longrightarrow 0.$$

Calculate $E_\infty^{0,1}$ and $E_\infty^{1,0}$. Note that $E_{r+1}^{0,1}$ is computed from complex

$$\begin{tikzcd}
E_r^{-r+1,1+r} & & \\
& E_r^{0,1} \arrow[ul] & \\
& & E_r^{r+1,1-r} \arrow[ul]
\end{tikzcd}$$

and $E_r^{-r+1,1+r} = E_r^{r+1,1-r} = 0$ for all $r \geq 2$, we have $E_\infty^{0,1} = E_2^{0,1}$. Similarly, since $E_{r+1}^{1,0}$ is computed from complex

$$\begin{tikzcd}
E_r^{2-r,r} & & \\
& E_r^{1,0} \arrow[ul, "d_r^{1,0}"'] & \\
& & E_r^{r,-r} \arrow[ul, "d_r^{r,-r}"']
\end{tikzcd}$$

and $E_r^{-r+1,1+r} = E_r^{2-r,1+r} = 0$ for all $r \geq 3$, hence $E_\infty^{1,0} = E_3^{1,0}$. Note that $E_2^{2,-2} = 0$ and from

$$\begin{tikzcd} E_2^{0,2} & & \\ & E_2^{0,1} \arrow[ul, "d_2^{1,0}"'] & \\ & & E_2^{2,-2} = 0, \arrow[ul, "d_2^{2,-2}"'] \end{tikzcd}$$

we have $E_3^{1,0} \cong \operatorname{Ker} d_2^{1,0} / \operatorname{Im} d_2^{2,-2} = \operatorname{Ker} d_2^{1,0}$. Hence we have exact sequence

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow \operatorname{Ker} d_2^{1,0} \longrightarrow 0,$$

since $\operatorname{Ker} d_2^{1,0} \subseteq E_2^{0,1}$, we have exact sequence

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2},$$

where exactness at $E_2^{1,0}$ is given by $\operatorname{Im}(H^1(E^\bullet) \to E_2^{1,0}) = \operatorname{Ker} d_2^{1,0}$. By (2.27), we have a filtration

$$E_\infty^{0,2} \xhookrightarrow{E_\infty^{1,1}} F_1 \xhookrightarrow{E_\infty^{2,0}} H^2(E^\bullet),$$

where $F_1 / E_\infty^{0,2} \cong E_\infty^{1,1}$ and $H^2(E^\bullet)/F_1 \cong E_\infty^{2,0}$. Now we compute $E_\infty^{0,2}$, note that

$$\begin{tikzcd} E_r^{-r+1,2+r} & & \\ & E_r^{0,2} \arrow[ul, "d_r^{0,2}"'] & \\ & & E_r^{r-1,2-r} \arrow[ul, "d_r^{r-1,2-r}"'] \end{tikzcd}$$

and $E_r^{-r+1,2+r} = E_r^{r-1,2-r} = 0$ for all $r \geq 3$, we have

$$E_\infty^{0,2} = E_3^{0,2} = \operatorname{Ker} d_2^{0,2} / \operatorname{Im} d_2^{1,0} = E_2^{0,2} / \operatorname{Im} d_2^{1,0}.$$

Hence we have exact sequence

$$E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow E_3^{0,2} \longrightarrow 0.$$

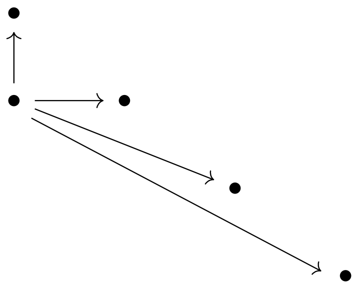
Since $E_3^{0,2} \hookrightarrow H^2(E^\bullet)$, the sequence

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^\bullet)$$

is exact. □

The other orientation

You may have observed that we could as well have done everything in the opposite direction, i.e., reversing the roles of horizontal and vertical morphisms. The the sequences of arrows giving the spectral sequence would look like this (compare to (2.26)).



This spectral sequence is denoted $_{\uparrow}E_{\bullet}^{\bullet,\bullet}$ ("with the **upward orientation**"). Then we would again get pieces of a filtration of $H^{\bullet}(E^{\bullet})$ (where we have to be a bit careful with which $_{\uparrow}E_{\infty}^{p,q}$ corresponds to the subquotients — it is the opposite order to that of (2.27) for $_{\rightarrow}E_{\infty}^{p,q}$). **Warning:** in general there is no isomorphism between $_{\rightarrow}E_{\infty}^{p,q}$ and $_{\uparrow}E_{\infty}^{p,q}$.

In fact, this observation that we can start with either the horizontal or vertical maps was our secret goal all along. Both algorithms compute information about the same thing ($H^{\bullet}(E^{\bullet})$), and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the other way.

2.6.3 Example

We are now ready to see how this is useful. The moral of these example is the following. In the past, you may have proved various facts involving various sorts of diagrams, by chasing elements around. Now, you will just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

Example 2.36 (Proving the Snake Lemma) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the rows are exact in the middle (at A, B, C, D, E, F) and the squares commute. (Normally the Snake Lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \longrightarrow \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma \longrightarrow 0. \tag{2.28}$$

We plug this into our spectral sequence machinery. We first compute the cohomology using the rightward orientation, i.e., using the order (2.26). Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_{\infty}^{p,q} = 0$.

We next compute this "0" in another way, by computing the spectral sequence using the upward orientation. Then $_{\uparrow}E_1^{\bullet,\bullet}$ (with its differentials) is:

$$0 \longrightarrow \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \longrightarrow 0.$$

Then ${}_{\uparrow}E_2^{\bullet,\bullet}$ is of the form:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We see that after ${}_{\uparrow}E_2$, all the terms will stabilize except for the double question marks — all maps to and form the single question marks are to and from 0-entires. And after ${}_{\uparrow}E_3$, even these two double-question-mark terms will stabilize. But in the end our complex must be the 0 complex (since ${}_{\rightarrow}E_\infty^{p,q} = 0$, by (2.27), we know that $H^\bullet(E^\bullet) = 0$). This means that in ${}_{\uparrow}E_2$, all the entries must be zero, except for the two double question marks (these two ?? are $\operatorname{Ker}(\operatorname{Coker}\alpha \to \operatorname{Coker}\beta)$ and $\operatorname{Coker}(\operatorname{Ker}\beta \to \operatorname{Ker}\gamma)$), and these two must be isomorphic. This means that $0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma$ and $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0$ are both exact (that comes from the vanishing of the vanishing of the single question marks), and

$$\operatorname{Coker}(\operatorname{Ker}\beta \to \operatorname{Ker}\gamma) \cong \operatorname{Ker}(\operatorname{Coker}\alpha \to \operatorname{Coker}\beta)$$

is an isomorphism (that comes from the equality of the double question marks). Taken together, we have proved the exactness of (2.28), and hence the Snake Lemma! (**Notice:** in the end we didn't really care about the double complex. We just used it as a prop to prove the Snake Lemma.)

Spectral sequences make it easy to see how to generalize results further. For example, if $A \to B$ is no longer assumed to be injective, then ${}_{\rightarrow}E_1^{\bullet,\bullet}$ (with its differentials) is:

$$\begin{tikzcd}
0 & 0 & 0 \\
\operatorname{Ker}(A \to B) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] & 0. \arrow[u]
\end{tikzcd}$$

Hence ${}_{\rightarrow}E_\infty^{0,0} = H^0(E^\bullet) = \operatorname{Ker}(A \to B)$, by (2.27). Note that ${}_{\uparrow}E_\infty^{0,0} = H^0(E^\bullet)$ ((2.27) again), ${}_{\uparrow}E_2^{\bullet,\bullet}$ is of the form:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
0 \arrow[drr] & 0 \arrow[drr] & \operatorname{Ker}(\operatorname{Coker}\alpha \to \operatorname{Coker}\beta) \arrow[drr] & 0 \arrow[drr] & 0 & & \\
& 0 & \operatorname{Ker}(A \to B) \arrow[drr] & 0 \arrow[drr] & \operatorname{Coker}(\operatorname{Ker}\beta \to \operatorname{Ker}\gamma) \arrow[drr] & 0 & \\
& & & & 0 & 0 & 0.
\end{tikzcd}$$

Since $E_\infty^{p,q} = 0$ for all $p \neq 0$ or $q \neq 0$ and $\operatorname{Ker}(\operatorname{Ker}\alpha \to \operatorname{Ker}\beta) = \operatorname{Ker}(A \to B)$, we have exact sequence

$$\operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \longrightarrow \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma \longrightarrow 0.$$

✍ **Exercise 2.40 Unimportant exercise (Grafting exact sequence, a variant of the snake Lemma)** Extend the

Snake Lemma as follows. Suppose we have a commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & A' \arrow[r] & \cdots \\
\cdots \arrow[r] & W \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "a"] & Y \arrow[r] \arrow[u, "b"] & Z \arrow[r] \arrow[u, "c"] & 0. \arrow[u] &
\end{tikzcd}$$

where the top and bottom rows are exact. Show that the top and bottom rows can be "grafted together" to an exact sequence

$$\cdots \longrightarrow W \longrightarrow \operatorname{Ker} a \longrightarrow \operatorname{Ker} b \longrightarrow \operatorname{Ker} c$$

$$\longrightarrow \operatorname{Coker} a \longrightarrow \operatorname{Coker} b \longrightarrow \operatorname{Coker} c \longrightarrow A' \longrightarrow \cdots.$$

Proof We first compute the cohomology using the rightward orientation. Then because the rows are exact, ${}_{\rightarrow}E_1^{p,q} = 0$, so the spectral sequence has already converged: ${}_{\rightarrow}E_\infty^{p,q} = 0$. By (2.27), we have $H^\bullet(E^\bullet) = 0$.

We next compute the cohomology using the upward orientation. ${}_{\uparrow}E_1^{\bullet,\bullet}$ is

$$\begin{array}{c} 0 \longrightarrow \operatorname{Coker} a \longrightarrow \operatorname{Coker} b \longrightarrow \operatorname{Coker} c \longrightarrow A' \longrightarrow \cdots \\ \\ \cdots \longrightarrow W \longrightarrow \operatorname{Ker} a \longrightarrow \operatorname{Ker} b \longrightarrow \operatorname{Ker} c \longrightarrow 0. \end{array} \tag{2.29}$$

And ${}_{\uparrow}E_2^{\bullet,\bullet}$ is:

$$\begin{tikzcd}
 & 0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & & & & \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? \arrow[drr] & ? \arrow[drr] & 0 & \\
 & 0 & ? \arrow[drr] & ? \arrow[drr] & ? \arrow[drr] & ?? \arrow[drr] & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0. &
\end{tikzcd}$$

We see that after page ${}_{\uparrow}E_2^{\bullet,\bullet}$, all the terms will stabilize except for the double question marks. They are $\operatorname{Ker}(\operatorname{Coker} a \to \operatorname{Coker} b)$ and $\operatorname{Coker}(\operatorname{Ker} b \to \operatorname{Ker} c)$. Since $H^\bullet(E^\bullet) = 0$, ${}_{\uparrow}E_\infty^{p,q} = 0$. Hence ? are all 0, and after page ${}_{\uparrow}E_3^{\bullet,\bullet}$, ?? terms will stabilize, moreover, they must be isomorphism, i.e.,

$$\operatorname{Ker}(\operatorname{Coker} a \to \operatorname{Coker} b) \cong \operatorname{Coker}(\operatorname{Ker} b \to \operatorname{Ker} c),$$

note that we have

$$\operatorname{Ker} b \longrightarrow \operatorname{Ker} c \twoheadrightarrow \operatorname{Coker}(\operatorname{Ker} b \to \operatorname{Ker} c) \xrightarrow{\sim} \operatorname{Ker}(\operatorname{Coker} a \to \operatorname{Coker} b) \hookrightarrow \operatorname{Coker} a,$$

so sequence $\operatorname{Ker} b \to \operatorname{Ker} c \to \operatorname{Coker} a$ is exact. Since ? are all 0, (2.29) exact. Taken together, sequence

$$\cdots \longrightarrow W \longrightarrow \operatorname{Ker} a \longrightarrow \operatorname{Ker} b \longrightarrow \operatorname{Ker} c$$

$$\longrightarrow \operatorname{Coker} a \longrightarrow \operatorname{Coker} b \longrightarrow \operatorname{Coker} c \longrightarrow A' \longrightarrow \cdots.$$

is exact. □

Example 2.37 (The five Lemma) Suppose

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\varepsilon"]
\end{tikzcd} \tag{2.30}$$

where the rows are exact and squares commute.

Suppose $\alpha, \beta, \delta, \varepsilon$ are isomorphisms. Show that γ is an isomorphism.

Proof We first compute the cohomology of the total complex using the rightward orientation (2.26). We choose this because we see that we will get lots of zeros. Then ${}_{\rightarrow}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ?. \arrow[u]
\end{tikzcd}$$

Then ${}_{\rightarrow}E_2^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
0 & & & & 0 & & \\
0 & & & & 0 & & \\
& ? \arrow[uul] & 0 & 0 & 0 & ? \arrow[uul] & \\
& ? \arrow[uul] & 0 & 0 & 0 & ? \arrow[uul] & \\
& & 0 \arrow[uul] & & & & 0 \arrow[uul] \\
& & 0 \arrow[uul] & & & & 0 \arrow[uul]
\end{tikzcd}$$

and therefore the sequence will converge by E_2, as we will never get any arrows between two nonzero entires in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, it vanishes in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this using the upward orientation. Then ${}_{\uparrow}E_1$ looks like this:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

and the spectral sequence converges at this step. We wish to show that those two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed were zero — so we are done! □

Theorem 2.6.1 (Five Lemma for modules)

Consider the following commutative diagram.

$$\begin{tikzcd}
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \\
A_1 \arrow[r] \arrow[u, "f_1"] & A_2 \arrow[r] \arrow[u, "f_2"'] & A_3 \arrow[r] \arrow[u, "f_3"'] & A_4 \arrow[r] \arrow[u, "f_4"'] & A_5 \arrow[u, "f_5"']
\end{tikzcd}$$

where the top and bottom rows are exact sequences.

(1) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.

(2) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.

(3) If f_2 and f_4 are isomorphisms, f_1 is surjective, and f_5 is injective, then f_3 is isomorphism.

Proof (2) is the dual statement of (1), we just prove (1), and (2) is similarly.

Suppose f_2 and f_4 are surjective and f_5 is injective, we want to show that f_3 is surjective.

We first compute the cohomology of the total complex using the rightward orientation. Then ${}_{\rightarrow}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
\operatorname{Ker}(B_1 \to B_2) & 0 & 0 & 0 & \operatorname{Coker}(B_4 \to B_5) \\
\operatorname{Ker}(A_1 \to A_2) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \operatorname{Coker}(A_4 \to A_5). \arrow[u]
\end{tikzcd}$$

Then ${}_{\rightarrow}E_2^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
0 & & & & 0 & & \\
0 & & & & 0 & & \\
& ? \arrow[uul] & 0 & 0 & 0 & ? \arrow[uul] & \\
& ? \arrow[uul] & 0 & 0 & 0 & ? \arrow[uul] & \\
& & 0 \arrow[uul] & & & & 0 \arrow[uul] \\
& & 0 \arrow[uul] & & & & 0 \arrow[uul]
\end{tikzcd}$$

and therefore the sequence will converge by ${}_{\rightarrow}E_2$. Hence by (2.27),

$$E_2^{0,3} \xhookrightarrow{E_2^{1,2}} F_1 \xhookrightarrow{E_2^{2,1}} F_2 \xhookrightarrow{E_2^{3,0}} H^3(E^\bullet),$$

we have $H^3(E^\bullet) = 0$.

We next compute this using the upward orientation. The ${}_{\uparrow}E_1$ looks like this:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Coker} f_1 \arrow[r] & 0 \arrow[r] & \operatorname{Coker} f_3 \arrow[r] & 0 \arrow[r] & \operatorname{Coker} f_5 \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker} f_1 \arrow[r] & \operatorname{Ker} f_2 \arrow[r] & \operatorname{Ker} f_3 \arrow[r] & \operatorname{Ker} f_4 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

And ${}_{\uparrow}E_2$ looks like this:

$$\begin{tikzcd}
0 \arrow[drr] & & 0 \arrow[drr] & & 0 \arrow[drr] & & & & \\
0 \arrow[drr] & 0 \arrow[drr] & \operatorname{Coker} f_1 \arrow[drr] & 0 \arrow[drr] & \operatorname{Coker} f_3 \arrow[drr] & 0 & \operatorname{Coker} f_5 \arrow[drr] & 0 & \\
0 & 0 & ? \arrow[drr] & ? \arrow[drr] & ? \arrow[drr] & ?? \arrow[drr] & 0 & 0 & 0 \\
& & & & 0 & 0 & 0 & 0. &
\end{tikzcd}$$

Hence ${}_{\uparrow}E_\infty^{2,1} = \operatorname{Coker} f_3$. By (2.27), i.e. filtration

$${}_{\uparrow}E_2^{3,0} \xhookrightarrow{{}_{\uparrow}E_2^{2,1}} {}_{\uparrow}F_1 \xhookrightarrow{{}_{\uparrow}E_2^{1,2}} {}_{\uparrow}F_2 \xhookrightarrow{{}_{\uparrow}E_2^{0,3}} H^3(E^\bullet),$$

we have $H^3(E^\bullet)/?? \cong \operatorname{Coker} f_3$, since $H^3(E^\bullet) = 0$, $\operatorname{Coker} f_3 = 0$, i.e., f_3 is surjective.

Now, we prove (3). Suppose f_2 and f_4 are isomorphism, and f_5 is injective, we want to show f_3 is isomorphism.

We first compute the cohomology of the total complex using the rightward orientation. Same as the proof of (1), we have $H^3(E^\bullet) = 0$ and $H^2(E^\bullet) = 0$.

We next compute this using the upward orientation. The ${}_\uparrow E_1$ looks like this:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} f_3 & \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} f_5 \\ \\ \operatorname{Ker} f_1 & \longrightarrow & 0 & \longrightarrow & \operatorname{Ker} f_3 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Hence the spectral sequence converges at page 1. By (2.27), i.e. filtration

$${}_\uparrow E_1^{3,0} \xhookrightarrow{{}_\uparrow E_1^{2,1}} \bullet \xhookrightarrow{{}_\uparrow E_1^{1,2}} \bullet \xhookrightarrow{{}_\uparrow E_1^{0,3}} H^3(E^\bullet)$$

and

$${}_\uparrow E_1^{2,0} \xhookrightarrow{{}_\uparrow E_1^{1,1}} \bullet \xhookrightarrow{{}_\uparrow E_1^{0,2}} H^2(E^\bullet),$$

we have $H^3(E^\bullet) \cong {}_\uparrow E_1^{2,1} = \operatorname{Coker} f_3$ and $H^2(E^\bullet) \cong \operatorname{Ker} f_3$, hence $\operatorname{Coker} f_3 = 0$ and $\operatorname{Ker} f_3 = 0$, i.e., f_3 is isomorphism, we are done! ☐

Proposition 2.6.2 (The mapping cone)

*Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes. Suppose $C^\bullet$ is the single complex associated to the double complex $A^\bullet \to B^\bullet$. ($C^\bullet$ is called the **mapping cone** of μ.) Then there is a long exact sequence of complexes:*

$$\cdots \longrightarrow H^{i-1}(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^{i+1}(C^\bullet) \longrightarrow \cdots.$$

In particular, μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

Remark There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.

Proof Consider the double complex as following.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \\ & & {\scriptstyle\mu^{n-1}}\uparrow & & {\scriptstyle\mu^{n}}\uparrow & & {\scriptstyle\mu^{n+1}}\uparrow & & \\ \cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \end{array}$$

We compute the cohomology of the total complex $C^\bullet$ using the rightward orientation. Then ${}_\rightarrow E_1^{\bullet,\bullet}$ looks like this:

$$\begin{array}{ccc} 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow \\ H^{n-1}(B^\bullet) & H^n(B^\bullet) & H^{n+1}(B^\bullet) \\ \uparrow & \uparrow & \uparrow \\ H^{n-1}(A^\bullet) & H^n(A^\bullet) & H^{n+1}(A^\bullet) \\ \uparrow & \uparrow & \uparrow \\ 0 & 0 & 0 \end{array}$$

And ${}_{\to}E_2^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
0 & & \\
0 & & \\
& \operatorname{Coker}(H^n(A^\bullet)\to H^n(B^\bullet)) \arrow[uul] & \\
& \operatorname{Ker}(H^n(A^\bullet)\to H^n(B^\bullet)) \arrow[uul] & \\
& & 0 \arrow[uul] \\
& & 0 \arrow[uul]
\end{tikzcd}$$

Hence the spectral sequence converges at page 2, and therefore ${}_{\to}C_\infty^{i,0} = {}_{\to}C_2^{i,0} = \operatorname{Ker}(H^i(A^\bullet)\to H^i(B^\bullet))$, ${}_{\to}C_\infty^{i,1} = C_2^{i,1} = \operatorname{Coker}(H^i(A^\bullet)\to H^i(B^\bullet))$, and ${}_{\to}C^{i,j} = 0$ for all $j\neq 0,1$. By (2.27), we have

$${}_{\to}C_2^{0,i} \xhookrightarrow{{}_{\to}C_2^{1,i-1}} \bullet \hookrightarrow \cdots \hookrightarrow \bullet \xhookrightarrow{{}_{\to}C_2^{i,0}} H^i(C^\bullet),$$

hence $H^i(C^\bullet)/{}_{\to}C_2^{i-1,1} \cong {}_{\to}C_2^{i,0}$. Then we have exact sequence

$$0 \longrightarrow \operatorname{Coker}(H^{i-1}(A^\bullet)\to H^{i-1}(B^\bullet)) \longrightarrow H^i(C^\bullet) \longrightarrow \operatorname{Ker}(H^i(A^\bullet)\to H^i(B^\bullet)) \longrightarrow 0. \tag{2.31}$$

Consider the sequence

$$\cdots \longrightarrow H^{i-1}(A^\bullet) \longrightarrow H^{i-1}(B^\bullet) \xrightarrow{\alpha_{i-1}} H^i(C^\bullet) \xrightarrow{\beta_i} H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow \cdots, \tag{2.32}$$

where $\alpha_{i-1} : H^{i-1}(B^\bullet)\to H^i(C^\bullet)$ is given by $H^{i-1}(B^\bullet) \twoheadrightarrow \operatorname{Coker}(H^{i-1}(A^\bullet)\to H^{i-1}(B^\bullet)) \hookrightarrow H^i(C^\bullet)$ and $\beta_i : H^i(C^\bullet)\to H^i(A^\bullet)$ is given by $H^i(C^\bullet) \twoheadrightarrow \operatorname{Ker}(H^i(A^\bullet)\to H^i(B^\bullet)) \hookrightarrow H^i(A^\bullet)$. We now check the exactness. Note that

$$\operatorname{Ker}\alpha \cong \operatorname{Im}(H^{i-1}(A^\bullet)\to H^{i-1}(B^\bullet)),$$

sequence (2.32) is exact at $H^{i-1}(B^\bullet)$. By exactness of (2.31)

$$\begin{aligned}
\operatorname{Im}\alpha_{i-1} &\cong \operatorname{Im}(\operatorname{Coker}(H^{i-1}(A^\bullet)\to H^{i-1}(B^\bullet))\to H^i) \\
&\cong \operatorname{Ker}(H^i(C^\bullet)\to \operatorname{Ker}(H^i(A^\bullet)\to H^i(B^\bullet))) \cong \operatorname{Ker}\beta_i,
\end{aligned}$$

(2.32) is exact at $H^i(C^\bullet)$. Finally, note that

$$\operatorname{Im}\beta \cong \operatorname{Ker}(H^i(A^\bullet)\to H^i(B^\bullet)),$$

(2.32) is exact at $H^i(A^\bullet)$. Hence sequence (2.32) is exact, as we desired.

In particular, μ induces an isomorphism on cohomology if and only if $\operatorname{Ker}(H^n(A^\bullet)\to H^n(B^\bullet)) = 0$ and $\operatorname{Coker}(H^n(A^\bullet)\to H^n(B^\bullet)) = 0$ if and only if $H^n(C^\bullet) = 0$ for all n if and only if the mapping cone $C^\bullet$ is exact. □

✍ **Exercise 2.41** Use spectral sequences to show that a short exact sequence of complexes give a long exact sequence in cohomology (Theorem 2.5.4). (This is a generalization of Proposition 2.6.2.)

Proof Suppose we have a short exact sequence of complexes.

$$
\begin{tikzcd}
0^\bullet & : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
C^\bullet \arrow[u] & : & \cdots \arrow[r] & C^{n-1} \arrow[r] \arrow[u] & C^n \arrow[r] \arrow[u] & C^{n+1} \arrow[r] \arrow[u] & \cdots \\
B^\bullet \arrow[u] & : & \cdots \arrow[r] & B^{n-1} \arrow[r] \arrow[u] & B^n \arrow[r] \arrow[u] & B^{n+1} \arrow[r] \arrow[u] & \cdots \\
A^\bullet \arrow[u] & : & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[u] & A^n \arrow[r] \arrow[u] & A^{n+1} \arrow[r] \arrow[u] & \cdots \\
0^\bullet \arrow[u] & : & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

We want to show that it induces a long exact sequence in cohomology.

$$
\begin{tikzcd}
& & \cdots \arrow[r] & H^{n-1}(C^\bullet) \arrow[r] & {} \\
H^n(A^\bullet) \arrow[r] & H^n(B^\bullet) \arrow[r] & H^n(C^\bullet) \arrow[r] & {} \\
H^{n+1}(A^\bullet) \arrow[r] & \cdots
\end{tikzcd}
$$

We first compute the cohomology of the total complex $E^\bullet$ using the upward orientation. Then ${}_{\uparrow}E_1^{\bullet,\bullet}$ looks like this:

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Hence ${}_{\uparrow}E_\infty^{p,q} = {}_{\uparrow}E_1^{p,q} = 0$. By (2.27), we have $H^k(E^\bullet) = 0$ for all k.

We next compute the cohomology of total complex $E^\bullet$ using the rightward orientation. Denote

$$
0^\bullet \longrightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \longrightarrow 0.
$$

Then ${}_{\rightarrow}E_1^{\bullet,\bullet}$ looks like this:

$$
\begin{tikzcd}
0 & 0 & 0 \\
H^{n-1}(C^\bullet) \arrow[u] & H^n(C^\bullet) \arrow[u] & H^{n+1}(C^\bullet) \arrow[u] \\
H^{n-1}(B^\bullet) \arrow[u, "\beta^{n-1*}"] & H^n(B^\bullet) \arrow[u, "\beta^{n*}"] & H^{n+1}(B^\bullet) \arrow[u, "\beta^{n+1*}"] \\
H^{n-1}(A^\bullet) \arrow[u, "\alpha^{n-1*}"] & H^n(A^\bullet) \arrow[u, "\alpha^{n*}"] & H^{n+1}(A^\bullet) \arrow[u, "\alpha^{n+1*}"] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

And ${}_{\rightarrow}E_2^{\bullet,\bullet}$ looks like:

$$
\begin{tikzcd}
0 & 0 & & & \\
 & 0 & 0 & 0 & \\
 & \operatorname{Coker}\beta^{n-1*} \arrow[uul] & \operatorname{Coker}\beta^{n*} \arrow[uul] & \operatorname{Coker}\beta^{n+1*} & \\
 & \operatorname{Ker}\beta^{n-1*}/\operatorname{Im}\alpha^{n-1*} & \operatorname{Ker}\beta^{n*}/\operatorname{Im}\alpha^{n*} \arrow[uul] & \operatorname{Ker}\beta^{n+1*}/\operatorname{Im}\alpha^{n+1*} \arrow[uul] & \\
 & \operatorname{Ker}\alpha^{n-1*} & \operatorname{Ker}\alpha^{n*} \arrow[uul] & \operatorname{Ker}\alpha^{n+1*} \arrow[uul] & \\
 & 0 & 0 \arrow[uul] & 0 \arrow[uul] & \\
 & & & 0 \arrow[uul] & 0 \arrow[uul]
\end{tikzcd}
$$

Hence ${}_{\rightarrow}E_\infty^{n,1} = {}_{\rightarrow}E_2^{n,1} = \operatorname{Ker}\beta^{n*}/\operatorname{Im}\alpha^{n*}$. In page 3, i.e., ${}_{\rightarrow}E_3^{\bullet,\bullet}$ looks like:

$$
\begin{tikzcd}
0 & & \\
0 & & \\
0 & & \\
 & \operatorname{Coker}(\operatorname{Ker}\alpha^{n+1*} \to \operatorname{Coker}\beta^{n*}) \arrow[uuul] & \\
 & \operatorname{Ker}\beta^{n*}/\operatorname{Im}\alpha^{n*} \arrow[uuul] & \\
 & \operatorname{Ker}(\operatorname{Ker}\alpha^{n*} \to \operatorname{Coker}\beta^{n-1*}) \arrow[uuul] & \\
 & & 0 \arrow[uuul] \\
 & & 0 \arrow[uuul] \\
 & & 0 \arrow[uuul]
\end{tikzcd}
$$

Hence ${}_{\rightarrow}E_\infty^{n,0} = {}_{\rightarrow}E_3^{n,0} = \operatorname{Ker}(\operatorname{Ker}\alpha^{n*} \to \operatorname{Coker}\beta^{n-1*})$, ${}_{\rightarrow}E_\infty^{n,2} = {}_{\rightarrow}E_3^{n,2} = \operatorname{Coker}(\operatorname{Ker}\alpha^{n+1*} \to \operatorname{Coker}\beta^{n*})$, and ${}_{\rightarrow}E_\infty^{n,j} = 0$ for all $j \neq 0, 1, 2$. By (2.27), we have a filtration

$$
{}_{\rightarrow}E^{0,i} \xhookrightarrow{{}_{\rightarrow}E^{1,i-1}} \bullet \xhookrightarrow{{}_{\rightarrow}E^{2,i-2}} \bullet \hookrightarrow \cdots \hookrightarrow \bullet \xhookrightarrow{{}_{\rightarrow}E^{i,0}} H^i(E^\bullet).
$$

Since $H^i(E^\bullet) = 0$, we know that ${}_{\to}E_\infty^{n,0} = {}_{\to}E_\infty^{n,1} = {}_{\to}E_\infty^{n,2} = 0$, i.e.,

$$\begin{aligned}
\operatorname{Ker}(\operatorname{Ker}\alpha^{n*} \to \operatorname{Coker}\beta^{n-1*}) &= 0\\
\operatorname{Ker}\beta^{n*} / \operatorname{Im}\alpha^{n*} &= 0\\
\operatorname{Coker}(\operatorname{Ker}\alpha^{n+1*} \to \operatorname{Coker}\beta^{n*}) &= 0
\end{aligned}$$

It follows that sequence

$$H^n(A^\bullet) \xrightarrow{\alpha^{n*}} H^n(B^\bullet) \xrightarrow{\beta^{n*}} H^n(C^\bullet)$$

is exact, and

$$\operatorname{Ker}\alpha^{n*} \cong \operatorname{Coker}\beta^{n-1*}.$$

Define $H^{n-1}(C^\bullet) \to H^n(A^\bullet)$ by setting

$$H^{n-1}(C^\bullet) \twoheadrightarrow \operatorname{Coker}\beta^{n-1*} \xrightarrow{\sim} \operatorname{Ker}\alpha^{n*} \hookrightarrow H^n(A^\bullet),$$

then sequence

$$H^{n-1}(C^\bullet) \longrightarrow H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

is exact. Taken together, the sequence

$$\begin{aligned}
&\cdots \longrightarrow H^{n-1}(C^\bullet) \longrightarrow\\
H^n(A^\bullet) \longrightarrow &H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \longrightarrow\\
H^{n+1}(A^\bullet) &\longrightarrow \cdots
\end{aligned}$$

is exact. □

The Grothendieck composition-of-functors spectral sequence (Chapter 24) will be an important example of a spectral sequence that specializes in a number of useful ways.

You are now ready to go out into the world and use spectral sequences to your heart's content.

2.6.4 Complete definition of spectral sequences, and proof

You should most definitely not read the precise definition of a spectral sequence, and the proof that they works as advertised, any time soon after reading the introduction to spectral sequences above. But after a suitable interval, you should at least flip through a construction and proof to convince yourself that nothing fancy is involved. The ideal is not as bad as you might think.

It is useful to notice that the proof implies that spectral sequences are functorial in the 0-th page: the spectral sequence formalism has good functorial properties in the double complex. Unfortunately, Grothendieck's terminology "spectral functor" did not catch on.

Chapter 3 Sheaves

It is perhaps surprising that geometric spaces are often best understood in terms of (nice) functions on them. For example, a differentiable mainfold that is a subset of $\mathbb{R}^n$ can be studied in terms of its smooth (C^∞) functions. Because "geometric spaces" can have few (everywhere-defined) functions, a more precise version of this insight is that the structure of the space can be well understood by considering all functions on all open subsets of the space. This information is encoded and organized in something called a **sheaf**. We will define sheaves and describe useful facts about them. We will begin with a motivating example to convince you that the notion is not so foreign.

One reason sheaves are slippery to work with is that they keep track of a huge amount of information, and there are some subtle local-to-global issues. There are also three different ways of getting a hold of them:

1. in terms of open sets — intuitive but in some ways the least helpful;
2. in terms of **stalks**;
3. in terms of a base of a topology.

(Some people strongly prefer the espace etale interpretation, as well.)

3.1 Motivating example: The sheaf of smooth functions

Consider smooth (C^∞) functions on the topological space $X = \mathbb{R}^n$ (or more generally on a manifold X). The sheaf of smooth functions on X is the data of all smooth functions on all open subsets on X. We will see how to manage these data, and observe some of their properties. On each open set $U \subseteq X$, we have a ring of smooth functions. We denote this ring of functions $\mathscr{O}(U)$.

Given a smooth function on an open set, you can restrict it to a smaller open set, obtaining a smooth function there. In other words, if $U \subseteq V$ is an inclusion of open sets, we have a "restriction map" $\operatorname{res}_{V,U} : \mathscr{O}(V) \to \mathscr{O}(U)$.

Take a smooth function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set. The result is the same as if you restrict the smooth function on the big open set directly to the small open set. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd} \mathscr{O}(W) \arrow[rr, "\operatorname{res}_{W,V}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & & \mathscr{O}(V) \arrow[dl, "\operatorname{res}_{V,U}"] \\ & \mathscr{O}(U) & \end{tikzcd}$$

Next take two smooth functions f_1 and f_2 on a big open set U, and an open cover of U by some collection of open subsets $\{U_i\}$. (We say $\{U_i\}$ **covers** U, or is an **open cover of** U, if $U = \bigcup U_i$.) Suppose that f_1 and f_2 agree on each of these U_i. Then they must have been the same function to begin with. In other words, if $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathscr{O}(U)$, and $\operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2$, then $f_1 = f_2$. Thus we can **identify** functions on an open set by looking at them on a covering by small open sets.

Finally, suppose you are given the same U and cover $\{U_i\}$, take a smooth function on each of the U_i — a function f_1 on U_1, a function f_2 on U_2, and so on — and assume they agree on the pairwise overlaps. Then they can be "glued together" to make one smooth function on all of U. In other words, given $f_i \in \mathscr{O}(U_i)$ for all i, such that $\operatorname{res}_{U_i, U_i \cap U_j} f_i = \operatorname{res}_{U_j, U_i \cap U_j} f_j$ for all i and j, then there is some $f \in \mathscr{O}(U)$ such that $\operatorname{res}_{U,U_i} f = f_i$ for all i.

The entire example above would have worked just as well with continuous functions, or real-analytic functions, or just plain real-valued functions. Thus all of these classes of "nice" functions share some common properties. We will soon formalize these properties in the notion of a sheaf.

Before we do, we first give another definition, that of the germ of smooth function at a point $p \in X$. Intuitively, it is a "shred" of a smooth function at p.

Definition 3.1.1 (The germ of a smooth function)

Germs *are objects of the form*

$$(f, \text{ open set } U) \quad \text{such that} \quad p \in U, f \in \mathscr{O}(U)$$

modulo the relation that $(f, U) \sim (g, V)$ *if there is some open set* $W \subseteq U, V$ *containing* p *where* $f|_W = g|_W$. *In other words, two functions that are the same in an open neighborhood of p (but may differ elsewhere) have the same germ.*

We call this set of germs the ***stalk*** *at p, and denote it* $\mathscr{O}_p$.

Remark

(1) The stalk is a ring.

We can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f+g$ defined on $U \cap V$. Moreover, $f+g$ is well-defined; if $\widetilde{f}$ has the same germ as f, meaning that there is some open set W containing p on which they agree, and $\widetilde{g}$ has the same germ as g, meaning they agree on some open W' containing p, then $\widetilde{f}+\widetilde{g}$ is the same function as $f+g$ on $U \cap V \cap W \cap W'$.

(2) The stalk is a colimit.

Notice also that if $p \in U$, you get a map $\mathscr{O}(U) \to \mathscr{O}_p$. For $V \subseteq U$, we have

$$\begin{tikzcd} \mathscr{O}_p & \\ \mathscr{O}(U) \arrow[u] \arrow[r, "\mathrm{res}_{U,V}"'] & \mathscr{O}(V). \arrow[ul] \end{tikzcd}$$

Proposition 3.1.1

The stalk $\mathscr{O}_p$ *is a local ring with maximal ideal* $\mathfrak{m}_p = \{f \in \mathscr{O}_p : f(p) = 0\}$.

Proof We can see that $\mathscr{O}_p$ is a local ring as follows. Consider those germs vanishing at p, which we denote $\mathfrak{m}_p \subseteq \mathscr{O}_p$, that is, $\mathfrak{m}_p = \{f \in \mathscr{O}_p : f(p) = 0\}$.

(i) $\mathfrak{m}_p$ is an ideal.

Let $f, g \in \mathfrak{m}_p$, then $f(p) = g(p) = 0$, hence, $\mathfrak{m}_p$ is closed under addition. Let $f \in \mathfrak{m}_p$ and $h \in \mathscr{O}_p$, then $hf(p) = 0$, and therefore $hf \in \mathfrak{m}_p$, that is, $\mathscr{O}_p\mathfrak{m}_p \subseteq \mathfrak{m}_p$. Hence, $\mathfrak{m}_p$ is an ideal.

(ii) $\mathfrak{m}_p$ is maximal.

Cosider the following sequence,

$$0 \longrightarrow \mathfrak{m}_p \longrightarrow \mathscr{O}_p \xrightarrow{\varphi} \mathbb{R} \longrightarrow 0,$$

where $\varphi : f \mapsto f(p)$. Clearly, above sequence is exact, and therefore $\mathscr{O}_p/\mathfrak{m}_p \cong \mathbb{R}$. Hence, $\mathscr{O}_p/\mathfrak{m}_p$ is a field, $\mathfrak{m}_p$ is maximal.

(iii) $\mathfrak{m}_p$ is the only maximal ideal of $\mathscr{O}_p$.

It suffices to show that each element in $\mathscr{O}_p - \mathfrak{m}_p$ is unit. Let $g \in \mathscr{O}_p - \mathfrak{m}_p$, then $g(p) \neq 0$, hence, $1/g(p) \neq 0$, this imples that g is a unit in $\mathscr{O}_p$. Hence, every ideal$\neq (1)$ is contained in $\mathfrak{m}$. Hence, $\mathfrak{m}_p$ is the only maximal ideal of $\mathscr{O}_p$.

□

Remark Note that we can interpret the value of a function at a point, or the value of germ at a point, as an element of the local ring modulo the maximal ideal. We will see that this doesn't work for more general sheaves, but does work for things behaving like sheaves of functions. This will be formalized in the notion of a **locally ringed space**.

Remark Notice that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a module over $\mathscr{O}_p/\mathfrak{m}_p \cong \mathbb{R}$, i.e., it is a real vector space. It turns out be naturally (whatever that means) the cotangent space to the differentiable or analytic manifold at p. In differential geometry, the cotangent space T_p^*M at p is the dual of the tangent space T_pM, whose elements are differential forms (e.g. df). Elements of $\mathfrak{m}_p$ are function germs that vanish at p; elements of $\mathfrak{m}_p^2$ are function germs that vanish at p with zero derivatives. Therefore, equivalence classes in $\mathfrak{m}_p/\mathfrak{m}_p^2$ represent "first-order infinitesimal changes" of functions at p, that is, derivative information. As a real rector space, $\mathfrak{m}_p/\mathfrak{m}_p^2$ is naturally isomorphic to the cotangent space T_p^*M. Specifically:

$$T_p^*M \cong \mathfrak{m}_p/\mathfrak{m}_p^2,$$

where the differential df corresponds to the equivalence class of the function germs $f - f(p)$. This insight will prove handy later, when we define tangent and cotangent spaces of schemes.

✍ **Exercise 3.1** **(Exercise for those with differential geometric backgroud.)** Prove this. Rhetorical question for experts: what goes wrong if the sheaf of continuous functions is substituted for the sheaf of smooth functions? What goes wrong if you use the sheaf of C^1 functions?

Solution

(i) *The sheaf of continuous functions.*
The maximal ideal $\mathfrak{m}_p$ for continuous functions contains germs vanishing at p, but these functions may not be differentiable. Hence, $\mathfrak{m}_p/\mathfrak{m}_p^2$ may become infinite-dimensional, failing to match the finite-dimensional cotangent space T_p^*M.

(ii) *The sheaf of C^1 functions.*
$C_p^1/\mathfrak{m}_p \cong \mathbb{R}$ also holds, and first derivatives are definable. However, higher-order derivatives are not guaranteed. The sheaf C^1 is not closed under repeated differentiation, limiting geometric constructions that require smoothness. For example, let $f(x) = x + x^2 sin(\frac{1}{x})$, then $f(x) \in \mathfrak{m}_0$, but $f'(0)$ not exist.

3.2 Definition of sheaf and presheaf

We now formalize these notions, by defining presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward. Sheaves are more complicated to defined, and some notions such as cokernel require more thought. But sheaves are more useful because they are in some vague sense more geometric, we can get information about a sheaf locally.

3.2.1 Definition of sheaf and presheaf on a topological space X

To be concrete, we will define sheaves of sets. However, in the definiton the category $\mathbf{Sets}$ can be replaced by any reasonable category, and other important examples are abelian groups $\mathbf{Ab}$, k-vector spaces $\mathbf{Vec}_k$, rings $\mathbf{Rings}$, modules over a ring $\mathbf{Mod}_A$, and more. Sheaves (and presheaves) are often written in calligraphic font.

The fact that $\mathscr{F}$ is a sheaf on a topological space X is often written as

$$\begin{array}{c}\mathscr{F}\\ \Big| \\ X.\end{array}$$

Definition 3.2.1 (Presheaf)

*A **presheaf of sets** $\mathscr{F}$ on a topological space X is the following data.*

*(1) To each open set $U \subseteq X$, we have a set $\mathscr{F}(U)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over** U. By convention, if the "U" is omitted, it is implicitly taken to be X: "**sections of $\mathscr{F}$**" means "sections of $\mathscr{F}$ over X". These are also called **global sections**.*

*(2) For each inclusion $U \hookrightarrow V$ of open sets, we have a **restriction map***

$$\operatorname{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U).$$

If $f \in \mathscr{F}(V)$, we often write

$$f|_U$$

for $\operatorname{res}_{V,U}(f)$.

The data is required to satisfy the following two conditions.

(1) The map $\operatorname{res}_{U,U}$ is the identity: $\operatorname{res}_{U,U} = \operatorname{id}_{\mathscr{F}(U)}$.

(2) If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then the restriction maps compose as you would expect, i.e., the diagram

$$\begin{tikzcd} \mathscr{F}(W) \arrow[rr, "\operatorname{res}_{W,U}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & & \mathscr{F}(V) \arrow[dl, "\operatorname{res}_{V,U}"] \\ & \mathscr{F}(U) & \end{tikzcd}$$

commutes.

Proposition 3.2.1

A presheaf is the same as a contravariant functor.

Proof By definition of presheaf, it is obvious. □

We define the stalk of a presheaf at a point in two equivalent ways. One will be hands-on, and the other will be as a colimit.

Definition 3.2.2 (Stalks and germs)

*Define the **stalk** of a presheaf $\mathscr{F}$ at a point p to be the set of **germs** of $\mathscr{F}$ at p, denote $\mathscr{F}_p$. Germs correspond to sections over some open set containing p, and two of these sections are considered the same if they agree on some smaller open set containing p. More precisely: the stalk is*

$$\{(f, \text{ open } U) : p \in U, f \in \mathscr{F}(U)\}$$

modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subseteq U, V$ where $p \in W$ and $\operatorname{res}_{U,W} f = \operatorname{res}_{V,W} g$.

Definition 3.2.3 (Stalk)

A useful equivalent definition of a stalk is as a colimit of all $\mathscr{F}(U)$ over all open sets U containing p:

$$\mathscr{F}_p = \varinjlim \mathscr{F}(U).$$

*If $p \in U$, and $f \in \mathscr{F}(U)$, then the image of f in $\mathscr{F}_p$ is called the **germ of f at** p.*

Remark The index category is filtered set (given any two such open sets, there is a third such set contained in both), so these two definitions are the same by Exercise 2.27. Hence we can define stalks for sheaves of sets, groups, rings, and other things for which colimits exist for directed sets. It is very helpful to keep both definitions of stalk in mind at the same time.

Remark Unlike Proposition 3.1.1, in general, the value of a section at a point doesn't make sense.

Definition 3.2.4 (Sheaf)

*A presheaf is a **sheaf** if it satisfies two more axioms. Notice that these axioms use the additional information of when some open sets cover another.*

*(1) **Identity axiom.** For any open set U, if $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$, and $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$.*
*(A presheaf satisfying the identity axiom is called a **separated presheaf**, but we will not use that notation in any essential way.)*

*(2) **Gluability axiom.** If $\{U_i\}_{i\in I}$ is an open cover of U, then given $f_i \in \mathscr{F}(U_i)$ for all i, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there is some $f \in \mathscr{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.*

Remark In mathematics, definitions often come paired: "at most one" and "at least one". In this case, identity means there is at most one way to glue, and gluability means that there is at least one way to glue.

Remark An additional axiom sometimes included is that $\mathscr{F}(\varnothing)$ is a one-element set, and in general, for a sheaf with values in a category, $\mathscr{F}(\varnothing)$ is required to be the final object in the category. This actually follows from the above definitions, assuming that the empty product is appropriately defined as the final object. For example, in the definition of sheaf of rings, $\mathscr{F}(\varnothing) = 0 -$ ring.

Example 3.1 If U and V are disjoint, then $\mathscr{F}(U \cap V) = \mathscr{F}(U) \times \mathscr{F}(V)$. Here we use the fact that $\mathscr{F}(\varnothing)$ is the final object.

Definition 3.2.5 (Stalk of a sheaf, germs of a section of a sheaf)

*The **stalk of a sheaf** at a point is just its stalk as a presheaf — the same definition applies — and similarly for the **germs of a section of a sheaf**.*

✍ **Exercise 3.2 (Presheaves that are not sheaves.)** Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves:

(a) bounded functions,

(b) holomorphic functios admitting a holomorphic square root.

Proof

(a) Let open set $U \subseteq \mathbb{C}$, and define $\mathscr{B}(U) = \{f : U \to \mathbb{C}$:f bounded on $U\}$. The restriction maps are function restrictions. Hence, $\mathscr{B}$ is a presheaf. We give a counterexample to show that $\mathscr{B}$ is not a sheaf. Let $U_n = D(n, 1)$ be an open disk, then $\{U_n\}$ is a countable open cover over $\mathbb{C}$. Let $f_n(z) = z$, then $f_n(z)$ bounded on each U_n, and therefore $f_n(z) \in \mathscr{B}(U_n)$. If $\mathscr{B}$ is a sheaf, there exists $f \in \mathscr{B}(\mathbb{C})$ such that $\mathrm{res}_{U,U_i} f = f_i$. This f must be $f(z)$, but $f(z)$ unbounded on $\mathbb{C}$, which a contradiction. Hence, $\mathscr{B}$ is not a sheaf.

(b) For each open set $U \subseteq \mathbb{C}$, define

$$\mathscr{G}(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic and there exists a holomorphic } g \text{ such that } g^2 = f\}.$$

The restriction maps are function restrictions. Hence, $\mathscr{G}$ is a presheaf. We give a counterexample to show that $\mathscr{G}$ is not a sheaf.

Let $U = \mathbb{C} \setminus \{0\}$, U is an open set of $\mathbb{C}$. Let $U_1 = \mathbb{C} \setminus [0, +\infty)$ and $U_2 = \mathbb{C} \setminus (-\infty, 0]$, so $\{U_1, U_2\}$ is the open cover of U. On U_1, define $f_1(z) = z$, with $g_1(z) = e^{\frac{1}{2}\mathrm{Log}(z)}$ (branch cut along $[0, +\infty)$), so $g_1^2 = z$. On U_2, define $f_2(z) = z$, with $g_2(z) = e^{\frac{1}{2}\mathrm{Log}(z)}$ (branch cut along $(-\infty, 0]$), so $g_2^2 = z$. On $U_1 \cap U_2 = \mathbb{C} \setminus \mathbb{R}$, $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2} = z$ and $g_1^2 = g_2^2 = z$. If there existed $f \in \mathscr{G}(\mathbb{C} \setminus \{0\})$ such that $f|_{U_i} = f_i$, then $f(z) = z$. However, $\sqrt{z}$ cannot be single-valued on $\mathbb{C} \setminus \{0\}$ (due to branch cut necessity), so no global holomorphic square root exists, leading to a contradiction. Hence, $\mathscr{G}$ is not a sheaf.

□

Definition 3.2.6 (Equalizer)

*Given two morphism $f, g : B \to C$, then their **equalizer** is an ordered pair (A, e) with $fe = ge$ that is universal with this property: if $q : X \to B$ satisfies $fq = gq$, then there exists a unique $q' : X \to B$ with $eq' = q$.*

$$\begin{tikzcd} A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\ X \arrow[u, dashed, "q'"] \arrow[ur, "q"'] & & \end{tikzcd}$$

More generally, if $(f_i : B \to C)_{i \in I}$ is a family of morphisms, then the equalizer is an ordered pair (A, e) with $f_i e = f_j e$ for all $i, j \in I$ that is universal with this property.

Example 3.2 In **Sets**, the equalizer is (E, e), where $E = \{b \in B : fb = gb\} \subseteq B$ and $e : E \to C$ is the inclusion. In $\mathbf{Mod}_R$, the equalizer of f and g is $(\mathrm{Ker}(f - g), e)$, where $e : \mathrm{Ker}(f - g) \to B$ is the inclusion. Hence, $\mathrm{Ker}\, f$ is the equalizer of f and 0.

Example 3.3 (**Interpretation in terms of the equalizer exact sequence.**) The two axioms for a presheaf to be a sheaf can be interpreted as "exactness" of the "equalizer exact sequence":

$$\cdot \longrightarrow \mathscr{F}(U) \longrightarrow \prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j).$$

Identity axiom ensures that $\mathscr{F}(U) \to \prod \mathscr{F}(U_i)$ is injective. Gluability axiom implies that the equalizer of the two maps $\prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j)$ (two maps are $(s_i) \mapsto (s_i|_{U_i \cap U_j})$ and $(s_i) \mapsto (s_j|_{U_i \cap U_j})$) must coincide exactly with the image of $\mathscr{F}(U)$.

Exercise 3.3 The identity and gluability axioms may be interpreted as saying that $\mathscr{F}\left(\bigcup_{i \in I} U_i\right)$ is a certain limit. What is that limit?

Proof $\mathscr{F}(\bigcup_{i \in I} U_i)$ is the equalizer of $\prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j)$, the equalizer is a limit (by the definition of equalizer, it is clear). □

3.2.2 Example of sheaves

Here are number of example of sheaves.

Sheaves of functions

Exercise 3.4

(a) Verify that the example of §3.1 are indeed sheaves (of smooth functions, or continuous functions, or real-analytic functionsm, or plain real-valued functions, on a manifold or $\mathbb{R}^n$).

(b) Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

Proof

(a) Let M be a smooth mainifold. For any open set $U \subseteq M$, define:

$$C^\infty(U) = \{f : U \to \mathbb{R} : f \text{ is smooth}\}.$$

The restriction maps are the natural restrictions of functions: $\mathrm{res}_{U,V} : C^\infty(U) \to C^\infty(V)$ for $V \subseteq U$.
Let $U \subseteq M$ be an open set, $\{U_i\}$ is an open cover of U, and $f \in C^\infty(U)$ satisfies $f|_{U_i} = 0$ for all i, we need to show that $f = 0$. Let $x \in U$, there exists some U_i containing x. Since $f|_{U_i} = 0$, $f(x) = 0$. Hence, $f = 0$.
Let $f_i \in C^\infty(U_i)$ satisfies $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, we shall to show that there exists a unique $f \in C^\infty(U)$ with $f|_{U_i} = f_i$.
For existence. Define $f : U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ when $x \in U_i$. The condition $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j ensures f is well-defined. For any $x \in U$, there exists some $i \in I$ such that $x \in U_i$. By assumption, $f|_{U_i}$ is smooth. Thus, there exists a coordinate chart (V_x, φ_x) containing x (with $V_x \subseteq U_i$ and $\varphi_x : V_x \to \varphi_x(V_x) \subseteq \mathbb{R}^n$ homeomorphism) such that:

$$f|_{U_i} \circ \varphi_x^{-1} : \varphi_x(V_x) \to \mathbb{R}$$

is smooth. Since M is a smooth manifold, its coordinate chart over the entire manifold. for each $x \in U$, choose the chart (V_x, φ_x) as above, where $V_x \subseteq U_i$ and $x \in V_x$. These charts $\{V_x\}_{x \in U}$ form an open cover of U. For any $x \in U$, the local representation of f in the chart (V_x, φ_x) is:

$$f \circ \varphi_x^{-1} = (f|_{U_i}) \circ \varphi_x^{-1},$$

and the right-hand side is smooth, by the previous discussion. Therefore, $f \circ \varphi_x^{-1}$ is smooth on $\varphi_x(V_x)$. Note that a function is smooth on a manifold if and only if it is smooth in every coordinate chart, $f \in C^\infty(U)$.
For uniqueness. If there is another $g \in C^\infty(U)$ with $g|_{U_i} = f_i$. For all $x \in U$,

$$g(x) = g|_{U_i}(x) = f_i(x) = f|_{U_i}(x) = f(x),$$

hence $f = g$.
Consequently, C^∞ is indeed a sheaf.

(b) Let X be a topological space X. For any open set $U \subseteq X$, define:

$$\mathscr{F}(U) = \{f : U \to \mathbb{R} : f \text{ is continuous}\}.$$

The restriction maps are the natural restriction of functions: $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ for $V \subseteq U$.
Let $U \subseteq X$ be an open set, $\{U_i\}$ is an open cover of U, and $f \in \mathscr{F}(U)$ satisfies $f|_{U_i} = 0$ for all i, we need to show that $f = 0$. Let $x \in U$, there exists some U_i containing x. Since $f|_{U_i} = 0$, $f(x) = 0$. Hence, $f = 0$.
Let $f_i \in \mathscr{F}(U_i)$ satisfies $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, we shall to show that there exists a unique $f \in \mathscr{F}(U)$ with $f|_{U_i} = f_i$.
For existence. Define $f : U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ when $x_i \in U_i$. The condition $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j ensures f is well-defined. Now, we need to show that f is continuous. Let $V \subseteq \mathbb{R}$

open set, note that $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$, since each f_i is continuous, $f_i^{-1}(V)$ is open, and therefore $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$ is open. Hence, f is continuous, i.e., $f \in \mathscr{F}(U)$.
For uniqueness. If there is another $g \in \mathscr{F}(U)$ with $g|_{U_i} = f_i$. For all $x \in U$,

$$g(x) = g|_{U_i}(x) = f_i(x) = f|_{U_i}(x) = f(x),$$

hence $f = g$.

□

Definition 3.2.7 (Restriction of a sheaf)

*Suppose $\mathscr{F}$ is a sheaf on X, and U is an open subset of X. Define the **restriction of $\mathscr{F}$ to** U, denoted $\mathscr{F}|_U$, to be the collection $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for all open subsets $V \subseteq U$. Clearly this is a sheaf on U.*

Remark In fact, we can restrict sheaves to arbitrary subsets.

The skyscraper sheaf

Example 3.4 **(The skyscraper sheaf.)** Suppose X is a topological space, with $p \in X$, and S is a set. Let $i_p : p \to X$ be the inclusion. Then $i_{p,*}S$ defined by

$$i_{p,*}S(U) = \begin{cases} S & \text{if } p \in U, \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf.

Let $V \subseteq U$, the restriction map defined by

$$f|_V = \begin{cases} e & \text{if } p \notin U, \\ \text{if } p \in U & \begin{cases} f & \text{if } p \in V \subseteq U, \\ e & \text{if } p \notin V \subseteq U. \end{cases} \end{cases}$$

It is easy to see that $i_{p,*}S$ is a presheaf. For any open set $U \subseteq X$, if $\{U_i\}_{i\in I}$ is an open cover of U, and $f_1, f_2 \in i_{p,*}U$, and $f_1|_{U_i} = f_2|_{U_i}$ for all i, by the definition of restriction map, $f_1 = f_2$. If $f_i \in i_{p,*}(U_i)$ for all i with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, also by the definition of restriction map, there is some $f \in i_{p,*}S(U)$ such that $f|_{U_i} = f_i$. Hence, $i_{p,*}S$ is a sheaf.

This is called a **skyscraper sheaf** supported at p, because the informal picture of it looks like a skyscarper at p. (Mild caution: this informal picture suggests that the only nontrivial stalk of a skyscraper sheaf is at p, which isn't case.)

There is an analogous definition for sheaves of abelian groups, except $i_{p,*}S(U)$ should be a final object.

Constant presheaf and constant sheaf

Definition 3.2.8 (Constant presheaf)

*Let X be a topological space, and S a set. Define $\underline{S}_{\text{pre}}(U) = S$ for all open sets U. $\underline{S}_{\text{pre}}$ forms a presheaf (the restriction map is the identity). This is called the **constant presheaf associated to** S.*

Remark This isn't (in general) a sheaf.

Definition 3.2.9 (Constant sheaf)

Let X be a topological space, and S a set. Let $\mathscr{F}(U)$ be the set of maps $U \to S$ that are ***locally constant****, i.e., for any point p in U, there is an open neighborhood of p where the function is constant. $\mathscr{F}$ is a sheaf. This is called the* ***constant sheaf*** *(with values in S). We denote this sheaf $\underline{S}$.*

(A better description is this: endow S with the discrete topology, and let $\mathscr{F}(U)$ be the continuous maps $U \to S$.)

Remark DO NOT CONFUSE IT WITH THE CONSTANT PRESHEAF!

✍ **Exercise 3.5** Show that constant sheaf is a sheaf.

Proof By the definition of constant sheaf, it is easy to see that constant sheaf is a presheaf. Let U be an open set of X, and $\{U_i\}$ is the open cover of U. If $s_i \in \mathscr{F}(U_i)$ for all i with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Defined $s(p) = s_i(p)$ when $p \in U_i$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ ensures that s is well-defined. We need to show that s is locally constant. For any $x \in U$, there is a U_i countaining x. Since $s_i \in \mathscr{F}(U_i)$, there exist $p \in V \subseteq U_i$ such that $s_i : V \to S$ is constant, and therefore $s : V \to S$ is constant. Hence, gluability axiom holds.

If $s_1|_{U_i} = s_2|_{U_i}$ for all i. For all $x \in U$ there exists U_i such that $x \in U_i$, and therefore $s_1(x) = s_1|_{U_i}(x) = s_2|_{U_i}(x) = s_2(x)$, which implies the identity axiom.

Hence, constant sheaf is a sheaf. □

Morphisms glue and sheaf of sections of a map

✍ **Exercise 3.6** **("Morphisms glue.")** Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of U to Y. Show that this forms a sheaf.

Proof Let U be an open set of X, and $\{U_i\}_i$ is the open cover of U. Define

$$\mathscr{F}(U) = \{f : U \to Y : f \text{ is continuous}\}.$$

The restriction maps are the natural restriction of functions: $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ for $V \subseteq U$.

Let $f \in \mathscr{F}(U)$ satisfies $f|_{U_i} = 0$ for all i, we need to show that $f = 0$. Let $x \in U$, there exists some U_i containing x. Since $f|_{U_i} = 0$, $f(x) = 0$. Hence, $f = 0$, which implies identity axiom.

Let $f_i \in \mathscr{F}(U_i)$ satisfies $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, we shall to show that there exists $f \in \mathscr{F}(U)$ with $f|_{U_i} = f_i$.

Define $f : U \to Y$ by setting $f(x) = f_i(x)$ when $x_i \in U_i$. The condition $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j ensures f is well-defined. Now, we need to show that f is continuous. Let $V \subseteq Y$ be an open set, then $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$. Since each f_i is continuous, each $f_i^{-1}(V)$ is open, and therefore $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$ is open. Hence, f is continuous, i.e., $f \in \mathscr{F}(U)$. Hence, gluability axiom holds.

Hence, $\mathscr{F}$ is a sheaf. □

Remark Exercise 3.4, with $Y = \mathbb{R}$, and Exercise 3.5, with $Y = S$ with the discrete topology, are both special cases.

This is a fancier version of the previous exercise.

✍ **Exercise 3.7**

(a) **(Sheaf of sections of a map.)** Suppose we are given a continuous map $\mu : Y \to X$. Show that "section of μ" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps $s : U \to Y$ such that $\mu \circ s = \mathrm{id}|_U$. Show that this forms a sheaf. This is motivation for the phrase "section of a sheaf". (vector bundles)

(b) Suppose that Y is a topological group. Show that continuous maps to Y form a sheaf of groups.

Proof

(a) Let $U \subseteq X$ be an open set, and $\{U_i\}_i$ is the open cover of U. Define $\mathscr{F}(U) = \{s : U \to Y : s \text{ continuous and } \mu \circ s = \mathrm{id}\,|_U\}$. The restriction maps are the natural restriction of functions.

Let $f \in \mathscr{F}(U)$ satisfies $f|_{U_i} = 0$ for all i, we need to show that $f = 0$. Let $x \in U$, there exists some U_i containing x, then $f(x) = f|_{U_i}(x) = 0$, which implies identity axiom.

Let $f_i \in \mathscr{F}(U_i)$ satisfies $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, we shall to show that there exists $f \in \mathscr{F}(U)$ with $f|_{U_i} = f$.

Define $f : U \to Y$ by setting $f(x) = f_i(x)$ when $x_i \in U_i$. The condition ensures f is well-defined. Now, we need to show that $f \in \mathscr{F}(U)$, that is, f is continuous and $\mu \circ f = \mathrm{id}_U$. Same as the proof in Exercise 3.6, f is continuous. Let $x \in U$, then there exists U_i containing x. Hence, $\mu \circ f(x) = \mu \circ f|_{U_i}(x) = x$, since $\mu \circ f|_{U_i} = \mathrm{id}_{U_i}$. This implies that $\mu \circ f = \mathrm{id}_U$, and therefore $f \in \mathscr{F}(U)$.

Consequently, $\mathscr{F}$ is a sheaf.

(b) It suffices to show that $\mathscr{F}(U)$ is a group and the restriction maps are group homomorphism. And apply the same method of part (a), $\mathscr{F}$ is a sheaf.

Define the multiplication on $\mathscr{F}(U)$ by setting $(f \cdot g)(x) = f(x)g(x)$ for all $x \in U$. The inverse of $f \in \mathscr{F}(U)$ given by $f^{-1}(x) = (f(x))^{-1}$. Let $e_Y : U \to Y$ defined by $e_Y(x) = e$, where e is the unite element in Y. Since $(e_Y \cdot f)(x) = e_Y(x)f(x) = e \cdot f(x) = f(x)$ and $(f \cdot e_Y)(x) = f(x)e_Y(x) = f(x) \cdot e = f(x)$, e_Y is the unite element in $\mathscr{F}(U)$. Hence, $\mathscr{F}(U)$ is a group.

Let $V \subseteq U$ and $f, g \in \mathscr{F}(U)$, then

$$\mathrm{res}_{U,V}(f \cdot g)(x) = (f \cdot g)|_V(x) = (f \cdot g)(x) = f(x)g(x) = f|_V(x)g|_V(x)$$

for all $x \in V$. Hence, the restriction maps are the group homomorphism.

□

The space of sections (espace étalé) of a (pre)sheaf

Depending on your background, you may prefer the following perspective on sheaves.

Definition 3.2.10 (Space of sections (espace étalé))

Suppose $\mathscr{F}$ is a presheaf of sets (e.g., a sheaf) on a topological space X. Construct a topological space F along with a continuous map $\pi : F \to X$ as follows: as a set, F is the disjoint union of all the stalks of $\mathscr{F}$, i.e.,

$$F = \coprod_{x \in X} \mathscr{F}_x.$$

*This naturally gives a map of sets $\pi : F \to X$ (see remark). Topologize F as follows. Each $s \in \mathscr{F}(U)$ determines a subset $\{(x, s_x) : x \in U\}$ of F. The topology on F is the weakest topology such that these subsets are open. The topological space F could be thought of as the **space of sections of** $\mathscr{F}$ (and in French is called the **espace étalé**).*

Remark

(1) Each element $s_x \in F$ must belongs to an unique $\mathscr{F}_x$, since F is the disjoint union of $\mathscr{F}_x$. Formally, each $s_x \in F$ can be written as (x, s_x), then this naturally gives a map of sets $\pi : F \to X$ by setting $\pi((x, s_x)) = x$.

(2) Subsets $\{(x, s_x) : x \in U\}$ form a base of the topology. For each $y \in F$, there is an open neighborhood V of y and an open neighborhood U of $\pi(y)$ such that $\pi|_V$ is a homeomorphism from V to U.

Proof Let $y \in F$, then $y = (x, s_x)$ for some $x \in U \subseteq X$. Hence, $y \in \{(x, s_x) : x \in U\}$. Let $B_s = \{(x, s_x) : x \in U\}$, $B_t = \{(x, t_x) : x \in V\}$, and $B_s \cap B_t \neq \varnothing$. Say $y = (x, s_x) = (x, t_x) \in B_s \cap B_t$, then as germs $s_x = t_x$, hence, there exists $W \subseteq U \cap V$ such that $s|_W = t|_W$. Let $B = \{(x, (s|_W)_x) : x \in W\}$, then $y \in B \subseteq B_s \cap B_t$. Hence, subsets $\{(x, s_x) : x \in U\}$ form a base of the topology.

Let $y \in F$, then $y = (x_0, s_{x_0})$ for some $x_0 = \pi(y)$ and $s_{x_0} \in \mathscr{F}_{x_0}$. Then there exist an open set $U \subseteq X$ containg x_0 such that s_{x_0} is the image of s in $\mathscr{F}_{x_0}$. Let $V = \{(x, s_x) : x \in U\}$, then V is an open neighborhood of y. We claim that $\pi|_V : V \to U$ is a homeomorphism. By the construction of V, $\pi|_V$ is a bijection. Let $V_0 \subseteq V$ be an open subset, then V_0 is the union some bases of the topology. Hence, $\pi|_V(V_0)$ is the union of some open sets of X, and therefore is open. Let $W \subseteq U$ be an open set, then $\pi|_V^{-1}(W) = \{(x, s_x) : x \in W\}$, it is an open set. Hence, $\pi|_V : V \to U$ is a homeomorphism. □

Remark We will not discuss this construction at any length, but it can have some advantages:

(a) It is always better to know as many ways as possible of thinking about a concept.

(b) "Inverse image" (informally, "pullback") has a natural interpretation in this language.

(c) Sheafification has a natural interpretation in this language.

The pushforward sheaf

Definition 3.2.11 (The pushforward sheaf or direct image sheaf)

Suppose $\pi : X \to Y$ is a continuous map, and $\mathscr{F}$ is a presheaf on X. Then define a presheaf $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$, where V is an open subset of Y. $\pi_*\mathscr{F}$ is a presheaf on Y, and is a sheaf if $\mathscr{F}$ is. This is called the ***pushforward (or direct image)*** *of $\mathscr{F}$. More precisely, $\pi_*\mathscr{F}$ is called the* ***pushforward of $\mathscr{F}$ by*** *π.*

Exercise 3.8 Show that $\pi_*\mathscr{F}$ is a presheaf on Y, and is a sheaf if $\mathscr{F}$ is.

Proof

(1) $\pi_*\mathscr{F}$ is a presheaf.

Let $V_1 \supseteq V_2$ be open sets of Y. We need to show that the restriction map $\mathrm{res}_{V_1,V_2} : \pi_*\mathscr{F}(V_1) \to \pi_*\mathscr{F}(V_2)$ is well-defined ($\mathrm{res}_{V_1,V_2}(f) = \mathrm{res}_{\pi^{-1}(V_1),\pi^{-1}(V_2)}(f)$, RHS as the restriction map of sheaf $\mathscr{F}$). Note that $\pi_*\mathscr{F}(V_1) = \mathscr{F}(\pi^{-1}(V_1))$ and $\pi_*\mathscr{F}(V_1) = \mathscr{F}(\pi^{-1}(V_2))$, since π is continuous, $\pi^{-1}(V_1)$ and $\pi^{-1}(V_2)$ are open, and $\pi^{-1}(V_1) \supseteq \pi^{-1}(V_2)$. Hence, the restriction map is well-defined, and therefore $\pi_*\mathscr{F}$ is a presheaf.

(2) If $\mathscr{F}$ is a sheaf, $\pi_*\mathscr{F}$ is a sheaf.

By part (1), we know $\pi_*\mathscr{F}$ is a presheaf. It suffices to check identity axiom and gluability axiom.

Let V be any open subset of Y, $\{V_i\}_{i\in I}$ is an open cover of V. Let $f_1, f_2 \in \pi_*\mathscr{F}(V)$ and $f_1|_{\pi^{-1}(V_i)} = f_2|_{\pi^{-1}(V_i)}$ for all i. Since $\mathscr{F}$ is a sheaf and $\{\pi^{-1}(V_i)\}_i$ is an open cover of $\pi^{-1}(V)$, $f_1 = f_2$ for $f_i \in \mathscr{F}\pi^{-1}(V) = \pi_*\mathscr{F}(V)$, which holds identity axiom.

Let $f_i \in \pi_*\mathscr{F}(V_i)$ for all i, such that $f_i|_{\pi^{-1}(V_i)\cap\pi^{-1}(V_j)} = f_j|_{\pi^{-1}(V_i)\cap\pi^{-1}(V_j)}$ for all i, j. Since $\mathscr{F}$ is a sheaf and $\{\pi^{-1}(V_i)\}_i$ is an open cover of $\pi^{-1}(V)$, there exists $f \in \mathscr{F}(\pi^{-1}(V)) = \pi_*\mathscr{F}(V)$ such that $\mathrm{res}_{\pi^{-1}(V),\pi^{-1}(V_i)}(f) = f_i$ for all i. Hence, $\pi_*\mathscr{F}$ is a sheaf.

□

Example 3.5 The skyscraper sheaf can be interpreted as the pushforward of the constant sheaf $\underline{S}$ on a one-point space p, under the inclusion morphism $i_p : \{p\} \to X$.

Proof Let $U \subseteq X$ be an open set, then

$$i_p^{-1}(U) = \begin{cases} \{p\} & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

Hence,

$$i_{p,*}\underline{S}(U) = \underline{S}(i_p^{-1}(U)) = \begin{cases} S & \text{if } p \in U \\ 0 & \text{if } 0 \notin U. \end{cases}$$

This is agree with the definition of the sky scraper sheaf. □

Remark Once we endow sheaves with the structure of a category, we will see that the pushforward is a functor from sheaves on X to sheaves on Y.

Proposition 3.2.2 (Pushforward induces maps of stalks)

Suppose $\pi : X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf of sets (or rings or A-modules) on X. If $\pi(p) = q$, there is a natural morphism of stalks $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$.

Proof In fact, $(\pi_*\mathscr{F})_q = \varinjlim_{V\ni q} \pi_*\mathscr{F}(V) = \varinjlim_{V\ni q} \mathscr{F}(\pi^{-1}(V))$. For all $q \in V \subseteq Y$ open set, $\pi^{-1}(V)$ is an open set of X and containing p, since π is continuous and $\pi(p) = q$. The open subset of Y which containing q given a partial ordered set by inclusion, then $\{\mathscr{F}(\pi^{-1}(V)), V \ni q\}$ with restriction maps give a direct system. For all $V \ni q$, define $\iota_V : \pi_*\mathscr{F}(V) \to \mathscr{F}_p$ by setting $\iota_V(f) = \overline{(f, \pi^{-1}(V))}$, it is easy to see this map is well-defined. Now, let $V' \supseteq V$, and consider the following diagram.

$$\begin{tikzcd}
\varinjlim_{V\ni q} \pi_*\mathscr{F}(V) \arrow[rr, dashed, "\theta"] & & \mathscr{F}_p \\
& \pi_*\mathscr{F}(V') \arrow[ul] \arrow[ur, "\iota_{V'}"'] \arrow[d, "\mathrm{res}_{V',V}"'] & \\
& \pi_*\mathscr{F}(V) \arrow[uul, bend left] \arrow[uur, bend right, "\iota_V"'] &
\end{tikzcd}$$

By the universal property of colimit, there exists unique morphism $\theta : (\pi_*\mathscr{F})_q \to \mathscr{F}_p$ maps the image of section in $(\pi_*\mathscr{F})_q$ to the image of section in $\mathscr{F}_p$. □

Ringed spaces, and $\mathscr{O}_X$-modules

Definition 3.2.12 (Ringed spaces, structure sheaf)

Suppose $\mathscr{O}_X$ is a sheaf of rings on a topological space X (i.e., a sheaf on X with values in the category of **Rings***). Then $(X, \mathscr{O}_X)$ is called a* ***ringed space****. The sheaf of rings is often denoted by $\mathscr{O}_X$. This sheaf is called the* ***structure sheaf*** *of the ringed space. Sections of the structure sheaf $\mathscr{O}_X$ over an open subset U are called* ***functions on*** *U. Functions on X are called* ***global functions****, or just* ***functions****.*

Remark what we call "functions", others sometimes call "regular functions". Furthermore, we will later define "rational functions" on schemes, which are not precisely function in this sense, they are a particular type of "partially-defined function".

Note *The symbol $\mathscr{O}_X$ will always refer to the structure sheaf of a ringed space X. The restriction $\mathscr{O}_X|_U$ of $\mathscr{O}_X$ to an open subset $U \subseteq X$ is denoted $\mathscr{O}_U$. (We will later call $(U, \mathscr{O}_U) \to (X, \mathscr{O}_X)$ an open embedding of ringed* *spaces.) The stalk of $\mathscr{O}_X$ at a point p is written "$\mathscr{O}_{X,p}$".*

Just as we have modules over a ring, we have $\mathscr{O}_X$-modules over a sheaf of rings $\mathscr{O}_X$. There is only one possible definition that could go with the name $\mathscr{O}_X$-module (or often $\mathscr{O}$-module) — a sheaf of abelian groups $\mathscr{F}$ with the following additional structure.

Definition 3.2.13 ($\mathscr{O}_X$-module)

Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups on X together with maps

$$\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$$

for each open $U \subseteq X$, giving each $\mathscr{F}(U)$ the structure of an $\mathscr{O}_X(U)$-module, and which are compatible with the restriction maps for open subsets $U \subseteq V \subseteq X$, i.e., diagram

$$\begin{tikzcd} \mathscr{O}_X(V) \times \mathscr{F}(V) \arrow[r, "action"] \arrow[d, "\mathrm{res}^{\mathscr{O}_X}_{V,U} \times \mathrm{res}^{\mathscr{F}}_{V,U}"'] & \mathscr{F}(V) \arrow[d, "\mathrm{res}^{\mathscr{F}}_{V,U}"] \\ \mathscr{O}_X(U) \times \mathscr{F}(U) \arrow[r, "action"] & \mathscr{F}(U) \end{tikzcd} \tag{3.1}$$

commutes.

Recall that the notion of A-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. Similarly, the notion of $\mathscr{O}_X$-module generalize the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module. Hence when we are proving things about $\mathscr{O}_X$-modules, we are also proving things about sheaves of abelian groups.

Exercise 3.9 If $(X, \mathscr{O}_X)$ is a ringed space, and $\mathscr{F}$ is an $\mathscr{O}_X$-module, describe how for each $p \in X$, $\mathscr{F}_p$ is an $\mathscr{O}_{X,p}$-module.

Proof Let $U \ni p$ be an open set of X, $r \in \mathscr{O}_X(U)$, $m \in \mathscr{F}(U)$, and denote r_p be the image of r in $\mathscr{O}_{X,p}$ and m_p be the image of m in $\mathscr{F}_p$. Since $\mathscr{F}$ is an $\mathscr{O}_X$-module, this gives action of $\mathscr{O}_X(U)$ on $\mathscr{F}(U)$, say $r \cdot m = rm$. Define the action of $\mathscr{O}_{X,p}$ on $\mathscr{F}_p$ by setting $r_p \cdot m_p = (rm)_p$. We need to show this action is well-defined. Let $r_p \sim r'_p$ and $m_p \sim m'_p$, where $r' \in \mathscr{O}_X(U')$ and $m' \in \mathscr{F}(U')$, then there exist $W \subseteq U \cap U', K \subseteq U \cap U'$ open sets which containing p such that $r|_W = r'|_W$ and $m|_K = m'|_K$. Let $V = W \cap K$, then $r|_V = r'|_V$ and $m|_V = m'|_V$. Hence, $(rm)|_V = (r'm')|_V$, and therefore $(rm)_p = (r'm')_p$, which implies that action is well-defined. The module axiom given by $\mathscr{O}_X$-module $\mathscr{F}$, hence, $\mathscr{F}_p$ is an $\mathscr{O}_{X,p}$-module. $\square$

Example 3.6 **(For those who know about vector bundles.)** The motivating example of $\mathscr{O}_X$-modules is the sheaf of sections of a vecter bundle. If $(X, \mathscr{O}_X)$ is a differentiable manifold (so $\mathscr{O}_X$ is the sheaf of smooth functions), and $\pi : V \to X$ is a vector bundle over X, then the sheaf of smooth sections $\sigma : X \to V$ is an $\mathscr{O}_X$-module.

Proof Let $U \subseteq X$ be any open set of X, the sheaf of sections of a vector bundle defined by

$$\mathscr{F}(U) = \{s : U \to V : \pi \circ s = \mathrm{id}_U, s \text{ is smooth}\}.$$

The restriction given by the restriction of section. By Exercise 3.7, $\mathscr{F}$ indeed a sheaf. Now, we give an $\mathscr{O}_X$-module structure on $\mathscr{F}$ as follow: for any open set $U \subseteq X$, define $\cdot : \mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ by setting $(f, s) \mapsto f \cdot s$, where $(f \cdot s)(x) = f(x) \cdot s(x)$ (RHS, is the scalar multiplication in the fiber $\pi^{-1}(x)$). We need to show that $\cdot$ is well-defined. Consider $\pi \circ (f \cdot s)$, let $x \in U$, then $\pi \circ (f \cdot s)(x) = \pi(f(x)s(x)) = x$ (since $f(x)s(x) \in \pi^{-1}(x)$), hence, $f \cdot s \in \mathscr{F}(U)$ (f is smooth), which implies that "$\cdot$" is well-defined. Then it is easy to check that $\mathscr{F}$ is an $\mathscr{O}_X$-module. $\square$

3.3 Morphisms of presheaves and sheaves

Whenever one defines a new mathematical object, category theory teaches to try to understand maps between them. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will describe the **category of presheaves** (of sets, abelian groups, etc.) and the **category of sheaves**.

3.3.1 Definition of morphisms of (pre)sheaves

Definition 3.3.1 (Morphism of presheaves)

A ***morphism of presheaves*** *of sets (or indeed of presheaves with values in any category) on* X, $\varphi : \mathscr{F} \to \mathscr{G}$, *is the data of maps* $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ *for all open sets* $U \subseteq X$ *behaving well with respect to restriction: if* $U \hookrightarrow V$, *then*

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \end{array}$$

commutes. (Notice: the underlying space of both $\mathscr{F}$ *and* $\mathscr{G}$ *is* X.)

Definition 3.3.2 (Morphism of sheaves)

A ***morphisms of sheaves*** *of sets on* X, $\varphi : \mathscr{F} \to \mathscr{G}$, *is the data of maps* $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ *for all open sets* $U \subseteq X$ *behaving well with respect to restriction: if* $U \hookrightarrow V$, *then*

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \end{array}$$

commutes.

Definition 3.3.3 (Morphisms of $\mathscr{O}_X$-modules)

An $\mathscr{O}_X$*-module homomorphism* $\mathscr{F} \to \mathscr{G}$ *between* $\mathscr{O}_X$*-modules* $\mathscr{F}, \mathscr{G}$ *on* X *is a sheaf morphism* $\mathscr{F} \to \mathscr{G}$ *such that for all open subsets* $U \subseteq X$, *the map* $\mathscr{F}(U) \to \mathscr{G}(U)$ *is a homomorphism of* $\mathscr{O}_X(U)$*-modules.*

Example 3.7 An example of a morphism of sheaves is the map from the sheaf of smooth functions on $\mathbb{R}$ to the sheaf of continuous functions. This is a "forgetful map": we are forgetting that these functions are differentiable, and remembering only that they are continuous.

Note *We may as well set some notation:*

(1) $\mathbf{Sets}_X$, $\mathbf{Ab}_X$, *etc. denote the category of sheaves of sets, abelian groups, etc. on a topological space* X

(2) Let $\mathbf{Mod}_{\mathscr{O}_X}$ *denote the category of* $\mathscr{O}_X$*-modules on a ringed space* $(X, \mathscr{O}_X)$.

(3) Let $\mathbf{Sets}_X^{pre}$, *etc. denote the category of presheaves of sets, etc. on* X.

Remark If we interpret presheaf on X as a contravariant functor (from the category of open sets), a morphism of presheaves on X is a natural transformation of functor.

Proposition 3.3.1 (Morphisms of (pre)sheaf induce morphisms of stalks)

If $\varphi : \mathscr{F} \to \mathscr{G}$ *is a morphism of (pre)sheaves on* X, *and* $p \in X$, *then* φ *induces a morphism of stalks* $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$. *Translation: taking the stalk at* p *induces a* ***stalkification functor*** $\mathbf{Sets}_X \to \mathbf{Sets}$.

Proof For all $U' \supseteq U$ open sets of X, consider the following diagram.

$$\begin{tikzcd} \mathscr{F}_p \arrow[rrr, dashed] & & & \mathscr{G}_p \\ & \mathscr{F}(U') \arrow[ul, "\pi^{\mathscr{F}}_{U'}"'] \arrow[r, "\varphi_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"'] & \mathscr{G}(U') \arrow[ur, "\pi^{\mathscr{G}}_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U',U}"] & \\ & \mathscr{F}(U) \arrow[uul, bend left, "\pi^{\mathscr{F}}_{U}"] \arrow[r, "\varphi_U"] & \mathscr{G}(U) \arrow[uur, bend right, "\pi^{\mathscr{G}}_{U}"'] & \end{tikzcd}$$

By the universal property of colimit, there exists unique morphism $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$. $\square$

Remark This proof will extend in obvious ways. For example, if φ is a morphism of $\mathscr{O}_X$-modules, then φ_p is a map of $\mathscr{O}_{X,p}$-modules.

✍ **Exercise 3.10** Suppose $\pi : X \to Y$ is a continuous map of topological spaces (i.e., a morphism in the category of topological spaces). Show that pushforward gives a functor $\pi_* : \mathbf{Sets}_X \to \mathbf{Sets}_Y$. Here $\mathbf{Sets}$ can be replaced by other categories.

Proof Let $\mathscr{F} \in \mathbf{Sets}_X$, then $\pi_*\mathscr{F} \in \mathbf{Sets}_Y$, by Exercise 3.8. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaf in $\mathbf{Sets}_X$. Define $\pi_*\varphi : \pi_*\mathscr{F} \to \pi_*\mathscr{G}$ by setting $\pi_*\varphi_V = \varphi_{\pi^{-1}(V)}$, where $V \subseteq Y$ be open set in Y. We need to show that $\pi_*\varphi$ is a morphism of sheaf between $\pi_*\mathscr{F}$ and $\pi_*\mathscr{G}$. Let $V' \supseteq V$, then $\pi^{-1}(V') \supseteq \pi^{-1}(V)$. Since φ is the morphism of sheaf, we have the following commutative diagram.

$$\begin{tikzcd} \mathscr{F}(\pi^{-1}(V')) \arrow[r, "\varphi_{\pi^{-1}(V')}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{\pi^{-1}(V'),\pi^{-1}(V)}"'] & \mathscr{G}(\pi^{-1}(V')) \arrow[d, "\mathrm{res}^{\mathscr{G}}_{\pi^{-1}(V'),\pi^{-1}(V)}"] \\ \mathscr{F}(\pi^{-1}(V)) \arrow[r, "\varphi_{\pi^{-1}(V)}"] & \mathscr{G}(\pi^{-1}(V)) \end{tikzcd}$$

Hence, diagram

$$\begin{tikzcd} \pi_*\mathscr{F}(V') \arrow[r, "\pi_*\varphi_{V'}"] \arrow[d, "\mathrm{res}^{\pi_*\mathscr{F}}_{V',V}"'] & \pi_*\mathscr{G}(V') \arrow[d, "\mathrm{res}^{\pi_*\mathscr{G}}_{V',V}"] \\ \pi_*\mathscr{F}(V) \arrow[r, "\pi_*\varphi_V"] & \pi_*\mathscr{G}(V) \end{tikzcd}$$

commutes. This implies that $\pi_*\varphi$ is a morphism of sheaf. Let $\mathrm{id}_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \in \mathbf{Sets}_X$ be identity morphism of sheaf, then $\pi_* \mathrm{id}_{\mathscr{F}} = \mathrm{id}_{\pi_*\mathscr{F}}$ is also an identity morphism of sheaf. Finally, it is easy to check π_* preserves composition. Hence, pushforward gives a functor $\pi_* : \mathbf{Sets}_X \to \mathbf{Sets}_Y$. $\square$

Definition 3.3.4 (Sheaf Hom)

Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X. (In fact, it will suffice that $\mathscr{F}$ is a presheaf.) Let $\mathscr{H}om(\mathscr{F},\mathscr{G})$ be the collection of data

$$\mathscr{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U).$$

*This is a sheaf of sets on X. This sheaf is called "**sheaf** $\mathscr{H}om$".*

Remark

(1) Strictly speaking, we should reserve Hom for when we are in an additive category, so this should possibly be called "sheaf Mor". But the terminology "sheaf $\mathscr{H}om$" is too established to uproot.

(2) It will be clear from your construction that, like Hom, $\mathscr{H}om(\square, \mathscr{G})$ is contravariant functor and $\mathscr{H}om(\mathscr{F}, \square)$ is covariant functor.

Remark $\mathscr{H}om$ does not commute with taking stalks. More precisely: it is not ture that $\mathscr{H}om(\mathscr{F},\mathscr{G})_p$ is isomorphic to $\mathrm{Mor}(\mathscr{F}_p, \mathscr{G}_p)$.

Example 3.8 **(Counterexample of above remark.)** Let H be a non-trivial abelian group. Let $\mathscr{G}$ be the constant sheaf H (i.e., $\mathscr{G} = \underline{H}$), and let $\mathscr{F}$ be the skyscraper sheaf at p with stalk H. We claim that $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is the zero sheaf. Let $U \subseteq X$ be any connected open set of X (if U is not connected, we only need to consider all connected components of U, as Exercise 3.12 (b)).

If $p \notin U$, for all $V \subseteq U$ open set, $p \notin V$, then $\mathscr{F}|_U(V) = \{0\}$. Hence,

$$\mathscr{H}om(\mathscr{F},\mathscr{G})(U) = \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U) = \{0\}.$$

If $p \in U$, we need to show that $\mathrm{Mor}_{\mathbf{Ab}}(\mathscr{F}|_U, \mathscr{G}|_U) = \{0\}$. Let $V \subseteq U$ be any open subset. If $V \ni p$, then $\mathscr{F}(V) = H, \mathscr{G}(V) = H$, and therefore $\varphi_V : H \to H$ is a group homomorphism. If $p \notin V$, then $\mathscr{F}(V) = 0$, φ_V must be zero homomorphism.

Let $p \in V \subseteq U$ and $p \notin W$ is an open subset of V. Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{V,W}"'] & \mathscr{G}(V) \arrow[d, "\mathrm{res}^{\mathscr{G}}_{V,W}"] \\
\mathscr{F}(W) \arrow[r, "\varphi_W"] & \mathscr{G}(W)
\end{tikzcd}$$

Pick $h \in \mathscr{F}(V) = H$, then

$$\mathrm{res}^{\mathscr{G}}_{V,W}(\varphi_V(h)) = \varphi_W(\mathrm{res}^{\mathscr{F}}_{V,W}(h)) = \varphi_W(0) = 0.$$

Since, $\mathscr{G}$ is a constant sheaf, restriction must be identity, hence, $\varphi_V(h) = 0$ for all $h \in H$, and therefore $\varphi_V = 0$ for all open subset $V \subseteq U$. Hence, $\mathscr{H}om(\mathscr{F},\mathscr{G})(U) = 0$, for all $U \ni p$.

Taking stalk at p, we have $\mathscr{H}om(\mathscr{F},\mathscr{G})_p = 0$. But $\mathrm{Mor}(\mathscr{F}_p, \mathscr{G}_p) = \mathrm{Mor}(H, H) \neq \{0\}$,i.e.,

$$\mathscr{H}om(\mathscr{F},\mathscr{G})_p \ncong \mathrm{Mor}(\mathscr{F}_p, \mathscr{G}_p).$$

Exercise 3.11 Show that sheaf $\mathscr{H}om$ is a sheaf of sets on X.

Proof Let $U' \supseteq U \subseteq X$ be opensets. The restriction map $\mathrm{res}_{U',U} : \mathscr{H}om(\mathscr{F},\mathscr{G})(U') \to \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$ given by $\varphi \in \mathrm{Mor}(\mathscr{F}|_{U'}, \mathscr{G}|_{U'}) \mapsto \varphi \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$. The composition of the restriction maps is easy to check, hence, $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is a presheaf. To show $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is also a sheaf, it suffices to check identity axiom and gluability axiom.

To show identity axiom. For any open set $U \subseteq X$, let $\{U_i\}_{i\in I}$ is an open cover of U, and $\varphi_1, \varphi_2 \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U)$, and $\varphi_1|_{U_i} = \varphi_2|_{U_i}$ for all i, we need to check $\varphi_1 = \varphi_2$. It suffices to show $(\varphi_1)_V = (\varphi_2)_V$ for all open subset $V \subseteq U$. Since $\{U_i\}_{i\in I}$ is an open cover of U, $\{U_i \cap V\}$ is an open cover of V. Let $s \in \mathscr{F}(V)$, since $\varphi_1|_{U_i} = \varphi_2|_{U_i}$ for all i, we have $(\varphi_1|_{U_i})_{U_i\cap V}(s|_{U_i\cap V}) = (\varphi_2|_{U_i})_{U_i\cap V}(s|_{U_i\cap V})$ for all i. Since $\mathscr{G}$ is a sheaf, by the identity axiom of $\mathscr{G}$, we have $(\varphi_1)_V(s) = (\varphi_2)_V(s)$ for all $s \in \mathscr{F}(V)$ for all open subsets $V \subseteq U$. Hence, $\varphi_1 = \varphi_2$.

To show gluability axiom. Let $\{U_i\}$ be an open cover of open set $U \subseteq X$, and given $\varphi_i \in \mathscr{H}om(\mathscr{F},\mathscr{G})(U_i)$ with $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$, we need to show there exists $\varphi \in \mathscr{F}(U)$ such that $\mathrm{res}_{U,U_i} \varphi = \varphi_i$ for all i.

We need to define φ. For all $V \subseteq U$ open set and section $s \in \mathscr{F}(V)$, $\varphi_V(s)$ must belong to $\mathscr{G}(V)$. Since $\{U_i \cap V\}$ is an open cover of V, by the condition, we have

$$\begin{aligned}
(\varphi_i|_{U_i\cap U_j})_{V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j}) =& (\varphi_i)_{V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j}) \\
=& ((\varphi_i)_{V\cap U_i}(s|_{V\cap U_i}))|_{V\cap U_i\cap U_j} \\
=& (\varphi_j|_{U_i\cap U_j})_{V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j}) \\
=& (\varphi_j)_{V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j}) \\
=& ((\varphi_j)_{V\cap U_j}(s|_{V\cap U_j}))|_{V\cap U_i\cap U_j},
\end{aligned}$$

where $(\varphi_i)_{V\cap U_i}(s|_{V\cap U_i}) \in \mathscr{G}(V \cap U_i)$ and $(\varphi_j)_{V\cap U_j}(s|_{V\cap U_j}) \in \mathscr{G}(V \cap U_j)$. By the gluability axiom of sheaf

$\mathscr{G}$, there exists $\widetilde{\varphi_V(s)} \in \mathscr{G}(V)$ such that $\widetilde{\varphi_V(s)}|_{V\cap U_i} = (\varphi_i)_{V\cap U_i}(s|_{V\cap U_i})$ for all i. Define $\varphi_V(s) = \widetilde{\varphi_V(s)} \in \mathscr{G}(V)$, for all $s \in \mathscr{F}(V)$ and for all $V \subseteq X$ open, then we defined φ. The gluability of $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ given by the definition of φ.

Hence, $\mathscr{H}om$ is a sheaf of sets on X. □

Note *We will use many variants of the definition of $\mathscr{H}om$. For example:*

(1) if $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups on X, then $\mathscr{H}om_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})$ is defined by taking $\mathscr{H}om_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})(U)$ to be the maps as sheaves of abelian groups $\mathscr{F}|_U \to \mathscr{G}|_U$ (Note that $\mathscr{H}om_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})$ has the structure of a sheaf of abelian groups in the natural way);

(2) if $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-modules, we define $\mathscr{H}om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{F}, \mathscr{G})$ in the analogous way (and it is an $\mathscr{O}_X$-module).

Obnoxiously, the subscripts $\mathbf{Ab}_X$ and $\mathbf{Mod}_{\mathscr{O}_X}$ are often dropped (here and in the literature), so be careful which category you are working in!

Definition 3.3.5 (Dual of $\mathscr{O}_X$-module)

We call $\mathscr{H}om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{F}, \mathscr{O}_X)$ the dual of the $\mathscr{O}_X$-module $\mathscr{F}$, and denote it $\mathscr{F}^\vee$.

Exercise 3.12

(a) If $\mathscr{F}$ is a sheaf of sets on X, then show that $\mathscr{H}om(\underline{\{p\}}, \mathscr{F}) \cong \mathscr{F}$, where $\underline{\{p\}}$ is the constant sheaf "with values in the one element set $\{p\}$".

(b) If $\mathscr{F}$ is a sheaf of abelian groups on X, then show that $\mathscr{H}om_{\mathbf{Ab}_X}(\underline{\mathbb{Z}}, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of sheaves of abelian groups).

(c) If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then show that $\mathscr{H}om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of $\mathscr{O}_X$-modules).

Remark (b) gives an example where taking stalks and sheaf $\mathscr{H}om$ commute, since

$$\mathscr{H}om_{\mathbf{Ab}_X}(\underline{\mathbb{Z}}, \mathscr{F})_p \cong \mathscr{F}_p \cong \mathrm{Hom}(\mathbb{Z}, \mathscr{F}_p) = \mathrm{Hom}(\underline{\mathbb{Z}}_p, \mathscr{F}_p).$$

Proof

(a) Let $U \subseteq X$ be any open set of X, then $\underline{\{p\}} = \{s : U \to \{p\} \text{ locally constant}\} = \{p\}$. Let $\varphi \in \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U)$. Let $W \subseteq V \subseteq U$ be any open sets of U, then $\varphi_V : \underline{\{p\}}(U) = \{p\} \to \mathscr{F}|_U(V) = \mathscr{F}(U)$, and we have the following commutative diagram,

$$\begin{tikzcd} \{p\} \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}^{\{p\}}_{V,W}"'] & \mathscr{F}(V) \arrow[d, "\mathrm{res}^{\mathscr{F}}_{V,W}"] \\ \{p\} \arrow[r, "\varphi_W"] & \mathscr{F}(W), \end{tikzcd}$$

that is, $\varphi_V(p)|_W = \varphi_W(p)$. For any $V \subseteq U$, let $\{U_i\}$ be an open cover of U, then $\varphi_{U_i}(p)|_{U_i\cap U_j} = \varphi_{U_j}(p)|_{U_i\cap U_j}$. Since $\mathscr{F}$ is a sheaf, there exists unique global section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = \varphi_{U_i}(p)$. This implies that for any $\varphi \in \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U)$, it determines unique global section $s \in \mathscr{F}(U)$. Conversely, for any section $s \in \mathscr{F}(U)$, we can define $\varphi \in \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U)$ by setting $\varphi_V(p) = s|_V$ for all $V \subseteq U$. Then we get a one-to-one correspondence between $\mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U)$ and $\mathscr{F}(U)$ for all open set $U \subseteq X$. Formally, for any open set $U \subseteq X$, there is a bijection

$$\theta_U : \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U) \to \mathscr{F}(U)$$

by setting $\theta_U(\varphi) = s, \varphi_V(p) = s|_V$ for all open set $V \subseteq U$ (hence, $s = \varphi_U(p)$).

Now, we need to check that $\theta : \mathscr{H}om(\underline{\{p\}}, \mathscr{F})$ is a morphism of sheaves. Let $U' \supseteq U$ be two open sets of X, it suffice to check the following diagram

$$\begin{tikzcd} \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U') \arrow[r, "\theta_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{H}om}_{U',U}"'] & \mathscr{F}(U') \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"] \\ \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U) \arrow[r, "\theta_U"] & \mathscr{F}(U) \end{tikzcd}$$

commutes. Let $\varphi \in \mathscr{H}om(\underline{\{p\}}, \mathscr{F})(U')$, then

$$\theta_U(\varphi|_U) = \varphi_U(p) = \varphi_{U'}(p)|_U = \theta_{U'}(\varphi)|_U,$$

which implies the diagram commutes. Hence, θ is a morphism of sheaves, and therefore,

$$\mathscr{H}om(\underline{\{p\}}, \mathscr{F}) \cong \mathscr{F}.$$

(b) Let $U \subseteq X$ be any open set of X, and $\pi_0(U)$ is the set of connected components of U, then

$$\underline{\mathbb{Z}}(U) = \prod_{C \in \pi_0(U)} \underline{\mathbb{Z}}(C).$$

If $V \subseteq U$, the restriction maps given by $\mathrm{res}^{\underline{\mathbb{Z}}}_{U,V}(s)|_{C\cap V} = s|_{C\cap V}$ for all $C \in \pi_0(U)$.
First, we define $\theta : \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F}) \to \mathscr{F}$. For all $U \subseteq X$ be any open set of X, for all sheaf morphism $\varphi \in \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U)$, define

$$\theta_U(\varphi) = s, \quad s|_C = \varphi_C(1_C),$$

where $1_C \in \underline{\mathbb{Z}}(C) = \mathbb{Z}$. Now, we need to check θ_U is well-defined for all $U \subseteq X$. It suffices to check two things $s \in \mathscr{F}(U)$ and θ is a morphism of sheaves. $\pi_0(U)$ is an open cover of U, note that for any C_i, C_j, $C_i \cap C_j = \varnothing$, then we have $(s|_{C_i})|_{C_i\cap C_j} = (s|_{C_j})|_{C_i\cap C_j}$, by the gluability axiom of $\mathscr{F}$, there exists unique $s \in \mathscr{F}(U)$ be the global section. Let $U' \supseteq U$, consider the following diagram.

$$\begin{tikzcd} \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U') \arrow[r, "\theta_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{H}om}_{U',U}"'] & \mathscr{F}(U') \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"] \\ \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U) \arrow[r, "\theta_U"] & \mathscr{F}(U) \end{tikzcd}$$

Let $\varphi \in \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U')$. In fact, for all $C' \in \pi_0(U')$, $\pi_0(U) = \{C' \cap U : C' \in \pi_0(U')\}$, then

$$\theta_U(\varphi|_U)|_{C'\cap U} = (\varphi|_U)_{C'\cap U}(1_{C'\cap U}) = \varphi_{C'\cap U}(1_{C'\cap U})$$

for all $C' \in \pi_0(U')$. On the other hand,

$$(\theta_{U'}(\varphi)|_{C'})|_U = \theta_{U'}(\varphi)|_{C'\cap U} = \varphi_{C'\cap U}(1_{C'\cap U})$$

for all $C' \in \pi_0(U')$. Hence, $\theta_U(\varphi|_U) = \theta_{U'}(\varphi)|_U$, and therefore θ is a sheaf morphism. Thus, θ is well-defined.
Next, we shall to check that θ is also a group homomorphism. For all open set $U \subseteq X$, let $\varphi, \psi \in \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U)$, then

$$\theta_U(\varphi + \psi)|_C = (\varphi + \psi)_C(1_C) = \varphi_C(1_C) + \psi_C(1_C)$$

for all $C \in \pi_0(U)$ (the last equality holds because $\mathscr{F}(U)$ is a group), that is, $\theta_U(\varphi+\psi) = \theta_U(\varphi)+\theta_U(\psi)$. Hence, θ is a group homomorphism.
We claim that θ_U is an isomorphism. Let $s \in \mathscr{F}(U)$, for all $C \in \pi_0(U)$, define $\varphi_C(1_C) = s|_C$, since each $C \in \pi_0(U)$ disjoint and $\mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})$ is a sheaf, we can glue all local sections φ_C into a unique global section $\varphi \in \mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})(U)$. Note that $\theta_U(\varphi)|_C = \varphi_C(1_C) = s|_C$, $\theta_U(\varphi) = s$, this implies that θ_U is a surjection. Let $\theta_U(\varphi) = 0$, then $\varphi_C(1_C) = 0$, hence, $\varphi_C = 0$, and therefore $\varphi = 0$, by the gluability axiom of $\mathscr{H}om(\underline{\mathbb{Z}}, \mathscr{F})$. Hence, θ_U is injection. Thus, θ_U is an isomorphism.

By above discussion, we know:

(i) θ is a well-defined sheaf homomorphism.

(ii) θ_U is a group isomorphism, for all open set $U \subseteq X$.

Hence, $\mathscr{H}om_{\mathbf{Ab}_X}(\mathbb{Z}, \mathscr{F}) \cong \mathscr{F}$.

(c) Note that $\mathscr{O}_X$ is a free $\mathscr{O}_X$-module with rank 1, and define $\theta_U : \mathscr{H}om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F})(U) \to \mathscr{F}(U)$ by setting $\theta_U(\varphi) = \varphi_U(1_U)$, where $1_U \in \mathscr{O}_X(U)$, then similar to (b), we have $\mathscr{H}om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$. □

3.3.2 Presheaves of abelian groups (and even "presheaf $\mathscr{O}_X$-modules") form an abelian category

We can make module-like constructions using presheaves of abelian groups on a topological space X. (Throughout this section, all (pre)sheaves are of abelian groups.) For example, we can clearly add maps of presheaves and get another map of presheaves: if $\varphi, \psi : \mathscr{F} \to \mathscr{G}$, then we define the map $\varphi + \psi$ by $(\varphi + \psi)(V) = \varphi(V) + \psi(V)$. (There is something small to check here: that the result is indeed a map of presheaves.) In this way, presheaves of abelian groups form an additive category (Definition 2.5.1: the morphisms between any two presheaves of abelian groups from an abelian group; there is a 0-object; and one can take finite products). For exactly that same reasons, sheaves of abelian groups also form an additive category.

Definition 3.3.6 (Presheaf kernel)

*If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, define the **presheaf kernel** $\operatorname{Ker}_{\mathrm{pre}} \varphi$ by*

$$(\operatorname{Ker}_{\mathrm{pre}} \varphi)(U) := \operatorname{Ker} \varphi_U,$$

for all open set $U \subseteq X$.

Proposition 3.3.2

$\operatorname{Ker}_{\mathrm{pre}} \varphi$ *is a presheaf.*

Proof Let $U' \supseteq U$ be two open sets of X, consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}_{\mathrm{pre}} \varphi(U') \arrow[r, "\operatorname{Ker}\varphi_{U'}"] \arrow[d, dashed, "\exists!"] & \mathscr{F}(U') \arrow[r, "\varphi_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"] & \mathscr{G}(U') \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U',U}"] \\
0 \arrow[r] & \operatorname{Ker}_{\mathrm{pre}} \varphi(U) \arrow[r, "\operatorname{Ker}\varphi_U"'] & \mathscr{F}(U) \arrow[r, "\varphi_U"'] & \mathscr{G}(U)
\end{tikzcd}$$

Note that $\mathrm{res}^{\mathscr{G}}_{U',U} \circ \varphi_{U'} \circ \operatorname{Ker} \varphi_{U'} = 0$, by the universal property of $\operatorname{Ker} \varphi_U$, there exist unique group homomorphism from $\operatorname{Ker}_{\mathrm{pre}} \varphi(U')$ to $\operatorname{Ker}_{\mathrm{pre}} \varphi(U)$, denote $\mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U',U}$. By the universal property of $\operatorname{Ker} \varphi_U$, $\mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U,U} = \mathrm{id}_{\operatorname{Ker}_{\mathrm{pre}} \varphi(U)}$. Now, consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}_{\mathrm{pre}} \varphi(U'') \arrow[r] \arrow[d, "\mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U'',U'}"'] & \mathscr{F}(U'') \arrow[r, "\varphi_{U''}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U'',U'}"'] & \mathscr{G}(U'') \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U'',U'}"] \\
0 \arrow[r] & \operatorname{Ker}_{\mathrm{pre}} \varphi(U') \arrow[r] \arrow[d, "\mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U',U}"'] & \mathscr{F}(U') \arrow[r, "\varphi_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"'] & \mathscr{G}(U') \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U',U}"] \\
0 \arrow[r] & \operatorname{Ker}_{\mathrm{pre}} \varphi(U) \arrow[r] & \mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U)
\end{tikzcd}$$

By the universal property of $\operatorname{Ker} \varphi_U$, $\mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U',U} \circ \mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U'',U'} = \mathrm{res}^{\operatorname{Ker}_{\mathrm{pre}}}_{U'',U}$. Hence, $\operatorname{Ker}_{\mathrm{pre}} \varphi$ is a presheaf. □

Definition 3.3.7

*If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaf, define the **presheaf cokernel** $\operatorname{Coker}_{\text{pre}} \varphi$ by*

$$(\operatorname{Coker}_{\text{pre}})(U) := \operatorname{Coker} \varphi_U,$$

for all open set $U \subseteq X$.

Proposition 3.3.3

$\operatorname{Coker}_{\text{pre}} \varphi$ is a presheaf.

Proof It is a presheaf by essentially the same (dual) argument. □

✍ **Exercise 3.13 (The cokernel deserves its name.)** Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Proof Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaf, let $\mathscr{K}$ be another sheaf with $\psi \circ \varphi = 0$, it suffices to show that there exists a unique presheaf morphism from $\operatorname{Coker}_{\text{pre}} \varphi$ to $\mathscr{K}$.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[drr, "0"'] & \mathscr{G} \arrow[r] \arrow[dr, "\psi"] & \operatorname{Coker}_{\text{pre}} \varphi \arrow[r] \arrow[d, dashed] & 0 \\ & & \mathscr{K} & \end{tikzcd}$$

Let $U \subseteq X$ be any open set, note that each presheaf is presheaf of abelian group, $\operatorname{Coker}_{\text{pre}} \varphi(U) = \mathscr{G}(U)/\operatorname{Im} \varphi_U$. For all section $s_U \in \mathscr{G}(U)$, say $\overline{s_U}$ be the image of s_U in $\operatorname{Coker}_{\text{pre}} \varphi(U)$. Define

$$\theta_U : \operatorname{Coker}_{\text{pre}} \varphi(U) \to \mathscr{K}$$

by setting $\theta_U(\overline{s_U}) = \psi_U(s_U)$. We need to check that θ_U is well-defined for all $U \subseteq X$. Let $\overline{s_U} = \overline{t_U}$, then $s_U - t_U \in \operatorname{Im} \mathscr{F}(U)$. Since $\psi \circ \varphi = 0$, $\psi_U(s_U - t_U) = 0$, and therefore $\psi_U(s_U) = \psi_U(t_U)$, that is, $\theta_U(\overline{s_U}) = \theta_U(\overline{t_U})$. Hence, θ_U is well-defined for all open set $U \subseteq X$. By the definition of θ_U, uniqueness is clearly.

Finally, we need to check θ is a presheaf morphism. Let $U' \supseteq U$, consider the following diagram.

$$\begin{tikzcd} \operatorname{Coker}_{\text{pre}} \varphi(U') \arrow[r, "\theta_{U'}"] \arrow[d, "\operatorname{res}^{\operatorname{Coker}_{\text{pre}}}_{U',U}"'] & \mathscr{K}(U') \arrow[d, "\operatorname{res}^{\mathscr{K}}_{U',U}"] \\ \operatorname{Coker}_{\text{pre}} \varphi(U) \arrow[r, "\theta_U"] & \mathscr{K}(U) \end{tikzcd}$$

Let $\overline{s_{U'}} \in \operatorname{Coker}_{\text{pre}} \varphi(U')$, then

$$\theta_U(\overline{s_{U'}}|_U) = \theta_U(\overline{s_U}) = \psi_U(s_U) = \psi_{U'}(s_{U'})|_U = \theta_{U'}(\overline{s_{U'}})|_U$$

for all open set $U \subseteq X$. Hence, θ is a presheaf morphism.

By the arbitrary of $\mathscr{K}$, presheaf cokernel is indeed a cokernels in the category of presheaves. □

Remark Similarly, $\operatorname{Ker}_{\text{pre}} \varphi \to \mathscr{F}$ satisfies the universal property for kernels in the category of presheaves.

It is not too tedious to verify that presheaves of abelian groups form an abelian category, we state the following result without proof here:

Theorem 3.3.1

$\mathbf{Ab}_X$ is an abelian category.

Remark The key idea is that all abelian-categorical notions may be defined and verified "open set by open set". We needn't worry about restriction maps — they "come along for the ride". Hence, we can define terms such as **subpresheaf**, **image presheaf** (or **presheaf image**) and **quotient presheaf** (or **presheaf quotient**), and they behave as you would expect. Construct kernels, quotients, cokernels, and images open set by open set. Homological algebra (exact sequences and so forth) works, and also "works open set by open set".

Exercise 3.14 Show that for a topological space X with open set U, $\mathscr{F} \mapsto \mathscr{F}(U)$ gives a functor from presheaves of abelian groups on X, $\mathbf{Ab}_X^{\mathrm{pre}}$, to abelian groups, $\mathbf{Ab}$. Then show that this functor is exact.

Proof Let $\Theta_U : \mathbf{Ab}_X^{\mathrm{pre}} \to \mathbf{Ab}$. Let $\varphi : \mathscr{F} \to \mathscr{G} \in \mathbf{Ab}_X^{\mathrm{pre}}$, define $\Theta_U(\varphi) : \Theta_U(\mathscr{F}) \to \Theta_U(\mathscr{G})$ by setting $\Theta_U(\varphi) = \varphi_U$. Note that φ is a presheaf morphism, and therefore φ_U is a group homomorphism. Hence, $\Theta_U(\varphi) \in \mathrm{Hom}_{\mathbf{Ab}}(\mathscr{F}(U), \mathscr{G}(U))$. For $\mathscr{F} \in \mathbf{Ab}_X^{\mathrm{pre}}$, $\Theta_U(\mathrm{id}_{\mathscr{F}}) = \mathrm{id}_{\mathscr{F}(U)}$. Let $\varphi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{G} \to \mathscr{K}$, then $\Theta_U(\psi \circ \varphi) = (\psi \circ \varphi)_U = \psi_U \circ \varphi_U = \Theta_U(\psi) \circ \Theta_U(\varphi)$. Hence, Θ_U is a functor.

Since $\mathrm{Ker}_{\mathrm{pre}}$, $\mathrm{Coker}_{\mathrm{pre}}$ defined "open set by open set", hence, Θ_U is exact. □

Proposition 3.3.4

A sequence of presheaves

$$0 \longrightarrow \mathscr{F}_1 \xrightarrow{f_1} \mathscr{F}_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \mathscr{F}_n \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow \mathscr{F}_1(U) \xrightarrow{(f_1)_U} \mathscr{F}_2(U) \xrightarrow{(f_2)_U} \cdots \xrightarrow{(f_{n-1})_U} \mathscr{F}_n(U) \longrightarrow 0$$

is exact for all U.

Proof This is an obvious observation. □

Remark The above discussion essentially carries over without change to presheaves with values in any abelian category.

However, we are interested in more geometric objects, sheaves, where things canbe understood in terms of their local behavior, thanks to the identity and gluing axioms. We will soon see that sheaves of abelian groups also form an abelian category, but a complication will arise that will force the notion of **sheafification** on us. Sheafification will be the answer to many of our prayers. We just haven't yet realized what we should be praying for.

Proposition 3.3.5

$\mathbf{Ab}_X$ *form an additive category.*

Proof As description of presheaves. □

Kernels work just as with presheaves:

Proposition 3.3.6

Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then the presheaf kernel $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$ is in fact a sheaf, and it satisfies the universal property of kernels.

Proof $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$ is a presheaf, by Proposition 3.3.2. It suffices to check the identity axiom and gluability axiom of $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$. Let $U \subseteq X$ be open set, and $\{U_i\}_{i\in I}$ is an open cover of U. Let sections $s, t \in \mathrm{Ker}_{\mathrm{pre}}\,\varphi(U)$ with $s|_{U_i} = t|_{U_i}$ for all i. Since each $\mathrm{Ker}_{\mathrm{pre}}\,\varphi(U_i)$ is a subgroup of $\mathscr{F}(U_i)$, $s|_{U_i}, t|_{U_i} \in \mathscr{F}(U_i)$. Since $\mathscr{F}$ is a sheaf, we have $s = t \in \mathscr{F}(U)$, and therefore $s = t \in \mathrm{Ker}_{\mathrm{pre}}\,\varphi(U)$, which implies the identity axiom. Let $s_i \in \mathrm{Ker}_{\mathrm{pre}}\,\varphi(U_i)$ with $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j, since $\mathrm{Ker}_{\mathrm{pre}}\,\varphi(U_i)$ is a subgroup of $\mathscr{F}(U_i)$ and $\mathscr{F}$ is a sheaf, by the gluability axiom of $\mathscr{F}$, there is a global section $s \in \mathscr{F}(U)$, such that $s|_{U_i} = s_i$. Now we need to check $s \in \mathrm{Ker}_{\mathrm{pre}}\,\varphi(U)$. Let $x \in U$, then $x \in U_i$ for some i, note that $s(x) = s|_{U_i}(x) = s_i(x) = 0$ for all i, $s = 0$, that is, $s \in \mathrm{Ker}_{\mathrm{pre}}\,\varphi(U)$, which implies the gluability axiom. Hence, $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$ is a sheaf.

The universal property of sheaf $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$ from the universal property of presheaf $\mathrm{Ker}_{\mathrm{pre}}\,\varphi$. $\square$

Definition 3.3.8 (Sheaf Kernel)

In Definition of presheaf kernel, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the ***sheaf of kernel*** $\mathrm{Ker}\,\varphi$ *by*

$$\mathrm{Ker}\,\varphi := \mathrm{Ker}_{\mathrm{pre}}\,\varphi.$$

The problem arises with the cokernel:

Exercise 3.15 Let X be $\mathbb{C}$ with the classical topology, let $\mathscr{O}_X$ be the sheaf of holomorphic funtions, and $\mathscr{F}$ be the presheaf of functions admitting a holomorphic logarithm. Describe an exact sequence of presheaves on X:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathscr{O}_X$ is the natural inclusion and $\mathscr{O}_X \to \mathscr{F}$ is given by $f \mapsto e^{2\pi i f}$. Show that $\mathscr{F}$ is not a sheaf.

Proof Let $U \subseteq X$ be any open set, then

$$\mathscr{F}(U) = \{g : U \to \mathbb{C} \setminus \{0\} : \exists \text{ holomorphic function } h : U \to \mathbb{C} \text{ such that } g = e^{2\pi i h}\}.$$

We need to check sequence of presheaf:

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ p\ } \mathscr{F} \longrightarrow 0$$

is exact.

(1) $\underline{\mathbb{Z}} \to \mathscr{O}_X$ is a monomorphism.
Note that for locally constant function over U is also a holomorphic function over U, hence, for all $U \subseteq X$, we have $\underline{\mathbb{Z}}(U) \hookrightarrow \mathscr{O}_X(U)$. Let $f_i : \mathscr{K} \to \underline{\mathbb{Z}}$ be sheaf morphisms with $i \circ f_1 = i \circ f_2$. Let V be an open subset of U, then, locally, $(i \circ f_j)_V = i \circ (f_j)_V = (f_j)_V$, hence, $(f_1)_V = (f_2)_V$ for all open subset $V \subseteq U$, and therefore $f_1 = f_2$. This implies that $\underline{\mathbb{Z}} \to \mathscr{O}_X$ is a monomorphism.

(2) $\mathrm{Ker}(\mathscr{O}_X \to \mathscr{F}) = \mathrm{Im}(\underline{\mathbb{Z}} \to \mathscr{O}_X)$.
It suffices to check that for any open subset $U \subseteq X$, we have $\mathrm{Ker}(\mathscr{O}_X(U) \to \mathscr{F}(U)) = \mathrm{Im}(\underline{\mathbb{Z}}(U) \to \mathscr{O}_X(U))$, i.e., $\mathrm{Im}\,i_U = \mathrm{Ker}\,p_U$. Unit element in $\mathscr{F}(U)$ is 1, hence, $1 = e^{2\pi i f}$ if and only if f is locally constant, that is, $f \in \underline{\mathbb{Z}}$. Thus, $\mathrm{Ker}(\mathscr{O}_X(U) \to \mathscr{F}(U)) = \mathrm{Im}(\underline{\mathbb{Z}}(U) \to \mathscr{O}_X(U))$ for all $U \subseteq X$, and therefore $\mathrm{Ker}(\mathscr{O}_X \to \mathscr{F}) = \mathrm{Im}(\underline{\mathbb{Z}} \to \mathscr{O}_X)$.

(3) $\mathscr{O}_X \to \mathscr{F}$ is a epimorphism.
It suffices to show that $\mathrm{Coker}(\mathscr{O}_X \to \mathscr{F}) = 0$. For all open subset $U \subseteq X$, consider $\mathrm{Coker}(\mathscr{O}_X(U) \to \mathscr{F}(U))$. Let $g \in \mathscr{F}(U)$, then there exists some holomorphic function h such that $g = e^{2\pi i h} = p_U(h)$, this implies that p_U is a surjective, and therefore $\mathrm{Coker}(\mathscr{O}_X(U) \to \mathscr{F}(U)) = 0$ for all open subset $U \subseteq X$. Hence, $\mathrm{Coker}(\mathscr{O}_X \to \mathscr{F}) = 0$.

Hence, the sequence of presheaf

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ p\ } \mathscr{F} \longrightarrow 0$$

is exact.

Now, we give a counterexample to show that $\mathscr{F}$ is not a sheaf. Let $U = \mathbb{C} \setminus \{0\}$, and consider the open cover $U_1 = \mathbb{C} \setminus (-\infty, 0]$ and $U_2 = \mathbb{C} \setminus [0, +\infty)$. Let $g(z) = z \in \mathscr{O}_X(U)$. Note that $g|_{U_1}(z) = e^{2\pi i \cdot \frac{\log z}{2\pi i}}$ and $g|_{U_2}(z) = e^{2\pi i \cdot (\frac{\log z}{2\pi i} + 1)}$, then $z \in \mathscr{F}(U_1)$ and $z \in \mathscr{F}(U_2)$. On $U_1 \cap U_2$, $(z|_{U_1})|_{U_1 \cap U_2} = (z|_{U_2})|_{U_1 \cap U_2}$, but $z \notin \mathscr{F}(U)$. This implies that $\mathscr{F}$ fails the gluing axiom. $\square$

We will have to put out hopes for understanding cokernels of sheaves on hold for a while. We will first learn to understand sheaves using stalks.

3.4 Properties determined at the level of stalks, and sheafification

3.4.1 Properties determined by stalks

We now come to the second way of understanding sheaves mentioned at the start of the chapter. In this section, we will see that lots of facts about sheaves can be checked "at the level of stalks". We call any property determined at the level of stalks **stalk-local**. This isn't true for presheaves, and reflects the local nature of sheaves. We will see that sections and morphisms are determined "by their stalks", and the property of a morphism being an isomorphism may be checked at stalks.

Proposition 3.4.1 (Sections are determined by germs)

A section of a sheaf of sets is determined by its germs, i.e., the natural map

$$\mathscr{F}(U) \longrightarrow \prod_{p\in U} \mathscr{F}_p$$

is injective.

Proof Consider the following diagram,

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[dr, dashed, "\theta"] \arrow[drr, "\iota_i"] \arrow[ddr, "\iota_j"'] & & \\
& \prod_{p\in U}\mathscr{F}_p \arrow[r, "\mathrm{pr}_i"'] \arrow[d, "\mathrm{pr}_j"] & \mathscr{F}_{p_i} \\
& \mathscr{F}_{p_j}, &
\end{tikzcd}$$

where ι_i, ι_j given by the definition of $\mathscr{F}_{p_i}, \mathscr{F}_{p_j}$. By the universal property of product there exists unique morphism $\theta : \mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$ defined by $s \mapsto (\iota_i(s))$. Now, we need to check θ is injective. Let $(s_i) = (t_i) \in \prod_{p\in U} \mathscr{F}_p$, then for all i exist open subset $(p \in) W_i \subseteq U$ such that $s_i|_{W_i} = t_i|_{W_i}$. Note that $\{W_i\}$ is an open cover of U, hence, by the identity axiom of $\mathscr{F}$, $s = t$, and therefore θ is injective □

Remark This proof will also apply to sheaves of groups, rings, etc. — to categories of "sets with additional structure". The same is true of many exercise in this section.

Proposition 3.4.1 suggests a question: which elements of the right side of $\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$ are in the image of the left side?

Definition 3.4.1 (Compatible germs)

*We say that an element $(s_p)_{p\in U}$ of the right side $\prod_{p\in U} \mathscr{F}_p$ of $\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$ consists of **compatible germs** if for all $p \in U$, there is some representative*

$$(\widetilde{s}_p \in \mathscr{F}(U_p), U_p \text{ open in } U)$$

for s_p (where $p \in U_p \subseteq U$) such that the germ of $\widetilde{s}_p$ at all $q \in U_p$ is s_q. Equivalently, there is an open cover $\{U_i\}$ of U, and section $f_i \in \mathscr{F}(U_i)$, such that if $p \in U_i$, then s_p is the germ of f_i at p.

Remark Clearly any section s of $\mathscr{F}$ over U gives a choice of compatible germs for U.

Proof Let $U \subseteq X$ is an open set, let section $s \in \mathscr{F}(U)$, then $(s_p)_{p\in U}$ consists of compatible germs. U_p open in U, and let $\widetilde{s}_p = s|_{U_p}$, then the germs of $\widetilde{s}_p$ at $q \in U_q$ is s_q, which implies that $(s_p)_{p\in U}$ consists of compatible germs. □

Proposition 3.4.2

Any choice of compatible germs for a sheaf of sets $\mathscr{F}$ over U is the image of a section of $\mathscr{F}$

Proof Let $(s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$ consists of compatible germs, then for all s_p there exists $\widetilde{s}_p \in \mathscr{F}(U_p)$ such that the germ of $\widetilde{s}_p$ at all $q \in U_p$ is s_q, for these q, there exists $\widetilde{s}_q \in \mathscr{F}(U_q)$ such that the germ of $\widetilde{s}_q$ is the germ of $\widetilde{s}_p$, hence, there exists an open subset $(p \in)W_p \subseteq U_p \cap U_q$ such that $\widetilde{s}_p|_{W_p} = \widetilde{s}_q|_{W_p}$ for all $q \in U_p$. Clearly, $\{W_p\}_{p\in U}$ is an open cover of U, and for each $\widetilde{s}_p \in \mathscr{F}(W_p)$ we have $\widetilde{s}_p|_{W_p\cap W_q} = \widetilde{s}_q|_{W_p\cap W_q}$, note that $\mathscr{F}$ is a sheaf, by gluability axiom of $\mathscr{F}$, there exists unique $s \in \mathscr{F}(U)$ such that $s|_{W_p} = \widetilde{s}_p$. Hence, $(s_p)_{p\in U}$ is the image of section $s \in \mathscr{F}(U)$. □

We have thus completely described the image of $\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$, in a way that will prove useful.

Remark This perspective motivates the agricultural terminology "sheaf": a sheaf is (the data of) a bunch of stalks, bundled together appropriately.

Now we throw morphisms into the mix. Recall Proposition 3.3.1: morphisms of (pre)sheaf induce morphisms of stalks.

Proposition 3.4.3 (Morphisms are determined by stalks)

If φ_1 and φ_2 are morphisms from a presheaf of sets $\mathscr{F}$ to a sheaf of sets $\mathscr{G}$ that induce the same maps on each stalk, then $\varphi_1 = \varphi_2$.

Proof Let $U \subseteq X$ be any open subset. Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{p\in U} \mathscr{F}_p \arrow[r] & \prod_{p\in U} \mathscr{G}_p
\end{tikzcd}$$

Say $\widetilde{\varphi} : \prod_{p\in U} \mathscr{F}_p \to \prod_{p\in U} \mathscr{G}_p$ induced by φ_1 and φ_2, by condition,

$$\widetilde{\varphi}((s_p)_{p\in U}) = (\varphi_{1,U}(s)_p)_{p\in U} = (\varphi_{2,U}(s)_p)_{p\in U}. \tag{3.2}$$

It suffices to show that $\varphi_{1,U}(s) = \varphi_{2,U}(s)$ for all $s \in \mathscr{F}(U)$, for all open subset $U \subseteq X$.

Let $p \in U$ and section $s \in \mathscr{F}(U)$, by (3.2), $\varphi_{1,U}(s)_p = \varphi_{2,U}(s)_p$. Hence, there exists an open subset $(p \in)V_p \subseteq U$ such that $\varphi_{1,U}(s)|_{V_p} = \varphi_{2,U}(s)|_{V_p}$. Clearly, $\{V_p\}_{p\in U}$ is an open cover of U, by the identity axiom of sheaf $\mathscr{G}$, $\varphi_{1,U}(s) = \varphi_{2,U}(s)$, then we done. □

Proposition 3.4.4 (Isomorphisms are determined by stalks)

A morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphisms of all stalks.

Proof "⇒": Let $V \subseteq U \subseteq X$ be open subsets of X. Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}_p \arrow[rr, dashed, shift left] & & \mathscr{G}_p \arrow[ll, dashed, shift left] \\
& \mathscr{F}(U) \arrow[ul] \arrow[r, hook, two heads] \arrow[d] & \mathscr{G}(U) \arrow[u] \arrow[d] \\
& \mathscr{F}(V) \arrow[uul, bend left] \arrow[r, hook, two heads] & \mathscr{G}(V) \arrow[uu, bend right]
\end{tikzcd}$$

By the universal property of $\mathscr{F}_p$ and $\mathscr{G}_p$, there exists unique morphisms $\mathscr{F}_p \to \mathscr{G}_p$ and $\mathscr{G}_p \to \mathscr{F}_p$, and therefore $\mathscr{F}_p \cong \mathscr{G}_p$.

"⇐": Say $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ induced by $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism for all $p \in X$. To show that φ is an isomorphism, it will be sufficient to show that $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for all U. First, we show φ_U is injective. Let $s \in \mathscr{F}(U)$, and suppose $\varphi_U(s) = 0$ in $\mathscr{G}(U)$. Then for every point $P \in U$, the image $\varphi_U(s)_P$ of $\varphi_U(s)$ in the stalk $\mathscr{G}_p$ is 0. Since φ_p is injective for each p, we have $s_p = 0$ in $\mathscr{F}_p$ for each $P \in U$. Hence, there exists an open subset subset $(p \in)W_p \subseteq W$ such that $s|_{W_p} = 0$. Note that $\{W_p\}$ is an open cover of U, by the gluability axiom of sheaf $\mathscr{F}$, $s = 0$, which implies that φ_U is an injective.

Next, we show that φ_U is surjective. Suppose we have a section $t \in \mathscr{G}(U)$. For each $p \in U$, let $t_p \in \mathscr{G}_p$ be its germ at P. Since φ_p is surjective, we can find $s_p \in \mathscr{F}_p$ such that $\varphi_p(s_p) = t_p$. Let s_p be represented by a section $\widetilde{s}_p$ on a neighborhood V_p of P. Then $\varphi_{V_p}(\widetilde{s}_p)$ and $t|_{V_p}$ are two elements of $\mathscr{G}(V_p)$, whose germs at P are the same. Hence, replacing V_p by a smaller neighborhood of P if necessary, we may assume that $\varphi(\widetilde{s}_P) = t|_{V_p}$ in $\mathscr{G}(V_p)$. Now U is covered by the open sets V_p, and on each V_p we have a section $\widetilde{s}_p \in \mathscr{F}(V_p)$. If p, q are two points, then $\widetilde{s}_p|_{V_p \cap V_q}$ and $\widetilde{s}_q|_{V_p \cap V_q}$ are two sections of $\mathscr{F}(V_p \cap V_q)$, which are both sent by φ to $t|_{V_p \cap V_q}$. Hence, by the injective of φ proved above, $\widetilde{s}_p|_{V_p \cap V_q} = \widetilde{s}_q|_{V_p \cap V_q}$, by the gluability axiom of sheaf $\mathscr{F}$, there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_p} = \widetilde{s}_p$. Finally, we have to check that $\varphi(s) = t$. Indeed, $\varphi_U(s)$, t are two sections of $\mathscr{G}(U)$, and for each $p \in U$, $\varphi_U(s)|_{V_p} = t|_{V_p}$, hence, by the identity axiom of sheaf $\mathscr{G}$, we conclude that $\varphi(s) = t$. □

Exercise 3.16

(a) Show that Proposition 3.4.1 is false for general presheaves.

(b) Show that Proposition 3.4.3 is false for general presheaves.

(c) Show that Proposition 3.4.4 is false for general presheaves.

Proof Let $X = \{p, q\}$ be a two points set with the discrete topology $\mathscr{T}_X = \{\{p\}, \{q\}, X, \varnothing\}$. Let $\mathscr{F}(X) = \{s, t\}$, $\mathscr{F}(\{p\}) = \{a\}$, and $\mathscr{F}(\{q\}) = \{b\}$, with restriction $s|_{\{p\}} = a$, $s|_{\{q\}} = b$, $t|_{\{p\}} = a$, and $t|_{\{q\}} = b$. Then $\mathscr{F}$ form a presheaf.

(a) Note that $s \mapsto (a, b)$ and $t \mapsto (a, b)$, $\mathscr{F}(X) \to \mathscr{F}_p \times \mathscr{F}_q$ is not injective.

(b) Let $\mathscr{G}(X) = \{m\}, \mathscr{G}(\{p\}) = \{a\}$, and $\mathscr{G}(\{q\}) = \{b\}$, then $\mathscr{G}$ form a presheaf, and its stalks are $\mathscr{G}_p = \{a\}$ and $\mathscr{G}_p = \{b\}$. Define $\varphi_1 : \mathscr{G} \to \mathscr{F}$ by setting $\varphi_{1,X}(m) = s$, $\varphi_{1,\{p\}}(a) = a$, and $\varphi_{1,\{q\}}(b) = b$. Define $\varphi_2 : \mathscr{G} \to \mathscr{F}$ by setting $\varphi_{2,X}(m) = t$, $\varphi_{2,\{p\}}(a) = a$, and $\varphi_{2,\{q\}}(b) = b$. Then φ_1 and φ_2 induced the same maps on each stalks, but $\varphi_1 \neq \varphi_2$.

(c) Let $\mathscr{G}(X) = \{m\}$, $\mathscr{G}(\{p\}) = \{a\}$, and $\mathscr{G}(\{q\}) = \{b\}$, then $\mathscr{G}$ form a presheaf, and its stalks are $\mathscr{G}_p = \{a\}$ and $\mathscr{G}_p = \{b\}$. Define $\varphi : \mathscr{G} \to \mathscr{F}$ by setting $\varphi_X(m) = s$, $\varphi_{\{p\}}(a) = a$, and $\varphi_{\{q\}}(b) = b$, then $\varphi_p : \mathscr{G}_p \to \mathscr{F}_p$ and $\varphi_q : \mathscr{G}_q \to \mathscr{F}_p$ are isomorphism, but φ is not isomorphism, since φ_X is not surjective.

□

3.4.2 Sheafification

Every sheaf is a presheaf (and indeed by definition sheaves on X form a full subcategory of the category of presheaves on X). Just as groupification (Example 2.32) gives an abelian group that best approximates a presheaf, with an analogous universal property. (One possible example to keep in mind is the sheafification of the pre sheaf of holomorphic functions admitting a square root on $\mathbb{C}$ with the classical topology.)

Definition 3.4.2 (Sheafification)

*If $\mathscr{F}$ is a presheaf on X, then a morphism of presheaf $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ on X is a **sheafification of** $\mathscr{F}$ if $\mathscr{F}^{\mathrm{sh}}$ is a sheaf, and for any sheaf $\mathscr{G}$, and any presheaf morphism $g : \mathscr{F} \to \mathscr{G}$, there exists a unique* *morphism of sheaves $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ making the diagram*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "g"'] & \mathscr{F}^{\mathrm{sh}} \arrow[d, dashed, "f"] \\ & \mathscr{G} \end{tikzcd}$$

commute.

Note ***Construction of $\mathscr{F}^{\mathrm{sh}}$.*** *Any presheaf of sets (or groups, rings, etc.) has a sheafification.*

Suppose $\mathscr{F}$ is a presheaf. Define $\mathscr{F}^{\mathrm{sh}}$ by defining $\mathscr{F}^{\mathrm{sh}}(U)$ as the set of "compatible (families of) germs" of the presheaf $\mathscr{F}$ over U. Explicitly:

$$\mathscr{F}^{\mathrm{sh}}(U) := \left\{ (f_p \in \mathscr{F}_p)_{p\in U} \;\middle|\; \begin{array}{l} \text{for all } p \in U, \text{ there exists an open neighborhood } V \text{ of } p, \\ \text{contained in } U, \text{ and } s \in \mathscr{F}(V) \text{ with } s_q = f_q \text{ for all } q \in V \end{array} \right\}.$$

Here s_q means the germ of s at q — the image of s in the stalk $\mathscr{F}_q$.

Proposition 3.4.5

If $\mathscr{F}$ is a presheaf on X, then $\mathscr{F}^{\mathrm{sh}}$ is a sheaf.

Proof Let $U \subseteq U' \subseteq X$ be any open subset of X. The restriction map given by $\mathrm{res}_{U',U}((f_p)_{p\in U'}) = (f_p)_{p\in U}$, where $(f_p)_{p\in U}$ consists of compatible germs. (This restriction is well-defined since it suffices to restrict s at $V' \cap U$ where $(p \in)V' \subseteq U'$.) By some easy check, $\mathscr{F}^{\mathrm{sh}}$ is a presheaf.

Let $\{U_i\}$ be an open cover of U, and $(f_p)_{p\in U} \in \mathscr{F}^{\mathrm{sh}}(U)$ with $(f_p)_{p\in U}|_{U_i} = (0)_{p\in U}|_{U_i}$, where $(f_p)_{p\in U}$ consists of compatible germs. We shall to show that $f_p = 0$ for all $p \in U$. Since $(f_p)_{p\in U_i} = (0)$, then exist $p \in W_i^p \in U_i$ such that $f|_{W_i^p} = 0$. Note that $\{W_i^p\}_{i,p}$ is an open cover of U, let $p \in U$, then $p \in W_i^p$ with $f|_{W_i^p} = 0$, hence, $f_p = 0$. This implies that $\mathscr{F}^{\mathrm{sh}}$ satisfies identity axiom.

Let $(f_p^i)_{p\in U_i} \in \mathscr{F}^{\mathrm{sh}}(U_i)$ with $(f_p^i)_{p\in U_i}|_{U_i\cap U_j} = (f_p^j)_{p\in U_j}|_{U_i\cap U_j}$, we shall to show there exists $(f_p)_{p\in U} \in \mathscr{F}^{\mathrm{sh}}(U)$ such that $(f_p)_{p\in U}|_{U_i} = (f_p^i)_{p\in U_i}$. Since $(f_p^i)_{p\in U_i\cap U_j}$ consists of compatible germs, for all $p \in U_i \cap U_j$, there exists an open neighborhood $(p \in)V_p^i \in U_i \cap U_j$, and $s_p^i \in \mathscr{F}(V_p^i)$ with $(s_p^i)_q = f_q^i$ for all $q \in V_p^i$, we may assume that $s_p^i|_{V_p^i} = f^i|_{V_p^i}$ for all $p \in U_i \cap U_j$. Similarly, we have $s_p^j|_{V_p^j} = f^j|_{V_p^j}$. Note that $\{V_p^i \cap V_p^j\}$ is also an open cover of U and $f^i|_{V_p^i\cap V_p^j} = f^j|_{V_p^i\cap V_p^j}$ for all $p \in U_i \cap U_j$. Hence, $f_p^i = f_p^j$ for all $p \in U_i \cap U_j$. Let $(f_p)_{p\in U} = (f_p^1)_{p\in U_1} \times (f_p^2)_{p\in U_2\setminus U_1} \times (f_p^3)_{p\in U_3\setminus(U_1\cap U_2)} \times \cdots$, then $(f_p)_{p\in U}$ consists of compatible germs with $(f_p)_{p\in U}|_{U_i} = (f_p^i)_{p\in U_i}$. This implies that gluability axiom holds.

Hence, $\mathscr{F}^{\mathrm{sh}}$ is a sheaf. □

Proposition 3.4.6

(1) Sheafification is unique up to unique isomorphism, assuming it exists.

(2) If $\mathscr{F}$ is a sheaf, then the sheafification is $\mathrm{id} : \mathscr{F} \to \mathscr{F}$.

Proof

(1) Let $\mathscr{F}$ is a presheaf on X. If there exists another morphism of presheaf $\theta : \mathscr{F} \to \mathscr{K}$ on X is a

sheafification of $\mathscr{F}$, the following diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[dr] & \mathscr{K} \arrow[loop above, dashed, "\mathrm{id}_{\mathscr{K}}"] \arrow[d, dashed, shift right, "\exists!"'] \\
& \mathscr{F}^{\mathrm{sh}} \arrow[u, dashed, shift right, "\exists!"'] \arrow[loop below, dashed, "\mathrm{id}_{\mathscr{F}^{\mathrm{sh}}}"]
\end{tikzcd}$$

commutes. Hence, $\mathscr{K} \cong \mathscr{F}^{\mathrm{sh}}$.

(2) Since $\mathscr{F}$ is a sheaf, clearly, any section s of $\mathscr{F}$ over U gives a choice of compatible germs for U, by Proposition 3.4.2, we have $\mathscr{F} \cong \mathscr{F}^{\mathrm{sh}}$. By part (1), the sheafification is $\mathrm{id} : \mathscr{F} \to \mathscr{F}$.

$\square$

Proposition 3.4.7 (Sheafification is a functor)

For any morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$, there is a natural induced morphism of sheaves $\varphi^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}^{\mathrm{sh}}$. Moreover, sheafification is a functor from presheaves on X to sheaves on X.

Proof Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{sh}"] \arrow[d, "\varphi"'] \arrow[dr, "\mathrm{sh} \circ \varphi"] & \mathscr{F}^{\mathrm{sh}} \arrow[d, dashed, "\exists!"] \\
\mathscr{G} \arrow[r, "\mathrm{sh}"'] & \mathscr{G}^{\mathrm{sh}}
\end{tikzcd}$$

By the universal property of sheafification, there exists unique morphism from $\mathscr{F}^{\mathrm{sh}}$ to $\mathscr{G}^{\mathrm{sh}}$, denote it φ^{sh}.

Now, we shall to check $\square^{\mathrm{sh}}$ is, indeed, a functor from presheaves on X to sheaves on X. By the definition of $\square^{\mathrm{sh}}$, let $\varphi \in \mathrm{Mor}(\mathscr{F}, \mathscr{G})$, then $\varphi^{\mathrm{sh}} \in \mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}^{\mathrm{sh}})$. Let $\mathscr{F}$ be a presheaf, then $\square^{\mathrm{sh}}(\mathrm{id}_{\mathscr{F}}) = \mathrm{id}_{\mathscr{F}^{\mathrm{sh}}}$, that is, $\square^{\mathrm{sh}}$ preserves identity morphisms. Let $\varphi_1 \in \mathrm{Mor}(\mathscr{F}, \mathscr{G})$ and $\varphi_2 \in \mathrm{Mor}(\mathscr{G}, \mathscr{K})$, consider the following diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d, "\varphi_1"'] \arrow[dr, no head] & \mathscr{F}^{\mathrm{sh}} \arrow[d] \arrow[dd, dashed, bend left, "\exists!"] \\
\mathscr{G} \arrow[r] \arrow[d, "\varphi_2"'] \arrow[dr] & \mathscr{G}^{\mathrm{sh}} \arrow[d] \\
\mathscr{K} \arrow[r, "\mathrm{sh}"] & \mathscr{K}^{\mathrm{sh}}
\end{tikzcd}$$

By the universal property of sheafification, we have $(\varphi_2 \circ \varphi_1)^{\mathrm{sh}} = \varphi_2^{\mathrm{sh}} \circ \varphi_1^{\mathrm{sh}}$. Hence, $\square^{\mathrm{sh}}$ is a functor. $\square$

Proposition 3.4.8

$\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{sh}$ *is a natural transformation.*

Proof Let $U \subseteq U' \subseteq X$ be two open subsets, consider following diagram,

$$\begin{tikzcd}
\mathscr{F}(U') \arrow[r, "\mathrm{res}^{\mathscr{F}}_{U',U}"] \arrow[d, "\mathrm{sh}_{U'}"'] & \mathscr{F}(U) \arrow[d, "\mathrm{sh}_U"] \\
\mathscr{F}^{\mathrm{sh}}(U') \arrow[r, "\mathrm{res}^{\mathscr{F}^{\mathrm{sh}}}_{U',U}"] & \mathscr{F}^{\mathrm{sh}}(U),
\end{tikzcd}$$

we shall to show this diagram is commute.

Define $\mathrm{sh}_U : \mathscr{F}(U) \to \mathscr{F}^{\mathrm{sh}}(U)$ by setting $s \mapsto (s_p)_{p\in U}$, where $s_p \in \mathscr{F}_p$. We need to check sh_U is well-defined. Let $(p \in)V \subseteq U$ be an open subset, consider $s|_V$, note that $(s|_V)_q = s_q$ for all $q \in V$. Hence, $(s_p)_{p\in U} \in \mathscr{F}^{\mathrm{sh}}(U)$. Let $s = t \in \mathscr{F}(U)$, then for all $p \in U$ exists $p \in V_s$ and $p \in V_t$ open subsets of U such that exists $\widetilde{s} \in \mathscr{F}(V_s)$ and $\widetilde{t} \in \mathscr{F}(V_t)$ with $\widetilde{s}_i = s_i$ for all $i \in V_s$ and $\widetilde{t}_j = t_j$ for all $j \in V_t$. Pick $\widetilde{s} = s|_{V_s}$ and $\widetilde{t} = t|_{V_t}$, then $\widetilde{s}|_{V_s\cap V_t} = \widetilde{t}|_{V_s\cap V_t}$, since $s = t \in \mathscr{F}(U)$. Hence, $s_i = t_i$ for all $i \in V_s \cap V_t$, and therefore $(s_p)_{p\in U} = (t_p)_{p\in U}$. This implies that sh_U is well-defined.

Let section $s \in \mathscr{F}(U')$, then

$$\mathrm{sh}_{U'}(s)|_U = (s_p)_{p\in U'}|_U = (s_p)_{p\in U}$$

and

$$\mathrm{sh}_U(s|_U) = ((s|_U)_p)_{p\in U} = (s_p)_{p\in U}.$$

Hence,

$$\mathrm{res}^{\mathscr{F}^{\mathrm{sh}}}_{U',U} \circ \mathrm{sh}_{U'} = \mathrm{sh}_U \circ \mathrm{res}^{\mathscr{F}}_{U',U}\,.$$

This implies that $sh : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ is a natural transformation. $\square$

Proposition 3.4.9

The natural transformation sh *satisfies the universal property of sheafification (Definition 3.4.2).*

Proof Let $\mathscr{G}$ be a sheaf with presheaf morphism $g : \mathscr{F} \to \mathscr{G}$. Consider the following diagram.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "g"'] & \mathscr{F}^{\mathrm{sh}} \arrow[d, dashed] \\ & \mathscr{G} \end{tikzcd}$$

Now, we will define $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$. Let $U \subseteq X$ be any open subset. Let $(s_p)_{p\in U} \in \mathscr{F}^{\mathrm{sh}}(U)$, then for all $p \in U$ exists an open neighborhood $p \in V_p \subseteq U$ such that exists a section $\widetilde{s}_p \in \mathscr{F}(V_p)$ with $(\widetilde{s}_p)_q = s_q$ for all $q \in V_p$. Then for all $p \in U$, we have $g(\widetilde{s}_p) \in \mathscr{G}(V_p)$ with $g_q((\widetilde{s}_p)_q) = (g(\widetilde{s}_p))_q = g_q(s_q) = (g(s))_q$ for all $q \in V_p$. This implies that $(g(s)_p)_{p\in U}$ consists of compatible germs. By Proposition 3.4.2, and note that $\mathscr{G}$ is a sheaf, $(g(s)_p)_{p\in U}$ determined a unique section t_U in $\mathscr{G}(U)$. Define $f_U : \mathscr{F}^{\mathrm{sh}}(U) \to \mathscr{G}(U)$ by setting $f_U((s_p)_{p\in U}) = t_U$. It is easy to see that f_U is well-defined and f is a sheaf morphism. And by the construction of f_U, above diagram is commutative. Hence, the natural transformation sh satisfies the universal property of sheafification. $\square$

Proposition 3.4.10

The sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X, i.e.,

$$\mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \cong \mathrm{Mor}(\mathscr{F}, F(\mathscr{G})),$$

where $\mathscr{F}$ is presheaf, $\mathscr{G}$ is sheaf, and $F(\square)$ is forgetful functor.

Proof Define $\tau_{\mathscr{F},\mathscr{G}} : \mathrm{Mor}(\mathscr{F}, F(\mathscr{G})) \to \mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G})$ by setting $\tau_{\mathscr{F},\mathscr{G}}(\varphi) = \widetilde{\varphi}$, where $\widetilde{\varphi} \circ \mathrm{sh} = \varphi$. $\tau_{\mathscr{F},\mathscr{G}}$ is well-defined, since the universal property of sheafification gives unique $\widetilde{\varphi}$.

Consider the following diagram,

$$\begin{tikzcd}
\mathrm{Mor}(\mathscr{F}', F(\mathscr{G})) \arrow[r, "g^*"] \arrow[d, "\tau_{\mathscr{F}',\mathscr{G}}"'] & \mathrm{Mor}(\mathscr{F}, F(\mathscr{G})) \arrow[r, "(Ff)_*"] \arrow[d, "\tau_{\mathscr{F},\mathscr{G}}"] & \mathrm{Mor}(\mathscr{F}, F(\mathscr{G}')) \arrow[d, "\tau_{\mathscr{F},\mathscr{G}'}"] \\
\mathrm{Mor}(\mathscr{F}'^{\mathrm{sh}}, \mathscr{G}) \arrow[r, "(g^{\mathrm{sh}})^*"] & \mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \arrow[r, "f_*"] & \mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}'),
\end{tikzcd} \tag{3.3}$$

where $f : \mathscr{G} \to \mathscr{G}'$ is sheaf morphism and $g : \mathscr{F} \to \mathscr{F}'$ is presheaf. We shall to check diagram (3.3) commutes.

Let $\varphi \in \mathrm{Mor}(\mathscr{F}', F(\mathscr{G}))$, then we have

$$(g^{\mathrm{sh}})^* \circ \tau_{\mathscr{F}',\mathscr{G}}(\varphi) = (g^{\mathrm{sh}})^*(\widetilde{\varphi}) = \widetilde{\varphi} \circ g^{\mathrm{sh}}$$

and

$$\tau_{\mathscr{F},\mathscr{G}} \circ g^*(\varphi) = \tau_{\mathscr{F},\mathscr{G}}(\varphi \circ g) = \widetilde{\varphi \circ g}.$$

Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "g"] \arrow[d] & \mathscr{F}' \arrow[r, "\varphi"] \arrow[d, dashed] & \mathscr{G} \\
\mathscr{F}^{\mathrm{sh}} \arrow[r, "g^{\mathrm{sh}}"'] \arrow[urr, dashed] & \mathscr{F}'^{\mathrm{sh}} \arrow[ur, dashed] &
\end{tikzcd}$$

Note that $\widetilde{\varphi} \circ g^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ and $\widetilde{\varphi \circ g} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$, by the universal property of sheafification, we have $\widetilde{\varphi} \circ g^{\mathrm{sh}} = \widetilde{\varphi \circ g}$.

Let $\psi \in \mathrm{Mor}(\mathscr{F}, F(\mathscr{G}))$, then we have

$$f_* \circ \tau_{\mathscr{F},\mathscr{G}}(\psi) = f_* \circ \widetilde{\psi} = f \circ \widetilde{\psi}$$

and

$$\tau_{\mathscr{F},\mathscr{G}'} \circ (Ff)_*(\psi) = \tau_{\mathscr{F},\mathscr{G}'}(Ff \circ \psi) = \widetilde{Ff \circ \psi}.$$

Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\psi"] \arrow[d] & \mathscr{G} \arrow[r, "f"] & \mathscr{G}' \\
\mathscr{F}^{\mathrm{sh}} \arrow[ur, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

Note that $f \circ \widetilde{\psi} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}'$ and $\widetilde{Ff \circ \psi} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}'$, by the universal property of sheafification, we have $f \circ \widetilde{\psi} = \widetilde{Ff \circ \psi}$.

Hence, diagram (3.3) commutes, and therefore τ is a natural transformation. Now, we give the inverse of τ. Let $\tau^{-1}_{\mathscr{F},\mathscr{G}} : \mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \to \mathrm{Mor}(\mathscr{F}, F(\mathscr{G}))$ defined by $\tau^{-1}_{\mathscr{F},\mathscr{G}}(\varphi) = \widehat{\varphi}$, where $\widehat{\varphi}^{\mathrm{sh}} = F(\varphi)$. Hence, τ is a natural isomorphism, and therefore

$$\mathrm{Mor}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \cong \mathrm{Mor}(\mathscr{F}, F(\mathscr{G})).$$

□

Proposition 3.4.11

$\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ *induces an isomorphism of stalks.*

Proof Let $x \subseteq X$ be any point. Now, we define $\mathscr{F}_x \to \mathscr{F}^{\mathrm{sh}}_x$ as follow: let $s_x \in$, then $s_x = \overline{(s, U)}$, where $x \in U$ and $s \in \mathscr{F}(U)$; apply sh_U, we have $\mathrm{sh}_U(s) = (s_p)_{p \in U} \in \mathscr{F}^{\mathrm{sh}}(U)$; take stalk at x, we have $(\mathrm{sh}_U(s))_x = ((s_p)_{p \in U})_x$. Define $\mathrm{sh}_x : \mathscr{F}_x \to \mathscr{F}^{\mathrm{sh}}_x$ by setting $s_x \mapsto ((s_p)_{p \in U})_x$. We need to check sh_x is well-defined. Let $s_x = t_x$, then there exists open neighborhood U of x contained in X, such that $s|_U = t|_U$, hence, $((s|_U)_p)_{p\in U} = ((t|_U)_p)_{p\in U}$, and therefore $(((s|_U)_p)_{p\in U})_x = (((t|_U)_p)_{p\in U})_x$, this implies that sh_x is well-defined.

Let $((s_p)_{p\in U})_x = ((t_p)_{p\in U})_x$, then there exists an open neighborhood V of x contained in X such that $(s_p)_{p\in V} = (t_p)_{p\in V}$. This implies that $s_p = t_p$ for all $p \in V$. In particular, let $p = x \in V$, we have $s_x = t_x$. Hence, sh_x is an injective.

Let $((s_p)_{p\in U})_x \in \mathscr{F}_x^{\mathrm{sh}}$, then there exists an open neighborhood W of x such that $(s_p)_{p\in W} \in \mathscr{F}^{\mathrm{sh}}(W)$. Since $(s_p)_{p\in W}$ consists of compatible germs, there exists $t \in \mathscr{F}(W)$ such that $t_p = s_p$ for all $p \in W$. Hence, $\mathrm{sh}_x(t_x) = ((t_p)_{p\in W})_x = ((s_p)_{p\in W})_x$, which implies that sh_x is a surjective.

Hence, $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ induces an isomorphism of stalks,

$$\mathscr{F}_x \overset{\sim}{\longleftrightarrow} \mathscr{F}_x^{\mathrm{sh}}.$$

□

Example 3.9 **The sheafification of a constant presheaf is the corresponding constant sheaf.** If X is a topological space and S is a set, then $(\underline{S}_{\mathrm{pre}})^{\mathrm{sh}}$ may be naturally identified with $\underline{S}$.

Proof Let $f \in \underline{S}_{\mathrm{pre}}(U)$, where $U \subseteq X$ be an open subset of X. Then for all $p \in U$, we have $s_p = S$, and therefore $\mathrm{sh}_U(s) = (s_p)_{p\in U}$. Hence, $(\underline{S}_{\mathrm{pre}})^{\mathrm{sh}}(U) = \prod S = \underline{S}(U)$ for all open subset $U \subseteq X$. □

Remark The "space of sections" (or espace étalé) construction (Definition 3.2.10) yields a different-sounding description of sheafification. The main ideal is identical: if $\mathscr{F}$ is a presheaf, let $\mathscr{F}$ is a presheaf, let F be the espace étalé of $\mathscr{F}$. In fact, if $\mathscr{F}$ is a presheaf, the sheaf of sections of $F \to X$ (Exercise 3.7) is the sheafification of $\mathscr{F}$. Constant sheaf may be interpreted as an example of this construction. The "espace étalé" construction of the sheafification is essentially the same as Construction

$$\mathscr{F}^{\mathrm{sh}}(U) := \left\{ (f_p \in \mathscr{F}_p)_{p\in U} \;\middle|\; \begin{array}{l} \text{for all } p \in U, \text{ there exists an open neighborhood } V \text{ of } p, \\ \text{contained in } U, \text{ and } s \in \mathscr{F}(V) \text{ with } s_q = f_q \text{ for all } q \in V \end{array} \right\}.$$

3.4.3 Subsheaves and quotient sheaves

We now discuss subsheaves and quotient sheaves from the perspective of stalks.

Proposition 3.4.12

Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. The following are equivalent.

(a) φ is a monomorphism in the category of sheaves.

(b) φ is injective on the level of stalks: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is injective for all $p \in X$.

(c) φ is injective on the level of open sets: $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subseteq X$.

Proof $(a) \Rightarrow (c)$. Since φ is a monomorphism in the category of sheaves, then we have sheaf of kernel $\operatorname{Ker}\varphi = 0$. Hence, for all open subset $U \subseteq X$, we have $\operatorname{Ker}\varphi(U) = \operatorname{Ker}\varphi_U = 0$. This implies that φ_U is injective.

$(c) \Rightarrow (b)$. Let $s_p \in \mathscr{F}_p$ with $\varphi_p(s_p) = 0$. We may assume that $s_p = \overline{(s, U)}$, where $(x \in) U \subseteq X$. Then $\varphi_p(s_p) = (\varphi_U(s))_p = 0$, hence, there exists open neighborhood V of p contained in U such that $\varphi_U(s)|_V = 0$. Note that $\varphi_V(s|_V) = \varphi_U(s)|_V = 0$ and φ_V is injective. We have $s|_V = 0$, and therefore $s_p = 0$, which implies that φ_p is injective for all $p \in X$.

$(b) \Rightarrow (a)$. Let $\psi_1 : \mathscr{E} \to \mathscr{F}$ and $\psi_2 : \mathscr{E} \to \mathscr{F}$ be two morphisms with $\varphi \circ \psi_1 = \varphi \circ \psi_2$, we shall to show that $\psi_1 = \psi_2$. Let $U \subseteq X$ be any open subset, consider the following diagram.

$$\begin{tikzcd}
\mathscr{E}(U) \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d] & \mathscr{F}(U) \arrow[r, "\varphi"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\prod_{p\in U} \mathscr{E}_p \arrow[r, shift left, "((\psi_1)_p)_{p\in U}"] \arrow[r, shift right, "((\psi_2)_p)_{p\in U}"'] & \prod_{p\in U} \mathscr{F}_p \arrow[r, "(\varphi_p)_{p\in U}"] & \prod_{p\in U} \mathscr{G}_p
\end{tikzcd}$$

Say $\widetilde{\varphi_1} = ((\psi_1)_p)_{p\in U}$, $\widetilde{\varphi_2} = ((\psi_2)_p)_{p\in U}$, and $\widetilde{\varphi} = (\varphi_p)_{p\in U}$. Since $\varphi \circ \psi_1 = \varphi_1 \circ \psi_2$, by the universal property of colimit, induced map satisfies $\widetilde{\varphi \circ \psi_i} = \widetilde{\varphi} \circ \widetilde{\psi_i}$ for $i = 1, 2$. Since each φ_p is injective, induced map $\widetilde{\varphi}$ is injective, and therefore $\widetilde{\psi_1} = \widetilde{\psi_2}$. By Proposition 3.4.3, $\psi_1 = \psi_2$, which implies that φ is a monomorphism. $\square$

Definition 3.4.3 (Subsheaf)

If $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on a topological space X which satisfies the conditions in Proposition 3.4.12, we say that $\mathscr{F}$ is a ***subsheaf*** *of $\mathscr{G}$ (where the "inclusion" φ is sometimes left implicit).*

Remark In fact, for any morphism of presheaves, if all maps of sections are injective, then all stalk maps are injective. And furthermore, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism from a separated presheaf to an arbitrary presheaf, then injectivity on the level of stalks implies that φ is a monomorphism in the category of presheaves.

Proposition 3.4.13

If $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of a sets on a topological space X. The following are equivalent.

(a) φ is an epimorphism in the category of sheaves.

(b) φ is surjective on the level of stalks: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective for all $p \in X$.

Proof $(a) \Rightarrow (b)$. Proof by contradiction. Suppose there exists $p_0 \in X$ such that $\varphi_{p_0} : \mathscr{F}_{p_0} \to \mathscr{G}_{p_0}$ is not surjective. Let $S = \{0, 1\}$, and consider the skyscraper sheaf $i_{p_0,*}S$. For all open subset $U \subseteq X$, $i_{p_0,*}S(U)$ defined by

$$i_{p_0,*}S(U) = \begin{cases} S & \text{if } p_0 \in U, \\ 0 & \text{if } p_0 \notin U. \end{cases}$$

Let $U \subseteq X$ be an open neighborhood of p_0, then $i_{p_0,*}S(U) = \{0, 1\}$. Let $s \in G(U)$, define

$$\alpha_U(s) = \begin{cases} 0 & \text{if } s_{p_0} \in \operatorname{Im}\varphi_{p_0} \\ 0 & \text{if } s_{p_0} \notin \operatorname{Im}\varphi_{p_0} \end{cases} \text{ and } \beta_U(s) = \begin{cases} 0 & \text{if } s_{p_0} \in \operatorname{Im}\varphi_{p_0} \\ 1 & \text{if } s_{p_0} \notin \operatorname{Im}\varphi_{p_0} \end{cases}.$$

Also, let $(p \notin) U \subseteq X$, for $s \in \mathscr{G}(U)$, define $\alpha_U(s) = \beta_U(s) = 0$. It is easy to see that α and β is sheaf morphism.

Now, we need to check that $\alpha_U \circ \varphi_U = \beta_U \circ \varphi_U$ for all open subset $U \subseteq X$. If $p_0 \in U$, note that $\varphi_{p_0}(t_{p_0}) \in \operatorname{Im}\varphi_{p_0}$ for all $t \in \mathscr{F}(U)$, we have

$$\alpha_U \circ \varphi_U(t) = \beta_U \circ \varphi_U(t) = 0;$$

if $p_0 \notin U$, we also have

$$\alpha_U \circ \varphi_U(t) = \beta_U \circ \varphi_U(t) = 0.$$

Hence, $\alpha_U \circ \varphi_U = \beta_U \circ \varphi_U$ for all open subset $U \subseteq X$.

But $\alpha \neq \beta$ as sheaf morphisms, which contradicts the fact that φ is an epimorphism.

$(b) \Rightarrow (a)$. Let $\alpha : \mathscr{G} \to \mathscr{E}$ and $\alpha : \mathscr{G} \to \mathscr{E}$ be two sheaf morphisms with $\alpha \circ \varphi = \beta \circ \varphi$, we shall to show that $\alpha = \beta$. Let $U \subseteq X$ be any open subset, consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi"] \arrow[d] & \mathscr{G}(U) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d] & \mathscr{E}(U) \arrow[d] \\
\prod_{p\in U} \mathscr{F}_p \arrow[r, "(\varphi_p)_{p\in U}"] & \prod_{p\in U} \mathscr{G}_p \arrow[r, shift left, "\tilde{\alpha}"] \arrow[r, shift right, "\tilde{\beta}"'] & \prod_{p\in U} \mathscr{E}_p
\end{tikzcd}$$

Use the same method as Proposition 3.4.12 $(b) \Rightarrow (a)$, we have $\alpha = \beta$. Hence, φ is an epimorphism. □

Definition 3.4.4 (Quotient sheaf)

Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of a sets on a topological space X, if the conditions in Proposition 3.4.13 hold, we say that $\mathscr{G}$ is a ***quotient sheaf*** *of $\mathscr{F}$.*

By Proposition 3.4.12 and Proposition 3.4.13, monomorphisms and epimorphisms — subsheafiness and quotient sheafiness — can be checked at the level of stalks.

Both propositions generalize readily to sheaves with values in any reasonable category, where "injective" is replaced by "monomorphism" and "surjective" is replaced by "epimorphism".

Notice that there was no part (c) to Proposition 3.4.13, and Example 3.10 shows why. (But there is a version of (c) that implies (a) and (b): surjective on all open sets in any base of a topology implies that the corresponding map of sheaves is an epimorphism.)

Example 3.10 Let $X = \mathbb{C}$ with the classical topology, and define $\mathscr{O}_X$ to be the sheaf of holomorphic functions, and $\mathscr{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 1.$$

(0 means the zero sheaf, for all open subset $U \subseteq X$, we have $0(U) = \{0\}$; 1 means the unit sheaf, for all open subset $U \subseteq X$, we have $1(U) = \{1\}$, where 1 is the unit of multiplication groups.) We will soon interpret this as an exact sequence of sheaves of abelian groups (the **exponential exact sequence**), although we don't yet have the language to do so.

Exercise 3.17 Show that $\exp : \mathscr{O}_X \to \mathscr{O}_X^*$ describes $\mathscr{O}_X^*$ as quotient sheaf of $\mathscr{O}_X$. Find an open set on which this map is not surjective.

Proof Let $\overline{(g, U)} \in \mathscr{O}_X^*$, then there exists a simply connected open set $(p \in)U \subseteq \mathbb{C}$ such that $g \in \mathscr{O}_X^*(U)$, that is, g is analytic and nowhere vanishing. Let

$$f(z) = \int_{z_0}^{z} \frac{g'(\xi)}{g(\xi)} d\xi + c,$$

where $c \in \mathbb{C}$ and $e^c = g(z_0)$. We need to show that $f(z)$ is analytic over U, and $\exp_U f(z) = g(z)$.

Since g nowhere zero and U is simply connected, $\frac{g'(z)}{g(z)}$ is holomorphic on U, and therefore $f'(z) = \frac{g'(z)}{g(z)}$ is holomorphic on U, hence, $f(z) \in H(D)$.

It is easy to see that $\exp_U f(z) = g(z)$. Hence, on the level of stalks $\exp_p : \mathscr{O}_{X,P} \to \mathscr{O}_X^*$ is a surjective for all $p \in \mathbb{C}$. By Proposition 3.4.13, $\exp : \mathscr{O}_X \to \mathscr{O}_X^*$ is an epimorphism.

Now, we give an example to show that there exists an open set U, such that $\mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$ is not surjective. Let $U = \mathbb{C} \setminus \{0\}$, then $g(z) = z \in \mathscr{O}_X^*(U)$. If there exists $f \in \mathscr{O}_X(U)$ with $e^f = z$, then $f = \log z$ would be a single-valued branch of $\log z$, contradicting the multivalued nature of $\log z$ on U. □

Remark This is a great example to get a sense of what "surjectivity" means for sheaves: nowhere vanishing holomorphic functions (such as the function z away from the origin) have logarithms locally, but they need not have logarithms globally.

3.5 Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get our hands on. We like the identity and gluability axioms, but they make proving things tricker than for presheaves. We have discussed how we can understand sheaves using stalks (using "compatible germs"). We now introducing the notion of a sheaf on a base. **Warning:** this way of understanding an entire sheaf from limited information is confusing. It may help to keep sight of the central insight that this partial information is enough to understand germs, and the notion of when they are compatible (with nearby germs).

3.5.1 Sheaf on a base

Definition 3.5.1 (Base of topology)

Suppose X be a topological space with topology $\mathscr{T}_X$, a ***base of a topology*** *is a subcollection of the open sets $\{B_i\} \subseteq \mathscr{T}_X$, such that each $U \in \mathscr{T}_X$ is a union of the B_i. We say $\mathscr{B} = \{B_i\}$.*

Example 3.11 Suppose $X = \mathbb{R}^n$. Then the way the classical topology is often first defined is by defining open balls $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and declearing that any union of open balls is open. So the balls form a base of the classical topology — we say they generate the classical topology.

As an application of how we use them, to check continuity of some map $\pi : X \to \mathbb{R}^n$, we need only thinking about the pullback of balls on $\mathbb{R}^n$ — part of the traditional $\varepsilon - \delta$ definition of continuity.

Now suppose we have a sheaf $\mathscr{F}$ on a topological space X, and a base $\{B_i\}$ of open sets on X. Then consider the information

$$\left(\{\mathscr{F}(B_i)\}, \{\mathrm{res}_{B_i,B_j} : \mathscr{F}(B_i) \to \mathscr{F}(B_j)\}\right),$$

which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

We can recover the entire sheaf from this information. This is because we can determine the stalks from this information, and we can determine when germs are compatible.

Exercise 3.18 Make this precise. How can you recover a sheaf $\mathscr{F}$ from this partial information?

Proof Let $\mathscr{B} = \{B_i\}$ be a base of topology space X. Say $B = \bigcup_i B_i$.

Let $\mathscr{F}_x = \varinjlim_{x \in B_i} \mathscr{F}(B_i)$. If $s_p = t_p \in \mathscr{F}_x$, we may assume $s \in \mathscr{F}(B_i)$ and $t \in \mathscr{F}(B_j)$, then there exists $x \in B_k$ with $B_k \subseteq B_i \cap B_j$ such that $s|_{B_k} = t|_{B_k}$.

Let $U \subseteq B$ be any open set, define

$$\mathscr{F}(U) = \{(s_p)_{p \in U} : \forall p \in U, \exists (U \supseteq) B_i \ni p \text{ and } \widetilde{s} \in \mathscr{F}(B_i) \text{ s.t. } \widetilde{s}_q = s_q, \forall q \in B_i\}.$$

We shall to show that $\mathscr{F}$ form a sheaf on B.

(1) $\mathscr{F}$ is a presheaf.

Let $U \subseteq U'$, be two open subsets of X, the restriction map $\mathrm{res}_{U',U} : \mathscr{F}(U') \to \mathscr{F}(U)$ given by

$$\mathrm{res}_{U',U}((s_p)_{p \in U'}) = (s_p)_{p \in U}.$$

By the definition of base of topology, the restriction map is well-defined.

Note that for all $(s_p)_{p \in U} \in \mathscr{F}(U)$, we have $\mathrm{res}_{U,U}((s_p)_{p \in U}) = (s_p)_{p \in U}$, i.e., $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$.

The transitivity is clearly by the definition of base of topology, and therefore $\mathscr{F}$ form a presheaf on X.

(2) $\mathscr{F}$ is a sheaf.

Let U be any open subset of X. Since $\{B_i\}$ is a base of topology, U is covered by some $B_i \in \mathscr{B}$, i.e., $U = \bigcup_i B_i$.

For identity axiom: Let $(s_p)_{p\in U}, (t_p)_{p\in U} \in \mathscr{F}(U)$ with $(s_p)_{p\in U}|_{B_i} = (t_p)_{p\in U}|_{B_i}$, i.e., $(s_p)_{p\in B_i} = (t_p)_{p\in B_i}$. Hence, for all $p \in B_i$, there exists $(p \in)V_{p,i} \subseteq B_i$ such that $s|_{V_{p,i}} = t|_{V_{p,i}}$, hence $s_p = t_p$ for all $p \in U$, that is, $(s_p)_{p\in U} = (t_p)_{p\in U}$, which implies that identity axiom holds.

For gluability axiom: Let $s_i \in \mathscr{F}(B_i)$, and $s_i|_{B_i\cap B_j} = s_j|_{B_i\cap B_j}$. Define $(s_p)_{p\in U}$ as follow: $(s_p)_{p\in U}|_{B_i} = (s_{ip})_{p\in B_i}$. Since $s_i|_{B_i\cap B_j} = s_j|_{B_i\cap B_j}$, $(s_p)_{p\in U}$ is well-defined. It suffices to show that $(s_p)_{p\in U} \in \mathscr{F}(U)$. For all $p \in U$, since $U = \bigcup_i B_i$, $p \in B_i$ for some i. Note that $(s_q)_{q\in U}|_{B_i} = (s_{iq})_{q\in B_i}$, we have $s_q = s_{iq}$ for all $q \in B_i$. Also note that $s_i \in \mathscr{F}(B_i)$, we have $(s_p)_{p\in U} \in \mathscr{F}(U)$.

Hence, $\mathscr{F}$ is a presheaf, which satisfies identity axiom and gluability axiom, and therefore $\mathscr{F}$ is a sheaf.

□

Definition 3.5.2 (Sheaf on a base)

A sheaf of sets (or abelian groups, rings, $\cdots$) on a base *$\{B_i\}$ is the following:*

(i) For each B_i in the base, we have a set $F(B_i)$. If $B_i \subseteq B_j$, we have maps

$$\operatorname{res}_{B_j,B_i} : F(B_j) \to F(B_i),$$

with $\operatorname{res}_{B_i,B_i} = \operatorname{id}_{F(B_i)}$. If $B_i \subseteq B_j \subseteq B_k$, then

$$\operatorname{res}_{B_k,B_i} = \operatorname{res}_{B_j,B_i} \circ \operatorname{res}_{B_k,B_j}.$$

*So far we have defined a **presheaf on a base** $\{B_i\}$.*

*(ii) **The base identity axiom:** If $B = \bigcup B_i$, then if $f, g \in F(B)$ are such that $\operatorname{res}_{B,B_i} f = \operatorname{res}_{B,B_i} g$ for all i, then $f = g$.*

*(iii) **The base gluability axiom:** If $B = \bigcup B_i$, and we have $f_i \in F(B_i)$ such that f_i agrees with f_j on any basic open set contained in $B_i \cap B_j$ (i.e., $\operatorname{res}_{B_i,B_k} f_i = \operatorname{res}_{B_j,B_k} f_j$, for all $B_k \subseteq B_i \cap B_j$) then there exists $f \in F(B)$ such that $\operatorname{res}_{B,B_i} f = f_i$ for all i.*

Theorem 3.5.1

Suppose $\{B_i\}$ is a base on X, and F is a sheaf of sets on this base. Then there is a sheaf $\mathscr{F}$ extending F with isomorphisms $\mathscr{F}(B_i) \xleftrightarrow{\sim} F(B_i)$ agreeing with the restriction maps. This sheaf $\mathscr{F}$ is unique up to unique isomorphism.

Proof We will define $\mathscr{F}$ as the sheaf of families of compatible germs of F. Defined the **stalk** of a presheaf F on a base at $p \in X$ by

$$F_p = \varinjlim F(B_i)$$

where the colimit is over all B_i (in the base) containing p.

We will say a family of germs in an open set U is compatible near p if there is a section s of F over some B_i containing p such that the germs over B_i are precisely the germs of s. More formally, define

$$\mathscr{F}(U) := \{(f_p \in F_p)_{p\in U} : \forall p \in U, \exists B \text{ with } p \in B \subseteq U \text{ and } s \in F(B), \text{ s.t. } s_q = f_q, \forall q \in B\}$$

where each B is in our base.

Same as Exercise 3.18, this is a sheaf. Now, we claim that if B is in our base, the natural map $F(B) \to \mathscr{F}(B)$ is an isomorphism.

Natural map $\theta : F(B) \to \mathscr{F}(B)$ is given by $s \mapsto (s_p)_{p\in B}$. By the definition of $\mathscr{F}(U)$, θ is well-defined. Let $(f_p)_{p\in B} \in \mathscr{F}(B)$, then for all $p \in B$ exists $B_p \subseteq U$ and $\widetilde{s}_p \in F(B)$ such that $(\widetilde{s}_p)_q = f_q$

for all $q \in B_p$. Hence, for all $q \in B_p$, there exists $q \in B_{p,q}(\in \mathscr{B})$ such that $\widetilde{s}_p|_{B_{p,q}} = f|_{B_{p,q}}$, and therefore $(f|_{B_{p,q_1}})|_{B_{p,q_1} \cap B_{p,q_2}} = (f|_{B_{p,q_2}})|_{B_{p,q_1} \cap B_{p,q_2}}$ for all $q_1, q_2 \in B_p$ for all $p \in B$, moreover exists $(\mathscr{B} \ni) B_{p,q_1,q_2} \subseteq B_{p,q_1} \cap B_{p,q_2}$ such that $(f|_{B_{p,q_1}})|_{B_{p,q_1,q_2}} = (f|_{B_{p,q_2}})|_{B_{p,q_1,q_2}}$. Since F is sheaf on a base, by the base gluability axiom, there exists $\widetilde{f} \in F(B)$ such that $\widetilde{f}|_{B_i} = f|_{B_i}$. Also, apply θ to $\widetilde{f}$, we have $\widetilde{f}_p = f_p$ for all $p \in B$. Define $\tau : \mathscr{F}(B) \to F(B)$ by setting

$$(f_p \in \mathscr{F}_p)_{p \in B} \mapsto \widetilde{f},$$

by above discussion τ is well-defined and $\theta \circ \tau = \mathrm{id}_{\mathscr{F}(B)}$. Let $s \in F(B)$, then

$$\tau \circ \theta(s) = \tau((s_p \in \mathscr{F}_p)_{p \in B}) = \widetilde{s},$$

by the base identity axiom $s = \widetilde{s}$, this implies that $\tau \circ \theta = \mathrm{id}_{F(B)}$. Hence, we have an isomorphism:

$$\mathscr{F}(B) \xleftrightarrow{\sim} F(B),$$

where $B \in \mathscr{B}$.

Clearly, the restriction maps for F are the same as the restriction maps of $\mathscr{F}$ (for elements of the base).

Finally, we shall to show that $\mathscr{F}$ is indeed unique up to unique isomorphism. Let $\mathscr{G}$ be a sheaf which extending F with isomorphisms $\mathscr{G}(B_i) \xleftrightarrow{\sim} F(B_i)$. It suffices to show that for all open subset $U \subseteq X$, we have $\mathscr{F}(U) \xleftrightarrow{\sim} \mathscr{G}(U)$. Let $f \in \mathscr{F}(U)$, since U is covered by some B_i and $\mathscr{G}(B_i) \xleftrightarrow{\sim} F(B_i)$, $f|_{B_i} \in \mathscr{G}(B_i)$, by the identity axiom of $\mathscr{G}$ and the gluability axiom of $\mathscr{G}$, we have unique $g \in \mathscr{G}(U)$ such that $g|_{B_i} = f|_{B_i}$. This gives a one-to-one correspondence between $\mathscr{F}(U)$ and $\mathscr{G}(U)$ for all open subset $U \subseteq X$. $\square$

Remark Theorem 3.5.1 shows that sheaves on X can be recovered from their "restriction to a base". It is clear from the argument that if $\mathscr{F}$ is a sheaf and F is the corresponding sheaf on the base B, then for any $p \in X$, $\mathscr{F}_p$ is naturally isomorphic to F_p, i.e., $\mathscr{F}_p \cong F_p$.

Theorem 3.5.1 is a statement about objects in a category, so we should hope for a similar statement about morphisms.

Theorem 3.5.2 (Morphisms of sheaves correspond to morphisms of sheaves on a base)

Suppose $\{B_i\}$ is a base for the topology of X. A morphism $F \to G$ of sheaves on the base is a collection of maps $F(B_k) \to G(B_k)$ such that the diagram

$$\begin{tikzcd} F(B_i) \arrow[r] \arrow[d, "\mathrm{res}^F_{B_i,B_j}"'] & G(B_i) \arrow[d, "\mathrm{res}^G_{B_i,B_j}"] \\ F(B_j) \arrow[r] & G(B_j) \end{tikzcd}$$

commutes for all $B_j \hookrightarrow B_i$.

(a) A morphism of sheaves is determined by the induced morphism of sheaves on the base.

(b) A morphism of sheaves on the base gives a morphism of the induced sheaves.

Proof

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{F} \to \mathscr{G}$ with $\varphi_{B_k} = \psi_{B_k}$. We shall to show that $\varphi = \psi$. It suffices to show that for all open subset $U \subseteq X$ we have $\varphi_U = \psi_U$.

Let U be any open subset of X, and therefore is covered by $\{B_k\}$. Consider the following commutative diagram.

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U,B_k}"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U,B_k}"] \\ \mathscr{F}(B_k) \arrow[r] & \mathscr{G}(B_k) \end{tikzcd}$$

Let $(f_p \in \mathscr{F}_p)_{p\in U} \in \mathscr{F}(U)$, then we have

$$\mathrm{res}^{\mathscr{G}}_{U,B_k} \circ \varphi_U((f_p)_{p\in U}) = \mathrm{res}^{\mathscr{G}}_{U,B_k}((\varphi_U(f)_p)_{p\in U}) = (\varphi_U(f)_p)_{p\in B_k},$$

similarly, we have $\mathrm{res}^{\mathscr{G}}_{U,B_k} \circ \psi_U((f_p)_{p\in U}) = (\psi_U(f)_p)_{p\in B_k}$. Since $\psi_{B_k} = \varphi_{B_k}$, we have

$$(\varphi_U(f)_p)_{p\in B_k} = (\psi_U(f)_p)_{p\in B_k},$$

hence, $\mathrm{res}^{\mathscr{G}}_{U,B_k} \circ \varphi_U((f_p)_{p\in U}) = \mathrm{res}^{\mathscr{G}}_{U,B_k} \circ \psi_U((f_p)_{p\in U})$. Since U is coved by $\{B_k\}$, by the identity axiom of $\mathscr{G}$, we have $\varphi_U((f_p)_{p\in U}) = \psi_U((f_p)_{p\in U})$, i.e., $\varphi_U = \psi_U$. Hence, $\varphi = \psi$.

(b) Let $\varphi : F \to G$ be a morphism of sheaves on the base. We need to show that there is a morphism of sheaves $\varphi^\sharp : \mathscr{F} \to \mathscr{G}$ induced by φ. Let U be any open set of X, and therefore U is covered by some base, i.e., $U = \bigcup_i B_i$.
Define $\varphi^\sharp : \mathscr{F} \to \mathscr{G}$ by setting

$$(f_p \in \mathscr{F}_p)_{p\in U} \mapsto (g_p \in \mathscr{G}_p)_{p\in U},$$

where $g_p = \varphi_{B_k}(f)_p$ if $p \in B_k$. We need to show that $\varphi^\sharp$ is well-defined. Let $(f_p)_{p\in U} \in \mathscr{F}(U)$, then for all $p \in U$, exists $B_k \subseteq U$ contained p and $\widetilde{f} \in \mathscr{F}(B_k)$ such that $\widetilde{f}_q = f_q$ for all $q \in B_k$. Hence, for all for all $p \in U$, exists $B_k \subseteq U$ contained p and $\varphi_{B_k}(\widetilde{f}) \in \mathscr{G}(B_k)$ such that $\varphi_{B_k}(\widetilde{f})_q = \varphi_{B_k}(f)_q$ for all $q \in B_k$. Note that $g_p = \varphi_{B_k}(f)_p$ if $p \in B_k$, $(g_p \in \mathscr{G}_p)_{p\in U} \in \mathscr{G}(U)$, which implies that $\varphi^\sharp$ is well-defined.
Now, we need to check $\varphi^\sharp$ is a morphism of sheaves. Let $U' \supseteq U$, and $U = \bigcup B_i$, $U' = \bigcup B'_i$, we may assume that $B'_i \supseteq B_i$. Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U') \arrow[r, "\varphi^\sharp_{U'}"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{U',U}"'] & \mathscr{G}(U') \arrow[d, "\mathrm{res}^{\mathscr{G}}_{U',U}"] \\
\mathscr{F}(U) \arrow[r, "\varphi^\sharp_U"] & \mathscr{G}(U)
\end{tikzcd}$$

Let $(f_p)_{p\in U'} \in \mathscr{F}(U')$, then

$$\varphi^\sharp_U(\mathrm{res}^{\mathscr{F}}_{U',U}((f_p)_{p\in U'})) = \varphi^\sharp_U((f_p)_{p\in U}) = (g_p)_{p\in U},$$

where $g_p = \varphi_{B_k}(f)_p$ if $p \in B_k$. On the other hand, we have

$$\mathrm{res}^{\mathscr{G}}_{U',U} \circ \varphi^\sharp_{U'}((f_p)_{p\in U'}) = \mathrm{res}^{\mathscr{G}}_{U',U}((g_p)_{p\in U'}) = (g_p)_{p\in U},$$

where $g_p = \varphi_{B'_k}(f)_p$ if $p \in B'_k$. Since $B'_k \supseteq B_k$, we have

$$\varphi^\sharp_U(\mathrm{res}^{\mathscr{F}}_{U',U}((f_p)_{p\in U'})) = \mathrm{res}^{\mathscr{G}}_{U',U} \circ \varphi^\sharp_{U'}((f_p)_{p\in U'}).$$

This implies that $\varphi^\sharp$ is a morphism of sheaves.

□

Remark The above constructions and arguments describe an equivalence of categories (Definition 2.1.10) between sheaves on X and sheaves on a given base of X. (Functors are restriction $\mathbf{Sets}_X \to \mathbf{Sets}_{\mathscr{B}}$ and extension $\mathbf{Sets}_{\mathscr{B}} \to \mathbf{Sets}_X$. By Theorem 3.5.2, identity holds.)

Remark It will be useful to extend these notion to $\mathscr{O}_X$-modules. It is easy to verify that there is a correspondence (really, equivalence of categories) between $\mathscr{O}_X$-modules and "$\mathscr{O}_X$-modules on a base". ($\mathscr{O}_X$-modules on a base: for each $B_i \in \mathscr{B}$, $\mathscr{F}(B_i)$ is an $\mathscr{O}_X(B_i)$-module, and for $B_j \subseteq B_i \in \mathscr{B}$, the restriction maps res_{B_i,B_j} are module homomorphisms satisfying the sheaf axioms.)

Proposition 3.5.1

Suppose a morphism of sheaves of sets $F \to G$ on a base $\{B_i\}$ is surjective for all B_i (i.e., $F(B_i) \to G(B_i)$ is surjective for all i). Then the corresponding morphism of sheaves (not on the base) is an epimorphism.

Proof Let $\mathscr{B} = \{B_i\}$ be a base of X. Say $\varphi^\sharp : \mathscr{F} \to \mathscr{G}$, where $\varphi : F \to G$ and $\mathscr{F}, \mathscr{G}$ corresponding sheaves of F, G respectively. By Proposition 3.4.13, it suffices to show that $\varphi^\sharp$ is surjective on the level of stalks, i.e., $\varphi^\sharp_q : \mathscr{F}_q \to \mathscr{G}_q$ is surjective for all $q \in X$. Let $s_q \in \mathscr{G}_q$, we may assume that s_q is the image of $(f_p)_{p\in U}$ for some U, then $(f_p)_{p\in U} \in \mathscr{G}(U)$. Hence, for all $p \in U$, exists $p \in B_p \subseteq U$ and $\widetilde{f_p} \in G(B_p)$ such that $(\widetilde{f_p})_q = f_q$ for all $q \in B_p$. Since $(\widetilde{f_p})_q = f_q$ for all $q \in B_p$, there exist $B_{p,q} \subseteq B_q (\in \mathscr{B})$ such that $\widetilde{f_p}|_{B_{p,q}} = f|_{B_{p,q}}$. Since $\varphi_{B_p} : F(B_p) \to G(B_p)$ is surjective, there exists $\widetilde{g_p} \in G(B_p)$ such that $\varphi_{B_p}(\widetilde{g_p}) = \widetilde{f_p}$. Hence, $\varphi_{B_p}(\widetilde{g_p})|_{B_{p,q}} = \widetilde{f}|_{B_{p,q}}$. Similarly, there exists $g \in F(B_p)$ such that $\varphi_{B_p}(g)|_{B_{p,q}} = f|_{B_{p,q}}$. Hence, $\varphi_{B_p}(\widetilde{g_p})|_{B_{p,q}} = \varphi_{B_p}(g)|_{B_{p,q}} = f|_{B_{p,q}}$, and therefore $f_q = \varphi_{B_p}(g)_q$ for all $q \in B_p$. This implies that $\varphi^\sharp_q$ is surjective for all $q \in U$ for any open subset $U \subseteq X$, then we done! $\square$

Remark The converse is not true (Exercise 3.17), unlike the case of for injectivity. This gives a useful sufficient criterion for epimorphism: a morphism of sheaves is an epimorphism if it is surjective for sections on base.

3.5.2 Gluing sheaves

We will repeatedly see the theme of constructing some object by gluing, in many different contexts. Keep an eye out for it! In each case, we carefully consider what information we need in order to glue.

Theorem 3.5.3 (Gluing sheaves)

Suppose $X = \bigcup U_i$ is an open cover of X, and we have sheaves $\mathscr{F}_i$ on U_i along with isomorphisms

$$\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i\cap U_j}$$

(with φ_{ii} the identity) that agree on triple overlaps, i.e.,

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$ (this is called the ***cocycle condition****, for reasons we ignore). Then these sheaves can be glued together into a sheaf $\mathscr{F}$ on X (unique up to unique isomorphism), with isomorphisms $\mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones.*

Proof Let V be any open subset of X, define

$$\mathscr{F}(V) = \left\{ (s_i) \in \prod \mathscr{F}_i(V \cap U_i) : \varphi_{ij}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j}, \text{ for all } i, j \right\}.$$

Now, we define restriction map as follow: let $V' \supseteq V$ be an open subset of X, define $\mathrm{res}_{V',V} : \mathscr{F}(V') \to \mathscr{F}(V)$ by setting

$$(s_i) \mapsto (s_i|_{V\cap U_i}).$$

Note that $s_i|_{V\cap U_i} \in \mathscr{F}_i(V \cap U_i)$ and

$$\begin{aligned}
\varphi_{ij}((s_i|_{V\cap U_i})|_{V\cap U_i\cap U_j}) &= \varphi_{ij}((s_i|_{V'\cap U_i\cap U_j})|_{V\cap U_i\cap U_j}) \\
&= (\varphi_{ij}(s_i|_{V'\cap U_i\cap U_j}))|_{V\cap U_i\cap U_j} \\
&= (s_j|_{V'\cap U_i\cap U_j})|_{V\cap U_i\cap U_j} \\
&= s_j|_{V\cap U_i\cap U_j},
\end{aligned}$$

the restriction map is well-defined. Also, it is easy to see that $\mathscr{F}$ is a presheaf.

(i) $\mathscr{F}$ is a sheaf.

Let V be any open subset of X, then V is covered by $U_i \cap V$, say $V_i = U_i \cap V$.

For identity axiom: Let $(s_i), (t_i) \in \mathscr{F}(V)$ with $(s_i)|_{V_k} = (t_i)|_{V_k}$ for all k. Since $(s_i)|_{V_k} = (s_i|_{V_k \cap U_i}) = (s_i|_{V \cap U_k \cap U_i}) = (s_i|_{V_i \cap U_k})$ and $(t_i)|_{V_k} = t_i|_{V_i \cap U_k}$, we have

$$s_i|_{V_i \cap U_k} = t_i|_{V_i \cap U_k}$$

for all k. Note that V_i is covered by $\{V_i \cap U_k\}_k$, by the identity axiom of $\mathscr{F}_i$, $s_i = t_i$, and therefore $(s_i) = (t_i)$, which implies that $\mathscr{F}$ satisfies the identity axiom holds.

For gluability axiom: Let $(s_i^{(n)}) \in \mathscr{F}(V_n)$ with $(s_i^{(n)})|_{V_n \cap V_m} = (s_i^{(m)})|_{V_n \cap V_m}$, then

$$\begin{aligned} s_i^{(n)}|_{V_i \cap V_n \cap V_m} &= s_i^{(n)}|_{V_i \cap (V_i \cap U_n) \cap (V_i \cap U_m)} \\ &= s_i^{(m)}|_{V_i \cap V_n \cap V_m} \\ &= s_i^{(m)}|_{V_i \cap (V_i \cap U_n) \cap (V_i \cap U_m)} \end{aligned}$$

for all i. Notice that V_i is covered by $\{V_i \cap U_n\}_n$, by the gluability axiom of $\mathscr{F}_i$, there exists unique $s_i \in \mathscr{F}_i(V_i)$ such that $s_i|_{V_i \cap U_n} = s_i^{(n)}$. Now, we need to check that $(s_i) \in \mathscr{F}(V)$. Note that

$$\varphi_{ij}(s_i|_{V \cap U_i \cap U_j}) = \varphi_{ij}(s_i^{(j)}) = s_j^{(j)} = s_j|_{V \cap U_i \cap U_j},$$

$(s_i) \in \mathscr{F}(V)$, which implies that $\mathscr{F}$ satisfies the gluability axiom.

(ii) $\mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$.

Let V be an open subset of U_k, then

$$\mathscr{F}|_{U_k}(V) = \mathscr{F}(V) = \left\{ (s_i) \in \prod \mathscr{F}_i(V \cap U_i) : \varphi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}, \text{ for all } i, j \right\}.$$

Define $\psi_k : \mathscr{F}|_{U_k} \to \mathscr{F}_k$ as follow:

$$(\psi_k)_V((s_i)) = s_k$$

for all V open subset of U_k. Since $\mathscr{F}_k(V \cap U_k) = \mathscr{F}_k(V)$, ψ_k is well-defined. We shall to check ψ_k is a morphism of sheaves for all k, and give the inverse of ψ_k^{-1}.

Let $V' \supseteq V$ be open subsets of U_k, consider the following diagram.

$$\begin{tikzcd} \mathscr{F}(V') \arrow[r, "(\psi_k)_{V'}"] \arrow[d, "\mathrm{res}_{V',V}"'] & \mathscr{F}_k(V') \arrow[d, "\mathrm{res}_{V',V}"] \\ \mathscr{F}(V) \arrow[r, "(\psi_k)_V"] & \mathscr{F}_k(V) \end{tikzcd}$$

Let $(s_i) \in \mathscr{F}(V')$, then

$$(\psi_k)_V \circ \mathrm{res}_{V',V}((s_i)) = (\psi_k)_V((s_i|_{V \cap U_i})) = s_k|_V$$

and

$$\mathrm{res}_{V',V} \circ (\psi_k)_{V'}(s_i) = \mathrm{res}_{V',V}(s_k) = s_k|_V.$$

Hence, $(\psi_k)_V \circ \mathrm{res}_{V',V} = \mathrm{res}_{V',V} \circ (\psi_k)_{V'}$. This implies that ψ_k is a morphism of sheaves.

Define $\psi_k^{-1} : \mathscr{F}_k \to \mathscr{F}|_{U_k}$ as follow:

$$(\psi_k^{-1})_V(s_k) = (\varphi_{k,1}(s_k|_{V \cap U_1}), \cdots, \varphi_{k,k-1}(s_k|_{V \cap U_{k-1}}), s_k, \varphi_{k,k+1}(s_k|_{V \cap U_{k+1}}), \cdots)$$

for all V is open subset of U_k. We need to check that ψ_k^{-1} is well-defined. Note that $\varphi_{j,i} \circ \varphi_{k,j} = \varphi_{k,i}$, hence, $(\varphi_{k,1}(s_k|_{V \cap U_1}), \cdots, \varphi_{k,k-1}(s_k|_{V \cap U_{k-1}}), s_k, \varphi_{k,k+1}(s_k|_{V \cap U_{k+1}}), \cdots) \in \mathscr{F}(V)$, i.e., ψ_k^{-1} is well-defined. It is easy to check that ψ_k^{-1} is a morphism of sheaves, and is the inverse of ψ_k. Hence,

$$\mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i.$$

(iii) $\mathscr{F}$ is unique up to unique isomorphism.

Let $\mathscr{F}'$ be another sheaf with isomorphism $\mathscr{F}'|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$, then $\mathscr{F}|_{U_i} \cong \mathscr{F}'|_{U_i}$ for all U_i, say isomorphism $\theta_i : \mathscr{F}|_{U_i} \to \mathscr{F}'|_{U_i}$. Let $p \in X$, we shall to show that $\mathscr{F}_p \cong \mathscr{F}'_p$ for all $p \in X$. Define $\theta_p : \mathscr{F}_p \to \mathscr{F}'_p$ as follow:

$$\theta_p(\overline{(s, V)}) = \overline{((\theta_i)_V(s), V)},$$

where $p \in U_i$, $V \subseteq U_i$, and $s \in \mathscr{F}|_{U_i}(V)$. It is easy to see θ_p is well-defined.

θ_p **is surjective:** Let $s'_p \in \mathscr{F}'_p$, since X is covered by $\{U_i\}$, $p \in U_i$ for some i, we may assume that $s'_p = \overline{(s', V)}$, where $V \subseteq U_i$ open subset. Since $(\theta_i)_V$ is isomorphism, there exists $s \in \mathscr{F}|_{U_i}(V)$ such that $(\theta_i)_V(s) = s'$, hence, $\theta_p(\overline{(s, V)}) = \overline{(s', V)} = s'_p$, which implies that $\mathscr{F}_p \to \mathscr{F}'_p$ is surjective.

θ_p **is injective:** Let $\theta_p(s_p) = \theta_p(t_p)$. We may assume that $p \in U_i$ and $s_p = \overline{(s, V_s)}$ where $V_s \subseteq U_i$, then $\theta_p(s_p) = \theta_p(\overline{(s, V_s)}) = \overline{((\theta_i)_{V_s}(s), V_s)}$. Similarly, $\theta_p(t_p) = \overline{((\theta_i)_{V_t}(t), V_t)}$, where $V_t \subseteq U_i$. Hence, there exists $V \subseteq V_s \cap V_t$ such that $(\theta_i)_{V_s}(s)|_V = (\theta_i)_{V_t}(t)|_V$. Since θ_i is an isomorphism, we have $s|_V = t|_V$, and therefore $s_p = t_p$, which implies that θ_p is injective.

Hence, $\mathscr{F}_p \cong \mathscr{F}'_p$ for all $p \in X$, by Proposition 3.4.4, $\mathscr{F} \cong \mathscr{F}'$.

□

Remark Thus we can "glue sheaves together", using limited patching information. Small observation: the hypothesis φ_{ii} is the identity is extraneous, as it follows from the cocycle condition ($\varphi_{ii} \circ \varphi_{ii} = \varphi_{ii}$, φ_{ii} must be identity).

Remark Theorem 3.5.3 almost says that the "set" of sheaves forms a sheaf itself, but not quite. Making this precise leads one to the notion of a **stack**.

3.6 Sheaves of abelian groups, and $\mathscr{O}_X$-modules, form abelian categories

We are now ready to see that sheaves of abelian groups, and their cousins, $\mathscr{O}_X$-modules, form abelian categories. In other words, we may treat them similarly to vector spaces, and modules over a ring. In the process of doing this, we will see that this is much stronger than an analogy; kernels, cokernels, exactness, and so forth can be understood at the level of stalks (which are just abelian groups), and the compatibility of the germs will come for free.

3.6.1 Sheaves of abelian groups form abelian category

The category of sheaves of abelian groups on a topological space X is clearly an additive category (Definition 2.5.1). In order to show that it is an abelian category, we must begin by showing that any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ has a kernel and a cokernel. We have already seen that φ has a kernel (Proposition 3.3.6): the presheaf kernel is a sheaf, and is a kernel in the category of sheaves.

Proposition 3.6.1 (The stalk of the kernel is the kernel of the stalks)

For all $p \in X$, there is a natural isomorphism

$$(\mathrm{Ker}(\mathscr{F} \to \mathscr{G}))_p \xleftrightarrow{\sim} \mathrm{Ker}(\mathscr{F}_p \to \mathscr{G}_p),$$

where $\mathscr{F}, \mathscr{G}$ are sheaves.

Proof Say $\varphi : \mathscr{F} \to \mathscr{G}$, then

$$(\mathrm{Ker}(\mathscr{F} \to \mathscr{G}))_p = (\mathrm{Ker}\,\varphi)_p = \varinjlim_{U \ni p} (\mathrm{Ker}\,\varphi)(U).$$

Let $\overline{(s,U)} \in (\operatorname{Ker}\varphi)_p$, then $s \in (\operatorname{Ker}\varphi)(U) \subseteq \mathscr{F}(U)$, i.e., $\varphi_U(s) = 0$. Hence, $(\varphi_U(s))_p = 0$. Note that $(\varphi_U(s))_p = \varphi_p(s_p)$, we have $\varphi_p(s_p) = 0$. Define $\theta : (\operatorname{Ker}\varphi)_p \to \operatorname{Ker}(\varphi_p)$ by setting

$$\overline{(s,U)} \mapsto s_p.$$

We need to check θ is well-defined. Let $\overline{(s,U_1)} \sim \overline{(t,U_2)}$, then there exists an open subset $U \subseteq U_1 \cap U_2$ such that $s|_U = t|_U$ in $\operatorname{Ker}\varphi_U \subseteq \mathscr{F}(U)$, hence, $s_p = t_p$ in $\mathscr{F}_p$. Note that $s_p \in \operatorname{Ker}\varphi_p$, θ is well-defined.

Next, we give the inverse of θ. Let $s_p \in \operatorname{Ker}(\varphi_p)$, then $\varphi_p(s_p) = (\varphi_U(s))_p = 0$ for some open subset $(p \in)U \subseteq X$. Hence, there exists $V \subseteq U$ such that $\varphi_U(s)|_V = 0$. Note that $\varphi_U(s)|_V = \varphi_V(s|_V)$, then $s|_V \in \operatorname{Ker}\varphi_V = (\operatorname{Ker}\varphi)(V)$. Define $\theta^{-1} : \operatorname{Ker}(\varphi_p) \to (\operatorname{Ker}\varphi)_p$ by setting

$$s_p \mapsto \overline{(s|_V, V)}.$$

We need to check that θ^{-1} is well-defined. Let $s_p = t_p$ in $\operatorname{Ker}(\varphi_p)$, then there exists an open subset $(p \in)U \subseteq X$ such that $s|_U = t|_U$. Also there exists an open subset $(p \in)V$ such that $s|_V, t|_V \in \operatorname{Ker}\varphi_V = (\operatorname{Ker}\varphi)(V)$, hence we have $\overline{(s|_{V\cap U}, V\cap U)} = \overline{(t|_{V\cap U}, V\cap U)}$, this implies that θ^{-1} is well-defined. By the definition of θ^{-1}, it is easy to see that θ^{-1} is, indeed, the inverse of θ.

Consider the following diagram.

$$\begin{tikzcd}
(\operatorname{Ker}\varphi)_p \arrow[r, dashed] & \operatorname{Ker}(\varphi_p) \arrow[rr, shift left] & & \mathscr{F}_p \arrow[ll, shift left] \arrow[rr, dashed, "\varphi_p"] & & \mathscr{G}_p \\
& & (\operatorname{Ker}\varphi)(U') \arrow[ull] \arrow[r] \arrow[d] & \mathscr{F}(U') \arrow[u] \arrow[r, "\varphi_{U'}"] \arrow[d] & \mathscr{G}(U') \arrow[ur] \arrow[d] & \\
& & (\operatorname{Ker}\varphi)(U) \arrow[uull] \arrow[r] & \mathscr{F}(U) \arrow[uu] \arrow[r, "\varphi_U"] & \mathscr{G}(U) \arrow[uur] &
\end{tikzcd}$$

θ is given by universal property of colimit, and therefore we have a natural isomorphism

$$(\operatorname{Ker}(\mathscr{F} \to \mathscr{G}))_p \xleftrightarrow{\sim} \operatorname{Ker}(\mathscr{F}_p \to \mathscr{G}_p).$$

$\square$

We next address the issue of the cokernel. Now $\varphi : \mathscr{F} \to \mathscr{G}$ has a cokernel in the category of presheaves, call it $\mathscr{H}_{\text{pre}}$ (where the subscript is meant to remind us that this is a presheaf). Let $\operatorname{sh} : \mathscr{H}_{\text{pre}} \to \mathscr{H}$ be its sheafification. Recall that the cokernel is defined using a universal property: it is the colimit of the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \\
0 &
\end{tikzcd} \tag{3.4}$$

in the category of presheaves.

Proposition 3.6.2

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be morphism of sheaves, $\mathscr{H}_{\text{pre}}$ be the cokernel of φ in the category of presheaves. Let $\operatorname{sh} : \mathscr{H}_{\text{pre}} \to \mathscr{H}$ be the sheafification of $\mathscr{H}_{\text{pre}}$. Then the composition $\mathscr{G} \to \mathscr{H}$ is the cokernel of φ in the category of sheaves.

Proof We show that it satisfies the universal property of cokernel. Given any sheaf $\mathscr{E}$ and a commutative diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \arrow[d] \\
0 \arrow[r] & \mathscr{E}.
\end{tikzcd}$$

We construct

$$
\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \arrow[d] \arrow[rrd, bend left] & & \\
0 \arrow[r] \arrow[rrrd, bend right] & \mathscr{H}_{\mathrm{pre}} \arrow[r, "\mathrm{sh}"] \arrow[rrd, dashed] & \mathscr{H} \arrow[rd, dashed] & \\
 & & & \mathscr{E}.
\end{tikzcd}
$$

We show that there is a unique morphism $\mathscr{H} \to \mathscr{E}$ making the diagram commute. As $\mathscr{H}_{\mathrm{pre}}$ is the cokernel in the category of presheaves, there is a unique morphism of presheaves $\mathscr{H}_{\mathrm{pre}} \to \mathscr{E}$ making the diagram commute. By the universal property of sheafification (Definition 3.4.2), there is a unique morphism of sheaves $\mathscr{H} \to \mathscr{E}$ making the diagram commute. □

Proposition 3.6.3 (The stalk of the cokernel is isomorphic to the cokernel of the stalk)

For all $p \in X$, there is a natural isomorphism

$$(\mathrm{Coker}(\mathscr{F} \to \mathscr{G}))_p \overset{\sim}{\longleftrightarrow} \mathrm{Coker}(\mathscr{F}_p \to \mathscr{G}_p),$$

where $\mathscr{F}, \mathscr{G}$ are sheaves.

Proof Consider the following diagram.

$$
\begin{tikzcd}
 & \mathscr{F}_p \arrow[rr] & & \mathscr{G}_p \arrow[rr] & & \mathrm{Coker}(\varphi_p) \arrow[rr, dashed, shift left] & & (\mathrm{Coker}\,\varphi)_p \arrow[ll, dashed, shift left] \\
\mathscr{F}(U') \arrow[ur] \arrow[rr] \arrow[d] & & \mathscr{G}(U') \arrow[ur] \arrow[rr] \arrow[d] & & \mathscr{H}_{\mathrm{pre}}(U') \arrow[rr, "\mathrm{sh}_{U'}"] \arrow[d] & & \mathscr{H}(U') \arrow[ur] \arrow[d] & \\
\mathscr{F}(U) \arrow[uur] \arrow[rr] & & \mathscr{G}(U) \arrow[uur] \arrow[rr] & & \mathscr{H}_{\mathrm{pre}}(U) \arrow[rr, "\mathrm{sh}_U"] & & \mathscr{H}(U) \arrow[uur] &
\end{tikzcd}
$$

Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian group, $\mathrm{Coker}(\varphi_p)$ is an abelian group, moreover, $\mathrm{Coker}(\varphi_p) = \mathscr{G}_p/\varphi_p(\mathscr{F}_p)$. Now, we shall to define $\theta_p : \mathrm{Coker}(\varphi_p) \to (\mathrm{Coker}\,\varphi)_p$. Let $\overline{s_p} \in \mathrm{Coker}(\varphi_p)$ be the image of s_p in $\mathrm{Coker}(\varphi_p)$, then there exists an open subset $U \ni p$ such that $s \in G(U)$. Let $[s]$ be the image of s in $\mathscr{H}_{\mathrm{pre}}(U)$, then $\mathrm{sh}_U([s]) = ([s]_q)_{q \in U} \in \mathscr{H}(U)$. Define $\theta_p : \mathrm{Coker}(\varphi_p) \to (\mathrm{Coker}\,\varphi)_p$ by setting

$$\overline{s_p} \mapsto (([s]_q)_{q \in U})_p.$$

(i) θ_p is well-defined.
Let $\overline{s_p} = \overline{t_p}$, then $s_p - t_p \in \varphi_p(\mathscr{F}_p)$. Let $s_p - t_p = \varphi_p(f_p)$, note that $\varphi_p(f_p) = (\varphi_U(f))_p$ for some open subset $U \ni p$, we have

$$s_p - t_p = (\varphi_U(f))_p.$$

Then there exist an open subset $(p \in) V \subseteq U$ such that $s|_V - t|_V = \varphi_U(f)|_V$. Hence, $[s] = [t]$ in $\mathscr{H}_{\mathrm{pre}}(V)$, and therefore $(([s]_q)_{q \in V})_p = (([t]_q)_{q \in V})_p$, which implies that $\theta_p(\overline{s_p}) = \theta_p(\overline{t_p})$. Hence, θ_p is well-defined.

(ii) θ_p is injective.
Let $(([s]_q)_{q \in U})_p = (([t]_q)_{q \in U})_p$ in $(\mathrm{Coker}\,\varphi)_p$, then there is an open subset of U, say $V \ni p$, such that $([s]_q)_{q \in V} = ([t]_q)_{q \in V}$. This implies that $[s]_q = [t]_q$ for all $q \in V$. We may assume that $[s] = [t]$ in $\mathscr{H}_{\mathrm{pre}}(V)$, then $s - t \in \varphi_V(\mathscr{F}(V))$. Taking stalk at p, we have $s_p - t_p \in \varphi_p(\mathscr{F}_p)$, hence, $\overline{s_p} = \overline{t_p}$, and therefore θ_p is injective.

(iii) θ_p is surjective.

Let $((s_q)_{q\in U})_p \in (\operatorname{Coker}\varphi)_p$, then there is an open subset $U \ni p$ such that $(s_q)_{q\in U} \in \mathscr{H}(U)$. For $q = p$, there exists $(p \in)V \subseteq U$ and $\widetilde{s} \in \mathscr{H}_{\text{pre}}(V)$ such that $\widetilde{s}_m = s_m$ for all $m \in V$. Consider $(s_q)_{q\in V} \in \mathscr{H}(V)$, we have $\operatorname{sh}_V(\widetilde{s}) = (s_q)_{q\in V}$. Let $\widetilde{s}$ be the image of $\widetilde{s}^* \in \mathscr{G}(V)$, then $\overline{\widetilde{s}^*_p} \in \operatorname{Coker}(\varphi_p)$, and therefore,

$$\theta_p(\overline{\widetilde{s}^*_p}) = (([\widetilde{s}^*]_q)_{q\in V})_p = ((s_q)_{q\in V})_p = ((s_q)_{q\in U})_p.$$

This implies that θ_p is a surjective.

Note that θ_p is given by universal property (see above diagram), θ_p is natural and is isomorphism. $\square$

We have now defined the notions of kernel and cokernel, and verified that they may be checked at the level of stalks. We have also verified that the properties of a morphism being a monomorphism or epimorphism are also determined at the level of stalks (Proposition 3.4.12 and Proposition 3.4.13). Hence we have proved the following:

Theorem 3.6.1

Sheaves of abelian groups on a topological space X form an abelian category.

That's all there is to it — what needs to be proved has been shifted to the stalks, where everything works because stalks are abelian groups!

And we see more: all structures coming from the abelian nature of this category may be checked at the level of stalks. For example:

Proposition 3.6.4

Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of abelian groups.

(a) The image sheaf $\operatorname{Im}\varphi$ is the sheafification of the image presheaf.

(b) The stalk of the image is the image of the stalk.

Proof

(a) First we define the presheaf image. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, define

$$\operatorname{Im}_{\text{pre}}\varphi(U) := \operatorname{Im}\varphi_U = \varphi_U(\mathscr{F}(U)),$$

for any open subset U of X. It is easy to see that $\operatorname{Im}_{\text{pre}}$ is a presheaf. Let $\operatorname{sh} : \operatorname{Im}_{\text{pre}}\varphi \to \operatorname{Im}\varphi$ be sheafification.

We constructs a commutative diagram,

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ 0 \arrow[r] & \operatorname{Coker}\varphi, \end{tikzcd}$$

where $\mathscr{E}$ be any sheaf with arrows $\mathscr{E} \to \mathscr{G}$ and $\mathscr{E} \to 0$.

Consider the following diagram.

$$
\begin{tikzcd}
\mathscr{E} \arrow[dr, dashed] \arrow[ddr, dashed] \arrow[ddrr, bend left] \arrow[dddr, bend right] & & \\
& \operatorname{Im}\varphi \arrow[dr, dashed] & \\
& \operatorname{Im}_{\mathrm{pre}}\varphi \arrow[u, "\mathrm{sh}"'] \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\
& 0 \arrow[r] & \operatorname{Coker}\varphi
\end{tikzcd}
$$

We show there is a unique morphism $\mathscr{E} \to \operatorname{Im}\varphi$ making the diagram commute. The universal property of presheaf kernel gives a unqiue morphism of presheaves $\mathscr{E} \to \operatorname{Im}_{\mathrm{pre}}\varphi$. And the universal property of sheafification gives a unique morphism of sheaves $\operatorname{Im}\varphi \to \mathscr{G}$. Define $\mathscr{E} \to \operatorname{Im}\varphi$ be the composition

$$\mathscr{E} \longrightarrow \operatorname{Im}_{\mathrm{pre}}\varphi \xrightarrow{\mathrm{sh}} \operatorname{Im}\varphi,$$

this morphism of sheaves make above diagram commute, and also it is unique.

(b) It suffices to show that

$$(\operatorname{Im}(\mathscr{F} \to \mathscr{G}))_p \xleftrightarrow{\sim} \operatorname{Im}(\mathscr{F}_p \to \mathscr{G}_p).$$

Note that $\operatorname{Im}(\mathscr{F} \to \mathscr{G}) = \operatorname{Ker}(\operatorname{Coker}(\mathscr{F} \to \mathscr{G})) = \operatorname{Ker}(\mathscr{G} \to \operatorname{Coker}(\mathscr{F} \to \mathscr{G}))$, we have

$$
\begin{aligned}
(\operatorname{Im}(\mathscr{F} \to \mathscr{G}))_p &= \operatorname{Ker}(\mathscr{G} \to \operatorname{Coker}(\mathscr{F} \to \mathscr{G}))_p \\
&\cong \operatorname{Ker}(\mathscr{G}_p \to \operatorname{Coker}(\mathscr{F} \to \mathscr{G})_p) \\
&\cong \operatorname{Ker}(\mathscr{G}_p \to \operatorname{Coker}(\mathscr{F}_p \to \mathscr{G}_p)) \\
&= \operatorname{Ker}(\mathscr{G}_p \to (\mathscr{G}_p / \operatorname{Im}(\mathscr{F}_p \to \mathscr{G}_p))) \\
&= \operatorname{Im}(\mathscr{F}_p \to \mathscr{G}_p),
\end{aligned}
$$

as we desired.

□

As a consequence, **exactness of a sequence of sheaves may be checked at the level of stalks**.

Proposition 3.6.5

Suppose $\alpha : \mathscr{F} \to \mathscr{G}$ and $\beta : \mathscr{G} \to \mathscr{H}$ are two morphisms of sheaves of abelian groups on X. Then

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is exact (at $\mathscr{G}$) if and only if for all $p \in X$,

$$\mathscr{F}_p \xrightarrow{\alpha_p} \mathscr{G}_p \xrightarrow{\beta_p} \mathscr{H}_p$$

is exact.

Proof Since sequence

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is exact, we have $\operatorname{Ker}\beta = \operatorname{Ker}(\mathscr{G} \to \mathscr{H}) = \operatorname{Im}\alpha = \operatorname{Im}(\mathscr{F} \to \mathscr{G})$. By Proposition 3.6.4 and Proposition 3.6.1,

$$\operatorname{Im}(\mathscr{F}_p \to \mathscr{G}_p) \cong \operatorname{Im}(\mathscr{F} \to \mathscr{G})_p = \operatorname{Ker}(\mathscr{G} \to \mathscr{H})_p \cong \operatorname{Ker}(\mathscr{G}_p \to \mathscr{H}_p),$$

this implies that the sequence

$$\mathscr{F}_p \xrightarrow{\alpha_p} \mathscr{G}_p \xrightarrow{\beta_p} \mathscr{H}_p$$

is exact.

Above process is reversible, then we done! □

Proposition 3.6.6 (Taking the stalk of a sheaf of abelian groups is an exact functor)

If X is a topological space and $p \in X$ is a point, then taking the stalk at p defines an exact functor $\mathbf{Ab}_X \to \mathbf{Ab}$.

Proof By Proposition 3.6.5, clearly. □

Exercise 3.19 Check that the exponential exact sequence (Example 3.10)

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 1.$$

is indeed an exact sequence of sheaves of abelian groups.

Proof It suffice to check exactness at the level of stalk, i.e., check seqeunce

$$0 \longrightarrow \underline{\mathbb{Z}}_p \xrightarrow{(\times 2\pi i)_p} \mathscr{O}_{X,p} \xrightarrow{\exp_p} {\mathscr{O}_{X,p}}^* \longrightarrow 1$$

is exact.

In fact, $\underline{\mathbb{Z}}_p = \mathbb{Z}$.

(a) $(\times 2\pi i)_p : \mathbb{Z} \to \mathscr{O}_{X,p}$ is injective.

Let $(\times 2\pi i)_p(n) = 0$, then $(\times 2\pi i)_p(n) = (2n\pi i)_p = 0$, hence, $n = 0$, which implies that $(\times 2\pi i)_p$ is injective.

(b) $\operatorname{Im}(\times 2\pi i)_p = \operatorname{Ker}\exp_p$.

Let $(2n\pi i)_p \in \operatorname{Im} \times 2\pi i)_p$, then $\exp_p((2n\pi i)_p) = e_p^{2n\pi i} = 1$. Hence, $\operatorname{Im}(\times 2\pi i)_p \subseteq \operatorname{Ker}\exp_p$.
Conversely, let $f_p \in \operatorname{Ker}\exp_p$, then $e_p^f = 0$. Hence, $f = 2n\pi i$, which implies that $f \in \operatorname{Im} \times 2\pi i)_p$, i.e., $\operatorname{Ker}\exp_p \subseteq \operatorname{Im} \times 2\pi i)_p$.

(c) $\exp_p : \mathscr{O}_{X,p} \to \mathscr{O}_{X,p}^*$ is surjective.

Let $g_p \in \mathscr{O}_{X,p}^*$, then there exists an open subset U such that g admits a holomorphic logarithm on U, i.e., $f = \log g \in \mathscr{O}_X(U)$. Hence, $f_p \in \mathscr{O}_{X,p}$, and $\exp_p(f_p) = g_p$, which implies that $\exp_p : \mathscr{O}_{X,p} \to \mathscr{O}_{X,p}^*$ is surjective.

Hence, sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_p \xrightarrow{(\times 2\pi i)_p} \mathscr{O}_{X,p} \xrightarrow{\exp_p} {\mathscr{O}_{X,p}}^* \longrightarrow 1$$

is exact. By Proposition 3.6.5, we done. □

Proposition 3.6.7 (Left-exactness of the functor of "sections over U")

Suppose $U \subseteq X$ is an open set, and $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of sheaves of abelian groups. Then sequence

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U)$$

is exact. Denote this functor by $\Gamma(U, \square)$. (i.e. $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$)

Proof Let $\square^\flat : \mathbf{Ab}_X \to \mathbf{Ab}_X^{\text{pre}}$ be forgetful functor, by Proposition 3.4.10, forgetful functor is right adjoint to the sheafification functor, i.e., $\square^\flat$ is left exact. Hence,

$$0 \longrightarrow \mathscr{F}^\flat \longrightarrow \mathscr{G}\flat \longrightarrow \mathscr{H}^\flat$$

By Proposition 3.3.4, for open subset U of X, sequence

$$0 \longrightarrow \mathscr{F}^\flat(U) \longrightarrow \mathscr{G}^\flat(U) \longrightarrow \mathscr{H}^\flat(U)$$

is exact.

Note that $\mathrm{Ker}_{\mathrm{pre}}(\mathscr{F}^\flat(U) \to \mathscr{G}^\flat(U)) = \mathrm{Ker}(\mathscr{F}(U) \to \mathscr{G}(U))$ and $\mathrm{Ker}_{\mathrm{pre}}(\mathscr{F}^\flat(U) \to \mathscr{G}^\flat(U)) = \mathrm{Im}_{\mathrm{pre}}(\mathscr{G}^\flat(U) \to \mathscr{H}^\flat(U)) = \mathrm{Im}_{\mathrm{pre}}(\mathscr{G} \to \mathscr{H}(U))$, by the universal property of sheafification, we have

$$\mathrm{Im}_{\mathrm{pre}}(\mathscr{G} \to \mathscr{H}(U)) \xleftrightarrow{\sim} \mathrm{Im}(\mathscr{G} \to \mathscr{H}(U)),$$

and therefore $\mathrm{Ker}(\mathscr{F}(U) \to \mathscr{G}(U)) = \mathrm{Im}(\mathscr{G}(U) \to \mathscr{H}(U))$.

Also $\mathscr{F} \to \mathscr{G}$ is a monomorphism, by Proposition 3.4.12, $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective. Hence, the sequence

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U)$$

is exact. □

✍ **Exercise 3.20 Show that the section functor need not be exact**: show that if $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of sheaves of abelian groups, then

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U) \longrightarrow 0$$

need not be exact.

Solution *Consider the sequence,*

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^* \longrightarrow 1,$$

by Exercise 3.19, this sequence is exact. But, by Exercise 3.17,

$$0 \longrightarrow \underline{\mathbb{Z}}(U) \longrightarrow \mathscr{O}_X(U) \longrightarrow \mathscr{O}_X^*(U) \longrightarrow 1,$$

may not exact.

Proposition 3.6.8 (Left exactness of pushforward)

Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of sheaves of abelian groups on X. If $\pi : X \to Y$ is a continuous map, then sequence

$$0 \longrightarrow \pi_*\mathscr{F} \longrightarrow \pi_*\mathscr{G} \longrightarrow \pi_*\mathscr{H}$$

is exact.

Proof Let $V \subseteq Y$ be any open subset of Y, since $\pi : X \to Y$ is continuous, $\pi^{-1}(V)$ is open in X. Since the functor of "sections over $\pi^{-1}(V)$" is a left exact functor (Proposition), the sequence

$$0 \longrightarrow \mathscr{F}(\pi^{-1}(V)) \longrightarrow \mathscr{G}(\pi^{-1}(V)) \longrightarrow \mathscr{H}(\pi^{-1}(V))$$

is exact, i.e., the sequence

$$0 \longrightarrow \pi_*\mathscr{F}(V) \longrightarrow \pi_*\mathscr{G}(V) \longrightarrow \pi_*\mathscr{H}(V)$$

is exact for all open subset $V \subseteq Y$. Hence, for all $p \in Y$, we have the exact sequence (look above sequence as exact sequence of presheaves, and use Proposition 3.4.11).

$$0 \longrightarrow \pi_*\mathscr{F}_p \longrightarrow \pi_*\mathscr{G}_p \longrightarrow \pi_*\mathscr{H}_p$$

By Proposition 3.6.5, the sequence

$$0 \longrightarrow \pi_*\mathscr{F} \longrightarrow \pi_*\mathscr{G} \longrightarrow \pi_*\mathscr{H}$$

is exact. □

Proposition 3.6.9 (Left exactness of $\mathscr{H}om$)

Suppose $\mathscr{F}$ is a sheaf of abelian groups on a topological space X. Then

(i) $\mathscr{H}om(\mathscr{F}, \square)$ is a left-exact covariant functor $\mathbf{Ab}_X \to \mathbf{Ab}_X$;

(ii) $\mathscr{H}om(\square, \mathscr{F})$ is a left-exact contravariant functor $\mathbf{Ab}_X \to \mathbf{Ab}_X$.

Proof

(i) Let

$$0 \longrightarrow \mathscr{G} \xrightarrow{\alpha} \mathscr{G}' \xrightarrow{\beta} \mathscr{G}''$$

be an exact sequence of sheaves. We need to show that the sequence,

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}) \xrightarrow{\alpha_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}') \xrightarrow{\beta_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}''),$$

is exact.

Let U be any open subset of X, consider the sequence of abelian groups.

$$0 \longrightarrow \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \xrightarrow{\alpha_*} \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}'|_U) \xrightarrow{\beta_*} \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}''|_U)$$

(1) $\alpha_* : \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \to \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}'|_U)$ is injective.

Let $\varphi \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ with $\alpha_*(\varphi) = 0$, then $\alpha \circ \varphi = 0$. Since α is monomorphism, $\varphi = 0$, which implies that α_* is injective.

(2) $\operatorname{Ker} \beta_* = \operatorname{Im} \alpha_*$.

Let $\psi \in \operatorname{Ker} \beta_* \subseteq \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}'|_U)$, then $\beta \circ \psi = 0$. Since $\operatorname{Im} \alpha = \operatorname{Ker} \beta$, for any open subset V of U and section $s \in \mathscr{F}(V)$, we have $\psi_V(s) \in \operatorname{Im} \alpha_V$. Then there exists unique $\theta_V(s) \in \mathscr{G}(V)$ such that $\alpha_V(\theta_V(s)) = \psi_V(s)$, i.e., $\alpha_V \circ \theta_V = \psi_V$ for any open subset V of U. Hence, $\alpha \circ \theta = \psi$, which implies that $\psi \in \operatorname{Im} \alpha_*$.

Conversely, let $\psi \in \operatorname{Im} \alpha_*$, then there exists $\theta \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ such that $\alpha \circ \theta = \psi$. Then for any open subset V of U, we have $\alpha_V \circ \theta_V = \psi_V$. Note that $\operatorname{Ker} \beta_V = \operatorname{Im} \alpha_V$, we have $\beta_V \circ \alpha_V = 0$, and therefore $\beta_V \circ \psi_V = 0$, i.e., $\psi_V \in \operatorname{Ker} \beta_V$ for all $V \subseteq U$. Hence, $\beta \circ \psi = 0$, i.e., $\psi \in \operatorname{Ker} \beta_*$.

Hence, the sequence

$$0 \longrightarrow \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \xrightarrow{\alpha_*} \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}'|_U) \xrightarrow{\beta_*} \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}''|_U)$$

is exact, i.e., the sequence

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})(U) \xrightarrow{\alpha_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}')(U) \xrightarrow{\beta_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}'')(U), \tag{3.5}$$

is exact for any open subset U of X. Look exact sequence 3.5 as exact sequence of presheaves, by Proposition 3.4.11. The sequence,

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G})_p \xrightarrow{\alpha_{*p}} \mathscr{H}om(\mathscr{F}, \mathscr{G}')_p \xrightarrow{\beta_{*p}} \mathscr{H}om(\mathscr{F}, \mathscr{G}'')_p,$$

is exact for any $p \in X$. By Proposition 3.6.5, the sequence

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{G}) \xrightarrow{\alpha_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}') \xrightarrow{\beta_*} \mathscr{H}om(\mathscr{F}, \mathscr{G}''),$$

is exact, and therefore $\mathscr{H}om(\mathscr{F}, \square)$ is a left-exact covariant functor $\mathbf{Ab}_X \to \mathbf{Ab}_X$.

(ii) Similar to (i).

□

3.6.2 $\mathscr{O}_X$-modules form abelian category

Theorem 3.6.2

If $(X, \mathscr{O}_X)$ is a ringed space, then $\mathscr{O}_X$-modules form an abelian category.

Proof Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be sheaves of $\mathscr{O}_X$-module. Let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be morphisms of sheaves of $\mathscr{O}_X$-modules. We define $\varphi + \psi : \mathscr{F} \to \mathscr{G}$ to be the map which on each open subset $U \subseteq X$ is the sum of the maps induced by φ, ψ, i.e.,

$$(\varphi + \psi)_U = \varphi_U + \psi_U.$$

This is clearly again a map of sheaves of $\mathscr{O}_X$-modules. It is also clear that composition of maps of $\mathscr{O}_X$-modules is bilinear with respect to this addition, i.e., for $\theta : \mathscr{E} \to \mathscr{F}$ and $\eta : \mathscr{G} \to \mathscr{H}$, we have

$$(\varphi + \psi) \circ \theta = \varphi \circ \theta + \psi \circ \theta$$

and

$$\eta \circ (\varphi + \psi) = \eta \circ \varphi + \eta \circ \psi.$$

We will denote 0 the sheaf of $\mathscr{O}_X$-modules which has constant value $\{0\}$ for all open subset $U \subseteq X$. Clearly this is both a final and an initial object of $\mathbf{Mod}_{\mathscr{O}_X}$.

Moreover, given a pair $\mathscr{F}, \mathscr{G}$ of sheaves of $\mathscr{O}_X$-modules we may define the direct sum as

$$\mathscr{F} \oplus \mathscr{G} = \mathscr{F} \times \mathscr{G}.$$

Thus $\mathbf{Mod}_{\mathscr{O}_X}$ is an additive category (Definition 2.5.1).

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules. We may define $\operatorname{Ker}\varphi$ to be the subsheaf of $\mathscr{F}$ with sections

$$\operatorname{Ker}\varphi(U) := \{s \in \mathscr{F}(U) : \varphi(s) = 0 \in \mathscr{G}(U)\}$$

for all open subset $U \subseteq X$. It is easy to see that this is indeed a kernel in the category of $\mathscr{O}_X$-modules. Moreover, on the level of stalks we have

$$\operatorname{Ker}(\varphi)_p \cong \operatorname{Ker}(\varphi_p)$$

for all $p \in X$.

On the other hand, we define $\operatorname{Coker}\varphi$ as the sheaf of $\mathscr{O}_X$-modules associated to the presheaf of $\mathscr{O}_X$-modules defined by the rule

$$U \mapsto \operatorname{Coker}_{\text{pre}}(\mathscr{G}(U) \to \mathscr{F}(U)) = \mathscr{F}(U)/\varphi_U(\mathscr{G}(U)),$$

i.e., $\operatorname{Coker}(\mathscr{F} \to \mathscr{G}) = (\operatorname{Coker}_{\text{pre}}(\mathscr{F} \to \mathscr{G}))^{\text{sh}}$. Since taking stalks commutes with taking sheafification (see Proposition 3.4.11 or Proposition 3.6.3), we see that

$$\operatorname{Coker}(\varphi)_p \cong \operatorname{Coker}(\varphi_p)$$

for all $p \in X$. Thus the map $\mathscr{G} \to \operatorname{Coker}\varphi$ is epimorphism, by Proposition 3.4.13. To show that this is a cokernel, consider the following diagram.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"'] \arrow[rr, bend left, "0"] \arrow[rrr, bend left=40, "0"] \arrow[rrd, "0"'] & \mathscr{G} \arrow[r] \arrow[rd, "\beta"'] & \operatorname{Coker}_{\text{pre}}\varphi \arrow[r, "\text{sh}"] \arrow[d, dashed] & \operatorname{Coker}\varphi \arrow[ld, dashed] \\
 & & \mathscr{H} &
\end{tikzcd} \tag{3.6}$$

In diagram (3.6), $\beta : \mathscr{G} \to \mathscr{H}$ is a morphism of $\mathscr{O}_X$-modules such that $\beta \circ \varphi$ is zero, then we get for every open subset $U \subseteq X$ a map induced by β from $\mathrm{Coker}_{\mathrm{pre}}\,\varphi(U)$ into $\mathscr{H}(U)$. By the universal property of sheafification we obtain a map $\mathrm{Coker}\,\varphi \to \mathscr{H}$ such that β is equal to the composition $\mathscr{G} \to \mathrm{Coker}\,\varphi \to \mathscr{H}$. The morphism $\mathrm{Coker}\,\varphi \to \mathscr{H}$ is unique because of $\mathscr{G} \to \mathrm{Coker}\,\varphi$ is surjective.

By the definition of abelian category (Definition 2.5.5) and above discussion, it suffice to show that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

(1) Every monomorphism is the kernel of its cokernel.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a monomorphism, it suffices to show that

$$\mathscr{F} \cong \mathrm{Ker}(\mathscr{G} \to \mathrm{Coker}\,\varphi).$$

By Proposition 3.4.4, it suffices to show that

$$\mathscr{F}_p \cong \mathrm{Ker}(\mathscr{G} \to \mathrm{Coker}\,\varphi)_p$$

for all $p \in X$.

Since $\varphi : \mathscr{F} \to \mathscr{G}$ is monomorphism, by Proposition 3.4.12, $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is injective for all $p \in X$, and therefore

$$\begin{aligned} \mathscr{F}_p &\cong \mathrm{Ker}(\mathscr{G}_p \to \mathrm{Coker}(\varphi_p)) \\ &\cong \mathrm{Ker}(\mathscr{G}_p \to (\mathrm{Coker}\,\varphi)_p) \\ &\cong (\mathrm{Ker}(\mathscr{G} \to \mathrm{Coker}\,\varphi))_p \end{aligned}$$

for all $p \in X$, then we done.

(2) Every epimorphism is the cokernel of its kernel.

Let $\psi : \mathscr{F} \to \mathscr{G}$ be an epimorphism, it suffices to show that

$$\mathscr{G} \cong \mathrm{Coker}(\mathrm{Ker}\,\psi \to \mathscr{F}).$$

By Propositon 3.4.4, it suffices to show that

$$\mathscr{G}_p \cong \mathrm{Coker}(\mathrm{Ker}\,\psi \to \mathscr{F})_p$$

for all $p \in X$.

Since $\psi : \mathscr{F} \to \mathscr{G}$ is epimorphism, by Proposition 3.4.13, $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective for all $p \in X$, and therefore

$$\begin{aligned} \mathscr{G}_p &\cong \mathrm{Coker}(\mathrm{Ker}(\varphi_p) \to \mathscr{F}_p) \\ &\cong \mathrm{Coker}((\mathrm{Ker}\,\varphi)_p \to \mathscr{F}_p) \\ &\cong \mathrm{Coker}(\mathrm{Ker}\,\psi \to \mathscr{F})_p \end{aligned}$$

for all $p \in X$.

Hence, $\mathscr{O}_X$-modules form an abelian category. □

Remark Many facts about sheaves of abelian groups carry over to $\mathscr{O}_X$-modules with out change, because a sequence of $\mathscr{O}_X$-modules is exact if and only if the underlying sequence of sheaves of abelian groups is exact. It is easily check that all of the statements of the earlier propositions in § 3.6 is also hold for $\mathscr{O}_X$-modules, when modified appropriately. For example:

Example 3.12 $\mathcal{H}om_{\mathscr{O}_X}(\square, \square)$ is a left-exact contravariant functor in its first argument and a left-exact covariant functor in its second argument.

Note ***Conclusion.*** *Just as presheaves of abelian groups on a topological space form an abelian category because all abelian-categorical notions make sense open set by open set, sheaves of abelian groups on a topological space form an abelian category because all abelian-categorical notions make sense stalk by stalk.*

3.6.3 Tensor product sheaf

We end with a useful construction using some of the idea in this section.

Definition 3.6.1 (Tensor product sheaf)

*Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. We define the **tensor product presheaf***

$$\mathscr{F} \otimes_{\mathrm{pre},\mathscr{O}_X} \mathscr{G}$$

as the rule which assigns to $U \subseteq X$ open the $\mathscr{O}_X(U)$-module $\mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.
*Having defined this we define the **tensor product sheaf** as the sheafification of the above:*

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = (\mathscr{F} \otimes_{\mathrm{pre},\mathscr{O}_X} \mathscr{G})^{\mathrm{sh}}.$$

Definition 3.6.2 (Bilinear maps of sheaves of $\mathscr{O}_X$-modules)

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be $\mathscr{O}_X$-modules. A bilinear map $f : \mathscr{F} \times \mathscr{G} \to \mathscr{H}$ of sheaves of $\mathscr{O}_X$-modules is a map of sheaves of sets as indicated such that for every open subset $U \subseteq X$ the induced map

$$\mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{H}(U)$$

is an $\mathscr{O}_X(U)$-bilinear map of modules.

Proposition 3.6.10

Tensor product of two $\mathscr{O}_X$-modules satisfies categorical definition.

Proof Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be $\mathscr{O}_X$-modules. Let U be open subset of X, and consider the following diagram,

$$\begin{tikzcd} \mathscr{F}(U) \times \mathscr{G}(U) \arrow[r, "t_U"] \arrow[dr, "f_U"'] & \mathscr{F}(U) \otimes_{\mathrm{pre},\mathscr{O}_X(U)} \mathscr{G}(U) \arrow[d, dashed] \\ & \mathscr{H}(U), \end{tikzcd}$$

where t_U and f_U are $\mathscr{O}_X(U)$-bilinear map. By the universal property of tensor product of modules, there exists unique $\mathscr{O}_X(U)$-map $\mathscr{F}(U) \otimes_{\mathrm{pre},\mathscr{O}_X(U)} \mathscr{G}(U) \to \mathscr{H}(U)$. Then we have a unique $\mathscr{O}_X$-map (presheaf) $\mathscr{F} \otimes_{\mathrm{pre},\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$. By the universal property of sheafification, there exists a unique $\mathscr{O}_X$-map (sheaf) $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})^{\mathrm{sh}} \to \mathscr{H}$, such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{F} \otimes_{\mathrm{pre},\mathscr{O}_X} \mathscr{G} \arrow[r, "\mathrm{sh}"] \arrow[dr] & \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \arrow[d, dashed] \\ & \mathscr{H} \end{tikzcd}$$

Hence the following diagram is commutative, and therefore tensor product of two $\mathscr{O}_X$-modules satisfies categorical definition.

$$\begin{tikzcd} \mathscr{F} \times \mathscr{G} \arrow[r, "t"] \arrow[dr, "f"'] & \mathscr{F} \otimes_{\mathrm{pre},\mathscr{O}_X} \mathscr{G} \arrow[r, "\mathrm{sh}"] \arrow[d] & \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \arrow[dl] \\ & \mathscr{H} & \end{tikzcd}$$

$\square$

Proposition 3.6.11 (The tensor product of stalks is the stalk of the tensor product) *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. Let $p \in X$. There is a canonical isomorphism of $\mathscr{O}_{X,p}$-modules*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_p \cong \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \mathscr{G}_p.$$

Proof Let $U' \supseteq U$ be open subsets of X. Consider the following diagram.

$$\begin{tikzcd}
\mathscr{F}(U') \otimes_{\mathrm{pre}, \mathscr{O}_X(U')} \mathscr{G}(U') \arrow[r, "\mathrm{sh}_{U'}"] \arrow[dr, dashed] & \mathscr{F}(U') \otimes_{\mathscr{O}_X(U')} \mathscr{G}(U') \arrow[dr, dashed] \arrow[drr] & & \\
& \mathscr{F}(U) \otimes_{\mathrm{pre}, \mathscr{O}_X(U)} \mathscr{G}(U) \arrow[r, "\mathrm{sh}_U"] & \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U) \arrow[r] & (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_p \arrow[d, dashed, shift right] \\
\mathscr{F}(U') \times \mathscr{G}(U') \arrow[uu] \arrow[rr] \arrow[d] \arrow[dr, dashed] & & \mathscr{F}(U') \arrow[d] \arrow[dr] & \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \mathscr{G}_p \arrow[u, dashed, shift right] \\
\mathscr{G}(U') \arrow[dr] \arrow[drr] & \mathscr{F}(U) \times \mathscr{G}(U) \arrow[uu] \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] & \mathscr{F}_p \\
& \mathscr{G}(U) \arrow[r] & \mathscr{G}_p & \mathscr{F}_p \times \mathscr{G}_p \arrow[l] \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

All dashed arrows given by the universal property, hence, there is a canonical isomorphism of $\mathscr{O}_{X,p}$-modules

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_p \cong \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \mathscr{G}_p.$$

□

3.7 The inverse image sheaf

We next describe a notion that is fundamental, but rather intricate. Suppose we have a continuous map $\pi : X \to Y$. If $\mathscr{F}$ is a sheaf on X, we have defined the pushforward (or direct image) sheaf $\pi_*\mathscr{F}$, which is a sheaf on Y. There is also a notion of **inverse image sheaf**. (We will not call it the pullback sheaf, reserving that name for a later construction for quasicoherent sheaves.) This is a covariant functor π^{-1} from sheaves on Y to sheaves on X. If the sheaves on Y have some additional structure (e.g., group or ring), then this structure is respected by π^{-1}.

3.7.1 Definition of the inverse image sheaf

Definition 3.7.1 (The inverse image sheaf) *We define the inverse image π^{-1} as the left adjoint to π_*.*

Remark This definition is given by adjoint: elegant but abstract.

This isn't really a definition; we need a construction to show that the adjoint exists. Note that we then get canonical maps $\pi^{-1}\pi_*\mathscr{F} \to \mathscr{F}$ (associated to the identity in $\mathrm{Mor}_Y(\pi_*\mathscr{F}, \pi_*\mathscr{F})$) and $\mathscr{G} \to \pi_*\pi^{-1}\mathscr{G}$ (associated to the identity in $\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \pi^{-1}\mathscr{G})$.

$$\begin{tikzcd} & \pi^{-1}\mathscr{G} \arrow[r] & \mathscr{F} \\ X \arrow[d, "\pi"'] \arrow[ur, dash] \arrow[urr, dash] & \mathscr{G} \arrow[r] & \pi_*\mathscr{F} \\ Y \arrow[ur, dash] \arrow[urr, dash] & & \end{tikzcd}$$

Remark How to get canonical maps $\pi^{-1}\pi_*\mathscr{F} \to \mathscr{F}$ and $\mathscr{G} \to \pi_*\pi^{-1}\mathscr{G}$?

By Definition 3.7.1, π^{-1} is left adjoint to π_*. Then we have canonical isomorphisms

$$\mathrm{Mor}_X(\pi^{-1}\pi_*\mathscr{F}, \mathscr{F}) \cong \mathrm{Mor}_Y(\pi_*\mathscr{F}, \pi_*\mathscr{F})$$

and

$$\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \pi^{-1}\mathscr{G}) \cong \mathrm{Mor}_Y(\mathscr{G}, \pi_*\pi^{-1}\mathscr{G}).$$

Let $\pi^{-1}\pi_*\mathscr{F} \to \mathscr{F}$ be the image of $\mathrm{id}_{\pi_*\mathscr{F}}$ in $\mathrm{Mor}_X(\pi^{-1}\pi_*\mathscr{F}, \mathscr{F})$, and $\mathscr{G} \to \pi_*\pi^{-1}\mathscr{G}$ be the image of $\mathrm{id}_{\pi^{-1}\mathscr{G}}$ in $\mathrm{Mor}_Y(\mathscr{G}, \pi_*\pi^{-1}\mathscr{G})$.

Construction: concrete but ungly.

Proposition 3.7.1

Define the temporary notation

$$\pi^{-1}_{\mathrm{pre}}\mathscr{G}(U) = \varinjlim_{V \supseteq \pi(U)} \mathscr{G}(V).$$

Then $\pi^{-1}_{\mathrm{pre}}\mathscr{G}$ defines a presheaf on X.

Remark Recall the explicit description of colimit: sections of π^{-1}_{pre} over U are sections on open sets containing $\pi(U)$, with an equivalence relation. Note that $\pi(U)$ won't be an open set in general.

Proof We first define the restriction map. Let $U' \supseteq U$ be open subsets of X. Define the restriction map $\mathrm{res}_{U',U} : \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U') \to \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U)$ by setting

$$s_{\pi(U')} \mapsto s_{\pi(U)}.$$

We shall to show the restriction map is well-defined. Let $s_{\pi(U')} = t_{\pi(U')}$ in $\pi^{-1}_{\mathrm{pre}}\mathscr{G}(U')$, then $s_{\pi(U')} = \overline{(s, V_s)}$ and $t_{\pi(U')} = \overline{(t, V_t)}$. Hence, there exists an open subset $(\pi(U') \subseteq) W \subseteq V_s \cap V_t$ such that $s|_W = t|_W$. Since $U' \supseteq U$, we have $\pi(U') \supseteq \pi(U)$, and therefore $\pi(U) \subseteq \pi(U') \subseteq W \subseteq V_s \cap V_t$. Hence, $s_{\pi(U)} = t_{\pi(U)}$, which implies that the restriction map is well-defined.

Consider $\mathrm{res}_{U,U} : \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U) \to \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U)$, clearly, $\mathrm{res}_{U,U} = \mathrm{id}_{\pi^{-1}_{\mathrm{pre}}\mathscr{G}(U)}$.

Let $U'' \supseteq U' \supseteq U$ be open sets of X, pick $s_{\pi(U'')} \in \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U'')$, then

$$\mathrm{res}_{U',U} \circ \mathrm{res}_{U'',U'}(s_{\pi(U'')}) = \mathrm{res}_{U',U}(s_{\pi(U')}) = s_{\pi(U)} = \mathrm{res}_{U'',U}(s_{\pi(U'')}),$$

i.e., $\mathrm{res}_{U',U} \circ \mathrm{res}_{U'',U'} = \mathrm{res}_{U'',U}$.

Hence, $\pi^{-1}_{\mathrm{pre}}\mathscr{G}$ defines a presheaf on X. $\square$

Exercise 3.21 Show that $\pi^{-1}_{\mathrm{pre}}\mathscr{G}$ needn't form a sheaf.

Proof Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. X endowed with the discrete topology, $\mathscr{P}(X) = \{\varnothing, X, \{x_1\}, \{x_2\}\}$, and Y endowed with the topology $\mathscr{T}_Y = \{\varnothing, \{y_1\}, Y\}$. Let $\pi : X \to Y$ by setting

$$\pi(x_1) = y_1, \quad \pi(x_2) = y_2.$$

Clearly, π is continuous.

Let $\mathscr{G} = \underline{\mathbb{Z}} \in \mathbf{Sets}_Y$ be the constant sheaves, then

$$\pi_{\text{pre}}^{-1}\mathscr{G}(\{x_1\}) = \varinjlim_{V \supseteq \{y_1\}} \mathscr{G}(V) = \mathbb{Z},$$
$$\pi_{\text{pre}}^{-1}\mathscr{G}(\{x_2\}) = \varinjlim_{V \supseteq \{y_2\}} \mathscr{G}(V) = \mathbb{Z},$$

and

$$\pi_{\text{pre}}^{-1}\mathscr{G}(X) = \varinjlim_{V \supseteq Y} \mathscr{G}(V) = \mathbb{Z}.$$

We verify that $\pi_{\text{pre}}^{-1}\mathscr{G}$ violates the gluability axiom. Choose $s_1 = 1 \in \pi_{\text{pre}}^{-1}\mathscr{G}(\{x_1\}) = \mathbb{Z}$ and $s_2 = 2 \in \pi_{\text{pre}}^{-1}\mathscr{G}(\{x_2\}) = \mathbb{Z}$. Since $\{x_1\} \cap \{x_2\} = \varnothing$, $s_1|_{\{x_1\}\cap\{x_2\}} = s_2|_{\{x_1\}\cap\{x_2\}}$. If $\pi_{\text{pre}}^{-1}\mathscr{G}$ is a sheaf, then there exist a global section $s \in \pi_{\text{pre}}^{-1}\mathscr{G}(X) = \mathbb{Z}$ such that $s|_{\{x_1\}} = 1$ and $s|_{\{x_2\}} = 2$. It is impossible!

Hence, $\pi_{\text{pre}}^{-1}\mathscr{G}$ needn't form a sheaf. $\square$

Definition 3.7.2 (The inverse image sheaf)

Suppose continuous map $\pi : X \to Y$, let $\mathscr{G}$ be a sheaf over Y, define the ***inverse image of*** $\mathscr{G}$ *by*

$$\pi^{-1}\mathscr{G} := (\pi_{\text{pre}}^{-1}\mathscr{G})^{\text{sh}}.$$

Note that π^{-1} is a functor from sheaves on Y to sheaves on X. The next theorem shows that π^{-1} is indeed left-adjoint to π_*.

Theorem 3.7.1 ((π^{-1}, π_*) are adjoint)

If $\pi : X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on Y, then we have a natural bijection

$$\operatorname{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \overset{\sim}{\longleftrightarrow} \operatorname{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F}).$$

Proof Let $\mathscr{F}$ and $\mathscr{F}'$ are sheaves on X, $\mathscr{G}$ and $\mathscr{G}'$ are sheaves on Y, $\varphi : \mathscr{F} \to \mathscr{F}'$ and $\psi : \mathscr{G} \to \mathscr{G}'$ are morphisms of sheaves. Define $\tau_{\mathscr{G},\mathscr{F}} : \operatorname{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \to \operatorname{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$ as follow: Let $\alpha \in \operatorname{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, let $V \subseteq Y$ be an open subset, then $\pi^{-1}(V) \subseteq X$ open subset and $V \supseteq \pi(\pi^{-1}(V))$, consider the following sequence,

$$\begin{array}{ccc} \mathscr{G}(V) \xrightarrow{\eta_V} \pi_{\text{pre}}^{-1}\mathscr{G}(\pi^{-1}(V)) = \varinjlim_{V' \supseteq \pi(\pi^{-1}(V))} G(V') & & \\ \big\downarrow \text{sh}_{\pi^{-1}(V)} & & \\ \pi^{-1}\mathscr{G}(\pi^{-1}(V)) & \xrightarrow{\quad\alpha_{\pi^{-1}(V)}\quad} & \mathscr{F}(\pi^{-1}(V)) = \pi_*\mathscr{F}(V), \end{array}$$

where η_V is natural embedding, let

$$\tau_{\mathscr{G},\mathscr{F}}(\alpha)_V = \alpha_{\pi^{-1}(V)} \circ \text{sh}_{\pi^{-1}(V)} \circ \eta_V.$$

Since $\alpha_{\pi^{-1}(V)}$, $\text{sh}_{\pi^{-1}(V)}$, and η_V are well-defined, $\tau_{\mathscr{G},\mathscr{F}}(\alpha)_V$ is well-defined. Hence we defined $\tau_{\mathscr{G},\mathscr{F}}$.

Next, we shall to define the inverse of $\tau_{\mathscr{F},\mathscr{G}}$. Let $U \subseteq X$ be any open subset of X, then $\pi(U) \subseteq Y$, for any

$V \supseteq \pi(U)$, we have $\pi^{-1}(V) \supseteq U$. Let $\beta \in \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$, consider the following diagram,

$$\begin{tikzcd}
\pi^{-1}_{\mathrm{pre}}\mathscr{G}(U) \arrow[rrr, dashed] & & & \mathscr{F}(U) \\
& \mathscr{G}(V') \arrow[ul] \arrow[r, "\beta_{V'}"] \arrow[d, "\mathrm{res}"'] & \pi_*\mathscr{F}(V') \arrow[ur, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] & \\
& \mathscr{G}(V) \arrow[uul, bend left] \arrow[r, "\beta_V"] & \pi_*\mathscr{F}(V) \arrow[uur, bend right, "\mathrm{res}"'] &
\end{tikzcd}$$

dashed arrow given by the universal property of colimit, say $\chi_{\mathrm{pre},\beta,U} : \pi^{-1}_{\mathrm{pre}}\mathscr{G}(U) \to \mathscr{F}(U)$. Then, we defined a morphism of presheaves $\chi_{\mathrm{pre},\beta} : \pi^{-1}_{\mathrm{pre}}\mathscr{G} \to \mathscr{F}$. Now, consider the following diagram,

$$\begin{tikzcd}
\pi^{-1}_{\mathrm{pre}}\mathscr{G} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\chi_{\mathrm{pre},\beta}"'] & \pi^{-1}\mathscr{G} \arrow[d, dashed, "\chi_\beta"] \\
& \mathscr{F},
\end{tikzcd}$$

where $\chi_\beta : \pi^{-1}\mathscr{G} \to \mathscr{F}$ given by the universal property of sheafification. Define $\theta_{\mathscr{G},\mathscr{F}} : \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F}) \to \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$ by setting

$$\theta_{\mathscr{G},\mathscr{F}}(\beta) = \chi_\beta.$$

By above discussion, $\theta_{\mathscr{G},\mathscr{F}}$ is well-defined. Now, we need to prove $\theta_{\mathscr{G},\mathscr{F}}$ is, indeed, the inverse of $\tau_{\mathscr{G},\mathscr{F}}$.

Let $\alpha \in \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, let U be any open subset of X, then $\pi(U) \subseteq Y$, for all open subset $V \supseteq \pi(U)$, we have

$$\tau_{\mathscr{G},\mathscr{F}}(\alpha)_V = \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V.$$

Consider the following diagram.

$$\begin{tikzcd}
\pi^{-1}\mathscr{G}(U) \arrow[rrr, shift left, "\chi_{\tau_{\mathscr{G},\mathscr{F}}(\alpha),U}"] \arrow[rrr, shift right, "\alpha_U"'] & & & \mathscr{F}(U) \\
\pi^{-1}_{\mathrm{pre}}\mathscr{G}(U) \arrow[u, "\mathrm{sh}_U"] \arrow[rrr, "\chi_{\mathrm{pre},\tau_{\mathscr{G},\mathscr{F}}(\alpha),U}"'] & & & \mathscr{F}(U) \arrow[u, equal] \\
& \mathscr{G}(V') \arrow[ul, "\eta_{V'}"] \arrow[r, "\tau_{\mathscr{G},\mathscr{F}}(\alpha)_{V'}"] \arrow[d, "\mathrm{res}"'] & \pi_*\mathscr{F}(V') \arrow[ur, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] & \\
& \mathscr{G}(V) \arrow[uul, bend left, "\eta_V"] \arrow[r, "\tau_{\mathscr{G},\mathscr{F}}(\alpha)_V"] & \pi_*\mathscr{F}(V) \arrow[uur, bend right, "\mathrm{res}"'] &
\end{tikzcd}$$

By the universal property of colimit, we have $\alpha_U \circ \mathrm{sh}_U = \chi_{\mathrm{pre},\tau_{\mathscr{G},\mathscr{F}}(\alpha),U}$, and therefore $\alpha \circ \mathrm{sh} = \chi_{\mathrm{pre},\tau_{\mathscr{G},\mathscr{F}}(\alpha)}$. By the universal property of sheafification, $\alpha = \chi_{\tau_{\mathscr{G},\mathscr{F}}(\alpha)}$. Hence,

$$\theta_{\mathscr{G},\mathscr{F}}(\tau_{\mathscr{G},\mathscr{F}}(\alpha)) = \chi_{\tau_{\mathscr{G},\mathscr{F}}(\alpha)} = \alpha,$$

i.e., $\theta_{\mathscr{G},\mathscr{F}} \circ \tau_{\mathscr{G},\mathscr{F}} = \mathrm{id}_{\mathrm{Mor}_X(\pi^{-1}\mathscr{G},\mathscr{F})}$.

Let $\beta \in \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$, then

$$\tau_{\mathscr{G},\mathscr{F}} \circ \theta_{\mathscr{G},\mathscr{F}}(\beta) = \tau_{\mathscr{G},\mathscr{F}}(\chi_\beta).$$

Let $V \subseteq Y$ be any open subset of Y, then $\pi^{-1}(V)$ open in X, and

$$\tau_{\mathscr{G},\mathscr{F}}(\chi_\beta)_V = \chi_{\beta,\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V.$$

Let $s \in \mathscr{G}(V)$, then

$$\begin{aligned}\chi_{\beta,\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V(s) =& \chi_{\beta,\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)}(s_{\pi^{-1}(V)}) \\ =& \chi_{\mathrm{pre},\beta,\pi^{-1}(V)}(s_{\pi^{-1}(V)}) \\ =& \beta_V(s),\end{aligned}$$

which implies that $\chi_{\beta,\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V = \beta_V$. Hence,

$$\tau_{\mathscr{G},\mathscr{F}} \circ \theta_{\mathscr{G},\mathscr{F}}(\beta) = \beta,$$

i.e., $\tau_{\mathscr{G},\mathscr{F}} \circ \theta_{\mathscr{G},\mathscr{F}} = \mathrm{id}_{\mathrm{Mor}_Y(\mathscr{G},\pi_*\mathscr{F})}$. Hence, $\theta_{\mathscr{G},\mathscr{F}}$ is, indeed, the inverse of $\tau_{\mathscr{G},\mathscr{F}}$.

Consider the following diagram,

$$\begin{tikzcd} \mathrm{Mor}_X(\pi^{-1}\mathscr{G}', \mathscr{F}) \arrow[r, "(\pi^{-1}(\psi))^*"] \arrow[d, "\tau_{\mathscr{G}',\mathscr{F}}"'] & \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \arrow[r, "\varphi_*"] \arrow[d, "\tau_{\mathscr{G},\mathscr{F}}"'] & \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}') \arrow[d, "\tau_{\mathscr{G},\mathscr{F}'}"] \\ \mathrm{Mor}_Y(\mathscr{G}', \pi_*\mathscr{F}) \arrow[r, "\psi^*"] & \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F}) \arrow[r, "(\pi_*(\varphi))_*"] & \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F}), \end{tikzcd} \tag{3.7}$$

where $\pi_*\varphi_V = \varphi_{\pi^{-1}(V)}$ for all open subset $V \subseteq Y$, and $\pi^{-1}(\psi)$ given by the universal property of sheafification, i.e., the following commutative diagram

$$\begin{tikzcd} \pi^{-1}\mathscr{G} \arrow[rrr, dashed, "\pi^{-1}(\psi)"] & & & \pi^{-1}\mathscr{G}' \\ \pi^{-1}_{\mathrm{pre}}\mathscr{G} \arrow[u, "\mathrm{sh}"] \arrow[rrr, dashed, "\pi^{-1}_{\mathrm{pre}}(\psi)"] & & & \pi^{-1}_{\mathrm{pre}}\mathscr{G}' \arrow[u, "\mathrm{sh}"'] \\ & \mathscr{G}(V') \arrow[ul, "\eta^{\mathscr{G}}_{V'}"] \arrow[r, "\psi_{V'}"] \arrow[d, "\mathrm{res}"'] & \mathscr{G}'(V') \arrow[ur, "\eta^{\mathscr{G}'}_{V'}"'] \arrow[d, "\mathrm{res}"] & \\ & \mathscr{G}(V) \arrow[uul, bend left, "\eta^{\mathscr{G}}_V"] \arrow[r, "\psi_V"] & \mathscr{G}'(V). \arrow[uur, bend right, "\eta^{\mathscr{G}'}_V"'] & \end{tikzcd}$$

Let $\alpha \in \mathrm{Mor}_X(\pi^{-1}\mathscr{G}', \mathscr{F})$, then we have

$$\psi^* \circ \tau_{\mathscr{G}',\mathscr{F}}(\alpha) = \tau_{\mathscr{G}',\mathscr{F}}(\alpha) \circ \psi$$

and

$$\tau_{\mathscr{G},\mathscr{F}} \circ (\pi^{-1}(\psi))^*(\alpha) = \tau_{\mathscr{G},\mathscr{F}}(\alpha \circ \pi^{-1}(\psi)).$$

Let $V \subseteq Y$ be any open subset, then

$$\begin{aligned}(\psi^* \circ \tau_{\mathscr{G}',\mathscr{F}}(\alpha))_V =& \tau_{\mathscr{G}',\mathscr{F}}(\alpha)_V \circ \psi_V \\ =& \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta^{\mathscr{G}'}_V \circ \psi_V\end{aligned}$$

and

$$\begin{aligned}(\tau_{\mathscr{G},\mathscr{F}} \circ (\pi^{-1}(\psi))^*(\alpha))_V =& \tau_{\mathscr{G},\mathscr{F}}(\alpha \circ \pi^{-1}(\psi))_V \\ =& (\alpha \circ \pi^{-1}(\psi))_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta^{\mathscr{G}}_V \\ =& \alpha_{\pi^{-1}(V)} \circ \pi^{-1}(\psi)_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta^{\mathscr{G}}_V \\ =& \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \pi^{-1}_{\mathrm{pre}}(\psi) \circ \eta^{\mathscr{G}}_V \\ =& \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta^{\mathscr{G}'}_V \circ \psi_V.\end{aligned}$$

Hence, $(\psi^* \circ \tau_{\mathscr{G}',\mathscr{F}}(\alpha))_V = (\tau_{\mathscr{G},\mathscr{F}} \circ (\pi^{-1}(\psi))^*(\alpha))_V$, for any open subset $V \subseteq Y$, for all $\alpha \in \mathrm{Mor}_X(\pi^{-1}\mathscr{G}', \mathscr{F})$, and therefore

$$\psi^* \circ \tau_{\mathscr{G}',\mathscr{F}} = \tau_{\mathscr{G},\mathscr{F}} \circ (\pi^{-1}(\psi))^*.$$

Hence, the left side of diagram (3.7) commutes.

Let $\alpha \in \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, let $V \subseteq Y$ be any open subset of Y, then

$$\begin{aligned}((\pi_*(\varphi))_* \circ \tau_{\mathscr{G},\mathscr{F}}(\alpha))_V =& \pi_*(\varphi)_V \circ \tau_{\mathscr{G},\mathscr{F}}(\alpha)_V \\ =& \pi_*(\varphi)_V \circ \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V^{\mathscr{G}}\end{aligned}$$

and

$$\begin{aligned}(\tau_{\mathscr{G},\mathscr{F}'} \circ \varphi_*(\alpha))_V =& \tau_{\mathscr{G},\mathscr{F}'}(\varphi \circ \alpha)_V \\ =& (\varphi \circ \alpha)_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V^{\mathscr{G}} \\ =& \varphi_{\pi^{-1}(V)} \circ \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V^{\mathscr{G}} \\ =& \pi_*(\varphi)_V \circ \alpha_{\pi^{-1}(V)} \circ \mathrm{sh}_{\pi^{-1}(V)} \circ \eta_V^{\mathscr{G}}.\end{aligned}$$

Hence, $((\pi_*(\varphi))_* \circ \tau_{\mathscr{G},\mathscr{F}}(\alpha))_V = (\tau_{\mathscr{G},\mathscr{F}'} \circ \varphi_*(\alpha))_V$, for any open subset $V \subseteq Y$, for all $\alpha \in \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, and therefore

$$(\pi_*(\varphi))_* \circ \tau_{\mathscr{G},\mathscr{F}} = \tau_{\mathscr{G},\mathscr{F}'} \circ \varphi_*.$$

Hence, the right side of diagram (3.7) commutes.

Consequently, diagram (3.7) is commutative, and therefore there is a natural bijection

$$\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \overset{\sim}{\longleftrightarrow} \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F}).$$

□

Remark **Alternative Proof Method:** Show that both side agree with the following third construction, which we denote $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F})$. A collection of maps $\Phi_{V,U} : \mathscr{G}(V) \to \mathscr{F}(U)$ (as U runs through all open sets of X, and V runs through all open sets of Y containing $\pi(U)$) is said to be **compatible** if for all open $U' \subseteq U \subseteq X$ and all open open $V' \subseteq V \subseteq Y$ with $\pi(U) \subseteq V$, $\pi(U') \subseteq V'$, the diagram

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, "\Phi_{V,U}"] \arrow[d, "\mathrm{res}_{V,V'}"'] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,U'}"] \\ \mathscr{G}(V') \arrow[r, "\Phi_{V',U'}"] & \mathscr{F}(U') \end{tikzcd} \tag{3.8}$$

commutes. Define $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F})$ to be the set of all compatible collections $\Phi = \{\Phi_{V,U}\}$.

Remark **("Stalk and skyscraper are an adjoint pair").** As a special case, if X is a point $p \in Y$, we see that $\pi^{-1}\mathscr{G}$ is the stalk $\mathscr{G}_p$ of $\mathscr{G}$, and maps from the stalk $\mathscr{G}_p$ to a set S are the same as maps of sheaves on Y from $\mathscr{G}$ to the skyscraper sheaf with set S supported at p.

Proof By Theorem 3.7.1, we have

$$\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Mor}_X(\mathscr{G}_p, \mathscr{F}) \cong \mathrm{Mor}_Y(\mathscr{G}, i_{p,*}(\mathscr{F})),$$

for all $p \in Y$, then we done. □

Proposition 3.7.2 (The stalks of $\pi^{-1}\mathscr{G}$ are the same as the stalks of $\mathscr{G}$)

If $\pi(p) = q$, there is a natural isomorphism

$$\mathscr{G}_q \xrightarrow{\sim} (\pi^{-1}\mathscr{G})_p.$$

Proof Let $i_p : \{p\} \in X$ and $i_q : \{q\} \in Y$, then we get skyscraper sheaves $i_{p,*}S$ and $i_{q,*}S$, where S is any set. Since stalk and skyscraper are an adjoint pair, we have

$$\mathrm{Mor}_{\mathbf{Sets}}(\mathscr{G}_q, S) \cong \mathrm{Mor}_Y(\mathscr{G}, i_{q,*}S),$$

and

$$\mathrm{Mor}_{\mathbf{Sets}}((\pi^{-1}\mathscr{G})_p, S) \cong \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, i_{p,*}S).$$

Since (π^{-1}, π^*) are adjoint, we have

$$\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, i_{p,*}S) \cong \mathrm{Mor}_Y(\mathscr{G}, \pi_*(i_{p,*}S)).$$

Now, let's calculate $\pi_*(i_{p,*}S)$. Let V be any open subset of Y, then

$$\begin{aligned}(\pi_*(i_{p,*}S))(V) = (i_{p,*}S)(\pi^{-1}(V)) &= \begin{cases} S & \text{if } p \in \pi^{-1}(V) \\ \{e\} & \text{if } p \notin \pi^{-1}(V) \end{cases} \\ &= \begin{cases} S & \text{if } q \in V \\ \{e\} & \text{if } q \notin V \end{cases} \\ &= (i_{q,*}S)(V).\end{aligned}$$

Hence,

$$\mathrm{Mor}_{\mathbf{Sets}}((\pi^{-1}\mathscr{G})_p, S) \cong \mathrm{Mor}_X(\pi^{-1}\mathscr{G}, i_{p,*}S) \cong \mathrm{Mor}_Y(\mathscr{G}, i_{q,*}S) \cong \mathrm{Mor}_{\mathbf{Sets}}(\mathscr{G}_q, S),$$

for all $S \in \mathbf{Sets}$, and therefore $\mathrm{Mor}_{\mathbf{Sets}}((\pi^{-1}\mathscr{G})_p, \square)$ and $\mathrm{Mor}_{\mathbf{Sets}}(\mathscr{G}_q, \square)$ are naturally isomorphic functor. By corollary of Yoneda's Lemma (Corollary 2.2.7 (iii)), we have

$$(\pi^{-1}\mathscr{G})_p \overset{\sim}{\longleftrightarrow} \mathscr{G}_q.$$

$\square$

Remark Proposition 3.7.2, along with the notion of compatible germs, may give you a simple way of thinking about (and perhaps visualizing) inverse image sheaves. Closely related: you can think of sections of the inverse image sheaf as, locally, inverse images of sections on the target. (Those preferring the "espace étalé" perspective, §3.2.2, can check that the "inverse image of the espace étalé" is the "espace étalé" of the inverse image.)

Proposition 3.7.3 (Useful)

If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathscr{G}$ is a sheaf on Y. Then $i^{-1}\mathscr{G}$ is isomorphic to $\mathscr{G}|_U$.

Proof Since $i : U \to Y$ is inclusion, by Proposition 3.4.4, it suffices to show that $(i^{-1}\mathscr{G})_p \cong (\mathscr{G}|_U)_p$. By Proposition 3.7.2, we done. $\square$

Proposition 3.7.4 (π^{-1} is an exact functor)

(i) π^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X.

(ii) π^{-1} is an exact functor from $\mathscr{O}_Y$-modules (on Y) to $(\pi^{-1}\mathscr{O}_Y)$-modules (on X).

Proof We only prove (i), essentially the same argument will show (ii).

Let $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ be an exact sequence in $\mathbf{Ab}_Y$, we shall to show that

$$\pi^{-1}\mathscr{F} \longrightarrow \pi^{-1}\mathscr{G} \longrightarrow \pi^{-1}\mathscr{H} \tag{3.9}$$

is exact. By Proposition 3.6.5, sequence (3.9) is exact if and only if

$$(\pi^{-1}\mathscr{F})_p \longrightarrow (\pi^{-1}\mathscr{G})_p \longrightarrow (\pi^{-1}\mathscr{H})_p \tag{3.10}$$

is exact for any $p \in X$. By Proposition 3.7.2, sequence (3.10) is isomorphic to sequence

$$\mathscr{F}_q \longrightarrow \mathscr{G}_q \longrightarrow \mathscr{H}_q, \tag{3.11}$$

for all $q = \pi(p)$. Since $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact, by Proposition 3.6.5, sequence (3.11) is exact. Hence, sequence (3.10) is exact, and therefore

$$\pi^{-1}\mathscr{F} \longrightarrow \pi^{-1}\mathscr{G} \longrightarrow \pi^{-1}\mathscr{H}$$

is exact. $\square$

3.7.2 The push-pull map

Definition 3.7.3 (The push-pull map)

Suppose

$$\begin{tikzcd} W \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd} \tag{3.12}$$

is a commutative (not necessarily Cartesian (Definition 2.2.11)!) diagram, and $\mathscr{F}$ is a sheaf on X. Define the ***push-pull map***

$$\beta^{-1}\alpha_*\mathscr{F} \longrightarrow \alpha'_*(\beta')^{-1}\mathscr{F} \tag{3.13}$$

of sheaves on Y as follows:

i. *Start with the identity*

$$(\beta')^{-1}\mathscr{F} \xrightarrow{\sim} (\beta')^{-1}\mathscr{F}$$

on W.

ii. *By adjointness of $((\beta')^{-1}, \beta'_*)$, this is the same as the data of a morphism*

$$\mathscr{F} \longrightarrow (\beta'_*)(\beta')^{-1}\mathscr{F}$$

on X.

iii. *Apply α_* to get a map*

$$\alpha_*\mathscr{F} \longrightarrow \alpha_*(\beta'_*)(\beta')^{-1}\mathscr{F}$$

on Z

iv. *By the commutativity of diagram (3.12), this is the map*

$$\alpha_*\mathscr{F} \longrightarrow \beta_*(\alpha'_*)(\beta')^{-1}\mathscr{F}$$

on Z.

v. *By adjointness of (β^{-1}, β_*), this yields a map (3.13).*

We observe that this entire construction is functorial in $\mathscr{F}$, i.e., given a map $\mathscr{F} \to \mathscr{G}$ of sheaves on X, we get a certain commutative diagram of sheaves on Y.

$$\begin{tikzcd} \beta^{-1}\alpha_*\mathscr{F} \arrow[r] \arrow[d] & \beta^{-1}\alpha_*\mathscr{G} \arrow[d] \\ \alpha'_*(\beta')^{-1}\mathscr{F} \arrow[r] & \alpha'_*(\beta')^{-1}\mathscr{G} \end{tikzcd}$$

We will later extend this to $\mathscr{O}$-modules, quasicoherent sheaves, and cohomology.

Exercise 3.22 (Surprisingly hard exercise.) We could have defined the push-pull map in a "dual way" starting with the identity $\alpha_*\mathscr{F} \to \alpha_*\mathscr{F}$ on Z, then using adjointness of (α^{-1}, α_*), and continuing from there. Why does this give the same definition of the push-pull map?

Proof See onRiv, Canonicity of the push-pull map for sheaves of sets. $\square$

3.7.3 The support of a sheaf, and the support of a section of a sheaf

Proposition 3.7.6 below gives us an excuse to introduce the notion of **support**, which we use repeatedly later.

Definition 3.7.4 (Support of the section)

Suppose $\mathscr{F}$ is a sheaf (or indeed separated presheaf) of abelian groups on X, and s is a global section of $\mathscr{F}$. Define the ***support of the section*** *s, denoted $\operatorname{Supp} s$, to be the set of points p of X where s has a nonzero germ:*

$$\operatorname{Supp} s := \{p \in X : s_p \neq 0 \text{ in } \mathscr{F}_p\}.$$

Remark

(1) We think of this as the subset of X where "the sections s lives" — the complement is the locus where s is the 0-section.

(2) We could define this even if $\mathscr{F}$ is a presheaf, but without the inclusion $\mathscr{F}(U) \hookrightarrow \prod_{p\in U} \mathscr{F}_p$ of Proposition 3.4.1, we could have the strange situation where we have a nonzero section that "lives nowhere", because it is 0 "near every point", i.e., is 0 in every stalk.

For example: let $U \subseteq X$ be any open subset of X, define $\mathscr{F}(U) = \mathbb{Z}$, and the restriction map defined by $\operatorname{res}_{U,V}(s) = s$ if $U = V$ and $\operatorname{res}_{U,V}(s) = 0$ if $U \supsetneq V$. Then $\mathscr{F}$ form a presheaf, also $\mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$ not injective. Let $s \in \mathscr{F}(X) = \mathbb{Z}$ be a global section, then $s_p = 0$ in $\mathscr{F}_p$. Hence $\operatorname{Supp} s = \varnothing$ for all $s \in \mathscr{F}(X)$.

Proposition 3.7.5 (The support of a section is closed)

$\operatorname{Supp} s$ is a closed subset of X.

Proof It suffices to show that $X \setminus \operatorname{Supp} s$ is an open subset of X. In fact,

$$X \setminus \operatorname{Supp} s = \{p \in X : s_p = 0 \text{ in } \mathscr{F}_p\},$$

say $U = X \setminus \operatorname{Supp} s$. Let $p \in U$, then there exists open subset $(p \in V_p) \subseteq X$ such that $s|_V = 0$. We shall to show that $V_p \subseteq U$. For any $q \in V_p$, there exists open subset $W \ni q$ such that $W \subseteq V_p$. Note that

$$\operatorname{res}_{V_p,W}(\operatorname{res}_{X,V_p}(s)) = 0 = \operatorname{res}_{X,W}(s) = s|_W,$$

we have $s_q = 0$. Hence, $y \in U$, and therefore $V_p \in U$. Since $U = \bigcup_{p\in U} V_p$ and each V_p is open in X, U is open in X, i.e., $\operatorname{Supp} s$ is a closed subset of X. $\square$

Remark **Caution: the locus where a continuous function is nonzero is open; the locus where the germ of a function is nonzero is closed.**

Basically by the definition of continuity, the locus where the value of a continuous function is nonzero is open. (More generally, the locus where the value of a function on a locally ringed space is nonzero is open.) In contrast, Proposition 3.7.5 shows that the locus where the germ of a function is nonzero is closed. We will try to avoid misunderstanding by using phrases like "f is 0 at p" (the value of f is zero, i.e., $f(p) = 0$) and "f is 0 near p" (the germ of f is zero, i.e., $f = 0$ in $\mathscr{O}_{X,p}$, or equivalently, f is zero in some neighborhood of p).

Definition 3.7.5 (Support of s sheaf)

Define the ***support of a sheaf*** *of groups* $\mathscr{G}$*, denoted* $\operatorname{Supp}\mathscr{G}$*, as the locus where the stalks are nontrivial:*

$$\operatorname{Supp}\mathscr{G} := \{p \in X : |\mathscr{G}_p| \neq 1\}.$$

Equivalently, $\operatorname{Supp}\mathscr{G}$ *is the union of supports of sections over all open sets:*

$$\operatorname{Supp}\mathscr{G} = \bigcup_{\substack{U\subseteq X \text{ open} \\ s\in\mathscr{G}(U)}} \operatorname{Supp} s$$

Remark Clearly support is a "stalk-local notion", and hence "commutes" with restriction to open sets, that is,

$$\operatorname{Supp}\mathscr{G}|_U = \operatorname{Supp}\mathscr{G} \cap U.$$

Irrelevant for us: more generally, if the sheaf has values in some category, such as the category of sets, the support can be defined as the points where the stalk is not the terminal object.

Proposition 3.7.6

(a) Suppose $Z \subseteq Y$ *is a closed subset, and* $i : Z \hookrightarrow Y$ *is the inclusion. If* $\mathscr{F}$ *is a sheaf of groups on* Z*, then the stalk* $(i_*\mathscr{F})_q$ *is the one-element group if* $q \notin Z$*, and* $(i_*\mathscr{F})_q = \mathscr{F}_q$ *if* $q \in Z$*.*

(b) Suppose $\operatorname{Supp}\mathscr{G} \subseteq Z$ *where* Z *is closed. Then the natural map* $\mathscr{G} \to i_*i^{-1}\mathscr{G}$ *is an isomorphism. Thus a sheaf supported on a closed subset can be considered a sheaf on that closed subset.*

Proof

(a) Let $V \subseteq Y$ be any open subset of Y, then $i^{-1}(V) = V \cap Z$. Hence,

$$i_*\mathscr{F}(V) = \mathscr{F}(i^{-1}(V)) = \mathscr{F}(V \cap Z).$$

Let $q \in Y$.

If $q \notin Z$, then $Y \setminus Z$ is an open neighborhood of q, and $(Y \setminus Z) \cap Z = \varnothing$, hence, $i_*\mathscr{F}(Y \setminus Z) = \mathscr{F}(\varnothing)$, and therefore $(i_*\mathscr{F})_q$ is the one-element group.

If $q \in Z$, let V be any open neighborhood of q, then $V \cap Z \neq \varnothing$. Note that the topology on Z can be induced by the topology on Y, each open neighborhood W of q in Z can be seen as $W = V \cap Z$, where V is the open neighborhood of q in Y. Hence, $(i_*\mathscr{F})_q = \mathscr{F}_q$.

(b) The natural map is given by the adjoint pair (i^{-1}, i_*), i.e., consider the identity $i^{-1}\mathscr{G} \xrightarrow{\sim} i^{-1}\mathscr{G}$, by adjointness of (i^{-1}, i_*), we get a natural map

$$\mathscr{G} \longrightarrow i_*i^{-1}\mathscr{G}.$$

Let $p \in Y$.

If $p \notin Z$, then $p \notin \operatorname{Supp}\mathscr{G}$, and therefore $|\mathscr{G}_p| = 1$, say $\mathscr{G}_p = \{e\}$ where e is unit element. Since $p \notin Z$, $Y \setminus Z$ is an open neighborhood of p, which disjoint Z, say $V = Y \setminus Z$. Hence,

$$i_*i^{-1}\mathscr{G}(V) = i^{-1}\mathscr{G}(\varnothing) = (\varinjlim_{W\supseteq i(\varnothing)} \mathscr{G}(W))^{\text{sh}} = \mathscr{G}(\varnothing).$$

Hence, $(i_*i^{-1}\mathscr{G})_p \cong G_p$.

If $p \in Z$, by part (a), we have $(i_*i^{-1}\mathscr{G})_p = (i^{-1}\mathscr{G})_p$. By Proposition 3.7.2, $\mathscr{G}_p \cong (i^{-1}\mathscr{G})_p$, and therefore $\mathscr{G}_p \cong (i_*i^{-1}\mathscr{G})_p$.

Hence, by Proposition 3.4.4, we have an isomorphism $\mathscr{G} \cong i_*i^{-1}\mathscr{G}$.

□

Extension by zero, an occasional left adjoint to the inverse image functor. In addition to always being a left adjoint, π^{-1} can sometimes be a right adjoint, when π is an inclusion of an open subset. We discuss this when we need it.

Part III

Schemes

Chapter 4 Toward affine schemes: the underlying set, and topological space

We are now ready to consider the notion of **scheme**, which is the type of geometric space central to algebraic geometry. We should first think through what we mean by "geometric space". You have likely seen the notion of manifold, and we wish abstract this notion so that it can be generalized to other settings, notably so that we can deal with nonsmooth and arithmetic objects.

4.1 Toward scheme

See Ravi Vakil, The Rising Sea: Foundations of Algebraic Geometry.

The key insight behind this generalization from the notion of something like a manifold to a more versatile notion of a "geometric space" is the following: we can understand a geometric space (such as manifold) well by understanding the functions on this space. More precisely, we will understand it though the sheaf of functions on the space. If we are interested in differentiable manifolds, we will consider smooth functions; if we are interested in analytic manifolds, we will consider real analytic functions; and so on.

Thus we will define a scheme to be the following data.

- The set: the points of the scheme
- The topology: the open sets of the scheme
- The structure sheaf: the sheaf of "algebraic functions" (a sheaf of rings) on the scheme.

Recall that a topological space with a sheaf of rings is called a ringed space.

We will try to draw picture throughout. Pictures can help develop geometric intuition, which can guide the algebraic development (and, eventually, vice versa).

We will try to make all three notions as intuitive as possible. For the set, in the key example of complex (affine) varieties (roughly, things cut out in $\mathbb{C}^n$ by polynomials), we will see that the points are the "traditional points" (n-tuples of complex numbers), plus some extra points that will be handy to have around. For the topology, we will require that "the subset where an algebraic function vanishes must be closed", and require nothing else. For the sheaf of algebraic functions (the structure sheaf), we will expect that in the complex plane $\mathbb{C}^2$, $(3x^2 + y^2)/(2x + 4xy + 1)$ should be an algebraic function on the open set consisting of points where the denominator doesn't vanish, and this will largely motivate our definition.

4.1.1 Example:Differentiable manifolds

Differentiable manifolds. As motivation, we return to our example of differentiable manifolds, reinterpreting them in this light. We will be quite informal in this discussion. Suppose X is a differentiable manifold. It is a topological space, and has a sheaf of smooth (C^∞) functions $\mathscr{O}_X$ (see §3.1). This gives X the structure of a ringed space. We have observed that evaluation at a point $p \in X$ gives a surjective map from the stalk to $\mathbb{R}$,

$$\mathscr{O}_{X,p} \twoheadrightarrow \mathbb{R}$$

so the kernel, the (germs of) functions vanish at p, is a maximal ideal $\mathfrak{m}_{X,p}$ (see Proposition 3.1.1).

We could define a differentiable real manifold as a topological space X with a sheaf of rings. We would require that there is a cover of X by open sets such that on each open set the ringed space is isomorphic to a ball around the origin in $\mathbb{R}^n$ (with the sheaf of smooth functions on that ball). With this definition, the ball is the basic patch, and a genera manifold is obtained by gluing these patches together. (Admittedly a great deal of geometry comes from how one chooses to patch the balls together!) In the algebraic setting, the basic bath is the notion of an affine scheme, which we will discuss soon. (In the definition of manifold, there is an additional requirement that the topological space be Hausdorff and second-countable, to avoid pathologies. Schemes are often required to be "separated" to avoid essentially the same pathologies.)

Functions are determined by their values at points. This is an obvious statement, but won't be true for schemes in general.

Morphisms of manifolds. How can we describe maps of differentiable manifolds $\pi : X \to Y$? They are certainly continuous maps — but which ones? We can pull back functions along continuous maps. Smooth functions pull back to smooth functions. More formally, we have a map $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. (The inverse image sheaf π^{-1} was defined in §3.7.) Inverse image is left-adjoint to pushforward, so we also get a map $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$.

Certainly given a map of differentiable manifolds, smooth functions pull back to smooth functions. It is less obvious that **this is a sufficient condition for a continuous map to be smooth.**

Proposition 4.1.1

Suppose that $\pi : X \to Y$ is a continuous map of differentiable manifolds (as topological spaces — not a priori smooth). Then π is smooth if smooth functions pull back to smooth functions, i.e., if pullback by π gives a map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$.

Proof Let V be any open set of Y, and (V, ψ) is a chart of Y, we defined the pullback $\pi_V^* : \mathscr{O}_Y(V) \to \pi_*\mathscr{O}_X(V) = \mathscr{O}_X(\pi^{-1}(V))$ by setting

$$f \mapsto f \circ \pi|_{\pi^{-1}(V)}.$$

We shall to show that $f \circ \pi|_{\pi^{-1}(V)}$ is smooth. Say $(U, \varphi) = (\pi^{-1}(V), \varphi)$, without loss of generally, we may assume that (U, φ) be a chart of X.

Let U is homeomorphic to $\mathbb{R}^n$ and V is homeomorphic to $\mathbb{R}^m$. Note that

$$f \circ \pi|_{\pi^{-1}(V)} \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \pi \circ \varphi^{-1}),$$

since $f \in \mathscr{O}_Y(V)$ (i.e., $f \circ \psi^{-1}$ is smooth), $f \circ \pi|_{\pi^{-1}(V)}$ is smooth if and only if π is smooth. Then we done. □

Proposition 4.1.2

A morphism of differentiable manifolds $\pi : X \to Y$ with $\pi(p) = q$ induces a morphism of stalks $\pi^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$. Then $\pi^\sharp(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$. In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p.

Proof In fact, $\mathfrak{m}_{Y,p} = \{f_q \in \mathscr{O}_{Y,q} : f_q(q) = 0\}$ and $\mathfrak{m}_{X,p} = \{f_p \in \mathscr{O}_{X,p} : f_p(p) = 0\}$. Define $\pi^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$ by setting

$$\pi^\sharp(f_q) = (\pi^*(f))_p = (f \circ \pi)_p.$$

We shall to show that $\pi^\sharp$ is well-defined. If $f_q = g_q$, then there exists $(q \in)V \subseteq Y$ such that $f|_V = g|_V$, hence $f \circ \pi|_{\pi^{-1}(V)} = g \circ \pi|_{\pi^{-1}(V)}$, where $\pi^{-1}(V)$ is an open neighborhood of p. Hence, $(\pi^*(f))_p = (\pi^*(g))_p$, this implies that $\pi^\sharp$ is well-defined.

Let $f_q \in \mathscr{O}_{Y,q}$, then there exists open subset $(q \in)V \subseteq Y$ such that $f \in \mathscr{O}_Y(V)$ with $f(q) = 0$. Apply pullback π_V^* to f, then we have $\pi_V^*(f) = f \circ \pi|_{\pi^{-1}(V)}$. Since $\pi(p) = q$, $\pi_V^*(f)(p) = 0$, which implies that $\pi_V^*(f)_p \in \mathfrak{m}_{X,p}$, i.e., $\pi^\sharp(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$. □

Remark **Aside.** Notice that π induces a map on tangent space

$$(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^\vee \longrightarrow (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^\vee.$$

This is the tangent map you would geometrically expect. Again, it is interesting that the cotangent map

$$\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2 \longrightarrow \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$$

is algebraically more natural than the tangent map (there are no "duals").

Remark Manifolds are covered by disks that are all isomorphic. This isn't true for schemes (even for "smooth complex varieties"). There are examples of two "smooth complex curves" (the algebraic version of Riemann surfaces) X and Y so that no nonempty open subset of X is isomorphic to a nonempty open subset of Y. And there is a Riemann surface X such that no two open subsets of X are isomorphic. Informally, this is because in the Zariski topology on schemes, all nonempty open sets are "huge" and have more "structure".

4.1.2 Other examples

If you are interested in differential geometry, you might be interested in differentiable manifolds, on which the functions under consideration are smooth functions. Similarly, if you are interested in topology, you will be interested in topological spaces, on which you will consider the continuous functions. If you are interested in complex geometry, you will be interested in complex manifolds (or possibly "complex analytic varieties"), on which the functions are holomorphic functions. In each of these cases of interesting "geometric spaces", the topological space and sheaf of functions is clear. The notion of scheme fits naturally into this family.

4.2 The underlying set of an affine scheme

See Ravi Vakil, The Rising Sea: Foundations of Algebraic Geometry.

For any ring A, we are going to define something called $\operatorname{Spec} A$, the **spectrum of** A. In this section, we will define a sheaf of rings on it (the structure sheaf). Such an object is called an **affine scheme**. Later $\operatorname{Spec} A$ will denote the set along with the topology, and sheaf of functions. But for now, as there is no possibility of confusion, $\operatorname{Spec} A$ will just be the set.

Definition 4.2.1 (Spectrum)

The set $\operatorname{Spec} A$ *is the set of prime ideals of* A. *The prime ideal* $\mathfrak{p}$ *of* A *when consider as an element of* $\operatorname{Spec} A$ *will be denoted* $[\mathfrak{p}]$, *to avoid confusion. Elements* $a \in A$ *will be called functions on* $\operatorname{Spec} A$, *and their* ***value*** *at the point* $[\mathfrak{p}]$ *will be* $a \pmod{\mathfrak{p}}$.

Remark This is weird: a function can take values in different rings at different points — the function 5 on $\operatorname{Spec} \mathbb{Z}$ takes the value $1 \pmod 2$ at $[(2)]$ and $2 \pmod 3$ at $[(3)]$.

Note *"An element* a *of the ring lying in a prime ideal* $\mathfrak{p}$*" translates to "a function* a *that is* 0 *at the point* $[\mathfrak{p}]$*" or "a function* a *vanishing at the point* $[p]$*", and we will use these phrases interchangeably.*

Notice that if you add or multiply two functions, you add or multiply their values at all points; this is a translation of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism.

If A is generated over a base field (or base ring) by elements $x_1, \ldots, x_r$, the elements $x_1, \ldots, x_r$ are often called **coordinates**, because we will later be able to reinterpret them as restrictions of "coordinates on r-space".

We are building up the beginning of a grand dictionary. At some point you will get the sense that we are slowly decoding some timeless Rosetta Stone whose etchings we struggle to make out.

Glimpses of the future. In Chapter 5: we will interpret functions on $\operatorname{Spec} A$ as global sections of the "structure sheaf", i.e., as a function on a ringed space. We repeat a caution from "ringed space": what we will call "functions", others may call "regular functions". And we will later define "rational functions", which are not precisely functions in this sense; they are a particular type of "partially-defined function".

The notion of "value of a function" will be later interpreted as a value of a function on a particular locally ringed space.

4.2.1 Some examples.

The complex affine line

The complex affine line: $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$. Let's find the prime ideals of $\mathbb{C}[x]$. As $\mathbb{C}[x]$ is an integral domain, 0 is prime. Also, $(x-a)$ is prime, for any $a \in \mathbb{C}$: it is even a maximal ideal, as the quotient by this ideal is a field:

$$0 \longrightarrow (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0$$

We now show that there are no other prime ideals. We use the fact that $\mathbb{C}[x]$ has a division algorithm, and is a UFD. Suppose $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p} \neq (0)$, then suppose $f(x) \in \mathfrak{p}$ is a nonzero element of smallest degree. It is not constant, as prime ideals can't contain 1. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. (Here we use that $\mathbb{C}$ is algebraically closed.) Then $g(x) \in \mathfrak{p}$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $x-a$ of $\mathfrak{p}$. Then we claim that $\mathfrak{p} = (x-a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x-a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x-a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction.

Thus we can and should (and must!) make a picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$ (see Figure 4.1).

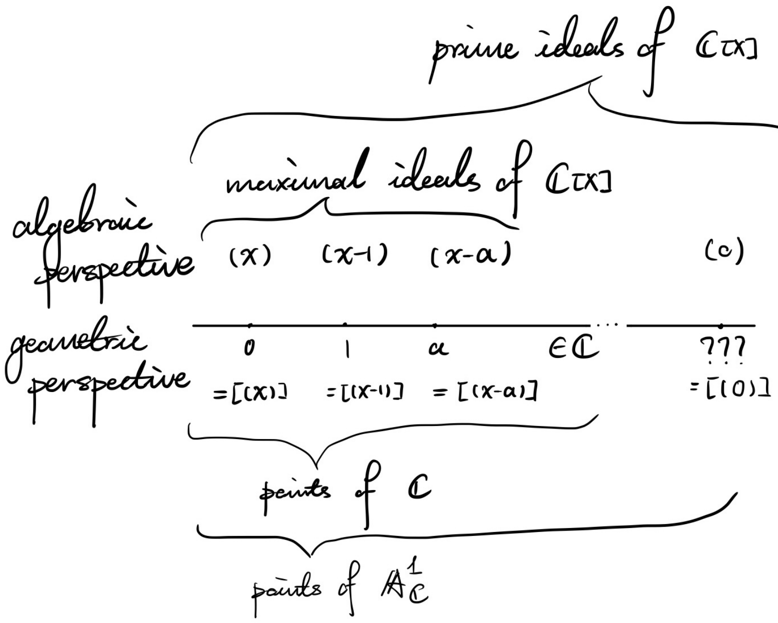


Figure 4.1: A picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$

There is one "traditional" point for each complex number, plus one extra ("bonus") point $[(0)]$. We can mostly picture $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$: the point $[(x-a)]$ we will reasonably associate to $a \in \mathbb{C}$. Where should we picture the point $[(0)]$? The best way of thinking about it is somewhat zen. It is somewhere on the complex line, but nowhere in particular. Because (0) is contained in all of these prime ideals, we will somehow associate it with this line passing through all the other points. This new point $[(0)]$ is called the "generic point" of line. (We will formally define "generic point" in §3.6.) It is "generically on the line" but you can't pin it down any further than that. It is not at any particular place on the line. (This is misleading too — we will see that it is "near" every point. So it is near everything, but located nowhere precisely.) We will place it far to the right for lack of anywhere better to put it. Notice that we sketch $\mathbb{A}^1_{\mathbb{C}}$ as one-(real-)dimensional (even though we picture it as an enhanced version of $\mathbb{C}$); this is to later remind ourselves that this will be a one-dimensional space, where dimensions are defined in an algebraic (or complex-geometric) sense. (Dimension will be defined in Chapter 13.)

To give you some feeling for this space, we make some statements that are currently undefined, but suggestive. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. $f(x) \pmod{x-1} = f(1)$ is $f(x)$'s value at point $[(x-1)]$. (What is its value at $[(0)]$? It is $f(x) \pmod 0$, which is just $f(x)$.)

Here is a more complicated example: $g(x) = (x-3)^3/(x-2)$ is a "rational function". It is defined everywhere but $x = 2$. (When we know what the structure sheaf is, we will be able to say that it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} \setminus \{2\}$.) We want to say that $g(x)$ has a triple zero at 3, and a single pole at 2, and we will be able to after Chapter 14.

The affine line over $k = \overline{k}$

The affine line over $k = \overline{k}$: $\mathbb{A}^1_k := \operatorname{Spec} k[x]$ where k is an algebraically closed field. This is called the affine line over k. All of our discussion in the previous example carries over without change. We will use the same picture, which is after all intended to just be a metaphor.

$\operatorname{Spec} \mathbb{Z}$

$\operatorname{Spec} \mathbb{Z}$. An amazing fact is that from our perspective, this will look a lot like the affine line $\mathbb{A}^1_{\overline{k}}$. The integers, like $\overline{k}[x]$, form a UFD, with a division algorithm. The prime ideals are: (0), and (p) where p is prime. Thus everything from "The complex affine line" carries over without change, even the picture. Our picture of $\operatorname{Spec} \mathbb{Z}$ is shown in Figure 4.2.

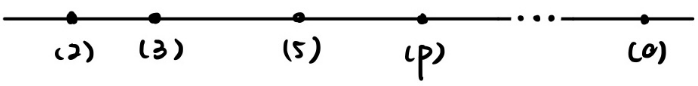


Figure 4.2: A "picture" of $\operatorname{Spec} \mathbb{Z}$, which looks suspiciously like Figure 4.1

Let's blithely carry over our discussion of functions to this space. 100 is a function on $\operatorname{Spec} \mathbb{Z}$. Its value at (3) is "$1 \pmod 3$". Its value at (2) is "$0 \pmod 2$", and in fact it has a double zero. $27/4$ is a "rational function" on $\operatorname{Spec} \mathbb{Z}$, defined away from (2). We want to say that it has a double pole at (2), and a triple zero at (3). Its value at (5) is

$$27 \times 4^{-1} \equiv 2 \times 4 \equiv 3 \pmod 5.$$

(We will gradually make this discussion precise over time.)

Silly but important examples, and the German word for bacon.

The set $\operatorname{Spec} k$ where k is any field is boring: one point. $\operatorname{Spec} 0$, where 0 is the zero-ring, is the empty set, as 0 has no prime ideals, i.e., $\operatorname{Spec} k = \varnothing$.

✍ **Exercise 4.1** A small exercise about small schemes.

(a) Describe the set $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. The ring $k[\varepsilon]/(\varepsilon^2)$ is called the ring of **dual numbers**, and will turn out to be quite useful. You should think of ε as a very small number, so small that its square is 0 (although it itself is not 0). It is a nonzero function whose value at all points is zero, thus giving our first example of functions not being determined by their values at points.

(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$. We will see this scheme again repeatedly. You might later think of it as a shred of a particularly nice "smooth curve".

Proof

(a) Each element in $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ has form $a + b\varepsilon$, where $a, b \in k$. We claim that $a + b\varepsilon$ is invertible if and only if $a \neq 0$.
If $a \neq 0$, note that

$$(a + b\varepsilon)(a^{-1} - a^{-2}b\varepsilon) = 1,$$

hence $a + b\varepsilon$ is invertible.
If $a = 0$, note that $(b\varepsilon)^2 = 0$, hence $a + b\varepsilon$ is not invertible.
Hence, the prime ideal of $k[\varepsilon]/(\varepsilon^2)$ is only (ε), and therefore $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) = \{(\varepsilon)\}$.
Moreover, $\varepsilon \pmod{(\varepsilon)} = 0$ in $(k[\varepsilon]/(\varepsilon^2))/(\varepsilon)$, i.e., ε is a nonzero function whose value at all points is zero.

(b) Using the consequence in commutative algebra, the prime ideal in $k[x]_{(x)}$ is one-to-one correspondence to the prime ideal in $k[x]$ which contained in (x). Assume $(0) \subsetneq \mathfrak{p} \subsetneq (x)$, since $k[x]$ is a PID, we may assume $\mathfrak{p} = (xa)$ for some $a \in k[x]$. However, $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$, $a \in \mathfrak{p}$. Then $xa \mid a$, this implies that x is a unit, contradiction. Hence, for $\mathfrak{p} \in \operatorname{Spec} k[x]$ with $\mathfrak{p} \subseteq (x)$. $\mathfrak{p} = (x)$. Also, note that $k[x]_{(x)}$ is integral domain, (0) is the prime ideal of $k[x]_{(x)}$. Hence, $\operatorname{Spec} k[x]_{(x)} = \{(0), (x)k[x]_{(x)}\}$.

□

In example of "The affine line over $k = \overline{k}$", we restricted to the case of algebraically closed fields for a reason: things are more subtle if the field is not algebraically closed.

The affine line over $\mathbb{R}$

The affine line over $\mathbb{R}$: $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$. Using the fact that $\mathbb{R}[x]$ is a Euclidean domain, similar arguments to those of examples above show that the prime ideals are $(0), (x - a)$ where $a \in \mathbb{R}$, and $(x^2 + ax + b)$ is an irreducible quadratic. The latter two are maximal ideals, i.e., their quotients are field. For example: $\mathbb{R}[x]/(x - 3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

✍ **Exercise 4.2** Show that for the last type of prime, of the form $(x^2 + ax + b)$, the quotient is always isomorphic to $\mathbb{C}$.

Proof By division algorithm, for all $f(x) \in \mathbb{R}[x]$, $f(x) = g(x)(x^2 + ax + b) + cx + d$. Since $x^2 + ax + c$ is irreducible, the two roots of $x^2 + ax + c$ are both imaginary numbers, say the root $r = r_1 + ir_2$ where $r_2 \neq 0$. Define a ring homomorphism $\nu_r : \mathbb{R}[x]/(x^2 + ax + d) \to \mathbb{C}$ by setting $cx + d \mapsto cr + d$. Let $\nu_r(cx + d) = cr + d = 0$, then $(cr_1 + d) + icr_2 = 0$, hence $c = 0$ and $d = 0$, this implies that ν_r is injective. Let $\alpha + i\beta \in \mathbb{C}$, note that $\nu_r(\frac{\beta}{r_2}x + \alpha - \beta\frac{r_1}{r_2}) = \alpha + i\beta$, ν_r is surjective. Hence, ν_r is isomorphism, and therefore $\mathbb{R}[x]/(x^2+ax+b) \cong \mathbb{C}$. $\square$

So we have the points that we would normally expect to see on the real line, corresponding to real numbers; the generic point 0; and new points which we may interpret as conjugate pairs of complex numbers (the roots of the quadratic). This last type of point should be seen as more akin to the real numbers than to the generic point. We can picture $\mathbb{A}^1_{\mathbb{R}}$ as the complex plane, folded along the real axis (see Figure 4.3). But the key point is that Galois-conjugate points (such as i and $-i$) are consider glued.

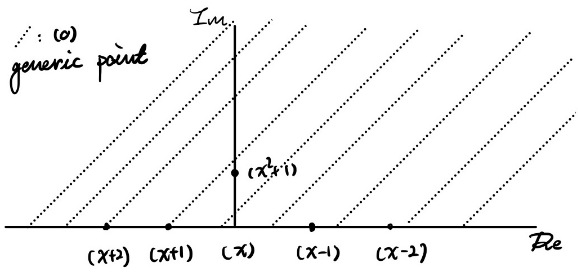


Figure 4.3: A "picture" of $\operatorname{Spec}\mathbb{R}[x]$

Let's explore functions on this space. Consider the function $f(x) = x^3 - 1$. Its value at the point $[(x-2)]$ is 7, or perhaps better, "$7 \pmod{x-2}$". How about at (x^2+1)? We get

$$x^3 - 1 \equiv -x - 1 \pmod{x^2+1},$$

which may be profitably interpreted as $-i-1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. It is more complicated, but we can recover much of the information we care about.

Exercise 4.3 Describe the set $\mathbb{A}^1_{\mathbb{Q}}$.

Solution *$\mathbb{Q}[x]$ is an integral domain, hence (0) is prime. Since $\mathbb{Q}[x]$ is also a Euclidean domain (and therefore PID), an ideal $(f(x))$ is prime if and only if $f(x)$ is irreducible. Hence,*

$$\operatorname{Spec}\mathbb{Q}[x] = \{(0), (f(x)) : f(x) \text{ is an irreducible polynomial}\}.$$

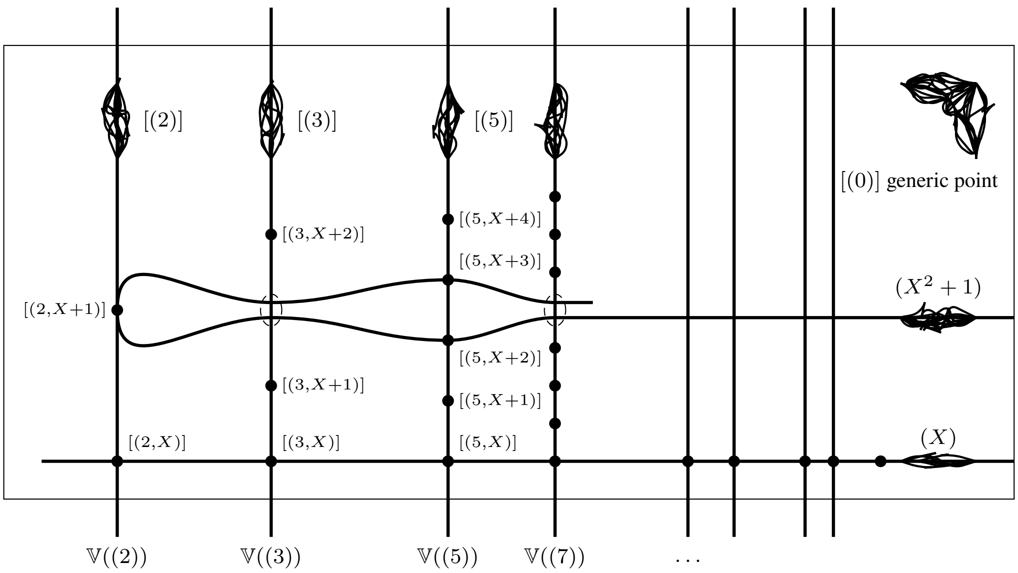


Figure 4.4: Picturing $\operatorname{Spec}\mathbb{Z}[x]$

The affine line over $\mathbb{F}_p$

The affine line over $\mathbb{F}_p$: $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$. As in the previous examples, $\mathbb{F}_p[x]$ is a Euclidean domain, so the prime ideals are of the form (0) or $(f(x))$ where $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}_p}$.

Note that $\operatorname{Spec} \mathbb{F}_p[x]$ has p points corresponding to the elements of $\mathbb{F}_p$, but also many more (infinitely more, see 4.4). For example, consider $\mathbb{F}_2 = \{0, 1\}$, $(x), (x-1)$ in $\operatorname{Spec} \mathbb{F}_2$ corresponding to the elements of $\mathbb{F}_p$, $(x^2 + x + 1)$ also in $\operatorname{Spec} \mathbb{F}_2$ but not corresponding to the elements of $\mathbb{F}_2$. This makes this space much richer than simply p points. For example, a polynomial $f(x)$ is not determined by its values at the p elements of $\mathbb{F}_p$, but it is determined by its values at the points of $\operatorname{Spec} \mathbb{F}_p[x]$. (As we have mentioned before, this is not ture for all schemes.)

You should think about this, even if you are a geometric person — this intuition will later turn up in geometric situations. Even if you think you are interested only in working over an algebraically closed field (such as $\mathbb{C}$), you will have nonalgebraically closed fields (such as $\mathbb{C}(x)$) forced upon you.

Exercise 4.4 If k is a field, show that $\operatorname{Spec} k[x]$ has infinitely many points.

Proof Since k is a field, the maximal ideals in $k[x]$ are principal, generated by irreducible polynomials. Hence, it suffices to show that there are infinitely many irreducible polynomials. We prove this by using Euclid's trick.

Suppose there are finitely many irreducible polynomials in $k[x]$, say $p_1, \ldots, p_r$. Consider a new polynomial

$$F(x) = p_1(x)p_2(x)\cdots p_r(x) + 1.$$

Since $k[x]$ is a UFD, there is an irreducible factor p, also, the irreducible polynomial p can't be any of $p_1, \cdots, p_r$. Then we get a new irreducible polynomial which is not in $p_1, \cdots, p_r$, a contradiction. $\square$

The complex affine plane

The complex affine plane: $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (As with examples "The complex affine line" and "The affine line over $k = \overline{k}$", our discussion will apply with $\mathbb{C}$ replaced by any algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a PID: (x, y) is not a principal ideal. We can quickly name some prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x-2, y-3)$ is prime, and indeed maximal, because $\mathbb{C}[x, y]/(x-2, y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x-a, y-b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g., $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime.

Exercise 4.5 Show that we have identified all the prime ideal of $\mathbb{C}[x, y]$.

Proof $\mathbb{C}[x, y]$ is a UFD, and therefore an integral domain, hence, (0) and $(f(x, y))$ where $f(x, y)$ is an irreducible polynomial are prime ideals in $\mathbb{C}[x, y]$.

Let $\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[x, y]$ that is not principal, and let $f(x, y), g(x, y) \in \mathfrak{p}$ with no common factor (if we can not find, $\mathfrak{p}$ must be principal). The greatest common divisor for $f(x, y)$ and $g(x, y)$ in $\mathbb{C}(x)[y]$ can be written in the form $\frac{a(x)c(x)}{b(x)}$, where $c(x)$ has no non-trivial factors in $\mathbb{C}[x, y]$. Since $\mathbb{C}[x, y]$ is a UFD, so this statement makes sense. Then for some $f'(x, y), p(x), g'(x, y)$, and $q(x)$, we have

$$f(x, y) = \frac{a(x)c(x)}{b(x)} \frac{f'(x, y)}{p(x)}$$

and

$$g(x, y) = \frac{a(x)c(x)}{b(x)} \frac{g'(x, y)}{q(x)},$$

which is to say that

$$a(x)c(x)f'(x,y) = b(x)p(x)f(x,y)$$
$$a(x)c(x)g'(x,y) = b(x)q(x)g(x,y).$$

Note that $c(x) \nmid b(x)$, $c(x) \nmid p(x)$, and $c(x) \nmid q(x)$, $c(x) \nmid f(x,y)$ and $c(x) \nmid g(x,y)$. Since, $f(x,y)$ and $g(x,y)$ have no common factor in $\mathbb{C}[x,y]$, $c(x)$ must be a constant, we may assume $c(x) = 1$. Hence, $gcd_{\mathbb{C}(x)[y]}(f(x,y), g(x,y)) = \frac{a(x)}{b(x)}$.

Using the Euclidean algorithm, it must be possible to write the gcd $\frac{a(x)}{b(x)}$ as follow:

$$\frac{a(x)}{b(x)} = \frac{u(x,y)}{t(x)} f(x,y) + \frac{v(x,y)}{w(x)} g(x,y),$$

clearing denominators,

$$a(x)t(x)w(x) = u(x,y)b(x)w(x)f(x,y) + v(x,y)b(x)t(x)g(x,y).$$

Define

$$h(x) := a(x)t(x)w(x),$$

we have a non-zero polynomial $h(x) \in (f(x,y), g(x,y)) \subseteq mfp$. Since $\mathbb{C}$ is algebraically closed, we may assume $h(x) = \prod(x - a_i)$. Since $\mathfrak{p}$ is prime ideal, there exists $x - a := x - a_i$ for some i such that $x - a \in \mathfrak{p}$. Similarly, there exists $y - b \in \mathfrak{p}$. Hence, $(x-a, y-b) \subseteq \mathfrak{p}$. Note that $\mathbb{C}[x,y]/(x-a, y-b) \cong \mathbb{C}$, $(x-a, y-b)$ is a maximal ideal, and therefore $\mathfrak{p} = (x-a, y-b)$. $\square$

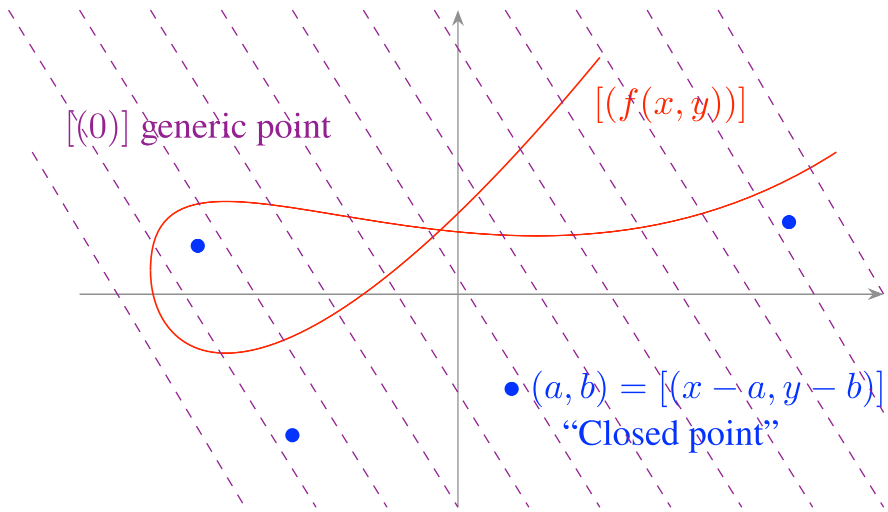


Figure 4.5: Picturing $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$

We now attempt to draw a picture of $\mathbb{A}^2_{\mathbb{C}}$ (see Figure 4.5). The maximal prime ideals of $\mathbb{C}[a,b]$ correspond to the traditional points in $\mathbb{C}^2$: $[(x-a, y-b)]$ corresponds to $(a,b) \in \mathbb{C}^2$. We now have to visualize the "bonus points". $[(0)]$ somehow lives behind all of the traditional points; it is somewhere on the affine plane, but nowhere in particular. So for example, it does not lie on the parabola $y = x^2$. The point $[(y - x^2)]$ lies on the parabola $y = x^2$, but nowhere in particular on it. (Figure 4.5 is a bit misleading. For example, the point $[(0)]$ isn't in the fourth quadrant; it is somehow near every other point, which is why it is depicted as a somewhat diffuse large dot.) You can see from this picture that we already are implicitly thinking about "dimension". The prime ideals $(x-a, y-b)$ are somehow of dimension 0, the prime ideals $(f(x,y))$ are of dimension 1, and (0) is of dimension 2. (All of our dimensions here are complex or algebraic dimensions. The complex plane $\mathbb{C}^2$ has real dimension 4, but complex dimension 2. Complex dimension are in general half of real dimensions.) We won't define dimension precisely until Chapter 13, but you should feel free to keep it in mind before then.

Note too that maximal ideals correspond to the "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal is a little confusing, and will remain so even once we have made the notions precise.

Complex affine n-space

We now come to the obvious generalization of example "The complex affine plane". Let $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$. (More generally, $\mathbb{A}^n_A$ is defined to be $\operatorname{Spec} A[x_1, \ldots, x_n]$, where A is an arbitrary ring. When the base ring is clear from context, the subscript A is often omitted. For pedants: the notation $\mathbb{A}^n_A$ implicitly includes the data of the n coordinate functions $x_1, \ldots, x_n$.) For concreteness, let's consider $n = 3$. We now have an interesting question in what at first appears to be pure algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$?

Analogously to before, $(x - a, y - b, z - c)$ is a prime ideal. This is a maximal ideal, because its residue ring is field $\mathbb{C}$; we think of these as "0-dimensional points". We will often write (a, b, c) for $[(x-a, y-b, z-c)]$ because of our geometric interpretation of these ideals. There are no more maximal ideals, by Hilbert's Weak Nullstellensatz.

Theorem 4.2.1 (Hilbert's Weak Nullstellensatz)

If k is an algebraically closed field, then the maximal ideals of $k[x_1, \ldots, x_n]$ are precisely those ideals of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $a_i \in k$.

We may as well state a slightly stronger version now.

Theorem 4.2.2 (Hilbert's Nullstellensatz)

If k is any field, every maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ has residue field $k[x_1, \ldots, x_n]/\mathfrak{m}$ a finite extension of k.

Translation: any field extension of k that is finitely generated as a k-algebra is necessarily also finitely generated as a k-vector space (i.e., is a finite extension of fields).

Remark This statement is also often called **Zariski's Lemma**.

Exercise 4.6 Show that the Nullstellensatz 4.2.2 implies the Weak Nullstellensatz 4.2.1.

Proof Let $\mathfrak{m}$ be the maximal ideal of $k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field extension of k. Let a_i be the image of x_i in $k[x_1, \ldots, x_n]/\mathfrak{m}$, then $k[x_1, \ldots, x_n]/\mathfrak{m} = k[a_1, \ldots, a_n]$ is finitely generated as a k-algebra. By Hilbert's Nullstellensatz, $k[a_1, \ldots, a_n]$ is also finite field extension of k. Since k is algebraically closed, $k \cong k[a_1, \ldots, a_n]$, and therefore we may assume each $a_i \in k$.

Define the k-algebra homomorphism

$$\begin{aligned} \varphi : k[x_1, \ldots, x_n] &\longrightarrow k[a_1, \ldots, a_n] = k \\ x_i &\longmapsto a_i, \end{aligned} \tag{4.1}$$

Clearly, $\mathfrak{m}$ is the kernel of φ. Note that $\operatorname{Ker} \varphi = (x_1 - a_1, \ldots, x_n - a_n)$, we have

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n).$$

□

The following fact is a useful accompaniment to the Nullstellensatz

Proposition 4.2.1

Any integral domain A which is a finite k-algebra (i.e., a k-algebra that is a finite-dimensional vector space over k) must be a field.

Proof Define k-algebraic homomorphism $\times f : A \to A$ by setting $g \mapsto gf$, where f be any nonzero element in A. It is easy to see that $\times f$ is well-defined and is a k-algebraic homomorphism. Note that A is a field if and only if k-algebra homomorphism $\times f$ is isomorphism. It suffices to show that $\times f$ is isomorphism.

Let $\times f(g) = gf = 0$, since A is integral domain and f is nonzero element, $g = 0$, which implies that $\times f$ is injective.

Since k is a field, A is also a finite k-vector space, then we have $\dim(A) = \dim \mathrm{Im}(\times f) + \mathrm{Ker}(\times f) = \mathrm{Im}(\times f)$, which implies that $\times f$ is surjective.

Hence, A is a field. □

Remark In combination the Nullstellensatz 4.2.2, we see that prime ideals of $k[x_1, \ldots, x_n]$ with finite-dimensional residue ring are the same as maximal ideals of $k[x_1, \ldots, x_n]$. More precisely, let $\mathfrak{p} \in \operatorname{Spec} k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]/\mathfrak{p}$ is an integral domain, if $k[x_1, \ldots, x_n]/\mathfrak{p}$ is also a finite k-algebra, by Proposition 4.2.1, $k[x_1, \ldots, x_n]/\mathfrak{p}$ is a field, and therefore $\mathfrak{p}$ is maximal ideal.

There are other prime ideals of $\mathbb{C}[x, y, z]$ too. We have (0), which corresponds to a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the "hypersurface" $f = 0$, so this is "2-dimensional" in nature. But we have not found them all! One clue: we have prime ideals of "dimension" 0, 2, and 3 — we are missing "dimension 1". Here is one such ideal: (x, y). We picture this as as the locus where $x = y = 0$, which is the z-axis. This is a prime ideal, as the corresponding quotient $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and should be interpreted as the functions on the z-axis). There are lots of "1-dimensional prime ideals", and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as irreducible curves. Thus remarkably the answer to the purely algebraic question ("what are the prime ideals of $\mathbb{C}[x, y, z]$") is fundamentally geometric!

The fact that the points of $\mathbb{A}^1_{\mathbb{Q}}$ corresponding to maximal ideals of the ring $\mathbb{Q}[x]$ (what we will soon call "closed points") can be interpreted as points of $\overline{\mathbb{Q}}$ where Galois-conjugates are glued together extends to $\mathbb{A}^n_{\mathbb{Q}}$. For example, in $\mathbb{A}^2_{\mathbb{Q}}$, $(\sqrt{2}, \sqrt{2})$ is glued to $(-\sqrt{2}, -\sqrt{2})$ but not to $(\sqrt{2}, -\sqrt{2})$. The following exercise will give you some idea of how this works.

✍ **Exercise 4.7**

(a) Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.
(b) Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.
(c) What are the residue fields in each case?

Solution

(a) Let $\mathfrak{m} = (x^2 - 2, x - y)$, note that

$$\mathbb{Q}[x, y]/(x^2 - 2, x - y) \cong \mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2}),$$

$\mathbb{Q}(\sqrt{2})$ is a field, and therefore $\mathfrak{m}$ is a maximal ideal. Note that $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\mathrm{id}, \tau\}$, where $\tau(\sqrt{2}) = -\sqrt{2}$. Hence, $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ are at the same orbit.

(b) Let $\mathfrak{m} = (x^2 - 2, x + y)$, then

$$\mathbb{Q}[x, y]/(x^2 - 2, x + y) \cong \mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2}),$$

hence, $\mathfrak{m}$ is maximal ideal. Note that $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\mathrm{id}, \tau\}$, where $\tau(\sqrt{2}) = -\sqrt{2}$. Hence, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ are at the same orbit.

(c) *They have the same residue field:* $\mathbb{Q}(\sqrt{2})$.

The description of "closed points" of $\mathbb{A}^2_{\mathbb{Q}}$ (those points corresponding to maximal ideals of the ring $\mathbb{Q}[x,y]$) as Galois-orbits of points in $\overline{\mathbb{Q}}^2$ can even be extended to other "nonclosed" points, as follows.

Exercise 4.8 Consider the map of sets $\varphi : \mathbb{C}^2 \to \mathbb{A}^2_{\mathbb{Q}}$ defined as follows. (z_1, z_2) is sent to the prime ideal of $\mathbb{Q}[x,y]$ consisting of polynomials vanishing at (z_1, z_2).

(a) What is the image of (π, π^2)?

(b) Show that φ is surjective.

Proof

(a) We claim that the image of (π, π^2) is $(y-x^2)$. $y-x^2$ is vanishing at (π, π^2). Also, $\mathbb{Q}[x,y]/(y-x^2) \cong \mathbb{Q}[x]$ is an integral domain, hence, $(y-x^2)$ is prime. Thus, the image of (π, π^2) is $(y-x^2)$.

(b) In fact,

$$\begin{aligned}\mathbb{A}^2_{\mathbb{Q}} =& \operatorname{Spec} \mathbb{Q}[x,y] \\ =& \{(0), \operatorname{Max} \mathbb{Q}[x,y], (f(x,y)) \text{ where } f(x,y) \text{ is irreducible polynomial in } \mathbb{Q}[x,y]\}.\end{aligned}$$

For $(0) \in \operatorname{Spec} \mathbb{Q}[x,y]$. Consider $(\pi, e) \in \mathbb{C}^2$. Note that there are no irreducible polynomials vanish at (π, e), also, $\pi, e \notin \mathbb{Q}$, hence, $\varphi(\pi, e) = (0)$.

For $\mathfrak{m} \in \operatorname{Max} \mathbb{Q}[x,y]$. Note that $\mathbb{Q}[x,y]/\mathfrak{m}$ is a field, and is the extension field of k, then there exists $(a,b) \in \mathbb{Q}[x,y]/\mathfrak{m}$ such that each element in $\mathfrak{m}$ vanish at (a,b), i.e., $\varphi(a,b) = \mathfrak{m}$.

For $(f(x,y))$, where $f(x,y)$ is irreducible polynomial in $\mathbb{Q}[x,y]$. Since $\mathbb{C}$ is algebraically closed, $f(x,y)$ has roots in $\mathbb{C}$, they given by

$$\operatorname{Gal}\left(\mathbb{Q}[x,y]/(f(x,y)) : \text{ extension of } \mathbb{Q}\right) \curvearrowright (r_1, r_2) \in \mathbb{C}^2,$$

where $(f(x,y))$ vanish at (r_1, r_2). Hence, $\varphi(r_1, r_2) = (f(x,y))$.

By above discussion, φ is a surjective.

□

4.2.2 Quotients and localization

Two natural ways of getting new rings from old — quotient and localizations — have interpretations in terms of spectra.

Quotients

Lemma 4.2.1

Suppose A is a ring, and I an ideal of A. Let $\varphi : A \to A/I$. Then φ^{-1} gives an inclusion-preserving bijection between prime ideals of A/I and prime ideals of A containing I.

Proof The proof of this lemma can be found in any standard textbook on abstract algebra; since it is straightforward, we omit it here. □

By the Lemma 4.2.1, we get following immediately.

Proposition 4.2.2

$\operatorname{Spec} A/I$ *as a subset of* $\operatorname{Spec} A$.

Thus we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$.

As an important motivational special case, you now have a picture of affine complex varieties. Suppose A is a finitely generated $\mathbb{C}$-algebra, generated by $x_1, \dots, x_n$, with relations $f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$. Then this description in terms of generators and relations naturally gives us an interpretation of $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think of as "traditional points" (n-tuples of complex numbers) along with some "bonus" points we haven't yet fully described. To see which of the traditional points are in $\operatorname{Spec} A$, we simply solve the equations $f_1 = \cdots = f_r = 0$. For example, $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$ may be pictured as shown in Figure 4.6. (Admittedly this is just a "sketch of the $\mathbb{R}$-points", but we will still find it helpful later.) This entire picture carries over (along with the Nullstellensatz) with $\mathbb{C}$ replaced by any algebraically closed field. Indeed, the picture of Figure 4.6 can be said to depict $\operatorname{Spec} k[x, y, z]/(x^2+y^2-z^2)$ for most algebraically closed fields k (although it is misleading in characteristic 2, because of the coincidence $x^2 + y^2 - z^2 = (x + y + z)^2$).

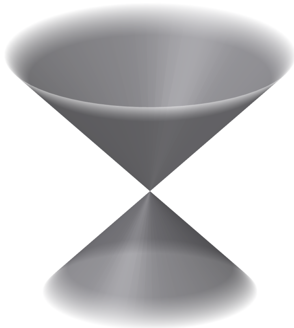


Figure 4.6: A "picture" of $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

Localizations

The following Proposition shows how prime ideals behave under localization.

Proposition 4.2.3

Suppose S is a multiplicative subset of A. The prime ideal of $S^{-1}A$ are one-to-one correspondence with the prime ideals of A which don't meet S, i.e.,

$$\mathfrak{p} \longleftrightarrow S^{-1}\mathfrak{p},$$

where $\mathfrak{p} \cap S = \varnothing$.

Proof Let $f : A \to S^{-1}A$ be a ring homomorphism which is defined by $f(x) = x/1$.

If $\mathfrak{q} \in \operatorname{Spec} S^{-1}A$, then $f^{-1}(\mathfrak{q}) := \mathfrak{q} \cap A \in \operatorname{Spec} A$, by Lemma 4.2.1.

Conversely, if $\mathfrak{p} \in \operatorname{Spec} A$, then $A/\mathfrak{p}$ is an integral domain. On the other hand, let $\overline{S}$ be the image of S in $A/\mathfrak{p}$. Note that localization commutes with quotient, we have $S^{-1}A/S^{-1}\mathfrak{p} \cong \overline{S}^{-1}(A/\mathfrak{p})$, which $\overline{S}^{-1}(A/\mathfrak{p})$ is either 0 or else is contained in the field of fraction of $A/\mathfrak{p}$, and therefore $S^{-1}\mathfrak{p}$ is either $S^{-1}A$ or prime ideal. Since every ideal in $S^{-1}A$ is an extended ideal, $S^{-1}\mathfrak{p} = (1)$ if and only if $\mathfrak{p} \cap S \neq \varnothing$. Then we done. $\square$

Hence, by Proposition 4.2.3, we get follow:

Proposition 4.2.4

$\operatorname{Spec} S^{-1}A$ *as a subset of* $\operatorname{Spec} A$.

Recall from §2.2.2 that there are two important flavors of localization. The first is $A_f = \{1, f, f^2, \dots\}^{-1}A$

where $f \in A$. A motivating example is $A = \mathbb{C}[x, y]$, $f = y - x^2$. The second is $A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1}A$, where $\mathfrak{p}$ is a prime ideal. A motivating example is $A = \mathbb{C}[x, y]$, $S = A \setminus (x, y)$.

If $S = \{1, f, f^2, \dots\}$, the prime ideals of $S^{-1}A$ are just those prime ideals not containing f — the points where "f doesn't vanish". (In §4.5, we will call this a distinguished open set, once we know what open sets are.) So to picture $\operatorname{Spec} \mathbb{C}[x, y]_{y-x^2}$, we picture the affine plane, and throw out those points on the parabola $y - x^2$ — the points (a, a^2) for $a \in \mathbb{C}$ (by which we mean $[(x - a, y - a^2)]$), as well as the "new kind of point" $[(y - x^2)]$.

It can be initially confusing to think about localization in the case where zero divisors are inverted, because localization $A \to S^{-1}A$ is not injective (Exercise 2.6). Geometric intuition can help. Consider the case $A = \mathbb{C}[x, y]/(xy)$ and $f = x$. What is the localization A_f? The space $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$ "is" the union of the two axes in the affine plane. Localizing means throwing out the locus where x vanishes. So we are left with the x-axis, minus the origin, so we expect $\operatorname{Spec} \mathbb{C}[x]_x$. So there should be some natural isomorphism

$$(\mathbb{C}[x, y]/(xy))_x \xrightarrow{\sim} \mathbb{C}[x]_x.$$

Exercise 4.9 Show that these two rings are isomorphic.

Proof In fact,

$$(\mathbb{C}[x, y]/(xy))_x = \left\{ \frac{f(x, y)}{x^k} : f(x, y) \in \mathbb{C}[x, y], k \geq 0, xy = 0 \right\}$$

and

$$\mathbb{C}[x]_x = \left\{ \frac{f(x)}{x^k} : f(x) \in \mathbb{C}[x], k \geq 0 \right\}.$$

Define the ring homomorphism $\varphi : (\mathbb{C}[x, y]/(xy))_x \to \mathbb{C}[x]_x$ by setting $\frac{f(x,y)}{x^k} \mapsto \frac{f(x,0)}{x^k}$. Clearly, φ is well-defined ring homomorphism, and is a surjective.

Let $\varphi(\frac{f(x,y)}{x^k}) = \frac{f(x,0)}{x^k} = 0$. Let $f(x, y) = \sum_i a_i x^i + \sum_j b_j y^j$, then $x^n f(x, 0) = x^n \sum_i a_i x_i^i = 0$ for some n in $\mathbb{C}[x]$, hence, $f(x, 0) = \sum_i a_i x_i^i = 0$, which implies that $f(x, y) = \sum_j b_j y^j$. Note that $\frac{f(x,y)}{x^k} = \frac{\sum_j b_j y^j}{x^k} = \frac{\sum_j b_j x^d y^j}{x^{k+d}}$, where d is the maximal degree of y, since $xy = 0$, we have $\frac{f(x,y)}{x^k} = 0$, which implies that φ is injective.

Hence,

$$(\mathbb{C}[x, y]/(xy))_x \xrightarrow{\sim} \mathbb{C}[x]_x.$$

□

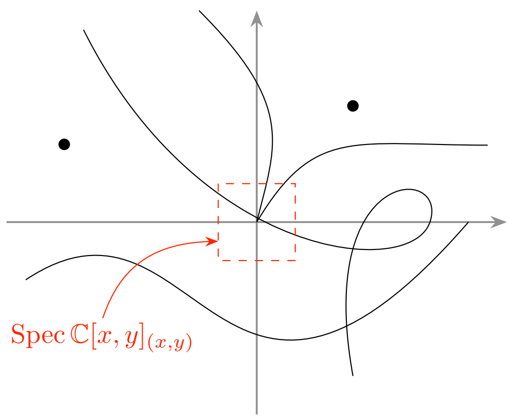


Figure 4.7: Picturing $\operatorname{Spec} \mathbb{C}[x, y]_{(x,y)}$ as a "shred of $\mathbb{A}^2_{\mathbb{C}}$". Only those points near the origin remain.

If $S = A \setminus \mathfrak{p}$, the prime ideals of $S^{-1}A$ are just the prime ideals of A contained in $\mathfrak{p}$. In our example $A = \mathbb{C}[x, y]$, $\mathfrak{p} = (x, y)$, we keep all those points corresponding to "things through the origin", i.e., the 0-dimension point (x, y), the 2-dimensional point (0), and those 1-dimensional points $(f(x, y))$ where $f(0, 0) = 0$, i.e., those "irreducible curves through the origin". You can think of this being a shred of the plane near the origin; anything not actually "visible" at the origin is discarded (see Figure 4.7).

Another example is when $A = k[x]$, and $\mathfrak{p} = (x)$ (or more generally when $\mathfrak{p}$ is any maximal ideal). Then $A_{\mathfrak{p}}$ has only two prime ideals (Exercise 4.1 (b), $\operatorname{Spec} A_{\mathfrak{p}} = \{(0), (x)k[x]_{(x)}\}$). You should see this as the germ of a "smooth curve", where one point is the "classical point", and the other is the "generic point of the curve". This is an example of a DVR, and indeed all DVRs should be visualized in such a way. We will discuss DVRs in Chapter 13. By then we will have justified the use of the words "smooth" and "curve".

4.2.3 Maps of rings induce maps of spectra (as sets)

We now make an observation that will later grow up to be the notion of morphisms of schemes.

Proposition 4.2.5

If $\varphi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of B.

Proof Let $ab \in f^{-1}(\mathfrak{p})$, then $f(ab) = f(a)f(b) \in \mathfrak{p}$, since $\mathfrak{p}$ is prime, $f(a) \in \mathfrak{p}$ or $f(b) \in \mathfrak{p}$, i.e., $a \in f^{-1}(\mathfrak{p})$ or $b \in f^{-1}(\mathfrak{p})$, which implies that $f^{-1}(\mathfrak{p})$ is prime. □

Hence a map of rings $\varphi : B \to A$ induces a map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ "in the opposite direction". This gives a contravariant functor from the category of rings to the category of sets: the composition of two maps of rings induces the composition of the corresponding maps of spectra.

Proposition 4.2.6

Let B be a ring.

(a) Suppose $I \subseteq B$ is an ideal. Then the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of Proposition 4.2.2.

(b) Suppose $S \subseteq B$ is a multiplicative set. Then the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion of Proposition 4.2.4.

Proof

(a) The elements in $\operatorname{Spec} B/I$ is one-to-one correspondence with the elements in $\operatorname{Spec} B$ which containing I, hence, the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion.

(b) The elements in $\operatorname{Spec} S^{-1}B$ is one-to-one correspondence with the elements in $\operatorname{Spec} B$ which do not meet S, and therefore, the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion.

□

Example 4.1 In the case of "affine complex varieties" (or indeed affine varieties over any algebraically closed field), the translation between maps given by explicit formulas and maps of rings is quite direct. For example, consider a map from the parabola in $\mathbb{C}^2$ (with coordinates a and b) given by $b = a^2$, to the "curve" in $\mathbb{C}^3$ (with coordinates x, y, and z) cut out by the equations $y = x^2$ and $z = y^2$. Suppose the map sends the point $(a, b) \in \mathbb{C}^2$ to the point $(a, b, b^2) \in \mathbb{C}^3$.

In our new language, we have a map

$$\operatorname{Spec} \mathbb{C}[a, b]/(b - a^2) \longrightarrow \operatorname{Spec} \mathbb{C}[x, y, z]/(y - x^2, z - y^2)$$

given by

$$\begin{aligned} \mathbb{C}[x,y,z]/(y-x^2, z-y^2) &\longrightarrow \mathbb{C}[a,b]/(b-a^2) \\ (x,y,z) &\longmapsto (a,b,b^2), \end{aligned}$$

i.e., $x \mapsto a$, $y \mapsto b$, and $z \mapsto b^2$.

✍ **Exercise 4.10** Consider the map of complex manifolds sending $\mathbb{C} \to \mathbb{C}$ via $x \mapsto y = x^2$. We interpret the "source" $\mathbb{C}$ as the "x-line", and the "target" $\mathbb{C}$ the "y-line". You can picture it as the projection of the parabola $y = x^2$ in the xy-plane to the y-axis (see Figure 4.8). Interpret the corresponding map of rings as given by $\mathbb{C}[y] \to \mathbb{C}[x]$ by $y \mapsto x^2$. Verify that the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$, using definition given above (identifying a with $[(y-a)]$, and $\sqrt{a}$ with $[(x-\sqrt{a})]$).

Proof Define $\varphi : \mathbb{C}[y] \to \mathbb{C}[x]$ by setting $\varphi(y) = x^2$. Point $a \in \mathbb{C}$ is corresponding to the maximal ideal $(y-a) \in \operatorname{Spec} \mathbb{C}[y]$, then $\varphi((y-a)) = (x^2-a) = (x-\sqrt{a})(x+\sqrt{a})$, which is corresponding to two points $\pm\sqrt{a} \in \mathbb{C}$. Hence, the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$. □

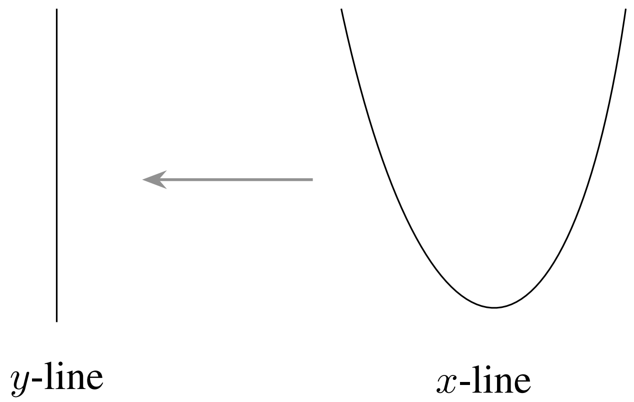


Figure 4.8: The map $\mathbb{C} \to \mathbb{C}$ given by $x \mapsto y = x^2$

✍ **Exercise 4.11** Suppose k is a field, and $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ are given. Let $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_m]$ be the morphism of k-algebras defined by $y_i \mapsto f_i$.

(a) Show that φ induces a map of sets $\operatorname{Spec} k[x_1, \ldots, x_m]/I \to \operatorname{Spec} k[y_1, \ldots, y_n]/J$ for any ideals $I \subseteq k[x_1, \ldots, x_m]$ and $J \subseteq k[y_1, \ldots, y_n]$ such that $\varphi(J) \subseteq I$.

(b) Show that the map of part (a) sends the point $(a_1, \ldots, a_m) \in k^m$ (or more precisely, $[(x_1 - a_1, \ldots, x_m - a_m)] \in \operatorname{Spec} k[x_1, \ldots, x_m]$) to

$$(f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)) \in k^n.$$

Proof

(a) Let $\mathfrak{p} \in \operatorname{Spec} k[x_1, \ldots, x_m]/I$, then $\mathfrak{p}$ is one-to-one correspondence with $\widetilde{\mathfrak{p}} \in \operatorname{Spec} k[x_1, \ldots, x_m]$ which is containing I, hence, $\widetilde{\mathfrak{p}} \supseteq I \supseteq \varphi(J)$, and therefore $\varphi^{-1}(\widetilde{\mathfrak{p}}) \in \operatorname{Spec} k[y_1, \ldots, y_n]$ with $\varphi^{-1}(\widetilde{\mathfrak{p}}) \supseteq J$. Denote $\mathfrak{q}$ be the image of $\varphi^{-1}(\widetilde{\mathfrak{p}})$ in $k[y_1, \ldots, y_n]/J$. Hence, φ induces a map of sets $\operatorname{Spec} k[x_1, \ldots, x_m]/I \to \operatorname{Spec} k[y_1, \ldots, y_n]/J$ which defined by $\mathfrak{p} \mapsto \mathfrak{q}$.

(b) Let $[(x_1 - a_1, \ldots, x_m - a_m)] \in \operatorname{Spec} k[x_1, \ldots, x_m]$. Denote $\overline{\varphi}$ be the induced map from φ. Then

$$\begin{aligned} \overline{\varphi}((x_1 - a_1, \ldots, x_m - a_m)) =& \varphi^{-1}((x_1 - a_1, \ldots, x_m - a_m)) \\ =& \{h \in k[y_1, \ldots, y_n] : \varphi(h) \in (x_1 - a_1, \ldots, x_m - a_m)\} \\ \in& \operatorname{Spec} k[y_1, \ldots, y_n]. \end{aligned}$$

We may assume $\varphi(h) = h(f_1, \ldots, f_n) = \sum_i h_i(x_i - a_i)$, where $h_i \in k[x_1, \ldots, x_m]$, then

$$h(f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)) = 0,$$

which implies that $h \in (y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m))$. Hence,

$$\overline{\varphi}((x_1 - a_1, \ldots, x_m - a_m)) \subseteq (y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m)).$$

Conversely, it is easy to see that

$$\varphi((y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m)) \subseteq (x_1 - a_1, \ldots, x_m - a_m),$$

and therefore

$$\overline{\varphi}((x_1 - a_1, \ldots, x_m - a_m)) \supseteq (y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m)).$$

Hence,

$$\overline{\varphi}((x_1 - a_1, \ldots, x_m - a_m)) = (y_1 - f_1(a_1, \ldots, a_m), \ldots, y_n - f_n(a_1, \ldots, a_m)),$$

i.e., the map of part (a) sends the point $(a_1, \ldots, a_m) \in k^m$ to

$$(f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)) \in k^n.$$

□

Exercise 4.12 Picturing $\mathbb{A}^n_{\mathbb{Z}}$. Consider the map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \ldots, x_n]$. If p is prime, describe a bijection between the fiber $\pi^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$. This exercise may give you a sense of how to picture maps (see 4.9), and in particular why you can think of $\mathbb{A}^n_{\mathbb{Z}}$ as an "$\mathbb{A}^n$-bundle" over $\operatorname{Spec} \mathbb{Z}$.

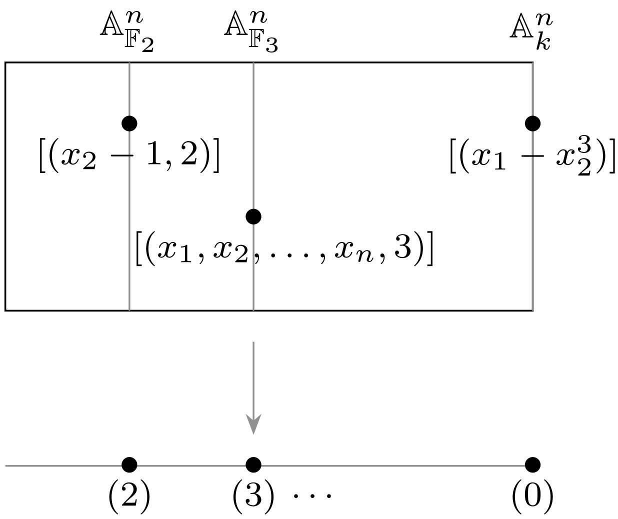


Figure 4.9: A picture of $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ as a "family of $\mathbb{A}^n$'s", or an "$\mathbb{A}^n$-bundle over $\operatorname{Spec} \mathbb{Z}$".

Solution $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is given by* $\mathfrak{p} \mapsto \mathfrak{p} \cap \mathbb{Z}$.

Let $(p) \in \operatorname{Spec} \mathbb{Z}$, *we need to describe a bijection between the fiber* $\pi^{-1}([(p)])$ *and* $\mathbb{A}^n_{\mathbb{F}_p}$. *In fact,*

$$\pi^{-1}([(p)]) = \{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n] : \mathfrak{p} \cap \mathbb{Z} = (p)\},$$

this implies that $\mathfrak{p}$ *is containing* (p). *Let* $\overline{\mathfrak{p}}$ *be the image of* $\mathfrak{p}$ *in* $\mathbb{Z}[x_1, \ldots, x_n]/(p) = \mathbb{F}_p[x_1, \ldots, x_n]$. *Define* $\varphi : \pi^{-1}([(p)]) \to \mathbb{A}^n_{\mathbb{F}_p}$ *by setting* $\varphi(\mathfrak{p}) = \overline{\mathfrak{p}}$, *where* $\overline{\mathfrak{p}} = \mathfrak{p} \pmod{(p)}$. *By above discussion* φ *is well-defined.*

Let $\overline{\mathfrak{p}} = \overline{\mathfrak{q}}$. *Suppose* $\mathfrak{p} \neq \mathfrak{q}$, *then there exists* $f \in \mathfrak{p}$ *but* $f \notin \mathfrak{q}$. *Note that* $f \pmod{(p)} \in \overline{\mathfrak{p}} = \overline{\mathfrak{q}}$, *we have* $f \in \mathfrak{q}$, *a contradiction. Hence,* $\mathfrak{p} = \mathfrak{q}$, *and therefore* φ *is injective.*

Let $\mathfrak{p} \in \mathbb{A}^n_{\mathbb{F}_p}$. Since p is prime, $\mathbb{F}_p$ is a field, hence, $\mathfrak{p} \cap \mathbb{F}_p = (0)$. Let

$$\mathfrak{q} = \left\{ \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Z}[x_1, \ldots, x_n] : \sum \overline{a_{i_1,\ldots,i_n}} x_1^{i_1} \cdots x_n^{i_n} \in \mathfrak{p} \right\},$$

then $\overline{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q} \cap \mathbb{Z} = (p)$, which implies that $\varphi(\mathfrak{q}) = \mathfrak{p}$. Hence, φ is surjective.

4.2.4 Functions are not determined by their values at points: the fault of nilpotents

We conclude this section by describing some strange behavior. We are developing machinery that will let us bring our geometric intuition to algebra. There is one serious point where your intuition will be false, so you should know now, and adjust your intuition appropriately. As noted by Mumford, "it is this aspect of schemes which was most scandalous when Grothendieck defined them."

Suppose we have a function (ring element) vanishing at all points. Then it is not necessarily the zero function! The translation of this question is: is the intersection of all prime ideals necessarily just 0? The answer is no, as is shown by the example of the ring of dual numbers $k[\varepsilon]/(\varepsilon^2) : \varepsilon \neq 0$, but $\varepsilon^2 = 0$. (We saw this ring in Exercise 4.1 (a).) Any function whose power is zero certainly lies in the intersection of all prime ideals.

Definition 4.2.2 (Nilpotent)

*Ring elements that have a power that is 0 are called **nilpotents**.*

Proposition 4.2.7

Let B be a ring, then:

*(a) the nilpotents of a ring B form an ideal. This ideal is called the **nilradical**, and is denoted $\mathfrak{N} = \mathfrak{N}(B)$;*

(b) if $I \subseteq \mathfrak{N}$ is an ideal of nilpotents, the inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection, thus nilpotents don't affect the underlying set.

Proof

(a) If $f \in \mathfrak{N}$, then $f^m = 0$ for some m, for all $b \in B$, we have $(bf)^m = b^m f^m = 0$, and therefore $B\mathfrak{N} \subseteq \mathfrak{N}$. We next show that $\mathfrak{N}$ is a subgroup of B. For all $f \in \mathfrak{N}$, we have $f^m = 0$ for some m, hence, $(-f)^m = (-1)^m f^m = 0$. Let $g \in \mathfrak{N}$, then $g^n = 0$ for some n. By the binomial theorem, $(x+y)^{m+n-1}$ is a sum of integer multiples of products $x^r y^s$, where $r + s = m + n - 1$; we cannot have both $r < m$ and $s < n$, hence each of these products vanishes and therefore $(x+y)^{m+n-1} = 0$. Hence, $x + y \in \mathfrak{N}$, and therefore $\mathfrak{N}$ is a subgroup of B.
By above discussion, $\mathfrak{N} \in \operatorname{Spec} B$.

(b) It suffices to show that I is contained in any prime ideal of B. Let $f \in I$, then $f^m = 0$ for some m. Let $\mathfrak{p} \in \operatorname{Spec} B$, then $0 = f^m \in \mathfrak{p}$, hence, $f \in \mathfrak{p}$ or $f^{m-1} \in \mathfrak{p}$. If $f \notin \mathfrak{p}$, we have $f^{m-2} \in \mathfrak{p}$, repeat this process, we have $f^2 \in \mathfrak{p}$, then $f \in \mathfrak{p}$, a contradiction. Hence, $f \in \mathfrak{p}$, and therefore $I \subseteq \mathfrak{p}$. Since the element in $\operatorname{Spec} B/I$ is one-to-one correspondence to the element in $\operatorname{Spec} B$ which is containing I, we have bijection

$$\operatorname{Spec} B/I \longleftrightarrow \operatorname{Spec} B.$$

□

Thus the nilradical is contained in the intersection of all the prime ideals. The converse is also true:

Theorem 4.2.3

The nilradical $\mathfrak{N}(A)$ is the intersection of all the prime ideals of A. Geometrically: a function on Spec A *vanishes at every point if and only if it is nilpotent.*

Proof Denote $\mathfrak{N}' = \bigcap_{\mathfrak{p}\in \operatorname{Spec} A} \mathfrak{p}$. By the proof in Proposition 4.2.7, the nilradical $\mathfrak{N}(A)$ is contained in all prime ideal of A, i.e., $\mathfrak{N} \subseteq \mathfrak{N}'$

Conversely, suppose that f is not nilpotent. Let

$$\Sigma = \{\mathfrak{a} \subseteq A : \mathfrak{a} \text{ is an ideal of } A \text{ and } f^n \notin \mathfrak{a} \text{ for all } n > 0\},$$

then Σ is not empty because $(0) \in \Sigma$. Apply Zorn's Lemma to the set Σ, ordered by inclusion, and therefore Σ has a maximal element. Let $\mathfrak{m}$ be the maximal element of Σ. Now, we shall to show that $\mathfrak{m}$ is a prime ideal.

Let $x, y \notin \mathfrak{m}$, then the ideals $(x) + \mathfrak{m}$, $(y) + \mathfrak{m}$ are containing $\mathfrak{m}$, and therefore $(x) + \mathfrak{m}, (y) + \mathfrak{m} \notin \Sigma$. Hence, exists n, m such that

$$f^m \in (x) + \mathfrak{m}, \quad f^n \in (y) + \mathfrak{m}.$$

It follows that $f^{m+n} = f^m f^n \in (xy) + \mathfrak{m}$, hence, $(xy) + \mathfrak{m} \notin \Sigma$, and therefore $xy \notin \mathfrak{m}$. Hence, $\mathfrak{m}$ is prime ideal. Note that $f \notin \mathfrak{m}$, $f \notin \mathfrak{N}'$, i.e., $\mathfrak{N}' \subseteq \mathfrak{N}$. Thus $\mathfrak{N} = \mathfrak{N}'$. □

In particular, although it is upsetting that functions are not determined by their values at points, we have precisely specified what the failure of this intuition is: two functions have the same values at points if and only if they differ by a nilpotent. You should think of this geometrically: a function vanishes at every point of the spectrum of a ring if and only if it has a power that is zero. And if there are no nonzero nilpotents — if $\mathfrak{N} = (0)$ — then functions are determined by their values at points.

Definition 4.2.3 (Reduced)

If a ring has no nonzero nilpotents, we say that it is ***reduced****.*

✍ **Exercise 4.13** **Derivatives without deltas and epsilons (or at least without deltas).** Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \varepsilon]/(\varepsilon^2)$. What then is $f(x + \varepsilon)$?

Solution *In fact,* $k[x, \varepsilon]/(\varepsilon^2) = \{a_0(x) + a_1(x)\varepsilon : a_i(x) \in k[x], i = 0, 1\}$. *Suppose* $f(x) = \sum_{i=0}^{n} a_i x^i$, *then in* $k[x, \varepsilon]/(\varepsilon^2)$ *we have*

$$\begin{aligned} f(x+\varepsilon) &= \sum_i a_i (x+\varepsilon)^i = \sum_{i=0}^{n} a_i (x^i + i x^{i-1}\varepsilon) \\ &= \sum_{i=0}^{n} a_i x^i + \varepsilon \sum_{i=0}^{n} i a_i x^{i-1} \\ &:= f(x) + \varepsilon f'(x). \end{aligned}$$

Remark This is a hint that nilpotents will be important in defining differential information (Chapter 22).

4.3 Visualizing schemes: Generic points

A heavy warning used to be given that pictures are note rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of cause, are not rigorous, but I should say most are (and I use them whenever possible myself).

— J.E.Littlewood

We all know that Art is not truth. Art is a lie that makes us realize truth, at least the truth that is given us to understand. The artist must know the manner whereby to convince others of the truthfulness of his lies.

— P.Picasso

For years, you have been able to picture $x^2 + y^2 = 1$ in the plane, and you now have an idea of how to picture $\operatorname{Spec} \mathbb{Z}$. If we are claiming to understand rings as geometric objects (through the Spec functor), then we should wish to develop geometric insight into them. To develop geometric intuition about schemes, it is helpful to have pictures in your mind, extending your intuition about geometric spaces you are already familiar with. As we go along, we will empirically develop some idea of what schemes should look like. This summarizes what we have gleaned so far.

Some mathematicians prefer to think completely algebraically, and never think in terms of pictures. Others will be disturbed by the fact that this is an art, not a science. And finally, this hand-waving will necessarily never be used in the rigorous development of the theory. For these reasons, you may wish to skip these sections. However, having the right picture in your mind can greatly help understanding what facts should be true, and how to prove them. Fitgerald's exhortation on the importance of stretching one's mind is essential advice to a mathematician.

The test of a first-rate intelligence is the ability to hold two opposed ideas in the mind at the same time, and still retain the ability to hold two opposed ideas in the mind at the same time, and still retain the ability to function.

— F.Scott Fitzgerald

Our starting point is the example of "affine complex varieties" (things cut out by equations involving a finite number of variables over $\mathbb{C}$), and more generally similar examples over arbitrary algebraically closed fields. We begin with notions that are intuitive ("traditional" points behaving the way you expect them to), and then add in the two features which are new and disturbing, generic points and nonreduced behavior. You can then extend this notion to seemingly different spaces, such as $\operatorname{Spec} \mathbb{Z}$.

Hilbert's Weak Nullstellensatz 4.2.1 shows that the "traditional points" are present as points of the scheme, and this carries over to any algebraically closed field. If the field is not algebraically closed, the traditional points are glued together into clumps by Galois conjugation, as in Example "the real affine line" and "the affine line over $\mathbb{F}_p$" in §4.2.1. This is a geometric interpretation of Hilbert's Nullstellensatz 4.2.2.

But we have some additional points to add to the picture. You should remember that they "correspond" to "irreducible" "closed" (algebraic) subsets. As motivation, consider the case of the complex affine plane §4.2.1: we had one for each irreducible polynomial, plus one corresponding to the entire plane. We will make "closed" precise when we define the Zariski topology (in the next section). You may already have an ideal of what "irreducible" should mean; we make that precise at the start of §4.6. By "correspond" we mean that each closed irreducible subset has a corresponding point sitting on it, called its **generic point** (defined in §4.6). It is a new point, distinct from all the other points in the subset. We don't know precisely where to draw the generic point, so we may stick it arbitrarily anywhere, but you should think of it as being "almost everywhere", and in particular, near every other point in the subset.

In §4.2.2, we saw how the points of $\operatorname{Spec} A/I$ should be interpreted as subsets of $\operatorname{Spec} A$. So for example, when you see $\operatorname{Spec} \mathbb{C}[x, y]/(x + y)$, you should picture this not just as a line, but as a line in the xy-plane; the choice of generators x and y of the algebra $\mathbb{C}[x, y]$ implies an inclusion into affine space.

In §4.2.2, we saw how the points of $\operatorname{Spec} S^{-1}A$ should be interpreted as subsets of $\operatorname{Spec} A$. The two points of $\operatorname{Spec} A$ where f doesn't vanish; we will later (§4.5) interpret this as a distinguished open set.

If $\mathfrak{p}$ is a prime ideal, then $\operatorname{Spec} A_{\mathfrak{p}}$ should be seen as a "shred of the space $\operatorname{Spec} A$ near the subset corresponding to $\mathfrak{p}$". The simplest nontrivial case of this is $A = k[x]$ and $\mathfrak{p} = (x) \subseteq A$ (see Exercise 4.1).

"If any of them can explain it", said Alice, (she had grown so large in the last few minutes that she wasn't a bit afraid of interrupting him), "I'll give him sixpence. I don't believe there's an atom of meaning in it."...

"If there's no meaning in it," said the King, "that saves a world of trouble, you know, as we needn't try to find any."

— Lewis Carroll

4.4 The underlying topological space of an affine scheme

We next introduce the **Zariski topology** on the spectrum of a ring. When you first hear the definition, it seems odd, but with a little experience it becomes reasonable. As motivation, consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, the complex (affine) plane (with a few extra points). In algebraic geometry, we will only be allowed to consider algebraic functions, i.e., polynomials in x and y. The locus where a polynomial vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these. In other words, it is the coarsest topology where these sets are closed.

4.4.1 Definition of Zariski topology

In particular although topologies are often described using open subsets, it will be more convenient for us to define this topology in terms of closed subsets.

Definition 4.4.1 (Vanishing set)

*If S is a subset of a ring A, define the **vanishing set** of S by*

$$V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\}.$$

It is the set of points on which all elements of S are zero. (It should now be second nature to equate "vanishing at a point" with "contained in a prime".) We declare that these — and no others — are the closed subsets.

For example, consider $V(xy, yz) \subseteq \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$. Which points are contained in this locus? We think of this as solving $xy = yz = 0$. Of the "traditional" points (interpreted as ordered triples of complex numbers, thanks to the Hilbert's Nullstellensatz), we have the points where $y = 0$ or $x = z = 0$: the xz-plane and the y-axis respectively. Of the "new" points, we have the generic point of the xz-plane (also known as the point $[(y)]$), and the generic point of the y-axis (also known as the point $[(x, z)]$). You might imagine that we also have a number of "one-dimensional" points contained in the xz-plane.

Exercise 4.14 Check that the x-axis is contained in $V(xy, yz)$.

Proof The x-axis is defined by $y = z = 0$. It is corresponding to point $[(y, z)]$. Since $\mathbb{C}[x, y, z]/(y, z) \cong \mathbb{C}[x]$ is an integral domain, (y, z) is a prime ideal. Note that $xy \in (y, z)$ and $yz \in (y, z)$, we have $\{xy, yz\} \subseteq (y, z)$, and therefore $[(y, z)] \in V(xy, yz)$. □

Let's return to the general situation. The following proposition lets us restrict attention to vanishing sst of ideals.

Proposition 4.4.1

If (S) is the ideal generated by S, then $V(S) = V((S))$.

Proof Let $[\mathfrak{p}] \in V(S)$, then we have $S \subseteq \mathfrak{p}$, and therefore $(S) \subseteq \mathfrak{p}$, hence, $\mathfrak{p} \in V((S))$, i.e., $V(S) \subseteq V((S))$. Conversely, if $\mathfrak{p} \in V((S))$, then $(S) \subseteq \mathfrak{p}$. Note that $S \subseteq (S)$, $S \subseteq \mathfrak{p}$, and therefore $\mathfrak{p} \in V(S)$. $\square$

Definition 4.4.2 (Zariski topology)

We define the ***Zariski topology*** *by declaring that $V(S)$ is closed for all S. (We may as well state here that* ***Zariski closure*** *means closure in the Zariski topology.)*

Let's check that the Zariski topology is indeed a topology.

Proposition 4.4.2 (Zariski topology is indeed a topology)

(a) $\varnothing$ and $\operatorname{Spec} A$ are both open subsets of $\operatorname{Spec} A$.

(b) If $\{I_i\}_i$ is a collection of ideals (as i runs over some index set), then $\bigcap_i V(I_i) = V(\sum_i I_i)$. Hence, the union of any collection of open sets is open.

(c) $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence, the intersection of any finite number of open sets is open.

By (a), (b), and (c), Zariski topology is indeed a topology

Remark The **product of two ideals** I_1 and I_2 of A are finite A-linear combinations of products of elements of I_1 and I_2, i.e., elements of the form $\sum_{j=1}^n i_{i,j} i_{2,j}$, where $i_{k,j} \in I_k$. Equivalently, it is the ideal generated by products of elements of I_1 and I_2. Also, products are associative, i.e., $(I_1 I_2) I_3 = I_1(I_2 I_3)$.

Proof

(a) Note that $V(\varnothing) = \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supseteq \varnothing\} = \operatorname{Spec} A$ and $V(\operatorname{Spec} A) = \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supseteq \operatorname{Spec} A\} = \varnothing$, φ and $\operatorname{Spec} A$ are both closed set, and therefore φ and $\operatorname{Spec} A$ are both open.

(b) Since $\sum_i I_i$ is the smallest ideal which contains each I_i, $\mathfrak{p}$ contains $\sum_i I_i$ if and only if $\mathfrak{p}$ contains I_i for all i. It follows that $\bigcap_i V(I_i) = V(\sum_i I_i)$.

(c) Certainly, if $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$, then $\mathfrak{p} \supseteq I_1 I_2$, which implies that $V(I_1) \cup V(I_2) \subseteq V(I_1 I_2)$. Conversely, if $\mathfrak{p} \supseteq I_1 I_2$ with $\mathfrak{p} \not\supseteq I_2$, then there exists $b \in I_2$ with $b \notin \mathfrak{p}$. Note that for any $a \in I_1$, we have $ab \in I_1 I_2 \subseteq \mathfrak{p}$, since $\mathfrak{p}$ is prime, $a \in \mathfrak{p}$, that is, $\mathfrak{p} \supseteq I_1$. Hence, $V(I_1) \cup V(I_2) \supseteq V(I_1 I_2)$. Consequently, $V(I_1) \cup V(I_2) = V(I_1 I_2)$.

$\square$

4.4.2 Properties of the "vanishing set" function $V(\cdot)$

The function $V(\cdot)$ is obviously inclusion-reversing: if $S_1 \subseteq S_2$, then $V(S_2) \subseteq V(S_1)$. Warning: We could have equality in the second inclusion without equality in the first.

Definition 4.4.3 (Radical)

If $I \subseteq A$ is an ideal, then define its ***radical*** *by*

$$\sqrt{I} := \{r \in A : r^n \in I \text{ for some } n \in \mathbb{Z}^{>0}\}.$$

We say an ideal is ***radical*** *if it equals its own radical.*

Example 4.2 The nilradical $\mathfrak{N}$ is $\sqrt{(0)}$.

Proposition 4.4.3

(a) $\sqrt{I}$ is an ideal.

(b) Prime ideals are radical.

(c) $\sqrt{\sqrt{I}} = \sqrt{I}$.
(d) $V(\sqrt{I}) = V(I)$.

Proof

(a) Let $r \in \sqrt{I}$, then $r^n \in I$ for some n. For any $a \in A$, we have $a^n r^n = (ar)^n \in I$, since I is an ideal. Hence, $ar \in \sqrt{I}$. Note that $(-1)^n r^n = (-r)^n \in I$, $-r \in \sqrt{I}$. Let $r' \in \sqrt{I}$, then there exists m such that $r'^m \in I$, hence, $(r + r')^{n+m-1} \in I$, and therefore $r + r' \in \sqrt{I}$. Hence $\sqrt{I}$ is an ideal.

(b) It suffices to show that $\mathfrak{p} = \sqrt{\mathfrak{p}}$, where $\mathfrak{p} \in \operatorname{Spec} A$. Clearly, $\mathfrak{p} \subseteq \sqrt{\mathfrak{p}}$. Let $r \in \sqrt{\mathfrak{p}}$, then $r^n \in \mathfrak{p}$ for some n. Hence, $r \in \mathfrak{p}$, it follows that $\mathfrak{p} = \sqrt{\mathfrak{p}}$.

(c) Clearly, $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Let $r \in \sqrt{\sqrt{I}}$, then $r^n \in \sqrt{I}$ for some n, by the definition of $\sqrt{I}$, $r^{nm} \in I$ for some m. Hence, $r \in \sqrt{I}$, it follows that $\sqrt{I} \supseteq \sqrt{\sqrt{I}}$.

(d) $V(\sqrt{I}) \subseteq V(I)$, since $I \subseteq \sqrt{I}$. Let $[\mathfrak{p}] \in V(I)$, then $\mathfrak{p} \supseteq I$, taking radical (use part (b)), we have $\mathfrak{p} \supseteq \sqrt{I}$. Hence, $[\mathfrak{p}] \in V(\sqrt{I})$, i.e., $V(\sqrt{I}) \supseteq V(I)$.

$\square$

Remark Here are two useful consequences.

(i) $(I \cap J)^2 \subseteq IJ \subseteq I \cap J$.

Proof Simple observation. $\square$

(ii) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.

Proof By (i), we have $(I \cap J)^2 \subseteq IJ \subseteq I \cap J$, taking radical, we have

$$\sqrt{I \cap J} = \sqrt{(I \cap J)^2} \subseteq \sqrt{IJ} \subseteq \sqrt{I \cap J},$$

hence, $\sqrt{IJ} = \sqrt{I \cap J}$. By Proposition 4.4.3 (d),

$$V(\sqrt{I \cap J}) = V(I \cap J) = V(IJ) = V(\sqrt{IJ}).$$

$\square$

(iii) $V(S) = V((S)) = V(\sqrt{(S)})$.

Proof By Proposition 4.4.1 and Proposition 4.4.3 (d). $\square$

Proposition 4.4.4 (Radicals commute with finite intersections)

If $I_1, \ldots, I_n$ are ideals of a ring A, then $\sqrt{\bigcap_{i=1}^n I_i} = \bigcap_{i=1}^n \sqrt{I_i}$.

Proof If we prove that $\sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$, then by induction $\sqrt{\bigcap_{i=1}^n I_i} = \bigcap_{i=1}^n \sqrt{I_i}$. Hence, it suffices to show that $\sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$

We already prove that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$. Let $f \in \sqrt{I_1 I_2}$, then $f^n \in I_1 \cap I_2$. Hence, $f^n \in I_1 \cap I_2 \subseteq I_1$ and $f^n \in I_1 \cap I_2 \subseteq I_2$, and therefore $f \in \sqrt{I_1}$ and $f \in \sqrt{I_2}$, i.e., $f \in \sqrt{I_1} \cap \sqrt{I_2}$. Thus, $\sqrt{I_1 I_2} \subseteq \sqrt{I_1} \cap \sqrt{I_2}$.

Conversely, let $f \in \sqrt{I_1} \cap \sqrt{I_2}$, then there exists n and m such that $f^n \in I_1$ and $f^m \in I_2$, and therefore $f^{nm} \in I_1 \cap I_2$. It follows that $f \in \sqrt{I_1 \cap I_2}$.

By above discussion, we have $\sqrt{I_1 I_2} = \sqrt{I_1} \cap \sqrt{I_2}$, hence

$$\sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}.$$

$\square$

Proposition 4.4.5

$\sqrt{I}$ is the intersection of all the prime ideals containing I.

Proof Let $\pi : A \to A/I$ be a natural homomorphism. We claim that $\pi(\sqrt{I}) = \mathfrak{N}(A/I)$. Let $f \in \sqrt{I}$, then $f^n \in I$ for some n, hence $\pi(f^n) = \pi(f)^n = 0$, it follows that $\pi(f) \in \mathfrak{N}(A/I)$, i.e., $\pi(\sqrt{I}) \subseteq \mathfrak{N}(A/I)$. Conversely, let $f \in \mathfrak{N}(A/I)$, then $f^n = 0$ for some n. Since π is surjective, $f = \pi(g)$ for some $g \in A$. Hence, $\pi(g)^n = \pi(g^n) = 0$, which implies that $g^n \in I$, and therefore $g \in \sqrt{I}$. Thus, $\pi(\sqrt{I}) = \mathfrak{N}(A/I)$.

Note that $\mathfrak{N}(A/I) = \bigcap_{\mathfrak{q}\in\operatorname{Spec} A/I} \mathfrak{q}$, say $\mathfrak{p} = \pi^{-1}(\mathfrak{q}) \in \operatorname{Spec} A$, we have

$$\sqrt{I} = \pi^{-1}\left(\bigcap_{\mathfrak{q}\in\operatorname{Spec} A/I} \mathfrak{q}\right) = \bigcap_{\mathfrak{q}\in\operatorname{Spec} A/I} \pi^{-1}(\mathfrak{q}) = \bigcap_{\mathfrak{p}\in\operatorname{Spec} A,\ \mathfrak{p}\supseteq I} \mathfrak{p},$$

as we desired. □

Example 4.3 Let's see how this meshes with our examples from the previous section.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "traditional" points (corresponding to maximal ideas, in bijection with $a \in \mathbb{C}$), and one "new" point $[(0)]$. The Zarisiki topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting: the open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "traditional points". The "new" point $[(0)]$ comes along for the ride, which is a good sign that it is harmless. Ignoring the "new" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology then the classical topology on $\mathbb{C}$.)

✍ **Exercise 4.15** Describe the topological space $\mathbb{A}^1_k$.

Proof In fact, $\mathbb{A}^1_k = \operatorname{Spec} k[x] = \{(0), \mathfrak{m}_a\ a \in k\}$, where $\mathfrak{m}_a$ is maximal ideal. Hence, the closed subsets of $\mathbb{A}^1_k$ are:

(i) the set of maximal ideals;

(ii) $\operatorname{Spec} k[x]$ and $\varnothing$.

The open subsets of $\mathbb{A}^1_k$ are:

(i) $\operatorname{Spec} k[x]$ minus a finite number of maximal ideals;

(ii) $\operatorname{Spec} k[x]$ and $\varnothing$.

□

Remark Notice that the strange new point $[(0)]$ is "near every other point" — every neighborhood of every point contains $[(0)]$.

The case of $\operatorname{Spec}\mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, $\operatorname{Spec}\mathbb{Z}$, and $\operatorname{Spec}\mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

Example 4.4 **Closed subsets of $\mathbb{A}^2_{\mathbb{C}}$.** The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. You should think through where the "one-dimensional prime ideals" fit into the picture. In Exercise 4.5, we identified all the prime ideals of $\mathbb{C}[x,y]$ (i.e., the points of $\mathbb{A}^2_{\mathbb{C}}$) as the maximal ideals $[(x-a, y-b)]$ (where $a, b \in \mathbb{C}$ — "zeor-dimensional points"), the "one-dimensional points" $[(f(x,y))]$ (where $f(x,y)$ is irreducible), and the "two-dimensional point" $[(0)]$.

Then the closed subsets are of the following form:

(a) the entire space (the closure of the "two-dimensional" $[(0)]$),

(b) a finite number (possibly none) of "curves" (each the closure of a "one-dimensional point" — the "one-dimensional point" along with the "zero-dimensional points" "lying on it") and a finite number (possibly none) of "zero-dimensional" points (what we will soon call "closed points".)

We will soon know enough to verify this using general theory.

4.4.3 Maps of rings induce continuous maps of topological spaces

We saw in §4.2.3 that a homomorphism of rings $\varphi : B \to A$ induces a map of sets $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$.

Proposition 4.4.6

Let $\varphi : B \to A$ be a homomorphism of rings, $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is induced by φ, then

(a) π is a continuous map;

(b) $\operatorname{Spec} \square$ can be interpreted as a contravariant functor **Rings** $\to$ **Top**.

Proof

(a) Let $V(I) \subseteq \operatorname{Spec} B$ be a closed subset, where I is an ideal of B. In fact, $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is given by $\pi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Consider $\pi^{-1}(V(I))$, let $\mathfrak{q} \in \pi^{-1}(V(I))$, then $\pi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) \in V(I)$. Since $V(I) = \{\mathfrak{p} \in \operatorname{Spec} B : \mathfrak{p} \supseteq I\}$, we have $\varphi^{-1}(\mathfrak{q}) \supseteq I$, and therefore $\mathfrak{q} \supseteq \varphi(I)$, it follows that $\pi^{-1}(V(I)) \subseteq V(\varphi(I)) \subseteq \operatorname{Spec} A$. We claim that $V(\varphi(I)) = \pi^{-1}(V(I))$. Let $\mathfrak{q} \in V(\varphi(I))$, then $\mathfrak{q} \supseteq \varphi(I)$, hence, $\varphi^{-1}(\mathfrak{q}) \supseteq I$. It follows that $\varphi^{-1}(\mathfrak{q}) \in V(I) \subseteq \operatorname{Spec} B$, i.e., $\pi(\mathfrak{q}) \in V(I)$, and therefore $\mathfrak{q} \in \pi^{-1}(V(I))$. Thus, $V(\varphi(I)) = \pi^{-1}(V(I)) \subseteq \operatorname{Spec} A$, and therefore $\pi^{-1}(V(I))$ is closed in $\operatorname{Spec} A$. Hence, π is a continuous map.

(b) Let $\varphi : A \to B \in$ **Rings**, then $\operatorname{Spec}(\varphi) = \varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$. Let $A \in$ **Rings**, then $\operatorname{Spec}(\mathrm{id}_A) = \mathrm{id}_A^{-1} = \mathrm{id}_{\operatorname{Spec} A}$, which implies that $\operatorname{Spec} \square$ preserves identity morphisms. Let $\varphi : A \to B$ and $\psi : B \to C$, then $\psi \circ \varphi : A \to C$. Consider $\operatorname{Spec}(\psi \circ \varphi)$, we have

$$\operatorname{Spec}(\psi \circ \varphi) = (\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1} = \operatorname{Spec} \varphi \circ \operatorname{Spec} \psi.$$

Hence, $\operatorname{Spec} \square$ is a contravariant functor.

$\square$

Remark Not all continuous maps arise in this way. Consider for example the continuous map on $\mathbb{A}^1_{\mathbb{C}}$ that is the identity except 0 and 1 (i.e., $[(x)]$ and $[(x-1)]$) are swapped; no polynomial can manage this marvelous feat.

Proof Let $\varphi : \mathbb{C}[x] \to \mathbb{C}[x]$ be a ring homomorphism, given by $x \mapsto p(x)$, where $p(x) \in \mathbb{C}[x]$. Then the induced map $\pi : \operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[x]$ is given by $\pi((x-a)) = \varphi^{-1}((x-a)) = (x - p(a))$.

If continuous map $\pi : \operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[x]$ is given by

$$\pi((x-a)) = \begin{cases} [(x-1)] & \text{if } a = 0 \\ [(x)] & \text{if } a = 1 \\ [(x-a)] & \text{others}, \end{cases}$$

then $p(0) = 1$, $p(1) = 0$, and $p(a) = a$ for $a \neq 0, 1$. Consider $q(x) = p(x) - x$, then $q(a) = 0$ for all $a \in \mathbb{C} \setminus \{0, 1\}$, i.e., polynomial $q(x)$ has infinite zeros, this contradicts the Fundamental Theorem of Algebra. Hence, not all continuous maps induced by a ring homomorphism. $\square$

In §4.2.2, we saw that $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are naturally subsets of $\operatorname{Spec} B$. It is natural to ask if the Zariski topology behaves well with respect to these inclusions, and indeed it does.

Proposition 4.4.7

Suppose that $I, S \subseteq B$ are an ideal and multiplicative subset respectively.

(a) $\operatorname{Spec} B/I$ is naturally a closed subset of $\operatorname{Spec} B$. If $S = \{1, f, f^2, \ldots\}$ ($f \in B$), then $\operatorname{Spec} S^{-1}B$ is naturally an open subset of $\operatorname{Spec} B$.

(b) The Zariski topology on $\operatorname{Spec} B/I$ (resp., $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$.

Proof

(a) Note that $\operatorname{Spec} B/I = \{\mathfrak{p} \in \operatorname{Spec} B : \mathfrak{p} \supseteq I\} = V(I) \subseteq \operatorname{Spec} B$, $\operatorname{Spec} B/I$ is a closed subset of

Spec B.

In fact,

$$\begin{aligned}\operatorname{Spec} B_f =&\{\mathfrak{p} \in \operatorname{Spec} B : \mathfrak{p} \cap \{1, f, f^2, \ldots\} = \varnothing\}\\ =&\{\mathfrak{p} \in \operatorname{Spec} B : f \notin \mathfrak{p}\}\\ =&\operatorname{Spec} B - \{\mathfrak{p} \in \operatorname{Spec} B : (f) \subseteq \mathfrak{p}\}\\ =&\operatorname{Spec} B - V(f).\end{aligned}$$

Hence, $\operatorname{Spec} S^{-1}B$ is an open subset of $\operatorname{Spec} B$.

(b) Let $S \subseteq B/I$ be an ideal, then exists $\widetilde{S} \supseteq I$ in $\operatorname{Spec} B$ such that the image of $\widetilde{S}$ in B/I is S. Then

$$\begin{aligned}V(S) =&\{\mathfrak{p} \in \operatorname{Spec} B/I : \mathfrak{p} \supseteq S\}\\ =&\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \supseteq \widetilde{S}\} \cap \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \supseteq I\}\\ =&V(\widetilde{S}) \cap \operatorname{Spec} B/I.\end{aligned}$$

It follows that the Zariski topology on $\operatorname{Spec} B/I$ is the subspace topology induced by inclusion in $\operatorname{Spec} B$. Let $T \subseteq B_f$ be an ideal, then exists ideal $\widetilde{T} \subseteq B$ with $\widetilde{T} \cap S = \varnothing$ such that $T = S^{-1}\widetilde{T} := \widetilde{T}_f$. Then

$$\begin{aligned}V(T) =&\{\mathfrak{p} \in \operatorname{Spec} B_f : \mathfrak{p} \supseteq T\}\\ =&\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \cap S = \varnothing, \mathfrak{q}_f \supseteq T = \widetilde{T}_f\}\\ =&\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \cap S = \varnothing\} \cap \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q}_f \supseteq \widetilde{T}_f\}\\ =&\operatorname{Spec} B_f \cap \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \supseteq \widetilde{T}\}\\ =&\operatorname{Spec} B_f \cap V(\widetilde{T}).\end{aligned}$$

It follows that the Zariski topology on $\operatorname{Spec} S^{-1}B$ is the subspace topology induced by inclusion in $\operatorname{Spec} B$.

$\square$

Remark For arbitrary S, $\operatorname{Spec} S^{-1}B$ need not be open or closed.

Proof Consider $\operatorname{Spec} \mathbb{Q} = \{(0)\} \subseteq \operatorname{Spec} \mathbb{Z}$. In fact, $\operatorname{Spec} \mathbb{Z} = \{(0), (p)$ where p is prime$\}$. Let $I = (n)$ be any ideal of $\mathbb{Z}$, then

$$\begin{aligned}V(I) =&\begin{cases}\{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} : \mathfrak{p} \supseteq (n)\} & \text{if } n \neq 0\\ \{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} : \mathfrak{p} \supseteq (0)\} & \text{if } n = 0\end{cases}\\ =&\begin{cases}\{(p) : p \mid n\} & \text{if } n \neq 0\\ \operatorname{Spec} \mathbb{Z} & \text{if } n = 0.\end{cases}\end{aligned}$$

It follows that (0) is neither open nor closed. $\square$

In particular, if $I \subseteq \mathfrak{N}$ is an ideal of nilpotents, the bijection $\operatorname{Spec} B/I \to \operatorname{Spec} B$ (Proposition 4.2.7 gives bijection, Proposition 4.4.6 gives continuous) is a homeomorphism. Thus nilpotents don't affect the topological space.

Proposition 4.4.8

Suppose $I \subseteq B$ is an ideal, f vanishes on $V(I)$ if and only if $f \in \sqrt{I}$.

Proof If f vanishes on $V(I)$, then for all $\mathfrak{p} \in V(I)$, we have $f \in \mathfrak{p}$. Note that $\sqrt{I}$ is the intersection of all prime ideals containing I, hence, $f \in \sqrt{I}$.

Conversely, if $f \in \sqrt{I}$, since $\sqrt{I}$ is the intersection of all prime ideals containing I, hence $f \in \mathfrak{p}$ where $\mathfrak{p}$

contains I, and therefore f vanishes on $V(I)$. $\square$

Exercise 4.16 Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

Proof In fact,

$$\begin{aligned}\operatorname{Spec} k[x]_{(x)} =&\{\mathfrak{p} \in \operatorname{Spec} k[x] : \mathfrak{p} \cap (\operatorname{Spec} k[x] - (x)) = \varnothing\}\\ =&\{\mathfrak{p} \in \operatorname{Spec} k[x] : \mathfrak{p} \subseteq (x)\}\\ =&\{(0), (x)\}.\end{aligned}$$

Let I be any ideal of $k[x]_{(x)}$, then

$$\begin{aligned}V(I) =&\{\mathfrak{p} \in \operatorname{Spec} k[x]_{(x)} : \mathfrak{p} \supseteq I\}\\ =&\begin{cases}\{(x)\} & I = (x)\\ \operatorname{Spec} k[x]_{(x)} & I = (0)\end{cases}\end{aligned}$$

Hence, the closed set in $\operatorname{Spec} k[x]_{(x)}$ are $\varnothing$, (x), and $\operatorname{Spec} k[x]_{(x)}$. $\square$

4.5 A base of the Zariski topological on Spec A: Distinguished open sets

Definition 4.5.1 (Distinguished open set)

If $f \in A$, define the distinguished open set

$$\begin{aligned}D(f) :=& \{[\mathfrak{p}] \in \operatorname{Spec} A : f \notin \mathfrak{p}\}\\ =& \{[\mathfrak{p}] : \operatorname{Spec} A : f([\mathfrak{p}]) \neq 0\}.\end{aligned}$$

It is the locus where f doesn't vanish.

We have already seen this set when discussing $\operatorname{Spec} A_f$ as a subset of $\operatorname{Spec} A$. For example, we have observed that the Zariski-topology on the distinguished open set $D(f) \subseteq \operatorname{Spec} A$ coincides with the Zariski topology on $\operatorname{Spec} A_f$ (Proposition 4.4.7).

The reason these sets are important is that they form a particularly nice base for the (Zariski) topology:

Proposition 4.5.1

The distinguished open sets form a base for the Zariski topology.

Proof Let $S \subseteq A$, then $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supseteq S\}$. Note that open subset $\operatorname{Spec} A - V(S) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\}$, we claim that $\{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\} = \bigcup_{f \in S} D(f)$.

Let $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\}$, then $\mathfrak{p} \not\supseteq S$, hence, there exists $f \in S$ such that $f \notin \mathfrak{p}$. It follows that $[\mathfrak{p}] \in D(f)$, and therefore $\{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\} \subseteq \bigcup_{f \in S} D(f)$.

Let $\mathfrak{p} \in \bigcup_{f \in S} D(f)$, then exists $f \in S$ such that $\mathfrak{p} \in D(f)$, hence, $f \notin \mathfrak{p}$, and therefore $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\}$. Hence, $\bigcup_{f \in S} D(f) \subseteq \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\}$.

Consequently, we have $\{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \not\supseteq S\} = \bigcup_{f \in S} D(f)$. Hence, $D(f)$ form a base for the Zariski topology on $\operatorname{Spec} A$. $\square$

Proposition 4.5.2

Suppose $f_i \in A$ as i runs over some index set J, then $\bigcup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(\{f_i\}_{i \in J}) = A$, or equivalently and very usefully, if there are a_i ($i \in J$), all but finitely many 0, such that $\sum_{i \in J} a_i f_i = 1$.

Proof If $(\{f_i\}_{i \in J}) = A$, then exists $a_i \in A$ such that $\sum_i a_i f_i = 1$ with all but finitely many a_i being zero.

Let $\mathfrak{p} \in \operatorname{Spec} A$, if $\mathfrak{p}$ contains each f_i, then $\mathfrak{p} = (1) = A$, this contradicts to $\mathfrak{p}$ is a prime ideal. Hence, there exists f_i such that $f_i \notin \mathfrak{p}$, and therefore $[\mathfrak{p}] \in D(f_i)$. Hence, $\operatorname{Spec} A \subseteq \bigcup_{i\in J} D(f_i)$. Clearly, we have $\bigcup_{i\in J} D(f_i) \subseteq \operatorname{Spec} A$. Hence, $\bigcup_{i\in J} D(f_i) = \operatorname{Spec} A$.

Conversely, if $\bigcup_{i\in J} D(f_i) = \operatorname{Spec} A$, then for all $\mathfrak{p} \in \operatorname{Spec} A$, there exists f_i such that $\mathfrak{p} \in D(f_i)$, i.e., $f_i \notin \mathfrak{p}$. Hence, there are no prime ideal contains all f_i. Note that maximal ideal is prime ideal, hence, $(\{f_i\}_{i\in J}) = A$. □

Proposition 4.5.3

If $\operatorname{Spec} A$ *is an infinite union of distinguished open sets* $\bigcup_{j\in J} D(f_j)$*, then in fact it is a union of a finite number of these, i.e., there is a finite subset* J' *so that* $\operatorname{Spec} A = \bigcup_{j\in J'} D(f_j)$.

Proof Since $1 = \sum_{i\in J} a_i f_i$ where J is finite set, we have $A = (f_i)_{i\in J}$, by Proposition 4.5.2, we done. □

Proposition 4.5.4

$D(f) \cap D(g) = D(fg)$*. In particular,* $D(f^n) = D(f)$ *for any* $n \in \mathbb{Z}$.

Proof

$$\begin{aligned} D(f) \cap D(g) =& \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}, g \notin \mathfrak{p}\} \\ \subseteq& \{\mathfrak{p} \in \operatorname{Spec} A : fg \notin \mathfrak{p}\} = D(fg). \end{aligned}$$

Conversely, let $[\mathfrak{p}] \in D(fg)$, then $fg \notin \mathfrak{p}$. If $f \in \mathfrak{p}$, since $\mathfrak{p}$ is an ideal, $fg \in \mathfrak{p}$, a contradiction. Hence, $f \notin \mathfrak{p}$, similarly, $g \notin \mathfrak{p}$. It follows that

$$D(fg) \subseteq D(f) \cap D(g).$$

□

Proposition 4.5.5

The following are equivalent:

(i) $D(f) \subseteq D(g)$.

(ii) $f^n \in (g)$ *for some* $n \geq 1$.

(iii) g *is an invertible element of* A_f.

Proof (i) $\Rightarrow$ (ii): Since $D(f) \subseteq D(g)$, then $V(f) \supseteq V(g)$. For any $\mathfrak{p} \in V(g)$, we have $\mathfrak{p} \ni f$, hence, $f \in \bigcap_{\mathfrak{p}\in V(g)} \mathfrak{p} = \sqrt{(g)}$. It follows that $f^n \in (g)$ for some $n \geq 1$.

(ii) $\Rightarrow$ (iii): Since $f^n \in (g)$ for some $n \geq 1$, exists $h \in A$ such that $f^n = hg$. Since f^n is an invertible element of A_f, we have $1 = (\frac{h}{f^n})g$. Hence, g is an invertible element of A_f.

(iii) $\Rightarrow$ (i): Since g is an invertible element of A_f, $g \in \{1, f, f^2, \ldots\}$. We may assume that $g = f^n$ for some $n \geq 1$. Let $\mathfrak{p} \in V(g)$, then $g = f^n \in \mathfrak{p}$, and therefore $f \in \mathfrak{p}$, which implies that $\mathfrak{p} \in V(f)$. Hence, $V(g) \subseteq V(f)$, i.e., $D(f) \subseteq D(g)$. □

Remark We will use Proposition 4.5.5 often. You can prove it thinking purely algebraically, but the following geometric interpretation may be helpful. Inside $\operatorname{Spec} A$, we have the closed subset $V(g) = \operatorname{Spec} A/(g)$, where g vanishes, and its complement $D(g)$, where g doesn't vanish. Then f is a function on this closed subset $V(g)$ (or more precisely, on $\operatorname{Spec} A/(g)$), and by assumption it vanishes at all points of the closed subset. Now any function vanishing at every point of the spectrum of ring must be nilpotent (Theorem 4.2.3). In other words, there is some n such that $f^n = 0$ in $A/(g)$, i.e., $f^n \equiv 0 \pmod g$ in A, i.e., $f^n \in (g)$.

Proposition 4.5.6

$D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

Proof Since $D(f) = \varnothing$, we have $V(f) = \operatorname{Spec} A$, it follows that f vanishing at every point of Spec, by Theorem 4.2.3, we done. □

Proposition 4.5.7 (Injective ring homomorphisms induce maps of spectra with dense image)

If $B \hookrightarrow A$, then the induced map of topological spaces $\operatorname{Spec} A \to \operatorname{Spec} B$ has dense image.

Proof Say $\varphi : B \hookrightarrow A$, then $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is given by $\pi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Suppose the image of $\operatorname{Spec} A$ not dense, then there exists a non-empty open subset of $\operatorname{Spec} B$, say U, such that $\pi(\operatorname{Spec} A) \cap U = \varnothing$. Since, each open subset of spectrum is the union of distinguished open set, we may assume $U = D(f)$, where $f \in B$.

Consider $\pi(\operatorname{Spec} A)$, we have $\pi(\operatorname{Spec} A) = \{\pi(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\} = \{\varphi^{-1}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}$. Since $D(f) \cap \{\varphi^{-1}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\} = \varnothing$, for all $\mathfrak{p} \in \operatorname{Spec} A$, we have $\varphi^{-1}(\mathfrak{p}) \notin D(f)$, i.e., $\varphi^{-1}(\mathfrak{p}) \in V(f)$. Hence, $f \in \varphi^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$, i.e., $\varphi(f) \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. By Theorem 4.2.3, $\varphi(f)$ is nilpotent, i.e., $\varphi(f)^n = \varphi(f^n) = 0$ for some n. Since φ is injective, we have $f^n = 0$. By Proposition 4.5.6, $D(f) = \varnothing$, a contradiction. □

Remark **Caution about notation.** There will be two variations on the notation $D(f)$: the projective distinguished open set $D_+(f)$ (the locus on a projective scheme where a "form f of positive degree" doesn't vansh), and X_f (the locus on a scheme X where a function f doesn't vanish). These are almost the same thing as $D(f)$, but not quite, so we keep the notation separate for reasons of hygiene.

4.6 Topological (and Noetherian) properties

Many topological notions are useful when applied to the topological space $\operatorname{Spec} A$, and later, to schemes.

4.6.1 Possible topological attributes of $\operatorname{Spec} A$: Connectedness, Irreducibility, Quasicompactness

Connectedness

Definition 4.6.1 (Connected)

A topological space X is ***connected*** *if it cannot be written as the disjoint union of two nonempty open sets.*

Example 4.5 below gives an example of a non-connected $\operatorname{Spec} A$, and the subsequent remark explains that all examples are of this form.

Example 4.5 If $A = A_1 \times A_2 \times \cdots \times A_n$, then there is a homeomorphism $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \coprod \cdots \coprod \operatorname{Spec} A_n \to \operatorname{Spec} A$ for which each $\operatorname{Spec} A_i$ is mapped onto a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. Thus $\operatorname{Spec} \prod_{i=1}^n A_i = \coprod_{i=1}^n \operatorname{Spec} A_i$ as topological spaces.

Remark $\operatorname{Spec} \prod_{i=1}^n A_i = \{A_1 \times A_2 \times \cdots \times \mathfrak{p}_i \times \cdots \times A_n : \mathfrak{p}_i \in \operatorname{Spec} A_i\}$.

The ideal of $\prod_{i=1}^n A_i$ is of the form $I_1 \times \cdots \times I_n$, where I_k is the ideal of A_k.

Proof Prove by induction. It suffices to show the case of $n = 2$, i.e., $\operatorname{Spec}(A_1 \times A_2) = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$. Define $\varphi_1 : \operatorname{Spec} A_1 \to \operatorname{Spec}(A_1 \times A_2)$ by setting $\mathfrak{p} \mapsto \mathfrak{p} \times A_2$, and define $\varphi_2 : \operatorname{Spec} A_2 \to \operatorname{Spec}(A_1 \times A_2)$

by setting $\mathfrak{p} \mapsto A_1 \times \mathfrak{p}$. Since φ_1 and φ_2 induced by projection $A \to A_i$, φ_i are both well-defined. Then we get a map:

$$\begin{aligned} \varphi : \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 &\longrightarrow \operatorname{Spec}(A_1 \times A_2) \\ (\mathfrak{p}, i) &\longmapsto \varphi_i(\mathfrak{p}). \end{aligned}$$

Let $f_1 = (1, 0)$ and $f_2 = (0, 1)$ in $A_1 \times A_2$, then we have

$$D(f_1) = \{\mathfrak{p} \in \operatorname{Spec} A : f_1 \notin \mathfrak{p}\}, \quad D(f_2) = \{\mathfrak{p} \in \operatorname{Spec} A : f_2 \notin \mathfrak{p}\}.$$

Note that φ_i is a bijection between $\operatorname{Spec} A_i$ and $D(f_i)$. We claim that φ_i is also a homeomorphism.

Let I be any ideal of $A_1 \times A_2$ with $f_i \notin I$. Then the closed subset of $D(f_i)$ has the form $V(I) = \{\mathfrak{p} \in D(f_i) : \mathfrak{p} \supseteq I\}$. Consider $\varphi_i^{-1}(V(I))$,

$$\begin{aligned} \varphi_i^{-1}(V(I)) =& \varphi_i^{-1}(\{\mathfrak{p} \in D(f_i) : \mathfrak{p} \supseteq I\}) \\ =& \{\mathfrak{p}_i \in \operatorname{Spec} A_i : \mathfrak{p}_i \supseteq I_i, 1 \notin I_i\} \\ =& V(I_i). \end{aligned}$$

Hence, $\varphi_i^{-1}(V(I))$ closed in $\operatorname{Spec} A_i$. It follows that φ_i is continuous for all $i = 1, 2$.

Let I_i be an ideal of A_i, consider $\varphi_i(V(I_i))$,

$$\begin{aligned} \varphi_i(V(I_i)) =& \varphi_i(\{\mathfrak{p} \in \operatorname{Spec} A_i : \mathfrak{p} \supseteq I_i\}) \\ =& \{\varphi_i(\mathfrak{p}) \in \operatorname{Spec}(A_1 \times A_2) : \varphi_i(\mathfrak{p}) \supseteq \varphi_i(I_i)\} \\ =& V(\varphi_i(I_i)), \end{aligned}$$

which is closed in $\operatorname{Spec}(A_1 \times A_2)$. Hence, φ_i^{-1} is continuous for all $i = 1, 2$. It follows that each φ_i is homeomorphism.

Note that $\operatorname{Spec}(A_1 \times A_2) = D(f_1) \coprod D(f_2)$ and each φ_i is homeomorphism, φ is a homeomorphism between $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$ and $\operatorname{Spec}(A_1 \times A_2)$, as we desired. □

Remark **The idempotent-connectedness package.** An extesion of Example 4.5 is that $\operatorname{Spec} A$ is not connected if and only if A is isomorphic to the product of non-zero rings A_1 and A_2. The key idea is to show that both conditions are equivalent to there existing non-zero $a_1, a_2 \in A$ for which $a_1^2 = a_1$, $a_2^2 = a_2$, $a_1 + a_2 = 1$, and hence $a_1 a_2 = 0$. An element $a \in A$ satisfying $a^2 = a$ is called an **idempotent**.

Irreducibility

Definition 4.6.2 (Irreducible)

*A topological space is said to be **irreducible** if it is nonempty, and it is not the union of two proper closed subsets. In other words, a nonempty topological space X is irreducible if whenever $X = Y \cup Z$ with Y and Z closed in X, we have $Y = X$ or $Z = X$.*

Equivalently (and helpfully): any two nonempty open subsets of X intersect.

This is a less useful notion in classical geometry — $\mathbb{C}^2$ is reducible (endow classical topology, $\mathbb{C}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_1) \geq 0\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_1) \leq 0\} := A \cup B$, A, B are closed), but we will see that $\mathbb{A}^2_{\mathbb{C}}$ is irreducible.

Proposition 4.6.1

(a) In an irreducible topological space, any non-empty open set is dense.

(b) If X is a topological space, and Z is a subset of X (with the subspace topology), then Z is irreducible if and only if the closure $\overline{Z}$ in X is irreducible.

Proof

(a) Let X be an irreducible topological space. If there exists an non-empty open set is not dense, say U, then $X \setminus U$ is a non-empty proper closed subset of X. Note that $X = \overline{U} \cup X \setminus U$ where $\overline{U}$ and $X \setminus U$ are proper closed, which contradicts the fact that X is irreducible.

(b) If Z is irreducible. Suppose that $\overline{Z}$ is not irreducible, then there exists two proper closed subsets of $\overline{Z}$, say Z_1, Z_2, such that $\overline{Z} = Z_1 \cup Z_2$. Hence, $Z = Z \cap (Z_1 \cup Z_2) = (Z \cap Z_1) \cup (Z \cap Z_2)$. Note that $Z \cap Z_i$ is closed in Z and Z is irreducible, $Z = Z \cap Z_1$ or $Z = Z \cap Z_2$. We may assume that $Z = Z \cap Z_1$, then $Z \subseteq Z_1$, and therefore $\overline{Z} \subseteq Z_1$, which contradicts the fact that Z_1 is a proper closed subset of $\overline{Z}$.

Conversely, if the closure $\overline{Z}$ in X is irreducible. Suppose that Z is reducible, i.e., $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are proper closed in Z. Since Z endowed the subspace topology of X, there exists $A, B \subseteq X$ closed sets such that $Z_1 = Z \cap A$ and $Z_2 = Z \cap B$. Hence, $Z \subseteq A \cup B$, taking closure, we have $\overline{Z} \subseteq \overline{A \cup B} = A \cup B$. Thus, $\overline{Z} = (A \cap \overline{Z}) \cup (B \cap \overline{Z})$, since $\overline{Z}$ is irreducible, we may assume that $\overline{Z} = \overline{Z} \cap A$, i.e., $\overline{Z} \subseteq A$. Hence, $Z \subseteq A \cap Z = Z_1$, which contradicts the fact that Z_1 is proper closed in Z.

$\square$

Remark For (a), we see that unlike in the classical topology, in the Zariski topology, non-empty open sets are all "huge".

Proposition 4.6.2

(i) If A is an integral domain, then $\operatorname{Spec} A$ is irreducible.

(ii) $\operatorname{Spec} A$ is irreducible if and only if $\mathfrak{N}(A)$ is prime.

Proof

(i) Suppose that $\operatorname{Spec} A$ is reducible, then there exists non-empty proper closed subsets, say $V(I_1), V(I_2)$ where I_i is ideal of A, such that $\operatorname{Spec} A = V(I_1) \cup V(I_2)$. Note that

$$\begin{aligned} V(I_1) \cup V(I_2) =& V(I_1 I_2) \\ =& \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supseteq I_1 I_2\} = \operatorname{Spec} A, \end{aligned}$$

i.e., any prime ideal of A must contain $I_1 I_2$. Hence, $I_1 I_2 \subseteq \mathfrak{N}(A)$. Since A is an integral domain, $\mathfrak{N}(A) = (0)$, hence, $I_1 I_2 = (0)$, and therefore $I_1 = (0)$ or $I_2 = (0)$. We may assume that $I_1 = (0)$, then $V(I_1) = \operatorname{Spec} A$, a contradiction.

(ii) If $\mathfrak{N}(A)$ is prime. Suppose $\operatorname{Spec} A = V(I_1) \cup V(I_2)$, then $\operatorname{Spec} A = V(I_1 I_2)$. Since $\mathfrak{N}(A)$ is prime, we have $I_1 I_2 \subseteq \mathfrak{N}(A)$, hence $I_1 \subseteq \mathfrak{N}(A)$ or $I_2 \subseteq \mathfrak{N}(A)$. By Theorem 4.2.3, we know that $V(\mathfrak{N}(A)) = \operatorname{Spec} A$. Hence $V(I_1) = \operatorname{Spec} A$ or $V(I_2) = \operatorname{Spec} A$, which implies that $\operatorname{Spec} A$ is irreducible.

Conversely, if $\operatorname{Spec} A$ is irreducible. Suppose $ab \in \mathfrak{N}(A)$, by Theorem 4.2.3, ab contained in any prime ideal of A. Consider $V(a) \cup V(b)$, since $V(a) \cup V(b) = V(ab) = \operatorname{Spec} A = V(\mathfrak{N}(A))$, by the irreducibility of $\operatorname{Spec} A$, we know that $V(a) = V(\mathfrak{N}(A))$ or $V(b) = V(\mathfrak{N}(A))$. It follows that $a \in \mathfrak{N}(A)$ or $b \in \mathfrak{N}(A)$, i.e., $\mathfrak{N}(A)$ is a prime ideal.

□

Proposition 4.6.3

An irreducible topological space is connected.

Proof If X is an irreducible topological space, then any two non-empty open subsets of X intersect, by the definition of Connected (Definition 4.6.1), X is connected. □

✍ **Exercise 4.17** Give an example of a ring A where $\operatorname{Spec} A$ is connected but reducible.

Solution *Let* $A = \mathbb{C}[x,y]/(x^2 - y^2)$, *then*

$$\begin{aligned}\operatorname{Spec} A =& \{\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[x,y] : \mathfrak{p} \supseteq (x^2 - y^2)\}\\ =& V(((x-y)(x+y)))\\ =& V(x-y) \cup V(x+y),\end{aligned}$$

where $V(x-y)$ *and* $V(x+y)$ *are proper closed subsets of* $\operatorname{Spec} A$. *Hence,* $\operatorname{Spec} A$ *is reducible.*

Since $V(x-y) = \operatorname{Spec} \mathbb{C}[x,y]/(x-y) \cong \operatorname{Spec} \mathbb{C}[x]$ *and* $V(x+y) = \operatorname{Spec} \mathbb{C}[x,y]/(x+y) \cong \operatorname{Spec} \mathbb{C}[x]$, *note that* $\mathbb{C}[x]$ *is an integral domain, by Proposition 4.6.2,* $V(x-y)$ *and* $V(x+y)$ *is irreducible. By Proposition 4.6.3,* $V(x-y)$ *and* $V(x+y)$ *are connected. Note that* $V(x-y) \cap V(x+y) = V((x-y, x+y)) = [(x-y, x+y)] \neq \varnothing$, $V(x-y) \cap V(x+y) = \operatorname{Spec} A$ *is connected.*

✍ **Exercise 4.18**

(a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subseteq k[w,x,y,z]$. Show that $\operatorname{Spec} k[w,x,y,z]/I$ is irreducible, by show that $k[w,x,y,z]/I$ is an integral domain. (We will later see this as the cone over the twisted cubic curve.)

(b) Note that the generators of the ideal of part (a) may be rewritten as the equations ensuring that

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \le 1,$$

i.e., as the determinants of the 2×2 submatrices. Generalize part (a) to the ideal of rank one $2 \times n$ matrices. This notion will correspond to the cone over the degree n rational normal curve.

Proof

(a) Define $\varphi : k[w,x,y,z] \to k[s,t]$ as follow:

$$\varphi := \begin{cases} w & \mapsto s^3, \\ x & \mapsto s^2 t, \\ y & \mapsto s t^2, \\ z & \mapsto t^3. \end{cases}$$

It is easy to check φ is a ring homomorphism. Clearly, $\varphi(k[w,x,y,z]) = k[s^3, s^2t, st^2, t^3]$. We claim that $I = \operatorname{Ker} \varphi$. Note that $\varphi(wz - xy) = s^3t^3 - s^3t^3 = 0$, $\varphi(wy - x^2) = s^4t^2 - s^4t^2 = 0$, and $\varphi(xz - y^2) = s^2t^4 - s^2t^4 = 0$, we have $I \subseteq \operatorname{Ker} \varphi$.
Let $f(w,x,y,z) \in \operatorname{Ker} \varphi$, then $\varphi(f) = f(s^3, s^2t, st^2, t^3) = 0$. Say $g_1 = wz - xy$, $g_2 = wy - x^2$, and $g_3 = xz - y^2$. Hence, we may assume that $f(w,x,y,z) = f_1 g_1 + f_2 g_2 + f_3 g_3 + r$, where the monomial of r none of which is divisible by wz, wy, xz. Hence, r only has forms: $w^l x^k$ with $k \ge 0, l \ge 1$, $y^m z^n$ with $m \ge 0, n \ge 1$, and $x^a y^b$ with $a \ge 0, b \ge 0$. Since $f(s^3, s^2t, st^2, t^3) = 0$, $r(s^3, s^2t, st^2, t^3) = 0$. The monomials (in s and t) of $r(s^3, s^2t, st^2, t^3)$ are of the following types: $s^{3l+2k}t^k$, $s^m t^{2m+3n}$, and

$s^{2a+b}t^{a+2b}$. Consider following equations,

$$\begin{cases} 3l+2k &= m \\ k &= 2m+3n \end{cases} \implies \begin{cases} l &= -m-n \\ k &= 2m+3n, \end{cases}$$

note that $l \geq 1$, hence above equations do not have solve. Hence $s^{3l+2k}t^k \neq s^m t^{2m+3n}$. Similarly, we have $s^{3l+2k}t^k \neq s^{2a+b}t^{a+2b}$ and $s^m t^{2m+3n} \neq s^{2a+b}t^{a+2b}$. Hence, there is no possible cancelation between these monomials, so $r = 0$, and therefore $f \in I$, i.e., $\operatorname{Ker}\varphi \subseteq I$.

Hence, $I = \operatorname{Ker}\varphi$, and therefore $k[s^3, s^2t, st^2, t^3] \cong k[w,x,y,z]/I$. Also, $k[s^3, s^2t, st^2, t^3]$ is a subring of integral domain $k[s,t]$, $k[s^3, s^2t, st^2, t^3]$ is integral domain. Hence, $k[w,x,y,z]/I$ is an integral domain, by Proposition 4.6.2, $\operatorname{Spec} k[w,x,y,z]/I$ is irreducible.

(b) Let

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

be a $2 \times n$ matrix with $\operatorname{rank} M \leq 1$ and $a_{2,i} = a_{1,(i+1)}$, then the determinants of the 2×2 submatrices are 0, i.e.,

$$\det \begin{pmatrix} a_{1i_1} & a_{1i_2} \\ a_{2i_1} & a_{2i_2} \end{pmatrix} = a_{1i_1}a_{2i_2} - a_{1i_2}a_{2i_1} = 0$$

for all $1 \leq i_1 < i_2 \leq n$.

For $n = 2$, all determinants of the 2×2 submatrices are the generators of I, as part (a).

For $n > 2$, let

$$I = \left(\det \begin{pmatrix} a_{1i_1} & a_{1i_2} \\ a_{2i_1} & a_{2i_2} \end{pmatrix} = 0, \forall 1 \leq i_1 < i_2 \leq n \right)$$

Define $\varphi : k[a_{11}, \ldots, a_{1n}, a_{2n}] \to k[s,t]$ by setting

$$\varphi := \begin{cases} a_{1k} & \mapsto s^{n+1-k}t^{k-1}, \forall 1 \leq k \leq n \\ a_{2n} & \mapsto t^n. \end{cases}$$

It is easy to check that $I \subseteq \operatorname{Ker}\varphi$. Let $f \in \operatorname{Ker}\varphi$, then $f(s^n, s^{n-1}t, \ldots, t^n) = 0$. Note that $\{s^n, s^{n-1}t, \ldots, t^n\}$ agree with

$$\det \begin{pmatrix} a_{1i_1} & a_{1i_2} \\ a_{2i_1} & a_{2i_2} \end{pmatrix} = 0, \forall 1 \leq i_1 < i_2 \leq n.$$

Hence, $I = \operatorname{Ker}\varphi$ (use the same method as part (a)), and therefore

$$k[a_{11}, \ldots, a_{1n}, a_{2n}]/I \cong k[s^n, s^{n-1}t, \ldots, t^n] \subseteq k[s,t].$$

Since $k[s,t]$ is an integral domain, $k[a_{11}, \ldots, a_{1n}, a_{2n}]/I$ is integral domain, by Proposition 4.6.2, we done.

□

Quasi-compactness

Definition 4.6.3 (Quasi-compact)

*A topological space X is **quasi-compact** if given any cover $X = \bigcup_{i\in I} U_i$ by open sets, there is a finite subset S of the index set I such that $X = \bigcup_{i\in S} U_i$. Informally: every open cover has a finite subcover.*

We will like this condition, because we are afraid of infinity. Depending on your definition of "compactness", this is the definition of compactness, minus possibly a Hausdorff condition. However, this isn't really the algebra-geometric analog of "compact" (we certainly wouldn't want $\mathbb{A}^1_{\mathbb{C}}$ to be compact) — the right analog is "properness".

Proposition 4.6.4

Spec A *is quasi-compact.*

Proof Let $\{U_i\}_{i\in J}$ be an open cover of Spec A, i.e., Spec $A = \bigcup_{i\in J} U_i$. Since each open set is the union of some $D(f_i)$, we may assume Spec $A = \bigcup_{i\in J} D(f_i)$. By Proposition 4.5.3, we can choose a finite subset $J' \subseteq J$ such that Spec $A = \bigcup_{i\in J'} D(f_i)$, which implies that Spec A is quasi-compact. □

Remark In general Spec A can have nonquasi-compact open sets.

Proof Let $A = k[x_1, x_2, x_3, \dots]$, $\mathfrak{m} = (x_1, x_2, x_3, \dots)$ is its maximal ideal. Consider Spec $A \setminus V(\mathfrak{m})$, we have

$$\operatorname{Spec} A \setminus V(\mathfrak{m}) = \operatorname{Spec} A \setminus \{\mathfrak{m}\} = \bigcup_{i=1}^{\infty} D(x_i).$$

We shall to show that we can not choose finite numbers of $D(x_i)$, such that $\bigcup_i D(x_i) = \operatorname{Spec} A \setminus V(\mathfrak{m})$.

Suppose Spec $A \setminus V(\mathfrak{m}) = \bigcup_{k=1}^{n} D(x_{i_k})$, where $\{x_{i_1}, \dots, x_{i_n}\} \subseteq \{x_1, x_2, \dots\}$ be a finite subset. Hence, there exists $x_t \in \{x_1, x_2, \dots\} \setminus \{x_{i_1}, \dots, x_{i_n}\}$. Let $\mathfrak{p} = (x_t, x_{i_1}, \dots, x_{i_n})$, then $\mathfrak{p}$ is a prime ideal, and $\mathfrak{p} \notin \bigcup_{k=1}^{n} D(x_{i_k})$, but $\mathfrak{p} \in \operatorname{Spec} A \setminus V(\mathfrak{m})$, a contradiction. Hence, Spec $A \setminus V(\mathfrak{m})$ is not quasi-compact. □

Proposition 4.6.5

(a) If X is a topological space that is a finite union of quasi-compact spaces, then X is quasi-compact.

(b) Every closed subset of a quasi-compact topological space is quasi-compact.

Proof

(a) Let $\{U_j\}_{j\in J}$ be any open cover of X. Since X is a finite union of quasi-compact spaces, we may assume $X = \bigcup_{i=1}^{n} X_i$, where X_i is quasi-compact spaces. Hence, $\{U_j \cap X_i\}_{j\in J}$ is an open cover of X_i. Since X_i is quasi-compact, we may assume $X_i = \bigcup_{k=1}^{m_i}(U_{i_k} \cap X_i)$. Hence,

$$X = \bigcup_{i=1}^{n}\bigcup_{k=1}^{m_i}(U_{i_k} \cap X_i) \subseteq \bigcup_{i=1}^{n}\bigcup_{k=1}^{m_i} U_{i_k},$$

and therefore X is quasi-compact.

(b) Let F be a closed subset of X, endow F with subspace topology. Let $V_{j_{j\in J}}$ be an open over of F, then $V_j = F \cap U_j$ where U_j is open subset of X. Since $F = \bigcup_j V_j \subseteq \bigcup_j U_j$, we have

$$X = \bigcup_j U_j \cup (X \setminus \bigcup_j U_j) \subseteq \bigcup_j U_j \cup (X \setminus F),$$

hence, $\{U_j\ (j \geq 1), X \setminus F\}$ is an open cover of X. Since X is quasi-compact, we may assume $X = \bigcup_{j=1}^{n} U_j \cup (X \setminus F)$. Note that $F \cap (X \setminus F) = \varnothing$, F must be covered by $\{U_j\}_{j=1}^{n}$. Hence,

$F = \bigcup_{j=i}^{n}(U_i \cap F)$, and therefore F is quasi-compact.

□

Remark Example 4.5 shows that $\coprod_{i=1}^{n} \operatorname{Spec} A_i \cong \operatorname{Spec} \prod_{i=1}^{n} A_i$, but this never holds if "n is infinite" and all A_i are nonzero, as Spec of any rings is quasi-compact (Proposition 4.6.4). This leads to an interesting phenomenon. We show that $\operatorname{Spec} \prod_{i=1}^{\infty} A_i$ is "strictly bigger" than $\coprod_{i=1}^{\infty} \operatorname{Spec} A_i$, where each A_i is isomorphic to the field k. First, we have an inclusion of sets $\coprod_{i=1}^{\infty} \operatorname{Spec} A_i \hookrightarrow \operatorname{Spec} \prod_{i=1}^{\infty} A_i$, as there is a maximal ideal of $\prod A_i$ correspongding to each i (precisely, those elements 0 in the i-th component). But there are other maximal ideals of $\prod A_i$. Cosider elements $\prod \mathfrak{a}_i$ that are "eventually zero", i.e., $\mathfrak{a}_i = 0$ for $i \gg 0$, $\prod \mathfrak{a}_i$ is a proper ideal not contained in any of these maximal ideals. This leads to the notion of **ultrafilters**, which are very useful, but irrelevant to our current discussion.

4.6.2 Possible topological properties of points of $\operatorname{Spec} A$

Definition 4.6.4 (Closed point)

*A point of a topological space $p \in X$ is said to be a **closed point** if $\{p\}$ is a closed subset.*

Example 4.6 In the classical topology on $\mathbb{C}^n$, all points are closed. In $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} k[t]$, all the points are closed except for $[(0)]$.

Proposition 4.6.6

The closed points of $\operatorname{Spec} A$ *correspond to the maximal ideals. In particular, nonempty affine schemes have closed points, as non-zero rings have maximal ideals.*

Proof Let $[\mathfrak{m}] \in \operatorname{Spec} A$ be a closed point, then $\{[\mathfrak{m}]\} = V(I)$ for some ideal $I \subseteq A$. Since $V(I) = \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supseteq I\} = \{[\mathfrak{m}]\}$, this implies $\mathfrak{m}$ is the only prime ideal which is containing I, note that I must contained in a maximal ideal, hence, $\mathfrak{m}$ must be maximal ideal. □

Connection to the classical theory of varieties

Hilbert's Nullstellensatz lets us interpret the closed points of $\mathbb{A}^n_{\mathbb{C}}$ as the n-tuples of complex numbers. More generally, the closed points of $\operatorname{Spec} \overline{k}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ (where $\overline{k}$ is an algebraically closed field) are naturally interpreted as those points in $\overline{k}^n$ satisfying the equations $f_1 = \cdots = f_r = 0$. Hence from now on we will say "closed point" instead of "traditional point" and "non-closed point" instead of "bonus" point when discussing subsets of $\mathbb{A}^n_{\overline{k}}$. The following proposition is the reason that on algebraic varieties, the set of closed points is dense.

Proposition 4.6.7

(a) *Suppose that k is a field, and A is a finitely generated k-algebra. Then the set of closed points of* $\operatorname{Spec} A$ *is dense.*

(b) *If A is a k-algebra that is not finitely generated, the set of closed points need not be dense.*

Proof

(a) By Proposition 4.6.6, the set of closed points of $\operatorname{Spec} A$ is $\operatorname{Max} A$. To show that $\operatorname{Max} A$ is dense, it suffices to show that $\operatorname{Max} A \cap D(f) \neq \varnothing$ for any $f \in A$ (see Wikipedia, Dense set, Definition), i.e., every nonempty $D(f)$ contains a closed point.

To this end, let $f \in A$ such that $D(f) \neq \varnothing$. Then, f is not nilpotent, and therefore A_f is not zero-ring. Hence, A_f has a maximal ideal. This is of the form $\mathfrak{p}_f$ for some prime ideal $\mathfrak{p} \subseteq A$ not containing f. In other words, $[\mathfrak{p}] \in D(f)$. We show that $[\mathfrak{p}]$ is a closed point by showing that $\mathfrak{p}$ is maximal (Proposition 4.6.6). It suffices to show that $A/\mathfrak{p}$ is a field. Note that $A/\mathfrak{p}$ is an integral domain, recall Proposition 4.2.1, it suffices to show that $A/\mathfrak{p}$ is a finite k-algebra (i.e., a k-algebra that is a finite-dimensional vector space over k).

Note that $A_f = A[1/f]$ is the image of $A[x]$ under map $x \mapsto 1/f$, hence, A_f is a finitely generated k-algebra, by Hilbert's Nullstenllensatz 4.2.2, $A_f/\mathfrak{p}_f$ is finitely generated as a k-vector space. Note that $(A/\mathfrak{p})_f \cong A_f/\mathfrak{p}_f$, $(A/\mathfrak{p})_f$ is finitely generated as a k-vector space. Since $A/\mathfrak{p}$ is integral domain, canonical homomorphism $A/\mathfrak{p} \to (A/\mathfrak{p})_f$ is injective. This homomorphism is also a k-linear map, hence, $A/\mathfrak{p}$ is a finite-dimensional vector space over k.

(b) Let $A = k[x_1, x_2, x_3, \dots]$ with maximal ideal $\mathfrak{m} = (x_1, x_2, x_3, \dots)$. Consider $A_\mathfrak{m}$, then $A_\mathfrak{m}$ is a k-algebra and is not finitely generated. Since $A_\mathfrak{m}$ is local ring, its maximal ideal is $\mathfrak{m}A_\mathfrak{m}$. Hence, the closure of $\operatorname{Max} A = \{[\mathfrak{m}A_\mathfrak{m}]\}$ is $\{[\mathfrak{m}A_\mathfrak{m}]\}$, hence, $\operatorname{Max} A = \{[\mathfrak{m}A_\mathfrak{m}]\}$ is not dense.

□

Proposition 4.6.8

Suppose k is an algebraically closed field, and $A = k[x_1, \dots, x_n]/I$ is a finitely generated k-algebra with $\mathfrak{N}(A) = \{0\}$. Consider the set $X = \operatorname{Spec} A$ as a subset of $\mathbb{A}_k^n$. The space $\mathbb{A}_k^n$ contains the "classical" points k^n. Then the functions on X are determined by their values on the closed points.

Proof We want to show: if $f, g \in A$ have the same value on X, then $f = g$.

Suppose f and g are different functions on X, then $f - g \neq 0$. Since $\mathfrak{N}(A) = 0$, $f - g \notin \mathfrak{N}(A)$, it follows that $D(f-g) \neq \varnothing$. By Proposition 4.6.7 (a), we have $\operatorname{Max} A \cap D(f-g) \neq \varnothing$, say $\mathfrak{m} \in \operatorname{Max} A \cap D(f-g)$, then $f - g \notin \mathfrak{m}$, a contradiction. □

Once we know what a variety is, this will immediately imply that a function on a variety over an algebraically closed field is determined by its values on the "classical points". (Before the advent of scheme theory, functions on "classical" points, and Proposition 4.6.8 basically shows that there is no harm in thinking of "traditional" varieties as a particular flavor of schemes.)

Specialization and generization, and generic point

Definition 4.6.5 (Specialization and generization)

*Given two points x, y of a topological space X, we say that x is a **specialization** of y, and y is a **generization** of x, if $x \in \overline{\{y\}}$.*

This (and Proposition 4.6.9) now makes precise our hand-waving about "one point containing another". It is of closure nonsense for a point to contain another. But it is not nonsense to say that the closure of a point contains another.

Example 4.7 For example, in $\mathbb{A}_\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, $[(y - x^2)]$ is a generization of $[(x-2, y-4)] = (2, 4) \in \mathbb{C}^2$, and $(2, 4)$ is a specialization of $[(y - x^2)]$ (see Figure 4.10).

Lemma 4.6.1

Let $X = \operatorname{Spec} A$, $[\mathfrak{p}] \in X$, then $\overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p})$.

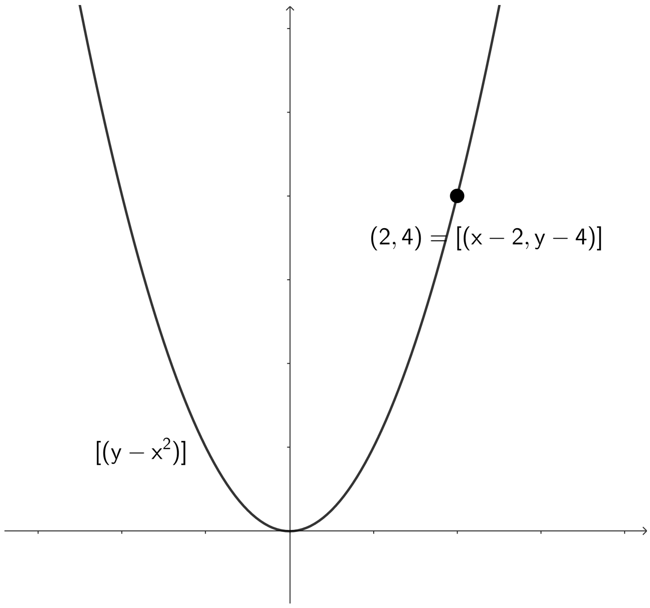


Figure 4.10: $(2, 4) = [(x - 2, y - 4)]$ is a specialization of $[(y - x^2)]$.
$[(y - x^2)]$ is a generization of $(2, 4)$.

Proof In fact,

$$\overline{\{[\mathfrak{p}]\}} = \bigcap_{V(I) \supseteq \{[\mathfrak{p}]\}} V(I) \subseteq V(\mathfrak{p}).$$

Let $[\mathfrak{q}] \in V(\mathfrak{p})$, then $\mathfrak{q} \supseteq \mathfrak{p}$. For all $V(I) \subseteq \{[\mathfrak{p}]\}$, $I \subseteq \mathfrak{p}$, hence, $\mathfrak{q} \supseteq I$, and therefore $[\mathfrak{q}] \in V(I)$. It follows that $[\mathfrak{q}] \in \bigcap_{V(I) \supseteq \{[\mathfrak{p}]\}} V(I) = \overline{\{[\mathfrak{p}]\}}$. Hence, $\overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p})$. □

Proposition 4.6.9

If $X = \operatorname{Spec} A$, then $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.

Proof By Lemma 4.6.1, clearly. □

Definition 4.6.6 (Generic point)

*We say that a point $p \in X$ is a **generic point** for a closed subset K if $\overline{\{p\}} = K$.*

This important notion predates Grothendieck. The early twentieth-century Italian algebraic geometers had a notion of "generic points" of a variety, by which they meant points with no special properties, so that anything proved of "a generic point" was true of "almost all" the points on that variety. The modern "generic point" has the same intuitive meaning. If something is "generically" or "mostly" true for the points of an irreducible subset, in the sense of being true for a dense open subset (for "almost all points"), then it is true for the generic point, and vice versa. (This is a statement of principle, not of fact. An interesting case is "reducedness", for which this principle does not hold in general, but does hold for "reasonable" schemes such as varieties.) For example, a function has value zero at the generic point of an irreducible scheme if and only if it has the value zero at all points. You should keep an eye out for other examples of this.

The phrase **general point** does not have the same meaning. The phrase "the general point of X satisfies such-and-such a property" means "every point of some dense open subset of X satisfies such-and-such a property". Be careful not to confuse "general" and "generic". But be warned that accepted terminology does not always follow this convention; witness "generic smoothness" and "generic flatness".

Exercise 4.19 Verify that $[(y-x^2)] \in \mathbb{A}^2_{\mathbb{C}}$ is a generic point for $V(y-x^2)$.

Proof Note that $\overline{\{[(y-x^2)]\}} = V((y-x^2)) = V(y-x^2)$ where $V(y-x^2)$ is a closed subset, $[(y-x^2)] \in \mathbb{A}^2_{\mathbb{C}}$ is a generic point for $V(y-x^2)$. □

As more motivation for the terminology "generic": we think of $[(y-x^2)]$ as being some non-specific point on the parabola (with the closed points $(a, a^2) \in \mathbb{C}^2$, i.e., $(x-a, y-a^2)$ for $a \in \mathbb{C}$, being "specific points"); it is "generic" in the conventional sense of the word. We might "specialize it" to a specific point of the parabola; hence for example $(2,4)$ is a specialization of $[(y-x^2)]$. (Again, see Figure 4.10.) To make this somewhat more precise:

Proposition 4.6.10

Suppose p is a generic point for the closed subset K. p is "near every point q of K" (every neighborhood of q contains p), and "not near any point r not in K" (there is a neighborhood of r not containing p).

Proof Since p is a generic point for the closed subset K, $\overline{\{p\}} = K$, i.e., $\{p\}$ is dense in K. Let U_q be any open neighborhood of q, then $K \cap U_q$ is an open neighborhood of q in K (endow K with subspace topology). Since $\{p\}$ is dense in K, $\{p\} \cap (U_q \cap K) \neq \varnothing$, and therefore $\{p\} \in U_q$. It follows that every neighborhood of q contains p.

Let X be a topological space, K is a closed subset of X. Let $r \in X \setminus K$, and U_r be any open neighborhood of r, then $(X \setminus K) \cap U_r$ is also an open neighborhood of r, say $V_r = (X \setminus K) \cap U_r$. Note that $p \notin V_r$, as we desired. □

We will soon see that there is a natural bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$, sending each point to its closure, and each irreducible closed subset to its (unique) generic point.

4.6.3 Irreducible and connected components, and Noetherian conditions

Irreducible component and connected components

Definition 4.6.7 (Irreducible component)

*An **irreducible component** of a topological space is a maximal irreducible subset (an irreducible subset not contained in any larger irreducible subset).*

Irreducible components are closed (*By Proposition 4.6.1 (b), the closure of irreducible component is irreducible, sine an irreducible subset not contained in any larger irreducible subset, they must be same, hence, closed*), and it can be helpful to think of irreducible components of a topological space X as maximal among the irreducible closed subsets of X. We think of these as the "pieces of X" (see Figure 4.11).

Definition 4.6.8 (Connected component)

*A subset Y of a topological space X is a **connected component** if it is a maximal connected subset (a connected subset not contained in any larger connected subset).*

Proposition 4.6.11 (Every topological space is the union of irreducible components)

Every point p of a topological space X is contained in an irreducible component of X.

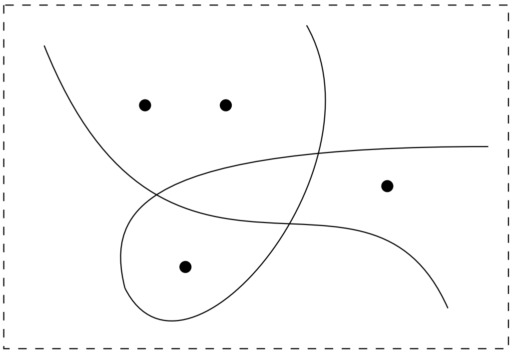


Figure 4.11: This closed subset of $\mathbb{A}^2_{\mathbb{C}}$ has six irreducible components

Proof Consider the partially ordered set

$$\mathscr{S}_p = \{\text{Irreducible subsets of } X \text{ containing } p\}.$$

We claim that $\mathscr{C}_p$ non-empty. Consider $\overline{\{p\}}$. If $\overline{\{p\}}$ is reducible, there exists proper closed subsets A and B of $\overline{\{p\}}$ such that $\overline{\{p\}} = A \cup B$. Hence, $\{p\} \subseteq A$ or $\{p\} \subseteq B$, we may assume that $\{p\} \subseteq A$. Note that $A \subsetneq \overline{\{p\}}$ proper closed, which contradicts to the fact that $\overline{\{p\}}$ is the smallest closed set containing $\{p\}$. Hence, $\overline{\{p\}}$ is irreducible, and therefore $\overline{\{p\}} \in \mathscr{S}_p$.

For any totally ordered subset $\{Z_\alpha\}$ of $\mathscr{S}_p$, Let $Z = \bigcup_\alpha Z_\alpha$, it is a upper bound of $\{Z_\alpha\}$, we want to show that $Z \in \mathscr{S}_p$. Consider the closure of Z, i.e., $\overline{Z}$. Suppose $\overline{Z}$ is reducible, i.e., exists proper closed subsets A and B of $\overline{Z}$ such that $\overline{Z} = A \cup B$, since $\{Z_i\}$ is totally ordered and each Z_i is irreducible, $Z \subseteq A$ and $Z \subseteq B$, we may assume that $Z \subseteq A$. Note that $A \subsetneq \overline{Z}$ proper closed, this contradicts to the fact that $\overline{Z}$ is the smallest closed set containing Z. Hence, $\overline{Z}$ is irreducible, by Proposition 4.6.1 (b), Z is irreducible, and therefore $Z \in \mathscr{S}$. Hence, every totally ordered subset has an upper bound belong to $\mathscr{S}_p$.

By Zorn's Lemma, there exists at least one maximal element in $\mathscr{S}_p$, say $X_p (\ni p)$, which is an irreducible component of X. $\square$

Remark Every point is contained in a connected component, and connected components are always closed. You can prove this now, but we deliberately postpone this until we need it.

On the other hand, connected components need not be open. An example of an affine scheme with connected components that are not open is $\operatorname{Spec}(\prod_1^\infty \mathbb{F}_2)$. ($\operatorname{Spec}(\prod_1^\infty \mathbb{F}_2) = \{[(0)], \text{maximal ideal}\}$, where each maximal ideal has form $\mathbb{F}_2 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_2 \times (0) \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$. $\operatorname{Spec}(\prod_1^\infty \mathbb{F}_2) = \coprod_{i=1}^\infty \mathfrak{m}_i$, each connected component $\mathfrak{m}_i$ is closed.)

Noetherian topological space and Noetherian induction

In the examples we have considered, the spaces have naturally broken up into a finite number of irreducible components. For example, the locus $xy = 0$ in $\mathbb{A}^2_{\mathbb{C}}$ we think of as having two "pieces" — the two axes. The reason for this is that their underlying topological spaces (as we shall soon establish) are **Noetherian**.

Definition 4.6.9 (Noetherian topological space)

A topological space X is called ***Noetherian*** *if it satisfies the* ***descending chain condition (d.c.c.)*** *for closed subsets: any sequence*

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

Here is a first example:

Example 4.8 Show that $\mathbb{A}^2_{\mathbb{C}}$ is a Noetherian topological space: any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Show that $\mathbb{C}^2$ with the classical topology is not a Noetherian topological space.

Solution *In fact,* $\operatorname{Spec} \mathbb{C}[x, y] = \{(0), (x - a, y - b)$ *where* $a, b \in \mathbb{C}, (f(x, y))$ $f(x, y)$ *is irreducible* $\}$. *Let*

$$V(I_1) \supseteq V(I_2) \supseteq \cdots \supseteq V(I_n) \supseteq \cdots$$

be any descending chain, by Hilbert's Nullstellensatz, then we have ascending chain of ideals

$$\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \cdots \subseteq \sqrt{I_n} \subseteq \cdots . \tag{4.2}$$

Since $\mathbb{C}[x, y]$ *is a Noetherian space, chain 4.2 is stationary. By Hilbert's Nullstellensatz, the chain of closed subset*

$$V(I_1) \supseteq V(I_2) \supseteq \cdots \supseteq V(I_n) \supseteq \cdots$$

is stationary. Hence, $\operatorname{Spec} \mathbb{C}[x, y]$ *is a Noetherian topological space.*

Next, we want show that $\mathbb{C}^2$ *with the classical topology is not a Noetherian space. Let* $F_n := \{z \in \mathbb{C}^2 : |z| \geq n$ *or* $|z| \leq \frac{1}{n}\}$, *clearly,* F_n *is closed in* $\mathbb{C}^2$, *but*

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

not stabilize. Hence, $\mathbb{C}^2$ *is not a Noetherian space.*

The following technique is called **Noetherian induction**.

Theorem 4.6.1 (Noetherian induction)

Let X be a Noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

Proof Suppose that property $\mathscr{P}$ not holds for X. Consider the following set:

$$\Sigma = \{Y \subseteq X : Y \text{ is closed}, \mathscr{P} \text{ not holds for } Y\},$$

Σ is not empty, since $X \in \Sigma$. Since X is a Noetherian space, Σ has minimal element, say Y_0. Let $Z \subsetneq Y_0$ be proper closed subset of Y_0, since Y_0 is minimal element in Σ, $Z \notin \Sigma$, and therefore Z satisfies property $\mathscr{P}$. Hence, $\mathscr{P}$ holds for every proper closed subset of Y_0, by hypothesis, $\mathscr{P}$ holds for Y_0, but $Y_0 \in \Sigma$, a contradiction. □

Proposition 4.6.12 (Any closed subset of Noetherian space has a finite number of "pieces")

Suppose X is a Noetherian topological space. Then every nonempty closed subset Z can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.

Proof Consider the collection of nonempty closed subsets of X that cannot be expressed as a finite union of irreducible closed subsets, say Σ. We will show that Σ is empty. Suppose Σ is not empty, let $Y_1 \in \Sigma$. If Y_1 contains another element in Σ, then choose one, and call it Y_2. If Y_2 properly contains another element in Σ, then choose one, and call it Y_3, and so on. By the descending chain condition, this must be stationary, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets. Since Y_r is not itself irreducible, so we may assume $Y_r = Y' \cup Y''$, where Y' and Y'' are both proper closed subsets. Since every closed subset properly contained in Y_r not belong to Σ, Y' and Y'' can be written as the union of a finite number of irreducible closed subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subseteq Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1/ \subseteq Z_1$. Similarly, $Z_1 \subseteq Z_a'$ for some a; but because $Z_1' \subseteq Z_1 \subseteq Z_a'$, and Z_1' is contained in no other Z_i', we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa. Hence, they must be the same list. □

Proposition 4.6.13

(a) Every connected component of a topological space X is the union of irreducible components of X.

(b) Any subset of X that is simultaneously open and closed must be the union of some of the connected components of X.

(c) If X is a Noetherian topological space, then the union of any subset of the connected components of X is always open and closed in X. In particular, connected components of Noetherian topological spaces are always open, which is not true for more general topological spaces (example: $\operatorname{Spec}(\prod_1^\infty \mathbb{F}_2)$).

Proof

(a) Every connected component of a topological space X is closed, endow connected component with subspace topology, by Proposition 4.6.11, connected component is the union of irreducible components of X.

(b) Let Y be a subset of X is simultaneously open and closed. Let C be any connected components of X. If $Y \cap C \neq \varnothing$ and $(X \setminus Y) \cap C \neq \varnothing$, since $X \setminus Y$ is open and C is connected, we have $C \subseteq X \setminus Y$ or $C \subseteq Y$. Hence, Y is the union of some of the connected components of X.

(c) By Proposition 4.6.12, X can be expressed uniquely as a finite union of irreducible closed subset, say $X = Z_1 \cup Z_2 \cup \cdots \cup Z_m$. Each Z_i belong to a connected component, and if $Z_i \cap Z_j \neq \varnothing$, Z_i and Z_j must belong to the same connected component, hence, X can be expressed uniquely as a finite disjoint union of connected components, say $X = \coprod_{i=1}^n C_i$, where each C_i is a finite union of some Z_j, hence, each C_i must be closed.

We claim that every connected component is open and closed in X. Let C_i be a connected component of X. Then $X \setminus C_i = \coprod_{\substack{j \neq i \\ 1 \leq j \leq n}} C_j$. Since each connected component is closed, $\coprod_{\substack{j \neq i \\ 1 \leq j \leq n}} C_j$ is open, i.e., C_i is open and closed in X.

Hence, the union of any subset of the connected components of X is always open and closed in X.

□

Noetherian rings

It turns out that all of the spectra we have considered (except $\operatorname{Spec} A[x_1, x_2, x_3, \dots]$) are Noetherian topological spaces, but that isn't true for the spectra of all rings. The Key characteristic all of our examples have had in common is that the rings were **Noetherian**.

Definition 4.6.10 (Noetherian ring)

*A ring is **Noetherian** if every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$. (This is called the **ascending chain condition** on ideals.)*

Here are some quick facts about Noetherian rings.

Example 4.9 Fields are Noetherian. $\mathbb{Z}$ is Noetherian.

Proposition 4.6.14 (Noetherianness is preserved by quotients)

If A is Noetherian, and I is any ideal of A, then A/I is Noetherian.

Proof Let $I_1 \subseteq I_2 \subseteq \cdots$ be any ascending sequence in A/I. Then I_i is corresponding to ideal $\widetilde{I_i} \subseteq A$, which contains I_1, and therefore we have ascending sequence in A,

$$\widetilde{I_1} \subseteq \widetilde{I_2} \subseteq \cdots .$$

Since A is Noetherian ring, this sequence must be stationary. Since this sequence is corresponding to sequence $I_1 \subseteq I_2 \subseteq \cdots$, $I_1 \subseteq I_2 \subseteq \cdots$ is stationary, and therefore A/I is Noetherian. $\square$

Proposition 4.6.15 (Noetherianness is preserved by localization)

if A is a Noetherian, and S is any multiplicative set, then $S^{-1}A$ is Noetherian.

Proof Let $S^{-1}I_1 \subseteq S^{-1}I_2 \subseteq \cdots$ be any ascending sequence in $S^{-1}A$. This sequence is corresponding to sequenec

$$\widetilde{I_1} \subseteq \widetilde{I_2} \subseteq \widetilde{I_3} \subseteq \cdots ,$$

where $\widetilde{I_i}$ is corresponding to I_i and $\widetilde{I_i} \cap S = \varnothing$. Since A is Noetherian ring, $S^{-1}I_1 \subseteq S^{-1}I_2 \subseteq \cdots$ is stationary, and therefore

$$\widetilde{I_1} \subseteq \widetilde{I_2} \subseteq \widetilde{I_3} \subseteq \cdots$$

is stationary. Hence, $S^{-1}A$ is Noetherian. $\square$

Proposition 4.6.16

PIDs are Noetherian rings.

Proof Let A be a PID. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be any ascending sequence in A, since A is PID, we may assume $I_i = (x_i)$. Since $(x_i) \subseteq (x_{i+1})$, we have $x_{i+1} \mid x_i$. Note that x_i can be decomposed into a product of finitely many irreducible elements, we may assume that $x_1 = up_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, then sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ must be stationary, and therefore A is Noetherian. $\square$

Proposition 4.6.17

A ring A is Noetherian if and only if every ideal of A is finitely generated.

Proof "$\Rightarrow$": Let $\mathfrak{a}$ be an ideal of A, and let Σ be the set of all finitely generated ideal of A. Since $0 \in \Sigma$, Σ is not empty, and therefore has a maximal element, say $\mathfrak{a}_0$. If $\mathfrak{a} \neq \mathfrak{a}_0$, consider the submodule $\mathfrak{a}_0 + Ax$, where $x \in \mathfrak{a}$ and $x \notin \mathfrak{a}_0$; this is finitely generated and strictly contains $\mathfrak{a}_0$, a contradiction. Hence, $\mathfrak{a} = \mathfrak{a}_0$, and therefore $\mathfrak{a}$ is finitely generated.

"$\Leftarrow$": Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$ be an ascending chain of ideals of A. Then $\mathfrak{a} = \sum_i \mathfrak{a}_i$ is an ideal of A, hence is finitely generated, say $\mathfrak{a} = (x_1, x_2, \dots, x_r)$. Say $x_i \in \mathfrak{a}_{n_i}$ and let $n = \max_{i=1}^r x_i$, then each $x_i \in \mathfrak{a}_n$, hence, $\mathfrak{a}_n = \mathfrak{a}$, and therefore the chain is stationary. Hecne, A is Noetherian. $\square$

Theorem 4.6.2 (The Hilbert Basis Theorem)

If A is Noetherian, then so is $A[x]$.

Proof We show that any ideal $I \subseteq A[x]$ is finitely generated. We inductively produce a set of generators $f_1, \dots$ as follows. For $n \geq 0$, if $I \neq (f_1, \dots, f_n)$, let f_{n+1} be any non-zero element of $I - (f_1, \dots, f_n)$ of lowest degree. Thus f_1 is any element of I of lowest degree, assuming $I \neq (0)$. If this procedure terminates, we are done. Otherwise, let $a_n \in A$ be the initial coefficient of f_n for all $n > 0$. As A is Noetherian, $(a_1, a_2, \dots) = (a_1, \dots, a_N)$ for some N. Say $a_{N+1} = \sum_{i=1}^N b_i a_i$. Then

$$f_{N+1} - \sum_{i=1}^{N} b_i f_i x^{\deg f_{N+1} - \deg f_i}$$

is an element of I that is not in $(f_1, \dots, f_N)$ (since $f_{N+1} \notin (f_1, \dots, f_N)$), and of lower degree than f_{N+1}, contradicting how we were supposed to have chosen f_{N+1}. $\square$

Remark By the results described above, any polynomial ring in finitely many variables over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely generated algebra over k or $\mathbb{Z}$, or any localization thereof, is Noetherian. Most "nice" rings are Noetherian, but not all rings are Noetherian: $k[x_1, x_2, \dots]$ is not, because $(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \cdots$ is a strictly ascending chain of ideals.

We now connect Noetherian rings and Noetherian topological spaces.

Proposition 4.6.18

If A is Noetherian, then $\operatorname{Spec} A$ *is a Noetherian topological space.*

Proof Let $V(I_1) \supset V(I_2) \supset V(I_3) \supseteq \cdots$ be any descending chain of closed subsets in $\operatorname{Spec} A$, by Hilbert's Nullstellensatz, we have

$$\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \sqrt{I_3} \subseteq \cdots$$

be a ascending sequence of ideals in A. Since A is a Noetherian ring, the sequence of radical ideals is stationary, by Hilbert's Nullstellensatz again, the descending sequence of closed set

$$V(I_1) \supseteq V(I_2) \supseteq V(I_3) \supseteq \cdots$$

is stationary. Hence, $\operatorname{Spec} A$ is a Noetherian topological space. $\square$

Exercise 4.20 Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

Proof Let $A = k[x_1, x_2, \dots]$, we claim that $\operatorname{Spec} A$ is not a Noetherian topological space. Consider the following descending chain of closed set,

$$V((x_1)) \supseteq V((x_1, x_2)) \supseteq V((x_1, x_2, x_3)) \supseteq \cdots \supseteq V((x_1, x_2, \dots, x_n)) \supseteq \cdots. \tag{4.3}$$

Suppose chain (4.3) is stationary, then $V((x_1, \dots, x_r)) = V((x_1, \dots, x_r, x_{r+1}))$ for some r. Note that $(x_1, \dots, x_r)$ is prime ideal, hence, $(x_1, \dots, x_r) \in V((x_1, \dots, x_r))$, but $(x_1, \dots, x_r) \subsetneq (x_1, \dots, x_r, x_{r+1})$, we have $(x_1, \dots, x_r) \notin V((x_1, \dots, x_r, x_{r+1}))$, a contradiction. Hence, chain of closed subset (4.3) is not stationary, and therefore $\operatorname{Spec} A$ is not a Noetherian topological space. $\square$

Remark If $\operatorname{Spec} A$ is a Noetherian topological space, A need not be Noetherian. One example is $A = k[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$. Then $\operatorname{Spec} A = \{[(x_1, x_2, \dots)]\}$ is a one point set, so is Noetherian. But A is not Noetherian as $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$ in A.

Proposition 4.6.19

Let X be a topological space. Then X is Noetherian space if and only if every open subset of X is quasi-compact.

Hence if A is Noetherian, every open subset of $\operatorname{Spec} A$ *is quasi-compact.*

Proof If X is Noetherian, let $U \subseteq X$ be an open subset. Let $\{U_i\}$ be an open cover of U. Let $V_n = \bigcup_{i=1}^n U_i$, then we get an ascending chain of open subsets,

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots, \tag{4.4}$$

hence, we have a descending chain of closed subsets,

$$X \setminus V_1 \supseteq X \setminus V_2 \supseteq X \setminus V_3 \cdots. \tag{4.5}$$

Since X is Noetherian, descending chain of closed subsets (4.5) is stationary, say $X \setminus V_r = X \setminus V_{r+1} = \cdots$, then $\bigcup_{i=1}^r U_i = \bigcup_{i=1}^{r+1} U_i = \cdots$. Hence, $U = \bigcup_{i=1}^r U_i$, which implies that U is quasi-compact.

Conversely, if every open subset of X is quasi-compact. Let $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ be an ascending chain of open subsets of X, let $U = \bigcup_{i=1}^\infty U_i$ is open. Note that $\{U_i\}$ is an open cover of X, by hypothesis, U is quasi-compact, we may assume that $U = \bigcup_{i=1}^n U_i$. Hence, $\bigcup_{i=1}^n U_i = \bigcup_{i=1}^{n+1} U_i = \cdots$, and therefore $U_n = U_{n+1} = U_{n+2} = \cdots$. Thus, $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is stationary, and therefore X is Noetherian space. □

Noetherian conditions for modules

Definition 4.6.11 (Noetherian module)

*If A is any ring, not necessarily Noetherian, we say an A-**module is Noetherian** if it satisfies the ascending chain condition (a.c.c.) for submodules.*

Example 4.10 A ring A is Noetherian if and only if it is a Noetherian A-module.

Proposition 4.6.20

M is a Noetherian A-module if and only if every submodule of M is finitely generated.

Proof The proof same as Proposition 4.6.17, or see [1]. □

Proposition 4.6.21

Let

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then M is Noetherian if and only if M' and M'' are Noetherian.

Proof "⇒": If M is Noetherian A-module. Let $M_1' \subseteq M_2' \subseteq \cdots$ be an ascending chain of submodules of M'. Then we have a chain

$$\alpha(M_1') \subseteq \alpha(M_2') \subseteq \cdots \subseteq \alpha(M_n') \subseteq \cdots.$$

Since M' is Noetherian, above chain is stationary, i.e., exists n such that $\alpha(M_n') = \alpha(M_{n+1}') = \cdots$. Noting that α is injective, we have $M_n' = M_{n+1}' = \cdots$.

Let $M''_1 \subseteq M''_2 \subseteq \cdots$ be an ascending chain of submodules of M''. Then we have a chain

$$\beta^{-1}(M''_1) \subseteq \beta^{-1}(M''_2) \subseteq \cdots \subseteq \beta^{-1}(M''_n) \subseteq \cdots.$$

Since M'' is Noetherian, above chain is stationary, i.e., exists n such that $\beta^{-1}(M''_n) = \beta^{-1}(M''_{n+1}) = \cdots$. Note that β is a surjective, we have $M''_n = M''_{n+1} = \cdots$.

"$\Leftarrow$": Let $(L_n)_{n\geq 1}$ be an ascending chain of submodules of M. Then $(\alpha^{-1}(L_n))$ is a chain in M', and $(\beta(L_n))$ is a chain in M''. For large enough n both these chains are stationary, i.e.,

$$\alpha^{-1}(L_n) = \alpha^{-1}(l_{n+1}) = \cdots, \quad \beta(L_n) = \beta(l_{n+1}) = \cdots.$$

We shall to prove $L_n = L_{n+1}$. Let $x \in L_{n+1}$, then $\beta(y) = \beta(x)$ for some $y \in L_n$. Note that $x - y \in \operatorname{Ker}\beta = \operatorname{Im}\alpha$, hence exists $z \in L_n$ such that $\alpha(z) = x - y$, and therefore $z \in \alpha^{-1}(L_{n+1}) = \alpha^{-1}(L_n)$. Hence, $x - y \in \alpha(\alpha^{-1}(L_n)) \subseteq L_n$. This implies $x \in L_n$, that is, $L_n = L_{n+1}$, by induction, we have

$$L_n = L_{n+1} = L_{n+2} = \cdots.$$

Hence, M is Noetherian. □

Corollary 4.6.1

If M_i $(1 \leq i \leq n)$ are Noetherian A-modules, then $\bigoplus_{i=1}^n M_i$ is Noetherian.

Proof By induction, we reduce immediately to the case that $M_1 \oplus M_2$. Consider exact sequence,

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{p} M_2 \longrightarrow 0,$$

which i is inclusion and p us projection. By Proposition 4.6.21, $M_1 \oplus M_2$ is Noetherian module. □

Corollary 4.6.2

If A is a Noetherian ring , then $A^{\oplus n}$ is a Noetherian A-module.

Proof By Corollary 4.6.1, clearly. □

Proposition 4.6.22

If A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module.

Proof Since M is a finitely generated A-module, then M is isomorphic to a quotient of $A^{\oplus n}$ for some integer $n > 0$. By Corollary 4.6.2, $A^{\oplus n}$ is a Noetherian A-module. We may assume that $M \cong A^{\oplus n}/K$, then we have a exact sequence,

$$0 \longrightarrow K \hookrightarrow A^{\oplus n} \twoheadrightarrow A^{\oplus n}/K \longrightarrow 0,$$

since $A^{\oplus n}$ is a Noetherian A-module, $M \cong A^{\oplus n}/K$ is Noetherian, by Proposition 4.6.21. □

Corollary 4.6.3

Any submodule of a finitely generated module over a Noetherian ring is Noetherian module.

Proof Let A be a Noetherian ring, M is a finitely generated A-module, let N be a submodule of M. By Proposition 4.6.20, N is finitely generated A-module, by Proposition 4.6.22, N is Noetherian. □

Why you should not worry about Noetherian hypotheses: Should you work hard to eliminate Noetherian hypotheses? Should you worry about Noetherian hypotheses? Should you stay up at night thinking about non-Noetherian rings? For the most past, the answer to these questions is "no". Most people will never need to worry about non-Noetherian rings, but there are reasons to be open to them. First, they can actually come up.

For example, fibered products of Noetherian schemes over Noetherian schemes (and even fibered products of Noetherian points over Noetherian points!) can be non-Noetherian, and the normalization of Noetherian rings can be non-Noetherian. You can either work hard to show that the rings or schemes you care about don't have this pathology, or you can just relax and not worry about it. Second, there is often no harm in working with schemes in general. Knowing when Noetherian conditions are needed will help you remember why results are true, because you will have some sense of where Noetherian conditions enter into arguments. Finally, for some people, non-Noetherian rings naturally come up. For example, adeles are not Noetherian. And many valuation rings that naturally arise in arithmetic and tropical geometry are not Noetherian.

4.7 The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

We now introduce a notion that is in some sense "inverse" to the vanishing set function $V(\cdot)$.

Definition 4.7.1 (Function $I(\cdot)$)

Given a subset $S \subseteq \operatorname{Spec} A$, $I(S)$ is the set of functions vanishing on S. In other words,

$$I(S) = \{f \in A : f \in \mathfrak{p}, \forall [\mathfrak{p}] \in S\} = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} \subseteq A.$$

We make three quick observations.

- $I(S)$ is clearly an ideal of A.
- $I(\cdot)$ is inclusion-reversing: if $S_1 \subseteq S_2$, then $I(S_1) \supseteq I(S_2)$.

  Proof Let $f \in I(S_2)$, then $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in S_2$. Since $S_1 \subseteq S_2$, $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in S_1$, i.e., $f \in I(S_1)$. Hence, $I(S_2) \subseteq I(S_1)$. □
- $I(\overline{S}) = I(S)$.

  Proof In fact, $S \subseteq \overline{S}$, then $I(S) \supseteq I(\overline{S})$. Since $\overline{S}$ is the smallest closed subset of $\operatorname{Spec} A$ which contains S, we have $\overline{S} = \bigcap_{V(I) \supseteq S} V(I)$. Hence,

  $$I(\overline{S}) = I\left(\bigcap_{V(I) \supseteq S} V(I)\right) = \left\{ f \in A : f \in \mathfrak{p}, \forall [\mathfrak{p}] \in \bigcap_{V(I) \supseteq S} V(I) \right\}$$

  Let $f \in I(S)$, then $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in S$. Hence, closed subset $V(f) \supseteq S$, and therefore $\bigcap_{V(I) \supseteq S} V(I) \subseteq V(f)$. It follows that f vanished at $\bigcap_{V(I) \supseteq S} V(I)$, and therefore $f \in I(\overline{S})$. Thus, $I(S) = I(\overline{S})$. □

✍ **Exercise 4.21** Let $A = k[x, y]$. If $S = \{[(y)], [(x, y-1)]\}$ (see Figure 4.12), then $I(S)$ consists of those polynomials vanishing on the x-axis, and at the point $(0, 1)$. Give generators for this ideal.

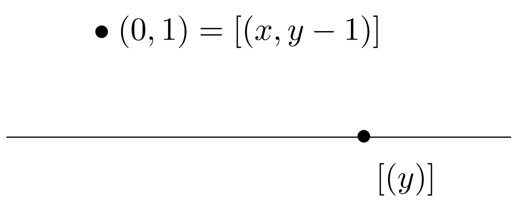


Figure 4.12: The set S of Exercise 4.21, pictured as a subset of $\mathbb{A}^2$

Proof By definition, $I(S) = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} = (y) \cap (x, y-1)$. Let $f \in I(S)$, then $f \in (y) \cap (x, y-1)$, i.e. f vanishing on points $[(y)]$ and $[(x, y-1)]$, i.e., f vanishing on the x-axis and at the point $(0, 1)$. Note that

$(y) + (x, y-1) = (1)$, $(y) \cap (x, y-1) = (y)(x, y-1) = (xy, y^2 - y)$. Hence, the generators of $I(S)$ is xy and $y^2 - y$. □

✍ **Exercise 4.22** Suppose $S \subseteq \mathbb{A}^3_{\mathbb{C}}$ is the union of the three axes. Give generators for the ideal $I(S)$. Be sure to prove it!

Proof $S \subseteq \mathbb{A}^3_{\mathbb{C}}$ is the union of the three axes, hence, $S = V((x,y)) \cup V((y,z)) \cup V((x,z))$. Then $I(S) = (x,y) \cap (y,z) \cap (x,z)$. We claim that $I(S) = (xy, xz, yz)$. Note that $xy, xz, yz \in (x,y) \cap (y,z) \cap (x,z)$, we have $(xy, xz, yz) \subseteq (x,y) \cap (y,z) \cap (x,z)$. Let $f \in (x,y) \cap (y,z) \cap (x,z)$. Hence, if the monomial of f has x, then this monomial must have y or z; if the monomial of f has y, then this monomial must have x or z; if the monomial of f has z, then this monomial must have x or y. This implies that f generated by xy, xz, and yz, and therefore $f \in (xy, xz, yz)$. Hence, $I(S) = (xy, xz, yz)$. □

Remark We will see that this ideal is not generated by less than three elements.

Proposition 4.7.1

Let A be a ring, and $S \subseteq \operatorname{Spec} A$, then $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

Proof Let $[\mathfrak{p}] \in S$, then $f \in \mathfrak{p}$ for all $f \in I(S)$, hence, $\mathfrak{p} \supseteq I(S)$, and therefore $S \subseteq V(I(S))$. Since $V(I(S))$ is closed, we have $\overline{S} \subseteq V(I(S))$.

Let $V(J)$ be any closed subset which contains S, then for all $[\mathfrak{p}] \in S$ we have $\mathfrak{p} \supseteq J$. It follows that

$$J \subseteq \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} = I(S),$$

and therefore $V(J) \supseteq V(I(S))$. Hence,

$$\overline{S} = \bigcap_{V(J) \supseteq S} V(J) \supseteq V(I(S)),$$

which implies that $V(I(S)) = \overline{S}$. □

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then $f^n \in I(S)$ for some n, this implies that f^n vanishes on S for some n, hence, f vanishes on S, so $f \in I(S)$. Hence, $I(S) = \sqrt{I(S)}$.

Proposition 4.7.2

If $J \subseteq A$ is an ideal, then $I(V(J)) = \sqrt{J}$.

Proof $f \in I(V(J))$ if and only if f vanishes on $V(J)$ if and only if $f \in \sqrt{J}$, since $\sqrt{J}$ is the intersection of prime ideals which contains J. □

Proposition 4.7.1 and Proposition 4.7.2 show that V and I are "almost" inverse. More precisely:

Theorem 4.7.1 (Hilbert's Nullstellensatz)

$V(\cdot)$ and $I(\cdot)$ give an inclusion-reversing bijection between closed subsets of Spec A *and radical ideals of A (where a closed subset gives a radical ideal by $I(\cdot)$, and a radical ideal gives a closed subset by $V(\cdot)$).*

✍ **Exercise 4.23** Let $J = (x^2 + y^2 - 1, y - 1)$. Find, with proof, an element of $I(V(J)) \setminus J$.

Proof By Hilbert's Nullstellensatz, $I(V(J)) = \sqrt{J} = \sqrt{(x^2+y^2-1, y-1)}$, we claim that $\sqrt{(x^2+y^2-1, y-1)} = (x, y-1)$. Clearly, $y - 1 \in (x^2+y^2-1, y-1)$, and therefore $y-1 \in \sqrt{(x^2+y^2-1, y-1)}$. Note that

$$x^2 + y^2 - 1 = x^2 + (y-1)^2 + 2(y-1),$$

$x^2 \in (x^2+y^2-1, y-1)$, hence, $(x, y-1) \subseteq \sqrt{(x^2+y^2-1, y-1)}$. Since $(x, y-1)$ is maximal ideal in

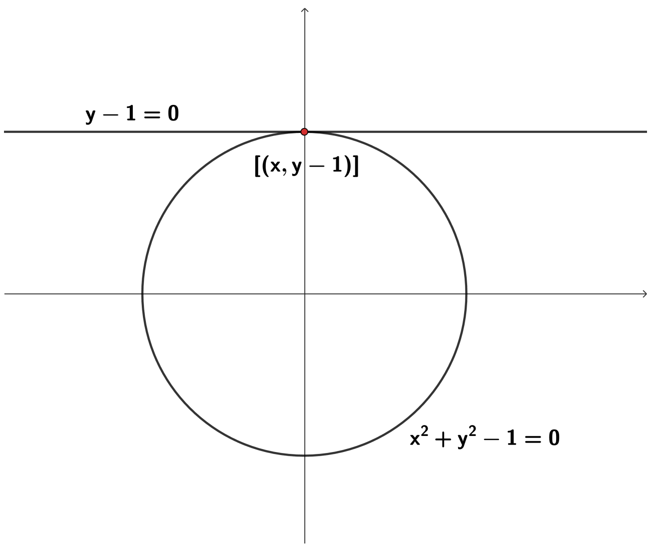


Figure 4.13: Picture of Exercise 4.23

$k[x,y]$, then $(x,y-1)=\sqrt{(x^2+y^2-1,y-1)}$.

Note that $x\notin(x^2+y^2-1,y-1)$, then we find an element $x\in I(V(J))$, but $x\notin J$, as we desired. $\square$

Proposition 4.7.3

$V(\cdot)$ and $I(\cdot)$ give a bijection between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals of A. Hence, in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$

Proof Let $V(\mathfrak{a})$ be an irreducible closed subsets of $\operatorname{Spec} A$, then $I(V(\mathfrak{a}))=\sqrt{\mathfrak{a}}$. Note that $I(V(\mathfrak{a}))=\bigcap_{[\mathfrak{p}]\in V(\mathfrak{a})}\mathfrak{p}$ is the intersection of prime ideals, therefore, $\sqrt{\mathfrak{a}}$ is a prime ideal.

Conversely, let $\mathfrak{p}$ be a prime ideal, consider $V(\mathfrak{p})$. Note that $V(\mathfrak{p})=\{[\mathfrak{q}]\in\operatorname{Spec} A:\mathfrak{q}\supseteq\mathfrak{p}\}=\operatorname{Spec} A/\mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is integral domain, hence, $\operatorname{Spec} A/\mathfrak{p}=V(\mathfrak{p})$ is irreducible closed subset of $\operatorname{Spec} A$. $\square$

Remark From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset of $\operatorname{Spec} A$ has precisely one generic point — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

Definition 4.7.2 (Minimal prime ideal)

*A prime ideal of a ring A is a **minimal prime ideal** (or more simply, **minimal prime**) if it is minimal with respect to inclusion.*

Example 4.11 The only minimal prime ideal of $k[x,y]$ is (0).

Proposition 4.7.4

If A is any ring, then the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal prime ideals of A. In particular, $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime ideal.

Proof Let Z be an irreducible component of $\operatorname{Spec} A$, then Z is an irreducible closed subset of $\operatorname{Spec} A$. By Proposition 4.7.3, $I(Z)$ is a prime ideal. If $I(Z)$ is not minimal prime ideal, there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}\subsetneq I(Z)$. By Proposition 4.7.3 again, $V(\mathfrak{p})(\supsetneq V(I(Z))=Z)$ is an irreducible closed subset, contradicts to the fact that Z is irreducible component. Hence, $I(Z)$ is the minimal prime ideal.

Conversely, if $\mathfrak{p}$ is a minimal prime ideal, then $V(\mathfrak{p})$ is an irreducible closed subset. If $V(\mathfrak{p})$ is not irreducible component, then there exists an irreducible closed subset $V(\mathfrak{a})$ such that $V(\mathfrak{a}) \supsetneq V(\mathfrak{p})$, by Proposition 4.7.3, we have prime ideal $\sqrt{a}(\subsetneq \mathfrak{p})$, contradicts to the fact that $\mathfrak{p}$ is a minimal prime. Hence, $V(\mathfrak{p})$ is an irreducible component of Spec A.

In particular, if Spec A is irreducible, then $I(\operatorname{Spec} A) = \bigcap_{[\mathfrak{p}]\in \operatorname{Spec} A} \mathfrak{p} = \mathfrak{N}(A)$ is prime ideal. Since it is contained in every prime ideal of A, $\mathfrak{N}(A)$ is the unique minimal ideal of A. Conversely, if A has only one minimal prime ideal, say $\mathfrak{a}$, then $\mathfrak{a}$ contains in all prime ideals of A. Hence, $\mathfrak{a} = \mathfrak{N}(A)$, and therefore $V(\mathfrak{a}) = V(\mathfrak{N}(A)) = \operatorname{Spec} A$ is irreducible. □

Proposition 4.6.12, Proposition 4.6.18, and Proposition 4.7.4 imply that every Noetherian ring has a finite number of minimal prime ideals. An algebraic fact is now revealed to be really a "geometric" fact!

Exercise 4.24 In $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$, the subset cut out by $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ should certainly have irreducible components corresponding to the distinct irreducible factor of f. Prove this.

Proof Let $f = ap_1^{e1}p_2^{e2}\cdots p_n^{e_n}$, where p_i is irreducible polynomial and $a \in k$. Consider $V((f))$. Note that $(f) = (p_1^{e_1}p_2^{e_2}\cdots p_n^{e_n}) = (p_1^{e_1})(p_2^{e_2})\cdots(p_n^{e_n})$. Hence,

$$\begin{aligned} V((f)) = \operatorname{Spec} k[x_1, \ldots, x_n]/(f) &= V((p_1^{e_1})) \cup V((p_2^{e_2})) \cup \cdots \cup V((p_n^{e_n})) \\ &= V(\sqrt{(p_1^{e_1})}) \cup \cdots \cup V(\sqrt{(p_n^{e_n})}) \\ &= V((p_1)) \cup \cdots \cup V((p_n)). \end{aligned}$$

We claim that each (p_i) is the minimal prime ideal in $\operatorname{Spec} k[x_1, \ldots x_n]/(f)$. If there exists prime ideal $\mathfrak{p} \subseteq k[x_1, \ldots x_n]/(f)$ such that $\mathfrak{p} \subseteq (p_i)$. Let $q \in \mathfrak{p}$ be an irreducible polynomial, then $p_i \mid q$, since p_i and q are irreducible polynomials, $q = p_i$, hence, $(p_i) = (q) \subseteq \mathfrak{p}$, and therefore $(p_i) = \mathfrak{p}$. Hence, each (p_i) is minimal ideal, by Proposition 4.7.4, $V((p_i))$ is an irreducible component. □

Exercise 4.25 What are the minimal prime ideals of $k[x, y]/(xy)$, where k is a field?

Solution *By Exercise 4.24, (x) and (y) are the minimal prime ideals.*

Beginning of a grand dictionary between algebra and geometry. We are now well on our away to building a grand dictionary between algebra and geometry.

Algebra	Geometry
Ring A	Affine scheme Spec A
$\mathfrak{p} = I(\mathfrak{p})$ prime ideal of A	$\mathfrak{p} = [\mathfrak{p}]$ point of Spec A
Element $f \in A$	Function f on Spec A
$f \pmod{\mathfrak{p}}$	$f(\mathfrak{p})$; value of the function f at $\mathfrak{p}$
$f \in \mathfrak{p}$	The function f vanishes (is zero) at $\mathfrak{p}$
Nilradical ideal $\mathfrak{N} = \sqrt{0} = \bigcap_{[\mathfrak{p}]\in \operatorname{Spec} A} \mathfrak{p}$ of A	Functions vanishing at every point
Maximal ideal of A	Irreducible closed subset of Spec A that is a point (=closed point of Spec A)
$\mathfrak{p} = I(Z)$ prime ideal of A	Irreducible closed subset $Z = V(\mathfrak{p})$ of Spec A
Minimal prime ideal of A	Irreducible component of Spec A
Radical ideal $I = \sqrt{I} = I(S)$ of A	Closed subset $S = V(I)$ of Spec A

Table 4.1: The correspondence between algebra and geometry.

Chapter 5 The structure sheaf, and the definition of schemes in general

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathscr{O}_{\operatorname{Spec} A}$, which we think of as the "sheaf of algebraic functions". You should keep in your mind the example of "algebraic functions" on $\mathbb{C}^n$, which you understand well. For example, for the plane $\mathbb{A}^2$, we expect that on the open set $D(xy)$ (away from the two axes), $(3x^4 + y + 4)/x^7y^3$ should be an algebraic function.

These functions will have values at points, but won't be determined by their values at points. But like all sections of sheaves, they will be determined by their germs.

5.1 The structure sheaf of an affine scheme

5.1.1 Definition of structure sheaf

We want now to give a ring of functions on any open set, but we know that it suffices to describe the functions on a base of the topology. We will see that the base of distinguished open sets will make things much more tractable than you might fear. So we now describe that structure sheaf as a sheaf of rings on the base of distinguished open sets (Theorem 3.5.1 and Proposition 4.5.1).

Definition 5.1.1 (Functions on distinguished open set)

Define $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ to be the localization of A at the multiplicative set of all functions that do not vanish outside of $V(f)$ (i.e., those $g \in A$ such that $V(g) \subseteq V(f)$, or equivalently $D(f) \subseteq D(g)$.) Formally,

$$\mathscr{O}_{\operatorname{Spec} A}(D(f)) := S^{-1}A,$$

where $S = \{g \in A : V(g) \subseteq V(f)\} = \{g \in A : D(g) \supseteq D(f)\}$.

Remark

(i) This definition depends only on $D(f)$, and not on f itself.

(ii) $\mathscr{O}_{\operatorname{Spec} A}(\varnothing) = \{0\}$.

Proof The multiplicative set $S = \{g \in A : D(g) \supseteq \varnothing\} = A$. By Proposition 4.5.6, $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}(A)$. Hence, $f^n = 0$ for some n, hence, $0 \in S$. It follows that $S^{-1}A = \{0\}$, i.e., $\mathscr{O}_{\operatorname{Spec} A}(\varnothing) = \{0\}$. □

Proposition 5.1.1

If A is a ring, $f \in A$, then there is an isomorphism of rings

$$\mathscr{O}_{\operatorname{Spec} A}(D(f)) \cong A_f.$$

In particular, if $f = 1$, then $\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \cong A$.

Proof Say $\mathscr{O}_{\operatorname{Spec} A}(D(f)) = S^{-1}A$, where $S = \{g \in A : D(g) \supseteq D(f)\}$. Define $\varphi : A \to S^{-1}A$ by setting $f \mapsto f/1$, clearly, it is a well-defined ring homomorphism. Let $s \in \{1, f, f^2, \dots\}$, we want to show that $\varphi(s)$ is the unit in $S^{-1}A$. We may assume $s = f^n$ for some n, then $\varphi(s) = f^n/1 = f^n$. Note that $D(f^n) \supseteq D(f)$, we have $f^n \in S$, i.e., $\varphi(s)$ is the unit in $S^{-1}A$. Consider the following diagram,

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & S^{-1}A \\ A_f, \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

where $\psi : A \to A_f$ is the canonical ring homomorphism, by the universal property of localization, there exists unique ring homomorphism $\theta : A_f \to S^{-1}A$ such that above diagram commutes.

Next, we want to show that $\psi(s)$ is the unit in A_f, where $s \in S$. Let $s \in S$, then $D(s) \supseteq D(f)$, by Proposition 4.5.5, s is an invertible element of A_f. Consider the following diagram,

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi"'] & A_f \\ S^{-1}A, \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

by the universal property of localization, there exists unique ring homomorphism $\eta : S^{-1}A \to A_f$. Hence, we have the isomorphism of rings

$$\mathscr{O}_{\operatorname{Spec} A}(D(f)) = S^{-1}A \cong A_f.$$

□

Definition 5.1.2 (The restriction map)

*If $D(f') \subseteq D(f)$, define **the restriction map***

$$\operatorname{res}_{D(f),D(f')} : \mathscr{O}_{\operatorname{Spec} A}(D(f)) \longrightarrow \mathscr{O}_{\operatorname{Spec} A}(D(f'))$$

in the obvious way: the latter ring is a further localization of the former ring.

The restriction maps obviously compose: this is a "presheaf on the distinguished base".

Theorem 5.1.1

The data just described give a sheaf on the distinguished base, and hence determine a sheaf on the topological space $\operatorname{Spec} A$.

Definition 5.1.3 (Structure sheaf)

In Theorem 5.1.1, the sheaf on the topological space $\operatorname{Spec} A$ *is called the **structure sheaf**, and will be denoted $\mathscr{O}_{\operatorname{Spec} A}$, or sometimes $\mathscr{O}$ if the subscript is clear from the context.*

The notation $\operatorname{Spec} A$ *will hereafter denote the data of a topological space with a structure sheaf. Such a topological space, with structure sheaf, will be called an **affine scheme**.*

Remark We continue to use the language of ringed space: functions on open sets, global functions, and so forth. An important lesson of Theorem 5.1.1 is not just that $\mathscr{O}_{\operatorname{Spec} A}$ is a sheaf, but also that the distinguished base provides a good way of working with $\mathscr{O}_{\operatorname{Spec} A}$. Notice also that we have justified interpreting elements of A as functions on $\operatorname{Spec} A$.

Now, let's prove Theorem 5.1.1.

Proof We must show that the base identity and base gluability axioms hold (Definition 3.5.2). Suppose $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$, or equivalently (Proposition 4.5.2) the ideal generated by the f_i is the entire ring A.

(Experts familiar with the equalizer exact sequence of Example 3.3 will realize that we are showing exactness of

$$0 \longrightarrow A \longrightarrow \prod_{i \in I} A_{f_i} \longrightarrow \prod_{i \neq j \in I} A_{f_i f_j}$$

where $\{f_i\}_{i \in I}$ is a set of functions with $(f_i)_{i \in I} = A$. Signs are involved in the right-hand map: the map $A_{f_i} \to A_{f_i f_j}$ is the localization map, and the map $A_{f_j} \to A_{f_i f_j}$ is the negative of the localization map. Base identity corresponds to injectivity at A, and gluability corresponds to exactness at $\prod_i A_{f_i}$.)

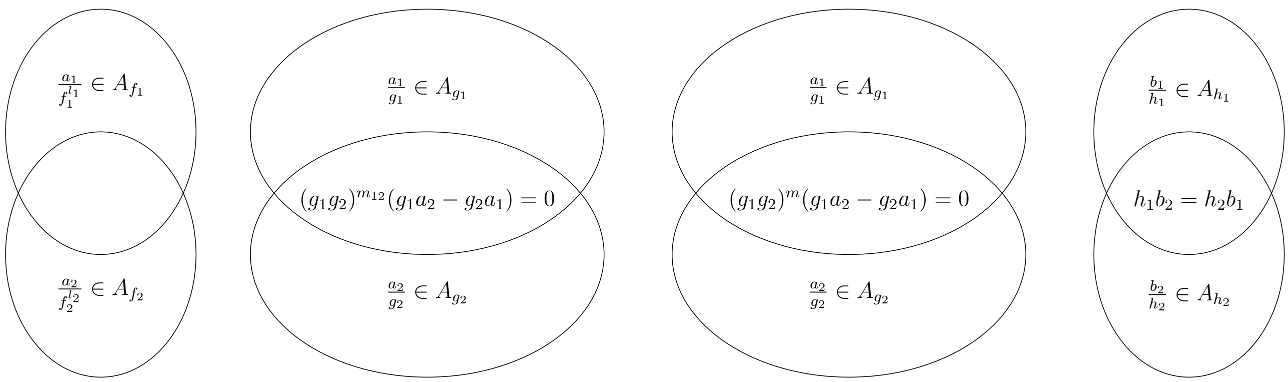
We check identity on the base. Suppose that $D(f) = \operatorname{Spec} A_f = \bigcup_{i \in I} D(f_i)$ where i runs over some index set I. Then there is some finite subset of I, which we name $\{1, 2, \ldots, n\}$, such that $D(f) = \operatorname{Spec} A_f = \bigcup_{i=1}^n D(f_i)$, i.e., $(f_1, \ldots, f_n) = A_f$ (quasi-compactness of $\operatorname{Spec} A_f$, Proposition 4.6.4; each f_i is the element in A_f). Suppose we are given $s \in A_f$ such that $\operatorname{res}_{D(f), D(f_i)} s = 0$ in $A_{f_i} (\cong \mathscr{O}_{\operatorname{Spec} A}(D(f_i)))$ for all i. We want to show that $s = 0$. The fact that $\operatorname{res}_{D(f), D(f_i)} s = 0$ in A_{f_i} implies that there is some m such that for each $i \in \{1, \ldots, n\}$, $f_i^m s = 0$. Since $\operatorname{Spec} A_f = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n D(f_i^m)$ (Proposition 4.5.4), we have $A_f = (f_1^m, f_2^m, \ldots, f_n^m)$, so there are $r_i \in A_f$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A_f, form which

$$s = \left(\sum r_i f_i^m \right) s = \sum r_i (f_i^m s) = 0.$$

Thus we have checked the base identity axiom.

We next show base gluability. (Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.) Suppose $D(f) = \bigcup_{i \in I} D(f_i)$, where I is an index set (possibly horribly infinite). Suppose we are given elements in each A_{f_i} that agree on the overlaps $A_{f_i f_j} (= \mathscr{O}_{\operatorname{Spec} A}(D(f_i f_j)) = \mathscr{O}_{\operatorname{Spec} A}(D(f_i) \cap D(f_j)))$ (Proposition 4.5.4).

Assume first that I is finite, say $I = \{1, \ldots, n\}$. We have elements $a_i / f_i^{l_i} \in A_{f_i}$ agreeing on overlaps $A_{f_i f_j}$ (see Figure 5.1(a)). Let $g_i = f_i^{l_i}$, note that $D(f_i) = D(f_i^m) = D(g_i)$, we can simplify notion by considering our elements as of the form $a_i / g_i \in A_{g_i}$ (Figure 5.1(b)).



(a) (b) (c) (d)

Figure 5.1: Base gluability of the structure sheaf

The fact that a_i / g_i and a_j / g_j "agree on the overlap" (i.e., in $A_{g_i g_j}$) means that for some m_{ij},

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

in A. By taking $m = \max m_{ij}$ (here we use the finiteness of I), we can simplify notation:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

for all i, j (Figure 5.1(c)). Let $b_i = a_i g_i^m$ for all i, and $h_i = g_i^{m+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more (Figure 5.1(d)): on each $D(h_i)$, we have a function b_i / h_i, and the overlap condition is

$$h_j b_i = h_i b_j. \tag{5.1}$$

Now $\bigcup_i D(h_i) = \operatorname{Spec} A_f$, implies that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A_f$. Define

$$r = \sum r_i b_i. \tag{5.2}$$

This will be the element of A_f that restricts to each b_j/h_j. Indeed, from the overlap condition (5.1),

$$r h_j = \sum_i r_i b_i h_j = \sum_i r_i h_i b_j = b_j.$$

We next deal with the case where I is infinite. Choose a finite subset $\{1, \ldots, n\} \subseteq I$ with $(f_1, \ldots, f_n) = A_f$ (or equivalently, use quasi-compactness of $\operatorname{Spec} A_f$ to choose a finite subcover by $D(f_i)$). Construct r as above, using (5.2). We will show that for any $z \in I - \{1, \ldots, n\}$, r restricts to the desired element $a_z/f_z^{l_z}$ of A_{f_z}. Repeat the entire process above with $\{1, \ldots, n, z\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in A_f$ which restricts to $a_i/f_i^{l_i}$ for $i \in \{1, \ldots, n, z\}$. Then by the base identity axiom, $r' = r$. (Note that we use base identity to prove base gluability. This is an example of how the identity axiom is somehow "prior" to the gluability axiom.) Hence, r restricts to the desired element $a_z/f_z^{l_z}$ of A_{f_z}.

By Definition 3.5.2, $\mathscr{O}_{\operatorname{Spec} A}$ is a sheaf on a base, by Theorem 3.5.1, $\mathscr{O}_{\operatorname{Spec} A}$ extend to a sheaf on topological space $\operatorname{Spec} A$. □

Remark Definition 5.1.1 and Theorem 5.1.1 suggests a potentially slick way of describing sections of $\mathscr{O}_{\operatorname{Spec} A}$ over any open subset: perhaps $\mathscr{O}_{\operatorname{Spec} A}(U)$ is the localization of A at the multiplicative set of all function that do not vanish at any point of U. This is not true. A counterexample (that you will later be able to make precise): let $\operatorname{Spec} A$ be two copies of $\mathbb{A}_k^2$ glued together at their origins and let U be the complement of the origin(s). Then the function which is 1 on the first copy of $\mathbb{A}_k^2 \setminus \{(0,0)\}$ and 0 on the second copy of $\mathbb{A}_k^2 \setminus \{(0,0)\}$ is not of this form.

5.1.2 $\mathscr{O}_{\operatorname{Spec} A}$-modules coming from A-modules

The following generalization of Theorem 5.1.1 will be essential in the definition of quasi-coherent sheaf in Chapter 7.

Definition 5.1.4 ($\mathscr{O}_{\operatorname{Spec} A}$-module)

Suppose M is an A-module. Define $\widetilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$, i.e.,

$$\widetilde{M}(D(f)) := S^{-1}M,$$

where $S = \{g \in A : V(g) \subseteq V(f)\} = \{g \in A : D(g) \supseteq D(f)\}$.
Define restriction maps $\operatorname{res}_{D(f),D(g)}$ in the analogous way to $\mathscr{O}_{\operatorname{Spec} A}$. This defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec} A$. Also, $\widetilde{M}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module.

Proof By Atiyah-Macdonald [1] Proposition 3.5, we have $\widetilde{M}(D(f)) = S^{-1}M \cong S^{-1}A \otimes_A M$, by Proposition 5.1.1, $S^{-1}A \cong A_f$, hence,

$$\widetilde{M}(D(f)) \cong A_f \otimes_A M \cong M_f.$$

We check identity on the base. Suppose that $D(f) = \operatorname{Spec} A_f = \bigcup_{i \in I} D(f_i)$, where I is an index set. Since $\operatorname{Spec} A_f$ is quasi-compact (Proposition 4.6.4), we may assume that $D(f) = \bigcup_{i=1}^n D(f_i)$, i.e., $(f_1, \ldots, f_n) = A_f$. Suppose we are given $s \in \widetilde{M}(D(f)) = M_f$ such that $\operatorname{res}_{D(f),D(f_i)} s = 0$ in $\widetilde{M}(D(f_i)) = M_{f_i}$ for all i. We want to show that $s = 0$. The fact that $\operatorname{res}_{D(f),D(f_i)} s = 0$ in M_{f_i}, implies that there is some m such that for each $i \in \{1, \ldots, n\}$, $f_i^m s = 0$. Since $\operatorname{Spec} A_f = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n D(f_i^m)$ (Proposition 4.5.4), we have $A_f = (f_1^m, \ldots, f_n^m)$, so there are $r_i \in A_f$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A_f, since M_f is A_f-module,

form which
$$s = \left(\sum r_i f_i^m\right) s = \sum r_i (f_i^m s) = 0.$$
Thus we have checked the base identity axiom.

We next show base gluability. Suppose $D(f) = \bigcup_{i \in I} D(f_i)$, where I is an index set. Suppose we are given elements in each $\widetilde{M}(D(f_i)) = M_{f_i}$ that agree on the overlaps $\widetilde{M}(D(f_i) \cap D(f_j)) = \widetilde{M}(D(f_i f_j)) = M_{f_i f_j}$.

Assume first that I is finite, say $I = \{1, \ldots, n\}$. We have elements $m_i / f_i^{l_i} \in M_{f_i}$, agreeing on overlaps $M_{f_i f_j}$. Let $g_i = f_i^{l_i}$, note that $D(f_i) = D(f_i^{l_i}) = D(g_i)$, we can simplify notion by considering our elements as of the form $m_i / g_i \in M_{g_i}$. The fact that m_i / g_i and m_j / g_j agree on the overlap (i.e., in $M_{g_i g_j}$) means that for some k_{ij},
$$(g_i g_j)^{k_{ij}} (g_j m_i - g_i m_j) = 0$$
in M. By taking $k = \max k_{ij}$ (we use the finiteness of I), we can simplify notation:
$$(g_i g_j)^k (g_j m_i - g_i m_j) = 0$$
for all i, j. Let $n_i = m_i g_i^k$ for all i, and $h_i = g_i^{k+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more: on each $D(h_i)$, we have a function n_i / h_i, and the overlap condition is
$$h_j n_i = h_i n_j. \tag{5.3}$$
Now $\bigcup_i D(h_i) = \operatorname{Spec} A_f$, implies that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A_f$. Define
$$r = \sum r_i n_i. \tag{5.4}$$
Note that, from the overlap condition (5.3),
$$r h_j = \sum_i r_i n_i h_j = \sum_i r_i h_i n_j = n_j,$$
which implies that
$$\operatorname{res}_{D(f), D(f_i)} r = n_j / h_j = m_j / f_j^{l_j}.$$

We next deal with the case where I is infinite. Choose a finite subset $\{1, \ldots, n\} \subseteq I$ with $(f_1, \ldots, f_n) = A_f$. Construct r as above, using (5.4). We will show that for any $z \in I - \{1, \ldots, n\}$, r restricts to the desired element $m_z / f_z^{l_z}$ of M_{f_z}. Repeat the entire process above with $\{1, \ldots, n, z\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in M_f$ which restricts to $m_i / f_i^{l_i}$ for $i \in \{1, \ldots, n, z\}$. Then by the base identity axiom, $r' = r$. Hence, r restricts to the desired element $m_z / f_z^{l_z}$ of M_{f_z}.

By Definition 3.5.2, $\widetilde{M}$ is a sheaf on a base, by Theorem 3.5.1, $\widetilde{M}$ extend to a sheaf on topological space $\operatorname{Spec} A$.

To show that $\widetilde{M}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module, it suffices to show that $\widetilde{M}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module on distinguished base. Note that $\widetilde{M}(D(f)) \cong M_f$ is A_f-module, it suffices to check the diagram (3.1) in Definition 3.2.13 commutes. Let $D(f) \supseteq D(g)$, consider the following diagram,
$$\begin{tikzcd}
\mathscr{O}_{\operatorname{Spec} A}(D(f)) \times \widetilde{M}(D(f)) \arrow[r, "\text{action}"] \arrow[d, "\operatorname{res}^{\mathscr{O}_{\operatorname{Spec} A}}_{D(f),D(g)} \times \operatorname{res}^{\widetilde{M}}_{D(f),D(g)}"'] & \widetilde{M}(D(f)) \arrow[d, "\operatorname{res}^{\widetilde{M}}_{D(f),D(g)}"] \\
\mathscr{O}_{\operatorname{Spec} A}(D(g)) \times \widetilde{M}(D(g)) \arrow[r, "\text{action}"] & \widetilde{M}(D(g)),
\end{tikzcd}$$
since $\mathscr{O}_{\operatorname{Spec} A}(D(h)) \cong A_h$ and $\widetilde{M}(D(h)) \cong A_h$ for any $h \in A$, it is equal to check that following diagram

commutes.

$$\begin{array}{ccc} A_f \times M_f & \xrightarrow{\text{action}} & M_f \\ {\scriptstyle \mathrm{res}^{\mathscr{O}_{\operatorname{Spec} A}}_{D(f),D(g)} \times \mathrm{res}^{\widetilde{M}}_{D(f),D(g)}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}^{\widetilde{M}}_{D(f),D(g)}} \\ A_g \times M_g & \xrightarrow{\text{action}} & M_g. \end{array}$$

Let $(a/f^l, m/f^k) \in A_f \times M_f$. Since $D(f) \supseteq D(g)$, by Proposition 4.5.5, f is an invertible element of A_g, hence,

$$\mathrm{res}^{\mathscr{O}_{\operatorname{Spec} A}}_{D(f),D(g)} \times \mathrm{res}^{\widetilde{M}}_{D(f),D(g)}(a/f^l, m/f^k) = (a/f^l, m/f^k)$$

and

$$\mathrm{res}^{\widetilde{M}}_{D(f),D(g)}(am/f^{l+k}) = am/f^{l+k},$$

which implies above diagram commutes. Hence, $\widetilde{M}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module on distinguished base, and therefore $\widetilde{M}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module. $\square$

Remark In the course of the proof in Definition 5.1.4, we show that if $(f_i)_{i\in I} \in A$,

$$0 \longrightarrow M \longrightarrow \prod_{i\in I} M_{f_i} \longrightarrow \prod_{i\neq j\in I} M_{f_i f_j} \tag{5.5}$$

is exact. In particular, M can be identified with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ is an inclusion. This will be useful later: we will want to show that if $(f_1, \ldots, f_n) = A$, and M_{f_i} have this property, then M does too. This ideal will be made precise in the Affine Communication Lemma in Chapter 6.

Note *By the proof in Theorem 3.5.1, we can provide precise descriptions of both $\mathscr{O}_{\operatorname{Spec} A}(U)$ and $\widetilde{M}(U)$, for $U \subseteq \operatorname{Spec} A$ open subset:*

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} A}(U) =& \{(\varphi_{[\mathfrak{p}]} \in \mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]})_{[\mathfrak{p}]\in U} : \\ & \forall [\mathfrak{p}] \in U, \exists D(f) \textit{ with } [\mathfrak{p}] \in D(f) \subseteq U \textit{ and } s \in A_f \textit{ s.t. } s_{[\mathfrak{q}]} = \varphi_{[\mathfrak{q}]}, \forall [\mathfrak{q}] \in D(f)\}, \end{aligned} \tag{5.6}$$

where $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = \varinjlim_{D(h)\ni[\mathfrak{p}]} \mathscr{O}_{\operatorname{Spec} A}(D(f)) = \varinjlim_{h\in A-\mathfrak{p}} A_h$,

$$\begin{aligned} \widetilde{M}(U) =& \{(\varphi_{[\mathfrak{p}]} \in \widetilde{M}_{[\mathfrak{p}]})_{[\mathfrak{p}]\in U} : \\ & \forall [\mathfrak{p}] \in U, \exists D(f) \textit{ with } [\mathfrak{p}] \in D(f) \subseteq U \textit{ and } s \in M_f \textit{ s.t. } s_{[\mathfrak{q}]} = \varphi_{[\mathfrak{q}]}, \forall [\mathfrak{q}] \in D(f)\}, \end{aligned} \tag{5.7}$$

where $\widetilde{M}_{[\mathfrak{p}]} = \varinjlim_{D(h)\ni[\mathfrak{p}]} \widetilde{M}(D(h)) = \varinjlim_{h\in A-\mathfrak{p}} M_h$.

Proposition 5.1.2

Suppose $\mathfrak{p}$ is a prime ideal of A. Then there is a canonical isomorphism

$$\widetilde{M}_{[\mathfrak{p}]} \xleftrightarrow{\sim} M_{\mathfrak{p}}$$

between the stalk of $\widetilde{M}$ and the localization of M. The stalk of $\mathscr{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the local ring $A_{\mathfrak{p}}$.

Proof In fact, $\widetilde{M}_{[\mathfrak{p}]} = \varinjlim_{D(h)\ni[\mathfrak{p}]} \widetilde{M}(D(h))$. Define $\theta : \varinjlim_{D(h)\ni[\mathfrak{p}]} \widetilde{M}(D(h)) \to M_{\mathfrak{p}}$ by setting

$$\overline{(m/f^k, D(f))} \mapsto m/f^k.$$

We shall to check that θ is well-defined $A_{\mathfrak{p}}$-map. If $\overline{(m/f^k, D(f))} = \overline{(n/g^l, D(g))}$, then $m/f^k = n/g^l$ on some $([\mathfrak{p}] \in)D(h) \subseteq D(f) \cap D(g)$. Since $\widetilde{M}(D(h)) \cong M_h$, $h^t(mg^l - nf^k) = 0$. Note that $h \notin \mathfrak{p}$, $h^t \notin \mathfrak{p}$, we have $m/f^k = n/g^i$ in $M_{\mathfrak{p}}$. Hence, θ is well-defined $A_{\mathfrak{p}}$-module homomorphism.

(a) Surjective.

Let $m/f \in M_\mathfrak{p}$, then $f \notin \mathfrak{p}$. Note that $\theta(\overline{(m/f, D(f))}) = m/f$, θ is surjective.

(b) Injective.

Let $m/f^k \in M_\mathfrak{p}$ such that $m/f^k = 0$. Note that $m/f^k \in \widetilde{M}(D(f))$, hence $\overline{(m/f^k, D(f))} = 0$. Thus θ is injective.

Hecne, we have the canonical isomorphism,

$$\widetilde{M}_{[\mathfrak{p}]} \xleftrightarrow{\sim} M_\mathfrak{p}.$$

Then we done.

In particular, $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]}$ is $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]}$-module itself, hence, we have isomorphism,

$$\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} \xleftrightarrow{\sim} A_\mathfrak{p}.$$

□

The following exercise shows how an important result can be understood quite differently from algebraic and geometric perspective.

Exercise 5.1 Suppose M is an A-module. Show that the natural map

$$M \longrightarrow \textstyle\prod_{\mathfrak{p}\in\operatorname{Spec} A} M_\mathfrak{p}$$

is an injection in two ways:

(a) by considering the kernel of the map, and show that any element of it must be 0.

(b) by applying Proposition 3.4.1 (sections of a sheaf are determined by germs) to the sheaf $\widetilde{M}$.

Thus an A-module is zero if and only if all its localizations at prime ideals are zero, and we see this as a simultaneously (a) algebraic and (b) geometric fact.

Proof

(a) Denote $\iota : M \to \prod_{\mathfrak{p}\in\operatorname{Spec} A} M_\mathfrak{p}$. Let $m \in \operatorname{Ker} \iota$, then $\iota(m) = (m, m, \dots) = (0, 0, \dots)$, where each $m \in M_\mathfrak{p}$. If $m \neq 0$, then $\operatorname{Ann}(m)$ is a proper ideal of A. By Atiyah-MacDonald [1] Corollary 1.4, there exists a maximal ideal of A containing $\operatorname{Ann}(m)$, say $\mathfrak{m}$. Since $m = 0$ in $M_\mathfrak{m}$, there exists $h \in A - \mathfrak{m}$ such that $hm = 0$ in A, i.e., $h \in \operatorname{Ann}(m)$ but $h \notin \mathfrak{m}$, a contradiction. Hence, $m = 0$. It follows that $\operatorname{Ker} \iota = \{0\}$, and therefore $M \hookrightarrow \prod_{\mathfrak{p}\in\operatorname{Spec} A} M_\mathfrak{p}$.

(b) Note that $M = M_1 = \widetilde{M}(\operatorname{Spec} A) = \widetilde{M}(D(1))$ and $\prod_{\mathfrak{p}\in\operatorname{Spec} A} M_\mathfrak{p} = \prod_{\mathfrak{p}\in\operatorname{Spec} A} \widetilde{M}_{[\mathfrak{p}]}$, consider

$$\widetilde{M}(\operatorname{Spec} A) \longrightarrow \prod_{\mathfrak{p}\in\operatorname{Spec} A} \widetilde{M}_{[\mathfrak{p}]}.$$

Apply Proposition 3.4.1, we done.

□

We write above exercise as proposition and corollary:

Proposition 5.1.3

Suppose M is an A-module. The natural map

$$M \longrightarrow \textstyle\prod_{\mathfrak{p}\in\operatorname{Spec} A} M_\mathfrak{p}$$

is an injection.

Corollary 5.1.1

A-module is zero if and only if all its localizations at prime ideals are zero.

Proposition 5.1.4

There is a bijection:

$$\{\text{maps of } A\text{-modules } M \to N\} \longleftrightarrow \left\{\text{maps of } \mathscr{O}_{\operatorname{Spec} A}\text{-modules } \widetilde{M} \to \widetilde{N}\right\}.$$

Fancy translation: $\mathbf{Mod}_A$ *as a full subcategory of* $\mathbf{Mod}_{\mathscr{O}_{\operatorname{Spec} A}}$.

Proof Let $\varphi : M \to N$ be a maps of A-modules. By Theorem 3.5.2, it suffice to consider the morphism of sheaves on distinguished base. Define $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ as follow: let $D(f)$ be distinguished open subset, define $\widetilde{\varphi}_f : M_f \to N_f$ by setting

$$m/f^k \mapsto \varphi(m)/f^k.$$

Clearly, it is a well-defined A_f-map. We need to check $\widetilde{\varphi}$ is a morphism of sheaves on distinguished base. Let $D(f) \supseteq D(g)$, consider following diagram.

$$\begin{tikzcd} M_f \arrow[r, "\widetilde{\varphi}_f"] \arrow[d, hook] & N_f \arrow[d, hook] \\ M_g \arrow[r, "\widetilde{\varphi}_g"] & N_g. \end{tikzcd}$$

(Since $D(f) \subseteq D(g)$, by Proposition 4.5.5, f is an invertible element of A_g, hence, we get the embedding $M_f \hookrightarrow M_g$ and $N_f \hookrightarrow N_g$)

Let $m/f^k \in M_f$, then

$$\widetilde{\varphi}_g(m/f^k) = \varphi(m)/f^k = \widetilde{\varphi}_f(m/f^k).$$

It follows that above diagram commutes, and therefore $\widetilde{\varphi}$ is a morphism of sheaves on distinguished base. By Theorem 3.5.2, $\widetilde{\varphi}$ can extend to a morphism of sheaves. Note that $\widetilde{\varphi}_f$ is $\mathscr{O}_{\operatorname{Spec} A}(D(f))$-module homomorphism, $\widetilde{\varphi}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module homomorphism over distinguished base, by Theorem 3.5.2, extended sheaves morphism is an $\mathscr{O}_{\operatorname{Spec} A}$-module homomorphism. We also denote $\widetilde{\varphi}$ as extended sheaves morphism.

Conversely, let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ be a map of $\mathscr{O}_{\operatorname{Spec} A}$-modules. Consider $\widetilde{\varphi}_1 : \widetilde{M}(\operatorname{Spec} A) \to \widetilde{N}(\operatorname{Spec} A)$, note that $\widetilde{M}(\operatorname{Spec} A) = M$, $\widetilde{N}(\operatorname{Spec} A) = N$, and $\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$, we have A-module homomorphism $\widetilde{\varphi}_1 : M \to N$, define $\varphi = \widetilde{\varphi}_1$.

Define

$$\tau : \{\text{maps of } A\text{-modules } M \to N\} \to \left\{\text{maps of } \mathscr{O}_{\operatorname{Spec} A}\text{-modules } \widetilde{M} \to \widetilde{N}\right\}$$

by setting

$$\varphi \mapsto \widetilde{\varphi}$$

and define inverse

$$\theta : \left\{\text{maps of } \mathscr{O}_{\operatorname{Spec} A}\text{-modules } \widetilde{M} \to \widetilde{N}\right\} \to \{\text{maps of } A\text{-modules } M \to N\}$$

by setting

$$\widetilde{\varphi} \mapsto \widetilde{\varphi}_{D(1)}.$$

Note that

$$\theta \circ \tau(\varphi) = \theta(\widetilde{\varphi}) = \widetilde{\varphi}_{D(1)} = \varphi$$

and

$$\tau \circ \theta(\widetilde{\varphi}) = \tau(\widetilde{\varphi}_{D(1)}) = \widetilde{\varphi},$$

there is a bijection:

$$\{\text{maps of } A\text{-modules } M \to N\} \longleftrightarrow \left\{\text{maps of } \mathscr{O}_{\operatorname{Spec} A}\text{-modules } \widetilde{M} \to \widetilde{N}\right\}.$$

□

Corollary 5.1.2

Let A be a ring and let $X = \operatorname{Spec} A$. The functor $\widetilde{\cdot} : M \to \widetilde{M}$ gives an exact, fully faithful functor from the category of A-modules to the category of $\mathscr{O}_X$-module.

Proof Consider the exact sequence of A-modules,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0,$$

by Proposition 5.1.4, we have sequence,

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{K} \longrightarrow 0. \tag{5.8}$$

To show that sequence (5.9) is exact, by Proposition 3.6.5, it suffices to show the exactness at the level of stalks. Let $p \in X$, then exists affine open subset $\operatorname{Spec} A_i \hookrightarrow X$ such that $p \in \operatorname{Spec} A$. By Proposition 3.6.5, we have exact sequence

$$0 \longrightarrow M_p \longrightarrow N_p \longrightarrow K_p \longrightarrow 0.$$

By Proposition 5.1.2, we have $M_p = \widetilde{M}_p$, $N_p = \widetilde{N}_p$, and $K_p = \widetilde{K}_p$, hence sequence

$$0 \longrightarrow \widetilde{M}_p \longrightarrow \widetilde{M}_p \longrightarrow \widetilde{M}_p \longrightarrow 0$$

is exact. By Proposition 3.6.5, sequence (5.9) is exact.

To say $\widetilde{\cdot}$ is fully faithful means that for any A-modules M and N, we have $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N})$. By Proposition 5.1.4, it is clear. □

Definition 5.1.5 (Support of a module)

Motivated by Proposition 5.1.2, and the notion of support of a section of a sheaf (Definition 3.7.4), define the ***support of*** $m \in M$ *by*

$$\operatorname{Supp} m := \left\{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0 \textit{ in } \widetilde{M}_{[\mathfrak{p}]}\right\} \subseteq \operatorname{Spec} A,$$

which is the support of m considered as a section of $\widetilde{M}$.
Similarly (following Definition 3.7.5), define the ***support of*** M *by*

$$\operatorname{Supp} M := \operatorname{Supp} \widetilde{M} = \{[\mathfrak{p}] \in \operatorname{Spec} A : M_{\mathfrak{p}} \neq 0\} \subseteq \operatorname{Spec} A.$$

Remark These notions will come up repeatedly. We will discuss support in more detail in Chapter 7.

Proposition 5.1.5

$\operatorname{Supp} m$ *is a closed subset of* $\operatorname{Spec} A$.

Proof By Proposition 3.7.5 and Proposition 5.1.2, clearly. □

5.1.3 Recurring counterexample

The example of two planes meeting at a point (§5.1.1) will appear many times as an example or a counterexample. It is important to have such examples (or counterexamples) at the back of your mind, as "boundary markers" reminding you of what is reasonable, and what to watch out for when you hear a new statement. The appearances of the following are listed in the index (under "recurring (counter)examples"), so you can see how often they come up, and where. You do not have the language to understand many of these yet, but you can come back to this later.

- two planes meeting at a point
- affine space minus the origin, and its inclusion in affine space
- affine space with doubled origin
- (the cone over) the quadric surface, $A = k[w,x,y,z]/(wz - xy)$
- an embedded point on a line, $k[x,y]/(y^2, xy)$
- x^2 in the projective plane
- infinite disjoint unions of schemes (especially $\coprod \operatorname{Spec} k[x]/(x^n)$)
- the morphism $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$
- $k(x) \otimes_k k(y)$
- $\prod^\infty \mathbb{F}_2$

5.2 Visualizing schemes: Nilpotents

In §4.3, we discussed how to visualize the underlying set of schemes, adding in generic points to our previous intuition of "classical" (or closed) points. Our later discussion of the Zariski Topology fit well with that picture. In our definition of the "affine scheme" $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$, we have the additional information of nilpotents, which are invisible on the level of points (§4.2.4), so now we figure out how to picture them. We will then readily be able to glue them together to picture schemes in general, once we have made the appropriate definitions. As we are building intuition, we cannot be rigorous of precise.

As motivation, note that we have inclusion-reversing bijections

$$\begin{aligned} \text{maximal ideals of } A &\longleftrightarrow \text{closed points of } \operatorname{Spec} A && \text{(Proposition 4.6.6)} \\ \text{prime ideals of } A &\longleftrightarrow \text{irreducible closed subsets of } \operatorname{Spec} A && \text{(Proposition 4.7.3)} \\ \text{radical ideals of } A &\longleftrightarrow \text{closed subsets of } \operatorname{Spec} A && \text{(Theorem 4.7.1)} \end{aligned}$$

If we take the things on the right as "pictures", our goal is to figure out how to picture ideals that are not radical:

$$\text{ideals of } A \longleftrightarrow \text{???}$$

(We will later fill this in rigorously in a different way with the notion of a closed subscheme, the scheme-theoretic version of closed subsets. But our goal now is create a picture.)

As motivation, when we see the expression $\operatorname{Spec} \mathbb{C}[x]/(x(x-1)(x-2)) = \{[(x)], [(x-1)], [(x-2)]\}$, we immediately interpret it as a closed subset of $\mathbb{A}^1_{\mathbb{C}}$, namely $\{0, 1, 2\}$. In particular, the map $\mathbb{C}[x] \to \mathbb{C}[x]/(x(x-1)(x-2))$ can be interpreted (via the Chinese Remainder Theorem, see Atiyah-MacDonald [1] Proposition 1.10) as: take a function on $\mathbb{A}^1_{\mathbb{C}}$, and restrict it to the three points 0, 1, and 2.

This will guide us in how to visualize a non-radical ideal. The simplest example to consider is $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ (Exercise 4.1 (a)). As a subset of $\mathbb{A}^1_{\mathbb{C}}$, it is just the origin $0 = [(x)]$, which we are used to thinking of as $\operatorname{Spec} \mathbb{C}[x]/(x)$ (i.e., corresponding to the ideal (x), not (x^2)). We want to enrich this picture in some way. We should picture $\mathbb{C}[x]/(x^2)$ in terms of the information the quotient remembers. The image of a polynomial $f(x)$ is the information of its value at 0, and its derivative (Exercise 4.13). We thus picture this as being the point, plus a little bit more — a little bit of infinitesimal "fuzz" on the point (see Figure 5.2). The sequence of restrictions $\mathbb{C}[x] \to \mathbb{C}[x]/(x^2) \to \mathbb{C}[x]/(x)$ should be interpreted as nested pictures.

$$\begin{array}{ccccc} \mathbb{C}[x] & \twoheadrightarrow & \mathbb{C}[x]/(x^2) & \twoheadrightarrow & \mathbb{C}[x]/(x) \\ \\ f(x) & & \longmapsto & & f(0), \end{array}$$

Similarly, $\mathbb{C}[x]/(x^3)$ remembers even more information — the second derivative as well. Thus we picture this as the point 0 with even more fuzz.

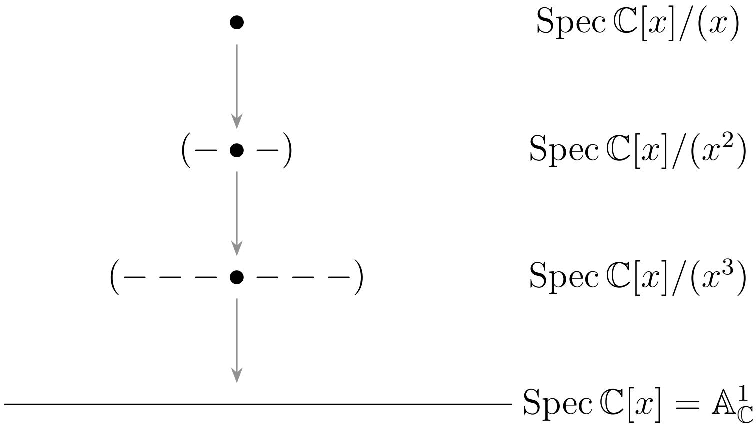


Figure 5.2: Picturing quotients of $\mathbb{C}[x]$

More subtleties arise in two dimensions (see Figure 5.3). Consider

$$\operatorname{Spec} \mathbb{C}[x, y]/(x, y)^2,$$

which is sandwiched between two rings we know well:

$$\begin{array}{ccccc} \mathbb{C}[x,y] & \twoheadrightarrow & \mathbb{C}[x,y]/(x,y)^2 & \twoheadrightarrow & \mathbb{C}[x,y]/(x,y) \\ \\ f(x,y) & & \longmapsto & & f(0,0). \end{array}$$

Again, taking the quotient by $(x, y)^2$ remembers the first derivative, "in all directions". We picture this as fuzz around the point, in the shape of a circle (no direction is privileged). Similarly, $(x, y)^3$ remembers the second derivative "in all directions" — bigger circular fuzz.

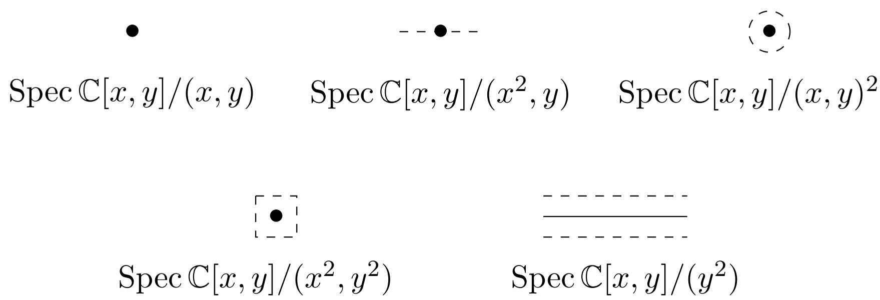


Figure 5.3: Picturing quotients of $\mathbb{C}[x, y]$

Consider instead the ideal (x^2, y). What it remembers is the derivative only in the x direction — given a polynomial, we remember its value at 0, and the coefficient of x. We remember this by the fuzz only in the x direction.

This gives us some handle on picturing more things of this sort, but now it becomes more an art than a science. For example, $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,y^2)$ we might picture as a fuzzy square around the origin. Since $(x,y)^3 \subseteq (x^2,y^2) \subseteq (x,y)^2$, we have $\mathbb{C}[x,y]/(x,y)^3 \supseteq \mathbb{C}[x,y]/(x^2,y^2) \supseteq \mathbb{C}[x,y]/(x,y)^2$, hence, this square is circumscribed by the circular fuzz $\operatorname{Spec}\mathbb{C}[x,y]/(x,y)^3$, and inscribed by the circular fuzz $\operatorname{Spec}\mathbb{C}[x,y]/(x,y)^2$. One feature of this example is that given two ideals I and J of a ring A (such as $\mathbb{C}[x,y]$), your fuzzy picture of $\operatorname{Spec} A/(I,J)$ should be the "intersection" of picture of $\operatorname{Spec} A/I$ and $\operatorname{Spec} A/J$ in $\operatorname{Spec} A$. For example, $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,y^2)$ should be the intersection of two thickened lines.

Example 5.1

(i) $\operatorname{Spec}\mathbb{C}[x,y]/(x^5,y^3) = \operatorname{Spec}\mathbb{C}[x,y]/(x^5) \cap \operatorname{Spec}\mathbb{C}[x,y]/(y^3)$.

(ii)
$$\begin{aligned}&\operatorname{Spec}\mathbb{C}[x,y]/(x^3,y^4,z^5,(x+y+z)^2)\\ =&\operatorname{Spec}\mathbb{C}[x,y]/(x^3) \cap \operatorname{Spec}\mathbb{C}[x,y]/(y^4) \cap \operatorname{Spec}\mathbb{C}[x,y]/(z^5) \cap \operatorname{Spec}\mathbb{C}[x,y]/((x+y+z)^2)\end{aligned}$$

(iii) $\operatorname{Spec}\mathbb{C}[x,y]/((x,y)^5,y^3) = \operatorname{Spec}\mathbb{C}[x,y]/(x,y)^5 \cap \operatorname{Spec}\mathbb{C}[x,y]/(y^3)$.

One final example that will motivate us in Chapter 7 is $\operatorname{Spec}\mathbb{C}[x,y]/(y^2,xy)$. Knowing what a polynomial in $\mathbb{C}[x,y]$ is modulo (y^2,xy) is the same as knowing its value on the x-axis, as well as first-order differential information around the origin (see Figure 5.4).

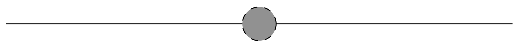


Figure 5.4: A picture of the scheme $\operatorname{Spec} k[x,y]/(y^2,xy)$. The fuzz at the origin indicates where "there is nonreducedness".

Our picture capture useful information that you already have some intuition for. For example, consider the intersection of the parabola $y = x^2$ and the x-axis (in the xy-plane), see Figure 5.5. You already have a sense that the intersection has multiplicity two. In terms of this visualization, we interpret this as intersecting (in $\operatorname{Spec}\mathbb{C}[x,y]$):

$$\begin{aligned}\operatorname{Spec}\mathbb{C}[x,y]/(y-x^2) \cap \operatorname{Spec}\mathbb{C}[x,y]/(y) &= \operatorname{Spec}\mathbb{C}[x,y]/(y-x^2,y)\\ &= \operatorname{Spec}\mathbb{C}[x,y]/(y,x^2)\\ &= \operatorname{Spec}\mathbb{C}[x]/(x^2)\end{aligned}$$

which we interpret as the fact that the parabola and line not just meet with multiplicity two, but that the "multiplicity 2" part is in the direction of the x-axis. We will make this example precise in Chapter 10.

Figure 5.5: The "scheme-theoretic" intersection of the parabola $y = x^2$ and the x-axis is a nonreduced scheme (with fuzz in the x-direction)

We will later make the location of the fuzz more precise when we discuss associated points. We will see that in reasonable circumstances, the fuzz is concentrated on closed subsets.

5.3 Definition of schemes

5.3.1 Schemes

We can now define **scheme** in general.

Definition 5.3.1 (Isomorphism of ringed spaces)

Define an ***isomorphism of ringed spaces*** $\varphi = (\pi, \pi^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *as*

(i) a homeomorphism $\pi : X \to Y$

(ii) an isomorphism of sheaves $\mathscr{O}_X$ *and* $\mathscr{O}_Y$*, considered to be on the same space via* π*. More precisely, is an isomorphism*

$$\pi^\sharp : \mathscr{O}_Y \xrightarrow{\sim} \pi_* \mathscr{O}_X$$

of sheaves on Y*, or equivalently, an isomorphism*

$$\pi^{-1} \mathscr{O}_Y \xrightarrow{\sim} \mathscr{O}_X$$

of sheaves on X*.*

Remark In other words, we have a "correspondence" of sets, topologies, and structure sheaves. Every isomorphism $\alpha : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ clearly has an "inverse isomorphism" $\alpha^{-1} : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$.

Definition 5.3.2 (Affine scheme, Scheme, and Zariski topology)

An ***Affine scheme*** *is a ringed space that is isomorphic to* $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ *for some* A*. A* ***scheme*** $(X, \mathscr{O}_X)$ *is a ringed space such that any point of* X *has an open neighborhood* U *such that* $(U, \mathscr{O}_X|_U)$ *is an affine scheme. The topology on a scheme is called the* ***Zariski topology****. The scheme can be denoted* $(X, \mathscr{O}_X)$*, although it is often denoted* X*, with the structure sheaf* $\mathscr{O}_X$ *left implicit.*

Definition 5.3.3 (An isomorphism of schemes)

An ***isomorphism of schemes*** $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *is an isomorphism as ringed spaces.*

Definition 5.3.4 (Functions)

Let $(X, \mathscr{O}_X)$ *be a scheme, if* $U \subseteq X$ *is an open subset, then the elements of* $\Gamma(U, \mathscr{O}_X) = \mathscr{O}_X(U)$ *are said to be the* ***functions on*** U*.*

Remark From the definition of the structure sheaf on an affine scheme, several things are clear. First of all, if we are told that $(X, \mathscr{O}_X)$ is an affine scheme, we may recover its ring (i.e., find the ring A such that $\operatorname{Spec} A = X$) by taking the ring of global sections, as $X = D(1)$, so:

$$\begin{aligned} \Gamma(X, \mathscr{O}_X) =& \Gamma(D(1), \mathscr{O}_{\operatorname{Spec} A}) \quad \text{as } D(1) = \operatorname{Spec} A \\ =& A. \end{aligned}$$

You can verify that we get more, and can "recognize X as the scheme $\operatorname{Spec} A$": we get an isomorphism $\pi : (\operatorname{Spec} \Gamma(X, \mathscr{O}_X), \mathscr{O}_{\operatorname{Spec} \Gamma(X, \mathscr{O}_X)}) \xrightarrow{\sim} (X, \mathscr{O}_X)$. For example, if $\mathfrak{m}$ is a maximal ideal of $\Gamma(X, \mathscr{O}_X)$, then $\{\pi([\mathfrak{m}])\} = V(\mathfrak{m})$. The following propositions will make these ideas rigorous — they are subtler than they first appear.

Remark **Caution.** It is not part of the definition that the overlap of two affine open subsets is also affine, although this is true in most cases of interest.

Proposition 5.3.1

There is a bijection between the isomorphisms $\operatorname{Spec} A \xrightarrow{\sim} \operatorname{Spec} A'$ *(of ringed spaces) and the ring isomorphisms* $A' \xrightarrow{\sim} A$*, i.e.,*

$$\left\{\textit{isomorphisms } \operatorname{Spec} A \xrightarrow{\sim} \operatorname{Spec} A'\right\} \longleftrightarrow \left\{\textit{the ring isomorphisms } A' \xrightarrow{\sim} A\right\}.$$

Proof Let $\varphi : A' \to A$ be a ring isomorphism. Define $\widetilde{\varphi} : \operatorname{Spec} A \to \operatorname{Spec} A'$ by setting $\widetilde{\varphi}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Clearly, it is a well-defined. $\widetilde{\varphi}^{-1} : \operatorname{Spec} A' \to \operatorname{Spec} A$ is given by $\widetilde{\varphi}^{-1}(\mathfrak{p}) = \varphi(\mathfrak{p})$. Since, φ is isomorphism of rings, it is easy to see that $\widetilde{\varphi}^{-1}$ is, indeed, the inverse of $\widetilde{\varphi}$.

First, we want to check this map is a homeomorphism. Let $V(I') = \operatorname{Spec} A'/I'$ be any closed subset of $\operatorname{Spec} A'$, then

$$\begin{aligned}
\widetilde{\varphi}^{-1}(\operatorname{Spec} A'/I') =& \widetilde{\varphi}^{-1}\left(\{\mathfrak{p} \in \operatorname{Spec} A' : \mathfrak{p} \supseteq I'\}\right)\\
=&\{\varphi(\mathfrak{p}) \in \operatorname{Spec} A : \varphi(\mathfrak{p}) \supseteq \varphi(I')\}\\
=&\operatorname{Spec} A/\varphi(I'),
\end{aligned}$$

where the last equal is given by φ is isomorphism, hence, $\widetilde{\varphi}^{-1}$ maps closed subset to closed subset, and therefore $\widetilde{\varphi}$ is continuous.

Conversely, let $V(I) = \operatorname{Spec} A/I$ be any close subset of $\operatorname{Spec} A$, then

$$\begin{aligned}
\widetilde{\varphi}(\operatorname{Spec} A/I) =& \widetilde{\varphi}\left(\{\mathfrak{p} \in \operatorname{Spec} A' : \mathfrak{p} \supseteq I\}\right)\\
=& \left\{\varphi^{-1}(\mathfrak{p}) \in \operatorname{Spec} A' : \varphi^{-1}(\mathfrak{p}) \supseteq \varphi^{-1}(I)\right\}\\
=& \operatorname{Spec} A'/\varphi^{-1}(I)
\end{aligned}$$

where the last equal is given by φ is isomorphism, hence, $\widetilde{\varphi}$ maps closed subset to closed subset, and therefore $\widetilde{\varphi}^{-1}$ is continuous. Hence, $\widetilde{\varphi}$ is a homeomorphism.

Next, we want to check that $\mathscr{O}_{\operatorname{Spec} A'} \xrightarrow{\sim} \widetilde{\varphi}_* \mathscr{O}_{\operatorname{Spec} A}$. It suffices to check this on the distinguished base. Define $\widetilde{\varphi}^\sharp : \mathscr{O}_{\operatorname{Spec} A'} \to \widetilde{\varphi}_* \mathscr{O}_{\operatorname{Spec} A}$ by setting $\widetilde{\varphi}^\sharp_{D(f)} : \mathscr{O}_{\operatorname{Spec} A'}(D(f)) \to \widetilde{\varphi}_* \mathscr{O}_{\operatorname{Spec} A}(D(f))$ as follow: since φ is isomorphism, we have

$$\begin{aligned}
\widetilde{\varphi}^{-1}(D(f)) =& \widetilde{\varphi}^{-1}(\{\mathfrak{p} \in \operatorname{Spec} A' : f \notin \mathfrak{p}\})\\
=&\{\varphi(\mathfrak{p}) \in \operatorname{Spec} A : \varphi(f) \notin \varphi(\mathfrak{p})\}\\
=&D(\varphi(f)),
\end{aligned}$$

hence, $\widetilde{\varphi}_* \mathscr{O}_{\operatorname{Spec} A}(D(f)) = \mathscr{O}_{\operatorname{Spec} A}(\widetilde{\varphi}^{-1}(D(f))) = \mathscr{O}_{\operatorname{Spec} A}(D(\varphi(f))) = A_{\varphi(f)}$, define $\widetilde{\varphi}^\sharp_{D(f)} : A'_f \to A_{\varphi(f)}$ with $\widetilde{\varphi}^\sharp_{D(f)}(a'/f^n) = \varphi(a')/\varphi(f)^n$. Clearly, it is a well-defined. Let $D(f) \supseteq D(g)$ in $\operatorname{Spec} A'$, consider the following diagram.

$$\begin{tikzcd}
A'_f \arrow[r, "\widetilde{\varphi}^\sharp_{D(f)}"] \arrow[d, "\operatorname{res}^{\operatorname{Spec} A'}_{D(f),D(g)}"'] & A_{\varphi(f)} \arrow[d, "\operatorname{res}^{\operatorname{Spec} A}_{D(\varphi(f)),D(\varphi(g))}"] \\
A'_g \arrow[r, "\widetilde{\varphi}^\sharp_{D(g)}"] & A_{\varphi(g)}
\end{tikzcd}$$

We want to check above diagram commutes. Let $a'/f^n \in A'_f$, then

$$\widetilde{\varphi}^\sharp_{D(g)} \circ \operatorname{res}^{\operatorname{Spec} A'}_{D(f),D(g)}(a'/f^n) = \widetilde{\varphi}^\sharp_{D(g)}(a'/f^n) = \varphi(a')/\varphi(f)^n$$

and

$$\operatorname{res}^{\operatorname{Spec} A}_{D(\varphi(f)),D(\varphi(g))} \circ \widetilde{\varphi}^\sharp_{D(f)}(a'/f^n) = \operatorname{res}^{\operatorname{Spec} A}_{D(\varphi(f)),D(\varphi(g))}(\varphi(a')/(\varphi(f))^n) = \varphi(a')/(\varphi(f))^n.$$

Hence, $\widetilde{\varphi}^\sharp_{D(g)} \circ \mathrm{res}^{\mathrm{Spec}\,A'}_{D(f),D(g)} = \mathrm{res}^{\mathrm{Spec}\,A}_{D(\varphi(f)),D(\varphi(g))} \circ \widetilde{\varphi}^\sharp_{D(f)}$, and therefore $\widetilde{\varphi}^\sharp$ is a morphism of sheaves. Since φ is an isomorphism, $\widetilde{\varphi}^\sharp_{D(f)}$ is isomorphism for all distinguished open subset $D(f)$, hence, $\widetilde{\varphi}^\sharp$ is an isomorphism of sheaves. Thus, $\varphi^\dagger := (\widetilde{\varphi}, \widetilde{\varphi}^\sharp)$ is an isomorphism of ringed spaces.

Let $\psi = (\pi, \pi^\sharp) : \mathrm{Spec}\,A \to \mathrm{Spec}\,A'$ be an isomorphism of ringed space. Then we have an isomorphism of sheaves,

$$\mathscr{O}_{\mathrm{Spec}\,A'} \xrightarrow{\sim} \psi_* \mathscr{O}_{\mathrm{Spec}\,A}.$$

Consider $D(1) \subseteq \mathrm{Spec}\,A'$, then we have an isomorphism of rings

$$\mathscr{O}_{\mathrm{Spec}\,A'}(D(1)) = A' \xrightarrow{\sim} \psi_* \mathscr{O}_{\mathrm{Spec}\,A}(D(1)).$$

In fact, $\psi_*\mathscr{O}_{\mathrm{Spec}\,A}(D(1)) = \mathscr{O}_{\mathrm{Spec}\,A}(\psi^{-1}(D(1)))$. Since ψ is also a homeomorphism between $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,A'$, we have $\psi^{-1}(D(1)) = \psi^{-1}(\mathrm{Spec}\,A') = \mathrm{Spec}\,A$, and therefore $\psi_*\mathscr{O}_{\mathrm{Spec}\,A}(D(1)) = A$. Hence, there is an isomorphism of rings

$$\pi^\sharp_{D(1)} : A' \to A.$$

Denote $\psi^\natural = \pi^\sharp_{D(1)}$.

At last, we want to check

$$\mathscr{A} = \left\{\text{isomorphisms } \mathrm{Spec}\,A \xrightarrow{\sim} \mathrm{Spec}\,A'\right\} \longrightarrow \left\{\text{the ring isomorphisms } A' \xrightarrow{\sim} A\right\} = \mathscr{B}.$$

is a bijection.

Let $\varphi = (\pi, \pi^\sharp) \in \mathscr{A}$, by above discussion, φ induces an isomorphisms of rings $\varphi^\natural : A' \to A$, where $\varphi^\natural = \pi^\sharp_{D(1)}$, and $\varphi^\natural$ induces an isomorphism of ringed space $\varphi^{\natural\dagger} = (\eta, \eta^\sharp) : \mathrm{Spec}\,A \to \mathrm{Spec}\,A'$, where $\eta : \mathrm{Spec}\,A \to \mathrm{Spec}\,A'$ induced by $\varphi^\natural$ and $\eta^\sharp : \mathscr{O}_{\mathrm{Spec}\,A'} \to \eta_*\mathscr{O}_{\mathrm{Spec}\,A}$. We want to show that $\varphi = \varphi^{\natural\dagger}$ as isomorphism of ringed space.

(i) On the level of point.

Let $[\mathfrak{p}] \in \mathrm{Spec}\,A$, then

$$\varphi^{\natural\dagger}([\mathfrak{p}]) = \eta([\mathfrak{p}]) = (\varphi^\natural)^{-1}([\mathfrak{p}]) = (\pi^\sharp_{D(1)})^{-1}([\mathfrak{p}]) = \pi([\mathfrak{p}]) = \varphi([\mathfrak{p}]),$$

hence, on the level of point $\varphi = \varphi^{\natural\dagger}$.

(ii) As maps of topological space.

It is clear, since φ and $\varphi^{\natural\dagger}$ are homeomorphisms and they are the same on the level of point, hence, $\varphi = \varphi^{\natural\dagger}$ as maps of topological space.

(iii) On the level of structure sheaves.

We need to check $\pi^\sharp = \eta^\sharp$. It suffices to check on the distinguished base. Let $D(f) \subseteq \mathrm{Spec}\,A'$ be any distinguished open subset, then for all $a'/f^n \in \mathscr{O}_{\mathrm{Spec}\,A'}(D(f)) = A'_f$, we have

$$\eta^\sharp_{D(f)}(a'/f^n) = \varphi^\natural(a')/\varphi^\natural(f)^n = \pi^\sharp_{D(1)}(a')/\pi^\sharp_{D(1)}(f)^n = \pi^\sharp_{D(f)}(a'/f^n),$$

(the last equal from the universal property of localization) i.e., $\eta^\sharp_{D(f)} = \pi^\sharp_{D(f)}$. Hence, $\eta^\sharp = \pi^\sharp$.

Conversely, let $\varphi \in \mathscr{B}$, φ induces an isomorphisms of ringed space $\varphi^\dagger = (\widetilde{\varphi}, \widetilde{\varphi}^\sharp)$, and $\varphi^\dagger$ induces an isomorphisms of rings $\varphi^{\dagger\natural} : A' \to A$, where $\varphi^{\dagger\natural} = \widetilde{\varphi}^\sharp_{D(1)}$. We want to show that $\varphi^{\dagger\natural} = \varphi$. Note that $\varphi^{\dagger\natural} = \widetilde{\varphi}^\sharp_{D(1)} = \varphi$, then we done.

By above discussion, we have a bijection,

$$\left\{\text{isomorphisms } \mathrm{Spec}\,A \xrightarrow{\sim} \mathrm{Spec}\,A'\right\} \longleftrightarrow \left\{\text{the ring isomorphisms } A' \xrightarrow{\sim} A\right\}.$$

$\square$

More generally, given $f \in A, \Gamma(D(f), \mathscr{O}_{\mathrm{Spec}\,A}) \cong A_f$. Thus under the natural inclusion of sets $\mathrm{Spec}\,A_f \hookrightarrow$

Spec A, the Zariski topology on Spec A restricts to give the Zariski topology on Spec A_f (Proposition 4.4.7), and the structure sheaf of Spec A restricts to the structure sheaf of Spec A_f, as the next proposition shows.

Proposition 5.3.2

Suppose $f \in A$. Then under the identification of $D(f)$ in Spec A with Spec A_f, there is a natural isomorphism of ringed spaces

$$(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}) \xleftrightarrow{\sim} (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}).$$

Proof Let $\mathrm{id} : D(f) \to \operatorname{Spec} A_f$, clearly, id is a homeomorphism.

Let $D_{\operatorname{Spec} A_f}(g) \subseteq \operatorname{Spec} A_f$ be a distinguished open subset, then

$$D_{\operatorname{Spec} A_f}(g) = \{[\mathfrak{p}] \in \operatorname{Spec} A_f : g \notin \mathfrak{p}\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : g \notin \mathfrak{p}, f \notin \mathfrak{p}\} = D_{\operatorname{Spec} A}(fg),$$

hence,

$$\begin{aligned} \mathrm{id}_* \mathscr{O}_{\operatorname{Spec} A}|_{D_{\operatorname{Spec} A}(f)}(D_{\operatorname{Spec} A_f(g)}) =& \mathscr{O}_{\operatorname{Spec} A}(D_{\operatorname{Spec} A}(f) \cap D_{\operatorname{Spec} A}(g)) \\ =& \mathscr{O}_{\operatorname{Spec} A}(D_{\operatorname{Spec} A}(fg)) \\ =& A_{fg}. \end{aligned}$$

On the other hand, we have

$$\mathscr{O}_{\operatorname{Spec} A_f}(D_{\operatorname{Spec} A_f}(g)) = (A_f)_g \cong A_{fg}.$$

It follows that $\mathrm{id}^\sharp : \mathscr{O}_{\operatorname{Spec} A_f} \xrightarrow{\sim} \mathrm{id}_* \mathscr{O}_{\operatorname{Spec} A}|_{D_{\operatorname{Spec} A}(f)}$. Hence,

$$(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}) \xleftrightarrow{\sim} (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}).$$

□

Proposition 5.3.3

If X is a scheme, and U is any open subset, then $(U, \mathscr{O}_X|_U)$ is also a scheme.

Proof It suffices to show that for all point of U has an open neighborhood V such that $(V, (\mathscr{O}_X|_U)|_V)$ is an affine scheme. Let $p \in U$, and $V \ni p$ is an open neighborhood of p in U. Note that $(\mathscr{O}_X|_U)|_V = \mathscr{O}_X|_{U\cap V}$ and $U \cap V$, we have $(V, (\mathscr{O}_X|_U)|_V) = (V, \mathscr{O}_X|_{U\cap V}) = (U \cap V, \mathscr{O}_X|_{U\cap V})$. Since X is a scheme, $(U \cap V, \mathscr{O}_X|_{U\cap V})$ is an affine scheme, and therefore $(V, (\mathscr{O}_X|_U)|_V)$ is an affine scheme. Hence, $(U, \mathscr{O}_X|_U)$ is a scheme. □

Definition 5.3.5 (Open subscheme)

*Suppose X is a scheme, and U is any open subset. We say $(U, \mathscr{O}_X|_U)$ is an **open subscheme of** X (Proposition 5.3.3, U is indeed a scheme). If U is also an affine scheme, we often say U is an **affine open subset**, or an **affine open subscheme**, or sometimes informally just an **affine open**.*

Example 5.2 $D(f)$ is an affine open subscheme of Spec A.

Remark It is not true that every affine open subscheme of Spec A is of the form $D(f)$.

Proposition 5.3.4

If X is a scheme, then the affine open sets form a base for the Zariski topology.

Proof Let $U \subseteq X$ be any open subset of X, then U is a scheme, by the definition of scheme, for all $p \in U$, exists an open neighborhood $V_p \subseteq U$ such that V_p is an affine scheme. Note that $U = \bigcap\limits_{\substack{\text{open } V_p \ni p \\ p \in U}} V_p$, where each V_p is affine open set. It follows that the affine open sets form a base for the Zariski topology. □

Definition 5.3.6 (The disjoint union of schemes)

The ***disjoint union of schemes*** *is the disjoint union of sets, with the expected topology (thus it is the disjoint union of topological spaces), with the expected sheaf. We use the symbol $X \coprod Y$ for the disjoint union of X and Y (because once we know what morphisms of schemes are, this construction will turn out to be the coproduct in the category of schemes).*

Proposition 5.3.5

(a) The disjoint union of a finite number of affine schemes is also an affine scheme.

(b) (A first example of a non-affine scheme) An infinite disjoint union of (non-empty) affine scheme is not an affine scheme.

Proof

(a) Let $I = \{1, 2, \ldots, n\}$ be a finite set, and $\{X_i\}_{i \in I}$ be a collection of affine schemes. Since each X_i is affine scheme, we may assume $X_i = \operatorname{Spec} A_i$ for all $i \in I$. Consider $\coprod_{i \in I} \operatorname{Spec} A_i$, by Example 4.5 , $\coprod_{i=1}^n \operatorname{Spec} A_i = \operatorname{Spec} \prod_{i=1}^n A_i$ as topological spaces, say $\pi : \coprod_{i=1}^n \operatorname{Spec} A_i \to \operatorname{Spec} \prod_{i=1}^n A_i$
Let $D(\prod_{i=1}^n f_i)$ be a distinguished open subset of $\operatorname{Spec} \prod_{i=1}^n A_i$. The structure of $D(\prod_{i=1}^n f_i)$ is,

$$\begin{aligned} D\left(\prod_{i=1}^n f_i\right) &= \left\{[\mathfrak{p}] \in \operatorname{Spec} \prod_{i=1}^n A_i : \prod_{i=1}^n f_i \notin \mathfrak{p}\right\} \\ &= \left\{[\mathfrak{p}] \in \operatorname{Spec} \prod_{i=1}^n A_i : \mathfrak{p} = A_1 \times \cdots \times A_{i-1} \times \mathfrak{p}_i \times A_{i+1} \times \cdots \times A_n, f_i \notin \mathfrak{p}_i, \forall i \in I\right\} \\ &= \coprod_{i=1}^n \{A_1 \times \cdots \times A_{i-1} \times \mathfrak{p}_i \times A_{i+1} \times \cdots \times A_n : f_i \notin \mathfrak{p}_i\} \\ &\cong \coprod_{i=1}^n D_{\operatorname{Spec} A_i}(f_i) \subseteq \coprod_{i=1}^n \operatorname{Spec} A_i. \end{aligned}$$

Hence, $\pi^{-1}(D(\prod_{i=1}^n f_i)) = \coprod_{i=1}^n D_{\operatorname{Spec} A_i}(f_i)$.
Note that

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} \prod_{i=1}^n A_i}\left(D\left(\prod_{i=1}^n f_i\right)\right) &= \left(\prod_{i=1}^n A_i\right)_{\prod_{i=1}^n f_i} \\ &\cong \prod_{i=1}^n (A_i)_{f_i} = \prod_{i=1}^n \mathscr{O}_{\operatorname{Spec} A_i}(D_{\operatorname{Spec} A_i}(f_i)) \\ &= \left(\prod_{i=1}^n \mathscr{O}_{\operatorname{Spec} A_i}\right)\left(\coprod_{i=1}^n D_{\operatorname{Spec} A_i}(f_i)\right) \\ &= \left(\prod_{i=1}^n \mathscr{O}_{\operatorname{Spec} A_i}\right)\left(\pi^{-1}\left(D\left(\prod_{i=1}^n f_i\right)\right)\right) \\ &= \left(\pi_* \prod_{i=1}^n \mathscr{O}_{\operatorname{Spec} A_i}\right)\left(D\left(\prod_{i=1}^n f_i\right)\right), \end{aligned}$$

and it is easy to check the compatibility of the ring isomorphism with the restriction maps, hence, we have an isomorphism of schemes

$$\coprod_{i=1}^n \operatorname{Spec} A_i \xrightarrow{\sim} \operatorname{Spec} \prod_{i=1}^n A_i.$$

It follows that the disjoint union of a finite number of affine schemes is also an affine scheme.

(b) By Proposition 4.6.4, affine scheme is quasi-compact, but infinite disjoint union of nonempty quasi-compact space is clearly not quasi-compact, and therefore infinite disjoint union of affine scheme is not an affine scheme.

□

Remark (**A first glimpse of closed subschemes.**) Open subsets of a scheme come with a natural scheme structure (Definition 5.3.6). For comparison, closed subsets can have many "natural" scheme structures. We will discuss this later in Chapter 10, but for now, it suffices for you know that a closed subscheme of X is, informally, a particular kind of scheme structure on a closed subset of X. As an example: if $I \subseteq A$ is an ideal, then $\operatorname{Spec} A/I$ endows the closed subset $V(I) \subseteq \operatorname{Spec} A$ with a scheme structure; but note that there can be different ideals with the same vanishing set (for example (x) and (x^2) in $k[x]$).

5.3.2 Stalks of the structure sheaf: germs, values at a point, and the residue field of a point

Like every sheaf, the structure sheaf has stalks, and unsurprisingly, they are interesting from an algebraic point of view. In fact, we have seen them before.

Definition 5.3.7 (Locally ringed space)

*We say a ringed space is a **locally ringed space** if its stalks are local ring.*

Example 5.3 (**Schemes are locally ringed spaces.**) By Proposition 5.1.2, the stalk of $\mathscr{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the local ring A_p. Hence Schemes are locally ringed spaces.

Example 5.4 (**Manifolds are locally ringed spaces.**) By Proposition 3.1.1, manifolds are locally ringed spaces.

In both cases, taking the quotient by the maximal ideal may be interpreted as evaluating at the point.

Definition 5.3.8 (Residue field, value of a function at a point)

*The maximal of the local ring $\mathscr{O}_{X,p}$ is denoted by $\mathfrak{m}_{X,p}$ or $\mathfrak{m}_p$, and the **residue field** $\mathscr{O}_{X,p}/\mathfrak{m}_p$ is denoted $\kappa(p)$. Each function f on an open subset U of a locally ringed space X has a value at each point p of U: the **value of** f **at** p (denoted $f(p)$) is the image of the germ of f at p under the canonical map $\mathscr{O}_{X,p} \to \kappa(p)$. We say that a function **vanishes** at a point p if its value at p is 0.*

This generalizes our notion of the value of a function on $\operatorname{Spec} A$ defined in Definition 4.2.1. Notice that we can't even make sense of the phrase "function vanishing" on ringed spaces in general.

Proposition 5.3.6

(a) If f is a function on a locally ringed space X, then the subset of X where f vanishes is closed.

(b) If f is a function on a locally ringed space that vanishes nowhere, then f is invertible.

Proof

(a) Let $f \in \mathscr{O}_X(X)$. Say $V = \{p \in X : f_p \in \mathfrak{m}_p\}$, we want to show that V is closed in X. It suffices to show that $X \setminus V = \{p \in X : f_p \notin \mathfrak{m}_p\} = \{p \in X : f_p \text{ is invertible in } \mathscr{O}_{X,p}\}$ is open. Let $p \in X \setminus V$, then there exists $g_p \in \mathscr{O}_{X,p}$ such that $f_p g_p = 1$. Hence, there exists an open subset $U_p \subseteq X$ such that $f|_{U_p} \cdot g = 1$, where $g \in \mathscr{O}_X(U_p)$. Hence, $f_q g_q = 1$ for all $q \in U_q$, it follows that for all $p \in X \setminus V$ exists an open neighborhood $U_p \ni p$ such that $U_p \subseteq X \setminus V$. Hence,

$$X \setminus V = \bigcup_{p \in X \setminus V} U_p,$$

which implies that $X \setminus V$ is an open set, and therefore V is closed in X.

(b) Let $f \in \mathscr{O}_X(X)$, if f vanishes nowhere, $f_p \notin \mathfrak{m}_p$ for all $p \in X$. It follows that for all p, there exists g_p such that $f_p g_p = 1$. Hence, there exists an open neighborhood U_p of p such that $f|_{U_p} \cdot \widetilde{g}_p = 1$, where $\widetilde{g}_p \in \mathscr{O}_X(U_p)$, so $\widetilde{g}_p = \frac{1}{f|_{U_p}}$.
Let $p, q \in X$, then we have $\widetilde{g}_p \in \mathscr{O}_X(U_p)$ and $\widetilde{g}_q \in \mathscr{O}_X(U_q)$ such that $\widetilde{g}_p = \frac{1}{f|_{U_p}}$ and $\widetilde{g}_q = \frac{1}{f|_{U_q}}$. Hence, we have

$$\widetilde{g}_p\big|_{U_p \cap U_q} = \frac{1}{f|_{U_p}}\bigg|_{U_p \cap U_q} = \frac{1}{f|_{U_q}}\bigg|_{U_p \cap U_q} = \widetilde{g}_q\big|_{U_p \cap U_q}.$$

Note that $\{U_p\}_{p \in X}$ is an open cover of X, by the gluability axiom of $\mathscr{O}_X$, there exists $g \in \mathscr{O}_X(X)$ such that $g|_{U_p} = \widetilde{g}_p \in \mathscr{O}_X(U_p)$. Hence, $f(x)g(x) = 1$ for all $x \in X$, it follows that f is invertible.

□

Consider a point $[\mathfrak{p}]$ of an affine scheme $\operatorname{Spec} A$. (Of course, any point of a scheme can be interpreted in this way, as each point has an affine open neighborhood.) The residue field at $[\mathfrak{p}]$ is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which is isomorphic to $K(A/\mathfrak{p})$, the fraction field of quotient (since $A/\mathfrak{p}$ is integral domain, its localization is its fraction field). It is useful to note that localization at $\mathfrak{p}$ and taking the quotient by $\mathfrak{p}$ "commute", i.e., the following diagram commutes.

$$\begin{tikzcd}
& A_{\mathfrak{p}} \arrow[dr, "\text{quotient}"] & & \\
A \arrow[ur, "\text{localize}"] \arrow[dr, "\text{quotient}"'] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r, leftrightarrow, "\sim"] & K(A/\mathfrak{p}) \\
& A/\mathfrak{p} \arrow[ur, "\text{localize, i.e., } K(\cdot)"'] & &
\end{tikzcd}$$

For example, consider the scheme $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, where k is a field of characteristic not 2. Then $(x^2 + y^2)/x(y^2 - x^5)$ is a function away from the y-axis and the curve $y^2 - x^5$. Its value at $(2, 4)$ (by which we mean $[(x - 2, y - 4)]$) is $(2^2 + 4^2)/(2(4^2 - 2^5))$, as

$$\frac{x^2 + y^2}{x(y^2 - x^5)} \equiv \frac{2^2 + 4^2}{2(4^2 - 2^5)}$$

in the residue field. And its value at $[(y)]$, the generic point of the x-axis, is $\frac{x^2}{-x^6} = -1/x^4$, which we see by setting y to 0. This is indeed an element of the fraction field of $k[x, y]/(y)$, i.e., $k(x)$. (If you think you care only about algebraically closed fields, let this example be a first waring: $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ won't be algebraically closed in general, even if A is a finitely generated $\mathbb{C}$-algebra!)

Example 5.5 $27/4$ is a function on $\operatorname{Spec} \mathbb{Z} \setminus \{[(2)], [(7)]\}$ or indeed on an even bigger open set. Its value at $[(5)]$ is $2/(-1) \equiv -2 \pmod 5$. Its value at the generic point $[(0)]$ is $27/4$. Its value vanishes at $[(3)]$.

Definition 5.3.9 (The fiber and the rank of an $\mathscr{O}_X$-module at a point)

*If $\mathscr{F}$ is an $\mathscr{O}_X$-module on a scheme X (or more generally, a locally ringed space), define the **fiber** (or fibre) **of $\mathscr{F}$ at a point** $p \in X$ by*

$$\mathscr{F}|_p := \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} \kappa(p).$$

The fiber is a vector space over $\kappa(p)$. The dimension of this vector space $\dim_{\kappa(p)} \mathscr{F}|_p$ is called the ***rank of*** $\mathscr{F}$ *at p, and is denoted* $\operatorname{rank}_p \mathscr{F}$.

Example 5.6 $\mathscr{O}_X|_p = \mathscr{O}_{X,p} \otimes_{\mathscr{O}_{X,p}} \kappa(p) = \mathscr{O}_{X,p} \otimes_{\mathscr{O}_{X,p}} (\mathscr{O}_{X,p}/\mathfrak{m}_p) \cong 1 \otimes_{\mathscr{O}_{X,p}} \kappa(p) \cong \kappa(p)$, and therefore $\operatorname{rank}_p \mathscr{O}_X = \dim_{\kappa(p)} \mathscr{O}_X|_p = 1$.

In the same way, we can cleanly and concisely define (various notions of) manifolds.

Definition 5.3.10 (Manifolds of various sorts)

A ***topological*** *(respectively,* ***differentiable****,* C^∞*(****smooth****)) (real)* ***manifold*** *is a ringed space $(X, \mathscr{O}_X)$, whose underlying topological space X is Hausdorff and second countable, such that any point of X has an open neighborhood U such that $(U, \mathscr{O}_X|_U)$ is isomorphic to an open ball in $\mathbb{R}^n$ with the sheaf of continuous (respectively, differentiable, C^∞) real-valued functions on it. The phrase* ***manifold*** *(without adjective) often means* ***smooth manifold****.*

A ***complex manifold*** *is a ringed space $(X, \mathscr{O}_X)$, whose underlying topological space X is Hausdorff and second countable, such that any point of X has an open neighborhood U such that $(U, \mathscr{O}_X|_U)$ is isomorphic to an open ball in $\mathbb{C}^n$ with the sheaf of holomorphic function on it.*

5.4 Three examples

We now give three extended examples. Our short-term goal is to see that we can really work with the structure sheaf, and can compute the ring of sections of interesting open sets that aren't just distinguished open sets of affine schemes. Our long-term goal is to meet interesting examples that will come up repeatedly in the future.

5.4.1 The (affine) plane minus the origin

This example will show you that the distinguished base is something that you can work with. Let $A = k[x, y]$, so $\operatorname{Spec} A = \mathbb{A}_k^2$. Let's work out the space of functions on the open set $U = \mathbb{A}_k^2 \setminus \{(0,0)\} = \mathbb{A}_k^2 \setminus \{[(x, y)]\}$.

It is not immediately obvious whether this is a distinguished open set. (In fact it is not, since (x, y) is not a principal ideal.) But in any case, we can describe it as the union of two things which are distinguished open sets: $U = D(x) \cup D(y)$. We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

The functions on $D(x)$ are, $\mathscr{O}_{\mathbb{A}_k^2}(D(x)) = A_x = k[x, y, 1/x]$. The functions on $D(y)$ are $A_y = k[x, y, 1/y]$. Note that A injects into its localizations (if 0 is not inverted), as it is an integral domain (Exercise 2.6), so A injects into both A_x and A_y, and both inject in to A_{xy} (and indeed into $k(x, y) = K(A)$). So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e., we are interpreting $A_x \cap A_y$ in A_{xy} (or in $k(x, y)$). Clearly those rational functions with only powers of x in the denominator, and also with only powers of y in the denominator, are polynomials. Translations: $A_x \cap A_y = A$. Thus we conclude:

$$\Gamma(U, \mathscr{O}_{\mathbb{A}_k^2}) \equiv k[x, y]. \tag{5.9}$$

In other words, we get no extra functions by removing the origin.

Remark Notice that any function on $\mathbb{A}_k^2 \setminus \{(0,0)\}$ extends over all of $\mathbb{A}_k^2$. This is an analog of **Hartogs's Lemma** in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are "smooth", but also if they are mildly singular — what we will call normal. We will make this precise in Chapter 14. This fact will be very useful for us.

We now show an interesting fact: $(U, \mathscr{O}_{\mathbb{A}^2}|_U)$ is a scheme ($U = D(x) \cup D(y)$), but it is not an affine scheme. Here's why: otherwise, if $(U, \mathscr{O}_{\mathbb{A}^2}|_U) \cong (\operatorname{Spec} A', \mathscr{O}_{\operatorname{Spec} A'})$, then we can recover A' by taking global sections:

$$A' = \Gamma(U, \mathscr{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (5.9) as $k[x, y]$. So if U is affine, then we have the isomorphism

$$U \overset{\sim}{\longleftrightarrow} \operatorname{Spec} A' = \operatorname{Spec} k[x, y] = \mathbb{A}^2_k.$$

But this bijection between prime ideals in a ring and points of the the spectrum is more constructive than that: *given the prime ideal I, you can recover the point as the generic point of the closed subset cut out by I, i.e., $V(I)$, and given the point p, you can recover the ideal as those functions vanishing at p, i.e., $I(p)$.* In particular, the prime ideal (x, y) of A' should cut out a point of $\operatorname{Spec} A'$. But on U, $V(x) \cap V(y) = \varnothing$. Conclusion: U is not an affine scheme. (If you are ever looking for a counterexample to something, and you are excepting one involving a non-affine scheme, keep this example in mind!)

5.4.2 Gluing two copies of $\mathbb{A}^1_k$ together in two different ways

We have now seen two examples of non-affine schemes: an infinite disjoint union of nonempty schemes (Proposition 5.3.5 (b)) and $\mathbb{A}^2 \setminus \{(0,0)\}$. We will give two more examples. They are important because they are the first examples of fundamental behavior, the first pathological , and second central.

First, we need to discuss how to glue two schemes together. Before that, we should review how to glue topological spaces together along isomorphic open sets. Give two topological spaces X and Y, and open subsets $U \subseteq X$ and $V \subseteq Y$ along with a homeomorphism $U \overset{\sim}{\longleftrightarrow} V$, we can create a new topological space W, that we think of as gluing X and Y together along $U \overset{\sim}{\longleftrightarrow} V$. It is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation $U \sim V$, where the quotient is given the quotient topology. Then X and Y are naturally (identified with) open subsets of W, and indeed cover W.

Now that we have discussed gluing topological spaces, let's glue schemes together. (This applies without change more generally to ringed spaces.) Suppose you have two schemes $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, and open subsets $U \subseteq X$ and $V \subseteq Y$, along with a homeomorphism $\pi : U \overset{\sim}{\longrightarrow} V$, and an isomorphism of structure sheaves $\mathscr{O}_V \overset{\sim}{\longrightarrow} \pi_* \mathscr{O}_U$ (i.e., an isomorphism of schemes $(U, \mathscr{O}_X|_U) \overset{\sim}{\longleftrightarrow} (V, \mathscr{O}_Y|_V)$). Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together using the isomorphism $U \overset{\sim}{\longleftrightarrow} V$. Then Theorem 3.5.3 shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has an open neighborhood that is an affine scheme.

Theorem 5.4.1 (Glue an arbitrary collection of scheme together)

We can glue an arbitrary collection of schemes together. Suppose we are given:

- *scheme X_i (as i runs over some index set I, not necessarily finite),*
- *open subschemes $X_{ij} \subseteq X_i$ with $X_{ii} = X_i$,*
- *isomorphisms $f_{ij} = (\pi_{ij}, \pi^\sharp_{ij}) : X_{ij} \overset{\sim}{\longrightarrow} X_{ji}$ with f_{ii} the identity*

such that

- *(the cocycle condition) the isomorphisms "agree on triple intersections", i.e.,*

$$f_{ik}|_{X_{ij}\cap X_{ik}} = f_{jk}|_{X_{ji}\cap X_{jk}} \circ f_{ij}|_{X_{ij}\cap X_{ik}}$$

(so implicitly, to make sense of the right side, $f_{ij}(X_{ik}\cap X_{ij}) \subseteq X_{jk}$).

Then there is a unique scheme X (up to unique isomorphism) along with open subschemes isomorphic to the X_i respecting this gluing data in the obvious sense.

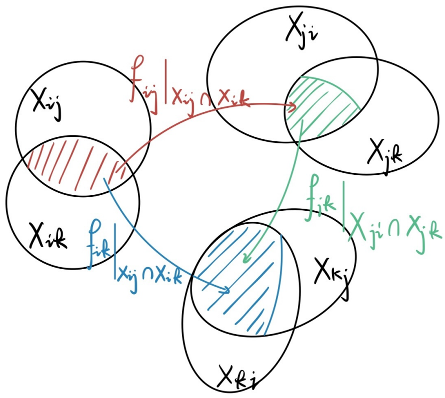


Figure 5.6: Picture of the cocycle condition

Remark The cocycle condition ensures that f_{ij} and f_{ji} are inverses (take $k = i$ in the cocycle condition). In fact, the hypothesis that f_{ii} is the identity also follows from the cocycle condition.

Proof

(i) Construct the underlying topological space X.

Consider the disjoint union $\coprod_{i\in I} X_i$, we define a relation $\sim$ by setting $(x,i)\sim(y_j)$ if exists $k\in I$ such that $\pi_{ik}(x) = \pi_{jk}(y)$. We want to check relation $\sim$ is a equivalence relation.

- **Reflexivity:** Note that $\pi_{ii} = \mathrm{id}$, so we have $(x,i)\sim(x,i)$.
- **Symmetry:** If $(x,i)\sim(y,j)$, then there exists $k\in I$ such that $\pi_{ik}(x) = \pi_{jk}(y)$. Hence, $(y,j)\sim(x,i)$.
- **Transitivity:** If $(x,i)\sim(y,j)$ and $(y,j)\sim(z,k)$, then there exists $m,n\in I$ such that $\pi_{im}(x) = \pi_{jm}(y)$ and $\pi_{jn}(y) = \pi_{kn}(z)$. Note that $\pi_{jn} = \pi_{mn}\circ\pi_{jm}$, we have $\pi_{mn}(\pi_{im}(x)) = \pi_{in}(x) = \pi_{mn}\pi_{jm}(y) = \pi_{jn}(y) = \pi_{kn}(z)$, i.e., $(x.i)\sim(z,k)$.

Hence, relation $\sim$ is a equivalence relation. Define

$$X = \left(\coprod X_i\right)\Big/_{\sim}$$

as the quotient space.

We now endow X with the quotient topology. Define $U\subseteq X$ be an open subset of X if and only if $q_i^{-1}(U)$ is open in X_i for all i, where $q_i : X_i \to X$ is the quotient map.

(ii) Construct the structure sheaf $\mathscr{O}_X$.

We claim that each quotient map q_i is open. Let $V\subseteq X_i$ be an open subset of X_i, consider $q_j^{-1}(q_i(V))$, note that

$$\begin{aligned} q_j^{-1}(q_i(V)) =& \{y \in X_j : q_j(y) = q_i(x), \text{ for some } x \in V\} \\ =& \{y \in X_j : \exists k \in I \text{ and } x \in V, \pi_{jk}(y) = \pi_{ik}(x)\} \\ =& \{y \in X_j : \exists k \in I \text{ and } x \in V, \pi_{kj}\pi_{jk}(y) = y = \pi_{kj}\pi_{ik}(x) = \pi_{ij}(x)\} \\ =& \{y \in X_j : \exists x \in V, y = \pi_{ij}(x)\} \\ =& \pi_{ij}(V \cap X_{ij}), \end{aligned} \tag{5.10}$$

and π_{ij} is a homeomorphism, we have $q_j^{-1}(q_i(V))$ is open in X_j. Hence, q_i is open.

Let $U_i = q_i(X_i)$, since q_i is open, U_i is open in X. Let $\overline{x} \in X$ be any point in X, then $x \in X_i$ for some i, and therefore $\overline{x} = q_i(x) \in q_i(X_i) \subseteq X$. Hence, $X = \bigcup_{i \in I} q_i(X_i) = \bigcup_{i \in I} U_i$.

Next, we define the structure sheaf on U_i. Since $q_i : X_i \to U_i$ is a homeomorphism, define

$$\mathscr{F}_{U_i} = q_{i,*}\mathscr{O}_{X_i}.$$

By (5.10), we have $q_j^{-1}(q_i(X_i)) = q_j^{-1}(U_i) = \pi_{ij}(X_i \cap X_{ij}) = \pi_{ij}(X_{ij}) = X_{ji}$, i.e.,

$$X_{ji} = q_j^{-1}(U_i).$$

Hence, we have the commutative diagram.

$$\begin{tikzcd} X_{ij} \arrow[rr, "\pi_{ij}"] \arrow[dr, "q_i|_{X_{ij}}"'] & & X_{ji} \arrow[dl, "q_j|_{X_{ji}}"] \\ & U_i \cap U_j & \end{tikzcd}$$

Consider $\mathscr{F}_{U_j}|_{U_i \cap U_j}$, let $V \subseteq U_i \cap U_j$, then

$$\begin{aligned} \mathscr{F}_{U_j}|_{U_i \cap U_j}(V) =& (q_{j,*}\mathscr{O}_{X_j})|_{U_i \cap U_j}(V) = \mathscr{O}_{X_{ji}}(q_j^{-1}(V)) \\ =& \mathscr{O}_{X_{ji}}(\pi_{ij} \circ q_i^{-1}(V)) = \mathscr{O}_{X_{ji}}(\pi_{ji}^{-1} \circ q_i^{-1}(V)) \\ =& \pi_{ji,*}\mathscr{O}_{X_{ji}}(q_i^{-1}(V)) \\ \cong& \mathscr{O}_{X_{ij}}(q_i^{-1}(V)) = (q_{j,*}\mathscr{O}_{X_j})|_{U_i \cap U_j}(V) \\ =& \mathscr{F}_{U_i}|_{U_i \cap U_j}(V), \end{aligned}$$

it follows that $\mathscr{F}_{U_i}|_{U_i \cap U_j} \overset{\sim}{\longleftrightarrow} \mathscr{F}_{U_j}|_{U_i \cap U_j}$.

Say $\varphi_{ij} : \mathscr{F}_{U_i}|_{U_i \cap U_j} \overset{\sim}{\longrightarrow} \mathscr{F}_{U_j}|_{U_i \cap U_j}$, it is easy to check that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$. By Theorem 3.5.3, $\mathscr{F}_{U_i}$ can be glued together into a sheaf on X, say $\mathscr{O}_X$, with isomorphism $\mathscr{O}|_{U_i} \overset{\sim}{\longrightarrow} \mathscr{F}_{U_i}$.

(iii) $(X, \mathscr{O}_X)$ is a scheme.

Since each $(U_i, \mathscr{O}_X|_{U_i}) \cong (X_i, \mathscr{O}_{X_i})$ and each $(X_i, \mathscr{O}_{X_i})$ is a scheme, $(U_i, \mathscr{O}_X|_{U_i})$ is therefore a scheme.

(iv) Uniqueness of $(X, \mathscr{O}_X)$.

Uniqueness given by the uniqueness of quotient topology and gluing sheaves (Theorem 3.5.3).

□

We will now give two non-affine schemes. Both are handy to know. In both cases, we will glue together two copies of the affine line $\mathbb{A}_k^1$. Let $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subseteq X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subseteq Y$. We will get both examples by gluing X and Y together along U and V. The difference will be in how we glue.

The affine line with the doubled origin

Consider the isomorphism $U \overset{\sim}{\longleftrightarrow} V$ via the isomorphism $k[t, 1/t] \overset{\sim}{\longleftrightarrow} k[u, 1/u]$ given by $t \longleftrightarrow u$ (Proposition 5.3.1). The resulting scheme is called the affine line with double origin. Figure 5.7 is a picture of it.

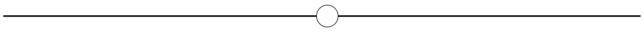


Figure 5.7: The affine line with doubled origin

As the picture suggests, intuitively this is an analog of a failure of Hausdorffness. Now $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we will want to make a definition that will prevent this from happening. This will be the notion of **separatedness** (to be dicussed in Chapter 12). (In fact, in some sense, separatedness is the right definition of Hausdorfness.) This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

✍ **Exercise 5.2** Show that the affine line with doubled origin is not affine.

Proof Let $X = \operatorname{Spec} k[t]$, $Y = \operatorname{Spec} k[u]$, $U = D(t)$, and $V = D(u)$. Let $Z = (X \coprod Y)/\sim$ be the affine line with doubled origin, where $\sim$ given by isomorphism $U \xleftrightarrow{\sim} V$ which induced by ring isomorphism $t \mapsto u$.

We first calculate the global section $\Gamma(Z, \mathscr{O}_Z)$. Let $s \in \Gamma(Z, \mathscr{O}_Z)$, then $(s|_X)|_{X\cap Y} = (s|_Y)|_{X\cap Y}$. In fact, $X \cap Y = U \cong V$, hence, $\Gamma(X\cap Y, \mathscr{O}_Z) \cong k[t]_t$. Since $(s|_X)|_{X\cap Y} = (s|_Y)|_{X\cap Y}$ in $k[t]_t$, exists n such that $t^n(s|_X - s|_Y) = 0$ in $k[t]$. Note that $\Gamma(X, \mathscr{O}_Z|_X) \cong k[t]$ and $\Gamma(Y, \mathscr{O}_Z|_Y) \cong k[t]$, also $k[t]$ is an integral domain, since $t \neq 0$, we have $s|_X = s|_Y$ in $k[t]$. It follows that $s \in k[t]$. Conversely, each element in $k[t]$ can be seen as the function over Z. Hence,

$$\Gamma(Z, \mathscr{O}_Z) \cong k[t].$$

Suppose Z is an affine scheme, then $(Z, \mathscr{O}_Z) \xleftrightarrow{\sim} (A, \operatorname{Spec} A)$, hence,

$$\Gamma(Z, \mathscr{O}_Z) \cong k[t] \cong \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A.$$

Note that $[(u)] \in Z$, but $[(u)] \notin k[t]$, a contradiction! Hence, Z is not an affine scheme. □

✍ **Exercise 5.3** Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You will have defined the **affine plane with doubled origin**. Describe two affine open subsets of this scheme whose intersection is not an affine open subset.

Proof Let $X = \operatorname{Spec} k[x_1, y_1]$ and $Y = \operatorname{Spec} k[x_2, y_2]$. Let $U = \operatorname{Spec} k[x_1, y_1] \setminus \{[(x_1, y_1)]\}$ and $V = \operatorname{Spec} k[x_2, y_2] \setminus \{[(x_2, y_2)]\}$, then $U = D(x_1) \cup D(y_1)$ and $V = D(x_2) \cup D(y_2)$. By §5.4.1, we have $\Gamma(U, \mathscr{O}_X) = k[x_1, y_1]$ and $\Gamma(V, \mathscr{O}_Y) = k[x_2, y_2]$. Let $\varphi : k[x_1, y_1] \to k[x_2, y_2]$ by setting $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$. Clearly, φ is a ring isomorphism. It induced an isomorphism of schemes $(U, \mathscr{O}_X|_U) \xleftrightarrow{\sim} (V, \mathscr{O}_Y|_V)$.

Now, we glue X and Y via $U \xleftrightarrow{\sim} V$. Let $Z = (X \coprod Y)/\sim$. We want to describe two affine open subsets of this scheme whose intersection is not an affine open subset. Consider X and Y, they are affine open subschemes of Z. In Z, the intersection of X and Y is $Z \setminus \{[(x_1, y_1)], [(x_2, y_2)]\}$. Clearly, $Z \setminus \{[(x_1, y_1)], [(x_2, y_2)]\}$ is isomorphic to $\operatorname{Spec} k[x, y] \setminus \{[(x, y)]\}$, by §5.4.1, $\operatorname{Spec} k[x, y] \setminus \{[(x, y)]\}$ is not affine, this implies that $X \cap Y$ is not affine. □

The projective line

Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \longleftrightarrow 1/u$. Figure 5.8 is a suggestive picture of this gluing. The resulting scheme is called the **projective line over the field** k, and is denoted $\mathbb{P}^1_k$.

Notice how the points glue. We assume that k is algebraically closed for convenience. On the first affine line, we have the closed points $[(t - a)]$, which we think of as "a on the t-line", and we have the generic point $[(0)]$. On the second affine line, we have closed points that are "b on the u-line", and generic point. Then a on

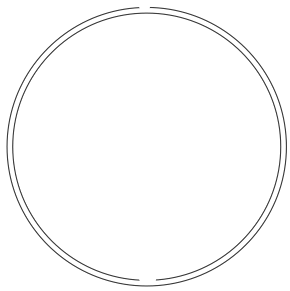


Figure 5.8: Gluing two affine lines together to get $\mathbb{P}^1$

the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of $k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \overset{\sim}{\longleftrightarrow} 1/u$).

If k is algebraically closed, we can interpret the closed points of $\mathbb{P}^1_k$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a, b]$, where a and b are not both zero, and $[a, b]$ is defined with $[ac, bc]$ where $c \in k^\times$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

Proposition 5.4.1

$\mathbb{P}^1_k$ is not affine.

Proof Let $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[u]$. $\mathbb{P}^1_k$ glued by X and Y via $t \longleftrightarrow 1/u$. We do this by calculating the ring of global sections. Let $s \in \Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k})$ with $(s|_X)|_{X\cap Y} = (s|_Y)|_{X\cap Y}$. Since $\Gamma(X, \mathscr{O}_{\mathbb{P}^1_k}) \cong k[t]$ and $\Gamma(Y, \mathscr{O}_{\mathbb{P}^1_k}) \cong k[u]$, we may assume $s|_X = f(t)$ and $s|_Y = g(u)$ are polynomials. Note that $\Gamma(X \cap Y, \mathscr{O}_{\mathbb{P}^1_k}) = k[t]_t$, restrict $f(t)$ to $X \cap Y$, we get something we can still call $f(t)$, and similarly for $g(u)$. Since $t \longleftrightarrow 1/u$ and $(s|_X)|_{X\cap Y} = (s|_Y)|_{X\cap Y}$, we have $f(t) = g(1/t)$. But the only polynomials in t that are at the same time polynomials in $1/t$ are the constants k. Thus $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = k$.

If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec}\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e., one point. But it isn't — it has lots of points. $\square$

Remark We have proved an analog of an important theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

5.4.3 Projective space

We now make a preliminary definition of **projective n-space over a field** k, denoted $\mathbb{P}^n_k$, by gluing together $n+1$ open sets each isomorphic to $\mathbb{A}^n_k$. Judicious choice of notation for these open sets will make our life easier. Our motivation is as follows. In the construction of $\mathbb{P}^1$ above, we thought of points of projective space as $[x_0 : x_1]$, where (x_0, x_1) are only determined up to scalars, i.e., (x_0, x_1) is considered the same as $(\lambda x_0, \lambda x_1)$. Then the first patch can be interpreted by taking the locus where $x_0 \neq 0$, and then we consider the points $[1 : t]$, and we think of t as x_0/x_1; even though x_0 and x_1 are not well-defined, x_1/x_0 is. The second corresponds to where $x_1 \neq 0$, and we consider the points $[u : 1]$, and we think of u as x_0/x_1. It will be useful to instead use the notation $x_{1/0}$ for t and $x_{0/1}$ for u.

For $\mathbb{P}^n$, we glue together $n+1$ open sets, one for each of $i = 0, \ldots, n$. The i-th open set U_i will have coordinates $x_{0/i}, \dots, x_{(i-1)/i}, x_{(i+1)/i}, \dots, x_{n/i}$. It will be convenient to write this as

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1) \tag{5.11}$$

(so we have introduced a "dummy variable" $x_{i/i}$ which we immediately set to 1). We glue the distinguished open set $D(x_{j/i})$ of U_i to the distinguished open set $D(x_{i/j})$ of U_j, by identifying these two schemes by describing the identification of rings

$$k[x_{0/i}, x_{1/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong k[x_{0/j}, x_{1/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies $x_{i/j}x_{j/i} - 1$). We need check that this gluing information agrees over triple overlaps.

Exercise 5.4 Check this, as painlessly as possible.

Proof Say $f_{ij} : D(x_{j/i}) \to D(x_{i/j})$, f_{ij} is induced by ring isomorphism $\varphi_{ij} : \Gamma(D(x_{i/j}), \mathscr{O}_{U_j}) \to \Gamma(D(x_{j/i}), \mathscr{O}_{U_i})$ which is given by $x_{k/j} \mapsto x_{k/i}/x_{j/i}$, and therefore f_{ij} is an isomorphism of affine open schemes.

We want to show that $f_{ik}|_{D(x_{j/i})\cap D(x_{k/i})} = f_{ik}|_{D(x_{i/j})\cap D(x_{k/j})} \circ f_{ij}|_{D(x_{j/i})\cap D(x_{k/i})}$. Consider the corresponding ring of $D(x_{j/i}) \cap D(x_{k/i})$, $D(x_{i/j}) \cap D(x_{k/j})$, and $D(x_{j/k}) \cap D(x_{i/k})$, they are all affine open subset, hence, we have

$$\begin{aligned}
\Gamma(D(x_{j/i}) \cap D(x_{k/i}), \mathscr{O}_{U_i}) &= k\left[x_{0/i}, \dots, x_{n/i}, \frac{1}{x_{j/i}}, \frac{1}{x_{k/i}}\right] \Big/ (x_{i/i} - 1), \\
\Gamma(D(x_{i/j}) \cap D(x_{k/j}), \mathscr{O}_{U_j}) &= k\left[x_{0/j}, \dots, x_{n/j}, \frac{1}{x_{i/j}}, \frac{1}{x_{k/j}}\right] \Big/ (x_{j/j} - 1), \\
\Gamma(D(x_{j/k}) \cap D(x_{i/k}), \mathscr{O}_{U_k}) &= k\left[x_{0/k}, \dots, x_{n/k}, \frac{1}{x_{i/k}}, \frac{1}{x_{j/k}}\right] \Big/ (x_{k/k} - 1).
\end{aligned}$$

Define $\varphi^e_{ij} : \Gamma(D(x_{i/j}x_{k/j}), \mathscr{O}_{U_j}) \to \Gamma(D(x_{j/i}x_{k/i}), \mathscr{O}_{U_i})$ by setting

$$\overline{\sum_{i_0,\dots,i_n,j_1,j_2} x_{0/j}^{i_0} \cdots x_{n/j}^{i_n} x_{i/j}^{-j_1} x_{k/j}^{-j_2}} \mapsto \overline{\sum_{i_0,\dots,i_n,j_1,j_2} \varphi_{ij}(x_{0/j})^{i_0} \cdots \varphi_{ij}(x_{n/j})^{i_n} \varphi_{ij}(x_{i/j})^{-j_1} \varphi_{ij}(x_{k/j})^{-j_2}}.$$

Clearly, φ^e_{ij} is a ring homomorphism, since φ_{ij} is an isomorphism, it is easy to check that φ^e_{ij} is an isomorphism. Hence, we have the following commutative diagram.

$$\begin{tikzcd}
\Gamma(D(x_{j/i}) \cap D(x_{k/i}), U_i) & & \Gamma(D(x_{i/j}) \cap D(x_{k/j}), U_j) \arrow[ll, "\varphi^e_{ij}"'] \\
& \Gamma(D(x_{i/k}) \cap D(x_{j/k}), U_k) \arrow[ul, "\varphi^e_{ik}"] \arrow[ur, "\varphi^e_{ik}"'] &
\end{tikzcd}$$

By Proposition 5.3.1, it follows that

$$f_{ik}|_{D(x_{j/i})\cap D(x_{k/i})} = f_{ik}|_{D(x_{i/j})\cap D(x_{k/j})} \circ f_{ij}|_{D(x_{j/i})\cap D(x_{k/i})}.$$

$\square$

Note that our definition does not use the fact that k is a field. Hence we may as well define $\mathbb{P}^n_A$ for any ring A. This will be useful later.

Definition 5.4.1 ($\mathbb{P}^n_A$)

Let A be a ring. The projective n-space over a ring A is defined by gluing together $n+1$ open sets each isomorphic to $\mathbb{A}^k_A$, denoted $\mathbb{P}^n_A$.

Proposition 5.4.2

The only functions on $\mathbb{P}_k^n$ are constants, i.e., $\Gamma(\mathbb{P}_k^n, \mathscr{O}) \cong k$, and hence that $\mathbb{P}_k^n$ is not affine if $n > 0$.

Proof Let $U_i = \operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$ be affine open subscheme of $\mathbb{P}_k^n$. Let $f \in \Gamma(\mathbb{P}_k^n, \mathscr{O})$, then $(f|_{U_i})|_{U_i \cap U_j} = (f|_{U_j})|_{U_i \cap U_j}$ in $\Gamma(U_i \cap U_j, \mathscr{O}_{\mathbb{P}_k^n})$.

We may assume that

$$f|_{U_i} = f_i(x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i})$$

and

$$f|_{U_j} = f_j(x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j}).$$

On the $D(x_{j/i}) = U_i \cap U_j$, we have the coordinate change:

$$x_{k/i} = x_{k/j}/x_{i/j},$$

hence,

$$\begin{aligned} f_i(x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i}) =& f_i\left(\frac{x_{0/j}}{x_{i/j}}, \ldots, \frac{x_{i-1/j}}{x_{i/j}}, \frac{x_{i+1/j}}{x_{i/j}}, \ldots, \frac{x_{n/j}}{x_{i/j}}\right) \\ =& \frac{1}{x_{i/j}^d} F(x_{0/j}, \ldots, x_{i-1/j}, x_{i/j}, x_{i+1/j}, \ldots, x_{n/j}) \\ =& f_j(x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j}), \end{aligned}$$

where $F(x_{0/j}, \ldots, x_{i-1/j}, x_{i/j}, x_{i+1/j}, \ldots, x_{n/j})$ is a homogeneous polynomial with degree d. Since f_j is a polynomial, d must be 0, and therefore $f|_{U_i} = f|_{U_j}$ must be constants for all i, j. Hence, $\Gamma(\mathbb{P}_k^n, \mathscr{O}) \cong k$. □

Remark There is even some geometric intuition behind this: the complement of the union of two open sets has codimension 2. But "Algebraic Hartogs' Lemma" says that any function defined on this union extends to be a function on all of projective space. Because we are expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.

Proposition 5.4.3

If k is algebraically closed, the closed points of $\mathbb{P}_k^n$ may be interpreted in the traditional way: the points are of the form $[a_0 : \cdots : a_n]$, where the a_i are not all zero, and $[a_0 : \cdots : a_n]$ is identified with $[\lambda a_0 : \cdots : \lambda a_n]$ where $\lambda \in k^\times$.

Proof $\mathbb{P}_k^n$ is covered by $U_i = \operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$. Let $p \in \mathbb{P}_k^n$ be a closed point, then $p \in U_i$ for some i. Since $\{p\}$ is closed in $\mathbb{P}_k^n$, $\{p\}$ is closed in U_i. Note that U_i is affine open subscheme, $\{p\}$ correspondence to a maximal ideal of $k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$, say $p = [(x_{0/i} - a_{0/i}, \ldots, x_{i-1/i} - a_{i-1/i}, x_{i+1/i} - a_{i+1/i}, \ldots, x_{n/i} - a_{n/i})] := [a_0 : \cdots : a_i : \cdots : a_n]$.

Next, we want to show that $[a_0 : \cdots : a_n]$ is identified with $[\lambda a_0 : \cdots : \lambda a_n]$, where $\lambda \in k^\times$. It correspondence to maximal ideal $(x_{0/i} - \lambda a_{0/i}, \ldots, x_{n/i} - \lambda a_{n/i})$. Note that

$$(x_{0/i} - \lambda a_{0/i}, \ldots, x_{n/i} - \lambda a_{n/i}) = \left(\frac{x_{0/i}}{\lambda} - a_{0/i}, \ldots, \frac{x_{n/i}}{\lambda} - a_{n/i}\right)$$

and

$$k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \cong k\left[\frac{x_{0/i}}{\lambda}, \ldots, \frac{x_{n/i}}{\lambda}\right] \Big/ \left(\frac{x_{i/i}}{\lambda} - 1\right),$$

$(x_{0/i} - \lambda a_{0/i}, \ldots, x_{n/i} - \lambda a_{n/i})$ and $(x_{0/i} - a_{0/i}, \ldots, x_{n/i} - a_{n/i})$ correspondence to the same point $[a_0 : \cdots : a_i : \cdots : a_n]$. Hence, $[a_0 : \cdots : a_i : \cdots : a_n]$ is identified with $[\lambda a_0 : \cdots : \lambda a_i : \cdots : \lambda a_n]$, where $\lambda \in k^\times$. □

Remark Helpful translation: we think of the closed points of $\mathbb{P}_k^n$ (where $k = \overline{k}$) as $[x_0 : x_1 : \cdots : x_n]$, with

$x_i \in k$ (not all x_i zero), and we identify $[x_0 : x_1 : \cdots : x_n]$ with $[\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$ for all $\lambda \in k^\times$. Then $[x_0 : x_1 : \cdots : x_n]$ corresponds to a line through the origin, and $(x_{0/i}, x_{1/i}, \ldots, x_{n/i})$ is where this line meets the hyperplane $x_i = 1$.

We will later give other definitions of projective space, and the x_i will be called **projective coordinates** there. Our first definition here will often be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from our current definition. Furthermore, as noted by Herman Weyl, " The introduction of numbers as coordinates is an act of violence."

5.4.4 The Chinese Remainder Theorem is a geometric fact

The Chinese Remainder Theorem is embedded in what we have done. We will see this in a single example, and will give the general statement.

Theorem 5.4.2 (Chinese Remainder Theorem)

Let A be a ring and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ ideals of A. Define a homomorphism

$$\varphi : A \longrightarrow \prod_{i=1}^{n} (A/\mathfrak{a}_i)$$

by the rule $\varphi(x) = (x + \mathfrak{a}_1, \ldots, x + \mathfrak{a}_n)$.

(i) If $\mathfrak{a}_i$, $\mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$.

(ii) φ is surjective if and only if $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$.

(iii) φ is injective if and only if $\bigcap \mathfrak{a}_i = (0)$.

Proof See Atiyah-MacDonald [1] Proposition 1.10. □

The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here is how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the prime ideals of this ring? Answer: those prime ideals of $\mathbb{Z}$ containing (60), i.e., those primes dividing 60, i.e., (2), (3), and (5). Figure 5.9 is a sketch of $\operatorname{Spec} \mathbb{Z}/(60)$. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology (each closed point is also open). What are the stalks? They are $(\mathbb{Z}/(60))_{(2)} = \mathbb{Z}/(4)$, $(\mathbb{Z}/(60))_{(3)} = \mathbb{Z}/(3)$ and $(\mathbb{Z}/(60))_{(5)} = \mathbb{Z}/(5)$. The nilpotents "at (2)" are indicated by the "fuzz" on that point. (We discussed visualizing nilpotents with "infinitesimal fuzz" in §5.2). So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/(60) \xrightarrow{\sim} \mathbb{Z}/(2^2) \times \mathbb{Z}/(3) \times \mathbb{Z}/(5).$$

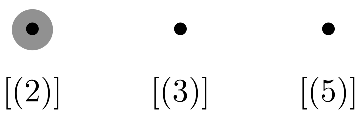


Figure 5.9: A picture of the scheme $\operatorname{Spec} \mathbb{Z}/(60)$

Example 5.7 Here is an example of a function on an open subset of a scheme with some surprising behavior. On $X = \operatorname{Spec} k[w, x, y, z]/(wz - xy)$, consider the open subset $D(y) \cap D(w)$. Clearly the function z/y on $D(y)$ agree with x/w on $D(w)$ on the overlap $D(y) \cap D(w)$. Hence they glue together to give a section. You may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function which has the description z/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition on different parts of the space.

This function has no "single description" as a well-defined expression in terms of w, x, y, z! There is a lot of interesting geometry here, and this scheme will be a constant source of (counter)example for us (§5.1.3). Here is a glimpse, in terms of words we have not yet defined. The space $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are working with the subset $X = \operatorname{Spec} k[w, x, y, z]/(wz - xy)$, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$. The open subset $D(y) \subseteq X$ is X minus some hypersurface, sow we are throwing away codimension 1 locus. You might think that the intersection of these two discarded loci is then codimension 2, and that failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs's Lemma-type theorem. But that's not true — $V(y) \cap V(w)$ is in fact codimension 1. Here is what is actually going on. The space $V(y)$ is obtained by throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(w)$ also involves throwing away the (cone over the) line l, which is a codimension 1 set. Remarkably, despite being "pure codimension 1" the cone over l is not cut out even set-theoretically by a single equation. This means that any expression $f(w, x, y, z)/g(w, x, y, z)$ for our function cannot correctly describe our function on $D(y) \cup D(w)$ — at some point of $D(y) \cup D(w)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(w)$, so g must vanish here. But g can't vanish just on the cone over l — it must vanish elsewhere too.

5.5 Projective schemes, and the Proj construction

Projective schemes are important for a number of reasons. Here are a few. Schemes that were of "classical interest" in geometry — and those that you would have cared about before knowing about schemes — are all projective or an open subset thereof (basically, quasiprojective). Moreover, most "schemes of interest" tend to be projective or quasiprojective. In fact, it is very hard to even give an example of a scheme satisfying basic properties — for example, finite type and "Hausdorff" ("separated") over a field — that is provably not quasiprojective. For complex geometers: it is hard to find a compact complex variety that is provably not projective, and it is quite hard to come up with a complex variety that is provably not an open subset of a projective variety. So projective schemes are really ubiquitous. Also, the notion of "projective k-scheme" is a good approximation of the algebra-geometric version of compactness ("properness").

Finally, although projective schemes may be obtained by gluing together affine schemes, and we know that keeping track of gluing can be annoying, there is a simple means of dealing with them worrying about gluing. Just as there is a rough dictionary between rings and affine schemes, we will have a (slightly looser) analogous dictionary between graded rings and projective schemes. Just as one can work with affine schemes by instead working with rings, one can work with projective schemes by instead working with graded rings.

5.5.1 Motivation from classical geometry

For geometric intuition, we recall how one thinks of projective space "classically" (in the classical topology, over the real numbers). $\mathbb{P}^n$ can be interpreted as the lines through the origin in $\mathbb{R}^{n+1}$. Thus subsets of $\mathbb{P}^n$ correspond to unions of lines through the origin of $\mathbb{R}^{n+1}$, and closed subsets correspond to such unions which are closed. (The same is not true with "closed" replaced by "open"!)

One often pictures $\mathbb{P}^n$ as being the "points at infinite distance" in $\mathbb{R}^{n+1}$, where the points infinitely far in one direction are associated with the points infinitely far in the opposite direction. We can make this more precise using the decomposition

$$\mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n$$

by which we mean that there is an open subset in $\mathbb{P}^{n+1}$ identified with $\mathbb{R}^{n+1}$ (the points with last "projective coordinate" nonzero), and the complementary closed subset identified with $\mathbb{P}^n$ (the points with last "projective coordinate" zero).

Then for example any equation cutting out sone set V of points in $\mathbb{P}^n$ will also cut out some set of points in $\mathbb{R}^{n+1}$ that will be a closed union of lines. We call this the **affine cone** of V. These equations will cut out some union of $\mathbb{P}^1$'s in $\mathbb{P}^{n+1}$, and we call this the **projective cone** of V. The projective cone is the disjoint union of the affine cone and V. For example, the affine cone over $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is just the "classical" picture of a cone in $\mathbb{R}^3$, see Figure 5.10. We will make this analogy precise in our algebraic setting in Chapter 10.

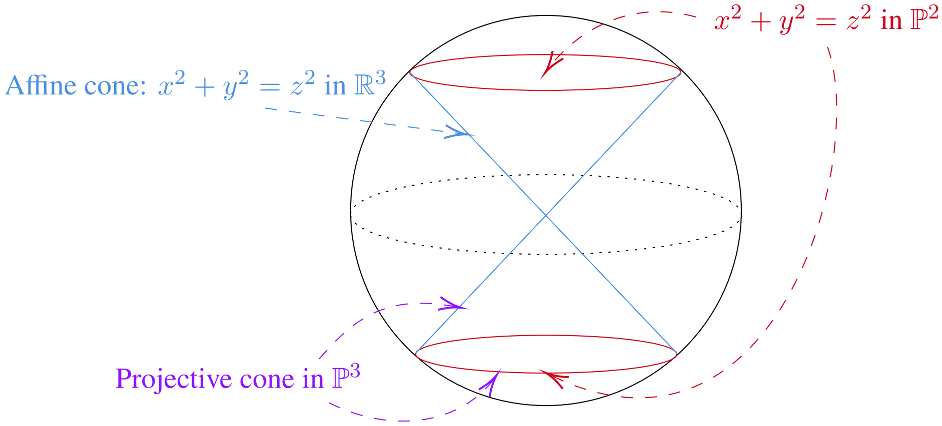


Figure 5.10: The affine and projective cone of $x^2 + y^2 = z^2$ in classical geometry

5.5.2 Projective schemes, a first description

We now describe a construction of projective schemes, which will help motivate the Proj construction. We begin by giving an algebraic interpretation of the cone just described, and more generally, getting some practice with transforming between projective coordinates, and coordinates on the standard affine open subsets. We switch coordinates from x, y, z to x_0, x_1, x_2 in order to use the notation of §5.4.3. For the next questions, $\mathbb{P}^2_k$, we take projective coordinates x_0, x_1, x_2. The big open set (or "coordinate chart") $U_0 = \{[x_0 : x_1 : x_2] : x_0 \neq 0\}$ has coordinates $x_{1/0}$ and $x_{2/0}$, which we interpret as x_1/x_0 and x_2/x_0. We have similar definitions for U_1 and U_2. It will be convenient to define $x_{0/0}$ as 1.

Exercise 5.5 Describe $(x_{0/1}, x_{2/1})$ in terms of $(x_{1/0}, x_{2/0})$.

Solution *Set $U_0 = \{[x_0 : x_1 : x_2] : x_0 \neq 0\}$ and $U_1 = \{[x_0 : x_1 : x_2] : x_1 \neq 0\}$. On $U_0 \cap U_1$, we have*

$$x_{0/1} = \frac{1}{x_{1/0}}, \quad x_{2/1} = \frac{x_{2/0}}{x_{1/0}}.$$

Exercise 5.6 Explain how to define a scheme that should be interpreted as $x_0^2 + x_1^2 - x_2^2 = 0$ "in $\mathbb{P}^2_k$".

Solution *Say X be a scheme which is interpreted as $x_0^2 + x_1^2 - x_2^2 = 0$ "in $\mathbb{P}^2_k$". Let*

$$X_2 := \mathrm{Spec}(k[x_{0/2}, x_{1/2}]/(x_{0/2}^2 + x_{1/2}^2 - 1)),$$
$$X_1 := \mathrm{Spec}(k[x_{0/1}, x_{2/1}]/(x_{0/1}^2 - x_{2/1}^2 + 1)),$$
$$X_0 := \mathrm{Spec}(k[x_{1/0}, x_{2/0}]/(x_{1/0}^2 - x_{2/0}^2 + 1)).$$

We claim that $X = (X_0 \coprod X_1 \coprod X_2) / \sim$, *where* $\sim$ *is given by* $x_{k/i} = x_{k/j}/x_{i/j}$ *and* $x_{k/j} = x_{k/i}/x_{j/i}$. *By Exercise 5.4, gluing information agrees over triple overlaps. By Theorem 5.4.1,* X_i *can glue together. Clearly, each* X_i *is an affine scheme, and therefore* X *is a scheme.*

✍ **Exercise 5.7** Consider the parabola $x_{2/0} = x_{1/0}^2$ (or, if you prefer, $y = x^2$). How does it meet the line at infinity? (What does this mean?) What is its description in the different coordinate patches U_1 and U_2?

Solution *The parabola* $x_{2/0} = x_{1/0}^2$ *lives in* U_0. *It can be written as* $x_1^2 - x_0x_2 = 0$. *The infinite line is* $x_0 = 0$, *consider the following equations.*

$$\begin{cases} x_1^2 - x_0x_2 &= 0, \\ x_0 &= 0. \end{cases}$$

We have $x_1 = 0$. *Hence,* $x_{2/0} = x_{1/0}^2$ *meets line at infinity at point* $[0:0:1] \in \mathbb{P}^2$. *Moreover,* $x_{2/0} = x_{1/0}^2$ *is tangent to the line at infinity at* $[0:0:1]$ *which intersection multiplicity is* 2 *(by Bezout's Theorem).*

On U_1, *the parabola on* U_0 *can be written as* $x_0x_2 = 1$, *it is a hyperbola. On* U_2, *the parabola can be written as* $x_1^2 = x_0$, *it is a parabola.*

Remark (**Degree d hypersurfaces in $\mathbb{P}^n$**.) The degree d homogeneous polynomials in $n+1$ variables over a field form a vector space of dimension $\binom{n+d}{d}$. It is almost true that two polynomials cut out the same subset of $\mathbb{P}_k^n$ if one is a nonzero scalar multiple of the other. Unfortunately, the examples of $x^2y = 0$ and $xy^2 = 0$ show that this isn't quite right. We will late be able to check that two polynomials cut out the same **closed subscheme** (whatever that means) if and only if one is a nonzero scalar multiple of the other. (This is some evidence that the notion of a "closed subscheme" is better than that of a "closed subset".) The zero polynomial doesn't really cut out a hypersurface in any reasonable sense of the word. Thus we informally imagine that "degree d hypersurfaces in $\mathbb{P}^n$ are parametrized by $\mathbb{P}^{\binom{n+d}{d}-1}$". This intuition will come up repeatedly (in special cases), and we will give it precise meaning in Chapter 26. (We will properly define **hypersurfaces** in Chapter 10, once we have the language of closed subschemes. At that time we will also define line, hyperplane, quadric hypersurfaces, conic curves, and other wondrous notions.)

✍ **Exercise 5.8** Consider $\mathbb{P}_A^n$, with projective coordinates $x_0, \ldots, x_n$. Given a collection of homogeneous polynomials $f_i \in A[x_0, \ldots, x_n]$, make sense of the scheme "cut out in $\mathbb{P}_A^n$ by the f_i." (This will later be made precise as an example of a "vanishing scheme".)

Solution *Let* $V_i = \operatorname{Spec}\left(A[x_{0/i}, \ldots, x_{n/i}] \middle/ \left(\frac{f_1}{x_i^{\deg f_1}}, \ldots, \frac{f_m}{x_i^{\deg f_m}}\right)\right)$, *define the scheme "cut out in* $\mathbb{P}_A^n$ *by the* f_i*" by setting*

$$V = \left(\coprod_i V_i\right) \Big/ \sim,$$

where $\sim$ *is given by* $x_{k/i} = x_{k/j}/x_{i/j}$ *and* $x_{k/j} = x_{k/i}/x_{j/i}$. *By the construction,* V *is indeed a scheme, and each* f_i *vanished on* V.

Remark This could be taken as the definition of a **projective A-scheme**.

5.5.3 Preliminaries on graded rings

The Proj construction produces a scheme out of a graded ring. We now give some background on graded rings.

Definition 5.5.1 ($\mathbb{Z}$-graded rings, Homogeneous ideals)

*A $\mathbb{Z}$-**graded ring** is a ring $S_\bullet = \bigoplus_{n\in\mathbb{Z}} S_n$ (the subscript is called the **grading**), where multiplication respects the grading, i.e., $S_m S_n \subseteq S_{m+n}$. Clearly S_0 is a subring, each S_n is an S_0-module, and $S_\bullet$ is a S_0-algebra. Those elements of some S_n are called **homogeneous elements** of $S_\bullet$.*
*An ideal I of $S_\bullet$ is a **homogeneous ideal** (or a **graded ideal**) if it is generated by homogeneous elements. Nonzero homogeneous elements have an obvious **degree**. An element of S_d is called a **form of degree** d.*

Proposition 5.5.1

(a) Ideal I is homogeneous if and only if it contains the degree n piece of each of its elements for each n. (Hence, we can decomposed into homogeneous pieces, $I = \bigoplus I_n$, and $S_\bullet/I$ has a natural $\mathbb{Z}$-grading. This is the reason for the name homogeneous ideal.)
(b) The set of homogeneous ideals of a given $\mathbb{Z}$-graded ring $S_\bullet$ is closed under sum, product, intersection, and radical.
(c) A homogeneous ideals $I \subseteq S_\bullet$ is prime if $I \neq S_\bullet$, and if for any homogeneous $a, b \in S_\bullet$, if $ab \in I$, then $a \in I$ or $b \in I$.

Proof

(a) Let I be an homogeneous ideal of graded ring $S_\bullet$, then I is generated by homogeneous elements, say $I = (f_i)_{i\in\mathscr{I}}$, where $\mathscr{I}$ is an index set and $f_i \in S_{d_i}$ for some d_i. Let $f \in I$, we may write $f = \sum_i g_i f_i$, where $g_i \in S_\bullet$. Since $g_i \in S_\bullet$, we may write $g_i = \sum_j g_{ij}$. Hence,

$$f = \sum_i \left(\sum_j g_{ij} \right) f_i = \sum_i \sum_j g_{ij} f_i,$$

clearly, each $g_{ij} f_i \in I$, then we done.
Conversely, if for all $f = \sum_i f_i \in I$ where $f_i \in S_i$, we have $f_i \in I$. We want to show that I is generated by homogeneous elements. Since each elements in I is generated by homogeneous elements which are all belong to I, I is generated by homogeneous elements, and therefore is a homogeneous ideal.
Say $I_n = I \cap S_n$, then we have $I = \bigoplus_n I_n$. Hence,

$$S_\bullet/I = \left(\bigoplus_n S_n \right) \Big/ \left(\bigoplus_n I_n \right) = \bigoplus_n (S_n/I_n).$$

Note that $(S_n/I_n)(S_m/I_m) \subseteq S_{n+m}/I_{n+m}$, $S_\bullet/I$ is a graded ring.

(b) Let $\mathscr{S}$ be the set of homogeneous ideals of a given $\mathbb{Z}$-graded ring $S_\bullet$. Let $I_1, I_2 \in \mathscr{S}$.
  (i) $I_1 + I_2$ is homogeneous ideal.
  Let $f + g \in I_1 + I_2$, where $f \in I_1$ and $g \in I_2$, then we may write $f = \sum_i f_i$ and $g = \sum_i g_i$, where $f_i, g_i \in S_i$. Hence, $f + g = \sum_i (f_i + g_i)$. Since I_1 and I_2 are homogeneous ideals, we have $f_i \in I_1$ and $g_i \in I_2$, and therefore $f_i + g_i \in I_1 + I_2$ for all i. By part (a), $I_1 + I_2$ is homogeneous ideal.
  (ii) $I_1 I_2$ is homogeneous ideal.
  Let $f = \sum_i f_i \in I_1$ and $g = \sum_i g_i \in I_2$, where $f_i, g_i \in S_i$, then

$$fg = \sum_{i,j} f_i g_j.$$

  Since Since I_1 and I_2 are homogeneous ideals, we have $f_i \in I_1$ and $g_i \in I_2$, and therefore $f_i g_j \in I_1 I_2$ for all i. By part (a), $I_1 I_2$ is homogeneous ideal.
  (iii) $I_1 \cap I_2$ is homogeneous ideal.

Let $f = \sum_i f_i \in I_1 \cap I_2$, since I_1 and I_2 are both homogeneous ideal $f_i \in I_1$ and $f_i \in I_2$. It follows that $f_i \in I_1 \cap I_2$. By part (a), $I_1 \cap I_2$ is homogeneous ideal.

(iv) $\sqrt{I}$ is homogeneous ideal, where $I \in \mathscr{S}$.

Let $f = \sum_i f_i \in \sqrt{I}$, then $f^n \in I$. Note that

$$f^n = \left(\sum_i f_i\right)^n = \sum_{i_0+i_1+\cdots=n} f_0^{i_0} f_1^{i_1} f_2^{i_2} \cdots ,$$

we want to show that each $f_0^{i_0} f_1^{i_1} \cdots$ is belong to $\sqrt{I}$. Let f_{n_0} is the first term which is not zero in f, then the lowest term in f^n is $f_{n_0}^n$. Since $f^n \in I$, $f_{n_0}^n \in I$, and therefore $f_{n_0} \in \sqrt{I}$. Hence, $f - f_{n_0} \in \sqrt{I}$. By induction each term of f belong to $\sqrt{I}$, by part (a), it follows that $\sqrt{I}$ is a homogeneous ideal.

(c) If I is not a homogeneous ideal, then there exists $f \notin I$ and $g \notin I$ such that $fg \in I$. We may assume that

$$f = \sum_i f_i, \quad g = \sum_j g_j,$$

then

$$fg = \sum_i \sum_j f_i g_j.$$

Since $f \notin I$, there exists $f_{i_0} \notin I$, similarly, exists $g_{j_0} \notin I$. Since $fg \in I$, we have $f_i g_j \in I$ for all i, j, in particular, $f_{i_0} g_{j_0} \in I$. By the hypothesis, we have $f_{i_0} \in I$ and $g_{j_0} \in I$, which a contradiction.

□

Proposition 5.5.2

If T is a multiplicative subset of $S_\bullet$ containing only homogeneous elements, then $T^{-1}S_\bullet$ has a natural structure as a $\mathbb{Z}$-graded ring.

Proof For each $\frac{a}{t} \in T^{-1}S_\bullet$, since $a \in S_\bullet$, we may write $a = \sum_i a_i$ where $a_i \in S_i$, hence,

$$\frac{a}{t} = \sum_i \frac{a_i}{t}.$$

Let

$$(T^{-1}S_\bullet)_n = \left\{\frac{a}{t} : a \in S_{n+\deg t}, t \in T\right\},$$

then $T^{-1}S_\bullet$ has a natural structure as a $\mathbb{Z}$-graded ring:

$$T^{-1}S_\bullet = \bigoplus_{n\in\mathbb{Z}} (T^{-1}S_\bullet)_n.$$

□

5.5.4 $\mathbb{Z}^{\geq 0}$-graded rings, graded ring over A, and finitely generated graded rings

Definition 5.5.2 ($\mathbb{Z}^{\geq 0}$-graded rings)

*A $\mathbb{Z}^{\geq 0}$**-graded ring** is a $\mathbb{Z}$-graded ring with no elements of negative degree.*

Note ***For the remainder of this note, graded ring will refer to a $\mathbb{Z}^{\geq 0}$-graded ring. Warning: this convention is nonstandard (for good reason).***

Note *From now on, unless otherwise stated, $S_\bullet$ is assumed to be a graded ring.*

Definition 5.5.3 (Graded ring over base ring)

*Fix a ring A, which we call the **base ring**. If $S_0 = A$, we say that $S_\bullet$ is a **graded ring over** A.*

Example 5.8 A key example of Definition 5.5.3 is $A[x_0, \ldots, x_n]$, or more generally $A[x_0, \ldots, x_n]/I$ where I is a homogeneous ideal with $I \cap S_0 = 0$. Here we take the conventional grading on $A[x_0, \ldots, x_n]$, where each x_i has weight 1 (i.e., $\deg x_i = 1$).

Definition 5.5.4 (Irrelevant ideal, Finitely generated graded ring)

*The subset $S_+ := \bigoplus_{i>0} S_i \subseteq S_\bullet$ is an ideal, called the **irrelevant ideal**. (The reason for the name "irrelevant" will be clearer in a few paragraphs.) If the irrelevant ideal S_+ is a finitely generated ideal, we say that $S_\bullet$ is a **finitely generated graded ring over** A. If $S_\bullet$ is generated by S_1 as an A-algebra, we say that $S_\bullet$ is **generated in degree** 1.*

Remark We will later find it useful to interpret "$S_\bullet$ is generated in degree 1" as "the natural map $\mathrm{Sym}^\bullet S_1 \to S_\bullet$ is a surjection". The symmetric algebra construction will be briefly discussed in Chapter 15.

Lemma 5.5.1

S_+ is a homogeneous ideal.

Proof Each element in S_+ can be written as $f = \sum_{i\geq 1} f_i$, note that each $f_i \in S_i$, hence, S_+ is a homogeneous ideal. $\square$

Proposition 5.5.3

(a) A graded ring $S_\bullet$ over A is a finitely generated graded ring (over A) if and only if $S_\bullet$ is a finitely generated graded A-algebra, i.e., generated over $A = S_0$ by a finite number of homogeneous elements of positive degree.

(b) A graded ring $S_\bullet$ over A is Noetherian if and only if $A = S_0$ is Noetherian and $S_\bullet$ is a finitely generated graded ring.

Proof

(a) If graded ring $S_\bullet$ over A is a finitely generated graded ring, by definition, $S_+ \subseteq S_\bullet$ is an finitely generated ideal, say $S_+ = (f_{i_1}, f_{i_2}, \ldots, f_{i_n})$, where $f_{i_j} \in S_{i_j}$ is homogeneous element. Let $g \in S_\bullet$, then x can be written as

$$g = \sum_{i\geq 0} g_i = g_0 + \sum_{i\geq 1} g_i.$$

Note that $\sum_{i\geq 1} g_i$ belong to S_+, we may assume that

$$\sum_{i\geq 1} g_i = \sum_{k=1}^{n} c_k f_{i_k},$$

where $c_k \in S_\bullet$. Each c_k can be written as

$$c_k = c_{k,0} + \sum_{t\geq 0} c_{k,t},$$

then we have

$$\begin{aligned} g =& g_0 + \sum_{k=1}^{n}(c_{k,0} + \sum_{t\geq 0} c_{k,t})f_{i_k} \\ =& g_0 + \sum_{k=1}^{n} c_{k,0}f_{i_k} + \sum_{k=1}^{n}\sum_{t\geq 0} c_{k,t}f_{i_k}. \end{aligned}$$

Repeat above process, since in $\{g_i\}_{i\geq 0}$ there are only finite terms not zero, the whole process stops after a finite number of steps. It follows that g can be written as a polynomial of $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$ with coefficient in A. It follows that $S_\bullet$ is a finitely generated graded A-algebra.

Conversely, if $S_\bullet$ is a finitely generated graded A-algebra. We want to show that $S_\bullet$ over A is a finitely generated ring, it suffices to show that S_+ is an finitely generated ideal of $S_\bullet$. Each element in $S_\bullet$ can be written as an element in $A[f_{i_1}, f_{i_2}, \ldots, f_{i_n}]$, where $f_{i_k} \in S_{i_k}$. Let $g \in S_+$, then

$$g = \sum_{i\geq 1} g_i,$$

where $g_i \in S_i$ and almost every g_i is zero. Each g_i can by written as a polynomial in $A[f_{i_1}, \ldots, f_{i_n}]$ with no terms of g_i belong to A, i.e.,

$$g_i = \sum_{k_1,\ldots,k_n} a_{k_1,\ldots,k_n} f_{i_1}^{k_1} \cdots f_{i_n}^{k_n} = h_1 f_{i_1} + h_2 f_{i_2} + \cdots + h_n f_{i_n},$$

where $h_t \in S_\bullet$. Hence, $S_+ \in (f_{i_1}, \ldots, f_{i_n})$. Clearly, $(f_{i_1}, \ldots, f_{i_n}) \subseteq S_+$, we have $S_+ = (f_{i_1}, \ldots, f_{i_n})$, and therefore $S_\bullet$ over A is a finitely generated graded ring.

(b) If graded ring $S_\bullet$ over A is Noetherian, then S_+ is a finitely generated ideal, and therefore $S_\bullet$ is a finitely generated graded ring over A. Note that $A = S_0 \cong S_\bullet/S_+$, since $S_\bullet$ is Noetherian, $A = S_0$ is Noetherian (Proposition 4.6.14).

Conversely, if $S_\bullet$ is a finitely generated graded ring, by part (a), $S_\bullet$ is a finitely generated graded A-algebra, we may assume that $S_\bullet = A[f_{i_1}, \ldots, f_{i_n}]$, where each f_{i_k} is homogeneous element. Note that A is Noetherian, by Hilbert's Basis Theorem, $S_\bullet$ is Noetherian.

□

5.5.5 The Proj construction

We now define a scheme $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a ($\mathbb{Z}^{\geq 0}$-)graded ring. Here are two examples, to provide a light at the end of the tunnel. If $S_\bullet = A[x_0, \ldots, x_n]$, we will recover $\mathbb{P}^n_A$; and if $S_\bullet = A[x_0, \ldots, x_n]/(f(x_0, \ldots, x_n))$ where f is homogeneous, we will construct something "cut out in $\mathbb{P}^n_A$ by the equation $f = 0$".

As we did with Spec of a ring, we will build $\operatorname{Proj} S_\bullet$ first as a set, then as a topological space, and finally as a ringed space. In our preliminary definition of $\mathbb{P}^n_A$, we glued together $n+1$ well-chosen affine pieces, but we don't want to make any choices, but we don't want to make any choices, so we do this by simultaneously consider "all possible" affine open sets. Our affine building blocks will be as follows. For each homogeneous $f \in S_+$, note that the localization $(S_\bullet)_f$ is naturally a $\mathbb{Z}$-graded ring, where $\deg(1/f) = -\deg f$. Consider

$$\operatorname{Spec}((S_\bullet)_f)_0, \tag{5.12}$$

where $((S_\bullet)_f)_0$ means the 0-graded piece of the graded ring $(S_\bullet)_f$. (These will be our affine building blocks, as f varies over the homogeneous elements of S_+.) The notation $((S_\bullet)_f)_0$ is admittedly horrible — the first and third subscripts refer to the grading, and the second refers to localization. As motivation for considering this construction: applying this to $S_\bullet = k[x_0, \dots, x_n]$, with $f = x_i$, we obtain the ring appearing in (5.11):

$$k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1).$$

(Before we begin the construction: another possible way of defining $\operatorname{Proj} S_\bullet$ is by gluing together affines of this form.)

Definition 5.5.5 (The points of $\operatorname{Proj} S_\bullet$)

*The **points** of $\operatorname{Proj} S_\bullet$ are the homogeneous prime ideals of $S_\bullet$ not containing the irrelevant ideal S_+ (the "relevant homogeneous prime ideals").*

Proposition 5.5.4

Suppose $f \in S_+$ is homogeneous.

(a) There is a bijection between $\operatorname{Spec}((S_\bullet)_f)_0$ and the homogeneous prime ideals of $(S_\bullet)_f$.

(b) $\operatorname{Spec}((S_\bullet)_f)_0$ is a subset of $\operatorname{Proj} S_\bullet$.

Proof

(a) Let $d = \deg f$. Let $\mathfrak{q}$ be the homogeneous prime ideals of $(S_\bullet)_f$, note that we have a natural embedding $((S_\bullet)_f)_0 \hookrightarrow (S_\bullet)_f$, then $\mathfrak{q} \cap ((S_\bullet)_f)_0$ is a prime ideal of $((S_\bullet)_f)_0$, i.e., $\mathfrak{q} \cap ((S_\bullet)_f)_0 \in \operatorname{Spec}((S_\bullet)_f)_0$. Conversely, let $\mathfrak{p} \in \operatorname{Spec}((S_\bullet)_f)_0$. Define $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$, where $\mathfrak{q}_i \subseteq ((S_\bullet)_f)_i$ and $a \in \mathfrak{q}_i$ if and only if $a^d/f^i \in \mathfrak{p}$. It is easy to see that $\mathfrak{p} = \mathfrak{q}_0$. We want to show that $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ is a homogeneous prime ideals of $(S_\bullet)_f$.

(i) $\mathfrak{q}$ is an ideal of $(S_\bullet)_f$.

Let $a \in \mathfrak{q}_i$, then $a \in \mathfrak{q}_i$ iff $\frac{a^d}{f^i} \in \mathfrak{p}$ iff $\left(\frac{a^d}{f^i}\right)^2 \in \mathfrak{p}$ iff $a^2 \in \mathfrak{q}_{2i}$.

Let $a_1, a_2 \in \mathfrak{q}_i$, we want to show that $a_1 + a_2 \in \mathfrak{q}_i$, it suffices to show that $(a_1 + a_2)^{2d}/f^{2i} \in \mathfrak{p}$. In fact,

$$\begin{aligned}\frac{(a_1+a_2)^{2d}}{f^{2i}} &= \frac{\sum_{k=0}^{2d} \binom{2d}{k} a_1^k a_2^{2d-k}}{f^{2i}} \\ &= \frac{a_2^{2d}}{f^{2i}} + \sum_{k=1}^{d} \binom{2d}{k} \frac{a_1^k a_2^{d-k}}{f^i} \cdot \frac{a_2^d}{f^i} + \sum_{k=d+1}^{2d-1} \binom{2d}{k} \frac{a_1^{k-d} a_2^{2d-k}}{f^i} \cdot \frac{a_1^d}{f^i} + \frac{a_1^{2d}}{f^{2i}}.\end{aligned}$$

For $1 \le k \le d$, $a_1^k a_2^{d-k} \in S_{ki} S_{(d-k)i} \subseteq S_{di}$, and therefore $\binom{2d}{k} \frac{a_1^k a_2^{d-k}}{f^i} \in (S_d)_f$. Similarly, for $d+1 \le k \le 2d-1$, $\binom{2d}{k} \frac{a_1^{k-d} a_2^{2d-k}}{f^i} \in (S_d)_f$. Note that $\frac{a_2^{2d}}{f^{2i}}, \frac{a_2^d}{f^i}, \frac{a_1^d}{f^i}, \frac{a_1^{2d}}{f^{2i}} \in \mathfrak{p}$, we have

$$\frac{(a_1+a_2)^{2d}}{f^{2i}} \in \mathfrak{p}.$$

Hence, $(a_1 + a_2)^2 \in \mathfrak{q}_{2i}$, and therefore $a_1 + a_2 \in \mathfrak{q}_i$.

Let $a \in \mathfrak{q}_i$, then $\frac{a^d}{f^i} \in \mathfrak{p}$, hence, $(-1)^d \frac{a^d}{f^i} = \frac{(-a)^d}{f^i} \in \mathfrak{p}$, i.e., $-a \in \mathfrak{q}_i$.

By above discussion, $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ is an ideal of $(S_\bullet)_f$.

(ii) $\mathfrak{q}$ is a homogeneous ideal of $(S_\bullet)_f$.

Since each $\mathfrak{q}_i \subseteq ((S_\bullet)_f)_i$, clearly, $\mathfrak{q}$ is a homogeneous ideal.

(iii) $\mathfrak{q}$ is prime ideal.

For any $a \in ((S_\bullet)_f)_i$ and $b \in ((S_\bullet)_f)_j$, suppose $ab \in \mathfrak{q}$. We want to show that $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. We may assume the $b \notin \mathfrak{q}$. Let $a = \frac{s_{i+n}}{f^n}$ and $b = \frac{s_{j+m}}{f^m}$, then $ab = \frac{s_{i+n}s_{j+m}}{f^{m+n}} \in \mathfrak{q}_{i+j}$, hence, $\frac{(ab)^d}{f^{i+j}} \in \mathfrak{p}$. Since $b \notin \mathfrak{q}$, we have $b \notin \mathfrak{q}_j$, i.e., $\frac{b^d}{f^j} \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, we have $\frac{a^d}{f^i} \in \mathfrak{p}$, i.e., $a \in \mathfrak{q}_i \subseteq \mathfrak{q}$.

By Proposition 5.5.1, we done!

Define φ as follow:

$$\begin{aligned}\varphi : \operatorname{Spec}((S_\bullet)_f)_0 &\longrightarrow \{\text{the homogeneous prime ideals of } (S_\bullet)_f\}\\ \mathfrak{p} &\longmapsto \mathfrak{q} = \bigoplus_i \mathfrak{q}_i,\end{aligned}$$

where $\mathfrak{q}_i \subseteq (S_{i+d})_f$ and $a \in \mathfrak{q}_i$ if and only if $\frac{a^d}{f^i} \in \mathfrak{p}$. Define ψ as follow:

$$\begin{aligned}\psi : \{\text{the homogeneous prime ideals of } (S_\bullet)_f\} &\longrightarrow \operatorname{Spec}((S_\bullet)_f)_0\\ \mathfrak{q} &\longmapsto \mathfrak{q} \cap ((S_\bullet)_f)_0.\end{aligned}$$

By above discussion, φ and ψ are all well-defined.

Let $\mathfrak{p} \in \operatorname{Spec}((S_\bullet))_0$, then

$$\psi \circ \varphi(\mathfrak{p}) = \psi(\cdots \oplus \mathfrak{q}_0 \oplus \cdots) = \mathfrak{q}_0 = \mathfrak{p}.$$

Let $\mathfrak{q} = \bigoplus_i \mathfrak{p}_i \in \{\text{the homogeneous prime ideals of } (S_\bullet)_f\}$, then

$$\varphi \circ \psi(\mathfrak{q}) = \varphi(\mathfrak{q}_0),$$

we want to show that $\varphi(\mathfrak{q}_0) = \mathfrak{q}$. It suffices to check that for each $\mathfrak{q}_i$, $a \in \mathfrak{q}_i$ if and only if $\frac{a^d}{f^i} \in \mathfrak{q}_0$. Let $a \in \mathfrak{q}_i$, then $a^d \in \mathfrak{q}_{id}$, and therefore $\frac{a^d}{f^i} \in \mathfrak{q}_0$. Conversely, if $\frac{a^d}{f^i} \in \mathfrak{q}_0 \subseteq \mathfrak{q}$, since $f \notin \mathfrak{q}$, $a^d \in \mathfrak{q}$. Note that $\mathfrak{q}$ is prime ideal, we have $a \in \mathfrak{q}$. Since $d \cdot \deg a - di = 0$, i.e., $\deg a = i$, hence, $a \in \mathfrak{q}_i$. Thus,

$$\varphi \circ \psi(\mathfrak{q}) = \mathfrak{q}.$$

It follows that there is a bijection

$$\{\text{the homogeneous prime ideals of } (S_\bullet)_f\} \longleftrightarrow \operatorname{Spec}((S_\bullet)_f)_0.$$

(b) It suffices to check each element in

$$\mathscr{S} = \{\text{the homogeneous prime ideals of } (S_\bullet)_f\}$$

not containing the irrelevant ideal S_+. Let $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i \in \mathscr{S}$. $\mathscr{S}$ as a subset of $\operatorname{Proj} S_\bullet$, $\mathfrak{q}$ must not containing $f \in S_+$, and therefore not containing S_+. It follows that

$$\operatorname{Spec}((S_\bullet)_f)_0 \longleftrightarrow \mathscr{S} \hookrightarrow \operatorname{Proj} S_\bullet,$$

as we desired.

□

The correspondence of the points of $\operatorname{Proj} S_\bullet$ with homogeneous prime ideals helps us picture $\operatorname{Proj} S_\bullet$. For example, if $S_\bullet = k[x, y, z]$ with the usual grading, then we picture the homogeneous prime ideal $(z^2 - x^2 - y^2)$ first as a subset of $\operatorname{Spec} S_\bullet$; it is a cone (see Figure 5.10). As in §5.5.1, we picture $\mathbb{P}^2_k$ as the "plane at infinity". Thus we picture this equation as cutting out a conic "at infinity" (in $\operatorname{Proj} S_\bullet$). We will make this intuition somewhat more precise in Chapter 10.

Motivated by the affine case, we can define **projective vanishing set** and **projective distinguished open set**:

Definition 5.5.6 (Projective vanishing set, projective distinguished open set)

*If T is a set of homogeneous elements of $S_\bullet$, define the **projective vanishing set of** T, $V(T) \subseteq \operatorname{Proj} S_\bullet$, to be those homogeneous prime ideals containing T but not S_+, i.e.,*

$$V(T) := \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : \mathfrak{p} \supseteq T, \mathfrak{p} \not\supseteq S_+\}.$$

Define $V(f) = \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : \mathfrak{p} \ni f, \mathfrak{p} \not\supseteq S_+\}$ if f is a homogeneous element of positive degree.

Let the ***projective distinguished open set*** $D_+(f) := \operatorname{Proj} S_\bullet \setminus V(f)$ *be the complement of* $V(f)$*. (The subscript is intended to distinguish this notation from the similar* $D(f)$*.)*

Remark Once we define a scheme structure on $\operatorname{Proj} S_\bullet$, we will (without comment) use $D_+(f)$ to refer to the *open subscheme*, not just the open subset. Although the definition of $D_+(f)$ makes sense even if f has degree 0, we deliberately allow only f of positive degree. For example, we will want the $D_+(f)$ to form an affine cover, and if f has degree 0, then $D_+(f)$ needn't be affine.

✍ **Exercise 5.9** Show that $D_+(f)$ "is" (or more precisely, "corresponds to") the subset $\operatorname{Spec}((S_\bullet)_f)_0$ you described in Proposition 5.5.4 (b). For example, in §5.4.3, the $D_+(x_i)$ are the standard open sets covering projective space.

Proof By the definition,

$$D_+(f) = \operatorname{Proj} S_\bullet \setminus V(f) = \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : f \notin \mathfrak{p}, \mathfrak{p} \not\supseteq S_+\}.$$

Since $f \notin \mathfrak{p}$, $\mathfrak{p}$ can be seen as a homogeneous prime ideal of $(S_\bullet)_f$, by Proposition 5.5.4 (a), $\mathfrak{p}$ corresponds to $\mathfrak{p} \cap ((S_\bullet)_f)_0$ in $\operatorname{Spec}((S_\bullet)_f)_0$, i.e.

$$D_+(f) \cong \operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet,$$

then we done. □

Definition 5.5.7 (Zariski topology on $\operatorname{Proj} S_\bullet$)

As in the affine case, the $V(I)$*'s satisfy the axioms of the closed sets of a topology and we call this* ***Zariski topology*** *on* $\operatorname{Proj} S_\bullet$.

Remark Other definitions given in the literature may look superficially different, but can be easily shown to be the same.

Many statements about the Zariski topology on Spec of a ring carry over to this situation with little extra work.

Lemma 5.5.2

$D_+(f) \cap D_+(g) = D_+(fg)$.

Proof

$$\begin{aligned} D_+(f) \cap D_+(g) =& \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : f \notin \mathfrak{p}, g \notin \mathfrak{p}, \mathfrak{p} \not\supseteq S_+\} \\ \subseteq& \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : fg \notin \mathfrak{p}, \mathfrak{p} \not\supseteq S_+\} \\ =& D_+(fg). \end{aligned}$$

Conversely, let $[\mathfrak{p}] \in D(fg)$, then $fg \notin \mathfrak{p}$ and $\mathfrak{p} \not\supseteq S_+$. If $f \in \mathfrak{p}$, since $\mathfrak{p}$ is an ideal, $fg \in \mathfrak{p}$, a contradiction. Hence, $f \notin \mathfrak{p}$, similarly, $g \notin \mathfrak{p}$. It follows that

$$D_+(f) \cap D_+(g) = D_+(fg).$$

□

Proposition 5.5.5

The projective distinguished open sets $D_+(f)$ *(as* f *runs through the homogeneous elements of* S_+*) form a base of the Zariski topology.*

Proof Let $U = \operatorname{Proj} S_\bullet \setminus V(T)$ be any open set, where $T \subseteq \operatorname{Proj} S_\bullet$. Then

$$\begin{aligned} U =& \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : \mathfrak{p} \not\supseteq T, \mathfrak{p} \not\supseteq S_+\} \\ =& \bigcup_{f \in T} \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : f \notin \mathfrak{p}, \mathfrak{p} \not\supseteq S_+\} \\ =& \bigcup_{f \in T} D_+(f). \end{aligned}$$

It follows that the projective distinguished open sets $D_+(f)$ (as f runs through the homogeneous elements of S_+) form a base of the Zariski topology. □

Proposition 5.5.6

Let $S_\bullet$ be a graded ring. Suppose I is any homogeneous ideal of $S_\bullet$ contained in S_+, and f is a homogeneous element of positive degree. Then f vanishes on $V(I)$ (i.e., $V(I) \subseteq V(f)$) if and only if $f^n \in I$ for some n. In particular, $D_+(f) \subseteq D_+(g)$ if and only if $f^n \in (g)$ for some n. (Here g is also homogeneous of positive degree.)

Proof Let f be a homogeneous element of positive degree and $I \subseteq S_+$ is any homogeneous ideal of $S_\bullet$.

If f vanishes on $V(I)$. Suppose that $f^n \notin I$ for any n, i.e., $f \notin \sqrt{I}$, hence, there exist a prime ideal $\mathfrak{p} \supseteq I$ such that $f \notin \mathfrak{p}$. Let $\mathfrak{p}^* =$ (homogeneous elements of $\mathfrak{p}$), clearly, $\mathfrak{p}^*$ is a homogeneous prime ideal which contains I. We want to show that $\mathfrak{p}^*$ is prime. Let a, b be two homogeneous elements such that $a \notin \mathfrak{p}^*$ and $b \notin \mathfrak{p}^*$, then $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$, since $\mathfrak{p}$ is prime, homogeneous element $ab \notin \mathfrak{p}$, and therefore $ab \notin \mathfrak{p}^*$, by Proposition 5.5.1 (c), $\mathfrak{p}^*$ is prime. Then we get a homogeneous prime ideal $\mathfrak{p}^*$ which contains I. Since f is a homogeneous element, $f \notin \mathfrak{p}^*$, and therefore $\mathfrak{p}^* \not\supseteq S_+$, hence, $\mathfrak{p}^* \in \operatorname{Proj} S_\bullet$. Since $\mathfrak{p}^* \supseteq I$, $[\mathfrak{p}^*] \in V(I)$. Since f vanishes on $V(I)$, we have $f \in \mathfrak{p}^*$, but $f \notin \mathfrak{p}^*$, a contradiction. Hence, $f \in \sqrt{I}$.

Conversely, if $f \in \sqrt{I}$. We want to show that $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in V(I)$. Note that

$$\sqrt{I} = \bigcap_{[\mathfrak{p}] \in \operatorname{Spec} I} \mathfrak{p} \subseteq \bigcap_{[\mathfrak{p}] \in V(I)} \mathfrak{p},$$

we have $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in V(I)$, i.e., f vanish on $V(I)$.

In particular, $D_+(f) \subseteq D_+(g)$ if and only if $V(f) \supseteq V(g)$ if and only if $f^n \in I$ for some n. □

Definition 5.5.8 (Function $I(\cdot)$)

Let $S_\bullet$ be a graded ring. Given a subset $Z \subseteq \operatorname{Proj} S_\bullet$, $I(Z)$ is the set of functions vanishing on S. In other words,

$$I(Z) := \{f \in S_\bullet : f \in \mathfrak{p}, \forall [\mathfrak{p}] \in Z\} = \bigcap_{[\mathfrak{p}] \in Z} \mathfrak{p} \subseteq S_\bullet.$$

Proposition 5.5.7

Let $S_\bullet$ be a graded ring.

(a) *If $Z \subseteq \operatorname{Proj} S_\bullet$, then $I(Z)$ is a homogeneous ideal of $S_\bullet$.*
(b) *For any two subset, $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.*
(c) *For any subset $Z \subseteq \operatorname{Proj} S_\bullet$, $V(I(Z)) = \overline{Z}$.*

Proof

(a) Let $f \in I(Z)$, then $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in Z$. Write f as $f = \sum_{i \geq 0} f_i$. Since each $[\mathfrak{p}] \in Z$ as an ideal is homogeneous prime ideal, each $f_i \in \mathfrak{p}$ for all $[\mathfrak{p}] \in Z$. Hence, $f_i \in I(Z)$, it follows that $I(Z)$ is a homogeneous ideal.

(b) Let Z_1, Z_2 be any two subset. Let $f \in I(Z_1 \cup Z_2)$, then $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in Z_1 \cup Z_2$, hence $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in Z_1$ and $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in Z_2$, that is, $I(Z_1 \cup Z_2) \subseteq I(Z_1) \cap I(Z_2)$.
Conversely, let $f \in I(Z_1) \cap I(Z_2)$, then $f \in \mathfrak{p}$ for all $\mathfrak{p} \in Z_1$ and $f \in \mathfrak{p}$ for all $\mathfrak{p} \in Z_2$, i.e., $f \in \mathfrak{p}$ for all $\mathfrak{p} \in Z_1 \cup Z_2$. Hence, $I(Z_1) \cap I(Z_2) \subseteq I(Z_1 \cup Z_2)$.
By above discussion, we have $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

(c) Let $[\mathfrak{p}] \in Z$, then $f \in \mathfrak{p}$ for all $f \in I(Z)$, hence, $\mathfrak{p} \supseteq I(Z)$, and therefore $Z \subseteq V(I(Z))$. Since $V(I(Z))$ is closed, we have $\overline{Z} \subseteq V(I(Z))$.
Let $V(J)$ be any closed subset which contains Z, then for all $[\mathfrak{p}] \in Z$ we have $\mathfrak{p} \supseteq J$. It follows that
$$J \subseteq \bigcap_{[\mathfrak{p}] \in Z} \mathfrak{p} = I(Z),$$
and therefore $V(J) \supseteq V(I(Z))$. Hence,
$$\overline{Z} = \bigcap_{V(J) \supseteq Z} V(J) \supseteq V(I(Z)),$$
which implies that $V(I(Z)) = \overline{Z}$.

□

Proposition 5.5.8

Let $S_\bullet$ be a graded ring, and $I \subseteq S_+$ be a homogeneous ideal. The following are equivalent.

(a) $V(I) = \varnothing$.

(b) For any homogeneous f_i (as i runs through some index set) generating I, $\bigcup D_+(f_i) = \operatorname{Proj} S_\bullet$.

(c) $\sqrt{I} \supseteq S_+$.

Proof Since I is a homogeneous ideal, I is generated by homogeneous elements, say $I = (f_i)_{i \in \mathscr{I}}$, where $\mathscr{I}$ is an index set.

(a) $\Rightarrow$ (b): Clearly, $D_+(f_i) \subseteq \operatorname{Proj} S_\bullet$, then $\bigcup_{i; \in \mathscr{I}} D_+(f_i) \subseteq \operatorname{Proj} S_\bullet$. Conversely, let $[\mathfrak{p}] \in \operatorname{Proj} S_\bullet$. Since $V(I) = \varnothing$, $\mathfrak{p} \not\supseteq I$, then exists f_i for some $i \in \mathscr{I}$ such that $f_i \notin \mathfrak{p}$, and therefore $\mathfrak{p} \in D_+(f_i)$. It follows that $\operatorname{Proj} S_\bullet \subseteq \bigcup_i D_+(f_i)$. Hence, $\operatorname{Proj} S_\bullet = \bigcup_{i \in \mathscr{I}} D_+(f_i)$.

(b) $\Rightarrow$ (c): If there exists a homogeneous element $f \notin \sqrt{I}$, then exists a prime ideal $\mathfrak{p} \supset I$ such that $f \notin \mathfrak{p}$. By the proof in Proposition 5.5.6, we can construct a homogeneous prime ideal $\mathfrak{p}^*$ from $\mathfrak{p}$ such that $f \notin \mathfrak{p}^*$ and $\mathfrak{p}^* \supseteq I$, hence $\mathfrak{p}^* \not\supseteq S_+$, and therefore $[\mathfrak{p}^*] \in \operatorname{Proj} S_\bullet$. Note that $\operatorname{Proj} S_\bullet = \bigcup_{i \in \mathscr{I}} D_+(f_i)$, $[\mathfrak{p}^*] \in D(f_{i_0})$ for some i_0, which implies that $f_{i_0} \notin \mathfrak{p}^*$, but $I \subseteq \mathfrak{p}^*$, a contradiction! Hence, $\sqrt{I} \supseteq S_+$.

(c) $\Rightarrow$ (a): Since $\sqrt{I} \supseteq S_+$, we have $V(\sqrt{I}) \subseteq V(S_+) = \varnothing$. It suffices to show that $V(I) = V(\sqrt{I})$. Since $I \subseteq \sqrt{I}$, we have $V(\sqrt{I}) \subseteq V(I)$. Let $[\mathfrak{p}] \in V(I)$, then $\mathfrak{p} \supseteq I$. Let $f \in \sqrt{I}$, then $f^n \in I$ for some n, hence, $f^n \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, $f \in \mathfrak{p}$, it follows that $\sqrt{I} \subseteq \mathfrak{p}$, and therefore $[\mathfrak{p}] \in V(\sqrt{I})$. Hence, $V(I) = V(\sqrt{I})$. □

Remark This is more motivation for the ideal S_+ being "irrelevant": any ideal whose radical contains it is "geometrically irrelevant".

We now construct $\operatorname{Proj} S_\bullet$ as a scheme.

Proposition 5.5.9

Let $f \in S_+$ is a homogeneous element. Via the inclusion

$$D_+(f) = \operatorname{Spec}((S_\bullet)_f)_0 \longrightarrow \operatorname{Proj} S_\bullet$$

of Exercise 5.9, then the Zariski topology on $\operatorname{Proj} S_\bullet$ *restricts to the Zariski topology on* $\operatorname{Spec}((S_\bullet)_f)_0$.

Proof $\operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet$ gives an induced topology on $\operatorname{Spec}((S_\bullet)_f)_0$. In fact, by Proposition 5.5.4,

$$\operatorname{Spec}((S_\bullet)_f)_0 \longleftrightarrow \{\text{the homogeneous prime ideals of } (S_\bullet)_f\},$$

the elements in $\operatorname{Spec}((S_\bullet)_f)_0$ can be seen as the homogeneous prime ideals of $(S_\bullet)_f$, moreover they can be seen as the elements in $\operatorname{Proj} S_\bullet$ which not meet f.

Let $V_{\operatorname{Spec}}(I)$ be any closed subset of $\operatorname{Spec}((S_\bullet)_f)_0$. We want to show exists a closed subset V_{Proj} of $\operatorname{Proj} S_\bullet$ such that $V_{\operatorname{Spec}}(I) = V_{\operatorname{Proj}} \cap \operatorname{Spec}((S_\bullet)_f)_0$. Consider $V_{\operatorname{Spec}}(I)$,

$$\begin{aligned} V_{\operatorname{Spec}}(I) =& \{[\mathfrak{p}] \in \operatorname{Spec}((S_\bullet)_f)_0 : \mathfrak{p} \supseteq I\} \\ =& \{[\mathfrak{p}] \in \operatorname{Proj}(S_\bullet)_f : \mathfrak{p} \supseteq \widetilde{I}\} \\ =& \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet : \mathfrak{p} \supseteq \widetilde{I} \cap S_\bullet, f \notin \mathfrak{p}\} \\ =& \operatorname{Proj} S_\bullet \cap V_{\operatorname{Proj}}(\widetilde{I} \cap S_\bullet), \end{aligned}$$

note that $V_{\operatorname{Proj}}(\widetilde{I} \cap S_\bullet)$ closed in $\operatorname{Proj} S_\bullet$, it follows that the topology on $\operatorname{Spec}((S_\bullet)_f)_0$ same as induced topology on $\operatorname{Spec}((S_\bullet)_f)_0$. $\square$

Now that we have defined $\operatorname{Proj} S_\bullet$ as a topological space, we are ready to define the structure sheaf. On $D_+(f)$, we wish it to be the structure sheaf of $\operatorname{Spec}((S_\bullet)_f)_0$. We will glue these sheaves together using Theorem 3.5.3 on gluing sheaves.

Proposition 5.5.10

If $f, g \in S_+$ are homogeneous and nonzero, there is an isomorphism between $\operatorname{Spec}((S_\bullet)_{fg})_0$ *and the distinguished open subset $D(g^{\deg f}/f^{\deg g})$ of* $\operatorname{Spec}((S_\bullet)_f)_0$.

Proof Say $A = S_\bullet$, $k = \deg f$, and $l = \deg g$. By Proposition 5.3.2,

$$D\left(\frac{g^k}{f^l}\right) \cong \operatorname{Spec}((A_f)_0)_{\frac{g^k}{f^l}}.$$

By Proposition 5.3.1, it suffice to show that $(A_{fg})_0 \cong ((A_f)_0)_{\frac{g^k}{f^l}}$.

In fact, we have

$$(A_{fg})_0 = \left\{\frac{a}{(fg)^n} \in A_{fg} : a \text{ homogeneous elements in } A, \deg a = n(k+l), \forall n \geq 0\right\}$$

and

$$(A_f)_0 = \left\{\frac{a}{f^n} \in A_f : a \text{ homogeneous elements in } A, \deg a = nk, \forall n \geq 0\right\}.$$

Define $\varphi : (A_f)_0 \to (A_{fg})_0$ by setting

$$\frac{a}{f^n} \longmapsto \frac{ag^n}{(fg)^n}.$$

We shall to show that φ is well-defined. Let $\frac{a}{f^n} = \frac{b}{f^m}$ in $(A_f)_0$, then there exists t such that $f^t(af^m - bf^n) = 0$, hence, $f^t(af^m g^{n+m} - bf^n g^{n+m}) = 0$, which implies that $\frac{ag^n}{(fg)^n} = \frac{bg^m}{(fg)^m}$, i.e., $\varphi\left(\frac{a}{f^n}\right) = \varphi\left(\frac{b}{f^m}\right)$. It follows that φ is well-defined. Clearly, φ is a ring homomorphism.

Consider $\varphi\left(\frac{g^k}{f^l}\right)$, we have

$$\varphi\left(\frac{g^k}{f^l}\right)=\frac{g^{k+l}}{(fg)^l}=\frac{g^k}{f^l}.$$

Note that $\frac{f^l}{g^k}=\frac{f^{l+k}}{(fg)^k}\in(A_{fg})_0$ and $\frac{g^k}{f^l}\cdot\frac{f^l}{g^k}=1$, $\frac{g^k}{f^l}$ is a unit in $(A_{fg})_0$. By the universal property of localization, there exists unique ring homomorphism $\widetilde{\varphi}:((A_f)_0)_{\frac{g^k}{f^l}}\to(A_{fg})_0$, it is defined by

$$\frac{\frac{a}{f^n}}{(\frac{g^k}{f^l})^m}\longmapsto\frac{\varphi\left(\frac{a}{f^n}\right)}{\varphi\left(\frac{g^k}{f^l}\right)^m},$$

where $\frac{a}{f^n}\in(A_f)_0$. We want to show that $\widetilde{\varphi}$ is a ring isomorphism.

Let $\frac{\varphi\left(\frac{a}{f^n}\right)}{\varphi\left(\frac{g^k}{f^l}\right)^m}=\frac{\frac{ag^n}{(fg)^n}}{\frac{g^{(k+l)m}}{(fg)^{lm}}}=\frac{af^{lm}}{f^ng^{km}}=\frac{af^{(k+l)m}g^n}{(fg)^{km+n}}=0$ in $(A_{fg})_0$, then exists s such that $a(fg)^sf^{(k+l)m}g^n=af^{s+(k+l)m}g^{n+s}=0$, hence, $af^{s+(k+l)m}g^{k(n+s)}=0$. It follows that $\frac{g^{k(n+s)}a}{f^{l(n+s)+n}}=\left(\frac{g^k}{f^l}\right)^{n+s}\left(\frac{a}{f^n}\right)=0$ in A_f. Hence, $\frac{\frac{a}{f^n}}{\left(\frac{g^k}{f^l}\right)^m}=0$, which implies that $\widetilde{\varphi}$ is injective.

Let $\frac{a}{(fg)^p}\in(A_{fg})_0$, where a is homogeneous element in A and $\deg a=p(k+l)$. It suffices to show that exists $b\in A$ such that

$$\frac{a}{(fg)^p}=\frac{b}{f^n}\cdot\frac{1}{\left(\frac{g^k}{f^l}\right)^m},$$

i.e.,

$$b=af^{n-ml-p}g^{mk-p}.$$

Pick $m=\lceil\frac{p}{k}\rceil$ and $n=\lceil ml+p\rceil$, we have

$$n-ml-p\geq0,\quad mk-p\geq0,$$

and therefore $b\in A$. Hence, $\widetilde{\varphi}$ is surjective.

Consequently, $(A_{fg})_0\cong((A_f)_0)_{\frac{g^k}{f^l}}$. □

Similarly, $\operatorname{Spec}((S_\bullet)_{fg})_0$ is identified with a distinguished open subset of $\operatorname{Spec}((S_\bullet)_g)_0$. We then glue the various $\operatorname{Spec}((S_\bullet)_f)_0$ (as f varies) altogether, using these pair wise gluings.

By Proposition 5.5.5, we have $\operatorname{Proj}S_\bullet=\bigcup\limits_{f\in S_\bullet}D_+(f)$ (as topological space). By Exercise 5.9, we have $D_+(f)\cong\operatorname{Spec}((S_\bullet)_f)_0$, hence, for each projective distinguished open subset we have structure sheaf $(D_+(f),\mathscr{O}_{\operatorname{Spec}((S_\bullet)_f)_0})$. Let $f_i,f_j\in S_+$, then $D_+(f_if_j)=\operatorname{Spec}((S_\bullet)_{f_if_j})_0$. By Proposition 5.5.10, we have isomorphisms

$$\operatorname{Spec}\left((S_\bullet)_{f_if_j}\right)_0\cong D\left(\frac{f_j^{\deg f_i}}{f_i^{\deg f_j}}\right)\subseteq\operatorname{Spec}((S_\bullet)_{f_i})_0=D_+(f_i)$$

and

$$\operatorname{Spec}\left((S_\bullet)_{f_if_j}\right)_0\cong D\left(\frac{f_i^{\deg f_j}}{f_j^{\deg f_i}}\right)\subseteq\operatorname{Spec}((S_\bullet)_{f_j})_0=D_+(f_j).$$

Hence we get an isomorphism,

$$\varphi_{ij}:D\left(\frac{f_j^{\deg f_i}}{f_i^{\deg f_j}}\right)\xrightarrow{\sim}D\left(\frac{f_i^{\deg f_j}}{f_j^{\deg f_i}}\right),$$

where $D\left(\frac{f_j^{\deg f_i}}{f_i^{\deg f_j}}\right)$ is an open subscheme of $D(f_i)$ and $D\left(\frac{f_i^{\deg f_j}}{f_j^{\deg f_i}}\right)$ is an open subscheme of $D(f_j)$.

Note *Say* $D_{ij} := D\left(\frac{f_j^{\deg f_i}}{f_i^{\deg f_j}}\right)$, *then*

$$\varphi_{ij} : D_{ij} \to D_{ji}.$$

Now, we shall to check the cocycle condition in the Theorem 5.4.1:

Exercise 5.10 Checking that these gluings behave well on the triple overlaps (Theorem 5.4.1).

Proof Let $f_i, f_j, f_k \in S_+$, then $D_+(f_i) \cap D_+(f_j) \cap D_+(f_k) = D_+(f_i f_j f_k) = \mathrm{Spec}((S_\bullet)_{f_i f_j f_k})_0$ by Proposition 5.5.10, we have isomorphisms

$$\mathrm{Spec}\left((S_\bullet)_{f_i f_j f_k}\right)_0 \cong D\left(\frac{(f_j f_k)^{\deg f_i}}{(f_i)^{\deg f_j + \deg f_k}}\right) \subseteq \mathrm{Spec}((S_\bullet)_{f_i})_0 = D_+(f_i),$$

$$\mathrm{Spec}\left((S_\bullet)_{f_i f_j f_k}\right)_0 \cong D\left(\frac{(f_i f_k)^{\deg f_j}}{(f_j)^{\deg f_i + \deg f_k}}\right) \subseteq \mathrm{Spec}((S_\bullet)_{f_j})_0 = D_+(f_j),$$

and

$$\mathrm{Spec}\left((S_\bullet)_{f_i f_j f_k}\right)_0 \cong D\left(\frac{(f_i f_j)^{\deg f_k}}{(f_k)^{\deg f_i + \deg f_j}}\right) \subseteq \mathrm{Spec}((S_\bullet)_{f_k})_0 = D_+(f_k).$$

Then we have

$$D\left(\frac{(f_j f_k)^{\deg f_i}}{(f_i)^{\deg f_j + \deg f_k}}\right) \cong D\left(\frac{(f_i f_k)^{\deg f_j}}{(f_j)^{\deg f_i + \deg f_k}}\right) \cong D\left(\frac{(f_i f_j)^{\deg f_k}}{(f_k)^{\deg f_i + \deg f_j}}\right),$$

that is,

$$D_{ij} \cap D_{ik} \cong D_{ji} \cap D_{jk} \cong D_{ki} \cap D_{kj},$$

which implies that the cocycle condition holds. □

By Theorem 5.4.1, each $D_+(f)$ can glue together, we call this scheme $\mathrm{Proj}\, \mathbf{S}_\bullet$.

Proposition 5.5.11

Let $S_\bullet$ be a graded ring, for any $[\mathfrak{p}] \in \mathrm{Proj}\, S_\bullet$, there is an isomorphism,

$$\mathscr{O}_{\mathrm{Proj}\, S_\bullet, [\mathfrak{p}]} \xleftrightarrow{\sim} ((S_\bullet)_\mathfrak{p})_0.$$

Proof In fact, $\mathscr{O}_{\mathrm{Proj}\, S_\bullet, [\mathfrak{p}]} = \varinjlim_{D_+(f) \ni [\mathfrak{p}]} \mathscr{O}_{\mathrm{Proj}\, S_\bullet}(D_+(f)) = \varinjlim_{f \notin \mathfrak{p}} ((S_\bullet)_f)_0$. Define $\theta : \mathscr{O}_{\mathrm{Proj}\, S_\bullet, [\mathfrak{p}]} \to ((S_\bullet)_\mathfrak{p})_0$ by setting

$$\overline{\left(\frac{s}{f^n}, D_+(f)\right)} \longmapsto \frac{s}{f^n},$$

where $\deg s = n \cdot \deg f$. We shall to check that θ is well-defined. If $\overline{\left(\frac{s}{f^n}, D_+(f)\right)} = \overline{\left(\frac{t}{g^m}, D_+(g)\right)}$, then $\frac{s}{f^n} = \frac{t}{g^m}$ on some $([\mathfrak{p}] \in) D_+(h) \subseteq D_+(f) \cap D_+(g)$. Since $\mathscr{O}_{\mathrm{Proj}\, S_\bullet}(D_+(h)) \cong ((S_\bullet)_h)_0$, there exists l such that $h^l(sg^m - tf^n) = 0$. Note that $h \notin \mathfrak{p}$, $h^l \notin \mathfrak{p}$, hence, $\frac{s}{f^n} = \frac{t}{g^m}$ in $((S_\bullet)_\mathfrak{p})_0$, Hence, θ is well-defined ring homomorphism.

Now, we want to show that θ is an isomorphism.

(i) Surjective.

Let $\frac{s}{f} \in ((S_\bullet)_\mathfrak{p})_0$, then $f \notin \mathfrak{p}$. Note that $\theta\left(\overline{\left(\frac{s}{f}, D_+(f)\right)}\right) = \frac{s}{f}$, which implies that θ is surjective.

(ii) Infective.

Let $\frac{s}{f^n} \in ((S_\bullet)_\mathfrak{p})_0$ such that $\frac{s}{f^n} = 0$. Note that $\frac{s}{f^n} \in ((S_\bullet)_f)_0$, hence $\overline{\left(\frac{s}{f^n}, D_+(f)\right)} = 0$. Thus θ is injective.

Hence, we have an isomorphism,

$$\mathscr{O}_{\mathrm{Proj}\, S_\bullet, [\mathfrak{p}]} \xleftrightarrow{\sim} ((S_\bullet)_\mathfrak{p})_0,$$

as we desired. □

Exercise 5.11 **(Some will find this essential, while others will prefer to ignore it.)** (Re)interpret the structure sheaf of $\operatorname{Proj} S_\bullet$ in terms of compatible germs.

Solution *Let $U \subseteq \operatorname{Proj} S_\bullet$ be any open subset, by the proof in Theorem 3.5.1, we have*

$$\begin{aligned}\Gamma(U, \mathscr{O}_{\operatorname{Proj} S_\bullet}) =&\{\left(s_{[\mathfrak{p}]} \in \mathscr{O}_{\operatorname{Proj} S_\bullet,[\mathfrak{p}]}\right)_{[\mathfrak{p}]\in U} : \\ &\qquad \forall [\mathfrak{p}] \in U, \exists D_+(f) \text{ with } [\mathfrak{p}] \in D_+(f) \subseteq U \text{ and } t \in \mathscr{O}_{\operatorname{Proj} S_\bullet}(D_+(f)), \\ &\qquad \text{s.t. } t_{[\mathfrak{q}]} = s_{[\mathfrak{q}]}, \forall [\mathfrak{q}] \in D_+(f)\} \\ =&\{\left(s_{[\mathfrak{p}]} \in ((S_\bullet)_\mathfrak{p})_0\right)_{[\mathfrak{p}]\in U} : \\ &\qquad \forall [\mathfrak{p}] \in U, \exists f \notin \mathfrak{p} \text{ and } t/f^n \in ((S_\bullet)_f)_0, \text{ s.t. } s_{[\mathfrak{q}]} = t/f^n, \forall [\mathfrak{q}] \in \operatorname{Spec}((S_\bullet)_f)_0\}.\end{aligned}$$

Definition 5.5.9 (Projective space)

*We (re)definition **projective space** (over a ring A) by*

$$\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \ldots, x_n].$$

*This definition involves no messy gluing, or special choice of patches. Note that the variables $x_0, \ldots, x_n$, which we call the **projective coordinates** on $\mathbb{P}^n_A$, are part of the definition. (They may have other names than x's, depending on the context.)*

Exercise 5.12 Check that this agrees with our earlier construction of $\mathbb{P}^n_A$ (§5.4.3).

Proof Let $S_\bullet = A[x_0, \ldots, x_n]$ and $D_+(x_i)$ is distinguished open subset of $\operatorname{Spec} S_\bullet$, then

$$D_+(x_i) = \operatorname{Spec}((S_\bullet)_{x_i})_0 = \operatorname{Spec}(A[x_0, \ldots, x_n]_{x_i})_0 = \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1).$$

$D(x_{j/i})$ is an open subscheme of $D_+(x_i)$, and

$$D(x_{j/i}) = \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1).$$

Clearly, we have $D(x_{j/i}) \cong D(x_{i/j})$. Hence, $\operatorname{Proj} A[x_0, \ldots, x_n]$ agree with construction of $\mathbb{P}^n_A$ in §5.4.3. □

Notice that with our old definition of projective space, it would have been a nontrivial exercise to show that $D_+(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2_k$ (the complement of a plane conic) is affine; with our new perspective, it is immediate — it is $\operatorname{Spec}(k[x, y, z]_{x^2+y^2-z^2})_0$.

Exercise 5.13 Suppose that k is an algebraically closed field. We know from Proposition 5.4.3 that the closed points of $\mathbb{P}^n_k$, as defined in §5.4.3 (Proposition 5.4.3), are in the bijection with the points of "classical" projective space. By Exercise 5.12, the scheme $\mathbb{P}^n_k$ as defined in §5.4.3 is isomorphic to $\operatorname{Proj} k[x_0, \ldots, x_n]$. Therefore, each point $[a_0, \ldots, a_n]$ of classical projective space corresponds to a homogeneous prime ideal of $k[x_0, \ldots, x_n]$. Which homogeneous prime ideal is it?

Solution *Let $[a_0 : \cdots : a_n]$ is closed point of $\mathbb{P}^n_k$, then the corresponding ideal in $\operatorname{Spec}(k[x_0, \ldots, x_n]_{x_i})_0$ is $\left(x_{0/i} - a_{0/i}, \ldots, x_{n/i} - a_{n/i}\right)$. In $S_\bullet = k[x_0, \ldots, x_n]$, this corresponds to the homogeneous prime ideal $(a_i x_0 - a_0 x_i, \ldots, a_i x_n - a_n x_i)$.*

We now figure out the "right definition" of the vanishing scheme, in analogy with the vanishing set $V(\cdot)$ (Definition 4.4.1). We will be defining a closed subscheme (properly defined in Chapter 10).

Definition 5.5.10 (Vanishing scheme)

Let $S_\bullet$ be a $\mathbb{Z}^{\geq 0}$-graded ring, which is generated as S_0-algebra by S_1, let $f \in S_+$ and let $V_+(f)$ the *vanishing scheme of f in* $\operatorname{Proj} S_\bullet$ *constructed as:*

$$V_+(f) := \operatorname{Proj} S_\bullet/(f).$$

Let I be any homogeneous ideal I of S_+. Define

$$V_+(I) := \operatorname{Proj} S_\bullet/I.$$

On every affine open set $U = \operatorname{Spec} R$, $V_+(f) \cap U$ is clearly a closed subset by Proposition 5.5.9, hence, $V_+(f) \cap U = \operatorname{Spec} R/J$ for some ideal J.

Remark This is the same notation as the "vanishing set" $V(\cdot)$, but now means something richer than a mere set.

Warning: in general, f isn't a function on $\operatorname{Proj} S_\bullet$. We will later interpret it as something close: a section of a line bundle.

5.5.6 Projective and quasi-projective schemes

Definition 5.5.11 (Projective and quasi-projective schemes)

We call a scheme of the form (i.e., isomorphic to) $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a finitely generated graded ring over A, a ***projective scheme over*** A, *or a* ***projective*** A***-scheme***.

A ***quasi-projective*** *A-scheme is a quasi-compact open subscheme of a projective A-scheme.*

Remark The "A" is omitted if it is clear from the context, often A is a field.

Remark

(i) Note that $\operatorname{Proj} S_\bullet$ makes sense even when $S_\bullet$ is not finitely generated. This can be useful.

(ii) The quasi-compact requirement in the definition of quasi-projectivity is of course redundant in the Noetherian case, which is all that matters to most.

Example 5.9 Note that $\mathbb{P}^0_A = \operatorname{Proj} A[T] = \{\text{the homogeneous prime ideal of } A[T] \text{ which not containing } (T)\}$, we have

$$\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong \operatorname{Spec} A.$$

Thus "$\operatorname{Spec} A$ is a projective A-scheme".

We can make this definition of projective space even more choice-free as follows.

Definition 5.5.12 (Projectivization of a vector space)

Let V be an $(n+1)$-dimensional vector space over k. (Here k can be replaced by any ring A and V by a free module as usual.) Define

$$\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots.$$

If for example V is the dual of the vector space with basis associated to $x_0, \ldots, x_n$ (i.e., $V = \operatorname{Span}\{x_0, \ldots, x_n\}^\vee$), we would have $\operatorname{Sym}^\bullet V^\vee = k[x_0, \ldots, x_n]$. Then we can define

$$\mathbb{P}V := \operatorname{Proj}(\operatorname{Sym}^\bullet V^\vee).$$

Remark In this language, we have an interpretation for $x_0, \ldots, x_n$: they are linear functionals on the underlying vector space V. **Warning:** some authors use the definition $\mathbb{P}V = \operatorname{Proj}(\operatorname{Sym}^\bullet V)$, so be cautious.

Proposition 5.5.12

Suppose k is algebraically closed. There is a natural bijection between one-dimensional subspaces of V and the closed points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space V.

Proof By Exercise 5.12 and Proposition 5.4.3, we done. □

Remark You may be surprised at the appearance of the dual in the definition of $\mathbb{P}V$. This is partially explained by the previous discussion. Most normal (traditional) people define the projectivization of a vector space V to be the space of one-dimensional subspaces of V. Grothendieck considered the projectivization to be the space of one-dimensional quotients. One motivation for this is that it gets rid of the annoying dual in the definition above. There are better reasons, that we won't go into here. In a nutshell, quotients tend to be better-behaved than subobjects for coherent sheaves, which generalize the notion of vector bundle. (Coherent sheaves are discussed in Chapter 15.)

Remark On another note related to Proposition 5.5.12: you can also describe a natural bijection between points of V and the closed points of $\operatorname{Spec}(\operatorname{Sym}^\bullet V^\vee)$. This construction respects the affine/projective cone picture of Chapter 10.

The Grassmannian. At this point, we could describe the fundamental geometric object known as the **Grassmannian**, and give the "wrong" (but correct) definition of it. We will instead wait until Chapter 8 to give the wrong definition, when we will know enough to sense that something is amiss. The right definition will be given in Chapter 17.

Chapter 6 Some properties of schemes

Now that we have defined the notion of a scheme, we can define some useful properties of schemes. As you see each definition, you should try it out in specific examples of your choice, such as your favorite schemes of the form $\operatorname{Spec}\mathbb{C}[x_1,\ldots,x_n]/(f_1,\ldots,f_r)$.

6.1 Topological properties

6.1.1 Some simple properties

The definition of connected (Definition 4.6.1), connected component (Definition 4.6.8), (ir)reducible (Definition 4.6.2), irreducible component (Definition 4.6.7), quasi-compact (Definition 4.6.3), generization (Definition 4.6.5), specialization (Definition 4.6.5), generic point (Definition 4.6.6), Noetherian topological space (Definition 4.6.9), and closed point (Definition 4.6.4), were given in §4.6.

Proposition 4.6.2 shows that $\mathbb{A}_k^n$ is irreducible. This argument "behaves well under gluing", yielding:

Proposition 6.1.1

$\mathbb{P}_k^n$ is irreducible.

Proof By Definition 4.6.2, it suffices to show that in $\mathbb{P}_k^n$ any two nonempty open subsets intersect. Let U_1, U_2 be any nonempty open subsets of $\mathbb{P}_k^n$, we want to show that $U_1 \cap U_2 \neq \varnothing$. Note that $\mathbb{P}_k^n = \bigcup_i D_+(x_i)$, where $D_+(x_i)$ is distinguished base, we have

$$\begin{aligned} U_1 \cap U_2 =& \mathbb{P}_n^k \cap U_1 \cap U_2 \\ =& \left(\bigcup_i D_+(x_i)\right) \cap U_1 \cap U_2 \\ =& \bigcup_i \left((D_+(x_i)\cap U_1) \cap (D_+(x_i)\cap U_2)\right). \end{aligned}$$

Since $D_+(x_i) \cap U_j$ is open in $D_+(x_i)$ for $j = 1,2$ and $D_+(x_i) \cong \mathbb{A}_k^n$ is irreducible (Proposition 4.6.2), we have

$$U_1 \cap U_2 = (D_+(x_i)\cap U_1) \cap (D_+(x_i)\cap U_2) \neq \varnothing$$

for some i. Hence, $U_1 \cap U_2 \neq \varnothing$, which implies that $\mathbb{P}_k^n$ is irreducible. $\square$

Proposition 6.1.2

There is a bijection between irreducible closed subsets and points for any schemes (the map sending a point p to the closed subset $\overline{\{p\}}$ is a bijection).

Proof Let $(X, \mathscr{O}_X)$ be a scheme. Let $p \in X$ be any point of X, clearly, $\{p\}$ is irreducible in X, by Proposition 4.6.1 (b), $\overline{\{p\}}$ is irreducible closed subset of X.

Conversely, let Z be any irreducible closed subset of X. Let U, V be any two affine open subscheme of X which joint Z nonempty, then $U \cap Z$ is an irreducible closed subset of U and $V \cap Z$ is an irreducible closed subset of V. Since U is affine open subscheme, by Proposition 4.7.3, there exists a unique generic point p_U for $U \cap Z$, i.e., the closure of p_U in U is $U \cap Z$. We want to show that p_U is also generic point for $V \cap Z$.

Note that the closure of p_U in X must contain $U \cap Z$, we claim that the closure of p_U is Z. If not, $\overline{\{p_U\}}$ is a proper closed subset of Z. Note that $Z = (Z \setminus U \cap Z) \cup \overline{\{p_U\}}$ and $Z \setminus U \cap Z$ is a proper closed subset of Z,

Z is the union of two proper closed subset, it contradicts to the fact that Z is irreducible. Hence, the closure of p_U in X is Z.

Since $\overline{\{p_U\}} = Z$ in X, $\{p_U\}$ is a dense subset of Z, hence $\{p_U\}$ contained in every nonempty open subset of Z, and therefore $p_U \in V \cap Z$. Note that the closure of p_U in $V \cap Z$ is $V \cap Z$, p_U is the generic point for $V \cap Z$. By Proposition 4.7.3, p_U is the unique generic point for $V \cap Z$.

By above discussion we get a bijection between irreducible closed subsets and points for any schemes, i.e.,

$$\{\text{points in Scheme}\} \longleftrightarrow \{\text{irreducible closed subsets of scheme}\}$$

$$p \longmapsto \overline{\{p\}}$$

$$\begin{array}{c}\text{generic point for } Z \cap U \\ \text{where U is any affine open subscheme}\end{array} \longleftarrow\!\shortmid \; Z.$$

□

Proposition 6.1.3

If X is a scheme that has a finite cover $X = \bigcup\limits_{i=1}^{n} \operatorname{Spec} A_i$ where A_i is Noetherian, then X is a Noetherian topological space.

Proof Since each A_i is Noetherian, by Proposition 4.6.18, $\operatorname{Spec} A_i$ is a Noetherian topological space. It suffices to show that a topological space that is a finite union of Noetherian subspaces is itself Noetherian.

Let $X = \bigcup X_i$ be a topological space, where each X_i is Noetherian topological space. Let

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots \tag{6.1}$$

be any sequence of closed subsets of X. We want to show this sequence is stationary. Note that

$$Z_1 \cap X_i \supseteq Z_2 \cap X_i \supseteq \cdots \supseteq Z_n \cap X_i \supseteq \cdots \tag{6.2}$$

is a sequence of closed subsets of X_i. Since X_i is Noetherian, sequence (6.2) is stationary, i.e., exists r_i such that $X_i \cap Z_{r_i} = X_i \cap Z_{r_i+1} = \cdots$. Let $r = \max_i r_i$. Note that $Z_j = \bigcup_i (Z_j \cap X_i)$, we have

$$Z_r = \bigcup_i (Z_r \cap X_i) = \bigcup_i (Z_{r+1} \cap X_i) = Z_{r+1},$$

which implies that sequence (6.1) is stationary. Hence, X is a Noetherian topological space. □

Corollary 6.1.1

$\mathbb{P}_k^n$ and $\mathbb{P}_{\mathbb{Z}}^n$ are Noetherian topological spaces.

Proposition 6.1.4

A scheme X is quasi-compact if and only if it can be written as a finite union of affine open subschemes (affine open subsets).

Proof Let X be a quasi-compact scheme. By the definition of scheme, we may assume that $X = \bigcup\limits_{q \in X} U_q$ where U_q is affine open subscheme. Since X is a quasi-compact scheme, we can choice finite numbers of affine open subscheme $\{U_i\}_{i=1}^n$ in $\{U_q\}_{q \in X}$ such that $X = \bigcup_{i=1}^n U_i$.

Conversely, if X is a finite union of affine open subscheme, say $X = \bigcup\limits_{i=1}^{n} U_i$, where U_i is affine open subscheme. Let $\{V_i\}$ is any open cover of X. By Proposition 4.6.4, each U_i is quasi-compact, we can choice finite numbers of V_i cover each U_i, and therefore X can be covered by finite numbers of V_i. Hence, X is quasi-compact.

□

Remark We will soon call a scheme with such a cover a Noetherian scheme.

Proposition 6.1.5 (Quasi-compact schemes have closed points)

If X is a quasi-compact scheme, then

(a) every point has a closed point in its closure;

(b) every nonempty closed subset of X contains a closed point of X.

In particular, every nonempty quasi-compact scheme has a closed point.

Proof It is enough to show that every closed subset Z of X has a closed point. Since X is a quasi-compact scheme, by Proposition 6.1.4, we may assume that $X = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$. Then $Z = \bigcup_{i=1}^{n}(Z \cap \operatorname{Spec} A_i)$. We may assume that each $Z \cap \operatorname{Spec} A_i \neq \varnothing$ and $Z \cap \operatorname{Spec} A_i \nsubseteq \bigcup_{j \neq i}(Z \cap \operatorname{Spec} A_j)$ (use quasi-compactness[2]), then $Z \cap \operatorname{Spec} A_i$ closed in $\operatorname{Spec} A_i$, and therefore $V(I_i) = Z \cap \operatorname{Spec} A_i$ for some ideal I_i. Since each $V(I_i)$ is open in Z, $Z \setminus (\bigcup_{i=2}^{n} V(I_i))$ is closed in $V(I_1)$, pick closed point $[\mathfrak{m}] \in Z \setminus (\bigcup_{i=2}^{n} V(I_i))$, then $\overline{\{[\mathfrak{m}]\}} = \{[\mathfrak{m}]\} \subseteq Z \setminus (\bigcup_{i=2}^{n} V(I_i)) \subseteq V(I_1)$. Since $Z \setminus (\bigcup_{i=2}^{n} V(I_i))$ is closed in Z, the closure of $\{[\mathfrak{m}]\}$ in Z is also $\{[\mathfrak{m}]\}$. □

Remark **Warning:** there exist nonempty schemes with no closed points, see Chapter 16.

Remark Proposition 6.1.5 will often be used in the following way. If there is some property $\mathscr{P}$ of points of a scheme that is "open" (if a point p has $\mathscr{P}$, then there is some open neighborhood U of p such that all the points in U have $\mathscr{P}$), then to check if all points of a quasi-compact scheme have $\mathscr{P}$, it suffices to check only the closed points. This provides a connection between schemes and the classical theory of varieties — the points of traditional varieties are the closed points of the corresponding scheme (essentially by the Nullstellensatz). In many good situations, the closed points are dense, but this is not true in some fundamental cases (Proposition 4.6.7 (b)).

Definition 6.1.1 (Open condition)

We call some property $\mathscr{P}$ of points of a scheme that is "open", if a point p has $\mathscr{P}$, then there is some open neighborhood U of p such that all the points in U have $\mathscr{P}$.

6.1.2 Quasi-separated schemes

Quasi-separatedness is a weird notion that comes in handy for certain people (**Warning:** we will later realize that this is a really a property of morphisms, not for schemes, Chapter 9.) Most people, however, can ignore this notion, as the schemes they will encounter in real life will all have this property.

Definition 6.1.2 (Quasi-separated)

*A topological space is **quasi-separated** if the intersection of any two quasi-compact open subsets is quasi-compact.*

Remark The motivation for the "separatedness" in the name is that it is a weakened version of separated, for which the intersection of any two affine open sets is affine, see Chapter 12.

Proposition 6.1.6

A scheme is quasi-separated if and only if the intersection of any two affine open subsets is a finite union of affine open subsets.

Proof Let X be a quasi-separated scheme. Let U_1, U_2 be any two affine subschemes of X, since X is quasi-separated and each affine scheme is quasi-compact (Proposition 4.6.4), $U_1 \cap U_2$ is quasi-compact. Since $U_1 \cap U_2$ open in X, by Proposition 5.3.3, $U_1 \cap U_2$ is a scheme, by Proposition 6.1.4, $U_1 \cap U_2$ can be written as a finite union of affine open subschemes.

Conversely, let U_1, U_2 be any two quasi-compact open subsets of scheme X, we want to show that $U_1 \cap U_2$ is quasi-compact. Since $U_1 \cap U_2$ is open in X, by Proposition 5.3.3, $U_1 \cap U_2$ is a scheme, by Proposition 6.1.4, it suffices to show that $U_1 \cap U_2$ can be written as a finite union of affine open subschemes.

Since U_1 and U_2 are quasi-compact open subschemes of X, we may assume that

$$U_1 = \bigcup_{i=1}^{n} \operatorname{Spec} A_i, \quad U_2 = \bigcup_{j=1}^{m} \operatorname{Spec} B_j,$$

then

$$U_1 \cap U_2 = \left(\bigcup_{i=1}^{n} \operatorname{Spec} A_i \right) \cap \left(\bigcup_{j=1}^{m} \operatorname{Spec} B_j \right) = \bigcup_{i,j} (\operatorname{Spec} A_i \cap \operatorname{Spec} B_j).$$

By the hypothesis, the intersection of any two affine open subschemes is quasi-compact, each $\operatorname{Spec} A_i \cap \operatorname{Spec} B_j$ can be written as the finite union of affine open subschemes. Hence, $U_1 \cap U_2$ can be written as a finite union of affine open subschemes, as we desired. □

Corollary 6.1.2

Affine schemes are quasi-separated.

Proof Let X be an affine schemes, by Proposition 4.5.3, we may assume that $X = \bigcup_{i=1}^{n} D(f_i)$. By Proposition 6.1.6, it suffices to show that the intersection of any two affine open subsets of X is a finite union of affine open subsets. Let U_1, U_2 be any two affine open subsets of X, by Proposition 4.6.4, U_1 and U_2 are quasi-compact, hence,

$$U_1 = \bigcup_{k=1}^{s} D(f_{i_k}), \quad U_2 = \bigcup_{l=1}^{t} D(f_{j_l}),$$

where $\{f_{i_k}\}_{k=1}^{s} \subseteq \{f_i\}$ and $\{f_{j_l}\}_{l=1}^{t} \subseteq \{f_i\}$. Then we have

$$U_1 \cap U_2 = \bigcup_{k=1}^{s} \bigcup_{l=1}^{t} (D(f_{i_k}) \cap D(f_{j_l})) = \bigcup_{k=1}^{s} \bigcup_{l=1}^{t} D(f_{i_k} f_{j_l}).$$

By Proposition 5.3.2, each $D(f_{i_k} f_{j_l})$ is affine open subsets, hence, $U_1 \cap U_2$ is a finite union of affine open subsets. □

Corollary 6.1.3

A scheme X is quasi-separated if and only if there exists a cover of X by affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

Proof If X is a quasi-separated scheme. Since X is a scheme, X can be covered by affine open subsets, by Proposition 6.1.6, the intersection of any two affine open subsets is a finite union of affine open subsets.

Conversely, suppose that $X = \bigcup_i \operatorname{Spec} A_i$, where $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ covered by a finite number of affine open subsets. Let U_1, U_2 be any two affine open subsets of X, we want to show that $U_1 \cap U_2$ is a finite union of affine open subsets. By Proposition 4.6.4, U_1 and U_2 are quasi-compact, we may assume that

$$U_1 = \bigcup_{k=1}^{s} \operatorname{Spec} A_{i_k}, \quad U_2 = \bigcup_{l=1}^{t} \operatorname{Spec} A_{j_l},$$

then

$$U_1 \cap U_2 = \bigcup_{k=1}^{s} \bigcup_{l=1}^{t} (\operatorname{Spec} A_{i_k} \cap \operatorname{Spec} A_{j_l}).$$

By the hypothesis, $\operatorname{Spec} A_{i_k} \cap \operatorname{Spec} A_{j_l}$ covered by a finite number of affine open subsets, hence, $U_1 \cap U_2$ is a finite union of affine open subsets. By Proposition 6.1.6, X is quasi-separated. □

We will see that quasi-separatedness will be a useful hypothesis in theorems (in conjunction with quasi-compactness), and that various interesting kinds of schemes (affine, locally Noetherian, separated) are quasi-separated, and this will allow us to state theorems more succinctly (e.g., "if X is a quasi-compact and quasi-separated" rather than "if X is quasi-compact, and either this or that or the other thing hold").

"Quasi-compact and quasi-separated" means something concrete:

Corollary 6.1.4

A scheme X is quasi-compact and quasi-separated if and only if X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

Proof By Proposition 6.1.4 and Proposition 6.1.6, we done. □

Note ***Qcqs.*** *When you see "quasi-compact and quasi-separated" as hypotheses in a theorem, you should take this as a clue that you will use this interpretation, and that finiteness will be used in an essential way. This combination of hypotheses is so common that people often describe it as* ***qcqs*** *for short.*

Proposition 6.1.7

All projective A-schemes are qcqs.

Proof Let $X = \operatorname{Proj} S_\bullet$ be any projective A-scheme, where $S_\bullet$ is a finitely generated graded ring over A. Hence, S_+ is a finitely generated ideal, say $S_+ = (f_1, \ldots, f_n)$. By Proposition 5.5.8, we have $\operatorname{Proj} S_\bullet = \bigcup_{i=1}^n D_+(f_i)$. By Proposition 5.5.9, each $D_+(f_i)$ is affine open subsets. By Lemma 5.5.2, $D_+(f_i) \cap D_+(f_j) = D_+(f_i f_j)$ is affine open subsets. By Corollary 6.1.4, X is qsqc. □

Example 6.1 **A nonquasi-separated scheme.** Let $X = \operatorname{Spec} k[x_1, x_2, \ldots]$, and let U be $X \setminus [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \ldots)$. Take two copies of X, glued along U ("affine ∞-space with a doubled origin"). The result is not quasi-separated.

Proof Let X_1 and X_2 denote the two copies of X and let Y denote the space obtained by gluing the two affine schemes along U. Then the image of X_1 and X_2 are still open in Y and they are all affine scheme, hence, they are quasi-compact (Proposition 4.6.4). Consider $X_1 \cap X_2$, we have

$$X_1 \cap X_2 \cong (\operatorname{Spec} k[x_1, x_2, \ldots]) \setminus \{[\mathfrak{m}]\},$$

by remark after Proposition 4.6.4, $X_1 \cap X_2$ is not quasi-compact, and therefore X is not quasi-separated. □

6.1.3 Dimension

One very important topological notion is **dimension**. (It is amazing that this is a topological idea.) But despite being intuitively fundamental, it is more difficult, so we postpone it until Chapter 13.

6.2 Reducedness and integrality

6.2.1 Reducedness

Recall that one of alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of nilpotents (§4.2.4).

Definition 6.2.1 (Reduced scheme)

*We say that a scheme X is **reduced** if its stalks $\mathscr{O}_{X,p}$ have no nonzero nilpotents for all p.*

Proposition 6.2.1

If f and g are two functions (global sections of $\mathscr{O}_X$) on a reduced scheme that agree at all points, then $f = g$.

Proof Let X be a reduced scheme, let $X = \bigcup_i \operatorname{Spec} A_i$ be an affine open cover of X. Let $f, g \in \mathscr{O}_X(X)$ with $f(p) = g(p)$ for all $p \in X$, say $h = f - g$, then $h(p) = 0$ for all $p \in X$, we want to show that $h = 0$ in $\mathscr{O}_X(X)$. For any $U_i := \operatorname{Spec} A_i$, by Proposition 3.4.1, we have an injective,

$$\mathscr{O}_X|_{U_i}(U_i) = A_i \longleftrightarrow \prod_{[\mathfrak{p}]\in U_i} \mathscr{O}_{\operatorname{Spec} A_i,[\mathfrak{p}]} = \prod_{[\mathfrak{p}]\in U_i} (A_i)_{\mathfrak{p}}. \tag{6.3}$$

Since $h(p) = 0$ for all $p \in X$, we have $(h|_{U_i})_{[\mathfrak{p}]} \in \mathfrak{p}(A_i)_{\mathfrak{p}}$ for all $[\mathfrak{p}] \in U_i$. Since (6.3) is injective, $h|_{U_i} \in \mathfrak{p}$ for all $[\mathfrak{p}] \in \operatorname{Spec} A_i$, i.e., $h|_{U_i} \in \mathfrak{N}(A_i)$. Note that X is reduced, $\mathfrak{N}((A_i)_{\mathfrak{p}}) = \mathfrak{N}(A_i)_{\mathfrak{p}} = 0$ for all $[\mathfrak{p}] \in U_i$. Hence, $(h|_{U_i})_{[\mathfrak{p}]} = 0$ for all $[\mathfrak{p}] \in U_i$, by the injective (6.3), $h|_{U_i} = 0$. By the identity axiom of sheaf $\mathscr{O}_X$, $h = 0$ in $\mathscr{O}_X(X)$, that is, $f = g$. □

Proposition 6.2.2

Let A be a ring, A is a reduced ring (Definition 4.2.3) if and only if $\operatorname{Spec} A$ is reduced.

Proof If A is a reduced ring, then $\mathfrak{N}(A) = 0$. For any $[\mathfrak{p}] \in \operatorname{Spec} A$, note that

$$\mathfrak{N}(A_{\mathfrak{p}}) = \bigcap_{[\mathfrak{q}]\in\operatorname{Spec} A_{\mathfrak{p}}} \mathfrak{q} = \bigcap_{\substack{[\mathfrak{q}]\in\operatorname{Spec} A \\ \mathfrak{q}\subseteq\mathfrak{p}}} \mathfrak{q} = \bigcap_{[\mathfrak{q}]\in\operatorname{Spec} A} \mathfrak{q}_{\mathfrak{p}} = \left(\bigcap_{[\mathfrak{q}]\in\operatorname{Spec} A} \mathfrak{q} \right)_{\mathfrak{p}} = \mathfrak{N}(A)_{\mathfrak{p}} = 0 \tag{6.4}$$

and $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = A_{\mathfrak{p}}$, $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]}$ has no nonzero nilpotents for all $[\mathfrak{p}] \in \operatorname{Spec} A$. Hence, $\operatorname{Spec} A$ is a reduced ring.

If $\operatorname{Spec} A$ is reduced, then $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = A_{\mathfrak{p}}$ have no nonzero nilpotents for all $[\mathfrak{p}] \in \operatorname{Spec} A$, that is, $\mathfrak{N}(A_{\mathfrak{p}}) = 0$. We want to show that $\mathfrak{N}(A) = 0$. Let $f \in \mathfrak{N}(A)$, by (6.4), $f_{\mathfrak{p}} = 0$ in $A_{\mathfrak{p}}$ for all $[\mathfrak{p}] \in \operatorname{Spec} A$. By Proposition 3.4.1, we have an injection

$$\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A \longleftrightarrow \prod_{[\mathfrak{p}]\in\operatorname{Spec} A} \mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = \prod_{[\mathfrak{p}]\in\operatorname{Spec} A} A_{\mathfrak{p}},$$

hence, $f = 0$ in A, and therefore $\mathfrak{N}(A) = 0$. Hence, A is a reduced ring. □

Corollary 6.2.1

$\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced.

Proof Note that $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$, by Proposition 6.2.2, it suffices to show that $k[x_1, \ldots, x_n]$ is a reduced ring. Since $k[x_1, \ldots, x_n]$ is an integral domain, $\mathfrak{N}(k[x_1, \ldots, x_n]) = 0$, and therefore $k[x_1, \ldots, x_n]$ is a reduced ring.

$\mathbb{P}_k^n$ is covered by $n+1$ copies of $\mathbb{A}_k^n$ and therefore is reduced. □

Proposition 6.2.3

A scheme X is reduced if and only if $\mathscr{O}_X(U)$ is reduced for every open set U of X.

Proof If X is reduced. Let $f \in \mathscr{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f^n in $\mathscr{O}_{X,p}$ is zero for all $p \in U$. Since X is reduced, $\mathfrak{N}(\mathscr{O}_{X,p}) = 0$, it follows that $f = 0$ in $\mathscr{O}_{X,p}$ for all $p \in U$. By Proposition 6.2.1, we have an injective,

$$\mathscr{O}_X(U) \hookrightarrow \prod_{p \in U} \mathscr{O}_{X,p}.$$

Hence, $f = 0$ in $\mathscr{O}_X(U)$, which implies that $\mathscr{O}_X(U)$ is reduced.

Conversely, it suffices to show that $\mathscr{O}_{X,p}$ has no nonzero nilpotents for all p. Suppose there exists p_0 such that $\mathfrak{N}(\mathscr{O}_{X,p_0}) \neq 0$, let nonzero $f_{p_0} \in \mathfrak{N}(\mathscr{O}_{X,p_0})$, then exists n such that $f_{p_0}^n = 0$. Let $f_{p_0} = \overline{(f, U_{p_0})}$, where U_{p_0} is an open neighborhood of p_0. Since $f_{p_0}^n = 0$ in $\mathscr{O}_{X,p_0}$, there exists an open neighborhood of p_0, say V, such that $V \subseteq U$ with $(f|_V)^n = 0$. By the hypothesis, $\mathfrak{N}(\mathscr{O}_X(V)) = 0$, hence, $f|_V = 0$, a contradiction. Hence, $\mathscr{O}_{X,p}$ has no nonzero nilpotents for all p, and therefore X is reduced. □

Example 6.2 The scheme $\operatorname{Spec} k[x,y]/(y^2, xy)$ is nonreduced. When we sketched it in Figure 5.4, we indicated that the fuzz represented nonreducedness at the origin. The following exercise is a first stab at making this precise.

✍ **Exercise 6.1**

(a) Show that $\left(k[x,y]/(y^2, xy)\right)_x$ has no nonzero nilpotent elements.

(b) Show that the only point of $\operatorname{Spec} k[x,y]/(y^2, xy)$ with a nonreduced stalk is the origin.

Proof

(a) We claim that

$$\left(k[x,y]/(y^2, xy)\right)_x \cong k[x,y]_x/(y^2, xy)_x \cong k[x]_x.$$

First isomorphism given by the localization commutes with quotient. In fact,

$$\begin{aligned} k[x,y]_x/(y^2, xy)_x &= \left\{ \frac{f(x,y)}{x^k} \in k[x,y]_x : \frac{y^2}{x^m} = 0, \frac{xy}{x^n} = 0, \forall m, n \right\} \\ &= \left\{ \frac{f(x,0)}{x^k} \in k[x]_x : \frac{f(x,y)}{x^k} \in k[x,y]_x \right\}, \end{aligned}$$

define $\varphi : k[x,y]_x/(y^2, xy)_x \to k[x]_x$ by setting

$$\overline{\frac{f(x,y)}{x^k}} \longmapsto \frac{f(x,0)}{x^k}.$$

Clearly, φ is well-defined ring homomorphism. By the construction of $k[x,y]_x/(y^2, xy)_x$, φ is surjective. Let $\frac{f(x,0)}{x^k} = 0$ in $k[x]_x$, since $k[x]$ is integral domain, we have $f(x,0) = 0$. In $k[x,y]_x/(y^2, xy)_x$, $\frac{y}{x^n} = \frac{xy}{x^{n+1}} = 0$, hence, $\frac{f(x,y)}{x^k} = \frac{f(x,0)+f(0,y)}{x^k} = \frac{f(x,0)}{x^k} = 0$ in $k[x,y]_x/(y^2, xy)_x$. It follows that φ is injective. Hence,

$$\left(k[x,y]/(y^2, xy)\right)_x \cong k[x,y]_x/(y^2, xy)_x \cong k[x]_x.$$

(b) Let $[\mathfrak{p}] \in \operatorname{Spec} k[x,y]/(y^2,xy)$, then $(y^2,xy) \subseteq \mathfrak{p} \subseteq k[x,y]$. Since $y^2 \in \mathfrak{p}$, note that $\mathfrak{p}$ is a prime ideal, we have $y \in \mathfrak{p}$. We have the following situations.

(i) $x \notin \mathfrak{p}$.

Consider $\big(k[x,y]/(y^2,xy)\big)_{\mathfrak{p}}$, since $x \notin \mathfrak{p}$, x is invertible in $\big(k[x,y]/(y^2,xy)\big)_{\mathfrak{p}}$. Hence

$$\big(k[x,y]/(y^2,xy)\big)_{\mathfrak{p}} \cong k[x]_{\mathfrak{p}}.$$

Since $\mathfrak{N}(k[x]_{\mathfrak{p}}) = \mathfrak{N}(k[x])_{\mathfrak{p}}$ and $\mathfrak{N}(k[x]) = 0$, we have $\mathfrak{N}(k[x]_{\mathfrak{p}}) = 0$. Hence, stalks $\big(k[x,y]/(y^2,xy)\big)_{\mathfrak{p}}$ are reduced.

(ii) $x \in \mathfrak{p}$, $y \in \mathfrak{p}$, i.e., $\mathfrak{p} = (x,y)$.

Consider $\big(k[x,y]/(y^2,xy)\big)_{(x,y)}$. Note that $\frac{y}{x} \in \big(k[x,y]/(y^2,xy)\big)_{(x,y)}$ and $\left(\frac{y}{x}\right)^2 = \frac{y^2}{x^2} = 0$, $\frac{y}{x}$ is a nilpotent in the stalk $\big(k[x,y]/(y^2,xy)\big)_{(x,y)}$, hence, $\big(k[x,y]/(y^2,xy)\big)_{(x,y)}$ is nonreduced.

By above discussion the only point of $\operatorname{Spec} k[x,y]/(y^2,xy)$ with a nonreduced stalk is the origin.

□

Proposition 6.2.4

If X is a quasi-compact scheme, then it suffices to check reducedness at closed point. More precisely, if $\mathscr{O}_{X,p}$ have no nonzero nilpotent for all closed point $p \in X$, then X is reduced.

Proof X is reduced if and only if $\mathscr{O}_{X,p}$ have no nonzero nilpotent for all $p \in X$. We want to show that if $\mathscr{O}_{X,p}$ have no nonzero nilpotent for all closed point $p \in X$, $\mathscr{O}_{X,p}$ have no nonzero nilpotent for all $p \in X$.

Suppose $[\mathfrak{q}]$ is not a closed point in X, since X is a quasi-compact scheme, by Proposition 6.1.5, $\overline{\{\mathfrak{q}\}}$ contains a closed point of X, say $[\mathfrak{p}]$. Let U be an affine open neighborhood of $[\mathfrak{p}]$, note that $[\mathfrak{q}]$ is a generic point for $\overline{\{\mathfrak{q}\}}$, by Proposition 4.6.10, U contains $[\mathfrak{q}]$. Since U is affine open subset, we may assume that $U = \operatorname{Spec} A$, then $\operatorname{Spec} A \cap \overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$. Note that $\mathfrak{p} \in V(\mathfrak{q})$, we have $\mathfrak{p} \supseteq \mathfrak{q}$. Note that $\mathscr{O}_{X,[\mathfrak{q}]} = A_{\mathfrak{q}}$, we want to show $\mathfrak{N}(A_{\mathfrak{q}}) = 0$. Claim that $(A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} = A_{\mathfrak{q}}$. Clearly, we have inclusions, $A_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{q}}$ and $A \hookrightarrow (A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}}$. Since $\mathfrak{p} \supseteq \mathfrak{q}$, we have $A - \mathfrak{p} \subseteq A - \mathfrak{q}$ and $A - \mathfrak{q} \subseteq A_{\mathfrak{p}} - \mathfrak{q}A_{\mathfrak{p}}$. Hence, the following diagrams commute.

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r, hook] \arrow[d, "\text{localization}"'] & A_{\mathfrak{q}} \\ (A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "\text{localization}"'] & (A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} \\ A_{\mathfrak{q}} \arrow[ur, dashed] & \end{tikzcd}$$

It follows that $(A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} = A_{\mathfrak{q}}$. Since $\mathscr{O}_{X,p}$ have no nonzero nilpotent for all closed point $p \in X$, we have $\mathfrak{N}(A_{\mathfrak{p}}) = 0$, hence,

$$\mathfrak{N}(A_{\mathfrak{q}}) = \mathfrak{N}((A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}}) = \mathfrak{N}(A_{\mathfrak{p}})_{\mathfrak{q}A_{\mathfrak{p}}} = 0.$$

It follows that $\mathscr{O}_{X,[\mathfrak{q}]}$ has no nonzero nilpotent, then we done. □

Remark We will see in Chapter 7 that reducedness is an open condition for locally Noetherian schemes. But in general, reducedness is not an open condition. You may able to identify the underlying topological space of

$$X = \operatorname{Spec} \mathbb{C}[x, y_1, y_2, \ldots]/(y_1^2, y_2^2, y_3^2, \cdots, (x-1)y_1, (x-2)y_2, (x-3)y_3, \cdots)$$

with that of $\operatorname{Spec} \mathbb{C}[x]$, and then to show that the nonreduced points of X are precisely the closed points corresponding to the positive integers. The complement of this set is not Zariski open.

Proof

(i) X is homeomorphic to $\operatorname{Spec} \mathbb{C}[x]$.

Let $\mathfrak{p} \in X$, since $y_i^2 = 0 \in \mathfrak{p}$, each y_i must belong to X. Define $\varphi : \operatorname{Spec} \mathbb{C}[x] \to X$ by setting

$$\mathfrak{p} \longmapsto \mathfrak{p} + (y_1, y_2, \cdots)$$

and define $\psi : X \to \operatorname{Spec}\mathbb{C}[x]$ by setting

$$\mathfrak{q} \longmapsto \mathfrak{q} \cap \mathbb{C}[x].$$

Clearly, φ and ψ is well-defined. Let $\mathfrak{p} \in X$, then

$$\psi \circ \varphi(\mathfrak{p}) = \psi(\mathfrak{p} + (y_1, y_2, \cdots)) = \mathfrak{p}.$$

Let $\mathfrak{q} \in X$, then

$$\varphi \circ \psi(\mathfrak{q}) = \varphi(\mathfrak{q} \cap \mathbb{C}[x]) = \mathfrak{q} \cap \mathbb{C}[x] + (y_1, y_2, \cdots) = \mathfrak{q}.$$

It follows that φ is a bijection. Let $V_X(I)$ be a closed set of X, then

$$\begin{aligned}\varphi^{-1}(V_X(I)) =& \psi(V_X(I)) \\ =& \psi(\{\mathfrak{q} \in X : \mathfrak{q} \supseteq I\}) \\ =& \{\mathfrak{q} \cap \operatorname{Spec}\mathbb{C}[x] \in \operatorname{Spec}\mathbb{C}[x] : \mathfrak{q} \cap \mathbb{C}[x] \supseteq I \cap \mathbb{C}[x]\} \\ =& \{\mathfrak{p} \in \operatorname{Spec}\mathbb{C}[x] : \mathfrak{p} \supseteq I \cap \mathbb{C}[x]\} \\ =& V_{\operatorname{Spec}\mathbb{C}[x]}(I \cap \mathbb{C}[x]),\end{aligned}$$

is a closed subset of $\operatorname{Spec}\mathbb{C}[x]$. Conversely, let $V_{\operatorname{Spec}\mathbb{C}[x]}(J)$ be a closed subset, then

$$\begin{aligned}\psi^{-1}(V_{\operatorname{Spec}\mathbb{C}[x]}(J)) =& \varphi(V_{\operatorname{Spec}\mathbb{C}[x]}(J)) \\ =& \varphi(\{\mathfrak{p} \in \operatorname{Spec}\mathbb{C}[x] : \mathfrak{p} \supseteq J\}) \\ =& \{\mathfrak{p} + (y_1, y_2, \cdots) \in X : \mathfrak{p} + (y_1, y_2, \cdots) \supseteq J + (y_1, y_2, \cdots)\} \\ =& V_X(J + (y_1, y_2, \cdots)),\end{aligned}$$

is a closed subset of X. Hence, φ is a homeomorphism.

(ii) The nonreduced points of X are precisely the closed points corresponding to the positive integers.

By part (i), each element in X have form $(x - a, y_1, y_2, \cdots)$ where $a \in \mathbb{C}$. Consider $\mathscr{O}_{X,[\mathfrak{p}]}$, then

$$\mathscr{O}_{X,[\mathfrak{p}]} = \left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_i^2, (x-n)y_n)_{n\geq 1}\right)_{(x-a,y_1,y_2,\cdots)}.$$

If $a \in \mathbb{Z}^{\geq 0}$, then each $x - n$, where $n \neq a$, is invertible. Since $(x - n)y_n = 0$, we have $y_n = 0$ for all $n \neq a$. Then we have

$$\left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_i^2, (x-n)y_n)_{n\geq 1}\right)_{(x-a,y_1,y_2,\cdots)} \cong \left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_a^2, (x-a)y_a)\right)_{(x-a,y_a)}.$$

Note that y_a is a nilpotent in $\left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_a^2, (x-a)y_a)\right)_{(x-a,y_a)}$, hence, $\mathscr{O}_{X,[\mathfrak{p}]}$ is nonreduced for all $a \in \mathbb{Z}^{\geq 0}$.

If $a \notin \mathbb{Z}^{\geq 0}$, then each $x - n$, where $n \in \mathbb{Z}^{\geq 0}$, is invertible. Hence, each $y_n = 0$ in $\left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_i^2, (x-n)y_n)_{n\geq 1}\right)_{(x-a,y_1,y_2,\cdots)}$, and therefore we have

$$\left(\mathbb{C}[x, y_1, y_2, \cdots]/(y_i^2, (x-n)y_n)_{n\geq 1}\right)_{(x-a,y_1,y_2,\cdots)} \cong \mathbb{C}[x]_{x-a}.$$

Note that $\mathfrak{N}(\mathbb{C}[x]_{x-a}) = \mathfrak{N}(\mathbb{C}[x])_{x-a} = 0$, $\mathscr{O}_{X,[\mathfrak{p}]}$ has no nonzero nilpotent, and therefore is a reduced ring.

(iii) The complement of the set of nonreduced point of X is not Zariski open.

It suffices to check the set of nonreduced point of X is not closed in X. Suppose this set is closed in X, say $V_X(I) = \{(x - n, y_1, y_2, \cdots)\}_{n\in\mathbb{Z}^{\geq 0}}$. Since $V_X(\sqrt{I}) = V_X(I)$, we may assume that I is a radical ideal. By Hilbert's Nullstellensatz, we have

$$I = I(V_X(I)) = \bigcap_{[\mathfrak{p}]\in V_X(I)} \mathfrak{p} = \bigcap_{n\in\mathbb{Z}^{\geq 0}} (x - n, y_1, y_2, \cdots) = (y_1, y_2, \dots).$$

But for all $a \notin \mathbb{Z}^{geq}$, we have $(x-a, y_1, y_2, \cdots) \supseteq (y_1, y_2, \dots)$, and therefore $(x-a, y_1, y_2, \cdots) \in V_X(I)$, a contradiction.

□

Proposition 6.2.5

A scheme X is reduced, then its ring of global section $\Gamma(X, \mathscr{O}_X)$ is reduced.

Proof Suppose $\Gamma(X, \mathscr{O}_X)$ is not reduced, then $\Gamma(X, \mathscr{O}_X)$ has nonzero nilpotent, say f, there exists some n such that $f^n = 0$ in $\Gamma(X, \mathscr{O}_X)$. Hence, $f_p^n = 0$ in $\mathscr{O}_{X,p}$ for all $p \in X$. Note that X is reduced, $\mathfrak{N}(\mathscr{O}_{X,p}) = 0$ for all $p \in X$, hence, $f_p = 0$. By Proposition, we have injection

$$\Gamma(X, \mathscr{O}_X) \hookrightarrow \textstyle\prod_{p \in X} \mathscr{O}_{X,p},$$

hence, $f = 0$ in $\Gamma(X, \mathscr{O}_X)$, a contradiction. □

However, the converse of Proposition 6.2.5 is not true:

Example 6.3 If X is the scheme cut out by $x^2 = 0$ in $\mathbb{P}_k^2$, then $\Gamma(X, \mathscr{O}_X) \cong k$, and that X is nonreduced.

Proof Since X is the scheme cut out by $x^2 = 0$ in $\mathbb{P}_k^2$, $X = \operatorname{Proj}\left(k[x,y,z]/(x^2)\right)$. In fact, $X = D_+(x) \cup D_+(y) \cup D_+(z)$, by Proposition 5.5.9, we have

$$D_+(x) = \operatorname{Spec}((k[x,y,z]/(x^2))_x)_0 = \operatorname{Spec} 0 = \varnothing,$$
$$D_+(y) = \operatorname{Spec}((k[x,y,z]/(x^2))_y)_0 = \operatorname{Spec} k[x/y, z/y]/((x/y)^2),$$

and

$$D_+(z) = \operatorname{Spec}((k[x,y,z]/(x^2))_z)_0 = \operatorname{Spec} k[x/z, y/z]/((x/z)^2),$$

hence, $X = \operatorname{Spec} k[x/y, z/y]/((x/y)^2) \cup \operatorname{Spec} k[x/z, y/z]/((x/z)^2)$. Consider the following sequence.

$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \mathscr{O}_X(D_+(y)) \times \mathscr{O}_X(D_+(z)) \rightrightarrows \mathscr{O}_X(D_+(y) \cap D_+(z)) \tag{6.5}$$

By Proposition 5.5.10, we have

$$D_+(y) \cap D_+(z) = D_+(yz) \cong D_{D_+(y)}(z/y) \cong D_{D_+(z)}(y/z).$$

Hence, from sequence (6.5) we have

$$\begin{tikzcd}
 & & & \left(k\left[\frac{x}{y}, \frac{z}{y}\right] \Big/ \left(\left(\frac{x}{y}\right)^2\right)\right)_{\frac{z}{y}} \arrow[dd, leftrightarrow, "\cong"] \\
0 \arrow[r] & \mathscr{O}_X(X) \arrow[r] & k\left[\frac{x}{y}, \frac{z}{y}\right] \Big/ \left(\left(\frac{x}{y}\right)^2\right) \times k\left[\frac{x}{z}, \frac{y}{z}\right] \Big/ \left(\left(\frac{x}{z}\right)^2\right) \arrow[ur] \arrow[dr] & \\
 & & & \left(k\left[\frac{x}{z}, \frac{y}{z}\right] \Big/ \left(\left(\frac{x}{z}\right)^2\right)\right)_{\frac{y}{z}}
\end{tikzcd}$$

Let $s \in \mathscr{O}_X(X)$, let $s|_{D_+(y)} = f\left(\frac{x}{y}, \frac{z}{y}\right)$ and $s|_{D_+(z)} = g\left(\frac{x}{z}, \frac{y}{z}\right)$. On $D_+(y) \cap D_+(z)$, we have coordinate change

$$\frac{x}{y} = \frac{\frac{x}{z}}{\frac{y}{z}}, \qquad \frac{z}{y} = \frac{1}{\frac{y}{z}}.$$

Since $(s|_{D_+(y)})|_{D_+(y) \cap D_+(z)} = (s|_{D_+(z)})|_{D_+(y) \cap D_+(z)}$, we have

$$f\left(\frac{x}{y}, \frac{z}{y}\right) = f\left(\frac{\frac{x}{z}}{\frac{y}{z}}, \frac{1}{\frac{y}{z}}\right) = \frac{1}{\left(\frac{y}{z}\right)^d} F\left(\frac{x}{z}\right) = \frac{\left(\frac{x}{z}\right)^d}{\left(\frac{y}{z}\right)^d} = g\left(\frac{x}{z}, \frac{y}{z}\right),$$

where $d = \deg g$ and $F\left(\frac{x}{z}\right)$ is a homogeneous polynomial with variable $\frac{x}{z}$ (and therefore $F\left(\frac{x}{z}\right) = \left(\frac{x}{z}\right)^d$). Since $g\left(\frac{x}{z}, \frac{y}{z}\right)$ is a polynomial with variable $\frac{x}{z}, \frac{y}{z}$, d must be 0, i.e., $s|_{D_+(y)} = s|_{D_+(z)}$ is constant, by the identity axiom, s is constant. It follows that $\Gamma(X, \mathscr{O}_X) \cong k$. Since f is a field, k is reduced, and therefore $\Gamma(X, \mathscr{O}_X)$ is reduced.

Note that $\mathscr{O}_X(D_+(y)) \cong k\left[\frac{x}{y}, \frac{z}{y}\right] \Big/ \left(\left(\frac{x}{y}\right)^2\right)$ has nonzero nilpotent $\frac{x}{y}$, $\mathscr{O}_X(D_+(y))$ is not reduced ring. By Proposition 6.2.3, X is nonreduced. □

Proposition 6.2.6

Suppose X is quasi-compact, and f is a function that vanishes at all points of X, then there is some n such that $f^n = 0$.

Proof Since X is a quasi-compact scheme, we may assume that $X = \bigcup_{i=1}^n \operatorname{Spec} A_i$. In each $\operatorname{Spec} A_i$, since f vanishes at all points of X, we have $D(f) = \varnothing$, where $D(f)$ is distinguished open subset of $\operatorname{Spec} A_i$, by Proposition 4.5.6, there exists k_i such that $f^{k_i} = 0$. Let $k = \max_{1 \le i \le n} k_i$, then $f^k = 0$, as we desired. □

Example 6.4 Proposition 6.2.6 may fail if X is not quasi-compact.

Proof Let $A_n = k[\varepsilon]/(\varepsilon^n)$, and $X = \coprod_{n=1}^{\infty} \operatorname{Spec} A_n$. Note that each $\operatorname{Spec} A_n = \{(\varepsilon)\}$, $X = \{((\varepsilon), n)\}_{n=1}^{\infty}$. Let $f = \varepsilon$, then f vanishes at all points of X. Suppose there exists n such that $f^n = \varepsilon^n = 0$, then $\varepsilon^n \in (\varepsilon^m)$ for all m, it is impossible. It follows that Proposition 6.2.6 fail. □

Remark Proposition 6.2.6 is less important, but shows why we like quasi-compactness, and gives a standard pathology when quasi-compactness doesn't hold.

6.2.2 Integrality

Definition 6.2.2 (Integral)

A scheme X is ***integral*** *if it is nonempty, and $\mathscr{O}_X(U)$ is an integral domain for every nonempty open subset U of X.*

Proposition 6.2.7

A scheme X is integral if and only if it is irreducible and reduced.

Proof If X is a integral scheme, then $\mathscr{O}_X(U)$ is an integral domain for every nonempty open subsets U of X. Hence, $\mathscr{O}_X(U)$ is reduced ring for any open subset U of X, by Proposition 6.2.3, X is reduced. Suppose X is not irreducible (Definition 4.6.2), then there exists disjoint open subsets of X, say U, V. Consider $\mathscr{O}_X(U \cup V)$. Note that we have exact sequence

$$0 \longrightarrow \mathscr{O}_X(U \cup V) \longrightarrow \mathscr{O}_X(U) \times \mathscr{O}_X(V) \rightrightarrows \mathscr{O}_X(U \cap V) = \mathscr{O}_X(\varnothing) = 0,$$

it follows that $\mathscr{O}_X(U \cup V) \cong \mathscr{O}_X(U) \times \mathscr{O}_X(V)$. Let $(s, 0), (0, t) \in \mathscr{O}_X(U) \times \mathscr{O}_X(V)$, then $(s, 0) \cdot (0, t) = 0$, i.e., $(s, 0)$ and $(0, t)$ are zero divisors, which implies that $\mathscr{O}_X(U \cup V)$ is not a integral domain, a contradiction.

Conversely, if X is irreducible and reduced. Let U be any open subset of X, we want show $\mathscr{O}_X(U)$ is an integral domain. Let $f, g \in \mathscr{O}_X(U)$ such that $fg = 0$. Define the closed subsets of U (Proposition 5.3.6 (a))

$$Z_f = \{p \in U : f_p \in \mathfrak{m}_p \subseteq \mathscr{O}_{X,p}\}, \quad Z_g = \{p \in U : g_p \in \mathfrak{m}_p \subseteq \mathscr{O}_{X,p}\},$$

where $\mathfrak{m}_p$ is the maximal ideal of local ring $\mathscr{O}_{X,p}$. Since $fg = 0$, for all $p \in U$, we have $f_p g_p = 0$ in $\mathscr{O}_{X,p}$. We claim that $\mathscr{O}_{X,p}$ is an integral domain. Let $V = \operatorname{Spec} A$ be an affine open neighborhood of p, since X is irreducible and reduced, V is also irreducible and reduced. Let $ab = 0$ in A, then $V(0) =$

$\operatorname{Spec} A = V(a) \cup V(b)$. Since $\operatorname{Spec} A$ is irreducible, $V(a) = \operatorname{Spec} A$ or $V(b) = \operatorname{Spec} A$, we may assume that $V(a) = \operatorname{Spec} A$, then $a \in \bigcap_{[\mathfrak{p}] \in \operatorname{Spec} A} \mathfrak{p} = \mathfrak{N}(A)$, hence, by Proposition 6.2.2, $\mathfrak{N}(A) = 0$, which implies that $a = 0$. Hence, A is an integral domain, and therefore $A_p = \mathscr{O}_{X,p}$ is an integral domain. Since $f_p g_p = 0$ in $\mathscr{O}_{X,p}$, we have $f_p = 0$ or $g_p = 0$ for all $p \in U$, that is, $U \subseteq Z_f \cup Z_g$. Clearly, we have $Z_f \cup Z_g \subseteq U$, hence, $U = Z_f \cup Z_g$. Since X is irreducible and reduced, so is U. Hence, $U = Z_f$ or $U = Z_g$. We may assume that $Z_f = U$, then f vanish at all points of U. By Proposition 6.2.1, we have $f = 0$, which implies that $\mathscr{O}_{X,p}$ is an integral domain. Hence, X is integral. □

Proposition 6.2.8

An affine scheme $\operatorname{Spec} A$ *is integral if and only if* A *is an integral domain.*

Proof If A is an integral domain, by Proposition 4.6.2, $\operatorname{Spec} A$ is irreducible. Since A is integral domain, we have $\mathfrak{N}(A) = 0$, that is, A is reduced ring, by Proposition 6.2.2, $\operatorname{Spec} A$ is reduced. By Proposition 6.2.7, $\operatorname{Spec} A$ is integral.

Conversely, if $\operatorname{Spec} A$ is integral, by Proposition 6.2.7, $\operatorname{Spec} A$ is irreducible and reduced. By Proposition 6.2.2, A is a reduced ring. We want to show A is integral domain. Let $fg = 0$ in A, then we have $\operatorname{Spec} A = V(0) = V(fg) = V(f) \cup V(g)$, since $\operatorname{Spec} A$ is irreducible, $\operatorname{Spec} A = V(f)$ or $\operatorname{Spec} A = V(g)$. We may assume that $\operatorname{Spec} A = V(f)$, then f belong to any prime ideal of A, it follows that $f \in \mathfrak{N}(A) = 0$, hence, $f = 0$, and therefore A is an integral domain. □

Proposition 6.2.9

Suppose X *is an integral scheme. Then* X *has a generic point* η*. Suppose* $\operatorname{Spec} A$ *is any nonempty affine open subset of* X*. Then* $\mathscr{O}_{X,\eta}$ *is naturally identified with* $K(A)$*, the fraction field of* A*.*

Proof By Proposition 6.1.2, there is a bijection between irreducible closed subsets and points for any scheme, hence, there exists a point $\eta \in X$ such that $\overline{\{\eta\}} = X$, that is, η is a generic point for X.

Let $\operatorname{Spec} A$ be any nonempty affine open subset of X, since η is a generic point of X, by Proposition 4.6.10, η must belong to $\operatorname{Spec} A$, hence, $\mathscr{O}_{X,\eta} = A_\eta$. Since X is integral, $\operatorname{Spec} A$ is integral, by Proposition 6.2.8, A is integral domain. Hence, $(0) \in \operatorname{Spec} A$, note that $V((0)) = \overline{\{(0)\}} = \operatorname{Spec} A$, (0) is the generic point for $\operatorname{Spec} A$. In fact, $\overline{\{(0)\}} = \overline{\{\eta\}} \cap \operatorname{Spec} A$, by Proposition 6.1.2, $\eta = (0)$, hence, $A_\eta = A_{(0)} = K(A)$, where $K(A)$ is the fraction field of A, that is,

$$\mathscr{O}_{X,\eta} \cong K(A).$$

□

By Proposition 6.2.9, we can give the definition of function field and rational functions:

Definition 6.2.3 (Function field, rational functions)

Suppose X *is an integral scheme. Say* η *is the generic point of* X*. Let* $\operatorname{Spec} A$ *be any nonempty affine open subset of* X*. The* ***function field*** $K(X)$ *of* X *is defined by*

$$K(X) := K(A) = \mathscr{O}_{X,\eta}.$$

It can be computed on any nonempty open set of X*, as any such open set contains the generic point. The elements of* $K(X)$ *are called* ***rational functions*** *on* X*.*

Proposition 6.2.10

Suppose X is an integral scheme, then the restriction maps $\operatorname{res}_{U,V} : \mathscr{O}_X(U) \to \mathscr{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any nonempty affine open subset of X (so A is an integral domain), then the natural map $\mathscr{O}_X(U) \to \mathscr{O}_{X,\eta} = K(A)$ (where U is any nonempty open subset and η is the generic point) is an inclusion.

Proof Let $X = \bigcup_i \operatorname{Spec} A_i$, then $U = \bigcup_i (U \cap \operatorname{Spec} A_i)$ and $V = \bigcup_i (V \cap \operatorname{Spec} A_i)$. Since $U \supseteq V$, we have $U \cap \operatorname{Spec} A_i \supseteq V \cap \operatorname{Spec} A_i$. Since each $U \cap \operatorname{Spec} A_i$ is open in $\operatorname{Spec} A_i$, let $U \cap \operatorname{Spec} A_i = \bigcup_j D(f_{i,j})$, where $D(f_{i,j})$ is distinguished open subset of $\operatorname{Spec} A_i$. Similarly, we may assume that $V \cap \operatorname{Spec} A_i = \bigcup_k D(g_{i,k})$. Hence,

$$V \cap \operatorname{Spec} A_i = (U \cap \operatorname{Spec} A_i) \cap (V \cap \operatorname{Spec} A_i) = \bigcup_j \bigcup_k (D(f_{i,j}) \cap D(g_{i,k})).$$

By the definition of sheaf we have following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X(U) \arrow[r, hook] \arrow[d, "\operatorname{res}_{U,V}"'] & \prod_i \prod_j (A_i)_{f_{i,j}} \arrow[d, hook] \\
0 \arrow[r] & \mathscr{O}_X(V) \arrow[r, hook] & \prod_i \prod_j \prod_k (A_i)_{f_{i,j} g_{i,k}},
\end{tikzcd}$$

where $\prod_i \prod_j (A_i)_{f_{i,j}} \hookrightarrow \prod_i \prod_j \prod_k (A_i)_{f_{i,j} g_{i,k}}$ is given by $(A_i)_{f_{i,j}} \hookrightarrow (A_i)_{f_{i,j} g_{i,k}}$ (note that each A_i is integral domain, since X is integral). Hence, $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ is inclusion.

Let U be any nonempty open subset and η is the generic point for X, then $\eta \in U$, by Proposition 4.6.10. Note that $U \cap \operatorname{Spec} A$ is open subset of $\operatorname{Spec} A$, and therefore it is covered by some distinguished open subsets, hence, by above discussion, we have inclusion $\mathscr{O}_X(U) \hookrightarrow \mathscr{O}_X(D(f)) = A_f$. Since $D(f) \ni \eta$, we have inclusion $A_f \hookrightarrow A_\eta$, that is,

$$\mathscr{O}_X(U) \hookrightarrow \mathscr{O}_X(D(f)) = A_f \hookrightarrow A_\eta = \mathscr{O}_{X,\eta} = K(A),$$

as we desired. □

Thus irreducible varieties (an important example of integral schemes defined later) have the convenient property that sections over different open sets can be considered subsets of the same ring. In particular, restriction maps (except to the empty set) are always inclusions, and gluing is easy: functions f_i on a cover U_i of U (as i runs over an index set) glue if and only if they are the same element of $K(X)$ (since $\mathscr{O}_X(U_i) \hookrightarrow \mathscr{O}_X(U_i \cap U_j) \hookrightarrow K(A)$). This is one reason why irreducible varieties are usually introduced before schemes.

Remark **Caution.** Integrality is not stalk-local (the disjoint union of two integral schemes is not integral, as $\operatorname{Spec} A \coprod \operatorname{Spec} B = \operatorname{Spec}(A \times A)$ by Example 4.5), but it almost is.

6.3 The Affine Communication Lemma, and properties of schemes that can be checked "affine-locally"

6.3.1 The Affine Communication Lemma

This section is intended to address something tricky in the definition of schemes. We have defined a scheme as a topological space with a sheaf of rings, that can be covered by affine schemes. Hence we have all of the affine open sets in the cover, but we don't know how to communicate between any two of them. Somewhat more explicitly, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over to your cover. The **Affine Communication Lemma** will provide a convenient machine for doing this.

Thanks to this lemma, we can define a host of important properties of schemes. All of these are “**affine-local**”, in that they can be checked on any affine cover, i.e., a covering by open affine sets. (We state this more formally after the statement of the Affine Communication Lemma.) We like such properties because we can check them using any affine cover we like. If the scheme in question is quasi-compact, then we need only check a finite number of affine open sets.

Proposition 6.3.1

Suppose $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X*. Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$*.*

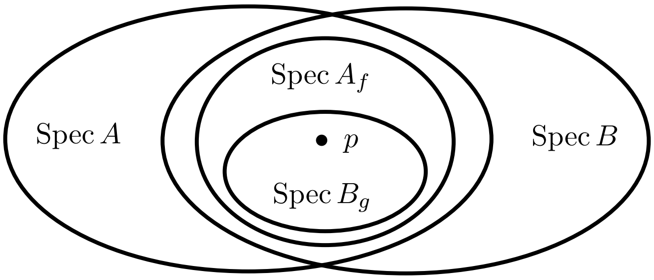


Figure 6.1: A trick to show that the intersection of two affine open sets may be covered by open sets that are simultaneously distinguished in both affine open sets.

Proof (See Figure 6.1.) Given any point $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of p in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$ and containing p. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$ and containing p. Then $g \in \Gamma(\operatorname{Spec} B, \mathscr{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathscr{O}_X) = A_f$. The points where g and g' vanish on $\operatorname{Spec} A_f$ are the same, so

$$\begin{aligned}\operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] \in \operatorname{Spec} A_f : g' \in \mathfrak{p}\}\\ &= \operatorname{Spec}(A_f)_{g'}.\end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

Definition 6.3.1 (Affine-local property)

Suppose X *is a scheme, an* ***affine-local property*** *is a property* $\mathscr{P}$ *of affine open subsets of* X *satisfying the following axioms. For any affine open subset* $\operatorname{Spec} A \hookrightarrow X$*,*

(i) if $\operatorname{Spec} A \hookrightarrow X$ *has property* $\mathscr{P}$ *then for any* $f \in A$*,* $\operatorname{Spec} A_f \hookrightarrow X$ *has property* $\mathscr{P}$*;*

(ii) if $(f_1, \ldots, f_n) = A$ *(i.e., the* $\operatorname{Spec} A_{f_i}$ *cover* $\operatorname{Spec} A$*, see Proposition* 4.5.3 *and Proposition* 4.5.2*), and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has* $\mathscr{P}$ *for all* i*, then* $\operatorname{Spec} A \hookrightarrow X$ *has property* $\mathscr{P}$*.*

Example 6.5 Reducedness is an affine-local property.

Proof If $\operatorname{Spec} A$ is reduced, by Proposition 6.2.2, A is reduced ring, i.e., $\mathfrak{N}(A) = 0$, hence, $\mathfrak{N}(A_f) = \mathfrak{N}(A)_f = 0$, by Proposition 6.2.2 again, $\operatorname{Spec} A_f$ is reduced.

Since $\operatorname{Spec} A$ is quasi-compact, by Proposition 4.5.3, we may assume $A = (f_1, \ldots, f_n)$. If $\operatorname{Spec} A_{f_i}$ is reduced, then A_{f_i} is reduced ring. Let $a \in \mathfrak{N}(A)$, then $a^n = 0$ in A for some n, Hence, $a^n = 0$ in A_{f_i}. Since each A_{f_i} is reduced, $\mathfrak{N}(A_{f_i}) = 0$, we have $a = 0$ in A_{f_i}. By the identity axiom of sheaf $\mathscr{O}_{\operatorname{Spec} A}$, $a = 0$ in A. Hence, $\mathfrak{N}(A) = 0$, that is, $\operatorname{Spec} A$ is reduced. □

The following easy result will be crucial for us.

Theorem 6.3.1 (Affine Communication Lemma)

Let X be a scheme and let $\mathscr{P}$ be an affine-local property of affine open subsets of X. If $X = \bigcup_{i \in I} \operatorname{Spec} A_i$ where $\operatorname{Spec} A_i$ has property $\mathscr{P}$. Then every affine open subset of X has $\mathscr{P}$ too.

Proof Let $\operatorname{Spec} A$ be an affine subscheme of X. Cover $\operatorname{Spec} A$ with a finite number of distinguished open sets $\operatorname{Spec} A_{g_j}$, each of which is distinguished in some $\operatorname{Spec} A_i$. This is possible by Proposition 6.3.1 and Proposition 4.6.4. By (i), each $\operatorname{Spec} A_{g_j}$ has $\mathscr{P}$. By (ii), $\operatorname{Spec} A$ has $\mathscr{P}$. □

Note that if U is an open subscheme of X, then U inherits any affine-local property of X. Note that also that any property that is stalk-local (a scheme has property P if and only if all its stalks have property Q) is necessarily affine-local (a scheme has property P if and only if all of its affine open sets have property R, where an affine scheme has property R if and only if all its stalks have property Q). But it is sometimes not so obvious what the right definition of Q is; see for example the discussion of normality in the next section.

6.3.2 Properties of schemes that can be checked "affine-locally": (Locally) Noetherian schemes, A-schemes, (Locally of) finite type over A, Affine varieties, and Projective varieties

By choosing the property P appropriately, we define some important properties of schemes.

Proposition 6.3.2

Suppose A is a ring, and $(f_1, \ldots, f_n) = A$.

(a) If A is a Noetherian ring, then so is A_{f_i}. If each A_{f_i} is Noetherian, then so is A.

(b) Suppose B is a ring, and A is a B-algebra. (Hence A_g is a B-algebra for all $g \in A$.) If A is a finitely generated B-algebra, then so is A_{f_i}. If each A_{f_i} is a finitely generated B-algebra, then so is A.

We will prove these shortly. But let's first motivate you to read the proof by giving some interesting definitions and results assuming Proposition 6.3.2 is true.

Locally Noetherian schemes, Noetherian shceme

Definition 6.3.2 (Locally Noetherian scheme, Noetherian scheme)

*Suppose X is a scheme. If X can be covered by affine open sets $\operatorname{Spec} A$ where A is Noetherian, we say that X is a **locally Noetherian scheme**. If in addition X is quasi-compact, or equivalently can be covered by finitely many such affine open sets (Proposition 6.1.4), we say that X is a **Noetherian scheme**.*

Remark By Proposition 6.3.2, being Noetherian is a local property. Hence, by Affine Communication Lemma 6.3.1, if X is locally Noetherian scheme, every affine open $U \subseteq X$ the ring $\mathscr{O}_X(U)$ is Noetherian.

Remark

(1) We will see a number of definitions of the form "if X has this property, we say that it is locally Q; if further X is quasi-compact, we say that it is Q."

(2) By Proposition 6.1.3, the underlying topological space of a Noetherian scheme is Noetherian. Hence by Proposition 4.6.19, all open subsets of a Noetherian scheme are quasi-compact.

Proposition 6.3.3

Locally Noetherian schemes are quasi-separated.

Proof Let X be a locally Noetherian scheme. We may assume that $X = \bigcup_i \operatorname{Spec} A_i$, where A_i is Noetherian. Let $\operatorname{Spec} B$ and $\operatorname{Spec} C$ be any two affine open subsets, we want to show that $\operatorname{Spec} B \cap \operatorname{Spec} C$ is a finite union of affine open subsets. In fact,

$$\operatorname{Spec} B = \operatorname{Spec} B \cap X = \bigcup_i (\operatorname{Spec} B \cap \operatorname{Spec} A_i),$$
$$\operatorname{Spec} C = \operatorname{Spec} C \cap X = \bigcup_i (\operatorname{Spec} C \cap \operatorname{Spec} A_i).$$

Note that $\operatorname{Spec} B \cap \operatorname{Spec} A_i$ and $\operatorname{Spec} C \cap \operatorname{Spec} A_i$ are open subsets of $\operatorname{Spec} A_i$ and affine scheme is quasi-compact, hence, they are finite union of distinguished open subsets of $\operatorname{Spec} A_i$. We may assume that $\operatorname{Spec} B \cap \operatorname{Spec} A_i = \operatorname{Spec}(A_i)_{f_i}$ and $\operatorname{Spec} C \cap \operatorname{Spec} A_i = \operatorname{Spec}(A_i)_{g_i}$, since $\operatorname{Spec} B$ and $\operatorname{Spec} C$ is quasi-compact, we may assume that $\operatorname{Spec} B = \bigcup_{i=1}^n \operatorname{Spec}(A_i)_{f_i}$ and $\operatorname{Spec} C = \bigcup_{j=1}^m \operatorname{Spec}(A_j)_{g_j}$. Hence,

$$\operatorname{Spec} B \cap \operatorname{Spec} C = \bigcup_{i=1}^n \bigcup_{j=1}^m (\operatorname{Spec}(A_i)_{f_i} \cap \operatorname{Spec}(A_j)_{g_j}).$$

By Proposition 6.3.1, each $\operatorname{Spec}(A_i)_{f_i} \cap \operatorname{Spec}(A_j)_{g_j}$ is the union of open sets that are simultaneously distinguished open subschemes of $\operatorname{Spec}(A_i)_{f_i}$ and $\operatorname{Spec}(A_j)_{g_j}$. Since A_i is Noetherian, by Proposition 4.6.15, each $(A_i)_{f_i}$ and $(A_j)_{g_j}$ are Noetherian. Note that $\operatorname{Spec}(A_i)_{f_i} \cap \operatorname{Spec}(A_j)_{g_j}$ is open subset of $\operatorname{Spec}(A_i)_{f_i}$, by Proposition 4.6.19, $\operatorname{Spec}(A_i)_{f_i} \cap \operatorname{Spec}(A_j)_{g_j}$ is quasi-compact. Hence, $\operatorname{Spec}(A_i)_{f_i} \cap \operatorname{Spec}(A_j)_{g_j}$ is the finite union of open sets that are simultaneously distinguished open subschemes of $\operatorname{Spec}(A_i)_{f_i}$ and $\operatorname{Spec}(A_j)_{g_j}$. Hence, $\operatorname{Spec} B \cap \operatorname{Spec} C$ is the finite union of some distinguished open sets, each distinguished open set is affine scheme, and therefore X is quasi-separated. □

Proposition 6.3.4

(a) A Noetherian scheme has a finite number of irreducible components.

(b) A Noetherian scheme has a finite number of connected components, each a finite union of irreducible components.

Proof

(a) Let X be a Noetherian scheme, by Proposition 6.1.3, then X is a Noetherian topological space, by Proposition 4.6.12 X can be expressed uniquely as a finite union of irreducible closed subsets, none contained in any other, then we done.

(b) By Proposition 4.6.13 (a), every connected component of X is the union of irreducible components of X. Since X is a finite union of irreducible components, hence, each connected components is the finite union of irreducible components of X. Since any two connected components disjoint, the number of connected components is finite.

□

Proposition 6.3.5

A Noetherian scheme X is integral if and only if X is nonempty and connected and all stalks $\mathscr{O}_{X,p}$ are integral domains.

Proof Let X be a Noetherian scheme. If X is integral, by Proposition 6.2.7, X is irreducible and reduced. By Proposition 4.6.3, X is connected. Each point $p \in X$ contained in some affine open subsets of X, say $\operatorname{Spec} A$. Since X is integral, A is integral domain, hence, localization $A_p = \mathscr{O}_{X,p}$ is integral domain.

Conversely, if a Noetherian ring X is nonempty and connected and all stalks $\mathscr{O}_{X,p}$ are integral domains, by Proposition 6.2.7, it suffices to show that X is reduced and irreducible. Since all stalks $\mathscr{O}_{X,p}$ are integral domain, $\mathfrak{N}(\mathscr{O}_{X,p}) = 0$, by the definition of reducedness, X is reduced.

If X is not irreducible, by Proposition 6.3.4, Noetherian scheme X is a finite number of irreducible components, say $X = \bigcup_{i=1}^n X_i$, where X_i is irreducible component. Since X is connected, the intersection of any two irreducible components not empty. Let $p \in X$, take all X_i which contains p, we obtain $\{X_{i_k}\}_{k=1}^m$ (i.e., no irreducible component X_i outside $\{X_{i_k}\}_{k=1}^m$ contains p), then $p \in \bigcap\limits_{k=1}^{m} X_{i_k}$. Since X is a scheme, exists an affine open subset $\operatorname{Spec} A \ni p$. In fact, $\operatorname{Spec} A = \bigcup_i (\operatorname{Spec} A \cap X_i)$, since each X_i is irreducible component, $\operatorname{Spec} A \cap X_i$ is also irreducible component of $\operatorname{Spec} A$. Since each X_i is closed (note that X_i is irreducible component), $\operatorname{Spec} A \cap X_i$ is irreducible closed subset of $\operatorname{Spec} A$. By Proposition 4.7.3, there exists prime ideal of A correspond to $\operatorname{Spec} A \cap X_i$, say $V(\mathfrak{p}_i) = \operatorname{Spec} A \cap X_i$. Then $p \in \bigcap\limits_{k=1}^{m} (X_{i_k} \cap \operatorname{Spec} A) = \bigcap\limits_{k=1}^{m} V(\mathfrak{p}_{i_k})$. Since there is no irreducible component X_i outside $\{X_{i_k}\}_{k=1}^m$ contains p, $\{\mathfrak{p}_{i_k}\}_{k=1}^m$ is all prime ideal which contained in p. Since $\operatorname{Spec} A \cap X_{i_k}$ is irreducible componet of $\operatorname{Spec} A$, $(\operatorname{Spec} A) \cap X_{i_k} \neq \operatorname{Spec} A$, it follows that each $\mathfrak{p}_{i_k}$ is not zero ideal. Hence, $(0) \notin \mathscr{O}_{X,p} = A_p$, and therefore A_p is not integral domain, it contradicts to $\mathscr{O}_{X,p}$ is integral domain. Hence, X is irreducible, and therefore is integral. □

Remark Thus, by Proposition 6.3.5, in "good situations", integrality is the union of local (stalks are integral domains) and global (connected) conditions.

Remark The ring of global section of a Noetherian scheme need not be Noetherian.

Schemes over a given field k, or more generally over a given ring A (A-schemes)

You may be particularly interested in working over a particular field, such as $\mathbb{C}$ or $\mathbb{Q}$, or over a ring such as $\mathbb{Z}$. Motivated by this we define the following notions:

Definition 6.3.3 (A-scheme (or scheme over A))

Suppose X is a scheme, A is a ring. We say X is A-scheme (or scheme over A), if $\mathscr{O}_X(U)$ is A-algebra for all open sets $U \subseteq X$, and all restriction maps are maps of A-algebras.

Remark Like some earlier notions such as quasi-separatedness, this will later in Definition 8.3.4 be properly understood as a "relative notion"; it is the data of a morphism $X \to \operatorname{Spec} A$.

Definition 6.3.4 (Locally of finite type over A, finite type over A)

*Suppose X is an A-scheme. If X can be covered by affine open sets $\operatorname{Spec} B_i$ where each B_i is a finitely generated A-algebra, we say that X is **locally of finite type over** A, or that it is a **locally finite type A-scheme**. (This is admittedly cumbersome terminology; it will make more sense later, once we know about morphisms in Chapter 9.) If furthermore X is quasi-compact, X is (of) **finite type over** A, or a*

finite type *A-scheme.*

Remark Note that a scheme locally of finite type over k or $\mathbb{Z}$ (or indeed any Noetherian ring) is locally Noetherian (apply Hilbert's Basis Theorem), and similarly a scheme of finite type over any Noetherian ring is Noetherian.

Example 6.6 As our key "geometric" examples:

(i) $\operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]/I$ is a finite type $\mathbb{C}$-scheme,

(ii) $\mathbb{P}^n_{\mathbb{C}}$ is a finite type $\mathbb{C}$-scheme. (The field $\mathbb{C}$ may be replaced by an arbitrary ring A.)

Proposition 6.3.6

*(a) (**Quasi-projective implies finite type.**) If X is a quasi-projective A-scheme (Definition 5.5.11), then X is of finite type over A. If A is furthermore assumed to be Noetherian, then X is a Noetherian scheme, and hence has a finite number of irreducible components.*

(b) Suppose U is an open subscheme of a projective A-scheme. Then U is locally of finite type over A. If A is Noetherian, then U is quasi-compact, and hence quasi-projective over A, and hence by (a) of finite type over A.

Proof

(a) Since X is a quasi-projective A-scheme, then X is a quasi-compact open subscheme of $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a finitely generated graded ring over $S_0 = A$. By Proposition 5.5.5, we may assume that $\operatorname{Proj} S_\bullet = \bigcup_i D_+(f_i)$, then $X = \bigcup_i (X \cap D_+(f_i))$. Since X is open subset of $\operatorname{Proj} S_\bullet$, $X \cap D_+(f_i)$ is open subset of $D_+(f_i) = \operatorname{Spec}((S_\bullet)_{f_i})_0$. Since $S_\bullet$ is a finitely generated graded ring over A, by Proposition 5.5.3 (a), $S_\bullet$ is a finitely generated graded A-algebra, by Proposition 6.3.2, $(S_\bullet)_{f_i}$ is finitely generated A-algebra, and therefore $((S_\bullet)_{f_i})_0$ is finitely generated A-algebra. Since $X \cap D_+(f_i)$ is open subset of $D_+(f_i)$, $X \cap D_+(f_i)$ covered by some distinguished open subsets, we may assume that $X \cap D_+(f_i) = D(g)$, then $X \cap D_+(f_i) = \operatorname{Spec}(((S_\bullet)_{f_i})_0)_g$, by Proposition 6.3.2 again, $(((S_\bullet)_{f_i})_0)_g$ is finitely generated A-algebra. By above discussion, we know that X is quasi-compact, and X can be covered by affine open subsets $\operatorname{Spec}(((S_\bullet)_{f_i})_0)_g$, where each $(((S_\bullet)_{f_i})_0)_g$ is a finitely generated A-algebra, hence, X is of finite type over A.

If A is furthermore assumed to be Noetherian, by Hilbert's Basis Theorem, each $(((S_\bullet)_{f_i})_0)_g$ is Noetherian, and therefore X is locally Noetherian scheme, also X is quasi-compact, X is Noetherian scheme, by Proposition 6.3.4, X has a finite number of irreducible components.

(b) Similar to the proof of (a), by removing the quasi-compactness condition, we conclude that U is locally of finite type over A.

If A is Noetherian. By the proof in (a) and Proposition 6.1.7, we may assume that $U = \bigcup_{i=1}^n (U \cap D_+(f_i))$. Since $U \cap D_+(f_i)$ is open in $\operatorname{Spec}((S_\bullet)_{f_i})_0$, since A is Noetherian, $((S_\bullet)_{f_i})_0$ is Noetherian, by Proposition 4.6.19, $\operatorname{Spec}((S_\bullet)_{f_i})_0$ is quasi-compact, and therefore $U \cap D_+(f_i)$ is quasi-compact. It follows that U is quasi-compact, and therefore U is quasi-projective over A, by part (a), U is of finite type over A.

□

✍ **Exercise 6.2** Show that in Proposition 6.3.6 (b), U may not be finite type over A if A is not Noetherian. Give an example of an open subscheme of a projective A-scheme that is not quasi-compact, necessarily for some non-Noetherian A.

Solution *Let $A = k[x_0, x_1, \dots]$, consider $\mathbb{P}^0_A$, by Example 5.9 , $\mathbb{P}^0_A \cong \operatorname{Spec} A$. Let $\mathfrak{m} = (x_0, x_1, \dots)$. By* *remark after Proposition 4.6.4, $U = \mathbb{P}^0_A \setminus V(\mathfrak{m})$ is not quasi-compact open subset of $\mathbb{P}^0_A$, and therefore U not be finite type over A.*

Affine varieties, Projective varieties

We now make a connection to the classical language of varieties.

Definition 6.3.5 (Affine variety, Projective variety)

*An affine scheme that is reduced and of finite type over k is said to be an **affine variety (over** k**)**, or an **affine** k**-variety**. A reduced (quasi)projective k-scheme is a **(quasi)projective variety (over** k**)**, or a **(quasi)projective** k**-variety**. (By Proposition 6.3.6, (quasi)projective k-scheme is already of finite type over A.)*

Remark Warning: in the literature, it is sometimes also assumed in the definition of variety that the scheme is irreducible, or that k is algebraically closed.

Proposition 6.3.7

(a) $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ is an affine k-variety if and only if $I \subseteq k[x_1, \ldots, x_n]$ is a radical ideal.

(b) Suppose $I \subseteq k[x_0, \ldots, x_n]$ is a radical graded ideal. Then $\operatorname{Proj} k[x_0, \ldots, x_n]/I$ is a projective k-variety.

Proof

(a) In fact, $k[x_1, \ldots, x_n]$ is finitely generated k-algebra, and therefore $k[x_1, \ldots, x_n]/I$ is finitely generated k-algebra. By Proposition 4.6.4, $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ is quasi-compact. Hence, $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$ is of finite type over k. It suffices to show that X is reduced if and only if I is a radical ideal.

If X is reduced, let $f\sqrt{I}$, then exists some n such that $f^n \in I$, hence, $f^n = 0$ in X. Since $\mathfrak{N}(X) = 0$, $f = 0$, which implies that $f \in I$, that is, $\sqrt{I} \subseteq I$. Hence I is radical ideal.

Conversely, if I is radical ideal, we want to show that X is reduced. Let $f \in \mathfrak{N}(\mathscr{O}_X(X)) = \mathfrak{N}(k[x_1, \ldots, x_n]/I)$, then $f^n \in I$, and therefore $f \in \sqrt{I} = I$, hence, $\mathfrak{N}(\mathscr{O}_X(X)) = 0$, it follows that $\mathscr{O}_X(X)$ is reduced ring. By Proposition 6.2.2, X is reduced.

(b) Let $S_\bullet = k[x_0, \ldots, x_n]/I$, then $S_\bullet$ is a finitely generated graded ring, by Proposition 5.5.3 (a), $S_\bullet$ is a finitely generated k-algebra. By Proposition 5.5.8 we have $\operatorname{Proj} S_\bullet = \bigcup_{i=0}^{n} D_+(x_i)$, where $D_+(x_i) = \operatorname{Spec}((S_\bullet)_{x_i})_0$. Since $S_\bullet$ is a finitely generated k-algebra, by Proposition 6.3.2 (b), $(S_\bullet)_{x_i}$ is finitely generated k-algebra, and therefore $((S_\bullet)_{x_i})_0$ is finitely generated k-algebra. By Proposition 6.1.7, $\operatorname{Proj} S_\bullet$ is quasi-compact, hence, $\operatorname{Proj} S_\bullet$ is of finite type over k. To show $\operatorname{Proj} S_\bullet$ is projective k-variety it suffices to show that $\operatorname{Proj} S_\bullet$ is reduced.

Since reducedness is affine-local property, it suffices to show that $D_+(x_i)$ is reduced. Note that $((S_\bullet)_{x_i})_0 \cong k[x_{1/i}, \ldots, x_{n/i}]/I_{x_i}$, it suffices to show that $k[x_{1/i}, \ldots, x_{n/i}]/I_{x_i}$ is reduced ring. We claim that I_{x_i} is radical ideal. Let $\frac{f}{x_i^n} \in \sqrt{I_{x_i}}$, then $\left(\frac{f}{x_i^n}\right)^m \in I_{x_i}$ for some m. Hence, $f^m \in I$, since I is radical, $f \in I$, and therefore $\frac{f}{x_i^n} \in I_{x_i}$, which implies that I_{x_i} is radical ideal. Let $f \in \mathfrak{N}(\mathscr{O}_{\operatorname{Proj} S_\bullet}(D_+(x_i))) = \mathfrak{N}(k[x_{1/i}, \ldots, x_{n/i}]/I_{x_i})$, then $f^n \in I_{x_i}$ for some n. Since I_{x_i} is radical, $f \in I_{x_i}$. It follows that $\mathfrak{N}(\mathscr{O}_{\operatorname{Proj} S_\bullet}(D_+(x_i))) = 0$. Hence, $D_+(x_i)$ is reduced by Proposition 6.2.2.

□

We will not define varieties in general until Chapter 12; we will need the notion of separatedness first, to exclude abominations like the line with the doubled origin (§5.4.2). But many of the statements we will make in this section about affine k-varieties will automatically apply more generally to k-varieties.

Proposition 6.3.8

A point of a locally finite type k-scheme is a closed point if and only if the residue field of the structure sheaf at that point is a finite extension of k. On a locally finite type k-scheme, the set of all closed points of X is dense (it follows that every neighborhood of a point contains a closed point).

Proof Let X be a locally finite type k-scheme, then $X = \bigcup_i \operatorname{Spec} A_i$ where A_i is finitely generated k-algebra.

If p is a closed point of X. Exists $\operatorname{Spec} A_i$ be an affine open neighborhood of p, since A_i is a finitely generated k-algebra, we may assume that $A_i = k[x_1, \ldots, x_r]$. Since p is a closed point, p is a maximal ideal of A_i. Consider $\mathscr{O}_{X,p}/\mathfrak{m}_p$, then

$$\mathscr{O}_{X,p}/\mathfrak{m}_p = (A_i)_p/\mathfrak{m}_p = (A_i)_p/p(A_i)_p \cong (A_i/p)_{(0)} \cong \operatorname{Frac}(A_i/p),$$

since p is maximal ideal of A_i, $\operatorname{Frac}(A_i/p) = (A_i)/p$, by Zariski's Lemma 4.2.2, A_i/p is a finite extension of k, and therefore $\mathscr{O}_{X,p}/\mathfrak{m}_p$ is a finite extension of k.

Conversely, let $p \in X$ such that $\kappa(p) = \mathscr{O}_{X,p}/\mathfrak{m}_p$ is a finite extension of k. Let $\operatorname{Spec} A_i$ be any affine open neighborhood of p, then $\kappa(p) \cong (A_i)_p/p(A_i)_p \cong \operatorname{Frac}(A_i/p)$ is a finite extension of k. Since A_i is finitely generated k-algebra, A_i/p is also finitely generated k-algebra, and therefore $k \subseteq A_i/p$. Note that $k \subseteq A_i/p \subseteq \operatorname{Frac}(A_i/p)$ where k and $\operatorname{Frac}(A_i/p)$ are both fields, A_i/p must be a field (see [3]), hence, A_i/p is a field, and therefore p is maximal ideal of A_i, which implies that p is a closed point in each $\operatorname{Spec} A_i$. Let $\operatorname{cl}(\{p\})$ be the closure of $\{p\}$ in X, note that

$$\operatorname{cl}(\{p\}) \cap \operatorname{Spec} A_i = \overline{\{p\} \cap \operatorname{Spec} A_i} = \overline{\{p\}} = \{p\},$$

hence,

$$\operatorname{cl}(\{p\}) = \bigcup_i (\operatorname{cl}(\{p\}) \cap \operatorname{Spec} A_i) = \bigcup_i \{p\} = \{p\},$$

it follows that $\{p\}$ is closed in X.

Let $\mathscr{S}$ be a set of all closed points of X, we want to show that $\operatorname{cl}_X(\mathscr{S}) = X$. By Proposition 4.6.7, $\operatorname{cl}_X(\mathscr{S}) \cap \operatorname{Spec} A_i = \operatorname{cl}_{\operatorname{Spec} A_i}(\mathscr{S} \cap \operatorname{Spec} A_i) = \operatorname{Spec} A_i$, hence,

$$\operatorname{cl}_X(\mathscr{S}) = \bigcup_i (\operatorname{cl}_X(\mathscr{S}) \cap \operatorname{Spec} A_i) = \bigcup_i \operatorname{Spec} A_i = X,$$

which implies that $\mathscr{S}$ dense in X. □

Remark Closed points need not be dense even on quite reasonable schemes, see Proposition 4.6.7 (b).

Definition 6.3.6

*The **degree of a closed point** p of a locally finite type k-scheme (e.g., a variety over k) is the degree of the field extension $\kappa(p)/k$.*

Example 6.7 In $\mathbb{A}_k^1 = \operatorname{Spec} k[t]$, the point $[(p(t))]$ ($p(t) \in k[t]$ irreducible) is $\deg p(t)$. If k is algebraically closed, the degree of every closed point is 1.

The proof of Proposition 6.3.2

Now, let's prove Proposition 6.3.2:

Proposition

Suppose A is a ring, and $(f_1, \ldots, f_n) = A$.

(a) If A is a Noetherian ring, then so is A_{f_i}. If each A_{f_i} is Noetherian, then so is A.

(b) Suppose B is a ring, and A is a B-algebra. (Hence A_g is a B-algebra for all $g \in A$.) If A is a finitely generated B-algebra, then so is A_{f_i}. If each A_{f_i} is a finitely generated B-algebra, then so is A.

Proof

(a) If A is a Noetherian ring, by Proposition 4.6.15, A_{f_i} is Noetherian ring for all f_i.
Conversely, if each A_{f_i} is Noetherian. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an increasing chain of ideals of A, then in each A_{f_i} we have correspondence increasing chain of ideals,

$$I_1 \otimes_A A_{f_i} \subseteq I_2 \otimes_A A_{f_i} \subseteq I_3 \otimes_A A_{f_i} \subseteq \cdots . \tag{6.6}$$

Since each A_{f_i} is Noetherian, chain (6.6) must be stationary, hence, for each f_i exists N_i such that for all $k \geq N_i$, $I_k \otimes_A A_{f_i} = I_{N_i} \otimes_A A_{f_i}$. Let $N = \max\limits_{1 \leq i \leq n} N_i$, we want to show that for all $k \geq N$, $I_k = I_N$.
Let $x \in I_k$, then $x \in I_k \otimes_A A_{f_i} = I_N \otimes_A A_{f_i}$, hence, exists m_i such that $f_i^{m_i} x \in I_N$. Let $m = \max\limits_{1 \leq i \leq n} m_i$.
Since $(f_1, \ldots, f_n) = A$, exists a_i such that

$$a_1 f_1 + \cdots + a_n f_n = 1,$$

and therefore

$$(a_1 f_1 + \cdots + a_n f_n)^{nm} := b_1 f_1^m + \cdots + b_n f_n^m = 1.$$

Hence,

$$x = (b_1 f_1^m + \cdots + b_n f_n^m) x = b_1 f_1^m x + \cdots + b_n f_n^m x \in I_N,$$

that is, $I_k \subseteq I_N$. Hence, $I_k = I_N$, it follows that A is Noetherian.

(b) If A is a finitely generated B-algebra, we may assume that $A = B[x_1, \ldots, x_n]$, then $A_{f_i} = B[x_1, \ldots, x_n]_{f_i} = B[x_1, \ldots, x_n, 1/f_i]$ is a finitely generated B-algebra.
Conversely, if each A_{f_i} is a finitely generated B-algebra, we may assume that $A_{f_i} = B[\frac{r_{i1}}{f_i^{k_1}}, \ldots, \frac{r_{in_i}}{f_i^{k_{n_i}}}]$.
Since $A = (f_1, \ldots, f_n)$, exists $a_i \in A$ such that $1 = \sum_{i=1}^n a_i f_i$. Let

$$\mathscr{S} = \{f_i\}_{i=1}^n \cup \{a_i\}_{i=1}^n \cup \{r_{ij}\}_{ij},$$

$\mathscr{S}$ is a finite set, we claim that $A = B[\mathscr{S}]$. Clearly, we have $B[\mathscr{S}] \subseteq A$. Let $r \in A$, then in A_{f_i} exists polynomial $p_i(\frac{r_{i1}}{f_i^{k_1}}, \ldots, \frac{r_{in_i}}{f_i^{k_{n_i}}}) \in A_{f_i}$ such that $r = p_i(\frac{r_{i1}}{f_i^{k_1}}, \ldots, \frac{r_{in_i}}{f_i^{k_{n_i}}})$, multiply both sides of the equation by a sufficiently large $f_i^{N_i}$, we obtain

$$f_i^{N_i} r = f_i^{N_i} p_i(\frac{r_{i1}}{f_i^{k_1}}, \ldots, \frac{r_{in_i}}{f_i^{k_{n_i}}}) \in B[\mathscr{S}].$$

Let $N = \max_{1 \leq i \leq n} N_i$, similar to the part (a) we have $1 = \sum_{i=1}^n b_i f_i^N$ for some $b_i \in B[\mathscr{S}]$. Hence,

$$r = (\sum_{i=1}^n b_i f_i^N) r = \sum_{i=1}^n b_i f_i^N r \in B[\mathscr{S}],$$

which implies that $A \subseteq B[\mathscr{S}]$, and therefore $A = B[\mathscr{S}]$, i.e., A is finitely generated B-algebra.

□

6.4 Normality and factoriality

6.4.1 Normality

We can now define a property of schemes that says that they are "not too far from smooth", called **normality**, which will come in very handy. We will see later that "locally Noetherian normal schemes satisfy Hartogs's Lemma" (Algebraic Hartogs's Lemma for Noetherian normal schemes): functions defined away from a set of codimension 2 extend over that set. (We saw a first glimpse of this in §5.4.1.) As a consequence, rational functions (Definition 6.2.3) that have no poles (certain sets of codimension one where the function isn't defined) are defined everywhere. We need definitions of dimension and poles to make this precise. See Chapter 14 and Chapter 27 for the fact that "smoothness" (really, "regularity") implies normality.

Definition 6.4.1 (Integral)

Let $\varphi : R \to S$ be a ring map.

(i) An element $s \in S$ is integral over R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^{\varphi}(s) = 0$, where $P^{\varphi}(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.

(ii) The ring map φ is integral if every $s \in S$ is integral over R.

In the following discussion, we will only consider integral domains.

Definition 6.4.2 (Integral closure, Integrally closed)

*Let $R \to S$ be a ring homomorphism. The ring $S' = \{s \in S : s \text{ is integral over } R\}$ is called the **integral closure** of R in S. If $R \subseteq S$ we say that R is **integrally closed** in S if $R = S'$.*

In this section we just use following definition of integrally closed:

Definition 6.4.3 (Integrally closed)

*An integral domain A is **integrally closed** if the only zeros in $K(A)$ (fraction field of A, i.e., $\mathrm{Frac}(A)$) to any monic polynomial in $A[x]$ must lie in A itself.*

Remark Consider the ring homomorphism $A \to K(A)$, Definition 6.4.3 agree with Definition 6.4.2.

Definition 6.4.4 (Normal)

*We say a scheme X is **normal** if all of its stalks $\mathscr{O}_{X,p}$ are normal, i.e., are integral domains, and integrally closed in their fraction fields $K(\mathscr{O}_{X,p})$.*

Remark

(1) By definition, normality is a stalk-local property, and therefore is affine-local property. Normal scheme is also reduced (Definition 6.2.1), since each stalks are integral domain.

(2) One might say that a ring A is **normal ring** if $\operatorname{Spec} A$ is normal, thereby extending the notion of "integral closure" to rings that are not integral domains.

Proposition 6.4.1 (Integral closure commutes with localization)

If $A \to B$ is a ring homomorphism, and $S \subseteq A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subseteq B$ is the integral closure of A in B.

Proof Since localization is exact we know that $S^{-1}B' \subseteq S^{-1}B$. Let $x/f \in S^{-1}B'$ where $x \in B'$ and $f \in S$, then

$$x^d + \sum_{i=1}^{d} a_i x^{d-i} = 0$$

in B for some $a_i \in A$. Hence, we have

$$(x/f)^d + \sum_{i=1}^{d} (a_i/f^i)(x/f)^{d-i} = 0$$

in $S^{-1}B$, it follows that x/f in the integral closure of $S^{-1}A$ in $S^{-1}B$, and therefore, $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$.

Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum_{i=1}^{d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \cdots f_d x)^d + \sum_{i=1}^{d} b_i a_i (f' f_1 \cdots f_d x)^{d-i} = 0,$$

for a suitable $f' \in S$. Hence, $f' f_1 \cdots f_d x \in B'$, and therefore $x/f \in S^{-1}B'$, which implies that the integral closure of $S^{-1}A$ in $S^{-1}B$ contained in $S^{-1}B'$. $\square$

Corollary 6.4.1 (Integrally closed domain behave well under localization)

If A is an integrally closed domain, and S is a multiplicative subset not containing 0, then $S^{-1}A$ is an integrally closed domain.

Proof Since A integrally closed, $K(A)' = A$, apply Proposition 6.4.1, then we done. $\square$

Corollary 6.4.2

If A is an integrally closed domain, then $\operatorname{Spec} A$ *is normal.*

Proof Since A is integrally closed, by Corollary 6.4.1, $A_{\mathfrak{p}}$ in integrally closed for all $[\mathfrak{p}] \in \operatorname{Spec} A$. $\square$

Corollary 6.4.3

For schemes that are quasi-compact or locally finite type over a field, normality can be checked at closed point. More precisely, if $\mathscr{O}_{X,p}$ is normal (integral domain and integrally closed in their fraction fields) for all closed point $p \in X$, then X is normal

Proof We want to show that $\mathscr{O}_{X,p}$ is normal for all $p \in X$. Suppose q is not a closed point in X, since X is quasi-compact or locally finite type over a field, by Proposition 6.1.5 or Proposition 6.3.8 (on a locally finite type k-scheme the set of all closed points of X is dense, which implies that every neighborhood of q intersects this set), $\overline{\{q\}}$ contains a closed point of X, say p. Let U be an affine open neighborhood of p, note that q is a generic point for $\overline{\{q\}}$, by Proposition 4.6.10, U contains q. Since U is affine open subset, we may assume that $U = \operatorname{Spec} A$, then $\operatorname{Spec} A \cap \overline{\{q\}} = V(q)$. Note that $p \in V(q)$, we have $p \supseteq q$. Note that $\mathscr{O}_{X,q} = A_q$, we want to show that $\mathscr{O}_{X,q} = A_q$ is integrally domain and is integrally closed in its fraction fields. Since $p \supseteq q$, we have $(A_p)_{qA_p} = A_q$ (see the proof in Proposition 6.2.4). Since $\mathscr{O}_{X,p} = A_p$ is normal, by Corollary 6.4.1, $(A_p)_{qA_p} = A_q$ is normal, as we desired. $\square$

Example 6.8 **It is not true that normal schemes are integral.** For example, the disjoint union of two normal schemes is normal. Thus $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not an integral domain, and therefore is not integral scheme.

Proof We give a proof of the fact that the disjoint union of two normal schemes is normal. Let X, Y be two normal schemes. Let $p \in X \coprod Y$, then $p \in X$ or $p \in Y$. Since X and Y are open in $X \coprod Y$, we have $\mathscr{O}_{X \coprod Y, p} = \mathscr{O}_{X,p}$ or $\mathscr{O}_{X \coprod Y, p} = \mathscr{O}_{Y,p}$. Since X and Y are normal, $\mathscr{O}_{X,p}$ and $\mathscr{O}_{Y,p}$ are normal, and therefore $\mathscr{O}_{X \coprod Y, p}$ is normal, it follows that $X \coprod Y$ is normal. □

Proposition 6.4.2

A Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes.

Proof Let X be a Noetherian scheme, if X is the finite disjoint union of integral Noetherian normal schemes, say $X = \coprod_{i=1}^{n} X_i$, where each X_i is integral Noetherian normal scheme. Let $p \in X$, then $p \in X_i$ for some i, hence, $\mathscr{O}_{X,p} = (\mathscr{O}_X|_{X_i})_p$ is normal, it follows that X is normal.

Conversely, if X is normal scheme, then X is reduced by the definition of normal scheme (Definition 6.4.4). Since X is Noetherian scheme, by Proposition 6.3.4, X has a finite number of irreducible components, say $X = \bigcup_{i=1}^{n} X_i$ where X_i is irreducible component. We claim that each X_i disjoint. Suppose that $X_i \cap X_j \neq \varnothing$, let $p \in X_i \cap X_j$, then there exists an affine open subset $\operatorname{Spec} A \ni p$. Hence, $p \in (\operatorname{Spec} A \cap X_i) \cap (\operatorname{Spec} A \cap X_j)$. Note that both $\operatorname{Spec} A \cap X_i$ and $\operatorname{Spec} A \cap X_j$ are irreducible components of $\operatorname{Spec} A$, by Hilbert's Nullstellensatz, $\operatorname{Spec} A \cap X_i = V(\mathfrak{q}_i)$ and $\operatorname{Spec} A \cap X_j = V(\mathfrak{q}_j)$ where $\mathfrak{q}_i$ and $\mathfrak{q}_j$ are nonzero minimal prime ideals of A. Also, since $p \in V(\mathfrak{q}_i) \cap V(\mathfrak{q}_j)$, we have $p \supseteq q_i$ and $p \supseteq q_j$, i.e., p has two different minimal prime ideals. It follows that (0) is not the prime ideal of $\mathscr{O}_{X,p} = A_p$ (since if $(0) \in \operatorname{Spec} A_p$, p must has unique minimal prime ideal (0)), hence, $\mathscr{O}_{X,p}$ is not domain, contradicts to the fact that $\mathscr{O}_{X,p}$ is normal domain. Thus each X_i is disjoint, and therefore $X = \coprod_{i=1}^{n} X_i$. Clearly, each X_i is a scheme, since X is normal, X_i is normal. It is easy to check X_i is Noetherian from X is Noetherian. Note that X_i is normal, irreducible, reduced, Noetherian scheme, by Proposition 6.2.7, X_i is integral Noetherian normal schemes. Hence, X is the finite disjoint union of integral Noetherian normal schemes. □

We are close to proving a useful result in commutative algebra, so we may as well go all the way.

Proposition 6.4.3

If A is an integral domain, then the following are equivalent.

(i) A is integrally closed.

(ii) $A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p} \subseteq A$.

(iii) $A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m} \subseteq A$.

Proof Corollary 6.4.1 shows that integral closedness is preserved by localization, so (i) implies (ii). Clearly, (ii) implies (iii).

It remains to show that (iii) implies (i). Suppose A is not integrally closed. We show that there is some $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{6.7}$$

(with $a_i \in A$) has a solution s in $K(A) \setminus A$. Let I be the **ideal of denominators of** s:

$$I := \{r \in A : rs \in A\}. \tag{6.8}$$

It is easy to see I is an ideal of A. Note that $1 \notin A$, we have $I \neq A$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. We claim that $s \notin A_{\mathfrak{m}}$. If not, let $s = a/f$ where $a \in A$ and $f \in A - \mathfrak{m}$. Since $\mathfrak{m} \supseteq I$, we have

$f \in A - I$, which implies that $fs \notin A$, a contradiction. Hence, $s \notin A_{\mathfrak{m}}$. So equation (6.7) in $A_{\mathfrak{m}}[x]$ shows that $A_{\mathfrak{m}}$ is not integrally closed as well, as we desired. □

Corollary 6.4.4

Spec A *is normal if and only if* A *is an integrally closed domain.*

Proof If A is an integrally closed domain, by Corollary 6.4.2, Spec A is normal.

Conversely, if Spec A is normal, then all of its stalks $\mathscr{O}_{X,[\mathfrak{p}]} = A_{\mathfrak{p}}$ is normal, by Proposition 6.4.3, A is normal. □

Proposition 6.4.4

If A is an integral domain, then $A = \bigcap_{\mathfrak{m} \in \operatorname{Max} A} A_{\mathfrak{m}}$.

Proof Since A is integral domain, we have inclusion $A \hookrightarrow A_{\mathfrak{m}}$. Hence, $A \subseteq \bigcap_{\mathfrak{m} \in \operatorname{Max} A} A_{\mathfrak{m}}$. Let $s \in \bigcap_{\mathfrak{m} \in \operatorname{Max} A} A_{\mathfrak{m}}$, suppose that $s \notin A$. Consider the ideal of denominators of s, i.e.,

$$I := \{r \in A : rs \in A\}.$$

By the hypothesis, $1 \notin I$, and therefore $I \neq A$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. Similar to the proof of Proposition 6.4.3, $s \notin A_{\mathfrak{m}}$, a contradiction. Hence $s \in A$, which implies that $A = \bigcap_{\mathfrak{m} \in \operatorname{Max} A} A_{\mathfrak{m}}$. □

✍ **Exercise 6.3** **(Unimportant exercise relating to the ideal of denominators.)** One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[w, x, y, z]/(wz - xy)$ which we first met in Example 5.7 , and which we will later recognize as the cone over the quadric surface, and $w/y = x/z \in K(A)$. Show that the ideal of denominators of this element of $K(A)$ is (y, z).

Proof Let I be the ideal of denominators of $w/y = x/z$, clearly $y, z \in I$, and therefore $(y, z) \subseteq I$. Conversely, let $s \in I$, then $sw/y \in A$ or $sx/z \in A$. Hence exists $b \in A$ such that $sw = yb$ or $sx = zb$, i.e., $sw \in (y)$ or $sx \in (z)$. Since both (y) and (z) are prime ideals, note that w, x not in (y) and (z), $s \in (y)$ or $s \in (z)$, hence, $s \in (y, z)$. It follows that $I = (y, z)$. □

We see that the I in Exercise 6.3 is not principal (In Chpater 14, you may be able to show it directly, using the fact that I is a homogeneous ideal of a graded ring). But we will also see that in good situations (Noetherian, normal), the ideal of denominators is "pure codimension 1" — this is the content of Algebraic Hartog's Lemma. In its proof, Chapter 14, we give a geometric interpretation of the ideal of denominators.

6.4.2 Factoriality

Definition 6.4.5 (Factorial)

We say a scheme X is factorial, if the stalk $\mathscr{O}_{X,p}$ is UFD (unique factorization domain) for all $p \in X$.

Remark The locus of points on an affine variety over an algebraically closed field that are factorial is an open subset.

Proposition 6.4.5

Any nonzero localization of a UFD is a UFD.

Proof Let A be a UFD, and $S \subseteq A$ be a multiplicative subset not containing 0. Let $\frac{a}{s} \in S^{-1}A$, where $a \in A$

and $s \in S$. Since A is UFD, we may assume that $a = up_1^{n_1} \cdots p_k^{n_k}$ and $s = vq_1^{m_1} \cdots q_r^{m_r}$ where u, v are units and each p_i, q_j are prime elements, then

$$\frac{a}{s} = \frac{u}{v} \cdot p_1^{n_1} \cdots p_k^{n_k} \cdot \frac{1}{q_1^{m_1}} \cdots \frac{1}{q_r^{m_r}}.$$

We claim that p_i and $\frac{1}{q_j}$ are prime element in $S^{-1}A$. Let $p_i = \frac{b_1}{t_1} \cdot \frac{b_2}{t_2} \in S^{-1}A$ where $b_1, b_2 \in A$ and $t_1, t_2 \in S$, then exists $l \in S$ such that $l(p_i t_1 t_2 - b_1 b_2) = 0$ in A, i.e., $lp_i t_1 t_2 = lb_1 b_2$, since A is UFD, $p_i \mid b_1$ or $p_i \mid b_2$. Without loss of generality, we may assume that $p_1 \mid b_1$, hence $b_1 = up_1$ for some unit u. Then $p_i = \frac{up_1}{t_1} \cdot \frac{b_2}{t_2}$, note that $\frac{up_1}{t_1}$ is unit in $S^{-1}A$, we have $p_i \mid \frac{b_2}{t_2}$. Hence, p_i is prime element. Similarly, it is easy to show that $\frac{1}{q_j}$ is prime element in $S^{-1}A$.

Next we want to show that the decomposition of $\frac{a}{s}$ is unique. Let

$$\frac{a}{s} = \frac{u}{v} \cdot p_1^{n_1} \cdots p_k^{n_k} \cdot \frac{1}{q_1^{m_1}} \cdots \frac{1}{q_r^{m_r}} = \frac{u'}{v'} \cdot p_1'^{n_1'} \cdots p_k'^{n_k'} \cdot \frac{1}{q_1'^{m_1'}} \cdots \frac{1}{q_r'^{m_r'}}. \tag{6.9}$$

where $\frac{u'}{v'}$ is unit in $S^{-1}A$, be another decomposition of $\frac{a}{s}$. Then exists w such that

$$wuv' p_1^{n_1} \cdots p_k^{n_k} q_1'^{m_1'} \cdots q_r'^{m_r'} = wu'v p_1'^{n_1'} \cdots p_k'^{n_k'} q_1^{m_1} \cdots q_r^{m_r},$$

by the unique factoriality of A, $p_i = p_i', q_j = q_j', n_i = n_i'$, and $m_j = m_j'$, it follows that the decomposition of $\frac{a}{s}$ is unique. Hence, $S^{-1}A$ is UFD. □

Corollary 6.4.5

If A is a UFD, then $\operatorname{Spec} A$ *is factorial.*

Proof By Proposition 6.4.5, the localization of A is UFD, hence, $\mathscr{O}_{\operatorname{Spec} A, [\mathfrak{p}]} = A_{\mathfrak{p}}$ is UFD for all $[\mathfrak{p}] \in \operatorname{Spec} A$, and therefore $\operatorname{Spec} A$ is factorial. □

Remark The converse need not hold. In fact, we will see that elliptic curves are factorial, yet NO affine open set is the Spec of a unique factorization domain. Hence, one can show factoriality by finding an appropriate affine cover, but there need not be such a cover of a factorial scheme. (One might reasonably call a ring A such that $\operatorname{Spec} A$ is factorial, **a factorial ring**; this is a strictly weaker notion than UFD. We won't need this terminology.)

Remark **How to check if a ring is a UFD ?** There are very few means of checking that a Noetherian integral domain is a UFD. Some useful ones are:

(i) By Gauss's Lemma, the polynomial rings over a UFD is UFD.
(ii) The localization of a unique factorization domain is also a unique factorization domain.
(iii) Height 1 prime ideals are principal.
(iv) Normal and $\mathrm{Cl} = 0$.
(v) Nagata's Lemma.

Caution: even if A is a UFD, $A[[t]]$ need not be UFD, see [4] page 165. The first example, due to P.Salmon, was $A = k(u)[[x, y, z]]/(z^2 + x^3 + uz^6)$.

One reason we like factoriality is that it implies normality.

Proposition 6.4.6 (Factoriality is that it implies normality)

UFDs are integrally closed. Hence, factorial schemes are normal.

Proof Let A be a UFD, and $\frac{a}{b} \in K(A)$ agree with equation

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{6.10}$$

in $A[x]$. Without loss of generality, we may assume that a, b do not have common irreducible factor. By (6.10), we have

$$a^n + c_{n-1}a^{n-1}b + \cdots + c_1ab^{n-1} + c_0b^n = 0,$$

hence,

$$b(c_{n_1}a^{n-1} + \cdots + c_0b^{n-1}) = -a^n.$$

Since a, b do not have common irreducible factor, $b \nmid a^n$, it follows that b is a unit, hence, $\frac{a}{b} = ab^{-1} \in A$, which implies that A is integrally closed. $\square$

Remark However, rings can be integrally closed without being UFD, as we will see in Exercise 6.9. Another example is given without proof in Exercise 6.11; in that example, Spec of the ring is factorial. A variation on Exercise 6.9 will show that schemes can be normal without being factorial.

Corollary 6.4.6

If A is UFD, then $\operatorname{Spec} A$ *is normal.*

Proof By Proposition 6.4.6, it is clearly. $\square$

6.4.3 Examples

Exercise 6.4 The following schemes are normal: $\mathbb{A}_k^n$, $\mathbb{P}_k^n$, $\operatorname{Spec}\mathbb{Z}$.

Proof Since $k[x_1, \ldots, x_n]$ and $\mathbb{Z}$ are both UFDs, by Corollary 6.4.6, $\mathbb{A}_k^n$ and $\operatorname{Spec}\mathbb{Z}$ are normal. Let $p \in \mathbb{P}_k^n$, then $d \in D_+(x_i)$ for some x_i, since $D_+(x_i) \cong \mathbb{A}_k^n$, $\mathscr{O}_{\mathbb{P}_k^n,p} \cong \mathscr{O}_{\mathbb{A}_k^n,p}$ is UFD. Hence, $\mathbb{P}_k^n$ is normal. $\square$

Exercise 6.5 **(Yielding many enlightening examples later.)** Suppose A is a UFD with 2 invertible, and $z^2 - f$ is irreducible in $A[z]$.

(a) Show that if $f \in A$ has no repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is normal.

(b) Show that if $f \in A$ has repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is not normal.

Proof

(a) Since A is UFD, then $A[z]$ is UFD. Since the irreducible element in UFD is prime element, and therefore $(z^2 - f)$ is prime ideal of $A[z]$. Hence, $B := A[z]/(z^2 - f)$ is indeed an integral domain. Suppose $\alpha \in K(B)$ integral over B, then exists a monic polynomial $F(T) \in B[T]$ such that $F(\alpha) = 0$. Replacing $F(T)$ by $\overline{F}(T)F(T)$, then $F(T) \in A[T]$. Note that $K(B) = K(A)(z)/(z^2 - f)$, we may assume that $\alpha = g + hz$ where $g, h \in K(A)$. Note that α is the solution of $Q(T) = T^2 - 2gT + (g^2 - h^2f) \in K(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K(A)[T]$. By multiplying $P(T)$ and $Q(T)$ by suitable elements $c_1, c_2 \in A$ to clear denominators, we obtain $c_1P(T) \in A[T]$ and $c_2Q(T) \in A[T]$ such that they are primitive. Hence, by Gauss's Lemma (see Atiyah-MacDonald[1], page 11, Exercises 2 (iv)), $c_1c_2P(T)Q(T) = c_1c_2F(T)$ is primitive, since $F(T)$ is primitive, $c_1c_2 = 1$, hence, $c_1 = c_2 = 1$. It follows that $Q(T) \in A[T]$, hence, $2g \in A$ and $g^2 - h^2f \in A$. Since 2 invertible, $g \in A$, and therefore $h^2f \in A$. Let $f = p_1p_2\cdots p_n \in A$ with distinct prime element p_i and $h = \frac{s}{t}$ with $\gcd(s,t) = 1$, then

$$h^2f = \frac{s^2}{t^2} \cdot p_1p_2\cdots p_n \in A.$$

Hence, $t^2 \mid s^2p_1p_2\cdots p_n$, since each p_i distinct, t must be unit, and therefore $h \in A$. Consequently, $\alpha \in B$, which implies that $B = A[z]/(z^2 - f)$ is integrally closed, by Corollary 6.4.2, $\operatorname{Spec} A[z]/(z^2 - f)$ is normal.

(b) Without loss of generality, let $f = p^2q$, where p is prime element and $p \nmid q$. Let $\alpha = \frac{z}{p} \in K(B)$, note

that
$$\alpha^2 = \frac{z^2}{p^2} = \frac{f}{p^2} = q \in A,$$
hence, α agree with $t^2 - q = 0$ in $B[t]$, i.e., α. We want to show that $\alpha \notin B$.
If $\alpha \in B$, then exists $c, d \in A$ such that
$$\frac{z}{p} = c + dz \Longrightarrow z = pc + pdz \Longrightarrow (1 - pd)z = pc.$$
Hence, $1 - pd = 0$ and $pc = 0$, note that p is a prime element, it is impossible. It follows that $\alpha \notin B$. Hence, B is not integrally closed, by Corollary 6.4.4, Spec $A[z]/(z^2 - f)$ is not normal.

□

Exercise 6.6 Show that the following schemes are normal:

(a) Spec $\mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to 3 modulo 4. **Caution:** the hypotheses of Exercise 6.5 do not apply, so you will have to do this directly. (Your argument may also show the result when 3 is replaced by 2. A similar argument shows that $\mathbb{Z}[(1 + \sqrt{n})/2]$ is integrally closed if $n \equiv 1 \pmod 4$ is square-free.)

(b) Spec $k[x_1, \ldots, x_n]/(x_1^2 + x_2^2 + \cdots + x_m^2)$ where char $k \neq 2$, $n \geq m \geq 3$.

(c) Spec $k[w, x, y, z]/(wz - xy)$ where char $k \neq 2$. This is our cone over a quadric surface example from Example 5.7 and Exercise 6.3. (The result also holds for char $k = 2$, but don't worry about this.)

Proof

(a) We claim that $x^2 - n$ is an irreducible element in $\mathbb{Z}[x]$. If $x^2 - n = (a_1x + b_1)(a_2x + b_2)$, then $a_1a_2 = 1$, $a_1b_2 + a_2b_1 = 0$, and $b_1b_2 = n$. Without loss of generality, we may assume that $a_1 = a_2 = 1$, then $n = b_1^2$, contradicts to the fact that n is a sequence. Hence, $x^2 - n$ is irreducible, since $\mathbb{Z}[x]$ is UFD, $(x^2 - n)$ is prime ideal. Hence, $\mathbb{Z}[x]/(x^2 - n)$ is integral domain.
Since n is a square-free number, we have $\mathbb{Z}[x]/(x^2-n) \cong \mathbb{Z}[\sqrt{n}]$. We want to show that $\mathbb{Z}[\sqrt{n}]$ is integrally closed. Let $\gamma = \alpha + \beta\sqrt{n} \in K(\mathbb{Z}[\sqrt{n}]) = \mathbb{Q}(\sqrt{n})$ where $\alpha, \beta \in \mathbb{Q}$ and γ integral over $\mathbb{Z}[\sqrt{n}]$, then there exists a monic polynomial $F(T) \in \mathbb{Z}[\sqrt{n}][T]$ such that $F(\gamma) = 0$. Replacing $F(T)$ by $\overline{F}(T)F(T)$, then $F(T) \in \mathbb{Z}[T]$. Note that γ is the solution of $Q(T) = T^2 - 2\alpha T + (\alpha^2 - n\beta^2) \in \mathbb{Q}[T]$, so we can factor $F(T) = P(T)Q(T)$ in $\mathbb{Q}[T]$. By multiplying $P(T)$ and $Q(T)$ by suitable elements $c_1, c_2 \in \mathbb{Z}$ to clear denominators, we obtain $c_1P(T) \in \mathbb{Z}[T]$ and $c_2Q(T) \in \mathbb{Z}[T]$ such that they are primitive, by Gauss's Lemma (Atiyah-MacDonald[1], page 11, Exercise 2 (iv)), $c_1c_2P(T)Q(T) = c_1c_2F(T)$ is primitive, since $F(T)$ is primitive, $c_1c_2 = 1$, hence, $c_1 = c_2 = 1$. It follows that $Q(T) \in \mathbb{Z}[T]$, hence, $2\alpha \in \mathbb{Z}$ and $\alpha^2 - n\beta^2 \in \mathbb{Z}$. Suppose $\alpha = \frac{a}{2}$ and $\beta = \frac{c}{d}$ with $\gcd(c, d) = 1$, then
$$\alpha^2 - n\beta^2 = \frac{a^2}{4} - n \cdot \frac{c^2}{d^2} \Longrightarrow 4d^2(\alpha^2 - n\beta^2) = a^2d^2 - 4nc^2.$$
Note that $a^2d^2 - 4nc^2 \equiv a^2d^2 \pmod 4$ and $4d^2(\alpha^2 - n\beta^2) \equiv 0 \pmod 4$, we have
$$a^2d^2 \equiv 0 \pmod 4,$$
i.e., $4 \mid a^2d^2$, i.e., $2 \mid ad$.
If d is odd, then $2 \mid a$, let $a = 2k$, then
$$\alpha^2 - n\beta^2 = \frac{4k^2}{4} - \frac{nc^2}{d^2} = k^2 - \frac{nc^2}{d^2} \in \mathbb{Z}.$$
Hence, $d^2 \mid nc^2$, since $gcd(c, d) = 1$, $d^2 \mid n$. It is impossible, since n is square-free.
If d is even, then $2 \mid d$, let $d = 2k$, then
$$\alpha^2 - n\beta^2 = \frac{a^2}{4} - \frac{nc^2}{4k^2} \in \mathbb{Z}.$$

Hence, exists m such that $m = \frac{a^2}{4} - \frac{nc^2}{4k^2}$, and therefore $a^2 - \frac{nc^2}{k^2} = 4m$, it follows that

$$a^2 - \frac{nc^2}{k^2} \equiv 0 \pmod 4.$$

Since $n \equiv 3 \pmod 4$, we have $\frac{nc^2}{k^2} \equiv \frac{3c^2}{k^2} \pmod 4$, hence,

$$a^2 \equiv \frac{3c^2}{k^2} \pmod 4 \Longrightarrow k^2a^2 \equiv 3c^2 \pmod 4.$$

If a is even, then $k^2a^2 \equiv 0 \pmod 4$, hence $3c^2 \equiv 0 \pmod 4$, it is impossible, since $c^2 \equiv 0 \pmod 4$ or $c^2 \equiv 1 \pmod 4$.

If a is odd, then $a^2 \equiv 1 \pmod 4$. If k is odd, then $k^2 \equiv 1 \pmod 4$, hence, $k^2a^2 \equiv 1 \pmod 4$. But $3c^2 \equiv 0$ or $3 \pmod 4$, a contradiction. If k is even, $k^2a^2 \equiv 0 \pmod 4$, hence, $3c^2 \equiv 0 \pmod 4$. It follows that c is even. Note that $\gcd(c, d) = 1$ and $d = 2k$, a contradiction.

By above discussion, $\frac{a}{2}$ and $\frac{c}{d}$ must belong to $\mathbb{Z}$, and therefore $\gamma \in \mathbb{Z}[\sqrt{n}]$. It follows that $\mathbb{Z}[\sqrt{n}]$ is integrally closed in $\mathbb{Q}(\sqrt{n})$. By Corollary 6.4.2, $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ is normal.

(b) Let $z = x_1$, $f = x_2^2 + \cdots + x_m^2$, and $A = k[x_2, \ldots, x_n]$, then $k[x_1, \ldots, x_n]/(x_1^2 + x_2^2 + \cdots + x_m^2) \cong A[z]/(z^2 + f)$. We want to show (i) A is UFD with 2 invertible; (ii) f has no repeated prime factors; (iii) $z^2 + f$ is irreducible in $A[z]$. Then by Exercise 6.5, $\operatorname{Spec} A[z]/(z^2 + f)$ is normal.

(i) A is UFD with 2 invertible.

Clearly, $A = k[x_2, \ldots, x_n]$ is a UFD. Since $\operatorname{char} k \neq 2$, 2 is invertible.

(ii) f has no repeated prime factors

We claim that f is irreducible in A. Note that $\deg f = 2$, if $f = gh$, then $\deg g = \deg h = 1$. Let $g = a_2x_2 + \cdots + a_mx_m$ and $h = b_2x_2 + \cdots + b_mx_m$, then

$$\begin{aligned} x_1^2 + \cdots + x_m^2 =&(a_2x_2 + \cdots + a_mx_m)(b_2x_2 + \cdots + b_mx_m) \\ =&\sum_{i=2}^{m} a_ib_ix_i^2 + \sum_{2 \le i < j \le m} (a_ib_j + a_jb_i)x_ix_j. \end{aligned}$$

Hence, $a_ib_i = 1$ and $a_ib_j + a_jb_i = 0$, thus, we have

$$\frac{a_i}{a_j} + \frac{a_j}{a_i} = \frac{a_i^2 + a_j^2}{a_ia_j} = 0 \Longrightarrow a_i^2 + a_j^2 = 0 \Longrightarrow a_i = \pm a_j\sqrt{-1}.$$

Sine $a_2 = \pm a_1\sqrt{-1}$ and $a_3 = \pm a_1\sqrt{-1}$, we have $a_2^2 + a_3^2 = -2a_1^2 = 0 \Rightarrow a_1 = 0$, by induction, $a_i = b_i = 0$ for all i, it is impossible. Hence f is irreducible in A, and therefore f has no repeated prime factor.

(iii) $z^2 + f$ is irreducible in $A[z]$.

By Gauss's Lemma, if $z^2 + f$ is irreducible in $K(A)[z]$ and $z^2 + f$ is primitive, then $z^2 + f$ is irreducible in $A[z]$. We claim that $z^2 + f$ is irreducible in $K(A)[z]$. If not, there exists $\frac{h}{g} \in K(A)$ where $h \in A$ and $g \in A$ such that $\frac{h^2}{g^2} = -f$, i.e., $h^2 = -fg^2$. Since f is irreducible, $f \mid h^2$, and therefore $f \mid h$. Let $h = fh'$, then $f^2h'^2 = -fg^2 \Rightarrow fh'^2 = -g^2$, by f is irreducible, $f \mid g^2 \Rightarrow f \mid g$. Let $g = fg'$, then we have $h'^2 = -fg'^2$. Repeat above discussion, we have ${h^{(n)}}^2 = -f{g^{(n)}}^2$ and

$$h = fh' = f^2h'' = \cdots = f^nh^{(n)} = \cdots, \quad g = fg' = f^2g'' = \cdots = f^ng^{(n)} = \cdots,$$

where $\deg h^{(n)} = \deg h^{(n-1)} - 1$ and $\deg g^{(n)} = \deg g^{(n-1)} - 1$. It is impossible, since h and g are finite degree. Hence, $z^2 + f$ is irreducible in $K(A)[z]$. Since it is primitive, by Gauss's Lemma, $z^2 + f$ is irreducible in $A[z]$.

By above discussion, apply Exercise 6.5, $\operatorname{Spec} A[z]/(z^2 + f) \cong \operatorname{Spec} k[x_1, x_2, \ldots, x_n]/(x_1^2 + \cdots + x_m^2)$

is normal.

(c) Note that

$$wz - xy = \left(\frac{w+z}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + \left(\frac{i(w-z)}{2}\right)^2 + \left(\frac{i(x+y)}{2}\right)^2,$$

apply part (b), we done.

□

✍ **Exercise 6.7** **(Diagonalizing quadratic forms.)** Suppose k is an algebraically closed field of characteristic not 2. (The hypothesis that k is algebraically closed is not necessary, so feel free to deal with this more general case.)

(a) Show that any quadratic form in n variables can be "diagonalized" by changing coordinates to be a sum of at most n squares. (e.g., $uw - v^2 = (\frac{u+w}{2})^2 + \left(\frac{i(u-w)}{2}\right)^2 + (iv)^2$, where the linear forms appearing in the squares are linearly independent.)

(b) Show that the number of squares appearing depends only on the quadratic form. For example, $x^2+y^2+z^2$ cannot be written as a sum of two squares. (Possible approach: given a basis $x_1, \ldots, x_n$ of the linear forms, write the quadratic form as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where M is a symmetric matrix. Determine how M transforms under a change of basis, and show that the rank of M is independent of the choice of basis.)

Proof This result comes from linear algebra and is presented without proof. You may see Wikipedia[5]. □

Remark The **rank** of the quadratic form is the number of ("linearly independent") squares needed. If the number of squares equals the number of variables, the quadratic form is said to be **full rank** or (of) **maximal rank**.

✍ **Exercise 6.8** **(Rings can be integrally closed but not UFD, arithmetic version.)** Show that $\mathbb{Z}[\sqrt{-5}]$ is integrally closed but not a UFD.

Proof In fact, $K(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Q}(\sqrt{-5})$. Let $\gamma = \alpha + \beta\sqrt{-5} \in \mathbb{Q}(\sqrt{-5})$ is integrally closed over $\mathbb{Z}[\sqrt{-5}]$, then exists $F(t) \in \mathbb{Z}[\sqrt{-5}][t]$ such that $F(\gamma) = 0$. Replacing $F(t)$ by $\overline{F}(t)F(t)$, then $F(t) \in \mathbb{Z}[t]$. Note that $\gamma = \alpha + \beta\sqrt{-5}$ agree with $Q(t) = t^2 - 2\alpha x + \alpha^2 + 5\beta^2 = 0$. Similar to the proof in Exercise 6.6 (a), $Q(t) \in \mathbb{Z}[t]$. Hence, $2\alpha \in \mathbb{Z}$ and $\alpha^2 + 5\beta^2 \in \mathbb{Z}$. Let $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{c}$ where $a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$, then

$$\frac{2a}{c} \in \mathbb{Z}, \quad \frac{a^2}{c^2} + \frac{5b^2}{c^2} = \frac{a^2+5b^2}{c^2} \in \mathbb{Z}.$$

We want to show $c = 1$. If $c > 1$, then exists prime number p such that $p \mid c$. Since $c \mid 2a$, we have $p \mid 2a$, and therefore $p = 2$ or $p \mid a$. Also, since $c^2 \mid (a^2 + 5b^2)$, we have $p^2 \mid (a^2 + 5b^2)$.

If $p = 2$, then we may assume that $c = 2m$, and therefore $m \mid a$. If $m \neq 1$. Let $a = km$, then we have

$$4m^2 \mid (k^2m^2 + 5b^2) \Longrightarrow 4 \mid (k^2 + 5 \cdot \left(\frac{b}{m}\right)^2).$$

Since $\gcd(a, b, c) = 1$, $m \nmid b$, hence, $\frac{b^2}{m^2} \notin \mathbb{Z}$, and therefore $\frac{5b^2}{m^2} \in \mathbb{Z}$. It is impossible, since 5 has no repeated prime factor. Hence, $m = 1$, it follows that $c = 2, a = k$, then $4 \mid (k^2 + 5b^2)$, i.e., $k^2 + 5b^2 \equiv 0 \pmod 4$. Thus,

$$k^2 + b^2 \equiv 0 \pmod 4,$$

which implies that $4 \mid k^2$ and $4 \mid b^2$, i.e., $2 \mid k$ and $2 \mid b$. Hence $\gcd(a,b,c) \geq 2 > 1$, a contradiction.

If $p \mid a$, then from $p^2 \mid (a^2+5b^2)$ we have $p^2 \mid 5b^2$. If $p \nmid 5$, then $p \mid b$, note that $p \mid c$, $\gcd(a,b,c) \geq p > 1$, a contradiction. Hence, $p \mid 5$, i.e., $p = 5$. Since $5 \mid b^2$, $5 \mid b$, i.e., $p \mid b$, hence, $\gcd(a,b,c) \geq p > 1$, a contradiction.

By above discussion, $c = 1$, hence, $\gamma = a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$, which implies that $\mathbb{Z}[\sqrt{-5}]$ is integrally closed. Consider $6 \in \mathbb{Z}[\sqrt{-5}]$, note that

$$6 = 2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5}),$$

$\mathbb{Z}[\sqrt{-5}]$ not a UFD. □

✍ **Exercise 6.9** **(Rings can be integrally closed but not UFD, geometric version.)** Suppose $\operatorname{char} k \neq 2$. Let $A = k[w,x,y,z]/(wz-xy)$, so $\operatorname{Spec} A$ is the cone over the smooth quadric surface (Example 5.7 and Exercise 6.3).

(a) Show that A is integrally closed.

(b) Show that A is not a UFD.

Proof

(a) By Exercise 6.6, $\operatorname{Spec} k[w,x,y,z]/(wz-xy)$ is normal, by Corollary 6.4.4, $k[w,x,y,z]/(wz-xy)$ is integrally closed domain.

(b) In $k[w,x,y,z]/(wz-xy)$, we have $wz = xy$. We want to show that w,z,x,y are irreducible elements. Without loss of generality, it suffices to show that w is irreducible, others are similar. Suppose $w = fg$, since $\deg w = 1$, $\deg f + \deg g = 1$, hence f or g must be constant, and therefore w is irreducible.

□

Remark Exercise 6.8 and Exercise 6.9 look similar, but there is a difference. The cone over the quadric surface is normal but not factorial. On the other hand, $\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$ is factorial — all of its stalks are UFD. (Since $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain).

✍ **Exercise 6.10** Suppose A is a k-algebra, and l is a finite field extension of k. (Your proof might not use finiteness; this hypothesis is included to avoid distraction by infinite-dimensional vector spaces.) Show that if $A \otimes_k l$ is a normal integral domain, then A is a normal integral domain as well. (Although we won't need this, a version of the converse is true if l is separable extension of k.)

Proof Since l is finite field extension of k, l is finitely generated k-module. Let $b_1 = 1, b_2, \ldots, b_d$ be the generators of l. Note that A is a k-module, hence, $A \otimes_k l$ is a finitely generated A-module, its generators are $1 \otimes b_1, \ldots, 1 \otimes b_d$. Clearly, we have the following injections

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & K(A) \arrow[d, hook] \\ A \otimes_k l \arrow[r, hook] & K(A) \otimes_k l. \end{tikzcd} \tag{6.11}$$

We claim that $K(A) \otimes_k l = K(A \otimes_k l)$. We first show that $K(A) \otimes_k l$ is a field. Since l is finite filed extension of k, $K(A) \otimes_k l$ is finitely generated $K(A)$-module, also $k(A) \otimes_k l$ is a ring, $k(A) \otimes_k l$ is a finite $K(A)$-algebra. We shall to show that $K(A) \otimes_k l$ is integral domain. Let $S = k - \{0\}$, then $S^{-1}A = K(A)$, $S^{-1}k = k$, and $S^{-1}l = l$. Hence, we have k-module isomorphism

$$K(A) \otimes_k l = S^{-1}A \otimes_{S^{-1}k} S^{-1}l \cong S^{-1}(A \otimes_k l).$$

Since $A \otimes_k l$ is integral domain, $S^{-1}(A \otimes_k l)$ is integral domain, and therefore $K(A) \otimes_k l$ is integral domain. By Proposition 4.2.1, $K(A) \otimes_k l$ is a field. Since $K(A \otimes_k l)$ is the minimal field which contains $A \otimes_k l$, by diagram (6.11) we have $A \otimes_k l \hookrightarrow K(A) \otimes_k l$, hence, $K(A \otimes_k l) \subseteq K(A) \otimes_k l$. We next show that

$K(A) \otimes_k l \hookrightarrow K(A \otimes_k l)$. Define map $\iota : K(A) \otimes_k l \to K(A \otimes_k l)$ by setting

$$\frac{a}{b} \otimes c \longmapsto \frac{a \otimes c}{b \otimes 1}.$$

Clearly, ι is well-defined field homomorphism, and therefore ι is injective. Hence, $K(A) \otimes_k l \subseteq K(A \otimes_k l)$. Consequently, $K(A) \otimes_k l = K(A \otimes_k l)$.

We claim that $(A \otimes_k l) \cap K(A) = A$. By diagram (6.11), we have $A \subseteq A \otimes_k l$ and $A \subseteq K(A)$, hence, $A \subseteq (A \otimes_k l) \cap K(A)$. Conversely, let $t = \sum_{i=1}^n a_i \otimes b_i \in (A \otimes_k l) \cap K(A)$, since $t \in K(A)$, we can multiply by some $u \in A$ such that $ut = u \cdot (\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n ua_i \otimes b_i \in A$. Hence, there exists $v \otimes 1 \in A$ such that $\sum_{i=1}^n ua_i \otimes b_i = v \otimes 1$, it follows that $a_j = 0$ for all $j \geq 2$. Hence, $t = a_1 \otimes b_1 = a_1 \otimes 1 \in A$. Thus $(A \otimes_k l) \cap K(A) = A$.

Finally, we show that A is integrally closed. Let $\gamma \in K(A)$ integral over A, then exists monic polynomial $F(T) \in A[T]$ such that $F(\gamma) = 0$. By diagram (6.11), $K(A) \hookrightarrow K(A) \otimes_k l$, we may write γ as $\gamma \otimes 1$. Since $K(A) \otimes_k l = K(A \otimes_k l)$, $\gamma \otimes 1$ in $K(A \otimes_k l)$. Since $\gamma \otimes 1$ is the foot of $F(T) \in (A \otimes_k l)[T]$, note that $A \otimes_k l$ integrally closed, we have $\gamma \otimes 1 \in A \otimes_k l$. Hence, $\gamma \in (A \otimes_k l) \cap K(A) = A$, which implies that A is integrally closed. $\square$

Exercise 6.11 **(UFD-ness is not affine-local.)** Let $A = \left(\mathbb{Q}[x,y]_{x^2+y^2}\right)_0$ denote the homogeneous degree 0 part of the ring $\mathbb{Q}[x,y]_{x^2+y^2}$. In other words, it consists of quotients $\frac{f(x,y)}{(x^2+y^2)^n}$, where f has pure degree $2n$.

(i) Show that the distinguished open sets $D\left(\frac{x^2}{x^2+y^2}\right)$ and $D\left(\frac{y^2}{x^2+y^2}\right)$ cover $\operatorname{Spec} A$.

(ii) Show that $A_{\frac{x^2}{x^2+y^2}}$ and $A_{\frac{y^2}{x^2+y^2}}$ are UFDs.

(iii) Finally, show that A is not a UFD.

Proof

(i) Note that $\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 1$, then $A = (\frac{x^2}{x^2+y^2}, \frac{y^2}{x^2+y^2})$. By Proposition 4.5.2, we have $\operatorname{Spec} A = D\left(\frac{x^2}{x^2+y^2}\right) \cup D\left(\frac{y^2}{x^2+y^2}\right)$, as we desired.

(ii) Define a ring homomorphism $\varphi : A \to \mathbb{Q}[t]_{t^2+1}$ by setting

$$\frac{f(x,y)}{(x^2+y^2)^n} \longmapsto \frac{f(x,tx)}{(x^2+t^2x^2)^n} = \frac{f(1,t)}{(1+t^2)^n}.$$

It is easy to check φ is well-defined. Consider $\varphi(\frac{x^2}{x^2+y^2})$,

$$\varphi(\frac{x^2}{x^2+y^2}) = \frac{x^2}{x^2+t^2x^2} = \frac{1}{1+t^2}$$

is the unit in $\mathbb{Q}[t]_{t^2+1}$. By the universal property of localization, there exists unique ring homomorphism $\varphi^\dagger : A_{\frac{x^2}{x^2+y^2}} \to \mathbb{Q}[t]_{t^2+1}$.

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\text{localization}"'] & \mathbb{Q}[t]_{t^2+1} \arrow[dl, dashed, shift left, "\psi^\dagger"] \\
A_{\frac{x^2}{x^2+y^2}} \arrow[ur, dashed, shift left, "\varphi^\dagger"] & \mathbb{Q}[t] \arrow[l, "\psi"] \arrow[u, "\text{localization}"']
\end{tikzcd}$$

Define $\psi : \mathbb{Q}[t] \to A_{\frac{x^2}{x^2+y^2}}$ by setting

$$f(t) \longmapsto f\left(\frac{y}{x}\right) = \frac{\frac{x^{2d} f(\frac{y}{x})}{(x^2+y^2)^d}}{\frac{x^{2d}}{(x^2+y^2)^d}},$$

where $d = \deg f$. It is easy to check ψ is a well-defined ring homomorphism. Consider $\psi(t^2+1)$, we have

$$\psi(t^2+1) = \frac{y^2}{x^2} + 1 = \frac{x^2+y^2}{x^2}$$

is the unit in $A_{\frac{x^2}{x^2+y^2}}$. By the universal property of localization, there exists unique ring homomorphism $\psi^\dagger : \mathbb{Q}[t]_{t^2+1} \to A_{\frac{x^2}{x^2+y^2}}$. Hence, we have

$$A_{\frac{x^2}{x^2+y^2}} \xleftrightarrow{\sim} \mathbb{Q}[t]_{t^2+1}.$$

Similarly, we have $A_{\frac{y^2}{x^2+y^2}} \xleftrightarrow{\sim} \mathbb{Q}[t]_{t^2+1}$.

(iii) In A, note that

$$\left(\frac{xy}{x^2+y^2}\right)^2 = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{y^2}{x^2+y^2}\right),$$

A is not a UFD.

□

Number theorists may prefer the example of Exercise 6.8: $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, but it turns out that you can cover it with two affine open subsets $D(2)$ and $D(3)$, each corresponding to UFDs. (For number theorists: to show that $\mathbb{Z}[\sqrt{-5}]_2$ and $\mathbb{Z}[\sqrt{-5}]_3$ are UFDs, first show that the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}/2$ using the geometry of numbers. Then show that the ideals $(1+\sqrt{-5}, 2)$ and $(1+\sqrt{-5}, 3)$ are not principal, using the usual norm in $\mathbb{C}$.) The ring $\mathbb{Z}[\sqrt{-5}]$ is an example of a Dedekind domain.

Remark For an example of k-algebra A that is not a UFD, but becomes one after a particular field extension, see Chapter 16.

Chapter 7 Rings are to modules as schemes are to $\cdots$

When considering the notion of rings for the first time, one is quickly led to define the notion of module. For example, any ring morphism $R \to S$ expresses S as an R-module. An ideal I of a ring is also an R-module, as is the quotient R/I; and the exact sequence $0 \to I \to R \to R/I \to 0$ is a first example of the utility of the abelian category structure of $\mathbf{Mod}_R$.

With a scheme X, you may think that we already have the right analog of modules: the category of $\mathscr{O}_X$-modules $\mathbf{Mod}_{\mathscr{O}_X}$, which also forms an abelian category (§3.6.2). But the right analog for a scheme X turns out to be better-behaved subset (subcategory!) of $\mathbf{Mod}_{\mathscr{O}_X}$, called **quasi-coherent $\mathscr{O}_X$-modules**, or **quasi-coherent sheaves**.

7.1 Quasi-coherent sheaves

7.1.1 Definition of quasi-coherent sheaf

Given an A-module M, we defined an $\mathscr{O}$-module $\widetilde{M}$ on $\operatorname{Spec} A$ in §5.1.2 — the sections over $D(f)$ were M_f. These are our "local models" for quasi-coherent sheaves. In the same way that a scheme is defined by "gluing together rings", a quasi-coherent sheaf over that scheme is obtained by "gluing together modules over those rings".

Definition 7.1.1 (The category of quasi-coherent sheaves)

If X is a scheme, then an $\mathscr{O}_X$-module $\mathscr{F}$ is a ***quasi-coherent sheaf*** *if for every affine open subset $\operatorname{Spec} A \subseteq X$, $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M. We make quasi-coherent sheaves on a scheme X into category $\mathbf{QCoh}_X$ by taking the morphisms to be morphisms as $\mathscr{O}_X$-modules.*

Theorem 7.1.1

Let X be a scheme, and $\mathscr{F}$ an $\mathscr{O}_X$-module. Suppose $\mathscr{P}$ is the property of affine open subschemes $\operatorname{Spec} A \subseteq X$ that $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M (in other words, if affine open subscheme $\operatorname{Spec} A \subseteq X$ satisfies $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M, we say $\operatorname{Spec} A$ has property $\mathscr{P}$). Then $\mathscr{P}$ satisfies the two hypotheses of the Affine-local property (Definition 6.3.1). Thus to check quasi-coherence of an $\mathscr{O}$-module, it suffices to check a collection of affine open sets covering X, by Affine Communication Lemma 6.3.1.

Remark For example, $\widetilde{M}$ is a quasi-coherent sheaf on $\operatorname{Spec} A$. Proposition 5.1.4 shows that the category $\mathbf{Mod}_A$ and the category $\mathbf{QCoh}_{\operatorname{Spec} A}$ are essentially the same (we have described an equivalence $\mathbf{Mod}_A \xleftrightarrow{\sim} \mathbf{QCoh}_{\operatorname{Spec} A}$).

Proof As usual, the first hypothesis of the Affine-local property 6.3.1 is easier: if $\operatorname{Spec} A$ has property $\mathscr{P}$, then $\operatorname{Spec} A_f$ has property $\mathscr{P}$: if M is an A-module, then $\widetilde{M}|_{\operatorname{Spec} A_f} \cong \widetilde{M_f}$ as sheaves of $\mathscr{O}_{\operatorname{Spec} A_f}$-modules (both sides agree on the level of distinguished open sets and their restriction maps), i.e., $\mathscr{F}|_{\operatorname{Spec} A_f} \cong \widetilde{M_f}$, hence, $\operatorname{Spec} A_f$ has property $\mathscr{P}$.

We next show the second hypothesis of the Affine-local property 6.3.1. We are given:

- an $\mathscr{O}_{\operatorname{Spec} A}$-module $\mathscr{F}$,
- elements $f_1, \ldots, f_n$ generating A,
- and isomorphisms $\varphi_i : \mathscr{F}|_{D(f_i)} \xrightarrow{\sim} \widetilde{M_i}$.

The isomorphisms

$$\begin{tikzcd}[column sep=small] & \mathscr{F}|_{D(f_i)\cap D(f_j)} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \widetilde{M_i}|_{D(f_i)\cap D(f_j)} & & \widetilde{M_j}|_{D(f_i)\cap D(f_j)} \end{tikzcd}$$

yield (by taking sections over $D(f_i) \cap D(f_j)$) isomorphisms $\varphi_{ij} : (M_i)_{f_j} \xrightarrow{\sim} (M_j)_{f_i}$ of $A_{f_i f_j}$-modules, satisfying the cocycle condition (Theorem 3.5.3). As suggestive notation, let $M_{ij} := (M_i)_{f_j}$ (identified with $(M_j)_{f_i}$ via φ_{ij}).

We seek an A-module M with an isomorphism $\widetilde{M} \xleftrightarrow{\sim} \mathscr{F}$, and (5.5) suggests that we should define M by

$$0 \longrightarrow M \longrightarrow M_1 \times \cdots \times M_2 \xrightarrow{\gamma} M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}. \tag{7.1}$$

(The discussion at (5.5) suggests how the map γ should be define, with appropriate signs.) Translation: $M = \operatorname{Ker}(\gamma)$. If we can describe an isomorphism $\widetilde{M}|_{D(f_i)} \xrightarrow{\sim} \widetilde{M_i}$ on each $D(f_i)$ (or equivalently, an isomorphism $M_{f_i} \xrightarrow{\sim} M_i$ of A_{f_i}-modules), such that triangle

$$\begin{tikzcd}[column sep=small] & \widetilde{M}|_{D(f_i)\cap D(f_j)} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \widetilde{M_i}|_{D(f_i)\cap D(f_j)} \arrow[rr, "\widetilde{\varphi_{ij}}", "\sim"'] & & \widetilde{M_j}|_{D(f_i)\cap D(f_j)} \end{tikzcd}$$

of sheaves on $\operatorname{Spec} A_{f_i f_j}$ commutes (for all i, j), then Theorem 3.5.3 (on gluing sheaves) gives our desired isomorphism $\widetilde{M} \xleftrightarrow{\sim} \mathscr{F}$.

By Proposition 5.1.4, we need only describe an isomorphism $M_{f_i} \xrightarrow{\sim} M_i$ such that the triangles

$$\begin{tikzcd}[column sep=small] & M_{f_i f_j} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ (M_i)_{f_j} \arrow[rr, "\varphi_{ij}"] & & (M_j)_{f_i} \end{tikzcd} \tag{7.2}$$

commute.

For notational convenience, we assume $i = 1$. Because localization is exact, form (7.1) we have an exact sequence.

$$0 \longrightarrow M_{f_1} \longrightarrow (M_1)_{f_1} \times \cdots \times (M_n)_{f_1} \xrightarrow{\gamma_{f_1}} (M_{12})_{f_1} \times (M_{13})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

We will show that

$$0 \longrightarrow M_1 \xrightarrow{\beta} (M_1)_{f_1} \times \cdots \times (M_n)_{f_1} \xrightarrow{\gamma_{f_1}} (M_{12})_{f_1} \times (M_{13})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1} \tag{7.3}$$

is an exact sequence, where β defined by $m \mapsto (m/1, 0, \ldots, 0)$. If sequence 7.3 is exact, we have

$$M_{f_1} \cong \operatorname{Im}(M_{f_i} \to (M_1)_{f_1} \times \cdots \times (M_n)_{f_1}) = \operatorname{Ker} \gamma_{f_1} = \operatorname{Im} \beta \cong M_1,$$

as we desired. We rewrite (7.3) as

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & (M_1)_{f_1} \times \cdots \times (M_n)_{f_1} \arrow[d, "\alpha"] \\ & & (M_1)_{f_1 f_2} \times \cdots \times (M_1)_{f_1 f_n} \times (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}. \end{tikzcd} \tag{7.4}$$

But this is precisely the exact sequence (5.5), except the ring A is replaced by A_{f_1}, and the module M is replaced by M_1. Hence, we have

$$M_{f_i} \xrightarrow{\sim} M_i.$$

Note that $(M_{f_i})_{f_j} \cong (M_{f_j})_{f_i}$, triangle (7.2) commutes. $\square$

Remark The proof of Theorem 7.1.1, phrased slightly more carefully, shows that quasi-coherent sheaves satisfy faithfully flat descent.

7.1.2 Examples of quasi-coherent sheaves

We give some examples to show that not every $\mathscr{O}_X$-module is a quasi-coherent sheaf.

Example 7.1 Suppose $X = \operatorname{Spec} k[t]$. Let $\mathscr{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$ and the usual $k[t]$-module structure. Then $\mathscr{F}$ is an $\mathscr{O}_X$-module that is not a quasi-coherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ is not generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in silly circumstances, this sheaf won't be quasi-coherent.)

Proof Clearly, the skyscraper sheaf is an $\mathscr{O}_X$-module. Consider $(\mathscr{F}|_{D(t+1)})_{[\mathfrak{p}]}$, note that

$$(\mathscr{F}|_{D(t+1)})_{[\mathfrak{p}]} = \begin{cases} k(t) & \text{if } [\mathfrak{p}] = [(t)] \\ 0 & \text{if } [\mathfrak{p}] \neq [(t)] \end{cases}$$

but $\widetilde{k(t)}_{[\mathfrak{p}]} = k(t)_{\mathfrak{p}}$ where $k(t)$ as $k[t]_{t+1}$-module, hence, $(\mathscr{F}|_{D(t+1)})_{[\mathfrak{p}]}$ not isomorphic to $\widetilde{k(t)}_{[\mathfrak{p}]}$, and therefore $\mathscr{F}|_{D(t+1)}$ not isomorphic to $\widetilde{k(t)}$. Hence, $\mathscr{F}$ is not a quasi-coherent sheaf. $\square$

Example 7.2 Suppose $X = \operatorname{Spec} k[t]$. Let $\mathscr{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$ and the usual $k[t]$-module structure. Then $\mathscr{F}$ is a quasi-coherent sheaf

Proof Clearly, $\mathscr{F}$ is an $\mathscr{O}_X$-module. We claim that $\mathscr{F} \cong \widetilde{k(t)}$, by Proposition 3.4.4, it suffices to check $\mathscr{F}_{[\mathfrak{p}]} \cong \widetilde{k(t)}_{[\mathfrak{p}]}$. Let U be any open affine neighborhood of $[\mathfrak{p}]$, by Proposition 4.6.10, $[(0)] \in U$, $\mathscr{F}(U) = k(t)$, and therefore $\mathscr{F}_{[\mathfrak{p}]} = k(t)$. On the other hand, $\widetilde{k(t)} = k(t)_{\mathfrak{p}} \cong k(t)$, since $k(t)$ is a field. Hence, we have $\mathscr{F}_{[\mathfrak{p}]} \cong \widetilde{k(t)}_{[\mathfrak{p}]}$, as we desired. It follows that $\mathscr{F}$ is a quasi-coherent sheaf. $\square$

Remark Example 7.2 will apply more generally, for example when X is an integral scheme with generic point η, and $\mathscr{F}$ is the skyscraper sheaf $i_{\eta,*}K(X)$.

7.1.3 Torsion-free sheaves (a stalk-local condition) and torsion sheaves

Definition 7.1.2 (Torsion-free)

*An A-module M is said to be **torsion-free** if $am = 0$ implies that either a is a zero-divisor in A or $m = 0$.*

Remark In the case where A is an integral domain, which is basically the only context in which we will use this concept, the definition of torsion-freeness can be restated as $am = 0$ only if $a = 0$ or $m = 0$.

Definition 7.1.3 (Torsion submodule)

*Let A be an integral domain, M is an A-module. The **torsion submodule** of M, denoted M_{tors}, consists of those elements of M annihilated by some nonzero element of A, i.e.,*

$$M_{\text{tors}} = \{m \in A : am = 0, \text{ for some nonzero } a \in A\}.$$

(If A is not an integral domain, this construction needn't yield an A-module.)

Remark Clearly M is torsion-free if and only if $M_{\text{tors}} = 0$.

Definition 7.1.4 (Torsion module)

We say a module M over an integral domain A is ***torsion*** *if $M = M_{\text{tors}}$, this is equivalent to $M \otimes_A K(A) = 0$.*

Definition 7.1.5 (Torsion-free scheme)

If X is a scheme, then an $\mathscr{O}_X$-module $\mathscr{F}$ is said to be ***torsion-free*** *if $\mathscr{F}_p$ is a torsion-free $\mathscr{O}_{X,p}$-module for all p.*

Motivated by the definition of M_{tors} above, we give the definition of **torsion quasi-coherent sheaf on a reduced scheme**:

Definition 7.1.6 (Torsion quasi-coherent sheaves on a reduced schemes)

We say that a quasi-coherent sheaf on a reduced scheme is ***torsion*** *if its stalk at the generic point of every irreducible component is* 0.

We will mainly use this for coherent sheaves on regular curves, where this notion is very simple indeed, but in the literature it comes up in more general situation.

7.2 Characterizing quasi-coherence using the distinguished affine base

Because quasi-coherent sheaves are locally of a very special form, in order to “know” a quasi-coherent sheaf, we need only know what the sections are over every affine open set, and how to restrict sections from an affine open set U to a distinguished affine open subset of U. We make this precise by defining what we will call the **distinguished affine base** of the Zariski topology — not a base in the usual sense. This is a refinement of the notion of a **sheaf on a base**. The point of this discussion is to give a useful characterization of quasi-coherence, but you may wish to just jump to §7.2.2.

7.2.1 Distinguished affine base

The open sets of the distinguished affine base are the affine open subsets of X. We have already observed that this forms a base. But forget that fact. We like distinguished open sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, and we don't really understand open embeddings of one random affine open subset in another. So we just remember the “nice” inclusions.

Definition 7.2.1 (Distinguished affine base)

The ***distinguished affine base*** *of a scheme X is the data of the affine open sets and the distinguished inclusion.*

In other words, we remember only some of the open sets (the affine open sets), and only some of the morphisms between them (the distinguished morphisms). If we think of a topology as a topology as a category (the category of open sets), we have described a subcategory.

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathscr{F}$ on X, we get a sheaf on the distinguished affine base. We will show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up, we can recover stalks as follows. (We will be implicitly using only the following fact. We have a collection of open subsets, and some inclusions among these subsets, such that if we have any $p \in U, V$ where U and V are in collection of open sets, there is some W containing p, and contained in U and V such that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key Proposition 6.3.1 that given any two affine open sets $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine open sets that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. In fancy language: the category of affine open sets, and distinguished inclusion, forms a filtered set 2.3.4.)

The stalk $\mathscr{F}_p$ is the colimit $\varinjlim(f \in \mathscr{F}(U))$ where the colimit is over all open sets contained in X. We compare this to $\varinjlim(f \in \mathscr{F}(U))$ where the colimit is over all affine open sets, and all distinguished inclusions. It is easy to check that the elements of one correspond to elements of the other (check by definition of germs).

Lemma 7.2.1

Let $\mathscr{F}$ be a sheaf on a scheme X, it gives a sheaf $\mathscr{F}^b$ on the distinguished base of X. Let $p \in X$ be a point, then there is a one-to-one correspondence between the elements of stalk $\mathscr{F}_p$ and the elements of stalk $\mathscr{F}^b_p$.

Proof Let $(f, U) \in \mathscr{F}_p$, where $U \ni p$ is an open subset of X. Then there exists an affine open subsets $\operatorname{Spec} A \subseteq U$ where $\operatorname{Spec} A \ni p$ such that $(f, U) \sim (f|_{\operatorname{Spec} A}, \operatorname{Spec} A)$, note that $(f|_{\operatorname{Spec} A}, \operatorname{Spec} A) \in \mathscr{F}^b_p$, define a map from $\mathscr{F}_p$ to $\mathscr{F}^b_p$ by setting $(f, U) \mapsto (f|_{\operatorname{Spec} A}, \operatorname{Spec} A)$ where $\operatorname{Spec} A \ni p$ contained in U. Conversely, let $(f|_{\operatorname{Spec} A}, \operatorname{Spec} A) \in \mathscr{F}^b_p$, clearly, $(f|_{\operatorname{Spec} A}, \operatorname{Spec} A) \in \mathscr{F}_p$. Hence, we get a one-to-one correspondence between the elements of stalk $\mathscr{F}_p$ and the elements of stalk $\mathscr{F}^b_p$. □

Proposition 7.2.1

A section of a sheaf on the distinguished affine base is determined by the section's germs.

Proof By Proposition 3.4.1 and Lemma 7.2.1, the natural map

$$\mathscr{F}(\operatorname{Spec} A) = \mathscr{F}^b(\operatorname{Spec} A) \longrightarrow \prod_{p \in \operatorname{Spec} A} \mathscr{F}_p = \prod_{p \in \operatorname{Spec} A} \mathscr{F}^b_p$$

is injective, then we done. □

Theorem 7.2.1

(a) A sheaf on the distinguished affine base $\mathscr{F}^b$ determines a unique (up to unique isomorphism) sheaf $\mathscr{F}$ which when restricted to the affine base is $\mathscr{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)

(b) A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.

(c) An $\mathscr{O}_X$-module "on the distinguished affine base" yields an $\mathscr{O}_X$-module.

Remark

(1) This proof is identical to our argument of §3.5 showing that sheaves are (essentially) the same as sheaves on a base, using the "sheaf of compatible germs" construction. The main reason for repeating it is to let you see that all that is needed is for the open sets to form a filtered set (or in the current case, that the category of open sets and distinguished inclusions is filtered).

(2) (a) and (b) are describing an equivalence of categories between sheaves on the Zariski topology of X and sheaves on the distinguished affine base of X.

Proof

(a) Suppose $\mathscr{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.
For any open set U of X, define the sheaf of compatible germs

$$\mathscr{F}(U) := \{(f_p \in \mathscr{F}^b_p)_{p \in U} : \text{ for all } p \in U, \text{ there exists } U_p \text{ with } p \in U_p \subseteq U, \\ s \in \mathscr{F}^b(U_p) \text{ such that } s_q = f_q \text{ for all } q \in U_p\},$$

where each U_p is in our base, and s_q means "the germ of s at q".
This really is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.
We next claim that if U is in our base, that $\mathscr{F}(U) = \mathscr{F}^b(U)$. We clearly have a map $\mathscr{F}^b(U) \to \mathscr{F}(U)$. This is an isomorphism on stalks, and hence an isomorphism by Proposition 3.4.4.

(b) Theorem 3.5.2.

(c) Remark of Theorem 3.5.2.

□

7.2.2 A characterization of quasi-coherent sheaves in terms of distinguished inclusions

We use this perspective to give a useful characterization of quasi-coherent sheaves among $\mathscr{O}_X$-modules. Suppose $\mathscr{F}$ is an $\mathscr{O}_X$-module, and $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \subseteq X$ is a distinguished open subscheme of an affine open subscheme of X. Let $\mathrm{res} : \Gamma(\operatorname{Spec} A, \mathscr{F}) \to \Gamma(\operatorname{Spec} A_f, \mathscr{F})$ be the restriction map. The source of res is an A-module, and the target is an A_f-module, so by the universal property of localization, res naturally factors as:

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[rr, "\mathrm{res}"] \arrow[dr, "\otimes_A A_f"'] & & \Gamma(\operatorname{Spec} A_f, \mathscr{F}) \\ & \Gamma(\operatorname{Spec} A, \mathscr{F})_f . \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Theorem 7.2.2

An $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for each such distinguished $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$,

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[rr, "\mathrm{res}"] \arrow[dr, "\otimes_A A_f"'] & & \Gamma(\operatorname{Spec} A_f, \mathscr{F}) \\ & \Gamma(\operatorname{Spec} A, \mathscr{F})_f , \arrow[ur, "\alpha"'] & \end{tikzcd}$$

α is an isomorphism.

Proof If $\mathscr{F}$ is quasi-coherent, then $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M. Hence

$$\Gamma(\operatorname{Spec} A_f, \mathscr{F}) = \mathscr{F}|_{\operatorname{Spec} A}(\operatorname{Spec} A_f) = \widetilde{M}(\operatorname{Spec} A_f) = M_f = \Gamma(\operatorname{Spec} A, \mathscr{F})_f,$$

as we desired.

Conversely, if for each such distinguished $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, there is isomorphism $\Gamma(\operatorname{Spec} A, \mathscr{F})_f \cong \Gamma(\operatorname{Spec} A_f, \mathscr{F})$, we want to show that $\mathscr{F}$ is quasi-coherent. It suffices to check that for any affine open subset $\operatorname{Spec} A \subseteq X$ we have $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$. Let $M = \Gamma(\operatorname{Spec} A, \mathscr{F})$, we claim that $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$. Since $\Gamma(\operatorname{Spec} A, \mathscr{F})_f \cong \Gamma(\operatorname{Spec} A_f, \mathscr{F})$, we have $\mathscr{F}|_{\operatorname{Spec} A_f} \cong \widetilde{M_f}$ for all $f \in A$. By Theorem 7.1.1, $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$, it follows that $\mathscr{F}$ is a quasi-coherent sheaf. □

Thus a quasi-coherent sheaf is (equivalent to) the data of one module for each affine open subset (a module over the corresponding ring), such that the module over a distinguished open set $\operatorname{Spec} A_f$ is given by localizing the module over $\operatorname{Spec} A$.

Example: the quasi-coherent sheaf of nilpotents

Proposition 7.2.2

If A is a ring, and $f \in A$, then $\mathfrak{N}(A_f) \cong \mathfrak{N}(A)_f$.

Proof In fact,

$$\mathfrak{N}(A_f) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A_f} \mathfrak{p} \cong \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ f \notin \mathfrak{p}}} \mathfrak{p} \cong \left(\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \right)_f = \mathfrak{N}(A)_f,$$

i.e., $\mathfrak{N}(A_f) \cong \mathfrak{N}(A)_f$. $\square$

Definition 7.2.2 (The quasi-coherent sheaf of nilpotents)

$\mathfrak{N}(A)$ is an A-module. Define

$$\widetilde{\mathfrak{N}(A)}(D(f)) = \mathfrak{N}(A)_f,$$

the restriction maps $\operatorname{res}_{D(f),D(fg)}$ defined by $\mathfrak{N}(A)_f \hookrightarrow \mathfrak{N}(A)_{fg}$. This defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec} A$. Also, $\widetilde{\mathfrak{N}(A)}$ is an $\mathscr{O}_{\operatorname{Spec} A}$-module.

*Let X be a scheme, then an $\mathscr{O}_X$-module $\mathfrak{N}_X$ is the **quasi-coherent sheaf of nilpotents** if for every affine open subset $\operatorname{Spec} A \subseteq X$, $\mathfrak{N}_X|_{\operatorname{Spec} A} \cong \widetilde{\mathfrak{N}(A)}$.*

Remark This is an example of an ideal sheaf of $\mathscr{O}_X$.

Proposition 7.2.3

The quasi-coherent sheaf of nilpotents is indeed a quasi-coherent sheaf.

Proof Let $\operatorname{Spec} A \subseteq X$ be any affine open subscheme of X. Note that

$$\Gamma(\operatorname{Spec} A, \mathfrak{N}_X) = \Gamma(\operatorname{Spec} A, \mathfrak{N}_X|_{\operatorname{Spec} A}) = \widetilde{\mathfrak{N}(A)}(\operatorname{Spec} A) = \mathfrak{N}(A),$$

by Proposition 7.2.2, for $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ we have $\mathfrak{N}(A_f) \cong \mathfrak{N}(A)_f$. Note that $\Gamma(\operatorname{Spec} A_f, \mathfrak{N}_X) = \mathfrak{N}(A_f)$, hence,

$$\Gamma(\operatorname{Spec} A_f, \mathfrak{N}_X) \cong \Gamma(\operatorname{Spec} A, \mathfrak{N}_X)_f.$$

By Theorem 7.2.2, $\mathscr{O}_X$-module $\mathfrak{N}_X$ is quasi-coherent. $\square$

Tensor product of quasi-coherent sheaves

Tensor product in the category of quasi-coherent sheaves on X can also be cleanly described in terms of affine open sets.

Proposition 7.2.4

If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent sheaves on a scheme X, then $\mathscr{F} \otimes \mathscr{G}$ is a quasi-coherent sheaf described by the following information: If $\operatorname{Spec} A \subseteq X$ is an affine open subset, and $\Gamma(\operatorname{Spec} A, \mathscr{F}) = M$ and $\Gamma(\operatorname{Spec} A, \mathscr{G}) = N$, then $\Gamma(\operatorname{Spec} A, \mathscr{F} \otimes \mathscr{G}) = M \otimes_A N$, and the restriction map $\Gamma(\operatorname{Spec} A, \mathscr{F} \otimes \mathscr{G}) \to \Gamma(\operatorname{Spec} A_f, \mathscr{F} \otimes \mathscr{G})$ is precisely the localization map $M \otimes_A N \to (M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$.

Proof Let $\operatorname{Spec} A \subseteq X$ be any affine open subsets of X, for all $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, note that

$$\begin{aligned}\Gamma(\operatorname{Spec} A_f, \mathscr{F} \otimes_{\mathrm{pre}, \mathscr{O}_X} \mathscr{G}) &= \mathscr{F}(\operatorname{Spec} A_f) \otimes_{A_f} \mathscr{G}(\operatorname{Spec} A_f) \\ &\cong M_f \otimes_{A_f} N_f \\ &\cong (M \otimes N)_f \\ &= \widetilde{M \otimes_A N}(\operatorname{Spec} A_f),\end{aligned}$$

since $\widetilde{M \otimes_A N}$ is a sheaf, $\mathscr{F} \otimes_{\mathrm{pre}, \mathscr{O}_X} \mathscr{G}$ is already a sheaf on the distinguished affine base. Hence, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = (\mathscr{F} \otimes_{\mathrm{pre}, \mathscr{O}_X} \mathscr{G})^{\mathrm{sh}} = \mathscr{F} \otimes_{\mathrm{pre}, \mathscr{O}_X} \mathscr{G}$ is a sheaf on the distinguished affine base. By Theorem 7.2.1, it determines unique sheaf on X. Also, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is clearly an $\mathscr{O}_X$-module, since $(\mathscr{F} \otimes \mathscr{G})|_{\operatorname{Spec} A} \cong \widetilde{M \otimes_A N}$, $\mathscr{F} \otimes \mathscr{G}$ is quasi-coherent. □

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of. This is a feature we will use often: *constructions involving quasi-coherent sheaves that involves sheafification for general sheaves don't require sheafification when consider on the distinguished affine base*. Along with the fact that injectivity, surjectivity, kernels, tensor products and so on may be computed on affine opens, this is the reason that it is particularly convenient to think about quasi-coherent sheaves in terms of affine open sets. (There is a slight caveat in the case of $\mathscr{H}om$.)

Remark **Elegant side remark, but perhaps too fancy for now.** In Proposition 7.2.4, the tensor product is in the category of quasi-coherent sheaves. But in fact this is even the tensor product in the category of $\mathscr{O}_X$-modules. To show that "$\mathscr{F} \otimes_{\mathbf{QCoh}_X} \mathscr{G}$ satisfies the universal property of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$", first show the following. Let $\mathscr{M}$ be a quasi-coherent sheaf on X, and $\mathscr{N}$ any $\mathscr{O}_X$-module. Let $\operatorname{Spec} A \subseteq X$ be an affine open subset, and $M = \mathscr{M}(\operatorname{Spec} A)$. Then the "global sections" map

$$\operatorname{Hom}_{\operatorname{Spec} A}(\mathscr{M}|_{\operatorname{Spec} A}, \mathscr{N}|_{\operatorname{Spec} A}) \longrightarrow \operatorname{Hom}_A(M, \mathscr{N}(\operatorname{Spec} A))$$

is an isomorphism. Our slogan: on an affine scheme, maps from a quasi-coherent sheaf to any $\mathscr{O}$-module are determined by global sections. You will hear an echo of this slogan in the parenthetical comment in Chapter 8.

Definition 7.2.3

Given a section s of $\mathscr{F}$ and a section t of $\mathscr{G}$, we have a section $s \otimes t$ of $\mathscr{F} \otimes \mathscr{G}$.

7.2.3 X_f and the Qcqs lemma

The next result will be useful in the future in showing certain constructions yield quasi-coherent sheaves.

Definition 7.2.4 (X_f)

If X is a scheme, and $f \in \Gamma(X, \mathscr{O}_X)$ is a function, let $X_f \subseteq X$ be the subset on which f doesn't vanish, i.e.,

$$X_f = \{p \in X : f(p) \neq 0\} = \{p \in X : f_p \notin \mathfrak{m}_p \subseteq \mathscr{O}_{X,p}\},$$

where f_p is the image of f in residue field $\kappa(p) = \mathscr{O}_{X,p}/\mathfrak{m}_p$.

Theorem 7.2.3 (The Qcqs Lemma)

Suppose X is a quasi-compact and quasi-separated scheme, $\mathscr{F}$ is a quasi-coherent sheaf on X, and $f \in \Gamma(X, \mathscr{O}_X)$ is a function on X. Then the restriction map

$$\operatorname{res}_{X_f \subseteq X} : \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X_f, \mathscr{F})$$

is precisely localization: there is an isomorphism $\Gamma(X,\mathscr{F})_f \xrightarrow{\sim} \Gamma(X_f,\mathscr{F})$ *making the following diagram commute.*

$$\begin{tikzcd} \Gamma(X,\mathscr{F}) \arrow[rr, "\mathrm{res}_{X_f\subseteq X}"] \arrow[dr, "\otimes_{\Gamma(X,\mathscr{O}_X)}(\Gamma(X,\mathscr{O}_X)_f)"'] & & \Gamma(X_f,\mathscr{F}) \\ & \Gamma(X,\mathscr{F})_f \arrow[ur, "\sim"'] & \end{tikzcd}$$

Remark We will use Five Lemma to prove the Qcqs Lemma:

Lemma 7.2.2 (Five Lemma)

Let $\mathscr{A}$ *be an abelian category. Consider a commutative diagram in* $\mathscr{A}$ *of the form*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

where the top and bottom rows are exact sequences.

(1) If f_2 *and* f_4 *are epimorphisms and* f_5 *is monomorphism, then* f_3 *is epimorphism.*

(2) If f_2 *and* f_4 *are monomorphisms and* f_1 *is epimorphism, then* f_3 *is monomorphism.*

(3) If f_2 *and* f_4 *are isomorphisms,* f_1 *is epimorphism, and* f_5 *is monomorphism, then* f_3 *is isomorphism.*

Proof Since X is qcqs, by the definition of sheaf, we have exact sequence,

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \bigoplus_{i=1}^{n}\Gamma(U_i,\mathscr{F}) \longrightarrow \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\Gamma(U_i\cap U_j,\mathscr{F}), \tag{7.5}$$

where $X=\bigcup_{i=1}^{n}U_i$. $U_i\cap U_j$ is quasi-compact, since X is quasi-separated, by Proposition 6.3.1, we may assume that $U_i\cap U_j=\bigcup_{k_{ij}=1}^{m_{ij}}U_{ijk_{ij}}$, where $U_{ijk_{ij}}$ is simultaneously distinguished open subscheme of U_i and U_j. By the definition of sheaf again, we have exact sequence.

$$0 \longrightarrow \Gamma(U_i\cap U_j,\mathscr{F}) \longrightarrow \bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma(U_{ijk_{ij}},\mathscr{F}) \tag{7.6}$$

Hence, we get a sequence from (7.5) and (7.6):

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \xrightarrow{d^0} \bigoplus_{i=1}^{n}\Gamma(U_i,\mathscr{F}) \xrightarrow{d^1} \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\Gamma(U_i\cap U_j,\mathscr{F}) \xrightarrow{\delta} \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma(U_{ijk_{ij}},\mathscr{F}). \tag{7.7}$$

By exact sequence (7.6), δ is an injection, hence, $\operatorname{Ker}\delta\circ d^1=\operatorname{Ker}d^1$, and therefore sequence (7.7) is exact.

Since localization is exact (Exercise 2.37), apply the exact functor $\otimes_{\Gamma(X,\mathscr{O}_X)}\Gamma(X,\mathscr{O}_X)_f$ to exact sequence (7.7), note that tensor product commutes with direct product (Proposition 2.2.7), we have a new exact sequence.

$$0 \longrightarrow \Gamma(X,\mathscr{F})_f \longrightarrow \bigoplus_{i=1}^{n}\Gamma(U_i,\mathscr{F})_f \longrightarrow \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma(U_{ijk_{ij}},\mathscr{F})_f. \tag{7.8}$$

Say $U_i=\operatorname{Spec}A_i$ and $U_{ijk_{ij}}=\operatorname{Spec}(A_i)_{g_{k_{ij}}}$, since $\mathscr{F}$ is quasi-coherent, by Theorem 7.2.2, we have

$$\Gamma(\operatorname{Spec}A_i,\mathscr{F})_f\cong\Gamma(\operatorname{Spec}(A_i)_f,\mathscr{F}),\quad \text{and } \Gamma(\operatorname{Spec}(A_i)_{g_{k_{ij}}},\mathscr{F})_f\cong\Gamma(\operatorname{Spec}(((A_i)_{g_{k_{ij}}})_f,\mathscr{F}). \tag{7.9}$$

Denote $(U_i)_f := \mathrm{Spec}(A_i)_f$ and $(U_{ijk_{ij}})_f := \mathrm{Spec}((A_i)_{g_{k_{ij}}})_f = \mathrm{Spec}(A_i)_{fg_{k_{ij}}}$.

On the other hand, note that

$$X_f = \bigcup_{i=1}^{n}(X_f \cap U_i) = \bigcup_{i=1}^{n}\mathrm{Spec}(A_i)_f = \bigcup_{i=1}^{n}(U_i)_f,$$

also note that

$$(X_f \cap U_i) \cap (X_f \cap U_j) = \bigcup_{k_{ij}=1}^{m_{ij}} (X_f \cap U_{ijk_{ij}}) = \bigcup_{k_{ij}=1}^{m_{ij}} \mathrm{Spec}(A_i)_{fg_{k_{ij}}} = \bigcup_{k_{ij}=1} (U_{ijk_{ij}})_f,$$

by the definition of sheaf, we have exact sequence

$$0 \longrightarrow \Gamma(X_f, \mathscr{F}) \longrightarrow \bigoplus_{i=1}^{n}\Gamma((U_i)_f, \mathscr{F}) \longrightarrow \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma((U_{ijk_{ij}})_f, \mathscr{F}). \tag{7.10}$$

By (7.9) we have

$$\bigoplus_{i=1}^{n}\Gamma((U_i)_f, \mathscr{F}) \cong \bigoplus_{i=1}^{n}\Gamma(U_i, \mathscr{F})_f, \quad \text{and} \quad \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma((U_{ijk_{ij}})_f, \mathscr{F}) \cong \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma(U_{ijk_{ij}}, \mathscr{F})_f, \tag{7.11}$$

hence, we get a diagram from (7.8), (7.10) and (7.11),

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathscr{F})_f \arrow[r] \arrow[d, dashed, "\sim"'] & \bigoplus_{i=1}^{n}\Gamma(U_i, \mathscr{F})_f \arrow[r] \arrow[d, "\sim"] & \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma(U_{ijk_{ij}}, \mathscr{F})_f \arrow[d, "\sim"] \\
0 \arrow[r] & \Gamma(X_f, \mathscr{F}) \arrow[r] & \bigoplus_{i=1}^{n}\Gamma((U_i)_f, \mathscr{F}) \arrow[r] & \bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\bigoplus_{k_{ij}=1}^{m_{ij}}\Gamma((U_{ijk_{ij}})_f, \mathscr{F}),
\end{tikzcd}$$

by Five Lemma 7.2.2, there is an isomorphism

$$\Gamma(X, \mathscr{F})_f \xrightarrow{\sim} \Gamma(X_f, \mathscr{F}).$$

By the universal property of localization, the following diagram commutes.

$$\begin{tikzcd}
\Gamma(X, \mathscr{F}) \arrow[rr, "\mathrm{res}_{X_f \subseteq X}"] \arrow[dr, "\otimes_{\Gamma(X,\mathscr{O}_X)}(\Gamma(X,\mathscr{O}_X)_f)"'] & & \Gamma(X_f, \mathscr{F}) \\
& \Gamma(X, \mathscr{F})_f \arrow[ur, "\sim"'] &
\end{tikzcd}$$

□

Example 7.3 **The Qcqs Lemma 7.2.3 is false without the quasi-compactness hypothesis.** Let $X = \coprod \mathbb{A}_k^1 = \coprod k[t]$, and let $f = (t, t, \dots) \in X$. In fact, $\Gamma(X_f, \mathscr{O}_X) = \prod k[t]_t$ and $\Gamma(X, \mathscr{O}_X)_f = (\prod k[t])_f$, note that $(\prod k[t])_f \subsetneq \prod k[t]_t$, hence the Qcqs Lemma 7.2.3 is false.

7.2.4 ⋆⋆ Grothendieck topologies

The distinguished affine base isn't a topology in the usual sense — the union of two affine sets isn't necessarily affine, for example. It is however a first new example of a generalization of a topology — the notion of a **site** or a **Grothendieck topology**. We give the definition to satisfy the curious, but we certainly won't use this notion. The idea is that we should abstract away only those notions we need to define sheaves. We need the notion of open set, but it turns out that we won't even need an underlying set, i.e., we won't even need the notion of points! Let's think through how little we need. For our discussion of sheaves to work, we needed to know what the open sets were, and what the (allowed) inclusions were, and these should "behave well", and in particular the data of the open sets and inclusions should form a category. (For example, the composition of an allowed inclusion with another allowed inclusion should be an allowed inclusion — in the distinguished affine base, a distinguished open set of a distinguished open set is a distinguished open set.) So we just require the data of this category. At this point, we can already define presheaf (as just a contravariant functor from this category of "open sets"). We saw this idea earlier in Proposition 3.2.1.

Definition 7.2.5 (Presheaf)

Let $\mathscr{C}$ be a category. A presheaf of sets on $\mathscr{C}$ is a contra-variant functor from $\mathscr{C}$ to **Sets**. *Morphisms of presheaves are transformations of functors. The category of presheaves of sets is denoted* $\mathrm{PSh}(\mathscr{C})$.

In order to extend this definition to that of a sheaf, we need to know more information. We want two open subsets of an open set to intersect in an open set, so *we want the category to be closed under fibered products (Exercise 2.11)*. For the identity and gluability axioms, we need to know *when some open sets cover another*, so we also remember this as part of the data of a Grothendieck topology. The data of the coverings satisfy some obvious properties. Every open set covers itself (i.e., *the identity map in the category of open sets is a covering*). Covering pull back: *if we have a map $Y \to X$, then any cover of X pulls back to a cover of Y*. Finally, *a cover of a cover should be a cover*. Such data (satisfying these axioms) is called a **Grothendieck topology** or a **site**. There are useful variants of this definition in the literature:

Note *Let $\mathscr{C}$ be a category. A family of morphisms with fixed target in $\mathscr{C}$ is given by an object $U \in \mathrm{obj}(\mathscr{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathscr{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.*

Remark It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

Definition 7.2.6 (Site (or Grothendieck topology))

A ***site*** *is given by a category $\mathscr{C}$ and a set $\mathrm{Cov}(\mathscr{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called* ***coverings of*** $\mathscr{C}$*, satisfying the following axioms:*

(1) *If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathscr{C})$.*
(2) *If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathscr{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathscr{C})$.*
(3) *If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ and $V \to U$ is a morphism of $\mathscr{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$.*

We can define the notion of a sheaf on a Grothendieck topology in the usual way, with no change:

Definition 7.2.7 (Sheaf on site)

Let $\mathscr{C}$ be a site, and let $\mathscr{F}$ be a presheaf of sets on $\mathscr{C}$. We say $\mathscr{F}$ is a sheaf if for every coving $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} \mathscr{F}(U_{i_0} \times_U U_{i_1}) \tag{7.12}$$

represents the first arrow as the equalizer (Definition 3.2.6) of pr_0^* and pr_1^*.

A **topos** is a scary name for a category of sheaves of sets on a Grothendieck topology.

Definition 7.2.8 (Topos, Topoi)

*A **topos** is the category $\mathbf{Sh}(\mathscr{C})$ of sheaves on a site $\mathscr{C}$.*

(1) *Let $\mathscr{C}$, $\mathscr{D}$ be sites. A **morphism of topoi** f from $\mathbf{Sh}(\mathscr{D})$ to $\mathbf{Sh}(\mathscr{C})$ is given by a pair of functors $f_* : \mathbf{Sh}(\mathscr{D}) \to \mathbf{Sh}(\mathscr{C})$ and $f^{-1} : \mathbf{Sh}(\mathscr{C}) \to \mathbf{Sh}(\mathscr{D})$ such that*

(a) *we have*

$$\mathrm{Mor}_{\mathbf{Sh}(\mathscr{D})}(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Mor}_{\mathbf{Sh}(\mathscr{C})}(\mathscr{G}, f_*\mathscr{F})$$

bifunctorially, and

(b) *the functor f^{-1} commutes with finite limits, i.e., is left exact.*

(2) *Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be sites. Given morphisms of topoi $f : \mathbf{Sh}(\mathscr{D}) \to \mathbf{Sh}(\mathscr{C})$ and $g : \mathbf{Sh}(\mathscr{E}) \to \mathbf{Sh}(\mathscr{D})$ the composition $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

Grothendieck topologies are used in a wide variety of contexts in and near algebraic geometry. Étale cohomology (using the étale topology), a generalization of Galois cohomology, is a central tool, as are more general flat topologies, such as the smooth topology. The definition of a Deligne-Mumford or Artin stack uses the étale and smooth topology, respectively. Tate developed a good theory of non-archimedean analytic geometry over totally disconnected ground fields such as $\mathbb{Q}_p$ using a suitable Grothendieck topology. Work in K-theory (related for example to Voevodsky's work) uses exotic topologies.

7.3 Quasi-coherent sheaves form an abelian category

Morphisms from one quasi-coherent sheaf on a scheme X to another are defined to be just morphisms as $\mathscr{O}_X$-modules. In this way, the quasi-coherent sheaves on a scheme X form a category, denoted $\mathbf{QCoh}_X$. (By definition it is a full subcategory of $\mod_{\mathscr{O}_X}$.) We now show that quasi-coherent sheaves on X form an abelian category.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. You have seen this idea before: there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it forms a subgroup.

You can look at back at the definition of an abelian category, and you will see that in order to check that a subcategory is an abelian subcategory, it suffices to check only the following:

(i) 0 is in the subcategory;

(ii) the subcategory is closed under finite sums;

(iii) the subcategory is closed under kernels and cokernels.

In our case of $\mathbf{QCoh}_X \subseteq \mathbf{Mod}_{\mathscr{O}_X}$, the first two are cheap: 0 is certainly quasi-coherent, and the subcategory is closed under finite sums: if $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, and over $\operatorname{Spec} A$, $\mathscr{F} \cong \widetilde{M}$ and $\mathscr{G} \cong \widetilde{N}$, then $\mathscr{F} \oplus_{\text{pre}} \mathscr{G} \cong \widetilde{M \oplus N}$, note that $\widetilde{M \oplus N}$ is already a sheaf, hence $\mathscr{F} \oplus \mathscr{G} \cong \widetilde{M \oplus N}$, so $\mathscr{F} \oplus \mathscr{G}$ is a quasi-coherent sheaf.

We now check (iii), using the characterization of Theorem 7.2.2. Suppose $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of quasi-coherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\operatorname{Ker} \alpha)(U) = \operatorname{Ker} \beta$ and $(\operatorname{Coker} \alpha)(U) = \operatorname{Coker} \beta$. Let $U = \operatorname{Spec} A$ and $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, we want to show that $\operatorname{Ker} \alpha$ is a quasi-coherent sheaf, note that $(\operatorname{Ker} \alpha)|_U(\operatorname{Spec} A_f) = (\operatorname{Ker} \alpha)|_{\operatorname{Spec} A_f}(\operatorname{Spec} A_f) =$

$\operatorname{Ker}(\beta_f)$ and $((\operatorname{Ker}\alpha)|_U(\operatorname{Spec} A))_f = (\operatorname{Ker}\beta)_f$, it suffices to show that $\operatorname{Ker}(\beta_f) \cong (\operatorname{Ker}\beta)_f$. Similarly, to show $\operatorname{Coker}\alpha$ is a quasi-coherent sheaf, it suffices to show that $\operatorname{Coker}(\beta_f) \cong (\operatorname{Coker}\beta)_f$. Note that we have an exact sequence,

$$0 \longrightarrow \operatorname{Ker}\beta \longrightarrow M \xrightarrow{\beta} N \longrightarrow \operatorname{Coker}\beta \longrightarrow 0,$$

since localization is exact, apply Five Lemma 7.2.2, we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d, "\otimes_A A_f"'] & M \arrow[r, "\beta"] \arrow[d, "\otimes_A A_f"] & N \arrow[r] \arrow[d, "\otimes_A A_f"] & \operatorname{Coker}\beta \arrow[r] \arrow[d, "\otimes_A A_f"] & 0 \\
0 \arrow[r] & (\operatorname{Ker}\beta)_f \arrow[r] \arrow[d, dashed, "\sim"'] & M_f \arrow[r, "\beta_f"'] \arrow[d, "\sim"] & N_f \arrow[r] \arrow[d, "\sim"] & (\operatorname{Coker}\beta)_f \arrow[r] \arrow[d, dashed, "\sim"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(\beta_f) \arrow[r] & M_f \arrow[r, "\beta_f"'] & N_f \arrow[r] & \operatorname{Coker}(\beta_f) \arrow[r] & 0
\end{tikzcd}$$

It follows that

$$(\operatorname{Ker}\beta)_f \cong \operatorname{Ker}(\beta_f), \quad \text{and } (\operatorname{Coker}\beta)_f \cong \operatorname{Coker}\beta_f,$$

i.e., $\operatorname{Ker}\alpha$ and $\operatorname{Coker}\alpha$ are quasi-coherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α (exactness can be checked stalk-locally). Thus the quasi-coherent sheaves indeed form an abelian category.

Proposition 7.3.1

A sequence of quasi-coherent sheaves $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ on X is exact if and only if it is exact on every open set in any given affine cover of X, i.e., if $X = \bigcup_i \operatorname{Spec} A_i$, sequence

$$\mathscr{F}|_{\operatorname{Spec} A_i} \longrightarrow \mathscr{G}|_{\operatorname{Spec} A_i} \longrightarrow \mathscr{H}|_{\operatorname{Spec} A_i}$$

is exact for any $\operatorname{Spec} A_i$.

(Thus we may check injectivity or surjectivity of a morphism of quasi-coherent sheaves by checking on an affine cover of our choice.)

Proof "$\Rightarrow$": Obviously.

"$\Leftarrow$": By Proposition 3.6.5, we may check the exactness of sheaves at the level of stalks. Let $X = \bigcup_i \operatorname{Spec} A_i$, say $U_i = \operatorname{Spec} A_i$, by the condition, we have exact sequence

$$\mathscr{F}|_{U_i} \longrightarrow \mathscr{G}|_{U_i} \longrightarrow \mathscr{H}|_{U_i}.$$

Let $p \in X$, then $p \in \operatorname{Spec} A_i$ for some i, by Proposition 3.6.5, sequence

$$(\mathscr{F}|_{U_i})_p \longrightarrow (\mathscr{G}|_{U_i})_p \longrightarrow (\mathscr{H}|_{U_i})_p$$

is exact. Note that $(\mathscr{F}|_{U_i})_p = \mathscr{F}_p$, $(\mathscr{G}|_{U_i})_p = \mathscr{G}_p$, and $(\mathscr{H}|_{U_i})_p = \mathscr{H}_p$, sequence

$$\mathscr{F}_p \longrightarrow \mathscr{G}_p \longrightarrow \mathscr{H}_p$$

is exact for all $p \in X$. By Proposition 3.6.5 again, sequence

$$\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H}$$

is exaxt. □

Remark

(i) In particular, taking sections over an affine open $\operatorname{Spec} A$ is an functor from the category of quasi-coherent sheaves on X to the category of A-modules. Recall that taking sections is only left-exact in general, see Proposition 3.6.7.

(ii) **Caution:** If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of quasi-coherent sheaves, then for any open set

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U)$$

is exact, and exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} A, \mathscr{F}) = 0$. This will indeed be the case.)

Proposition 7.3.2

An $\mathscr{O}_X$-module $\mathscr{F}$ on a scheme X is quasi-coherent if and only if there exists an open cover $\{U_i \hookrightarrow X\}_i$ such that on each U_i, $\mathscr{F}|_{U_i}$ is isomorphic to the cokernel of a map of two "free sheaves":

$$\mathscr{O}_{U_i}^{\oplus I} \longrightarrow \mathscr{O}_{U_i}^{\oplus J} \longrightarrow \mathscr{F}|_{U_i} \longrightarrow 0$$

is exact.

Proof If $\mathscr{O}_X$-module $\mathscr{F}$ on a scheme X is quasi-coherent, then we have an affine open cover $\{\operatorname{Spec} A_i \hookrightarrow X\}_i$ such that $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$. Consider M_i, M_i is an A_i-module, since every A_i-module is a quotient of a free A_i-module $A_i^{\oplus J}$ (see Rotman[6], page 58, Theorem 2.35), hence we have an exact sequence

$$K_i \hookrightarrow A_i^{\oplus J} \longrightarrow M_i \longrightarrow 0, \tag{7.13}$$

where $K_i = \operatorname{Ker}(A_i^{\oplus J} \to M_i)$. Note that K_i is also an A_i-module, we have the following exact sequence.

$$A_i^{\oplus I} \longrightarrow K_i \longrightarrow 0 \tag{7.14}$$

Combining exact sequences (7.13) and (7.14) yields the following sequence.

$$\begin{tikzcd} A_i^{\oplus I} \arrow[r] \arrow[d] & A_i^{\oplus J} \arrow[r] & M_i \arrow[r] & 0 \\ K_i \arrow[ur] \end{tikzcd} \tag{7.15}$$

We want show that (7.15) is exact, it suffices to check that $\operatorname{Im}(A_i^{\oplus I} \to K_i \hookrightarrow A_i^{\oplus J}) = \operatorname{Ker}(A_i^{\oplus J} \to M_i)$. By the exactness of sequence (7.13), we have $\operatorname{Ker}(A_i^{\oplus J} \to M_i) = \operatorname{Im}(K_i \hookrightarrow A_i^{\oplus J}) = K_i$. Since $K_i \hookrightarrow A_i^{\oplus J}$, we have $\operatorname{Im}(A_i^{\oplus I} \to K_i) = \operatorname{Im}(A_i^{\oplus I} \to K_i \hookrightarrow A_i^{\oplus J})$, hence,

$$\operatorname{Im}(A_i^{\oplus I} \to K_i \hookrightarrow A_i^{\oplus J}) = \operatorname{Im}(A_i^{\oplus I} \to K_i) = K_i = \operatorname{Ker}(A_i^{\oplus J} \to M_i),$$

it follows that (7.15) is an exact sequence. By Corollary 5.1.2, $\widetilde{\cdot}$ is an exact functor, hence the sequence

$$\widetilde{A_i^{\oplus I}} \longrightarrow \widetilde{A_i^{\oplus J}} \longrightarrow \widetilde{M_i} \longrightarrow 0$$

is exact. Note that $\widetilde{A}(\operatorname{Spec} A_f)^{\oplus T} = A_f^{\oplus T} = (A^{\oplus T})_f = \widetilde{A^{\oplus T}}(\operatorname{Spec} A_f)$ (localization commutes with arbitrary direct sums), we have $\widetilde{A}^{\oplus T} \cong \widetilde{A^{\oplus T}}$ by Theorem 7.2.1, hence the following sequence,

$$\widetilde{A_i}^{\oplus I} = \mathscr{O}_{\operatorname{Spec} A_i}^{\oplus I} \longrightarrow \widetilde{A_i}^{\oplus J} = \mathscr{O}_{\operatorname{Spec} A_i}^{\oplus J} \longrightarrow \widetilde{M_i} = \mathscr{F}|_{\operatorname{Spec} A_i} \longrightarrow 0,$$

is exact, as we desired.

Conversely, suppose $\{U_i \hookrightarrow X\}_i$ be an open cover of X such that on each U_i, sequence

$$\mathscr{O}_{U_i}^{\oplus I} \longrightarrow \mathscr{O}_{U_i}^{\oplus J} \longrightarrow \mathscr{F}|_{U_i} \longrightarrow 0 \tag{7.16}$$

is exact. By Theorem 7.2.2, we may assume that $X = \operatorname{Spec} A$. Let $M = \Gamma(X, \mathscr{F})$. By the assumption, we may shrink U to some distinguished open subsets, say $D(f)$, such that sequence

$$\mathscr{O}_{D(f)}^{\oplus I} \longrightarrow \mathscr{O}_{D(f)}^{\oplus J} \longrightarrow \mathscr{F}|_{D(f)} \longrightarrow 0$$

is exact. Hence, sequence

$$A_f^{\oplus I} \longrightarrow A_f^{\oplus J} \longrightarrow \Gamma(D(f), \mathscr{F}) \longrightarrow 0$$

is exact. Let $M_{D(f)} = \Gamma(D(f), \mathscr{F})$, then

$$\mathscr{F}|_{D(f)} \cong \widetilde{M_{D(f)}}.$$

Let $\{D(f_i) \hookrightarrow \operatorname{Spec} A\}_i$ be an affine open cover of X, we may write $M_i = M_{D(f_i)}$, then $M_i = \Gamma(D(f_i), \mathscr{F})$. By Theorem 7.2.2, to show $\mathscr{F}$ is quasi-coherent, it suffices to show that $\Gamma(X, \mathscr{F})_{f_i} \cong \Gamma(D(f_i), \mathscr{F})$, i.e., $M_{f_i} \cong M_i$. Consider the restriction map $\operatorname{res}_{X,D(f_i)} : M \to M_i$, it induces an A_{f_i}-module homomorphism

$$\widehat{\operatorname{res}_{X,D(f_i)}} : M_{f_i} \longrightarrow M_i,$$

we want to show that $M_{f_i} \cong M_i$.

For injectivity. Suppose that $\widehat{\operatorname{res}_{X,D(f_i)}}\left(\frac{m}{f_i^n}\right) = 0$, then $\widehat{\operatorname{res}_{X,D(f_i)}}\left(\frac{m}{f_i^n}\right) = \frac{\operatorname{res}_{X,D(f_i)}(m)}{f_i^n} = 0$, hence $\operatorname{res}_{X,D(f_i)}(m) = 0$ in M_i. On $D(f_i) \cap D(f_j) = D(f_i f_j)$, we have

$$\mathscr{F}|_{D(f_i f_j)} = (\mathscr{F}|_{D(f_i)})|_{D(f_i f_j)} = (\mathscr{F}|_{D(f_j)})|_{D(f_i f_j)} = \widetilde{M_j}|_{D(f_i f_j)} = \widetilde{(M_j)_{f_i f_j}} \tag{7.17}$$

Denote $\operatorname{res}_i := \operatorname{res}_{X,D(f_i)}$. Since $\operatorname{res}_j(m)|_{D(f_i f_j)} = \operatorname{res}_i(m)|_{D(f_i f_j)} = 0$ in $\Gamma(D(f_i f_j), \mathscr{F}) = (M_j)_{f_i f_j}$, there exists $n_j > 0$ such that $(f_i f_j)^{n_j} \operatorname{res}_j(m) = 0$ in M_j. Since f_j is unit in M_j, we have $f_i^{n_j} \operatorname{res}_j(m) = \operatorname{res}_j(f_i^{n_j} m) = 0$ in M_j. We may pick $N \gg 0$ such that

$$\operatorname{res}_j(f_i^{n_j} m) = 0 \in M_j, \ \forall j.$$

By the identity axiom of $\mathscr{F}$, $f_i^{n_j} m = 0$ in M. Hence $\frac{m}{f_i^n} = 0$ in M_{f_i}.

For surjectivity. Let $s_i \in M_i$, then $s_i|_{D(f_i f_j)} \in \Gamma(D(f_i f_j), \mathscr{F})$, by (7.17),

$$\Gamma(D(f_i f_j), \mathscr{F}) = \Gamma(D(f_i f_j), \mathscr{F}|_{D(f_i f_j)}) = \Gamma(D(f_i f_j), \widetilde{(M_j)_{f_i f_j}}) = (M_j)_{f_i f_j},$$

hence exists n_j such that $(f_i f_j)^{n_j} s_i \in M_j$. Pick $N \gg 0$, since f_j is unit in M_j, we have $f_i^N s_i \in M_j$ for all j. By the identity axiom of $\mathscr{F}$, exists $s \in M$ such that $\operatorname{res}_i(s) = f_i^N s_i$, hence $s_i = \frac{\operatorname{res}_i(s)}{f_i^N} = \widehat{\operatorname{res}_i}\left(\frac{s}{f_i^N}\right)$ in M_i. Hence, $\widehat{\operatorname{res}_i}$ is surjective.

Hence $M_{f_i} \cong M_i$, by Theorem 7.2.2, $\mathscr{F}$ is quasi-coherent sheaf. □

Another proof by Piye Yang[7].

Proof For every $x \in X$ we have a neighborhood $U \subseteq X$ such that $\mathscr{F}|_U$ is the cokernel of a morphism between free $\mathscr{O}_X|_U$-modules. Shrinking U to make $U = \operatorname{Spec} A$ is affine, we know on U, free $\mathscr{O}_X|_U$-module is of the form $\widetilde{A}^n$ for any cardfinality n, i.e., free $\mathscr{O}_X|_U$-module is quasi-coherent, so the cokernel is quasi-coherent. Hence $\mathscr{F}|_U = \widetilde{M}$ for some A-module M. Hence $\mathscr{F}$ is quasi-coherent. Moreover, if $\mathscr{F}|_U$ is the cokernel of a morphism between free $\mathscr{O}_X|_U$-modules of finite rank, i.e., cokernel of a morphism between two coherent sheaves, it is clearly coherent. □

Proposition 7.3.2 is the definition of a quasi-coherent sheaf on a ringed space in general. It is useful in many circumstances, for example in complex analytic geometry.

Definition 7.3.1 (Quasi-coherent sheaves on ringed space)

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules if for every point $x \in X$ there exists an open neighborhood $x \in U \subseteq X$ such that *sequence*

$$\mathscr{O}_U^{\oplus I} \longrightarrow \mathscr{O}_U^{\oplus J} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

is exact, i.e., $\mathscr{F}|_U \cong \operatorname{Coker}(\mathscr{O}_U^{\oplus I} \to \mathscr{O}_U^{\oplus J})$.

7.4 Finite type quasi-coherent, finitely presented, and coherent sheaves

Here are three natural finiteness conditions on an A-module M. If A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the simplest: a module could be **finitely generated**. In other words, there is a surjection $A^{\oplus p} \to M \to 0$.

The second is reasonable too. It could be finitely presented — it could have a finite number of generators with a finite number of relations. Translation: there exists a **finite presentation**, i.e., an exact sequence

$$A^{\oplus q} \longrightarrow A^{\oplus p} \longrightarrow M \longrightarrow 0.$$

The third notion is frankly a bit surprising:

Definition 7.4.1 (Coherent module)

We say that an A-module M is ***coherent*** *if*

(i) *it is finitely generated;*

(ii) *for any map $A^{\oplus p} \to M$ (not necessarily surjective!), the kernel is finitely generated.*

Proposition 7.4.1

If A is Noetherian ring, the following are the same:

(i) *A-module M is finitely generated;*

(ii) *A-module M is finite presentation;*

(iii) *A-module M is coherent.*

Proof Clearly coherent implies finitely presented, which in turn implies finitely generated. So suppose M is finitely generated. Take any $\alpha : A^{\oplus p} \to M$. Then $\operatorname{Ker} \alpha$ is a submodule of a finitely generated module over A, and thus finitely generated by Corollary 4.6.3. Thus M is coherent. □

Hence most people can think of these three notions as the same.

Proposition 7.4.2

The coherent A-module form an abelian subcategory of the category of A-modules.

Remark The proof in general is a series of short exercise in §7.7. You should try the only if you are particularly curious.

Proof If the ring A is Noetherian. It suffices to check three things:

(i) The 0-module is coherent.

(ii) The category of coherent modules is closed under finite sums (i.e., direct sums).

(iii) The category of coherent modules is closed under kernels and cokernels.

The first two are clear. For (iii), suppose that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{Coker} f$ is finitely generated (it is the image of N), and $\operatorname{Ker} f$ is too (it is a submodule of a finitely generated module over a Noetherian ring, Proposition 4.6.3). □

Remark Finitely generated modules over a PID. We record for future reference that finitely generated modules over a PID are finite direct sums of cyclic modules. We only mention it here because it needs mentioning somewhere.

We now extend the definitions of these three classes of modules to quasi-coherent sheaves.

Proposition 7.4.3 (Finite generation is an affine-local property 6.3.1)

(a) If $f \in A$, and M is a finitely generated A-module, then M_f is a finitely generated A_f-module.

(b) If $(f_1, \ldots, f_n) = A$, and M_{f_i} is a finitely generated A_{f_i}-module for all i, then M is a finitely generated A-module.

Proof

(a) Suppose that M is generated by $x_1, \ldots, x_2$. Note that $M_f = M \otimes_A A_f$, M_f is generated by $x_1 \otimes 1, \ldots, x_n \otimes 1$, it follows that M_f is a finitely generated A_f-module.

(b) Since each M_{f_i} is finitely generated A_{f_i}-module, we may assume that M_{f_i} is generated by $x_{i1}, \ldots, x_{in_i}$. Let $m \in M$, then $m \in M_{f_i}$ for all i. Suppose that

$$m = \sum_{j=1}^{n_i} a_{ij} x_{ij},$$

where $a_{ij} \in A_{f_i}$. Eliminate the denominator using f_i, then we have

$$f_i^{k_i} m = \sum_{j=1}^{n_i} b_{ij} y_{ij},$$

where $b_{ij} \in A$ and $y_{ij} \in M$. Pick $k = \max\limits_{1 \le i \le n} \{k_i\}$, then

$$f_i^k m = \sum_{j=1}^{n_i} b_{ij} y_{ij},$$

for all i. Since $A = (f_1, \ldots, f_n)$, we have $1 = c_1 f_1 + \cdots + c_n f_n$, hence

$$1 = (c_1 f_1 + \cdots + c_n f_n)^{nk} = d_1 f_1^k + d_2 f_2^k + \cdots + d_n f_n^k,$$

where $d_i \in A$. Let $m \in M$, then

$$\begin{aligned} m =& 1 \cdot m = (d_1 f_1^k + d_2 f_2^k + \cdots + d_n f_n^k) m \\ =& d_1 f_1^k m + d_2 f_2^k m + \cdots + d_n f_n^k m \\ =& \sum_{i=1}^{n} \sum_{j=1}^{n_i} d_i b_{ij} y_{ij}, \end{aligned}$$

it follows that any element of M is finitely generated by $\{y_{ij}\}$, and therefore M is a finitely generated A-module.

□

For the finite presentation case, the following result is handy, and versions apply in other situations.

Lemma 7.4.1 (Finitely presented implies always finitely presented (module version).)

*Suppose M is a finitely presented A-module (for **some** surjection from a finite free module, the kernel is finitely generated), and $\varphi : A^{\oplus p} \to M$ is **any** surjection. Then $\operatorname{Ker} \varphi$ is finitely generated.*

Proof Choose a finite presentation of M:

$$\bigoplus_{k=1}^{r} Af_k \xrightarrow{\beta} \bigoplus_{i=1}^{n} Ax_i \xrightarrow{\alpha} M \longrightarrow 0$$

so $\alpha(x_1), \dots, \alpha(x_n)$ generate M. Write the surjection of the hypothesis as $\varphi : \bigoplus_{j=1}^{p} Ay_i \to M$. Choose lifts $g_j \in \bigoplus Ax_i$ of $\varphi(y_j) \in M$, i.e., $\alpha(g_j) = \varphi(y_j)$. Then we can write

$$\begin{aligned} M &= \bigoplus Ax_i / (\beta(f_1), \dots, \beta(f_r)) \\ &= \left(\left(\bigoplus_i Ax_i \right) \oplus \left(\bigoplus_j Ay_j \right) \right) \Big/ (\beta(f_1), \dots, \beta(f_r), y_1 - g_1, \dots, y_p - g_p), \end{aligned}$$

the last equal is given by homomorphism $(m, n) \mapsto \alpha(m) + \varphi(n)$ where $m \in \bigoplus_i Ax_i$ and $n \in \bigoplus_j Ay_j$.

As the $\{\varphi(y_j)\}$ generate M, for each $i = 1, \dots, n$ we can choose $h_i \in \bigoplus_{j=1}^{p} Ay_j$ such that $\alpha(x_i) = \varphi(h_i)$. Define $h : \bigoplus_{i=1}^{n} Ax_i \to \bigoplus_{j=1}^{p} Ay_j$ by setting $h(x_i) = h_i$. Thus

$$\begin{aligned} M &= \left(\left(\bigoplus_i Ax_i \right) \oplus \left(\bigoplus_j Ay_j \right) \right) \Big/ (\beta(f_1), \dots, \beta(f_r), y_1 - g_1, \dots, y_p - g_p, x_1 - h_1, \dots, x_n - h_n) \\ &= \bigoplus_j Ay_j / (h(\beta(f_1)), \dots, h(\beta(f_r)), y_1 - h(g_1), \dots, y_p - h(g_p)), \end{aligned}$$

which implies that $\operatorname{Ker} \varphi$ is finitely generated. □

Proposition 7.4.4 (Finite presentation is an affine-local property 6.3.1)

(a) If $f \in A$, and if M is a finitely presented A-module, then M_f is a finitely presented A_f-module.

(b) If $(f_1, \dots, f_n) = A$, and M_{f_i} is a finitely presented A_{f_i}-module for all i, then M is finitely presented A-module.

Proof

(a) Since M is a finitely presented A-module, then there exists an exact sequence

$$A^{\oplus q} \longrightarrow A^{\oplus p} \longrightarrow M \longrightarrow 0.$$

Since localization is exact functor and it is commutes with arbitrary direct sums, we have exact sequence,

$$A_f^{\oplus q} \longrightarrow A_f^{\oplus p} \longrightarrow M_f \longrightarrow 0,$$

which implies that M_f is finitely presented.

(b) Since M_{f_i} is a finitely generated A_{f_i}-module for all i, hence each M_{f_i} is a finitely generated A_{f_i}-module. By Proposition 7.4.3, M is a finitely generated A-module, so we may write $M = \operatorname{Coker}(\alpha : N \hookrightarrow A^{\oplus n})$ for some submodule $N \subseteq A^{\oplus n}$. To show that M is finitely presentation, it suffices to show that N is finitely generated. Consider the following exact sequence.

$$N \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0$$

Localize at each f_i, we get exact sequence

$$N_{f_i} \longrightarrow A_{f_i}^{\oplus n} \longrightarrow M_{f_i} \longrightarrow 0.$$

Note that $N_{f_i} = \operatorname{Ker}(A_{f_i}^{\oplus n} \to M_{f_i})$ and each M_{f_i} is finitely presented, by Lemma 7.4.1, N_{f_i} is finitely generated for all i. Since finite generation is an affine-local property (Proposition 7.4.3), N is finitely generated, and therefore M is finitely presented.

□

Proposition 7.4.5 (Coherence is an affine-local property 6.3.1)

(a) If $f \in A$, and M is a coherent A-module, then M_f is a coherent A_f-module.

(b) If $(f_1, \dots, f_n) = A$, and M_{f_i} is a coherent A_{f_i}-module for all i, then M is a coherent module.

Proof

(a) Since M is coherent, M is finitely generated, by Proposition 7.4.3, M_f is finitely generated. Let $A_f^{\oplus p} \to M_f$ be any map, note that $\operatorname{Ker}(A_f^{\oplus p} \to M_f) = \operatorname{Ker}(A^{\oplus p} \to M)_f$ and $\operatorname{Ker}(A^{\oplus p} \to M)$ is finitely generated (since M is coherent), by Proposition 7.4.3, $\operatorname{Ker}(A^{\oplus p} \to M)_f$ is finitely generated. Hence M_f is coherent.

(b) By Proposition 7.4.3, M is finitely generated. Let $\varphi : A^{\oplus p} \to M$ be any map, note that $(\operatorname{Ker}\varphi)_{f_i} = \operatorname{Ker}\varphi_{f_i}$, since M_{f_i} is coherent, $\operatorname{Ker}\varphi_{f_i}$ is finitely generated, by Proposition 7.4.3 again, $\operatorname{Ker}\varphi$ is finitely generated, and therefore M is coherent.

□

Definition 7.4.2 (Finite type, finitely presented, coherent)

*A quasi-coherent sheaf $\mathscr{F}$ is **finite type** (resp., **finitely presented**, **coherent**) if for every affine open* $\operatorname{Spec} A$, $\Gamma(\operatorname{Spec} A, \mathscr{F})$ *is a finitely generated (resp., finitely presented, coherent) A-module.*

Remark

(1) Note that coherent sheaves are always finite type, and that on a locally Noetherian scheme, all three notions are the same (by Proposition 7.4.1). Proposition 7.4.2 implies that the coherent sheaves on X form an abelian category, which we denote $\mathbf{Coh}_X$.

(2) By the Affine Communication Lemma 6.3.1, and Proposition 7.4.3, 7.4.4, and 7.4.5, it suffices to check "finite typeness" (resp., finite presentation, coherence) on the open sets in a single affine cover.

Remark **Warning.** It is not uncommon in the later literature to incorrectly define coherent as finitely generated. Please only use correct definition, as the wrong definition cause confusion. Besides doing this for reasons of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

Why coherence? Proposition 7.4.2 is a good motivation for the definition of coherence: it gives a small (in a non-techincal sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition, rather than living in a Noetherian world where coherent means finite type. Complex geometers should care. They consider consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathscr{O}_X$-module in a way analogous to this. Then Oka's Theorem states that the structure sheaf of $\mathbb{C}^n$ (hence of any complex manifold) is coherent, and this is very hard.

The second sort of people who should care are the sort of arithmetic people who may need to work with non-Noetherian rings, or work in non-achimedean analytic geometry.

Remark **Quasi-coherent and coherent sheaves on ringed spaces.** We will discuss quasi-coherent and coherent sheaves on schemes, but they can be defined more generally (see Definition 7.3.1). Many of the results we state will hold in greater generality, but because the proofs look slightly different, we restrict ourselves to scheme to avoid distraction.

Example 7.4 **Coherence is not a good notion in smooth geometry.** The following example from B.Conrad shows that in quite reasonable (but less "rigid") situations, the structure sheaf is not coherent over itself. Consider the ring $\mathscr{O}_0$ of germs of smooth (C^∞) function at $0 \in R$, with coordinate x. Now $\mathscr{O}_0$ is a local ring. Its maximal ideal $\mathfrak{m}$ is generated by x. (Key idea: suppose $f \in \mathfrak{m}$, and suppose f has a representative defined on $(-\varepsilon, \varepsilon)$. Then for $t \in (-\varepsilon, \varepsilon)$, $f(t) = \int_0^t f'(u)du = t\int_0^1 f'(tv)dv$. By "differentiating under the integral sign" repeatedly, we may check that $\int_0^1 f'(tv)dv$ is smooth.)

Let $\varphi \in \mathscr{O}_0$ be the germ of a smooth function that is 0 for $x \leq 0$, and positive for $x > 0$ (such as $\varphi(x) = e^{1\frac{1}{x^2}}$ for $x > 0$). Consider the map $\times\varphi : \mathscr{O}_0 \to \mathscr{O}_0$. The kernel is the ideal I_φ of functions vanishing for $x \geq 0$. Clearly I_φ is nonzero (since $\varphi(-x) \in I_\varphi$), but as $\mathfrak{m} = (x)$, $I_\varphi = (x)\varphi$, by Nakayama's Lemma, I_φ cannot be finitely generated. (Essentially the same argument shows that the sheaf of smooth functions on $\mathbb{R}$ is not coherent.) This is why coherence has no useful meaning for smooth manifolds.

7.5 Algebraic interlude: The Jordan-Hölder package

7.5.1 Jordan-Hölder Theorem

The Jordan-Hölder Theorem in group theory is part of a more fundamental and somehow simpler story. The Jordan-Hölder "yoga" is why you can often factor some sort of algebraic object into primes or irreducibles, uniquely (in an appropriate sense), where each prime/irreducible appears the same number of times no matter how you factor. From this point of view, it generalizes unique factorization of integers; well-definedness of dimension of vector spaces; classification of finitely generated abelian groups; unique factorization of ideals in a Dedekind domain; and the traditional Jordan-Hölder Theorem in group theory.

Jordan-Hölder Theorem for abelian category

We will be mostly interested in modules over a ring, but there is no harm in working in a general abelian category $\mathscr{C}$. (This can be readily generalized further.)

Definition 7.5.1 (Simple (or irreducible) module, composition series, finite length)

*We say an object $M \in \mathscr{C}$ is **simple** (or **irreducible**) if its only subobjects are 0 and itself. A **composition series** for M is a (finite) filtration*

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n = M \tag{7.18}$$

*such that the quotients X_{i+1}/X_i are all simple. If M has a composition series, we say that M has **finite length**.*

Theorem 7.5.1 (The Jordan-Hölder Theorem)

Let $\mathscr{C}$ be an abelian category. Let M be an object of $\mathscr{C}$ which has a finite composition series. Given any two filtrations

$$A_\bullet : 0 = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n = M$$

and

$$B_\bullet : 0 = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_m = M$$

with $S_i = A_i/A_{i-1}$ and $T_j = B_j/B_{j-1}$ simple objects, we have $n = m$ and there exists a permutation σ of $\{1, \dots, n\}$ such that $S_i \cong T_{\sigma(i)}$ for all $i \in \{1, \dots, n\}$.

Definition 7.5.2 (Length of object)

We call the length of any of the composition series for M the ***length*** *of M, denoted $l(M)$. Length of object is well-defined by the Jordan-Hölder Theorem. But we even have a refined notion: we have the multiplicity with which each simple object appears in any composition series for M. If M is not of finite length, we say $l(M) = \infty$.*

Example 7.5 In the category of abelian groups, the finite-length objects are the finite ableian groups. The Jordan-Hölder Theorem in this case, applied to $\mathbb{Z}/n\mathbb{Z}$, can be used to give the unique factorization of n.

Proof of the Jordan-Hölder Theorem 7.5.1.

Proof Let j be the smallest index such that $A_1 \subseteq B_j$. Consider the map $S_1 = A_1 \hookrightarrow B_j \to B_j/B_{j-1} = T_j$, we claim that $S_1 \cong T_j$. Note that $\mathrm{Ker}(A_1 \hookrightarrow B_j \to B_j/B_{j-1}) = A_1 \cap B_{j-1}$, since $A_1 \nsubseteq B_{j-1}$ and A_1 is simple, $A_1 \cap B_{j-1} = 0$, which implies that $S_1 \to T_j$ is monomorphism. Since T_j is simple, we have $A_1 \cong T_j$. Moreover, the object $M/A_1 = A_n/A_1 = B_m/A_1$ has the two filtrations

$$A'_\bullet : 0 \subseteq A_2/A_1 \subseteq A_3/A_1 \subseteq \cdots \subseteq A_n/A_1$$

and

$$B'_\bullet : 0 \subseteq (B_1 + A_1)/A_1 \subseteq \cdots \subseteq (B_{j-1} + A_1)/A_1 \subseteq B_j/A_1 \subseteq B_{j+1}/A_1 \subseteq \cdots \subseteq B_m/A_1.$$

Note that

$$(B_{j-1} + A_1)/A_1 \cong B_{j-1}/(A_1 \cap B_{j-1}) \cong B_{j-1} \cong B_j/A_1,$$

we have

$$B'_\bullet : 0 \subseteq (B_1 + A_1)/A_1 \subseteq \cdots \subseteq (B_{j-1} + A_1)/A_1 = B_j/A_1 \subseteq B_{j+1}/A_1 \subseteq \cdots \subseteq B_m/A_1.$$

We want to show that $((B_i + A_1)/A_1)/((B_{i-1} + A_1)/A_1) \cong (B_i + A_1)/(B_{i-1} + A_1) \cong B_i/B_{j-1}$, for all $i \leq j-1$. Since $B_{i-1} \subseteq B_i$, we have $B_i + B_{i-1} + A_1 = B_i + A_1$. Note that $B_i \cap (B_{i-1} + A_1) = B_{i-1} + A_1 \cap B_i$, since A_1 is simple $A_1 \cap B_i$ must be 0, hence

$$B_i \cap (B_{i-1} + A_1) = B_{i-1} + A_1 \cap B_i = B_{i-1}.$$

By the Fundamental Theorem of Isomorphism, we have

$$\frac{B_i + A_1}{B_{i-1} + A_1} = \frac{B_i + (B_{i-1} + A_1)}{B_{i-1} + A_1} \cong \frac{B_i}{(B_{i-1} + A_1) \cap B_i} = B_i/B_{i-1}.$$

Hence $l(A_n/A_1) \leq n - 1$ and $l(B_m/A_1) \leq m - 1$. We may assume that $l(A_n/A_1) = n - 1$ and $l(B_m/A_1) = m - 1$. Let $A'_i = A_i/A_1$, $B'_i = (B_i + A_1)/A_1$ for $i \leq j - 1$ and $B'_i = B_i/A_1$ for $i \geq j$, then we have two composition series for M/A_1

$$A'_\bullet : 0 \subsetneq A'_2 \subsetneq A'_3 \subsetneq \cdots \subsetneq A'_n$$

and

$$B'_\bullet : 0 \subsetneq B'_1 \subsetneq \cdots \subsetneq B'_{j-2} \subsetneq B'_j \subsetneq \cdots \subsetneq B'_m.$$

with $S'_i = A'_i/A'_{i-1} \cong S_i$ and $T'_i = B'_i/B'_{i-1} \cong T_i$

Let j' be the smallest index such that $A'_2 \subseteq B'_{j'}$. Then $S_2 \cong T_{j'}$. By induction then we done. □

Proposition 7.5.1

Every subquotient of a finite-length object M is finite length.

Proof Since M is finite-length, say $l(M) = n$, let chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M \tag{7.19}$$

be a composition series for M. Let $M' \subseteq M'' \subseteq M$, we want to show that $l(M''/M') < \infty$. By chain (7.19), we have a chain for M''

$$0 = M_0 \cap M'' \subseteq M_1 \cap M'' \subseteq \cdots \subseteq M_n \cap M'' = M''.$$

We claim that $(M_i \cap M'')/(M_{i-1} \cap M'') \cong M_i/M_{i-1}$. Consider the map $M_i \cap M'' \hookrightarrow M_i \xrightarrow{\pi} M_i/M_{i-1}$, then

$$\operatorname{Ker}(M_i \cap M'' \hookrightarrow M_i \xrightarrow{\pi} M_i/M_{i-1}) = M_i \cap M'' \cap M_{i-1} = M'' \cap M_{i-1},$$

which implies that $(M_i \cap M'')/(M_{i-1} \cap M'') \cong M_i/M_{i-1}$.

Merge equal terms, we get a composition series for M'', say

$$0 = M''_0 \subsetneq M''_1 \subsetneq \cdots \subsetneq M''_m = M'',$$

where $m \leq n$, it follows that M' and M'' are finite length. Let j be the smallest index such that $M' \subseteq M''_j$, since M' is finite length, we have composition series

$$0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_k = M' \subseteq M''_j \subseteq \cdots \subseteq M''_m = M'',$$

where M'_i/M'_{i-1} and M''_i/M''_{i-1} are simple. Since $M'_k \subseteq M'$,

$$0 = 0 = \cdots = 0 = M'/M' \subseteq M''_j/M' \subseteq \cdots \subseteq M''_m/M' = M''/M',$$

note that

$$(M''_i/M')/(M''_{i-1}/M') \cong M''_i/M''_{i-1}$$

is simple, we have

$$l(M''/M') \leq m - j,$$

which implies that subquotient of M is finite length. □

Proposition 7.5.2

Length is additive in exact sequences: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then

$$l(M) = l(M') + l(M'').$$

Proof Let

$$0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_n = M'$$

be a composition series for M' and

$$0 = M''_0 \subsetneq M''_1 \subsetneq \cdots \subsetneq M''_m = M''$$

be a composition series for M''. Let

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

Since α is monomorphism, we have

$$0 = \alpha(M'_0) \subsetneq \alpha(M'_1) \subsetneq \cdots \subsetneq \alpha(M'_n) = \alpha(M').$$

Since β is epimorphism, we have

$$\beta^{-1}(0) = \beta^{-1}(M_0'') \subsetneq \beta^{-1}(M_1'') \subsetneq \cdots \subsetneq \beta^{-1}(M_m'') = \beta^{-1}(M'').$$

Since $\operatorname{Im}(M' \to M) = \operatorname{Ker}(M \to M'')$, we have a chain for M

$$\begin{aligned} 0 =& \alpha(M_0') \subsetneq \cdots \subsetneq \alpha(M_n') = \alpha(M') = \beta^{-1}(0) = \beta^{-1}(M_0'') \\ & \subsetneq \beta^{-1}(M_1') \subsetneq \cdots \subsetneq \beta^{-1}(M_m'') = \beta^{-1}(M'') = M. \end{aligned} \tag{7.20}$$

We want to show that chain (7.20) is a composition series for M. Since α is monomorphism, we have $M_i' \cong \alpha(M_i')$, hence $M_i'/M_{i-1}' \cong \alpha(M_i')/\alpha(M_{i-1}')$ is simple. Consider $\beta|_{\beta^{-1}(M_i'')} : \beta^{-1}(M_i'') \to M_i''$, since β is epimorphism, by the Fundamental Theorem of Isomorphism, we have $\beta^{-1}(M_i'')/\operatorname{Ker}\beta|_{\beta^{-1}(M_i'')} \cong M_i''$. Note that $0 \in M_i$ for all i, we have $\operatorname{Ker}\beta = \operatorname{Ker}\beta|_{\beta^{-1}(M_i'')} = \beta^{-1}(0)$, hence $\beta^{-1}(M_i'')/\beta^{-1}(0) \cong M_i''$, and therefore

$$\beta^{-1}(M_i'')/\beta^{-1}(M_{i-1}'') \cong \frac{\beta^{-1}(M_i'')/\beta^{-1}(0)}{\beta^{-1}(M_{i-1}'')/\beta^{-1}(0)} \cong M_i/M_{i-1}$$

is simple. Hence by chain (7.20) we have

$$l(M) = l(M') + l(M'').$$

□

Proposition 7.5.3

Any filtration of a finite length module can be refined into a composition series.

Proof Let

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = M$$

be any filtration of M. If N_i/N_{i-1} not simple, then exists a nonzero proper subobject, say $\widetilde{N_{i-1}'}$. Hence, exists $N_{i-1}' \subsetneq N_i$ which contains N_{i-1} such that $\widetilde{N_{i-1}'}$ be the image of N_{i-1}' in N_i/N_{i-1}. The we get a new filtration of M.

$$0 = N_0 \subseteq \cdots \subseteq N_{i-1} \subseteq N_{i-1}' \subsetneq N_i \subseteq \cdots \subseteq N_n = M$$

Repeat above process, since M is finite length, this process will be terminated. Then we get a filtration of M,

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M,$$

where each M_i/M_{i-1} is simple, i.e., it is a composition series for M, as we desired. □

Proposition 7.5.4

The finite length objects in $\mathscr{C}$ form a full subcategory of $\mathscr{C}$.

Proof Clearly. □

Jordan-Hölder Theorem for groups

The category of groups does not form an abelian category, so Theorem 7.5.1 can't immediately imply the traditional Jordan-Hölder Theorem for groups. However, the same proof applies without change, with only one additional input.

Definition 7.5.3 (Subnormal series, Normal series)

*A **subnormal series** of a group G is a sequence of subgroups, each a normal subgroup of the next one. In a standard notation*

$$1 = A_0 \lhd A_1 \lhd \cdots \lhd A_n = G.$$

*There is no requirement made that A_i be a normal subgroup of G, only a normal subgroup of A_{i+1}. The quotient groups A_{i+1}/A_i are called the **factor groups** of the series.*
If in addition each A_i is normal in G, then the series is called a normal series.

Definition 7.5.4 (Isomorphic for subnormal series)

Suppose

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G, \quad 1 = H_0 \lhd H_1 \lhd \cdots \lhd H_m = G$$

are two subnormal series of G. We say they are isomorphic, if $n = m$ and there exists a permutation σ of $\{1, \ldots, n\}$ such that $G_i/G_{i-1} \cong H_{\sigma(i)}/H_{\sigma(i)-1}$ for all $i \in \{1, \ldots, n\}$.

Theorem 7.5.2 (Jordan-Hölder Theorem for groups)

If G is finite group, then any two composition series of G are isomorphic.

Proof Our proof is by induction on $|G|$. The base case, $|G| = 1$, is trivial. We use the symbol $\sim$ to denote "are rearrangements of each other".

Let

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{r-1} \lhd G_r = G, \quad 1 = H_0 \lhd H_1 \lhd \cdots \lhd H_{s-1} \lhd H_s = G$$

be any two composition series of G. If $G_{r-1} = H_{r-1}$ then we are done by induction, applying the inductive hypothesis to the group G_{r-1}. So we may assume that $G_{r-1} \neq H_{s-1}$. We set

$$M = G_{r-1} \qquad N = H_{s-1} \qquad K = M \cap N.$$

We can find a composition series

$$0 = K_0 \lhd K_1 \lhd \cdots \lhd K_t = K$$

for K.

Then $0 = G_0 \lhd \cdots \lhd G_{r-1} = M$ and $0 = K_0 \lhd \cdots \lhd K_t = K \lhd M$ are both composition series for M. By induction, we have $r - 1 = t + 1$ and

$$(G_1/G_0, \ldots, G_{r-1}/G_{r-2}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, M/K). \tag{7.21}$$

Similarly, we have $s - 1 = t + 1$ and

$$(H_1/H_0, \ldots, H_{s-1}/H_{s-2}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, N/K). \tag{7.22}$$

From the equalities $r - 1 = t + 1 = s - 1$ we deduce $r = s$. Taking (7.21) and appending the quotient G/M to both lists, we have

$$(G_1/G_0, \ldots, G_{r-1}/G_{r-2}, G/G_{r-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, M/K, G/M).$$

Similarly (7.22) gives,

$$(H_1/H_0, \ldots, H_{s-1}/H_{s-2}, G/H_{s-1}) \sim (K_1/K_0, K_2/K_1, \ldots, K_t/K_{t-1}, N/K, G/N).$$

Note that the right hand sides of the above equations are identical except for the last two elements, and by the Fundamental Theorem for Isomorphism, we have $(M/K, G/M) \sim (N/K, G/N)$, as we desired. $\square$

7.5.2 Additional facts particular to modules over a ring.

We now apply these concepts specifically to the category $\mathbf{Mod}_A$.

Proposition 7.5.5

The simple object of $\mathbf{Mod}_A$ are precisely the objects of the form $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of A.

Proof Let M be a simple module. Pick nonzero element $x \in M$, then Ax is a submodule of M, since M is a simple module, $Ax = M$. We want to show that $A/\mathfrak{m}$ is simple. Let $\mathfrak{a} \subseteq A/\mathfrak{m}$ be an ideal of $A/\mathfrak{m}$, then $\mathfrak{a}$ corresponds to ideal of $A/\mathfrak{m}$ which contains $\mathfrak{m}$, since $\mathfrak{m}$ is maximal ideal, $\mathfrak{a} = (0)$. Hence $A/\mathfrak{m}$ is simple. Next, we want to show that $Ax \cong A/\mathfrak{m}$, where $\mathfrak{m} = \operatorname{Ann}(x)$. Define $\varphi : A \to Ax$ be setting

$$a \longmapsto ax.$$

Clearly φ is a surjective. By the Fundamental Theorem for Isomorphism, we have $Ax \cong A/\operatorname{Ann}(x)$. We claim that $\operatorname{Ann}(x)$ is a maximal ideal. If not, there exists an ideal, say I, such that $\operatorname{Ann}(x) \subsetneq I \subsetneq A$, hence $I/\operatorname{Ann}(x)$ is an ideal of $A/\operatorname{Ann}(x)$. Note that Ax is simple, $I = A$, a contradiction! Hence $\operatorname{Ann}(x)$ is a maximal ideal of A. $\square$

Proposition 7.5.6

Suppose M is a finite length A-module, and

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

is a composition series with $M_i/M_{i-1} \cong A/\mathfrak{m}_i$ (where the $\mathfrak{m}_i$ are maximal ideals, not necessarily distinct, Proposition 7.5.5). Then M is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_n$. Equivalent, M is an $A/(\mathfrak{m}_1 \ldots \mathfrak{m}_n)$-module.

Proof Prove by induction on $l(M)$. If $l(M) = 1$, then our composition series for M is

$$0 = M_0 \subsetneq M_1 = M,$$

i.e., M is simple, i.e., $M = Ax$ for some x. By Proposition 7.5.5, $M \cong A/\mathfrak{m}_1$ where $\mathfrak{m}_1 = \operatorname{Ann}(x)$ is maximal ideal. It follows that $M = Ax$ is annihilated by $\mathfrak{m}_1$, and also M is an $A/\mathfrak{m}_1$-module.

If Proposition 7.5.6 holds for $l(M) < n$, we want to show that Proposition 7.5.6 holds for $l(M) = n$. Consider the composition series for M,

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M.$$

Note that

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1}$$

is the composition for M_{n-1} with $l(M_{n-1}) = n - 1$, by the hypothesis, M_{n-1} is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}$. We want to show that $M = M_n$ is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathfrak{m}_n$. Note that $M_n/M_{n-1} \cong A/\mathfrak{m}_n$, by the proof in Proposition 7.5.5, $M_n/M_{n-1} = Ax$ where $x \in M_n/M_{n-1}$ nonzero and $Ax \cong A/\operatorname{Ann}(x)$, hence $\operatorname{Ann}(x) \cdot M_n \subseteq M_{n-1}$, i.e., $\mathfrak{m}_n \cdot M_n \subseteq M_{n-1}$.

Let $a_1 \cdots a_{n-1}a_n \in \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathfrak{m}_n$ where $a_i \in \mathfrak{m}_i$. For any $m \in M = M_n$, note that

$$a_1 \cdots a_{n-1}a_n m \in a_1 \cdots a_{n-1}M_{n-1} = 0,$$

it follows that M is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathfrak{m}_n$, as we desired. $\square$

Suppose now that the list $(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)$ consists of the distinct maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$, appearing with multiplicity $l_1, \ldots, l_s$. (There are the "refined" lengths mentioned in Definition 7.5.2.) Since each $\mathfrak{n}_i$ is maximal ideal, we have $\mathfrak{n}_i + \mathfrak{n}_j = A$ whenever $i \neq j$, then exists $n_i \in \mathfrak{n}_i$ and $n_j \in \mathfrak{n}_j$ such that $a_i n_i + a_j n_j = 1$, hence

$(a_i n_i + a_j n_j)^{l_i+l_j} = b_i n_i^{l_i} + b_j n_j^{l_j} = 1$, which implies that $\mathfrak{n}_i^{l_i} + \mathfrak{n}_j^{l_j} = A$, hence $\mathfrak{n}_i^{l_i}$, $\mathfrak{n}_j^{l_j}$ are coprime whenever $i \neq j$. By Chinese Remainder Theorem 5.4.2 (ii), we have a surjective

$$\varphi : A \twoheadrightarrow A/\mathfrak{n}_1^{l_1} \times \cdots \times A/\mathfrak{n}_s^{l_s}. \tag{7.23}$$

Consider $\operatorname{Ker}\varphi$, note that $\operatorname{Ker}\varphi = \bigcap_{i=1}^{s} \mathfrak{n}_i^{l_i}$, since $\mathfrak{n}_i^{l_i}$ pairwise coprime, by Chinese Remainder Theorem 5.4.2 (i), we have $\bigcap_{i=1}^{s} \mathfrak{n}_i^{l_i} = \mathfrak{n}_1^{l_1} \cdots \mathfrak{n}_s^{l_s}$. By the Fundamental Theorem for Isomorphism, we have an isomorphism

$$A/(\mathfrak{n}_s^{l_s} \cdots \mathfrak{n}_s^{l_s}) \xleftrightarrow{\sim} A/\mathfrak{n}_1^{l_1} \times \cdots \times A/\mathfrak{n}_s^{l_s}.$$

For $1 \leq i \leq s$, let $e_i \in A$ be an element of A such that $e_i \equiv 1 \pmod{n_i^{l_i}}$ and $e_i \equiv 0 \pmod{n_j^{l_j}}$ for $i \neq j$. The e_i exist by (7.23).

Proposition 7.5.7

Suppose M is a finite length A-module, then M is a direct sum of pieces, each with composition series with only one type of "simple factor", i.e.,

$$M \cong e_1 M \oplus \cdots \oplus e_s M,$$

each $e_i M$ is a finite-length module where all the simple quotients are $A/\mathfrak{n}_i$. Moreover, $\mathfrak{n}_i$ are distinct maximal ideal of A, and exists l_i such that $e_i \equiv 1 \pmod{\mathfrak{n}_i^{l_i}}$ and $e_i \equiv 0 \pmod{\mathfrak{n}_j^{l_j}}$ for $i \neq j$.

Proof Since M is a finite length A-module, let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M \tag{7.24}$$

be a composition series with $M_i/M_{i-1} \cong A/\mathfrak{m}_i$. By Proposition 7.5.6, M is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_n$, and therefore can be seen as $A/(\mathfrak{m}_1 \cdots \mathfrak{m}_n)$-module. Suppose the list $(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)$ consists of the distinct maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$, appearing with multiplicity $l_1, \ldots, l_s$. By the discussion before Proposition 7.5.7, $\mathfrak{n}_i^{l_i}$ and $\mathfrak{n}_j^{l_j}$ are coprime whenever $i \neq j$. Apply Chinese Remainder Theorem 5.4.2, we have surjective

$$\varphi : A \twoheadrightarrow A/\mathfrak{n}_1^{l_1} \times \cdots \times A/\mathfrak{n}_s^{l_s}.$$

For $1 \leq i \leq s$, let $e_i \in A$ be an element of A such that $e_i \equiv 1 \pmod{\mathfrak{n}_i^{l_i}}$ and $e_i \equiv 0 \pmod{\mathfrak{n}_j^{l_j}}$ for $i \neq j$, (by φ is surjective). Hence $e_i(e_i - 1) \equiv 0 \pmod{\mathfrak{n}_k^{l_k}}$, $e_i e_j \equiv 0 \pmod{\mathfrak{n}_k^{l_k}}$, and $e_1 + \cdots + e_s - 1 \equiv 0 \pmod{\mathfrak{n}_k^{l_k}}$ for all k. Hence $e_i(e_i - 1), e_i e_j, e_1 + \cdots + e_s - 1 \in \operatorname{Ker}\varphi = \bigcap_{i=1}^{s} \mathfrak{n}_i^{l_i} = \mathfrak{n}_1^{l_1} \cdots \mathfrak{n}_s^{l_s}$. In fact, $\mathfrak{n}_1^{l_1} \cdots \mathfrak{n}_s^{l_s} = \mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \operatorname{Ann}(M)$. It follows that for any $m \in M$, we have

(i) $e_i(e_i - 1)m = 0 \Longrightarrow e_i^2 m = e_i m$;

(ii) $e_i e_j m = 0$ whenever $i \neq j$;

(iii) $(e_1 + \cdots + e_s - 1)m = 0 \Longrightarrow (e_1 + \cdots + e_s)m = m$.

By (iii) we know that

$$M \cong e_1 M + \cdots e_s M_s.$$

Consider $e_j M \cap \left(\sum_{i \neq j} e_i M\right)$, let $m \in e_j M \cap \left(\sum_{i \neq j} e_i M\right)$ then $m = e_j m_j$ for some $m_j \in M$. Since $m \in \sum_{i \neq j} e_i M$, we may assume that

$$m = e_j m_j = e_1 m_1 + \cdots + e_{j-1} m_{j-1} + e_{j+1} m_{j+1} + \cdots + e_s m_s. \tag{7.25}$$

Apply e_j to both sides of equation (7.25), by (i) and (ii),

$$e_j^2 m_j = e_j m_j = m = 0,$$

hence $e_j M \cap \left(\sum_{i \neq j} e_i M\right) = \{0\}$, which implies that

$$M \cong e_1 M \oplus \cdots \oplus e_s M.$$

Next we want to show that each $e_i M$ is finite length module where all the simple quotients are $A/\mathfrak{n}_i$. By (7.24) we have a filtration for $e_i M$,

$$0 = e_i M_0 \subsetneq e_i M_1 \subsetneq \cdots \subsetneq e_i M_{n-1} \subsetneq e_i M_n = e_i M. \tag{7.26}$$

We claim that $e_i M_k / e_i M_{k-1} \cong A/\mathfrak{m}_k$ or 0 for all $1 \leq k \leq n$ for some maximal ideal $\mathfrak{m}_k$. Note that $M_{k-1} = \bigoplus_{t=1}^{s} e_t M_{k-1}$, hence $e_i M_{k-1} \cap M_{k-1} = e_i M_{k-1}$. Since $e_i M_{k-1} \subseteq e_i M_k$, we have $e_i M_{k-1} \subseteq e_i M_k \cap M_{k-1}$. Conversely, let $e_i m_k \in e_i M_k \cap M_{k-1}$, then $e_i m_k \in M_{k-1}$. Suppose $e_i m_k = \sum_{t=1}^{s} e_t m_{k-1,t}$ where $m_{k-1,t} \in e_t M_{k-1}$. Apply e_i to both sides, we get $e_i m_k = e_i m_{k-1,i} \in e_i M_{k-1}$. Hence $e_i M_{k-1} = e_i M_k \cap M_{k-1}$. By Proposition 7.5.5, we have

$$\frac{e_i M_k}{e_i M_{k-1}} = \frac{e_i M_k}{e_i M_k \cap M_{k-1}} \cong \frac{M_{k-1} + e_i M_k}{M_{k-1}} \subseteq \frac{M_k}{M_{k-1}}.$$

Since M_k / M_{k-1} is simple, $e_i M_k / e_i M_{k-1}$ must be 0 or M_k / M_{k-1}. If $e_i M_k / e_i M_{k-1} \neq 0$, by Proposition 7.5.5, $e_i M_k / e_i M_{k-1} \cong M_k / M_{k-1} \cong A/\mathfrak{m}_k$ for some maximal ideal $\mathfrak{m}_k$. By combining the equal terms in (7.26), we obtain a composition series for $e_i M$, which implies that each $e_i M$ is of finite length.

Say

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{p-1} \subsetneq N_p = e_i M$$

be the composition series for $e_i M$, where $N_k / N_{k-1} \cong A/\mathfrak{m}_k$ for some maximal ideal $\mathfrak{m}_k$, moreover, by the proof of Proposition 7.5.5, $\mathfrak{m}_k = \operatorname{Ann}(N_k / N_{k-1})$. We want to show that $\mathfrak{m}_k$ are the same. Since $e_i \equiv 0 \pmod{\mathfrak{n}_j^{l_j}}$ for all $i \neq j$, $e_i \in \bigcap_{j \neq i} \mathfrak{n}_j^{l_j} = \prod_{j \neq i} \mathfrak{n}_j^{l_j}$, and note that M can be seen as $A/(\mathfrak{n}_1^{l_1} \cdots \mathfrak{n}_s^{l_s})$-module, hence $e_i M$ annihilated by $\mathfrak{n}_i^{l_i}$. Hence $\mathfrak{n}_i^{l_i}$ annihilates N_k, it follows that $\mathfrak{n}_i^{l_i} \subseteq \operatorname{Ann}(N_k / N_{k-1}) = \mathfrak{m}_k$. Taking radical both sides, we have $\mathfrak{n}_i = \sqrt{\mathfrak{n}_i^{l_i}} \subseteq \sqrt{\mathfrak{m}_k} = \mathfrak{m}_k$. Since $\mathfrak{n}_i$ is maximal ideal, we have $\mathfrak{m}_k = \mathfrak{n}_i$ for all k, as we desired. $\square$

Lemma 7.5.1

Suppose M is an A-module. M has a composition series if and only if M satisfies both chain condition.

Proof If M has composition series, then any chain in M are of bounded length, hence both a.c.c. and d.c.c..

Conversely, if M satisfies both chain condition. Since $M = M_0$ satisfies the maximum condition. It has a maximal submodule $M_1 \subseteq M_0$. Similarly M_1 has a maximal submodule $M_2 \subseteq M_1$ and so on. Thus we have a strictly descending chain $M_0 \supseteq M_1 \supseteq \cdots$ which by d.c.c. must be finite, and hence is a composition series of M. $\square$

Proposition 7.5.8

Suppose M is a finite length A-module. Then M is finitely generated, and $\operatorname{Supp} M$ *(Definition 5.1.5) consists of finitely many points of* $\operatorname{Spec} A$*, all closed. We thus have a notion of the "length of M at each of these closed points".*

Proof By Lemma 7.5.1, M satisfies a.c.c., hence M is Noetherian module, and therefore is finitely generated.

Let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M \tag{7.27}$$

be the composition series of M with $M_i/M_{i-1} \cong A/\mathfrak{m}_i$.

Let $[\mathfrak{p}] \in \operatorname{Supp} M$, then $M_\mathfrak{p} \neq 0$. By chain (7.27), we have a filtration for $M_\mathfrak{p}$

$$0 = (M_0)_\mathfrak{p} \subseteq (M_1)_\mathfrak{p} \subseteq \cdots \subseteq (M_{n-1})_\mathfrak{p} \subseteq (M_n)_\mathfrak{p} = M_\mathfrak{p}.$$

Since $M_\mathfrak{p} \neq 0$, there exists k such that $(M_k)_\mathfrak{p}/(M_{k-1})_\mathfrak{p} \neq 0$. Since localization commutes with quotient,

$$0 \neq (M_k)_\mathfrak{p}/(M_{k-1})_\mathfrak{p} \cong (M_k/M_{k-1})_\mathfrak{p} \cong (A/\mathfrak{m}_k)_\mathfrak{p} \cong A_\mathfrak{p}/(\mathfrak{m}_k)_\mathfrak{p}.$$

Since $(\mathfrak{m}_k)_\mathfrak{p} = \mathfrak{m}_k A_\mathfrak{p}$ is an ideal of $A_\mathfrak{p}$, we must have $\mathfrak{m}_k \subseteq \mathfrak{p}$, note that $\mathfrak{m}_k$ is maximal ideal, $\mathfrak{p} = \mathfrak{m}_k$. It follows that $\operatorname{Supp} M \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, i.e., $\operatorname{Supp} M$ consists of finitely many points of $\operatorname{Spec} A$, all closed. □

Remark A converse under Noetherian hypotheses will be proved in Proposition 7.6.29.

Definition 7.5.5 (Artinian ring, Artinian local ring)

If A has finite length as a module over itself, we say A is an ***Artinian ring****. If A is furthermore a local ring, we say A is an* ***Artin*** *or* ***Artinian local ring****.*

7.5.3 Applying to schemes

Now applying above language to schemes. We next consider the category $\mathbf{QCoh}_X$ of quasi-coherent sheaves on a scheme X. We have the notion of the length $l(\mathscr{F})$ of a finite-length quasi-coherent sheaf on X.

Proposition 7.5.9

Suppose X is quasi-compact (why need quasi-compact, see Stacks project [8] Lemma 01PE). The simple objects of $\mathbf{QCoh}_X$ are those isomorphism to the structure sheaves of closed points.

Proof Let $\mathscr{F} \in \mathbf{QCoh}_X$ be a simple object, then for any affine open subsets $\operatorname{Spec} A \subseteq X$, $\mathscr{F}|_{\operatorname{Spec} A} = \widetilde{M}$ where M is simple ($\mathscr{F}|_{\operatorname{Spec} A}$ is simple, if X is quasi-compact, see MSE [9]). By Proposition 7.5.5, we have $M \cong A/\operatorname{Ann}(M)$ where $\operatorname{Ann}(M)$ is maximal ideal. Say $\mathfrak{m} = \operatorname{Ann}(M)$, we have

$$\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{A/\mathfrak{m}} \cong \mathscr{O}_{\operatorname{Spec}(A/\mathfrak{m})}.$$

Let $[\mathfrak{p}] \in \operatorname{Spec} A$, then

$$(\mathscr{F}|_{\operatorname{Spec} A})_{[\mathfrak{p}]} \cong (A/\mathfrak{m})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{m}A_\mathfrak{p} = \begin{cases} A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} = \kappa([\mathfrak{m}]) & \text{if } \mathfrak{p} = \mathfrak{m} \\ 0 & \text{if } \mathfrak{p} \neq \mathfrak{m}. \end{cases}$$

It follows that $\mathscr{F}|_{\operatorname{Spec} A} \cong i_{[\mathfrak{m}],*}\underline{\kappa([\mathfrak{m}])}$ where $i : \{[\mathfrak{m}]\} \hookrightarrow X$. In other words, for all affine open subsets $\operatorname{Spec} A \subseteq X$ exists closed point $x \in \operatorname{Spec} A$ such that $\mathscr{F}|_{\operatorname{Spec} A} \cong i_{x,*}\underline{\kappa(x)}$ where $i : \{x\} \hookrightarrow X$. Also, $\operatorname{Supp} \mathscr{F}|_{\operatorname{Spec} A} = \{x\}$.

Claim that $\operatorname{Supp} \mathscr{F} = \{x\}$. Suppose there exists $y \in \operatorname{Supp} \mathscr{F}$ such that $y \neq x$, We first show that y is a closed point. Let V be an affine open subset which contains y, then there exists closed point $z \in V$ such that $\operatorname{Supp} \mathscr{F}|_V = \{z\}$. Since $\operatorname{Supp} \mathscr{F}|_V = \operatorname{Supp} \mathscr{F} \cap V$, we have $z = y$, and therefore y is a closed point. Consider $j : X \setminus \{x\} \hookrightarrow X$. Note that $\mathscr{F}$ is a quasi-coherent sheaf, we have an isomorphism

$$j^*\mathscr{F} \xrightarrow{\sim} j^*$$

on $X \setminus \{x\}$. By adjointness of (j^*, j_*), we have a morphism

$$\mathscr{F} \longrightarrow j_*j^*\mathscr{F}.$$

Consider $\mathscr{G} := \mathrm{Ker}(\mathscr{F} \to j_*j^*\mathscr{F}) \hookrightarrow \mathscr{F}$. We next calculate $\mathscr{G}_x$ and $\mathscr{G}_y$.

Let $U = \operatorname{Spec} A \subseteq X$ be any affine open neighborhood of x, then

$$(j_*j^*\mathscr{F})(U) = j^*\mathscr{F}(U \setminus \{x\}) = j^{-1}\mathscr{F}(U \setminus \{x\}) \otimes_{f^{-1}\mathscr{O}_X(U\setminus\{x\})} \mathscr{O}_{X\setminus\{x\}}(U \setminus \{x\}).$$

Note that

$$j_{\mathrm{pre}}^{-1}\mathscr{F}(U \setminus \{x\}) = \varinjlim_{\text{open } V \supseteq j(U\setminus\{x\})} \mathscr{F}(V) = \mathscr{F}(U \setminus \{x\}),$$

where last equal given by $\{x\}$ is a closed point and therefore $U \setminus \{x\}$ is open. Hence

$$\begin{aligned}(j_*j^*\mathscr{F})_x = ((j_*j^*\mathscr{F})|_U)_x &= ((j^{-1}\mathscr{F})|_{U\setminus\{x\}} \otimes_{(f^{-1}\mathscr{O}_X)|_{U\setminus\{x\}}} \mathscr{O}_{U\setminus\{x\}})_x \\ &= (\mathscr{F}|_{U\setminus\{x\}})_x \otimes_{((f^{-1}\mathscr{O}_X)|_{U\setminus\{x\}})_x} (\mathscr{O}|_{U\setminus\{x\}})_x \\ &= 0 \otimes (\mathscr{O}|_{U\setminus\{x\}})_x \\ &= 0.\end{aligned}$$

Thus

$$\mathscr{G}_x = \mathrm{Ker}(\mathscr{F} \to j_*j^*\mathscr{F})_x = \mathrm{Ker}(\mathscr{F}_x \to (j_*j^*\mathscr{F})_x) = \mathrm{Ker}(\mathscr{F}_x \to 0) = \mathscr{F}_x \neq 0.$$

Let U be an affine open neighborhood of y such that $U \subseteq X \setminus \{x\}$, then $(j_*j^*\mathscr{F})(U) = j^*\mathscr{F}(U)$, hence

$$\begin{aligned}(j_*j^*\mathscr{F})_y =&(j_*j^*\mathscr{F}|_U)_y = (j^*\mathscr{F}|_U)_y \\ =&(j^{-1}\mathscr{F}|_U)_y \otimes_{(j^{-1}\mathscr{O}_X|_U)_y} (\mathscr{O}_U)_y \\ =&\mathscr{F}_{j(y)} \otimes_{(\mathscr{O}_U)_y} (\mathscr{O}_U)_y \\ =&\mathscr{F}_y \otimes_{(\mathscr{O}_U)_y} (\mathscr{O}_U)_y \cong \mathscr{F}_y.\end{aligned}$$

Thus $\mathscr{G}_y = \mathrm{Ker}(\mathscr{F}_y \to (j_*j^*\mathscr{F})_y) = \mathrm{Ker}(\mathscr{F}_y \to \mathscr{F}_y) = 0$. Since $y \in \operatorname{Supp}\mathscr{F}$, $\mathscr{F}_y \neq 0$. It follows that $\mathscr{G} \hookrightarrow \mathscr{F}$ is a nonzero subobject of $\mathscr{F}$, contradicts to the fact that $\mathscr{F}$ is simple. Hence $\operatorname{Supp}\mathscr{F} = \{x\}$, and therefore $\mathscr{F} \cong \mathscr{O}_{\operatorname{Spec}(A/\mathfrak{m})}$, where $\mathfrak{m}$ is maximal ideal of A. □

Proposition 7.5.10

Suppose X is quasi-compact (why need quasi-compact, see Stacks project [8] Lemma 01PE). Suppose that $\mathscr{F}$ is a finite length element of $\mathbf{QCoh}_X$, then $\mathscr{F}$ is finite type, and $\operatorname{Supp}\mathscr{F}$ consists of finitely many points of X, all closed. Thus we can define the length of a $\mathscr{F}$ at one of the points of $\operatorname{Supp}\mathscr{F}$.

Proof Let $\operatorname{Spec} A \subseteq X$ be any affine open subset of X, then $\mathscr{F}|_U \cong \widetilde{M}$ for some A-module M. Since $\mathscr{F}$ is finite length, $\mathscr{F}|_U$ must be finite length, hence M is finite length. By Proposition 7.5.8, M is finitely generated, and therefore $\mathscr{F}$ is finite type. Let

$$0 = \mathscr{F}_0 \hookrightarrow \mathscr{F}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{F}_{n-1} \hookrightarrow \mathscr{F}_n = \mathscr{F} \tag{7.28}$$

be the composition series of $\mathscr{F}$. Since each $\mathrm{Coker}(\mathscr{F}_{i-1} \hookrightarrow \mathscr{F}_i)$ is simple, by Proposition 7.5.9, $\mathrm{Coker}(\mathscr{F}_{i-1} \hookrightarrow \mathscr{F}_i) \cong \widetilde{A_i/\mathfrak{m}_i}$ for some $\mathfrak{m}_i$ which is the maximal ideal of A_i. Let $x \in \operatorname{Supp}\mathscr{F}$, then $\mathscr{F}_x \neq 0$. By composition series (7.29), we have a filtration for $\mathscr{F}_x$,

$$0 = (\mathscr{F}_0)_x \hookrightarrow (\mathscr{F}_1)_x \hookrightarrow \cdots \hookrightarrow (\mathscr{F}_{n-1})_x \hookrightarrow (\mathscr{F}_n)_x = \mathscr{F}_x. \tag{7.29}$$

Since $\mathscr{F}_x \neq 0$, there exists k such that $\mathrm{Coker}((\mathscr{F}_{k-1})_x \hookrightarrow (\mathscr{F}_k)_x) \neq 0$. Since taking stalks commutes with Coker, we have

$$\mathrm{Coker}((\mathscr{F}_{k-1})_x \hookrightarrow (\mathscr{F}_k)_x) \cong \mathrm{Coker}(\mathscr{F}_{k-1} \hookrightarrow \mathscr{F}_k)_x \cong (\widetilde{A_k/\mathfrak{m}_k})_x \cong A_x/(\mathfrak{m}_k A_x).$$

Hence, $x = [\mathfrak{m}_k]$. It fullows that $\operatorname{Supp}\mathscr{F} \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ where each $\mathfrak{m}_k$ is closed. □

Remark A converse under Noetherian hypotheses will be proved in Proposition 7.6.29.

Definition 7.5.6 (Length of scheme)

The ***length of a scheme*** X *is the length of the structure sheaf* $\mathscr{O}_X$ *(in* $\mathbf{QCoh}_X$*). A scheme* X *is* ***finite length*** *or* ***Artinian*** *if* $\mathscr{O}_X$ *is finite length.*

7.6 Visualizing schemes: Associated points and zero-divisors

The theory of **associated points** of a module refines the notion of support (Definition 5.1.5). Associated points will help us understand and visualize nilpotents, and generalize the notion of "rational functions" to non-integral schemes. They are useful in ways we won't use, for example through their connection to primary decomposition. They might be most useful for us in helping us understand and visualize (non-)zerodivisors, which will come up repeatedly, through effective Cartier divisors and line bundles, regular sequences, depth and Cohen-Macaulayness, and more.

7.6.1 Motivation

Figure 7.1 is a sketch of a scheme X. We see two connected components, and three irreducible components. The irreducible components of X have dimensions 2, 1, and 1, although we won't be able to make sense of "dimension" until Chapter 13. Both connected components are nonreduced.

We see a little more in this picture, which we will make precise in this section, in terms of "associated points". The reducible connected component seems to have different amounts of nonreduced behavior on different loci. The scheme X has six associated points, which are the generic points of the irreducible subsets "visible" in the picture. A function on X is a zerodivisor if its zero locus contains any of these six irreducible subvarieties.

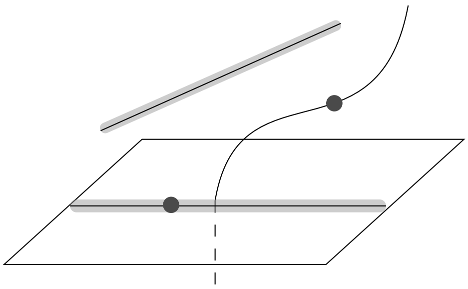


Figure 7.1: This scheme has six associated points, of which three are embedded points. A function is a zerodivisor if it vanishes at any of these six points.

Suppose M is a finitely generated module over a Noetherian ring A. For example, M could be A. Then there are some special points of $\operatorname{Spec} A$ that are particularly crucial to understanding M. These are the **associated points** of M (or equivalently, the **associated prime ideals** of M — we will use these terms interchangeably). As motivation, we give a zillion properties of associated points, and leave it to you to verify them from the theory developed in the rest of this section.

As you read this section, you may wish to keep in mind

$$M = A = k[x,y]/(y^2, xy)$$

(Figure 5.4) as a running example.

A zillion properties of associated points

Here are some of the properties of associated points that we will prove.

Introduction

- *There are finitely many associated points* $\operatorname{Ass}_A M \subseteq \operatorname{Spec} A$.
- *The support of M is the closure of the associated points of M: $\operatorname{Supp} M = \overline{\operatorname{Ass}_A M}$. The support of any submodule of M is the closure of some subset of the associated points of M. The support of any element of M is the closure of some subset of the associated points.*
- *The associated points of M are precisely the generic points of irreducible components of $\operatorname{Supp} m$ for all $m \in M$. The associated points of M are precisely the generic points of those $\operatorname{Supp} m$ which are irreducible (as m runs over the elements of M). The associated primes of M are precisely those prime ideals that are annihilators of some element of M.*
- *Taking "associated points" commutes with localization. Hence this notion is "geometric in nature", which will allow us to extend the notion to coherent sheaves on locally Noetherian schemes.*
- *Associate points behave **fairly** well in exact sequence. For example, the associated points of a submodule are a subset of the associated points of the module.*
- *If $I \subseteq A$ is an ideal, the associated primes $\mathfrak{p}$ of A/I are precisely those $\mathfrak{p}$ such that a $\mathfrak{p}$-primary ideal appears in the primary decomposition of I.*
- ***Important!** An element of A is a zerodivisor if and only if it vanishes at an associated point.*
- *An element of A is nilpotent if and only if it vanishes at every associated point. The locus of points $[\mathfrak{p}]$ of $\operatorname{Spec} A$ where the stalk $A_{\mathfrak{p}}$ is nonreduced is the closure of some subset of the associated points.*
- *An associated point that is in the closure of another associated point is said to be an **embedded point**. If A is reduced, then $\operatorname{Spec} A$ has no embedded points. Hypersurfaces in $\mathbb{A}_k^n$ have no embedded points. We will later see that complete intersections have no embedded points (Chapter 27).*
- *Elements of M are determined by their localization at the associated points. Sections of the corresponding sheaf $\widetilde{M}$ are determined by their germs at the associated points.*

This discussion immediately implies a notion of **associated point** for a coherent sheaf on a locally Noetherian scheme, with all the good properties described here. The phrase **associated point of a locally Noetherian scheme** X (without explicit mention of a coherent sheaf) means "associated point of $\mathscr{O}_X$", and similarly for **embedded points**.

We now establish these zillion facts.

7.6.2 More on the notion of support

Support and annihilator ideal

The notion of associated points of an A-module M refines the notion of support (in the case where M is finitely generated over a Noetherian ring A). (In what follows, we make no assumptions that A is Noetherian or that M is finitely generated until we need to.) To set this up, recall Definition 5.1.5 that the support of $m \in M$,

$$\operatorname{Supp} m = \{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0\},$$

is a closed subset of Spec A (Proposition 5.1.5), and thus of the form $V(I)$ for some I. Proposition 7.6.1 gives the "best such" I.

Definition 7.6.1 (Annihilator ideal)

*Define the **annihilator ideal*** $\operatorname{Ann}_A m \subseteq A$ *of an element* m *of an* A*-module* M *by:*

$$\operatorname{Ann}_A m := \{a \in A : am = 0\} = \operatorname{Ker}(A \xrightarrow{\sim} M).$$

The subscript A *is omitted if it is clear from context.*

Proposition 7.6.1

Suppose M *is an* A*-module, then* $\operatorname{Supp} m = V(\operatorname{Ann} m)$ *for all* $m \in M$*. Moreover, if* M *is finitely generated* A*-module, then* $\operatorname{Supp}(M) = V(\operatorname{Ann}_A M)$.

Proof Note that

$$\begin{aligned}\operatorname{Supp} m =& \{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0\} \\ =& \{[\mathfrak{p}] \in \operatorname{Spec} A : (A - \mathfrak{p})m \neq 0\} \\ =& \{[\mathfrak{p}] \in \operatorname{Spec} A : (A - \mathfrak{p}) \cap \operatorname{Ann} m = \varnothing\} \\ =& \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supseteq \operatorname{Ann} m\} \\ =& V(\operatorname{Ann} m),\end{aligned}$$

as we desired.

Since M is finitely generated A-module, we may assume that $M = Ax_1 + \cdots Ax_n$, then

$$\begin{aligned}V(\operatorname{Ann}_A M) =& V\left(\bigcap_{i=1}^n \operatorname{Ann}_A x_i\right) = V\left(\prod_{i=1}^n \operatorname{Ann}_A x_i\right) \\ =& \bigcup_{i=1}^n V(\operatorname{Ann}_A x_i) = \bigcup_{i=1}^n \operatorname{Supp}_A x_i \\ =& \operatorname{Supp}_A M,\end{aligned}$$

(we use the fact that $\operatorname{Ann} x_i$ are pairwise coprime, and apply CRT) as we desired. ☐

Recall (Definition 3.7.5) that

$$\operatorname{Supp} \widetilde{M} = \{p \in \operatorname{Spec} A : \widetilde{M}_p \neq 0\},$$

and the analogous Definition 5.1.5 of the support of the module M,

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A : M_{\mathfrak{p}} \neq 0\}, \tag{7.30}$$

so $\operatorname{Supp} M = \operatorname{Supp} \widetilde{M}$. If M is a principal module generated by $m \in M$, then

$$\operatorname{Supp} M = \operatorname{Supp} Am = \operatorname{Supp} m = V(\operatorname{Ann} m).$$

The notions of support and associated points behave well in exact sequences, and under localization. We begin to explain this now.

The notion of support behaves well in exact sequences

Proposition 7.6.2

Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules.

(a) $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.

(b) If M is a finitely generated module, then $\operatorname{Supp} M$ *is a closed subset of* $\operatorname{Spec} A$.

Proof

(a) Let $\mathfrak{p} \in \operatorname{Supp} M$, then $M_{\mathfrak{p}} \neq 0$. Since localization is exact, we have an exact sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0. \tag{7.31}$$

Since $M_{\mathfrak{p}} \neq 0$, $M'_{\mathfrak{p}} \neq 0$ or $M''_{\mathfrak{p}} \neq 0$, which implies that $\mathfrak{p} \in \operatorname{Supp} M' \cup \operatorname{Supp} M''$. Conversely, let $\mathfrak{p} \in \operatorname{Supp} M' \cup \operatorname{Supp} M''$, then $M'_{\mathfrak{p}} \neq 0$ or $M''_{\mathfrak{p}} \neq 0$. By the exactness of sequence (7.31), $M_{\mathfrak{p}} \neq 0$, i.e., $\mathfrak{p} \in \operatorname{Supp} M$. Hence $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.

(b) Prove by induction on the number of generators of M. Suppose M is generated by $x_1, \cdots, x_n$. If $n = 1$, then M is generated by x_1, i.e., $M = Ax_1$. Hence $\operatorname{Supp} M = \operatorname{Supp} Ax_1 = \operatorname{Supp} x_1$, by Proposition 5.1.5, $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$. Assume the result holds for the case of $n-1$ generators, we want to show that $\operatorname{Supp} M$ is closed in $\operatorname{Spec} A$, where $M = \bigoplus_{i=1}^{n} Ax_n$. Consider the following exact sequence.

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} Ax_i \longrightarrow M \longrightarrow Ax_n \longrightarrow 0$$

By part (a), $\operatorname{Supp} M = \operatorname{Supp} \bigoplus_{i=1}^{n-1} Ax_i \cup \operatorname{Supp} Ax_n$. By the induction hypothesis, $\operatorname{Supp} \bigoplus_{i=1}^{n-1} Ax_i$ is closed in $\operatorname{Spec} A$, also note that $\operatorname{Supp} Ax_n = \operatorname{Supp} x_n$ is closed in $\operatorname{Spec} A$ (Proposition 5.1.5), hence $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$.

□

Remark **Warning:** $\operatorname{Supp} M$ need not be closed in general; consider $A = \mathbb{Z}$ and $M = \bigoplus_{q \text{ prime}} \mathbb{Z}/(q)$. Let $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$, then

$$M_{\mathfrak{p}} = \bigoplus_{q \text{ prime}} (\mathbb{Z}/(q))_{\mathfrak{p}}.$$

If $\mathfrak{p} = (0)$, then $(\mathbb{Z}/(q))_{(0)} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(q) = 0$ for all prime number q. If $\mathfrak{p} = (p) \neq (0)$, then $(\mathbb{Z}/(p))_{(p)} \cong \mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)} \cong \operatorname{Frac}(\mathbb{Z}/(p)) = \mathbb{Z}/(p)$ and $(\mathbb{Z}/(q))_{(p)} = 0$ for all $q \neq p$ (since $q \notin (p)$, $q \in \mathbb{Z} - (p)$, but $q = 0$ in $\mathbb{Z}/(q)$, which implies that $(\mathbb{Z}/(q))_{(p)} = 0$). Hence $M_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} \setminus \{(0)\}$, it follows that $\operatorname{Supp} M = \operatorname{Spec} \mathbb{Z} \setminus \{(0)\}$, it is not a closed subset of $\operatorname{Spec} \mathbb{Z}$ since the closure $\operatorname{cl}(\operatorname{Supp} M) = \operatorname{Spec} \mathbb{Z} \neq \operatorname{Supp} M$.

Proposition 7.6.3

Suppose M is a finitely generated A-module, and $x \in A$ has value 0 at all the points of $\operatorname{Supp} M$*, i.e., x is contained in all of the primes where M is supported. Then some power x^n of x annihilates every element of M.*

Proof Since M is finitely generated, we may assume that $M = \bigoplus_{i=1}^{n} Am_i$. Let $\mathfrak{p} \in \operatorname{Supp} M$, then $M_{\mathfrak{p}} \neq 0$, hence

$$M_{\mathfrak{p}} = \left(\bigoplus_{i=1}^{n} Am_i \right)_{\mathfrak{p}} = \bigoplus_{i=1}^{n} (A_{\mathfrak{p}} \otimes_A Am_i) \neq 0,$$

it follows that $A_{\mathfrak{p}} \otimes_A Am_i \neq 0$ for some i. Hence $\mathfrak{p} \in \operatorname{Supp} m_i$ for some i, which implies that $\operatorname{Supp} M \subseteq \bigcup_{i=1}^{n} \operatorname{Supp} m_i$. Clearly, we have $\bigcup_{i=1}^{n} \operatorname{Supp} m_i \subseteq \operatorname{Supp} M$. Hence $\operatorname{Supp} M = \bigcup_{i=1}^{n} \operatorname{Supp} m_i$. By Proposition 7.6.1, $\operatorname{Supp} M = \bigcup_{i=1}^{n} V(\operatorname{Ann} m_i) = V(\prod_{i=1}^{n} \operatorname{Ann} m_i)$. By the condition, we know that $x \in \bigcap_{[\mathfrak{p}] \in \operatorname{Supp} M} \mathfrak{p}$, hence

$$x \in \bigcap_{[\mathfrak{p}] \in V(\prod_{i=1}^{n} \operatorname{Ann} m_i)} \mathfrak{p} = \sqrt{\prod_{i=1}^{n} \operatorname{Ann} m_i}.$$

Next we want to show that $\prod_{i=1}^{n} \operatorname{Ann} m_i = \bigcap_{i=1}^{n} \operatorname{Ann} m_i$, it suffices to show that $\operatorname{Ann} m_i$ pairwise coprime. By the proof in Proposition 7.5.5, each $\operatorname{Ann} m_i$ is maximal ideal of A, and therefore $\operatorname{Ann} m_i$ pairwise coprime, so

$$x \in \sqrt{\bigcap_{i=1}^{n} \operatorname{Ann} m_i}.$$

Hence $x^n \in \bigcap_{i=1}^{n} \operatorname{Ann} m_i$ for some n, which implies that x^n annihilates every element of M. □

Proposition 7.6.4

Suppose M is a finitely generated A-module, and $x \in A$. Then $\operatorname{Supp}(M/xM) = (\operatorname{Supp} M) \cap V(x)$. *Here M/xM is defined by the exact sequence*

$$M \xrightarrow{\times x} M \longrightarrow M/xM \longrightarrow 0.$$

Proof Let $\mathfrak{p} \in \operatorname{Supp}(M/xM)$, then $(M/xM)_{\mathfrak{p}} \neq 0$. Since localization commutes with quotient, we have $(M/xM)_{\mathfrak{p}} = M_{\mathfrak{p}}/(xM)_{\mathfrak{p}} \neq 0$, hence $M_{\mathfrak{p}} \neq 0$ and $(xM)_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. We claim that $x \in \mathfrak{p}$. If not, $x \notin \mathfrak{p}$, i.e. $x \in A - \mathfrak{p}$, then

$$(xM)_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A (xM) = A_{\mathfrak{p}} \otimes M = M_{\mathfrak{p}},$$

contradicts to the fact that $(xM)_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. Hence $\mathfrak{p} \in (\operatorname{Supp} M) \cap V(x)$.

Conversely, let $\mathfrak{p} \in (\operatorname{Supp} M) \cap V(x)$. Consider the exact sequence

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/xM \longrightarrow 0.$$

Since localization is exact, the sequence

$$0 \longrightarrow (xM)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow (M/xM)_{\mathfrak{p}} \longrightarrow 0 \tag{7.32}$$

is exact. Since $\mathfrak{p} \in V(x)$, we know that $(xM)_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. Since $\mathfrak{p} \in \operatorname{Supp} M$, $M_{\mathfrak{p}} \neq 0$. By the exactness of (7.32), we know that $(M/xM)_{\mathfrak{p}} \neq 0$, which implies that $\mathfrak{p} \in \operatorname{Supp}(M/xM)$, i.e., $(\operatorname{Supp} M) \cap V(x) \subseteq \operatorname{Supp}(M/xM)$.

By above discussion, we have $\operatorname{Supp}(M/xM) = (\operatorname{Supp} M) \cap V(x)$. □

7.6.3 Definition: Associated points and associated primes

Definition 7.6.2 (Associated points and associated primes)

*Define the **associated prime ideals** of an A-module M to be those prime ideals of A of the form $\operatorname{Ann}_A(m)$ for some $m \in M$.*

*Define the **associated points** of M to be the corresponding points of $\operatorname{Spec} A$; we use the terminology "associated points" and "associated primes" interchangeably.*

The set of associated points is denoted $\operatorname{Ass}_A M \subseteq \operatorname{Spec} A$. The subscript A is dropped if it is clear from the context.

Definition 7.6.3 (Associated primes of a ring)

*The **associated primes of a ring** A are the associated primes of A considered as an A-module (i.e., $M = A$ in Definition 7.6.2)*

Proposition 7.6.5 (Associated points of integral domains)

If A is an integral domain, then $\operatorname{Ass} A = \{[(0)]\}$ — the zero ideal is the only associated prime.

Proof Let $\operatorname{Ann}(a) \in \operatorname{Ass} A$, then $\operatorname{Ann}(a) \in \operatorname{Spec} A$ and $\operatorname{Ann}(a) \cdot a = 0$. Since A is an integral domain, A has nonzero divisor and $(0) \in \operatorname{Spec} A$, we have $\operatorname{Ann}(a) = (0)$, as we desired. □

Proposition 7.6.6 (Associated points of hypersurfaces)

Given $f \in k[x_1, \ldots, x_n]$, the associated primes of $k[x_1, \ldots, x_n]/(f)$ are those principal ideals generated by the prime factors of f.

Proof Say $A = k[x_1, \ldots, x_n]$ and $R = A/(f)$. Since A is UFD, we may assume that $f = f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}$ where each f_i is the prime factors of f. Let $\mathfrak{p} \in \operatorname{Ass} R$, then $\mathfrak{p} = \operatorname{Ann}(\overline{h})$ where $\overline{h}$ is the image of $h \in A$ in R. Say $\mathfrak{q} \supseteq (f)$ be the corresponding prime ideal of $\mathfrak{p}$ in A, in fact,

$$\mathfrak{q} = \{g \in A : gh \in (f)\}.$$

Since $f \in \mathfrak{q}$, exists f_i such that $f_i \in \mathfrak{q}$, then $f_i h \in (f)$. It follows that exists $c \in A$ such that $f_i h = cf$, then

$$h = c f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_n^{e_n},$$

where $f_i \nmid h$. Now, we calculate $\operatorname{Ann}(\overline{h})$,

$$\begin{aligned}
\operatorname{Ann}(\overline{h}) =& \{\overline{g} \in R : \overline{g} \cdot \overline{h} = 0\} \\
=& \{g \in A : gh = bf \text{ for some } b \in A\} \\
=& \{g \in A : cg = bf_i \text{ for some } b \in A\}.
\end{aligned}$$

Since $f_i \nmid h$, $c \mid b$, hence $g = b' f_i$, which implies that $\operatorname{Ann}(\overline{h}) = (\overline{f_i}) = (f_i)$, as we desired. □

Remark Above argument will apply more generally to any $f \in A$ where A is a UFD.

Lemma 7.6.1

Suppose M is an A-module, $[\mathfrak{p}] \in \operatorname{Ass}_A(M)$ if and only if there is an injection $A/\mathfrak{p} \hookrightarrow M$ of A-modules.

Proof If $[\mathfrak{p}] \in \operatorname{Ass}_A(M)$, then $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M$. Consider homomorphism $A \xrightarrow{\times m} Am$, then $A/\operatorname{Ann}(m) \cong Am \hookrightarrow M$.

Conversely, say $i : A/\mathfrak{p} \hookrightarrow M$ and $i(1) = m$. Consider $\times m : A \to Am$, we want to show that $A/\mathfrak{p} \cong Am$.

Clearly, $A/\operatorname{Ann}_A(m) \cong Am$, it suffices to show that $\mathfrak{p} = \operatorname{Ann}_A(m)$. Note that

$$\operatorname{Ann}_A(m) = \{a \in A : am = 0\} = \{a \in A : i(a) = 0\} = \mathfrak{p},$$

as we desired. □

Proposition 7.6.7

Let M and M' are both A-module.

(a) Suppose $M' \subseteq M$, then $\operatorname{Ass} M' \subseteq \operatorname{Ass} M$.

(b) $\operatorname{Ass} M \subseteq \operatorname{Supp} M$.

Proof

(a) Let $[\mathfrak{p}] \in \operatorname{Ass}_A(M')$, by Lemma 7.6.1, there is an injection $A/\mathfrak{p} \hookrightarrow M'$ of A-module. Since $M' \subseteq M$, then $A/\mathfrak{p} \hookrightarrow M$, by Lemma 7.6.1 again, $[\mathfrak{p}] \in \operatorname{Ass}_A(M)$. (The corresponding statement for "support" is implicit in Proposition 7.6.2 (a).)

(b) Let $[\mathfrak{p}] \in \operatorname{Ass} M$, then we have an exact sequence.

$$0 \longrightarrow A/\mathfrak{p} \hookrightarrow M$$

Since localization is exact, we have

$$0 \longrightarrow (A/\mathfrak{p})_\mathfrak{p} \hookrightarrow M_\mathfrak{p}.$$

Since $(A/\mathfrak{p})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ and $\mathfrak{p}A_\mathfrak{p}$ is the maximal ideal of $A_\mathfrak{p}$, $(A/\mathfrak{p})_\mathfrak{p} \neq 0$, and therefore $M_\mathfrak{p} \neq 0$. it follows that $[\mathfrak{p}] \in \operatorname{Supp} M$.

□

If M is finitely generated, then $\operatorname{Supp} M$ is closed (Proposition 7.6.2), so $\overline{\operatorname{Ass} M} \subseteq \operatorname{Supp} M$. Equality will be shown later, when A is Noetherian.

7.6.4 Nonzero modules over Noetherian rings have associated points

Suppose m is a nonzero element of an A-module M. Observe that for any nonzero multiple of xm of m, $\operatorname{Ann} m \subseteq \operatorname{Ann} xm \subsetneq A$.

Proposition 7.6.8

Suppose A is Noetherian, then there is some multiple $n = xm$ such that any nonzero multiple $yn \neq 0$ of n satisfies $\operatorname{Ann} yn = \operatorname{Ann} n$. Moreover, $\operatorname{Ann} n$ is prime ideal.

Proof Clearly, we have $\operatorname{Ann}(m) \subseteq \operatorname{Ann}(xm)$ for all $x \in A$. Then we get a chain

$$\operatorname{Ann}(m) \subseteq \operatorname{Ann}(x_1 m) \subseteq \operatorname{Ann}(x_2 m) \subseteq \cdots \subseteq \operatorname{Ann}(x_k m) \subseteq \cdots .$$

Since A is Noetherian ring, above chain must be stationary, hence exists x_n such that $\operatorname{Ann}(x_n m) = \operatorname{Ann}(x_{n+1} m) = \cdots$. Let $n = x_n m$, then $\operatorname{Ann}(n) = \operatorname{Ann}(yn)$ for any nonzero multiple $yn \neq 0$.

Next, we want to show that $\operatorname{Ann}(n)$ is prime ideal. Suppose $ab \in \operatorname{Ann}(n)$, so $abn = 0$. Then either $bn = 0$ (in which case $b \in \operatorname{Ann}(b)$), or else $a \in \operatorname{Ann}(bn) = \operatorname{Ann}(n)$ by above discussion. Hence $\operatorname{Ann}(n)$ is prime ideal. □

We have thus proved the following.

Proposition 7.6.9 (Nonzero modules over Noetherian rings have associated primes)

If M is a nonzero module over a Noetherian ring A, then $\operatorname{Ass}_A M$ is nonempty. More precisely, for any $m \neq 0$ in M, there is an associated prime $\mathfrak{p}$ containing $\operatorname{Ann} m$, and $\mathfrak{p} = \operatorname{Ann} xm$ for some $x \in A$.

7.6.5 Localizations at the associated primes

Recall the useful fact that $M \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} M_{\mathfrak{p}}$ is an injection (Proposition 5.1.3). Our current situation is much better: we can take the product over only the localization at associated primes.

Proposition 7.6.10

Suppose M is a module over a Noetherian ring A. Then the natural map

$$M \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}} \tag{7.33}$$

is an injection.

Proof If the kernel K is nonzero, by Proposition 7.6.9, K has an associated prime $\mathfrak{p}$, which is the annihilator of some $m \in K \subseteq M$, i.e., $\mathfrak{p} = \operatorname{Ann} m$. Consider $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$, $m = 1 \otimes m \neq 0$ in $M_{\mathfrak{p}}$, a contradiction. □

Remark Clearly we need only the maximal among the associated primes in (7.33).

7.6.6 Zerodivisors = elements of associated primes

Proposition 7.6.11

Suppose $f \in A$, with A Noetherian. Then f a zerodivisor on M if and only if f vanishes at an associated point of M. Translation: the set of zerodivisors is the union of the associated prime ideals.

Remark Again, we need only the maximal among the associated primes. For example, if $(A, \mathfrak{m})$ is a local ring, then $\mathfrak{m}$ is an associated prime if and only if every element of $\mathfrak{m}$ is a zerodivisor.

Proof If f vanishes at an associated point of M, then $f \in \operatorname{Ann}(m)$ for some $m \in M$, hence $fm = 0$, i.e., f a zerodivisor on M.

Conversely, if f vanishes at no associated point, i.e., $f \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$. Consider the commuting diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\times f"'] & \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}} \arrow[d, "\times f"] \\ M \arrow[r, hook] & \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}} \end{tikzcd}$$

where the rows are the maps of (7.33). The vertical arrow on the right (multiplication by f) is an injection by hypothesis, so the vertical arrow on the left must be an injection, hence f is not a zerodivisor on M, a contradiction. □

7.6.7 Associated points behave fairly well in exact sequences

Proposition 7.6.12

Suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{7.34}$$

is a short exact sequence of A-modules. Then

$$\operatorname{Ass} M' \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} M' \cup \operatorname{Ass} M''. \tag{7.35}$$

Proof The first inclusion of (7.35) was shown in Proposition 7.6.7 (a).

Suppose $[\mathfrak{p}] \in \operatorname{Ass} M$, so there is some $m \in M$ with $\operatorname{Ann} m = \mathfrak{p}$, then $Am \cong A/\mathfrak{p}$. We wish to find a submodule of M' or M'' isomorphic to $A/\mathfrak{p}$ (Lemma 7.6.1). If this proposition were true, we would expect to find such a submodule in the "part of (7.34) spanned by m". So we consider instead the exact sequence

$$0 \longrightarrow Am \cap M' \longrightarrow Am \longrightarrow Am/(Am \cap M') \longrightarrow 0,$$

noting that the three modules appearing here are submodules of the corresponding modules in (7.34). So by Proposition 7.6.7 it suffices to prove the result in this "special case", which can be rewritten as ($Am \cong A/\mathfrak{p}$)

$$0 \longrightarrow I/\mathfrak{p} \longrightarrow A/\mathfrak{p} \longrightarrow A/I \longrightarrow 0$$

where I is the annihilator of m considered as an element of the module $Am/(Am \cap M')$. For convenience, let $B = A/\mathfrak{p}$ (an integral domain) and $J = I/\mathfrak{p}$, so we rewrite the exact sequence further as the top row of

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & B/J \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0.
\end{tikzcd}$$

Now localize the top row of B-modules at $(0) \subseteq B$, so it becomes an exact sequence of vector spaces over the fraction field $K(B)$, and the central element is one-dimensional:

$$0 \longrightarrow J \otimes K(B) \longrightarrow K(B) \longrightarrow (B/J) \otimes K(B) \longrightarrow 0.$$

Thus one of the outside terms $J \otimes K(B)$ and $(B/J) \otimes K(B)$ has a nonzero element.

If $\dim_{K(B)} J \otimes_B K(B) = 1$, then exists $x \in J$ such that $\operatorname{Ann}_B(x) = 0$. (if not, for all $x \in J$, $\operatorname{Ann}_B(x) \neq 0$. Let $x \otimes b \in J \otimes_B K(B)$ and $c \in \operatorname{Ann}_B(x)$, then $x \otimes_B b = cx \otimes (b/c) = 0$, a contradiction.) Note that $B = A/\mathfrak{p}$, then $\operatorname{Ann}_A(x) = \mathfrak{p}$, which implies that $\mathfrak{p} \in \operatorname{Ass} J \subseteq \operatorname{Ass} M'$.

If $\dim_{K(B)}(B/J) \otimes K(B) = 1$, then exists $y \in B/J$ such that $\operatorname{Ann}_B(y) = 0$. Note that $B = A/\mathfrak{p}$, then $\operatorname{Ann}_A(y) = \mathfrak{p}$, which implies that $\mathfrak{p} \in \operatorname{Ass} B/J \subseteq \operatorname{Ass} M''$.

Hence, $\mathfrak{p} \in \operatorname{Ass} M' \cup \operatorname{Ass} M''$, and therefore $\operatorname{Ass} M \subseteq \operatorname{Ass} M' \cup \operatorname{Ass} M''$. □

Example 7.6 **Cautionary example.** The short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Note that $\operatorname{Ass} \mathbb{Z} = \{(0)\}$ and $\operatorname{Ass}(\mathbb{Z}/2) = \{(2)\}$, in this case $\operatorname{Ass} M \subsetneq M = \operatorname{Ass} M' \cup \operatorname{Ass} M''$. However, sometimes we can still ensure some associated primes of M'' lift to associated primes of M.

7.6.8 Finitely generated modules over Noetherian rings have finitely many associated points (primes)

Proposition 7.6.13

Suppose that M is a finitely generated module over a Noetherian ring A. Then M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \tag{7.36}$$

where $M_{i+1}/M_i \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Also, every associated prime of M appears as one of the $\mathfrak{p}_i$.

Proof Since M is finitely generated module over a Noetherian ring A, M is Noetherian module. We proceed by induction on n. The $n = 0$ case is trivial. Assume we have chosen $0 = M_0 \subseteq \cdots \subseteq M_{k-1}$ with $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ where $\mathfrak{p}_i \in \operatorname{Spec} A$. If $M_{k-1} \neq M$, then $M/M_{k-1} \neq 0$. Since M is Noetherian, M/M_{k-1} is Noetherian, by Proposition 7.6.9, $\operatorname{Ass}_A(M/M_{k-1})$ is not empty, so exists $\mathfrak{p}_k \in \operatorname{Ass}_A(M/M_{k-1}) \subseteq \operatorname{Spec} A$. By Lemma 7.6.1, $A/\mathfrak{p}_k \hookrightarrow M/M_{k-1}$. Pick a submodule of M, named M_k, such that $M_k/M_{k-1} \cong A/\mathfrak{p}_k$, then we get a chain in M:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-1} \subseteq M_k,$$

where $M_i/M_{i-1} \cong A/\mathfrak{p}_i$. Repeat this procedure, which terminates after a finite number of steps, since M is Noetherian.

By Proposition 7.6.12, from chain (7.36), we have

$$\begin{aligned}
\operatorname{Ass} M &\subseteq \operatorname{Ass}(M_{n-1}) \cup \operatorname{Ass}(M/M_{n-1}) \\
&\subseteq \operatorname{Ass}(M_{n-2}) \cup \operatorname{Ass}(M_{n-1}/M_{n-2}) \cup \operatorname{Ass}(M/M_{n-1}) \\
&\subseteq \cdots \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}(M_i/M_{i-1}) \\
&= \bigcup_{i=1}^{n} \operatorname{Ass}(A/\mathfrak{p}_i).
\end{aligned}$$

We now calculate $\operatorname{Ass}(A/\mathfrak{p}_i)$. Let $\mathfrak{q} \in \operatorname{Ass}(A/\mathfrak{p}_i)$, then exists nonzero $a \in A/\mathfrak{p}_i$ such that $\mathfrak{q} = \operatorname{Ann}_A(a)$. $\mathfrak{q}$ can be seen as an ideal of A which contains $\mathfrak{p}_i$. Hence $\mathfrak{q} \cdot a \subseteq \mathfrak{p}_i$, since $\mathfrak{p}_i$ is prime, $\mathfrak{q} \subseteq \mathfrak{p}_i$, and therefore $\mathfrak{q} = \mathfrak{p}_i$. It follows that $\operatorname{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$. Hence $\bigcup_{i=1}^{n} \operatorname{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, and therefore $\operatorname{Ass} M \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, as we desired. □

Corollary 7.6.1

Suppose an A-module M has a finite filtration (7.36), with no assumptions of finite generation or Noetherianity. Then every associated prime of M appears as one of the $\mathfrak{p}_i$.

Proof Recall the proof of Proposition 7.6.13, we don't use assumptions of finite generation or Noetherianity, hence the proof works in this Corollary, then we done. □

Corollary 7.6.2

If M is finitely generated over a Noetherian ring, M has finitely many associated points (primes).

Proof Proposition 7.6.13. □

Remark **Caution: Non-associated prime ideals may unavoidably appear among the quotients in (7.36).** Example 7.6 shows that the non-associated prime ideals may be among the quotients in (7.36), although in that case it is because the filtration was chosen unwisely. But better choices will not always remedy the problem:

Example 7.7 Consider the module $M = (x, y) \subseteq A = k[x, y]$ over the ring A. Then any filtration (7.36) of M contains a quotient $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is not an associated prime.

Proof We first calculate $\operatorname{Ass} M$. Let nonzero element $m \in M$, consider $\operatorname{Ann}(m) \subseteq A$. Let $a \in \operatorname{Ann}(m)$, then $am = 0$. Since $k[x, y]$ is integral domain and $m \neq 0$, $a = 0$. Hence $\operatorname{Ann}(m) = (0)$ for all nonzero $m \in M$. Since $k[x, y]$ is integral domain, $(0) \in \operatorname{Spec} A$, and therefore $\operatorname{Ass}(M) = \{(0)\}$.

Let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

be any filtration of M, where $M_i/M_{i-1} \cong A/\mathfrak{p}_i$. If for any quotient $A/\mathfrak{p}_i$ prime $\mathfrak{p}_i$ is associated point, we have $A/\mathfrak{p}_i = A/(0) = A$. Hence $M_1/M_0 = M_1 \cong A$. But $M_n = (x, y) \subsetneq A$, a contradiction. it follows that any filtration of M contains a quotient $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is not an associated prime. $\square$

Remark However, not all is lost: Proposition 7.6.19 will show that for any quotient $A/\mathfrak{p}_i$ in any filtration (7.36) of M, $\mathfrak{p}_i$ must contain an associated prime of M.

7.6.9 Associated points behave fairly well in exact sequences, continued

Proposition 7.6.14

Suppose A is a Noetherian ring. Consider the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of A-modules. Suppose $\mathfrak{p} \in \operatorname{Ass} M''$, but $\mathfrak{p} \notin \operatorname{Supp} M'$ (a stronger hypothesis than $\mathfrak{p} \notin \operatorname{Ass} M'$). Then $\mathfrak{p} \in \operatorname{Ass} M$.

Proof Let $\mathfrak{p} \in \operatorname{Ass} M'$, then $\mathfrak{p} = \operatorname{Ann} m''$ for some $m'' \in M''$. Choose a lift of $m \in M$ of $m'' \in M''$. Consider the "inclusion of short exact sequences"

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(Am \to Am'') \arrow[r] \arrow[d, hook] & Am \arrow[r] \arrow[d, hook] & Am'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0.
\end{tikzcd}$$

As $\operatorname{Supp}(\operatorname{Ker}(Am \to Am'')) \subseteq \operatorname{Supp} M'$, $\operatorname{Ass}(Am) \subseteq \operatorname{Ass} M$, and $\operatorname{Ass}(Am'') \subseteq \operatorname{Ass} M''$ (Proposition 7.6.2 and Proposition 7.6.7), we have reduced to considering the top row instead of the bottom row. The top row can be conveniently rewritten as

$$0 \longrightarrow \mathfrak{p}/I \longrightarrow A/I \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

(here $I = \operatorname{Ann}(m)$) where our hypothesis translates to $[\mathfrak{p}] \notin \operatorname{Supp}(\mathfrak{p}/I)$. For convenience, let $B = A/I$ so we may now consider the sequence

$$0 \longrightarrow \mathfrak{q} \longrightarrow B \longrightarrow B/\mathfrak{q} \longrightarrow 0,$$

where $\mathfrak{q} = \mathfrak{p}/I$ is prime of B, with $[\mathfrak{p}] \notin \operatorname{Supp}_A(\mathfrak{q})$. Since $B_\mathfrak{q} \cong \frac{\mathfrak{p}A_\mathfrak{p}}{IA_\mathfrak{p}}$, we have $[\mathfrak{q}] \notin \operatorname{Supp}_B \mathfrak{q}$.

From $[\mathfrak{q}] \notin \operatorname{Supp}_B \mathfrak{q}$, there is an element b of B that vanishes on $\operatorname{Supp} \mathfrak{q}$ but doesn't vanish at $[\mathfrak{q}]$. Translation: (i) b lies in all the primes of $\operatorname{Supp} \mathfrak{q}$, but (ii) $b \notin \mathfrak{q}$. Then from (i) there is some power b^n of b that annihilates all elements of $\mathfrak{q}$ (Proposition 7.6.3, which applies because $\mathfrak{q}$ is finitely generated, which in turn follows from Noetherianity of A). From (ii), $b^n \notin \mathfrak{q}$.

Then $\operatorname{Ann}(b^n)$ contains $\mathfrak{q}$ from (i), i.e., $\mathfrak{q} \subseteq \operatorname{Ann}(b^n)$. If exists nonzero $x \in \operatorname{Ann}(b^n) \setminus \mathfrak{q}$, then $xb^n = 0$. Since $\mathfrak{q}$ is prime, note that $b^n \notin \mathfrak{q}$ (from (ii)) and $x \notin \mathfrak{q}$, we have $xb^n = 0 \notin \mathfrak{p}$, a contradiction. Hence $\mathfrak{q} = \operatorname{Ann}(b^n)$, and therefore $\mathfrak{q}$ is an associated prime of B, which (unwinding our argument) shows that $\mathfrak{p}$ is an associated prime of M. □

7.6.10 Minimal primes are associated

Proposition 7.6.15 (Minimal primes ("irreducible components") are associated)

Suppose M is a finitely generated module over Noetherian A, and $\mathfrak{p} \subseteq A$ is a prime ideal corresponding to an irreducible component of $\operatorname{Supp} M \subseteq \operatorname{Spec} A$*. Then* $[\mathfrak{p}] \in \operatorname{Ass} M$.

Proof By Proposition 7.6.13, M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

where $M_{i+1}/M_i \cong A/\mathfrak{p}_i$. Then we have an exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow A/\mathfrak{p}_i \longrightarrow 0.$$

By Proposition 7.6.2, we have $\operatorname{Supp} M_i = \operatorname{Supp} M_{i-1} \cup \operatorname{Supp}(A/\mathfrak{p}_i)$. Hence $\operatorname{Supp} M = \bigcup_{i=1}^n \operatorname{Supp}(A/\mathfrak{p}_i)$. Consider $\operatorname{Supp}(A/\mathfrak{p}_i)$, let $\mathfrak{q} \in \operatorname{Supp}(A/\mathfrak{p}_i)$, then $\frac{A_\mathfrak{q}}{\mathfrak{p}_i A_\mathfrak{q}} \neq 0$, it follows that $\mathfrak{p}_i A_\mathfrak{q}$ must be an ideal of $A_\mathfrak{q}$, hence $\mathfrak{p}_i \subseteq \mathfrak{q}$, which implies that $\operatorname{Supp}(A/\mathfrak{p}_i) \subseteq V(\mathfrak{p}_i)$. Conversely, let $\mathfrak{q} \in V(\mathfrak{p}_i)$, then $\mathfrak{p}_i \subseteq \mathfrak{q}$. Hence, $\mathfrak{p}_i A_\mathfrak{q}$ is an ideal of $A_\mathfrak{q}$, and therefore $(A/\mathfrak{p}_i)_\mathfrak{q} = \frac{A_\mathfrak{q}}{\mathfrak{p}_i A_\mathfrak{q}} \neq 0$. It follows that $\mathfrak{q} \in \operatorname{Supp}(A/\mathfrak{p}_i)$. Hence $\operatorname{Supp}(A/\mathfrak{p}_i) = V(\mathfrak{p}_i)$, and therefore $\operatorname{Supp} M = \bigcup_{i=1}^n V(\mathfrak{p}_i)$. Since $\mathfrak{p} \in \operatorname{Spec} A$ corresponding to an irreducible component of $\operatorname{Supp} M \subseteq \operatorname{Spec} A$, by Proposition 4.7.4, $\mathfrak{p}$ is minimal prime in $\operatorname{Supp} M$, hence $\mathfrak{p} = \mathfrak{p}_i$ for some i. Let's assume that i is taken minimally. We have an exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow A/\mathfrak{p} \longrightarrow 0. \tag{7.37}$$

We want to show: (i) $\mathfrak{p} \in \operatorname{Ass}(A/\mathfrak{p})$; (ii) $\mathfrak{p} \notin \operatorname{Supp} M_{i-1}$.

For (i), let $\operatorname{Ann}(\overline{a}) \in \operatorname{Ass}(A/\mathfrak{p}) \subseteq \operatorname{Spec} A$, where $\overline{a} \in A/\mathfrak{p}$ nonzero, then $\operatorname{Ann}(\overline{a}) \cdot a \subseteq \mathfrak{p}$. Since $a \notin \mathfrak{p}$, and $\mathfrak{p}$ is prime, we have $\operatorname{Ann}(\overline{a}) \subseteq \mathfrak{p}$. Since $\mathfrak{p} \cdot a \subseteq \mathfrak{p}$, we have $\operatorname{Ann}(\overline{a}) = \mathfrak{p}$. Hence $\operatorname{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$. For (ii), by the assumption "i is taken minimally", we know that $\mathfrak{p} \notin \operatorname{Supp} M_{i-1}$.

By Proposition 7.6.14, $[\mathfrak{p}] \in \operatorname{Ass} M$, as we desired. □

Remark **Non-Noetherian Remark.** By combining Proposition 7.6.11 with Proposition 7.6.15, we see that if A is a Noetherian ring, then any element of any minimal prime $\mathfrak{p}$ is a zerodivisor. This is true without Noetherian hypotheses: suppose $s \in \mathfrak{p}$. Then by minimality of $\mathfrak{p}$, $\mathfrak{p}A_\mathfrak{p}$ is the unique prime ideal in $A_\mathfrak{p}$, so the element $s/1$ of $A_\mathfrak{p}$ is nilpotent (because it is contained in all prime ideals of $A_\mathfrak{p}$, Theorem 4.2.3). Thus for some $t \in A \setminus \mathfrak{p}$, $ts^n = 0$, so s is a zerodivisor in A. We will use this in Chapter 13.

Proposition 7.6.16

Suppose M is a finitely generated module over a Noetherian ring A. Then $\operatorname{Supp} M = \overline{\operatorname{Ass} M}$.

Proof By Proposition 7.6.2 (b), $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$. By Proposition 7.6.7 (b) and , we have $\overline{\operatorname{Ass} M} \subseteq \operatorname{Supp} M$. By Corollary 7.6.2, M has finitely many associated points, say $\operatorname{Ass} M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, hence

$$\overline{\operatorname{Ass} M} = \overline{\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}} = \bigcup_{i=1}^n \overline{[\mathfrak{p}_i]}.$$

Let $\mathfrak{q} \in \operatorname{Supp} M$, then $\mathfrak{q}$ must belong to an irreducible component of M, by Proposition 7.6.15, the corresponding minimal ideal of irreducible component is associated ideal, which implies that $\mathfrak{q} \in \overline{[\mathfrak{p}_i]}$ for some i. Hence

$\mathfrak{q} \in \overline{\operatorname{Ass} M}$. Hence $\operatorname{Supp} M = \overline{\operatorname{Ass} M}$. □

Proposition 7.6.17

The locus of points $[\mathfrak{p}]$ where $A_\mathfrak{p}$ is nonreduced (Definition 4.2.3) is the support of the nilradical $\operatorname{Supp} \mathfrak{N}$. Hence the "reduced locus" of a locally Noetherian scheme is open.

Proof Let $\mathfrak{R} = \{[\mathfrak{p}] \in \operatorname{Spec} A : A_\mathfrak{p} \text{ is nonreduced}\}$. Let $[\mathfrak{p}] \in \mathfrak{R}$, then $A_\mathfrak{p}$ has nonzero nilpotents, which implies that $\mathfrak{N}(A_\mathfrak{p}) = \mathfrak{N}(A)_\mathfrak{p} \neq 0$. Hence $[\mathfrak{p}] \in \operatorname{Supp} \mathfrak{N}(A)$. Conversely, if $[\mathfrak{p}] \in \operatorname{Supp} \mathfrak{N}(A)$, then $\mathfrak{N}(A)_\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p}) \neq 0$, which implies that $A_\mathfrak{p}$ is reduced. Hence $\mathfrak{R} = \operatorname{Supp} \mathfrak{N}(A)$.

Let X be a locally Noetherian scheme, then $X = \bigcup_i \operatorname{Spec} A_i$ where A is Noetherian. Say $\mathfrak{R} = \{[\mathfrak{p}] \in X : \mathfrak{N}(\mathscr{O}_{X,[\mathfrak{p}]}) = 0\}$ be the reduced locus of X. Note that

$$\mathfrak{R} = \bigcup_i \{[\mathfrak{p}] \in \operatorname{Spec} A_i : \mathfrak{N}((A_i)_\mathfrak{p}) = 0\} := \bigcup_i \mathfrak{R}_i,$$

by above discussion, we know that $\mathfrak{R}_i = \operatorname{Spec} A_i - \operatorname{Supp} \mathfrak{N}(A_i)$ which is open in $\operatorname{Spec} A_i$, and therefore is open in X. Hence $\mathfrak{R}$ is open. □

The following justifies a simple way of thinking about associated primes of a ring.

Proposition 7.6.18

A prime ideal $\mathfrak{p} \subseteq A$ is an associated prime of a Noetherian ring A if and only if there is $f \in A$ such that $\operatorname{Supp} f = V(\mathfrak{p}) = \overline{[\mathfrak{p}]}$.

Proof Let $\mathfrak{p}$ be an associated prime of A, then $\mathfrak{p} = \operatorname{Ann}(f)$ for some $f \in A$. By Proposition 7.6.1 and Lemma 4.6.1, $\operatorname{Supp} f = V(\operatorname{Ann}(f)) = V(\mathfrak{p}) = \overline{[\mathfrak{p}]}$. Conversely, if there is $f \in A$ such that $\operatorname{Supp} f = V(\mathfrak{p})$. By Proposition 7.6.1, we have $\operatorname{Supp} f = V(\operatorname{Ann}(f))$. By the proof in Proposition 7.5.5, $\operatorname{Ann}(f)$ is maximal ideal of A, and therefore is prime. Hence $\mathfrak{p} = \operatorname{Ann}(f)$, which implies that $\mathfrak{p} \in \operatorname{Ass} A$. □

Proposition 7.6.19

For each quotient in the filtration (7.36) of M, $\operatorname{Supp} A/\mathfrak{p}_i = \overline{[\mathfrak{p}_i]} \subseteq \operatorname{Supp} M$, and that every $\mathfrak{p}_i$ contains a minimal ideal (associated ideal).

Proof Let $\mathfrak{q} \in \operatorname{Supp} A/\mathfrak{p}_i$, then $(A/\mathfrak{p}_i)_\mathfrak{q} = A_\mathfrak{q}/\mathfrak{p}_i A_\mathfrak{q} \neq 0$, hence $\mathfrak{p}_i A_\mathfrak{q}$ must be an ideal of $A_\mathfrak{q}$. Hence $\mathfrak{p}_i \subseteq \mathfrak{q}$, which implies that $\operatorname{Supp}(A/\mathfrak{p}_i) = V(\mathfrak{p}_i) = \overline{[\mathfrak{p}_i]}$. Since $(M_i/M_{i-1})_{\mathfrak{p}_i} \cong (M_i)_{\mathfrak{p}_i}/(M_{i-1})_{\mathfrak{p}_i} \cong A_{\mathfrak{p}_i}/\mathfrak{p}_i A_{\mathfrak{p}_i} \neq 0$, $(M_i)_{\mathfrak{p}_i} \neq 0$, it follows that $\mathfrak{p}_i \in \operatorname{Supp} M_i \subseteq \operatorname{Supp} M$. By Proposition 7.6.2 (b), $\operatorname{Supp} M$ is closed subset of $\operatorname{Spec} A$, hence $\overline{[\mathfrak{p}_i]} \subseteq \operatorname{Supp} M$. Since $\mathfrak{p}_i \in \operatorname{Supp} M$, $\mathfrak{p}_i$ must belong to an irreducible components of $\operatorname{Supp} M$, say $V(\mathfrak{q})$, by Proposition 7.6.15, $[\mathfrak{q}] \in \operatorname{Ass} M$. Since $\mathfrak{p}_i \in V(\mathfrak{q})$, we have $\mathfrak{q} \subseteq \mathfrak{p}_i$. Since $V(\mathfrak{q})$ is irreducible component of $\operatorname{Supp} M$, by Proposition 4.7.4, $\mathfrak{q}$ is also minimal prime in $\operatorname{Supp} M$. □

7.6.11 "Support" and "associated points" commute with localization

Suppose S is a multiplicative subset of A, and $\mathfrak{p} \subseteq A$ is a prime ideal not meeting S, so (abusing notation slightly) $[\mathfrak{p}] \in \operatorname{Spec} S^{-1}A \subseteq \operatorname{Spec} A$ (Proposition 4.2.3).

Proposition 7.6.20 (Supp **commutes with localization)**

For any A-module M, $\operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A = \operatorname{Supp}_{S^{-1}A} S^{-1}M$.

Proof Let $S^{-1}\mathfrak{p} \in \operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A$. Consider $(S^{-1}M)_{S^{-1}\mathfrak{p}}$, since localization commutes with tensor product, we have

$$\begin{aligned}(S^{-1}M)_{S^{-1}\mathfrak{p}} &\cong (S^{-1}A)_{S^{-1}\mathfrak{p}} \otimes_{S^{-1}A} (S^{-1}A \otimes_A M)\\ &\cong ((S^{-1}A)_{S^{-1}\mathfrak{p}} \otimes_{S^{-1}A} S^{-1}A) \otimes_A M\\ &\cong (S^{-1}A)_{S^{-1}\mathfrak{p}} \otimes_A M.\end{aligned}$$

Note that $S \subseteq A - \mathfrak{p}$, the image of S in $A_\mathfrak{p}$ is also S. Hence $(S^{-1}A)_{S^{-1}\mathfrak{p}} \cong S^{-1}(A_\mathfrak{p}) \cong A_\mathfrak{p}$. It follows that

$$(S^{-1}M)_{S^{-1}\mathfrak{p}} \cong (S^{-1}A)_{S^{-1}\mathfrak{p}} \otimes_A M \cong A_\mathfrak{p} \otimes_A M \cong M_\mathfrak{p} \neq 0.$$

Hence $\operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A \subseteq \operatorname{Supp}_{S^{-1}A} S^{-1}M$.

Conversely, let $\mathfrak{q} \in \operatorname{Supp}_{S^{-1}A} S^{-1}M$, we may assume that $\mathfrak{q} = S^{-1}\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec} A$. By above discussion,

$$M_\mathfrak{p} \cong (S^{-1}M)_{S^{-1}\mathfrak{p}} \neq 0,$$

which implies that $S^{-1}\mathfrak{p} \in \operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A$. Hence $\operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A = \operatorname{Supp}_{S^{-1}A} S^{-1}M$. □

Lemma 7.6.2

Let M be a finitely generated A-module, S is a multiplicatively closed subset of A. Then $S^{-1}(\operatorname{Ann} M) = \operatorname{Ann}(S^{-1}M)$.

Proof If this is true for two A-modules M, N. Then it is true for $M + N$:

$$\begin{aligned}S^{-1}(\operatorname{Ann}(M+N)) =& S^{-1}(\operatorname{Ann} M + \operatorname{Ann} N)\\ =& S^{-1}(\operatorname{Ann} M) \cap S^{-1}(\operatorname{Ann} N)\\ =& \operatorname{Ann}(S^{-1}M) \cap \operatorname{Ann}(S^{-1}N)\\ =& \operatorname{Ann}(S^{-1}M + S^{-1}N)\\ =& \operatorname{Ann}(S^{-1}(M+N))\end{aligned}$$

Since M is finitely generated A-module, M can be written as $M = Ax_1 + \cdots + Ax_n$. Hence it suffices to show that Lemma holds for M generated by a single element. By Proposition 7.5.5, we know that $M = Ax \cong A/\operatorname{Ann}(x)$. Hence

$$\operatorname{Ann}(S^{-1}M) = \operatorname{Ann}(S^{-1}A/S^{-1}\operatorname{Ann}(x)) = S^{-1}\operatorname{Ann}(x) = S^{-1}\operatorname{Ann}(M),$$

as we desired. □

Proposition 7.6.21 (Ass **commutes with localization)**

Let M be a finitely generated module over a Noetherian ring A. Then $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A = \operatorname{Ass}_{S^{-1}A} S^{-1}M$.

Proof We first show that $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A \subseteq \operatorname{Ass}_{S^{-1}A} S^{-1}M$. If $\mathfrak{p} \in \operatorname{Ass}_A M$, by Lemma 7.6.1, then we have an injection $A/\mathfrak{p} \hookrightarrow M$. Since $\mathfrak{p} \in \operatorname{Spec} S^{-1}A$, $\mathfrak{p} \cap S = \varnothing$, hence the image of S in $A/\mathfrak{p}$ is also S. Localizing by S (which preserves injectivity), we have $S^{-1}(A/\mathfrak{p}) \cong S^{-1}A/S^{-1}\mathfrak{p} \hookrightarrow S^{-1}M$. By Lemma 7.6.1, $S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}A} S^{-1}M$. Hence $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A \subseteq \operatorname{Ass}_{S^{-1}A} S^{-1}M$.

We next show that $\operatorname{Ass}_{S^{-1}A} S^{-1}M \subseteq \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$. Suppose $\mathfrak{q} := S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}A} S^{-1}M$, so $\mathfrak{q} = \operatorname{Ann}_{S^{-1}A}(m/s)$, for some $s \in S$, and $m \in M$. Since the elements of S are the units in $S^{-1}A$, we have that $\mathfrak{q} = S^{-1}\mathfrak{p} = \operatorname{Ann}_{S^{-1}A} m/1$, by Lemma 7.6.2, $\mathfrak{q} = S^{-1}(\operatorname{Ann}_A m)$, moreover, $\operatorname{Ann}_A m$ is prime. By Proposition 7.6.1, we have $V(\mathfrak{q}) = V(\operatorname{Ann}_{S^{-1}A} m/1) = \operatorname{Supp}_{S^{-1}A} m/1$. By Proposition 7.6.20, $V(\mathfrak{q}) = \operatorname{Supp}_{S^{-1}A} m/1 = \operatorname{Supp}_A m \cap \operatorname{Spec} S^{-1}A$. Hence $\mathfrak{p} \in \operatorname{Supp}_A m$, we claim that $\mathfrak{p}$ is minimal prime in $\operatorname{Supp}_A m$. By Proposition 7.6.1, $\operatorname{Supp}_A m = V(\operatorname{Ann}_A m)$, then $\mathfrak{p} \in V(\operatorname{Ann}_A m)$. If exists $\mathfrak{p}' \subsetneq \mathfrak{p}$ in $V(\operatorname{Ann}_A m)$, then $\mathfrak{p}' \cap S \neq \varnothing$ (if not $\mathfrak{p}' \cap S = \varnothing$, then $S^{-1}\mathfrak{p} = S^{-1}(\operatorname{Ann}_A m) \subseteq S^{-1}\mathfrak{p}' \subsetneq S^{-1}\mathfrak{p}$, a contradiction). But $\mathfrak{p}' \cap S \subseteq \mathfrak{p} \cap S = \varnothing$, a contradiction. Hence $\mathfrak{p}$ is minimal prime in $\operatorname{Supp}_A m$. By Proposition 4.7.4, $\overline{[\mathfrak{p}]}$ is irreducible component of $\operatorname{Supp}_A m$. By Proposition 7.6.15, $[\mathfrak{p}] \in \operatorname{Ass} Am \subseteq \operatorname{Ass} M$, which implies that $\operatorname{Ass}_{S^{-1}A} S^{-1}M \subseteq \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$. □

7.6.12 Embedded points/primes

Definition 7.6.4 (Embedded points)

The associated points that are not the generic points of irreducible components of $\operatorname{Supp} M$ *are called* ***embedded points****, and their corresponding primes are called* ***embedded primes****.*

The motivation for this language is their geometric incarnation as embedded points. For example, the scheme of Figure7.1 has three embedded points (the corresponding ring has three embedded primes).

Remark Proposition 7.6.6 translate to "hypersurfaces in $\mathbb{A}_k^n$ have no embedded points". More generally, if A is a UFD, then $\operatorname{Spec} A/(f)$ has no embedded points for any $f \in A$. Generalizing in a different direction, we will see that "complete intersections have no embedded points".

Proposition 7.6.22

Suppose A is a reduced ring (i.e., $\mathfrak{N}(A) = 0$). Then $\operatorname{Spec} A$ *has no embedded primes.*

Proof Let $\mathfrak{p} = \operatorname{Ann} a \in \operatorname{Ass} A$ be an embedded point, then $\overline{[\mathfrak{p}]}$ is not an irreducible components of $\operatorname{Supp} A$, and therefore $\mathfrak{p}$ is not minimal prime of A (since $\operatorname{Supp} A = V(\operatorname{Ann} A) = V((0)) = \operatorname{Spec} A$). We want to show that there exists an element b of $\mathfrak{p}$ not contained in any of the minimal primes of A. It suffices to show that $\mathfrak{p} - \left(\bigcup_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}\right) \neq \varnothing$. If not, we have $\mathfrak{p} - \left(\bigcup_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}\right) = \varnothing$, i.e., $\mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}$. By Atiyah-MacDonald [1] Proposition 1.11, $\mathfrak{p} \subseteq \mathfrak{q}$ for some minimal prime $\mathfrak{q}$, a contradiction. Hence $\mathfrak{p} - \left(\bigcup_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}\right) \neq \varnothing$. Pick $b \in \mathfrak{p} - \left(\bigcup_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}\right)$, since $b \in \operatorname{Ann} a$, $ab = 0$. Since b not contained in any minimal primes $\mathfrak{q}$ of A, note that $ab = 0 \in \mathfrak{q}$, we have $a \in \mathfrak{p}$, which implies that $a \in \bigcap_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q}$. Since

$$\mathfrak{N}(A) = \bigcap_{\mathfrak{q} \in \operatorname{Spec} A} \mathfrak{q} = \bigcap_{\mathfrak{q} \text{ is minimal prime}} \mathfrak{q} = 0,$$

$a = 0$, and therefore $\mathfrak{p} = \operatorname{Ann} 0$, a contradiction. Hence $\mathfrak{p}$ is not an embedded point. □

Remark Thus reduced rings have no embedded primes. In the proof of Proposition 7.6.22, if A is not reduced ring, let $\mathfrak{p} = \operatorname{Ann} a$ be an embedded point, then $a \in \mathfrak{N}(A)$. Hence, we have:

Corollary 7.6.3

The only elements of a ring that an embedded prime can annihilate are nilpotents. More precisely, if A is nonreduced ring, any embedded point $\mathfrak{p}$ is annihilator of a nilpotent, i.e., $\mathfrak{p} = \operatorname{Ann} a$ where $a \in \mathfrak{N}(A)$.

Remark The converse to Proposition 7.6.22 is false. Rings without embedded primes can still have nilpotents — witness $k[x]/(x^2)$. Only associated points of $k[x]/(x^2)$ is $[(x)]$, note that $\overline{[(x)]} = [(x)]$, $[(x)]$ is not embedded point. But $\mathfrak{N}(k[x]/(x^2)) = (x^2) \neq 0$.

Proposition 7.6.23

If $\mathfrak{p}$ is an embedded prime of a ring A, then $A_{\mathfrak{p}}$ is nonreduced.

Proof If $A_{\mathfrak{p}}$ is reduced, by Proposition 7.6.22, $\operatorname{Spec} A_{\mathfrak{p}}$ has no embedded prime. In particular, $\mathfrak{p}A_{\mathfrak{p}}$ is minimal prime. Note that $\mathfrak{p}A_{\mathfrak{p}}$ is maximal ideal, $\operatorname{Spec} A_{\mathfrak{p}} = \{[\mathfrak{p}A_{\mathfrak{p}}]\}$. Hence $\mathfrak{p}$ must be minimal prime, contradicts to the fact that $\mathfrak{p}$ is an embedded prime of A. □

7.6.13 Get your hands dirty: Explicit algebraic exercises

✍ **Exercise 7.1** (see Figure 5.4)

(a) Suppose f is a function on $\operatorname{Spec} k[x,y]/(y^2,xy)$, i.e., $f \in k[x,y]/(y^2,xy)$. Show that $\operatorname{Supp} f$ is either the empty set, or the origin, or the entire space. Hence find the associated points of $\operatorname{Spec} k[x,y]/(y^2,xy)$.

(b) Show explicitly by hand that $f \in k[x,y]/(y^2,xy)$ is a zerodivisor if and only if $f(0,0)=0$.

Proof

(a) By Proposition 7.6.1, we have $\operatorname{Supp} f = V(\operatorname{Ann} f)$. Since $f \in k[x,y]/(y^2,xy)$, we may assume that $f = p + qy$ where $p \in k[x]$ and $q \in k$. Now, we calculate $\operatorname{Supp} f$,

(i) if $f=0$, then $\operatorname{Ann} f = k[x,y]/(y^2,xy)$, hence $\operatorname{Supp} f = V\left(k[x,y]/(y^2,xy)\right) = \varnothing$;

(ii) if $f = p+qy \neq 0$. Let $g \in \operatorname{Ann} f$, say $g = a+by$, then

$$fg = (p+qy)(a+by) = ap + (aq+bp)y = 0$$

in $k[x,y]/(y^2,xy)$. Hence $ap=0$ and $aq+bp \in (x,y)$.

If $p=0$, then $fg = aqy = 0$, since $q \in k$, we have $a \in (x)$. Hence

$$\operatorname{Ann} f = \{a+by \in k[x,y]/(y^2,xy) : a \in (x), b \in k\} = (x,y).$$

We have $\operatorname{Supp} f = V((x,y)) = (x,y)$.

If $p \neq 0$, then $a=0$, hence $bpy=0$. If $p \in (x)$, then $\operatorname{Ann}(f) = (y)$; if $p \notin (x)$, then $b=0$, which implies that $\operatorname{Ann} f = (0)$.

Hence $\operatorname{Supp} f = V((0)) = \operatorname{Spec} k[x,y]/(y^2,xy)$ or $\operatorname{Supp} f = V((y)) = \operatorname{Spec} k[x,y]/(y^2,xy)$.

By above discussion, we know that $\operatorname{Ass} k[x,y]/(y^2,xy) = \{(y),(x,y)\}$.

(b) If f is zerodivisor, by Proposition 7.6.11, f vanishes at $[(y)]$ or $[(x,y)]$, i.e., $f \in (y)$ or $f \in (x,y)$. Hence $f(0,0)=0$.

Conversely, if $f(0,0)=0$. We may write $f = p(x)+qy$ where $q \in k$. Hence $f(0,0) = p(0) = 0$, it follows that $p(x) = xp'(x)$. Pick $y \in k[x,y]/(y^2,xy)$, then

$$yf = xyp'(x) + qy^2 = 0$$

in $k[x,y]/(y^2,xy)$, which implies that f is zerodivisor.

□

✍ **Exercise 7.2** Suppose $X = \operatorname{Spec} \mathbb{C}[x,y]/I$, and that the associated points of X are $[(y-x^2)]$, $[(x-1,y-1)]$, and $[(x-2,y-2)]$. (Exercise 7.2 will verify that such an X actually exists.)

(a) Sketch X as a subset of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, including fuzz.

(b) Do you have enough information to know if X is reduced?

(c) Do you have enough information to know if $x+y-2$ is a zerodivisor? How about $x+y-3$? How about $y-x^2$?

(d) Let $I = (y-x^2)^3 \cap (x-1,y-1)^{15} \cap (x-2,y-2)$. Show that $X = \operatorname{Spec} \mathbb{C}[x,y]/I$ satisfies the hypotheses of this exercise. (Rhetorical question: Is there a "smaller" example? Is there a "smallest"?)

Solution

(a) *By the condition, we know that* $\operatorname{Ass}\mathbb{C}[x,y]/I = \{[(y-x^2)], [(x-1,y-1)], [(x-2,y-2)]\}$. *By Proposition 7.6.16, we have*

$$\begin{aligned}\operatorname{Supp}\mathbb{C}[x,y]/I =&\overline{\{[(y-x^2)], [(x-1,y-1)], [(x-2,y-2)]\}}\\ =&\overline{[(y-x^2)]}\cup\overline{[(x-1,y-1)]}\cup\overline{[(x-2,y-2)]}\\ =&\overline{[(y-x^2)]}\cup\{[(x-2,y-2)]\},\end{aligned}$$

it follows that $\overline{[(y-x^2)]}$ *and* $\{[(x-2,y-2)]\}$ *are two irreducible component of* $\operatorname{Supp}\mathbb{C}[x,y]/I$. *Hence* $[(x-1,y-1)]$ *is the only embedded points of* X. *By Proposition 7.6.23,* $(\mathbb{C}[x,y]/I)_{(x-1,y-1)}$ *is nonreduced. Hence we draw fuzz at* $[(x-1,y-1)]$.

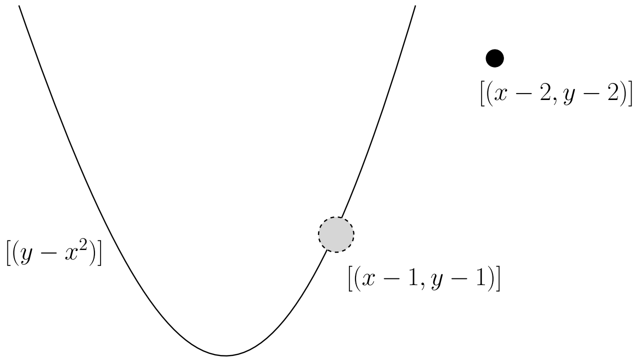


Figure 7.2: Picture of X

(b) *If* X *is reduced, by Proposition 6.2.2,* $\mathbb{C}[x,y]/I$ *is reduced ring, by Proposition 7.6.22,* X *has no embedded primes, contradicts to the fact that* $[(x-1,y-1)]$ *is embedded prime. Hence* X *is nonreduced.*

(c) *By Proposition 7.6.11,* $f\in\Gamma(X,\mathscr{O}_X)$ *is a zerodivisor if and only if* f *vanished at an associated point of* X. *Note that* $x+y-2 = x-1+y-1\in(x-1,y-1)$, $y-x^2\in(y-x^2)$, $x+y-2$ *and* $y-x^2$ *are zerodivisors of* $\Gamma(X,\mathscr{O}_X)$. *But* $x+y-3\notin(x-1,y-1)\cup(x-2,y-2)\cup(y-x^2)$, $x+y-3$ *not zerodivisor.*

(d) *Let* $\mathfrak{p}_1=(y-x^2)$, $\mathfrak{p}_2=(x-1,y-1)$, $\mathfrak{p}_3=(x-2,y-2)$, *then* $I=\mathfrak{p}_1^3\cap\mathfrak{p}_2^{15}\cap\mathfrak{p}_3$. *Hence, in* $\mathbb{C}[x,y]/I$, *we have* $(0)=\mathfrak{p}_1^3\cap\mathfrak{p}_2^{15}\cap\mathfrak{p}_3$, *it is a minimal primary decomposition of* (0) *in* $\mathbb{C}[x,y]/I$. *By* 1*st uniqueness theorem (see Atiyah-MacDonald [1] Theorem 4.5) and Atiyah-MacDonald [1] Proposition 7.17 (note that* $\mathbb{C}[x,y]/I$ *is indeed Noetherian ring), each* $\mathfrak{p}_i$ *is of the form* $\operatorname{Ann}(f_i)$ *where* $f_i\in\mathbb{C}[x,y]/I$. *Hence* $\mathfrak{p}_i$ *is associated prime.*

$I'=\mathfrak{p}_1^2\cap\mathfrak{p}_2\cap\mathfrak{p}_3$ *is another example. But there is no smallest* I, *since* $I''=\mathfrak{p}_1\cap\mathfrak{p}_2^2\cap\mathfrak{p}_3$ *also holds the hypotheses.*

7.6.14 Geometric definitions

Associated points

Definition 7.6.5 (Associated points of quasi-coherent sheaf)

Let X be a scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X.

(1) *We say $x \in X$ is associated to $\mathscr{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$, i.e., $\mathfrak{m}_x = \operatorname{Ann}(f)$ for some $f \in \mathscr{F}_x$.*

(2) *We denote $\operatorname{Ass}(\mathscr{F})$ or $\operatorname{Ass}_X(\mathscr{F})$ the set of associated points of $\mathscr{F}$.*

(3) *The associated points of X are the associated points of $\mathscr{O}_X$.*

Remark These definitions are most useful when X is locally Noetherian and $\mathscr{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Stacks Project Example 31.2.7 [8]. In the non-Noetherian case it may be more convenient to use weakly associated points, see Stacks Project Section 31.5 [8].

Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

Proposition 7.6.24

Let X be a scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $\operatorname{Spec}(A) = U \hookrightarrow X$ be an affine open, and set $M = \Gamma(U, \mathscr{F})$. Let $x \in U$, and let $\mathfrak{p} \in \operatorname{Spec} A$ be the corresponding prime.

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathscr{F}$.*

(2) *If $\mathfrak{p}$ is finitely generated, then the converse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in \operatorname{Ass}(M) \iff x \in \operatorname{Ass}(\mathscr{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof If $\mathfrak{p}$ is associated to M, then $\mathfrak{p} = \operatorname{Ann}(m) = \operatorname{Ann}(Am)$. By Lemma 7.6.2, we have $\mathfrak{p}A_\mathfrak{p} = \operatorname{Ann}_A(Am)_\mathfrak{p} = \operatorname{Ann}_{A_\mathfrak{p}}(A_\mathfrak{p} \otimes_A Am) = \operatorname{Ann}_{A_\mathfrak{p}}(m/1)$. Note that $\mathfrak{p}A_\mathfrak{p}$ is the maximal ideal of $A_\mathfrak{p}$, $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p}) \cong \operatorname{Ass}(\mathscr{F}_x)$, which implies that x is associated to $\mathscr{F}$.

Conversely, assume that x is associated to $\mathscr{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathscr{F}$ there exists an element $m' \in M_\mathfrak{p}$ such that $\mathfrak{p}A_\mathfrak{p} = \operatorname{Ann}_{A_\mathfrak{p}}(m')$. Write $m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. By Lemma 7.6.2, we have

$$\mathfrak{p}A_\mathfrak{p} = \operatorname{Ann}_{A_\mathfrak{p}}(A_\mathfrak{p}(m/f)) = \operatorname{Ann}_{A_\mathfrak{p}}(A_\mathfrak{p}(m/1)) = \operatorname{Ann}_A(m)_\mathfrak{p}.$$

Hence $\operatorname{Ann}_A(m) \subseteq \mathfrak{p}$, and $(\mathfrak{p}/I)_\mathfrak{p} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A - \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence $g\mathfrak{p} \subseteq I$, it is easy to check that $\mathfrak{p} = \operatorname{Ann}_A(gm)$, which implies that $\mathfrak{p} \in \operatorname{Ass}(M)$.

If X is locally Noetherian, then A is Noetherian and $\mathfrak{p}$ is always finitely generated. □

Proposition 7.6.25

Let X be a locally Noetherian scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $\operatorname{Ass}(\mathscr{F}) \cap U$ is finite for every quasi-compact open $U \subseteq X$.

Proof Since $\Gamma(U, \mathscr{F})$ is a finitely generated $\mathscr{O}_X(U)$-module over Noetherian ring $\mathscr{O}_X(U)$. By Corollary 7.6.2, $\operatorname{Ass}(\Gamma(U, \mathscr{F}))$ is finite, by Proposition 7.6.24, $\operatorname{Ass}(\mathscr{F}) \cap U = \operatorname{Ass}(\Gamma(U, \mathscr{F}))$ is finite. □

By Proposition 7.6.24 and Proposition 7.6.25, we may define the **associated points of a coherent sheaf on a locally Noetherian scheme**.

Definition 7.6.6 (Associated points of a coherent sheaf on a locally Noetherian scheme)

Let X be a locally Noetherian scheme, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. We say $x \in X$ is ***associated*** *if $x \in \operatorname{Ass}(\Gamma(U, \mathscr{F}))$ for any affine open $U \ni x$. Denote $\operatorname{Ass}(\mathscr{F})$ the set of associated points of $\mathscr{F}$. The associated points of X are the associated points of $\mathscr{O}_X$, i.e., $\operatorname{Ass}(X) := \operatorname{Ass}(\mathscr{O}_X)$.*

Proposition 7.6.26

Suppose X is a locally Noetherian scheme, and $U \hookrightarrow X$ is an open subscheme. Then the natural map

$$\Gamma(U, \mathscr{O}_X) \longrightarrow \prod_{p \in \operatorname{Ass} X \cap U} \mathscr{O}_{X,p} \tag{7.38}$$

is an injection.

Proof Since $U \hookrightarrow X$ is an open subscheme, we may assume that $U = \bigcup_i \operatorname{Spec} A_i$ where A_i is Noetherian. Hence, by Proposition 7.6.10 and Definition of sheaves, we have sequecne

$$0 \longrightarrow \Gamma(U, \mathscr{O}_X) \hookrightarrow \prod_i A_i \hookrightarrow \prod_i \prod_{\mathfrak{p} \in \operatorname{Ass}(A_i)} (A_i)_{\mathfrak{p}} \text{ (remove the same items)}.$$

Consider $\prod_i \prod_{\mathfrak{p} \in \operatorname{Ass}(A_i)} (A_i)_{\mathfrak{p}}$ (remove the same items) , by Proposition 7.6.24, we know that $\operatorname{Ass}(A_i) = \operatorname{Ass}(\mathscr{O}_X) \cap \operatorname{Spec} A_i = \operatorname{Ass}(X) \cap \operatorname{Spec} A_i$. Hence

$$\begin{aligned} &\prod_i \prod_{\mathfrak{p} \in \operatorname{Ass}(A_i)} (A_i)_{\mathfrak{p}} \text{ (remove the same items)} \\ =&\prod_i \prod_{\mathfrak{p} \in \operatorname{Ass}(X) \cap \operatorname{Spec} A_i} (A_i)_{\mathfrak{p}} \text{ (remove the same items)} \\ \cong& \prod_{p \in \operatorname{Ass} X \cap U} \mathscr{O}_{X,p}, \end{aligned}$$

as we desired. □

Embedded points

Definition 7.6.7 (Embedded points of quasi-coherent sheaf)

Let X be a scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X.

(1) *An embedded associated point of $\mathscr{F}$ is an associated point which is not maximal among the associated points of $\mathscr{F}$, i.e., it is the specialization of another associated point of $\mathscr{F}$.*
(2) *A point x of X is called an embedded point if x is an embedded associated point of $\mathscr{O}_X$.*
(3) *An embedded component of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.*

In the Noetherian case when $\mathscr{F}$ is coherent we have the following.

Proposition 7.6.27

Let X be a locally Noetherian scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then

(1) *the generic points of irreducible components of $\operatorname{Supp}(\mathscr{F})$ are associated points of $\mathscr{F}$, and*
(2) *an associated point of $\mathscr{F}$ is embedded if and only if it is not a generic point of an irreducible component of $\operatorname{Supp}(\mathscr{F})$.*

In particular an embedded point of X is an associated point of X which is not a generic point of an *irreducible component of X*

Proof Note that $\operatorname{Supp}(\mathscr{F}) = \bigcup_{\operatorname{Spec} A_i \hookrightarrow X} \operatorname{Supp}(\mathscr{F}|_{\operatorname{Spec} A_i})$, since $\mathscr{F}$ is coherent, $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$ where M_i is coherent A_i-module. Hence

$$\operatorname{Supp}(\mathscr{F}) \cong \bigcup_i \operatorname{Supp}(\widetilde{M_i}) \cong \bigcup_i \operatorname{Supp}(M_i)$$

Let η be a generic point, then $\overline{\{\eta\}} \subseteq \operatorname{Supp}(M_i)$ for some i. Hence $\overline{\{\eta\}}$ is irreducible component of $\operatorname{Supp}(M_i)$. Since M_i is coherent, M_i is finitely generated A_i-module, also since X is locally Noetherian, A_i is Noetherian. By Proposition 7.6.15, $\eta \in \operatorname{Ass} M_i$. By Proposition 7.6.24, $\eta \in \operatorname{Ass}(\mathscr{F})$. Part (2) is clearly from Part (1) and Definition 7.6.7. □

By Proposition 7.6.27, we may define the **embedded points of a coherent sheaf on a locally Noetherian scheme**.

Definition 7.6.8 (Embedded points of a coherent sheaf on a locally Noetherian scheme)

Let X be a locally Noetherian scheme, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. An associated point $x \in X$ is called embedded if it is not generic point of an irreducible component of $\operatorname{Supp} \mathscr{F}$.

Scheme-theoretic density (and the adjective "scheme-theoretic" more generally).

Definition 7.6.9 (Scheme-theoretically dense)

*An open subscheme U of a scheme X is said to be **scheme-theoretically dense** if any function on any open set V is 0 if it restricts to 0 on $U \cap V$, i.e., $f|_{U\cap V} = 0 \Longrightarrow f = 0$ in V, for any V.*

This is stronger than density in the usual (topological) sense. For example:

Proposition 7.6.28

If X is locally Noetherian, then an open subscheme $U \subseteq X$ is scheme-theoretically dense if and only if it contains all the associated points of X.

Proof Let U be scheme-theoretically dense, suppose exists $[\mathfrak{p}] \in \operatorname{Ass}(X)$ such that $[\mathfrak{p}] \notin U$. Since $[\mathfrak{p}] \in \operatorname{Ass}(X)$, by definition, $[\mathfrak{p}] \in \operatorname{Ass}(A)$ for any affine open $\operatorname{Spec} A \ni [\mathfrak{p}]$. Hence $\mathfrak{p} = \operatorname{Ann}(f)$ for some $f \in A$. We claim that any point $\mathfrak{q} \in U \cap \operatorname{Spec} A$ does not contain $\mathfrak{p}$. Since $[\mathfrak{p}] \notin U$, we have $[\mathfrak{p}] \in X - U$, note that we have $\overline{\{\mathfrak{p}\}} \subseteq X - U$, since $X - U$ is closed. Hence any element in $U \cap \operatorname{Spec} A$ does not contain $\mathfrak{p}$. Let $a \in \mathfrak{p} - \mathfrak{q}$, then $af = 0 \in \mathfrak{q}$, since $a \notin \mathfrak{q}$ and $\mathfrak{q}$ is prime, we have $f \in \mathfrak{q}$, which implies that f vanishes on $U \cap \operatorname{Spec} A$, i.e., $f|_{U\cap \operatorname{Spec} A} = 0$. Since U is scheme-theoretically dense, $f = 0$ in $\operatorname{Spec} A$, hence $\mathfrak{p} = \operatorname{Ann}_A(0) = A$, a contradiction.

Conversely, let V be any open subset of X, then $V = \bigcup_i \operatorname{Spec} A_i$, hence $U \cap V = \bigcup_i (U \cap \operatorname{Spec} A_i)$. Suppose $f|_{U\cap V} = 0$, where $f \in \Gamma(V, \mathscr{O}_X)$. By Proposition 7.6.24, we have

$$\operatorname{Ass} X \cap V = \bigcup_i (\operatorname{Ass} X \cap \operatorname{Spec} A_i) = \bigcup_i \operatorname{Ass} A_i.$$

Since $U \supseteq \operatorname{Ass} X$, we have $f|_{\operatorname{Ass} X\cap V} = f|_{\bigcup_i \operatorname{Ass} A_i} = 0$, and therefore $f|_{\operatorname{Ass} A_i} = 0$, hence $f = 0$ in $\mathscr{O}_{X,p}$ for all $p \in \operatorname{Ass} A_i$ for all i. By Proposition 7.6.27 (7.38), the natural map

$$\Gamma(V, \mathscr{O}_X) \longrightarrow \prod_{p \in \operatorname{Ass} X \cap V} \mathscr{O}_{X,p} = \prod_{p \in \bigcup_i \operatorname{Ass} A_i} \mathscr{O}_{X,p}$$

is an injection. Hence $f|_V = 0$, which implies that U is scheme-theoretically dense. □

Remark **Why scheme-theoretically dense is stronger then topological dense?** If U is topological dense, U meet very irreducible components of X, hence U contains all generic point of X (Proposition 4.6.10), but U may not contains embedded point. If U is scheme-theoretically dense, U contains $\operatorname{Ass} X$, hence U contains all embedded points.

More generally, the adjective **scheme-theoretic** typically indicates an ideal-theoretic definition, enriching and refining a more naive set-theoretic definition. Example will include:

- scheme-theoretically dense
- scheme-theoretic intersection
- scheme-theoretic support
- scheme-theoretic image
- scheme-theoretic closure
- scheme-theoretic preimage
- scheme-theoretic fiber

7.6.15 Revisiting the notion of length

Proposition 7.6.29

(a) (Noetherian converse to Proposition 7.5.8) Suppose A is a Noetherian ring. Then any finitely generated module whose support consists of finitely many points of $\operatorname{Spec} A$*, all closed, has finite length.*

(b) (Noetherian converse to Proposition 7.5.10) Suppose $\mathscr{F}$ is a finite type quasi-coherent sheaf on a locally Noetherian scheme X, supported at finitely many points of X, all closed. Then $\mathscr{F}$ has finite length.

Proof

(a) Note that a finitely generated module over Noetherian (resp. Artin) ring is Noetherian (resp. Artin) module, hence this module satisfies both a.c.c. and d.c.c., apply Lemma 7.5.1, this module has a composition series, and therefore has finite length. Hence, it suffices to show that finitely generated module M can be seen as a module over Artin ring.
Let M is a finitely generated module over Noetherian ring A. By condition, we may assume that $\operatorname{Supp}(M) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ where $\mathfrak{m}_i \in \operatorname{Max} A$. By Proposition 7.6.1, $\operatorname{Supp} M = V(\operatorname{Ann}_A M)$, hence $\operatorname{Ann} M \subseteq \mathfrak{m}_i$ for all i. Note that M can be seen as an $A/\operatorname{Ann}(M)$-module, we want to show that $A/\operatorname{Ann}(M)$ is Artin ring. Consider $\operatorname{Spec} A/\operatorname{Ann}(M)$, $\operatorname{Spec} A/\operatorname{Ann}(M) = V(\operatorname{Ann}(M)) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, hence Krull dimension $\dim A/\operatorname{Ann}(M) = 0$, and therefore $A/\operatorname{Ann}(M)$ is Artin ring, as we desired.

(b) Let $\operatorname{Supp} X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, then $X \setminus \{\mathfrak{m}_1, \dots, \mathfrak{m}_{i-1}, \mathfrak{m}_{i+1}, \dots, \mathfrak{m}_n\}$ is open. Let $U_i \hookrightarrow X \setminus \{\mathfrak{m}_1, \dots, \mathfrak{m}_{i-1}, \mathfrak{m}_{i+1}, \dots, \mathfrak{m}_n\}$ be an affine open neighborhood of x_i. Let $U = \bigcup_{i=1}^n U_i$. Define $\mathscr{F}_i := j_{i,!}(\mathscr{F}|_{U_i})$, where $j_i : U_i \hookrightarrow U$. We claim that $\mathscr{F}|_U \cong \prod\limits_{i=1}^n \mathscr{F}_i$. Check it at the stalk level, let $\mathfrak{m}_i \in \operatorname{Supp} X$, since $\mathfrak{m}_i \notin U_j$ for all $i \neq j$, we have

$$(\mathscr{F}|_U)_{\mathfrak{m}_i} \cong (\mathscr{F}|_{U_i})_{\mathfrak{m}_i} \cong \left(\prod_{i=1}^n \mathscr{F}_i\right)_{\mathfrak{m}_i},$$

hence $\mathscr{F}|_U \cong \prod_{i=1}^{n} \mathscr{F}_i$. We next show that $\mathscr{F}|_U$ is finite length. Since $\mathbf{QCoh}_X$ is abelian category, we have $\prod_{i=1}^{n} \mathscr{F}_i \cong \bigoplus_{i=1}^{n} \mathscr{F}_i$, hence $l(\mathscr{F}|_U) = \sum_{i=1}^{n} l(\mathscr{F}_i)$. Consider $l(\mathscr{F}_i)$, since $\mathscr{F}_i = j_{i,!}(\mathscr{F}|_{U_i})$, $l(\mathscr{F}|_{U_i}) = l(\mathscr{F}_i)$. Since $\mathscr{F}$ is finite type quasi-coherent sheaf on locally Noetherian scheme, $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is finitely generated $\mathscr{O}_X(U_i)$-module and $\mathscr{O}_X(U_i)$ is Noetherian ring. By Part (1), $l(M_i) < \infty$, and therefore $l(\mathscr{F}|_{U_i}) < \infty$. Hence $l(M) = \sum_{i=1}^{n} l(M_i) < \infty$, which implies that $\mathscr{F}|_U$ is finite length.
Since $\mathrm{Supp}(X) \cap X \setminus U = \varnothing$, we know that $\mathscr{F}|_{X\setminus U} = 0$, and therefore $\mathscr{F} = j_!(\mathscr{F}|_U)$ where $j : U \hookrightarrow X$. Hence $l(\mathscr{F}) = l(\mathscr{F}|_U) < \infty$, which implies that $\mathscr{F}$ is finite length.

$\square$

Proposition 7.6.30

(a) Each finite length quasi-compact scheme (Definition 7.5.6) has a finite number of points, with the discrete topology.

(b) If X is finite type over $\overline{k}$, of finite length l, then $\dim_{\overline{k}} \Gamma(X, \mathscr{O}_X) = l$.

Proof

(a) Let X be a finite length quasi-compact scheme. Since X is finite length, $\mathscr{O}_X$ is finite length, then we have a composition series

$$0 = \mathscr{F}_0 \hookrightarrow \mathscr{F}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{F}_n = \mathscr{O}_X$$

with $\mathrm{Coker}(\mathscr{F}_{i-1} \hookrightarrow \mathscr{F}_i)$ is simple object. By Proposition 7.5.9, $\mathrm{Coker}(\mathscr{F}_{i-1} \hookrightarrow \mathscr{F}_i) \cong \widetilde{A_i/\mathfrak{m}_i}$ where $\mathfrak{m}_i$ is maximal ideal of A_i. We claim that $X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$.
Let $x \in X \setminus \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, take stalk at x, we have

$$0 = (\mathscr{F}_0)_x \hookrightarrow (\mathscr{F}_1)_x \hookrightarrow \cdots \hookrightarrow (\mathscr{F}_n)_x = \mathscr{O}_{X,x}$$

with $\mathrm{Coker}((\mathscr{F}_{i-1})_x \hookrightarrow (\mathscr{F}_i)_x) \cong \mathrm{Coker}(\mathscr{F}_{i-1} \hookrightarrow \mathscr{F}_i)_x = 0$. Hence $\mathscr{O}_{X,x} = 0$, it is impossible, which implies that $X \setminus \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\} = \varnothing$, and therefore $X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$.
Since each $\mathfrak{m}_i$ is closed point, hence the topology on X is discrete topology.

(b) Since $l(X) = l$, by Part (a), we may assume that $X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_l\}$, where each $\mathfrak{m}_i$ is closed point. We may write $X = \bigcup_{i=1}^{l} \mathrm{Spec}(A_i/\mathfrak{m}_i)$ where $\mathfrak{m}_i$ is maximal ideal of A_i. Since each A_i is finitely generated $\overline{k}$-algebra, $A_i/\mathfrak{m}_i$ is a field which isomorphic to $\overline{k}$. By the definition pf sheaves, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \prod_{i=1}^{l} \Gamma(\mathrm{Spec}\, A_i/\mathfrak{m}_i, \mathscr{O}_X) \longrightarrow \prod_{i \neq j} \Gamma((\mathrm{Spec}\, A_i/\mathfrak{m}_i) \cap (\mathrm{Spec}\, A_j/\mathfrak{m}_j), \mathscr{O}_X).$$

Note that each $(\mathrm{Spec}\, A_i/\mathfrak{m}_i) \cap (\mathrm{Spec}\, A_j/\mathfrak{m}_j) = \varnothing$, we get an isomorphism

$$\Gamma(X, \mathscr{F}) \cong \prod_{i=1}^{l} \Gamma(\mathrm{Spec}(A_i/\mathfrak{m}_i)) \cong \prod_{i=1}^{l} A_i/\mathfrak{m}_i \cong \bigoplus_{i=1}^{l} A_i/\mathfrak{m}_i \cong \overline{k}^{\oplus l},$$

it follows that $\dim_{\overline{k}} \Gamma(X, \mathscr{O}_X) = l$.

$\square$

7.6.16 Generalizing the fraction field: the total fraction ring

Definition 7.6.10 (Rational functions on locally Noetherian schemes)

*A **rational function** on a locally Noetherian scheme is an element of the image of* $\Gamma(U, \mathscr{O}_U)$ *in (7.38) for some* U *containing all the associated points (by Proposition 7.6.28,* U *is scheme-theoretically dense). Equivalent, the set of rational functions is the colimit of* $\mathscr{O}_X(U)$ *over all open sets containing the associated points, i.e.,* $\varinjlim_{U \supseteq \operatorname{Ass} X} \mathscr{O}_X(U)$.

Example 7.8 On $\operatorname{Spec} k[x,y]/(y^2, xy)$ (Figure 5.4), by Exercise 7.1, $\frac{x-2}{(x-1)(x-3)}$ is a rational function, since it belong to $(k[x,y]/(y^2,xy))_{(y)}$ and $(k[x,y]/(y^2,xy))_{(x,y)}$. But $\frac{x-2}{x(x-1)}$ is not, since $\frac{x-2}{x(x-1)} \notin (k[x,y]/(y^2,xy))_{(x,y)}$.

A rational has a maximal **domain of definition**, because any two actual functions on an open set (i.e., sections of the structure sheaf over that open set) that agree as "rational functions" (i.e., on small enough open sets containing associated points) must be the same function, by the injectivity of (7.38).

Definition 7.6.11 (Regular at a point)

We say that a rational function f *is **regular at a point*** $\mathfrak{p}$ *if* $\mathfrak{p}$ *contained in this maximal domain of definition (or equivalently, there is some open set containing* $\mathfrak{p}$ *where* f *is defined).*

Example 7.9 On $\operatorname{Spec} k[x,y]/(y^2, xy)$, then rational function $\frac{x-2}{(x-1)(x-3)}$ has domain of definition consisting of everything but 1 and 3 (i.e., $[(x-1)]$ and $[(x-3)]$), and is regular away from those two points.

Definition 7.6.12 (Regular)

*A rational function is **regular** if it is regular at all points.*

Remark Unfortunately, "regular" is a regularly overused word in mathematics, and in algebraic geometry in particular.

Definition 7.6.13 (Indeterminacy locus)

The complement of the domain of definition of a rational function f *is called the **indeterminacy locus of*** f.

Definition 7.6.14 (Total fraction ring)

*The rational functions form a ring, called the **total fraction ring** or **total quotient ring** of* X*. If* $X = \operatorname{Spec} A$ *is affine, then this ring is called the **total fraction (or quotient) ring** of* A.

Remark If $X = \operatorname{Spec} A$ is affine, the total fraction ring is the function field $K(X)$ — the stalk at the generic point (since $\operatorname{Ass}(X) = \{(0)\}$, and apply Proposition 6.2.9) — so this extends our earlier Definition 6.2.3 of $K(\cdot)$.

Proposition 7.6.31

The ring of rational functions on a locally Noetherian scheme is the colimit of the functions over all open sets containing the associated points:

$$\varinjlim_{\{U : \operatorname{Ass} X \subseteq U\}} \mathscr{O}(U).$$

Slightly better (but slightly different): a rational function is the data of a function f defined on an open set containing all associated points, where two such data (U, f) and (U', f') define the same rational function if and only if $f_{U\cap U'} = f'|_{U\cap U'}$.
If X is reduced, then this is the same as requiring that they are defined on an open set of each of the irreducible components.

Proof If X is reduced, reducedness is affine local property (Example 6.5), we may assume that $X = \operatorname{Spec} A$. By Proposition 7.6.22, X has no embedded prime, then we done. □

Remark **Associated points of integral schemes.** In order for some of our discussion elsewhere to make sense in non-Noetherian settings, we note that the notion of associated points for integral schemes works perfectly, because it works for integral domains — only the generic point is associated. In particular, the definition above of rational functions on an integral scheme X agree with Definition 6.2.3, as precisely the elements of the function field $K(X)$.

7.7 ⋆⋆ Coherent modules over non-Noetherian rings

Part IV

Morphisms of schemes

Chapter 8 Morphisms of Schemes

We now describe the morphisms between schemes. We will define some easy-to-state properties of morphisms, but leave more subtle properties for late.

Recall that a scheme is (i) a set, (ii) with a topology, (iii) and a (structure) sheaf of rings, and that it is some times helpful to think of the definition as having three steps. In the same way, the notion of morphism of schemes $X \to Y$ may be defined (i) as a map of sets, (ii) that is continuous, and (iii) with some further information involving the sheaf of functions. In the case of affine schemes, we have already seen the map of sets (§4.2.3) and later saw that this map is continuous (Proposition 4.4.6).

8.1 Motivations for the "right" definition of morphism of schemes

Here are two motivations for how morphisms should behave. The first is algebraic, and the second is geometric.

Algebraic motivation. We will want morphisms of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ to be precisely the ring maps $B \to A$. We have already seen that ring maps $B \to A$ induce maps of topological spaces in the opposite direction (Proposition 4.4.6); the main new ingredient will be to see how to add the structure sheaf of functions into the mix. Then a morphism of schemes should be something that "on the level of affine open sets, look like this".

Geometric motivation. Motivated by the theory of manifolds (§4.1.1), which like schemes are ringed spaces, we want morphisms of schemes at the very least to be morphisms of ringed spaces; we now motivate what these are. (We will formalize this in the next section.) Notice that if $\pi : X \to Y$ is a map of differentiable manifolds, then a smooth function on Y pulls back to a smooth function on X. More precisely, given an open subset $U \subseteq Y$, there is a natural map $\Gamma(U, \mathscr{O}_Y) \to \Gamma(\pi^{-1}(U), \mathscr{O}_X)$. This behaves well with respect to restriction (restricting a function to a smaller open set and pulling back yields the same result as pulling back and then restricting), so in fact we have a map of sheaves on Y, i.e., $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$. Similarly a morphism of schemes $\pi : X \to Y$ should induce a map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$. But in fact in the category of differentiable manifolds a continuous map $\pi : X \to Y$ is a map of differentiable manifolds precisely when smooth functions on Y pullback to smooth functions on X (i.e., the pullback map from smooth functions on Y to functions on X in fact lies in the subset of smooth functions, i.e., the continuous map π induces a pullback of smooth functions, which can be interpreted as a map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$), so this map of sheaves characterizes morphisms in the differentiable category. So we could use this as the definition of morphism in the differentiable category (see Proposition 4.1.1).

But how do we apply this to the category of schemes? In the category of differentiable manifolds, a continuous map $\pi : X \to Y$ induces a pullback of (the sheaf of) functions, and we can ask when this induces a pullback of smooth functions. However, function are odder on schemes, and we can't recover the pullback map just from the map of topological spaces. The right path is to hardwire this into the definition of morphism, i.e., to have a continuous map $\pi : X \to Y$, along with a pullback map $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$. This leads to the definition of the category of ringed spaces.

On might hope to define morphisms of schemes as morphisms of ringed spaces. This isn't quite right, as then algebraic motivation isn't satisfied: as desired, to each morphism $A \to B$ there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, but there can be addition morphisms of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ not arising in this way. A revised definition as morphisms of sorting out the details, we take a different approach, using locally ringed spaces, which corresponds to asking not just that functions pullback, but also that values of functions pullback. However, we will check that our eventual definition actually is equivalent to this.

We begin by formally defining morphisms of ringed space.

8.2 Morphisms of ringed spaces

8.2.1 Definition: morphisms of ringed spaces

Definition 8.2.1 (Morphism of ringed spaces)

A ***morphism of ringed spaces*** *from* $(X, \mathscr{O}_X)$ *to* $(Y, \mathscr{O}_Y)$ *is a pair* $(\pi, \pi^\sharp)$ *of a continuous morphism* $\pi : X \to Y$ *and a map* $\pi^\sharp : \mathscr{O}_Y \to \pi_* \mathscr{O}_X$ *of sheaves of rings on* Y.

Remark By adjointness (Theorem 3.7.1), this is the same as a map $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

Remark There is an obvious notion of composition of morphisms, so ringed spaces form a category. Hence we have notion of automorphisms and isomorphism. It is easy to verify that an isomorphism of ringed spaces means the same thing as it did before (Definition 5.3.1).

If $U \subseteq Y$ is an open subset, then there is a natural morphism of ringed spaces $(U, \mathscr{O}_Y|_U) \to (Y, \mathscr{O}_Y)$. More precisely:

Definition 8.2.2 (Open embedding (or open immersion))

If $U \to Y$ *is an isomorphism of* U *with an open subset* V *of* Y*, and we are given an isomorphism* $(U, \mathscr{O}_U) \xrightarrow{\sim} (V, \mathscr{O}_Y|_V)$ *(via the isomorphism* $U \xrightarrow{\sim} V$*), then the resulting map of ringed spaces is called an* ***open embedding*** *(or* ***open immersion****) of ringed spaces, and morphism* $U \to Y$ *is often written* $U \hookrightarrow Y$.

Proposition 8.2.1 (Morphisms of ringed spaces glue)

Suppose $(X, \mathscr{O}_X)$ *and* $(Y, \mathscr{O}_Y)$ *are ringed spaces,* $X = \bigcup_i U_i$ *is an open cover of* X*, and we have morphisms of ringed spaces* $\pi_i : U_i \to Y$ *that "agree on the overlaps", i.e.,* $\pi_i|_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$. *Then there is a unique morphism of ringed spaces* $\pi : X \to Y$ *such that* $\pi|_{U_i} = \pi_i$.

Proof Define $\pi : X \to Y$ by setting $\pi|_{U_i} = \pi_i$. We first check π is well-defined. Let $x \in U_i \cap U_j$, since $\pi_i|_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$, we have $\pi(x) = \pi_i(x) = \pi_j(x)$. Hence π is well-defined. We next show that π is continuous. Let $V \subseteq Y$, then $\pi^{-1}(V) = \bigcup_i \pi_i^{-1}(V)$. Since each π_i is continuous $\pi_i^{-1}(V)$ is open in X, and therefore $\pi^{-1}(V)$ is open in X. Hence π is continuous.

We now check π along with a map $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$. Let $V \subseteq Y$ be any open subset of Y, pick $f \in \mathscr{O}_Y(V)$, we want to show that $f \circ \pi \in \pi_* \mathscr{O}_X(V) = \mathscr{O}_X(\pi^{-1}(V))$. By the condition, we know that $(f \circ \pi)|_{U_i} = f \circ \pi_i \in (\pi_i)_* \mathscr{O}_X(V) = \mathscr{O}_X(\pi_i^{-1}(V))$, with $((f \circ \pi)|_{U_i})|_{U_i \cap U_j} = ((f \circ \pi)|_{U_j})|_{U_i \cap U_j}$. By the gluability axiom of $\mathscr{O}_X$, there exists $s \in \mathscr{O}_X(\pi^{-1}(V))$ such that $s|_{\pi_i^{-1}(V)} = (f \circ \pi)|_{U_i}$. By the identity axiom of $\mathscr{O}_X$, we know that $f \circ \pi = s \in \mathscr{O}_X(\pi^{-1}(V))$. It follows that π along with $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$. $\square$

Proposition 8.2.2

Open immersions of ringed spaces are monomorphisms (in the category of ringed space).

Proof Let $i : U \hookrightarrow Y$ be an open immersion. Suppose there exists two morphisms $\mu_1 : Z \to U$ and $\mu_2 : Z \to U$ with $i \circ \mu_1 = i \circ \mu_2$. We want to show that $\mu_1 = \mu_2$. Look i as a morphism of topological space, i is injection, hence, as morphisms of topological spaces, $i \circ \mu_1 = i \circ \mu_2$ implies $\mu_1 = \mu_2$. As morphism of sheaves, we have $\mathscr{O}_Y \to (i \circ \mu_1)_*\mathscr{O}_Z$ and $\mathscr{O}_Y \to (i \circ \mu_2)_*\mathscr{O}_Z$ are the same. Note that $(i \circ \mu_i)_*\mathscr{O}_Z = (i_* \circ (\mu_i)_*)\mathscr{O}_Z$, $\mathscr{O}_Y \to (i_* \circ (\mu_1)_*)\mathscr{O}_Z$ and $\mathscr{O}_Y \to (i_* \circ (\mu_2)_*)\mathscr{O}_Z$ are the same. By adjointness (Theorem 3.7.1), $i^{-1}\mathscr{O}_Y \to (\mu_1)_*\mathscr{O}_Z$ and $i^{-1}\mathscr{O}_Y \to (\mu_2)_*\mathscr{O}_Z$ are the same. Since $i : U \hookrightarrow Y$ is open immersion, then we get an isomorphism of ringed spaces $(U, \mathscr{O}_U) \xrightarrow{\sim} (V, \mathscr{O}_Y|_V)$. By Proposition 3.7.3, we know that $i^{-1}\mathscr{O}_Y \cong \mathscr{O}_Y|_V$, hence $i^{-1}\mathscr{O}_Y \cong \mathscr{O}_U$. Hence $\mathscr{O}_U \to (\mu_1)_*\mathscr{O}_Z$ and $\mathscr{O}_U \to (\mu_2)_*\mathscr{O}_Z$ are the same. It follows that $\mu_1 = \mu_2$ as morphisms of ringed spaces, and therefore $i : U \hookrightarrow Y$ is monomorphism in the category of ringed space. □

8.2.2 Pushing Forward $\mathscr{O}$-module, pulling back $\mathscr{O}$-module

Pushing forward $\mathscr{O}$-module

Proposition 8.2.3 (Pushing Forward $\mathscr{O}$-modules)

Given a morphism of ringed spaces $\pi : X \to Y$, then sheaf pushforward induces a functor $\pi_* : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_Y}$.

Proof Let $\mathscr{F} \in \mathbf{Mod}_{\mathscr{O}_X}$ be an $\mathscr{O}_X$-module. Then $\pi_*\mathscr{F}$ is an $\pi_*\mathscr{O}_X$-module. Since we have a morphism of ringed spaces $\pi : X \to Y$, there is a morphism of sheaves, $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, hence $\pi_*\mathscr{F}$ can be seen as $\mathscr{O}_Y$ module, i.e., define the action $\mathscr{O}_Y(V) \times \pi_*\mathscr{O}_X(V)$ by setting

$$(f, s) \longmapsto (f \circ \pi) \cdot s,$$

where $V \subseteq Y$ open subset.

Let $\mathscr{F}_1, \mathscr{F}_2 \in \mathbf{Mod}_{\mathscr{O}_X}$, and morphism $\varphi \in \mathrm{Mor}_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{F}_1, \mathscr{F}_2)$. Define $\pi_*\varphi$ by setting $(\pi_*\varphi)_V = \varphi_{\pi^{-1}(V)}$ for any open subset $V \subseteq Y$, it is easy to see that $\pi_*\varphi \in \mathrm{Mor}_{\mathbf{Mod}_{\mathscr{O}_Y}}(\pi_*\mathscr{F}_1, \pi_*\mathscr{F}_2)$. By the definition, it is easy to verify that π_* preserves identity morphisms and π_* preserves composition. Hence $\pi_* : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_Y}$ is a functor. □

Pulling back $\mathscr{O}$-module

Suppose $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. A slight variation on the inverse image allows us to **pull back an $\mathscr{O}_Y$-module** $\mathscr{G}$ to get an $\mathscr{O}_X$-module $\pi^*\mathscr{G}$.

Lemma 8.2.1

Suppose $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. Then $(X, \pi^{-1}\mathscr{O}_Y)$ is a ringed space, and that the morphism of ringed spaces π factors into

$$(X, \mathscr{O}_X) \longrightarrow (X, \pi^{-1}\mathscr{O}_Y) \longrightarrow (Y, \mathscr{O}_Y).$$

Proof Clearly $\pi = \pi \circ \mathrm{id}$. Since $\pi : X \to Y$ is a morphism of ringed spaces, we have a morphism of sheaves of rings on Y,

$$\mathscr{O}_Y \longrightarrow \pi_*\mathscr{O}_X.$$

By adjointness (Theorem 3.7.1), we have a morphism of sheaves on X,

$$\pi^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X.$$

Note that $\mathscr{O}_X$ can be seen as $\mathrm{id}_* \mathscr{O}_X$ where $\mathrm{id} : X \to X$. Then we get a morphism of ringed spaces from $(X, \mathscr{O}_X)$ to $(X, \pi^{-1}\mathscr{O}_Y)$. Consider isomorphism $\pi^{-1}\mathscr{O}_Y \xrightarrow{\sim} \pi^{-1}\mathscr{O}_Y$, by adjointness (Theorem 3.7.1), we have

$$\mathscr{O}_Y \longrightarrow \pi_* \pi^{-1} \mathscr{O}_Y.$$

Then we get a morphism of ringed spaces from $(X, \pi^{-1}\mathscr{O}_Y)$ to $(Y, \mathscr{O}_Y)$. Clearly, $(\pi \circ \mathrm{id})_* \mathscr{O}_X = (\pi_* \mathrm{id}_*)\mathscr{O}_X$, then π factors into

$$(X, \mathscr{O}_X) \longrightarrow (X, \pi^{-1}\mathscr{O}_Y) \longrightarrow (Y, \mathscr{O}_Y).$$

□

Lemma 8.2.2

Suppose $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. Let $\mathscr{G}$ be an $\mathscr{O}_Y$-module, then $\pi^{-1}\mathscr{G}$ is a $\pi^{-1}\mathscr{O}_Y$-module. Moreover, $\pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is an $\mathscr{O}_X$-module.

Proof Let $U \subseteq X$ be any open subset of X, then $\pi^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq \pi(U)} \mathscr{G}(V)$ and $\pi^{-1}\mathscr{O}_Y(U) = \varinjlim_{V \supseteq \pi(U)} \mathscr{O}_Y(V)$. Define the action $\pi^{-1}\mathscr{O}_Y(U) \times \pi^{-1}\mathscr{G}(U) \to \pi^{-1}\mathscr{G}(U)$ same as action $\mathscr{O}_Y(U) \times \mathscr{G}(U) \to \mathscr{G}(U)$, and therefore $\pi^{-1}\mathscr{G}$ is a $\pi^{-1}\mathscr{O}_Y$-module.

π gives a morphism of sheaves on Y, $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$, by adjointness, we have $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, hence $\mathscr{O}_X$ can be seen as $\pi^{-1}\mathscr{O}_Y$-module, and therefore $\pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is an $\mathscr{O}_X$-module. □

Definition 8.2.3 (Pullback)

*Suppose $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. Let $\mathscr{G}$ be an $\mathscr{O}_Y$-module, then $\pi^{-1}\mathscr{G}$ is a $\pi^{-1}\mathscr{O}_Y$-module. We define $\pi^*\mathscr{G}$ be the tensor product $\pi^*\mathscr{G} := \pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$, call it the **pullback** of $\mathscr{G}$.*

Remark Lemma 8.2.1 and Lemma 8.2.2 make the definition of pullback meaningful.

Proposition 8.2.4

Pullback π^* is a covariant functor $\mathbf{Mod}_{\mathscr{O}_Y} \to \mathbf{Mod}_{\mathscr{O}_X}$.

Proof Let $\mathscr{G}_1, \mathscr{G}_2 \in \mathbf{Mod}_{\mathscr{O}_Y}$, and morphism $\varphi \in \mathrm{Mor}_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{G}_1, \mathscr{G}_2)$. Define $\pi^*\varphi : \pi^*\mathscr{G}_1 \to \pi^*\mathscr{G}_2$ by setting $\pi^{-1}\varphi \otimes \mathrm{id}_{\mathscr{O}_X}$, where $\pi^{-1}\varphi$ given by the universal property of colimit. Hence $\pi^*\varphi \in \mathrm{Mor}_{\mathbf{Mod}_{\mathscr{O}_X}}(\pi^*\mathscr{G}_1, \pi^*\mathscr{G}_2)$.

For any $\mathscr{G} \in \mathbf{Mod}_{\mathscr{O}_Y}$, consider $\pi^*(\mathrm{id}_{\mathscr{G}})$, we have $\pi^*(\mathrm{id}_{\mathscr{G}}) = \pi^{-1}(\mathrm{id}_{\mathscr{G}}) \otimes \mathrm{id}_{\mathscr{O}_X} = \mathrm{id}_{\pi^{-1}(\mathscr{G})} \otimes \mathrm{id}_{\mathscr{O}_X} = \mathrm{id}_{\pi^*\mathscr{G}}$. It follows that π^* preserves identity morphisms.

Let $\mathscr{G}_1 \xrightarrow{\varphi} \mathscr{G}_2 \xrightarrow{\psi} \mathscr{G}_3$. Note that

$$\pi^*(\psi \circ \varphi) = \pi^{-1}(\psi \circ \varphi) \otimes \mathrm{id}_{\mathscr{O}_X} = (\pi^{-1}(\psi) \otimes \mathrm{id}_{\mathscr{O}_X}) \circ (\pi^{-1}(\varphi) \otimes \mathrm{id}_{\mathscr{O}_X}) = \pi^*(\psi) \circ \pi^*(\varphi),$$

π^* preserves composition. Hence π^* is a functor. □

Proposition 8.2.5

Suppose $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $\rho : (W, \mathscr{O}_W) \to (X, \mathscr{O}_X)$ are morphisms of ringed spaces, then there is a natural isomorphism of functors

$$\rho^*\pi^* \xleftrightarrow{\sim} (\pi \circ \rho)^*.$$

Proof Let $\mathscr{G}$ be an $\mathscr{O}_Y$-module. Consider $(\rho^*\pi^*)\mathscr{G}$ and $(\pi \circ \rho)^*\mathscr{G}$, we want to show that $((\rho^*\pi^*)\mathscr{G})_p \cong$

$((\pi\circ\rho)^*\mathscr{G})_p$ for all $p\in W$. In fact,

$$\begin{aligned}(\pi\circ\rho)^*\mathscr{G} =&(\pi\circ\rho)^{-1}\mathscr{G}\otimes_{(\pi\circ\rho)^{-1}\mathscr{O}_Y}\mathscr{O}_W\\ =&(\rho^{-1}\circ\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\circ\pi^{-1})\mathscr{O}_Y}\mathscr{O}_W,\end{aligned}$$

where last equal by $(\pi\circ\rho)^{-1}\xleftrightarrow{\sim}\rho^{-1}\circ\pi^{-1}$ (check at the level of stalks, use Proposition 3.4.3). On the other hand, we have

$$\begin{aligned}(\rho^*\pi^*)\mathscr{G} =&\rho^*(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)\\ =&\rho^{-1}(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)\otimes_{\rho^{-1}\mathscr{O}_X}\mathscr{O}_W.\end{aligned}$$

We claim that $\rho^{-1}(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)\cong(\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X$. Check it at the level of stalks, let $p\in W$, then by Proposition 3.7.2 and Proposition 3.6.11, we have

$$\begin{aligned}\rho^{-1}(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)_p &\cong(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)_{\rho(p)}\\ &\cong(\pi^{-1}\mathscr{G})_{\rho(p)}\otimes_{(\pi^{-1}\mathscr{O}_Y)_{\rho(p)}}\mathscr{O}_{X,\rho(p)}\\ &\cong\mathscr{G}_{(\pi\circ\rho)(p)}\otimes_{\mathscr{O}_{Y,(\pi\circ\rho)(p)}}\mathscr{O}_{X,\rho(p)}\end{aligned}$$

and

$$\begin{aligned}((\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X)_p &\cong(\rho^{-1}\pi^{-1})\mathscr{G}_p\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_{Y,p}}\rho^{-1}\mathscr{O}_{X,p}\\ &\cong\mathscr{G}_{(\pi\circ\rho)(p)}\otimes_{\mathscr{O}_{Y,(\pi\circ\rho)(p)}}\mathscr{O}_{X,\rho(p)},\end{aligned}$$

hence

$$\rho^{-1}(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)_p\cong((\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X)_p,$$

which implies that (Proposition 3.4.3)

$$\rho^{-1}(\pi^{-1}\mathscr{G}\otimes_{\pi^{-1}\mathscr{O}_Y}\mathscr{O}_X)\cong(\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X.$$

Hence

$$(\rho^*\pi^*)\mathscr{G}\cong((\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X)\otimes_{\rho^{-1}\mathscr{O}_X}\mathscr{O}_W.$$

Note that $\mathscr{O}_W$ can be seen as $\rho^{-1}\mathscr{O}_X$-module, we have

$$\begin{aligned}(\rho^*\pi^*)\mathscr{G}&\cong((\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\rho^{-1}\mathscr{O}_X)\otimes_{\rho^{-1}\mathscr{O}_X}\mathscr{O}_W\\ &\cong(\rho^{-1}\pi^{-1})\mathscr{G}\otimes_{(\rho^{-1}\pi^{-1})\mathscr{O}_Y}\mathscr{O}_W\\ &\cong(\pi\circ\rho)^*\mathscr{G},\end{aligned}$$

which implies that

$$\rho^*\pi^*\xleftrightarrow{\sim}(\pi\circ\rho)^*.$$

□

Theorem 8.2.1 ((π^*,π_*) is adjoint pair)

π^* left-adjoint to π_*. More precisely, for any $\mathscr{F}\in\mathbf{Mod}_{\mathscr{O}_X}$, there is a natural isomorphism

$$\mathrm{Hom}_{(X,\mathscr{O}_X)}(\pi^*\mathscr{G},\mathscr{F})\xleftrightarrow{\sim}\mathrm{Hom}_{(Y,\mathscr{O}_Y)}(\mathscr{G},\pi_*\mathscr{F})$$

that is functorial in both $\mathscr{F}\in\mathbf{Mod}_{\mathscr{O}_X}$ and $\mathscr{G}\in\mathbf{Mod}_{\mathscr{O}_Y}$.

Proof Since $\pi:(X,\mathscr{O}_X)\to(Y,\mathscr{O}_Y)$ is morphism of ringed spaces, we have a morphism of sheaves $\pi^\sharp:\mathscr{O}_Y\to\pi_*\mathscr{O}_X$. By adjointness, we have $\pi^{-1}\mathscr{O}_Y\to\mathscr{O}_X$, hence $\mathscr{F}$ can be seen as $\pi^{-1}\mathscr{O}_Y$-module, denote

$\mathscr{F}_{\pi^{-1}\mathscr{O}_Y}$. By Proposition 2.4.1 and adjointness of (π^{-1}, π_*), we have

$$\begin{aligned}\mathrm{Hom}_{(X,\mathscr{O}_X)}(\pi^*\mathscr{G}, \mathscr{F}) &= \mathrm{Hom}_{\mathbf{Mod}_{\mathscr{O}_X}}(\pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X, \mathscr{F}) \\ &\cong \mathrm{Hom}_{\mathbf{Mod}_{\pi^{-1}\mathscr{O}_Y}}(\pi^{-1}\mathscr{G}, \mathscr{F}_{\pi^{-1}\mathscr{O}_Y}) \\ &\cong \mathrm{Hom}_{(Y,\mathscr{O}_Y)}(\mathscr{G}, \pi^*\mathscr{F}).\end{aligned}$$

□

The push-pull map for $\mathscr{O}$-modules

Definition 8.2.4 (The push-pull map)

Suppose

$$\begin{tikzcd} W \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd} \tag{8.1}$$

is a commutative (not necessarily Cartesian (Definition 2.2.11)) diagram of ringed spaces, and $\mathscr{F}$ is an $\mathscr{O}_X$-module. Define the ***push-pull map***

$$\beta^*\alpha_*\mathscr{F} \longrightarrow \alpha'_*\beta'^*\mathscr{F} \tag{8.2}$$

of $\mathscr{O}_Y$-modules as follows:

i. *Start with the identity*

$$\beta'^*\mathscr{F} \xrightarrow{\sim} \beta'^*\mathscr{F}$$

on W.

ii. *By adjointness of (β'^*, β'_*), this is the same as the data of a morphism*

$$\mathscr{F} \longrightarrow \beta'_*\beta'^*\mathscr{F}$$

on X.

iii. *Apply α_* to get a map*

$$\alpha_*\mathscr{F} \longrightarrow \alpha_*\beta'_*\beta'^*\mathscr{F}$$

on Z.

iv. *By the commutative of diagram (8.1), this is the map*

$$\alpha_*\mathscr{F} \longrightarrow \beta_*\alpha'_*\beta'^*\mathscr{F}$$

on Z.

v. *By adjointness of (β^*, β_*), this yields a map (8.2).*

8.2.3 Properties

Proposition 8.2.6

Given a morphism of ringed spaces $\pi : X \to Y$ with $\pi(p) = q$, then there is a map of stalks $\mathscr{O}_{Y,q} \longrightarrow \mathscr{O}_{X,p}$.

Proof By $\pi : X \to Y$, we have a morphism of sheaves $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, by adjointness of (π^{-1}, π_*), we have $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. By Proposition 3.3.1, we have $(\pi^{-1}\mathscr{O}_Y)_p \to \mathscr{O}_{X,p}$. By Proposition 3.7.2, we have

$\mathscr{O}_{Y,q} \cong (\pi^{-1}\mathscr{O}_Y)_p$, hence there is a map of stalks

$$\mathscr{O}_{Y,q} \longrightarrow \mathscr{O}_{X,p}.$$

□

Definition 8.2.5 (Key Definition)

Suppose $\pi^\sharp : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given in Proposition 4.4.6. To describe a morphism of sheaves $\mathscr{O}_{\operatorname{Spec} B} \to \pi_*\mathscr{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subseteq \operatorname{Spec} B$, we define

$$\mathscr{O}_{\operatorname{Spec} B}(D(g)) \longrightarrow \mathscr{O}_{\operatorname{Spec} A}(\pi^{-1}D(g)) = \mathscr{O}_{\operatorname{Spec} A}(D(\pi^\sharp g))$$

by $B_g \to A_{\pi^\sharp g}$.

✍ **Exercise 8.1** Verify Definition 8.2.5 makes sense (e.g., is independent of g).

Proof We first show that $B_g \to A_{\pi^\sharp g}$ is independent of choice of g. Suppose $D(g_1) = D(g_2)$, by Proposition 4.5.5, g_1 is an invertible element of B_{g_2} and g_2 is an invertible element of B_{g_1}. Define $B_{g_1} \to B_{g_2}$ by setting $\frac{h}{g_1^n} \mapsto \frac{g_1^{-n}h}{1}$. Conversely, define the inverse $B_{g_2} \to B_{g_1}$ by setting $\frac{h}{g_2^m} \mapsto \frac{g_2^{-m}h}{1}$. Hence $B_{g_1} \cong B_{g_2}$, which implies that $B_g \to A_{\pi^\sharp g}$ is independent of choice of g.

We next show that $\pi^{-1}D_{\operatorname{Spec} B}(g) = D_{\operatorname{Spec} A}(\pi^\sharp g)$. Let $\mathfrak{q} \in \pi^{-1}D(g) \subseteq \operatorname{Spec} A$, then $\pi(\mathfrak{q}) \in D(g) \subseteq \operatorname{Spec} B$. Hence $g \notin (\pi^\sharp)^{-1}(\mathfrak{q})$, and therefore $\pi^\sharp(g) \notin \mathfrak{q}$, which implies that $\mathfrak{q} \in D(\pi^\sharp g)$. Conversely, let $\mathfrak{q} \in D(\pi^\sharp g)$, then $\pi^\sharp g \notin \mathfrak{q}$, i.e., $g \notin (\pi^\sharp)^{-1}(\mathfrak{q}) = \pi(\mathfrak{q})$. Hence $\pi(\mathfrak{q}) \in D(g) \subseteq \operatorname{Spec} B$, and therefore $\mathfrak{q} \in \pi^{-1}D(g)$

By Theorem 5.1.1, $\mathscr{O}_{\operatorname{Spec} B}$ and $\mathscr{O}_{\operatorname{Spec} A}$ are sheaves on distinguished base. To describe a morphism of sheaves $\mathscr{O}_{\operatorname{Spec} B} \to \pi_*\mathscr{O}_{\operatorname{Spec} A}$, it suffices to show that $B_g \to A_{\pi^\sharp g}$ commutes with restriction. Let $D(f) \supseteq D(g)$. By Proposition 4.5.5, f is invertible element in B_g and $\pi^\sharp f$ is invertible element in $A_{\pi^\sharp g}$. Define $\varphi : B_f \to B_g$ by setting $\frac{h}{f^n} \mapsto \frac{f^{-n}h}{1}$, and define $\psi : A_{\pi^\sharp f} \to A_{\pi^\sharp g}$ by setting $\frac{t}{(\pi^\sharp f)^m} \mapsto \frac{(\pi^\sharp f)^{-m}t}{1}$. Also $\pi^\sharp_{D(f)} : B_f \to A_{\pi^\sharp f}$ is given by $\frac{h}{f^n} \mapsto \frac{\pi^\sharp(h)}{\pi^\sharp(f)^n}$. It is easy to see they are all well-defined. We want to show that the following diagram is commutes.

$$\begin{tikzcd} B_f \arrow[r, "\pi^\sharp_{D(f)}"] \arrow[d, "\varphi"'] & A_{\pi^\sharp f} \arrow[d, "\psi"] \\ B_g \arrow[r, "\pi^\sharp_{D(g)}"] & A_{\pi^\sharp g} \end{tikzcd}$$

Let $\frac{h}{f^n} \in B_f$, then we have

$$\pi^\sharp_{D(g)} \circ \varphi\left(\frac{h}{f^n}\right) = \pi^\sharp_{D(g)}\left(\frac{f^{-n}h}{1}\right) = \frac{\pi^\sharp(f)^{-n}\pi^\sharp(h)}{1}$$

and

$$\psi \circ \pi^\sharp_{D(f)}\left(\frac{h}{f^n}\right) = \psi\left(\frac{\pi^\sharp(h)}{\pi^\sharp(f)^n}\right) = \frac{\pi^\sharp(f)^{-n}\pi^\sharp(h)}{1},$$

hence $\pi^\sharp_{D(g)} \circ \varphi = \psi \circ \pi^\sharp_{D(f)}$, which implies that $B_g \to A_{\pi^\sharp g}$ commutes with restriction. □

Remark Definition 8.2.5 describes a morphism of sheaves on the distinguished base. Definition 8.2.5 is the third in a series of exercise. We saw that a morphism of rings induces a map of set in §4.2.3, a map of topological spaces in Proposition 4.4.6, and now a map of ringed spaces here.

The map of ringed spaces of Key Definition 8.2.5 is really not complicated. Here is an example.

Example 8.1 Consider the ring map $\mathbb{C}[y] \to \mathbb{C}[x]$ given by $y \mapsto x^2$ (see Figure 4.8). We are mapping the affine line with coordinate x to the affine line with coordinate y, i.e., $\operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[y]$. The map is (on closed points) $a \mapsto a^2$. For example, where does $[(x-3)]$ go to? Answer: $[(y-3)]$, i.e., $3 \mapsto 9$. What is the preimage of $[(y-4)]$? Answer: those prime ideals in $\mathbb{C}[x]$ containing $[(x^2-4)]$, i.e., $[(x-2)]$ and $[(x+2)]$, so the preimage of 4 is indeed ± 2. This is just about the map of sets, which is old news (§4.2.3), so let's now think about functions pulling back. What is the pullback of the function $\frac{3}{y-4}$ on $D(y-4) = \mathbb{A}^1 = \{4\}$? Of course it is $\frac{3}{x^2-4}$ on $\mathbb{A}^1 \setminus \{-2, 2\}$.

The construction of Key Definition 8.2.5 will soon be an example of a morphism of schemes. In fact we could make that definition right now. Before we do, we point out (via the next example) that not every morphism of ringed spaces between affine schemes is of the form of Ker Definition 8.2.5. In the language of §8.3, this morphism of ringed spaces is not a morphism of locally ringed spaces. We won't use this example, but it answers a question that everyone will have.

Example 8.2 **Not every morphism of ringed spaces between affine schemes is of the form of Key Definition 8.2.5.** Recall (Exercise 4.16) that $\operatorname{Spec} k[y]_{(y)}$ has two points, $[(0)]$ and $[(y)]$, where the second point is closed, and the first is not. Describe a map of ringed spaces $\operatorname{Spec} k(x) \to \operatorname{Spec} k[y]_{(y)}$ sending the unique point of $\operatorname{Spec} k(x)$ to the closed point $[(y)]$, where the pullback map on global sections sends k to k by the identity, and sends y to x. Show that this map of ringed spaces is not of the form described in Key Definition 8.2.5.

Proof Say $X = \operatorname{Spec} k(x)$ and $Y = \operatorname{Spec} k[y]_{(y)}$. Define $\pi : X \to Y$ by setting $\pi((0)) = [(y)]$. Define $\pi^\sharp_Y : k[y]_{(y)} \to k(x)$ by setting $y \mapsto x$ and $k \mapsto k$. We want to show that π is not of the form described in Key Definition 8.2.5. Note that $(\pi^\sharp_Y)^{-1}((0)) = (0)$, but $\pi([(0)]) = [(y)]$, which implies that π is not of the form described in Key Definition 8.2.5. $\square$

8.2.4 Tentative Definition we won't use

A morphism of schemes $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces that "locally looks like" the maps of affine schemes described in Key Definition 8.2.5. Precisely, for each choice of affine open sets $\operatorname{Spec} A \subseteq X$, $\operatorname{Spec} B \subseteq Y$, such that $\pi(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$, the induced map of ringed spaces should be of the form shown in Exercise 8.1.

We would like this definition to be checkable on any affine cover, and we might hope to use the Affine Communication Lemma to develop the theory in this way. This works, but it will be more convenient to use a clever trick: in the next section, we will use the notion of locally ringed spaces, and then once we have used it, we will discard it like yesterday's garbage.

8.3 From locally ringed spaces to morphisms of schemes

8.3.1 Morphisms of locally ringed spaces and morphisms of schemes

In order to prove that morphisms behave in a way we hope, we will use the notion of a locally ringed space. It will not be used later, although it is useful elsewhere in geometry. The notion of locally ringed spaces (and maps between them) is inspired by what we know about manifolds (see Proposition 4.1.2). If $\pi : X \to Y$ is a morphism of manifolds, with $\pi(p) = q$, and f is a function on Y vanishing at q, then the pulled back function $\pi^\sharp(f)$ on X should vanish on p. Put differently: germs of functions (at $q \in Y$) vanishing at should pull back to germs of functions (at $p \in X$) vanishing at p.

Recall (Definition 5.3.7) that a locally ringed space is a ringed space $(X, \mathscr{O}_X)$ such that the stalks $\mathscr{O}_{X,p}$ are all local rings.

Definition 8.3.1 (local homomorphism of local rings)

*A **local homomorphism of local rings** is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "**local ring map** $\varphi : R \to S$" means that φ is a local homomorphism of local rings.*

Definition 8.3.2 (Morphism of locally ringed spaces)

*A **morphism of locally ringed spaces** $(\pi, \pi^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces such that for all $p \in X$ the induced ring map (Proposition 8.2.6) $\pi_p^\sharp : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a local ring map (Definition 8.3.1), i.e., $\pi_p^\sharp(\mathfrak{m}_{Y,\pi(p)}) \subseteq \mathfrak{m}_{X,p}$.*

Remark In particular that locally ringed spaces form a category.

Definition 8.3.2 means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." You would also want: "if $p \mapsto q$, and g is a function not vanishing at q, then it will pull back to a function not vanishing at p." This is a consequence of the following Proposition.

Proposition 8.3.1

If $\pi : X \to Y$ is a morphism of locally ringed spaces, and $\pi(p) = q$, then π induces an inclusion $\kappa(q) \hookrightarrow \kappa(p)$ of residue fields. This captures that fact pullback sends the locus where a function is zero (resp., nonzero) to the locus where the pulled back function is zero (resp., nonzero). In particular, by Proposition 5.3.6 (b), the pullback of invertible functions are invertible functions.

Proof By Proposition 8.2.6, we have a map of stalks $\pi_p^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$. Since $\pi : X \to Y$ is a morphism of locally ringed spaces, we have $\pi_p^\sharp(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$. It induces a morphism of residue fields $\widetilde{\pi_p^\sharp} : \kappa(q) \to \kappa(p)$ which is given by $f + \mathfrak{m}_{Y,q} \mapsto \pi_p^\sharp(f) + \mathfrak{m}_{X,p}$. We want to show that $\widetilde{\pi_p^\sharp}$ is injective. Let $\widetilde{\pi_p^\sharp}(f + \mathfrak{m}_{Y,q}) = 0$, then $\pi_p^\sharp(f) \in \mathfrak{m}_{Y,q}$. From $\pi_p^\sharp(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$, we have $\mathfrak{m}_{Y,q} \subseteq (\pi_p^\sharp)^{-1}(\mathfrak{m}_{X,p})$, since $\mathscr{O}_{Y,q}$ is local ring, $\mathfrak{m}_{Y,q} = (\pi_p^\sharp)^{-1}(\mathfrak{m}_{X,p})$, and therefore $f \in \mathfrak{m}_{Y,q}$, which implies that $f + \mathfrak{m}_{Y,q} = 0$ in $\kappa(q)$. Hence $\kappa(q) \to \kappa(p)$ is an inclusion, i.e., $\kappa(q) \hookrightarrow \kappa(p)$. $\square$

To summarize: we use the notion of locally ringed space only to define morphisms of schemes, and to show that morphisms have reasonable properties. **The main things you need to remember about locally ringed spaces are**

(i) that the functions have values at points,

(ii) that given a map of locally ringed spaces, values of functions "pull back", and in particular, zeros of functions "pull back".

Proposition 8.3.2 (Morphism of locally ringed spaces glue)

Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are locally ringed spaces, $X = \bigcup_i U_i$ is an open cover of X, and we have morphisms of locally ringed spaces $\pi_i : U_i \to Y$ that "agree on the overlaps", i.e., $\pi_i|_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$. Then there is a unique morphism of locally ringed spaces $\pi : X \to Y$ such that $\pi|_{U_i} = \pi_i$.

Proof By Proposition 8.2.1, we may define $\pi : X \to Y$ by setting $\pi|_{U_i} = \pi_i$, then we get a unique morphisms of ringed spaces from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$. To show π is a morphism of locally ringed spaces, it suffices to show that for all $p \in X$ the induced ring map $\pi_p^\sharp : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a local ring map.

Pick $p \in X$, since $X = \bigcup_i U_i$, then $p \in U_i$ for some i. Hence $\mathscr{O}_{X,p} \cong \mathscr{O}_{U_i,p}$ and $\pi_p^\sharp = (\pi_i)_p^\sharp$. By condition, $(\pi_i)_p^\sharp$ is a local ring map, we have $\pi_p^\sharp : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is local ring map. Hence $\pi : X \to Y$ is morphism of locally ringed spaces. □

Proposition 8.3.3 (Important!)

Recall that Example 5.3 $\operatorname{Spec} A$ *is a locally ringed space. Then the morphism of ringed spaces* $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ *defined by a ring morphism* $\pi^\sharp_{\operatorname{Spec} B} : B \to A$ *(Definition 8.2.5, for simply we write* $\pi^\sharp = \pi^\sharp_{\operatorname{Spec} B}$*) is a morphism of locally ringed spaces.*

Proof Let $[\mathfrak{p}] \in \operatorname{Spec} A$ be any point of $\operatorname{Spec} A$, consider the induced map $\pi^\sharp_{[\mathfrak{p}]} : B_{(\pi^\sharp)^{-1}(\mathfrak{p})} \to A_\mathfrak{p}$, we want to show this map is local ring map. Say $\mathfrak{q} = (\pi^\sharp)^{-1}(\mathfrak{p})$.

Consider $\pi^\sharp_{[\mathfrak{p}]}(\mathfrak{q}B_\mathfrak{q})$, let $\frac{bf}{s} \in \mathfrak{q}B_\mathfrak{q}$ where $b \in B$, $f \in \mathfrak{q}$, and $s \in B \setminus \mathfrak{q}$, then

$$\pi^\sharp_{[\mathfrak{p}]}\left(\frac{bf}{s}\right) = \frac{\pi^\sharp(b)\pi^\sharp(f)}{\pi^\sharp(s)}.$$

Note that $\pi^\sharp(b) \in A$, $\pi^\sharp(f) \in \mathfrak{p}$ and $\pi^\sharp(s) \in \pi^\sharp(B \setminus \mathfrak{q}) \subseteq A \setminus \mathfrak{p}$, we have $\frac{\pi^\sharp(b)\pi^\sharp(f)}{\pi^\sharp(s)} \in \mathfrak{p}A_\mathfrak{p}$, i.e., $\pi^\sharp_{[\mathfrak{p}]}(\mathfrak{q}B_\mathfrak{q}) \subseteq \mathfrak{p}A_\mathfrak{p}$. It follows that $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is a locally ringed space. □

Remark **Small remark:** Why $\pi^\sharp(B \setminus \mathfrak{q}) \subseteq A \setminus \mathfrak{p}$? If not, there exists $s \in B \setminus \mathfrak{q}$ such that $\pi^\sharp(s) \in \mathfrak{p}$, hence $s \in (\pi^\sharp)^{-1}(\mathfrak{p}) = \mathfrak{q}$, a contradiction.

Proposition 8.3.4 (Key Proposition)

If $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ *is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map* $\pi^\sharp_{\operatorname{Spec} B} : B = \Gamma(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) \to \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) = A$ *as in Proposition 8.3.3.*

Remark Proposition 5.3.1 is a special case of Key Proposition 8.3.4.

Proof Suppose $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces. We wish to show that it is determined by its map on global sections $\pi^\sharp_{\operatorname{Spec} B} : B \to A$. We first need to check that the map of points is determined by global sections. Now a point p of $\operatorname{Spec} A$ can be identified with the prime ideal of global functions vanishing on it. By Proposition 8.3.1, we have inclusion $\kappa(\pi(p)) \hookrightarrow \kappa(p)$, note that $\kappa(\pi(p)) \cong K(B/\pi(p))$, we get a map

$$B \longrightarrow B/\pi(p) \hookrightarrow K(B/\pi(p)) \xrightarrow{\sim} \kappa(\pi(p)) \hookrightarrow \kappa(p), \tag{8.3}$$

hence $\operatorname{Ker}(B \to \kappa(p)) = \pi(p)$. (Here we use the fact that π is a map of locally ringed space.) Also note that

$$B \xrightarrow{\pi^\sharp_{\operatorname{Spec} B}} A \longrightarrow A/p \hookrightarrow K(A/p) \xrightarrow{\sim} \kappa(p), \tag{8.4}$$

we have $\operatorname{Ker}(B \to \kappa(p)) = (\pi^\sharp_{\operatorname{Spec} B})^{-1}(p)$. It is easy to check that (8.3) and (8.4) defined the same map, hence $\pi(p) = (\pi^\sharp_{\operatorname{Spec} B})^{-1}(p)$. It follows that π is the map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by a ring map $B \to A$.

By the proof in Exercise 8.1, we have $\pi^{-1}(D(g)) = D(\pi^\sharp_{\operatorname{Spec} B} g)$ if $g \in B$.

It remains to show that $\pi^\sharp : \mathscr{O}_{\operatorname{Spec} B} \to \pi_* \mathscr{O}_{\operatorname{Spec} A}$ is the morphism of sheaves given by Definition 8.2.5. It suffices to check this on the distinguished base. We now want to check that for any map of locally ringed spaces inducing the map of sheaves $\mathscr{O}_{\operatorname{Spec} B} \to \pi_* \mathscr{O}_{\operatorname{Spec} A}$, the map of sections on any distinguished open set $D(g) \subseteq \operatorname{Spec} B$ $(g \in B)$ is determined by the map of global sections $B \to A$.

Consider the commutative diagram

$$\begin{tikzcd}
B \arrow[r, equal] & \Gamma(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) \arrow[r, "\pi^\sharp_{\operatorname{Spec} B}"] \arrow[d, "\operatorname{res}_{\operatorname{Spec} B, D(g)}"'] & \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \arrow[r, equal] \arrow[d, "\operatorname{res}_{\operatorname{Spec} A, D(\pi^\sharp g)}"] & A \\
B_g \arrow[r, equal] & \Gamma(D(g), \mathscr{O}_{\operatorname{Spec} B}) \arrow[r, "\pi^\sharp_{D(g)}"'] & \Gamma(D(\pi^\sharp g), \mathscr{O}_{\operatorname{Spec} A}) \arrow[r, equal] & A_{\pi^\sharp g} = A \otimes_B B_g.
\end{tikzcd}$$

The vertical arrows (restrictions to distinguished open sets) are localizations by g, so the lower horizontal map $\pi^\sharp_{D(g)}$ is determined by the upper map (it is just localization by g). □

We are ready for our definition.

Definition 8.3.3 (Morphism of schemes)

If X and Y are schemes, then a morphism $(\pi, \pi^\sharp) : X \to Y$ as locally ringed spaces is called a ***morphism of schemes****. We have thus defined the category of schemes, which we denote* **Sch***. We then have notions of* ***isomorphism*** *— just the same as Definition 5.3.3 — and* ***automorphism****. The target Y of π is sometimes called the* ***base scheme*** *or the* ***base*** *(this may become clearer once we have defined the fibers of morphisms in Chapter 11.)*

The definition in terms of locally ringed spaces easily implies Tentative Definition §8.2.4:

Proposition 8.3.5 (Important!)

A morphism of schemes $\pi : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affine schemes. Precisely:

(1) If $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $\pi(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes.

(2) Suppose $\pi : X \to Y$ is a morphism of ringed space, such that $X = \bigcup_i \operatorname{Spec} A_i$ and $Y = \bigcup_i \operatorname{Spec} B_i$ with the property that $\pi(\operatorname{Spec} A_i) \subseteq \operatorname{Spec} B_i$ for each i. Then π is a morphism of schemes if and only if the restriction of π to each $\operatorname{Spec} A_i$ gives a morphism of affine schemes from $\operatorname{Spec} A_i$ to $\operatorname{Spec} B_i$.

Proof

(1) Since $\pi : X \to Y$ is a morphism of schemes, then by definition it is a morphism of locally ringed space. If $\operatorname{Spec} A \subseteq X$ and $\operatorname{Spec} B \subseteq Y$ are affine opens with $\pi(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$, then the restriction of π to $\operatorname{Spec} A$ gives a map of locally ringed space, i.e., $\pi|_{\operatorname{Spec} A} : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes.

(2) If π is a morphism of schemes, by Part (1), the restriction of π to each $\operatorname{Spec} A_i$ gives a morphism of affine schemes from $\operatorname{Spec} A_i$ to $\operatorname{Spec} B_i$.

Conversely, if the restriction of π to each $\operatorname{Spec} A_i$ gives a morphism of affine schemes from $\operatorname{Spec} A_i$ to $\operatorname{Spec} B_i$. Pick $p \in X$, since $X = \bigcup_i \operatorname{Spec} A_i$, $p \in \operatorname{Spec} A_i$ for some i, since $\pi(\operatorname{Spec} A_i) \subseteq \operatorname{Spec} B_i$, we also have $\pi(p) \in \operatorname{Spec} B_i$. We want to show that $\pi^\sharp_p : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a local ring map. Note that $\mathscr{O}_{X,p} \cong \mathscr{O}_{\operatorname{Spec} A_i, p}$ and $\mathscr{O}_{Y,\pi(p)} \cong \mathscr{O}_{\operatorname{Spec} B_i, \pi(p)}$. Since $\pi|_{\operatorname{Spec} A} : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes, $\mathscr{O}_{\operatorname{Spec} B_i, \pi(p)} \to \mathscr{O}_{\operatorname{Spec} A_i, p}$ is a local ring map, and therefore $\pi^\sharp_p : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a local ring map, which implies that π is a morphism of schemes.

□

In practice, we will use the affine cover interpretation, and forget completely about locally ringed spaces.

In particular, put imprecisely, the category of affine schemes is the category of rings with the arrows reversed. More precisely:

Proposition 8.3.6

The category of rings $\mathbf{Rings}$ *and the opposite category of affine schemes* $\mathbf{Aff.Sch}^{\mathrm{op}}$ *are equivalent.*

Proof Define the functor $F : \mathbf{Rings} \to \mathbf{Aff.Sch}^{\mathrm{op}}$ as follow. Let $A \in \mathbf{Rings}$, set $F(A) = \operatorname{Spec} A$. Let $\pi^{\sharp}_{\operatorname{Spec} B} \in \operatorname{Mor}_{\mathbf{Rings}}(B, A)$, set $F(\pi^{\sharp}_{\operatorname{Spec} B}) = \pi$ where $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by $\pi(\mathfrak{p}) = (\pi^{\sharp}_{\operatorname{Spec} B})^{-1}(\mathfrak{p})$. By Proposition 8.3.4, this is well-defined. It is easy to see that $F : \mathbf{Rings} \to \mathbf{Aff.Sch}^{\mathrm{op}}$ is a functor.

Define the functor $G : \mathbf{Aff.Sch}^{\mathrm{op}} \to \mathbf{Rings}$ as follow. Let $\operatorname{Spec} A \in \mathbf{Aff.Sch}^{\mathrm{op}}$, set $G(\operatorname{Spec} A) = A$. Let $\pi \in \operatorname{Mor}_{\mathbf{Aff.Sch}^{\mathrm{op}}}(\operatorname{Spec} A, \operatorname{Spec} B)$, set $G(\pi) = \pi^{\sharp}_{\operatorname{Spec} B}$. By the definition of morphism of schemes, G is well-defined. Also, it is easy to see that $G : \mathbf{Aff.Sch}^{\mathrm{op}} \to \mathbf{Rings}$ is a functor.

Next, we shall to show that $F \circ G \cong \mathrm{id}_{\mathbf{Aff.Sch}^{\mathrm{op}}}$. Let $\operatorname{Spec} A, \operatorname{Spec} B \in \mathbf{Aff.Sch}^{\mathrm{op}}$, consider the following diagram,

$$\begin{array}{ccc} F \circ G(\operatorname{Spec} A) & \xrightarrow{F \circ G(\pi)} & F \circ G(\operatorname{Spec} B) \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \xrightarrow{\quad \pi \quad} & \operatorname{Spec} B, \end{array}$$

note that $F \circ G(\operatorname{Spec} A) = \operatorname{Spec} A$, $F \circ G(\operatorname{Spec} B) = \operatorname{Spec} B$, and $F \circ G(\pi) = F(\pi^{\sharp}_{\operatorname{Spec} B}) = \pi$ (by Proposition 8.3.3), clearly, the diagram commutes and $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathbf{Aff.Sch}^{\mathrm{op}}}$.

Similarly, we can show that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathbf{Rings}}$. Hence $\mathbf{Rings}$ and $\mathbf{Aff.Sch}^{\mathrm{op}}$ are equivalent. □

In particular, there can be many different maps from one point to another! For example, here are two different maps from the point $\operatorname{Spec} \mathbb{C}$ to the point $\operatorname{Spec} \mathbb{C}$: the identity (corresponding to the identity $\mathbb{C} \to \mathbb{C}$), and complex conjugation. (There are even more such maps!)

It is clear (from the corresponding facts about locally ringed spaces) that morphisms of schemes glue (Proposition 8.3.2), and the composition of two morphisms is a morphism. Isomorphism in this category are precisely what we defined them to be earlier (Definition 5.3.3).

Exercise 8.2 Enlightening Exercise. This exercise can give you some practice with understanding morphisms of schemes by cutting up into affine open sets. Make sense of the following sentence: "$\mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$ given by

$$(x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \cdots : x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you may have to divide these up into affines, and describe the maps, and check that they glue.

Solution *In fact,* $\mathbb{A}^{n+1}_k \setminus \{0\} = \bigcup_{i=0}^{n} D(x_i)$ *and* $\mathbb{P}^n_k = \bigcup_{i=0}^{n} D_+(x_i)$. *By §5.4.3, we have*

$$D_+(x_i) \cong \operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \cong \operatorname{Spec} k[x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i}].$$

Also we have $D(x_i) \cong \operatorname{Spec} k[x_0, \ldots, x_n]_{x_i}$. *Define the maps of ring*

$$(\pi_i)^{\sharp}_{D_+(x_i)} : k[x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i}] \to k[x_0, \ldots, x_n]_{x_i}$$

by setting $x_{j/i} \mapsto \frac{x_j}{x_i}$, *then by Proposition 8.3.3, it gives a morphism of affine schemes* $\pi_i : D(x_i) \to D_+(x_i)$ *which defined by* $\pi_i(p) = ((\pi_i)^{\sharp}_{D_+(x_i)})^{-1}(p)$ *as continuous map and morphism of sheaves as Definition 8.2.5. In fact* $\pi_i|_{D(x_i x_j)} : D(x_i x_j) \to D_+(x_i x_j)$ *and* $\pi_j|_{D(x_i x_j)} : D(x_i x_j) \to D_+(x_i x_j)$, *we want to* *show that $\pi_i|_{D(x_ix_j)} = \pi_j|_{D(x_ix_j)}$ as morphisms of schemes. It suffices to check that $(\pi_i|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)} = (\pi_j|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)}$ as ring maps. By Proposition 5.5.10, we have*

$$\begin{aligned}(\pi_i|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)} : k[x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i}]_{x_{j/i}} &\longrightarrow k[x_0, \ldots, x_n]_{x_ix_j} \\ \frac{f(x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i})}{(x_{j/i})^n} &\longmapsto \frac{x_i^{2n} f(x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i})}{(x_ix_j)^n}\end{aligned}$$

and

$$\begin{aligned}(\pi_j|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)} : k[x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j}]_{x_{i/j}} &\longrightarrow k[x_0, \ldots, x_n]_{x_ix_j} \\ \frac{f(x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j})}{(x_{i/j})^n} &\longmapsto \frac{x_j^{2n} f(x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j})}{(x_ix_j)^n}.\end{aligned}$$

Note that on $D_+(x_i) \cap D_+(x_j)$ we have coordinate change

$$x_{j/i} = \frac{1}{x_{i/j}} \quad \text{and} \quad x_{k/i} = \frac{x_{k/j}}{x_{i/j}},$$

hence

$$k[x_{0/i}, \ldots, x_{i-1/i}, x_{i+1/i}, \ldots, x_{n/i}]_{x_{j/i}} \cong k[x_{0/j}, \ldots, x_{j-1/j}, x_{j+1/j}, \ldots, x_{n/j}]_{x_{i/j}},$$

and therefore $(\pi_i|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)}$ and $(\pi_j|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)}$ define the same ring maps, i.e., $(\pi_i|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)} = (\pi_j|_{D(x_ix_j)})^\sharp_{D_+(x_ix_j)}$. Hence $\pi_i|_{D(x_ix_j)} = \pi_j|_{D(x_ix_j)}$ as morphisms of affine schemes. Let $\iota_i : D_+(x_i) \hookrightarrow \mathbb{P}^n_k$ be open immersion, then $(\iota_i \circ \pi_i)|_{D(x_ix_j)} = (\iota_j \circ \pi_j)|_{D(x_ix_j)}$. By Morphism of locally ringed spaces glue 8.3.2, there is a unique morphism of schemes $\pi : \mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$ such that $\pi|_{D(x_i)} = \iota_i \circ \pi_i$. By above discussion, it is easy to see that π is indeed given by

$$(x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \cdots : x_n].$$

8.3.2 Morphisms to affine schemes

The following result shows that it is easy to describe morphisms to an affine scheme without working hard to cover the source with affine open sets.

Let X, Y be two schemes. We have a canonical map

$$\rho : \mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Rings}}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X)), \tag{8.5}$$

which to $(\pi, \pi^\sharp)$ associates $\pi^\sharp_Y : \mathscr{O}_Y(Y) \to \pi_*\mathscr{O}_X(Y) = \mathscr{O}_X(X)$. This map is "functorial" in X in the sense that for any morphism of schemes $\eta : Z \to X$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mor}(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Rings}}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \arrow[d] \\ \mathrm{Mor}(Z, Y) \arrow[r] & \mathrm{Hom}_{\mathbf{Rings}}(\mathscr{O}_Y(Y), \mathscr{O}_Z(Z)) \end{tikzcd} \tag{8.6}$$

the vertical arrow on the left being the composition with η, i.e., $\square \circ \eta$; the vertical arrow on the right being the composition with $\eta^\sharp_X : \mathscr{O}_X(X) \to \mathscr{O}_Z(Z)$, i.e., $\eta^\sharp_X \circ \square$.

Lemma 8.3.1

Let X, Y be affine schemes. Then the map (8.5) defined above is bijection.

Proof Apply Proposition 8.3.6, we done. $\square$

Proposition 8.3.7

Let Y be an affine scheme. For any scheme X, the canonical map

$$\rho : \operatorname{Mor}(X,Y) \longrightarrow \operatorname{Hom}_{\mathbf{Rings}}(\Gamma(Y,\mathscr{O}_Y),\Gamma(X,\mathscr{O}_X))$$

is bijection.

Proof We may assume that $X = \bigcup_i U_i$ where the U_i are affine open subsets. Using the commutative diagram (8.6) above with each $U_i \hookrightarrow X$, we obtain a new commutative diagram:

$$\begin{array}{ccc} \operatorname{Mor}(X,Y) & \xrightarrow{\rho} & \operatorname{Hom}(\Gamma(Y,\mathscr{O}_Y),\Gamma(X,\mathscr{O}_X)) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \prod_i \operatorname{Mor}(U_i,Y) & \xrightarrow{\gamma} & \prod_i \operatorname{Hom}(\Gamma(Y,\mathscr{O}_Y),\Gamma(U_i,\mathscr{O}_X)). \end{array}$$

By Lemma 8.3.1, γ is bijective. As α is clearly injective (Morphism of locally ringed spaces glue 8.3.2), it follows that ρ is injective. Let $\varphi \in \operatorname{Hom}(\Gamma(Y,\mathscr{O}_Y),\Gamma(X,\mathscr{O}_X))$. Say $\iota_i : U_i \hookrightarrow X$, then we have ring maps $(\iota_i)^\sharp_X : \Gamma(X,\mathscr{O}_X) \to \Gamma(U_i,\mathscr{O}_X)$. In fact, we have $\beta(\varphi) = ((\iota_i)^\sharp_X \circ \varphi)_i$. Since $\gamma = (\gamma_i)_i$ is bijection, say $\pi_i := \gamma_i^{-1}((\iota_i)^\sharp_X \circ \varphi)$. Indeed, for every affine open $V \subseteq U_i \cap U_j$, $\pi_i|_V = \pi_j|_V$ have the same image in $\operatorname{Hom}(\mathscr{O}_Y(Y),\mathscr{O}_X(V))$, and therefore $\pi_i|_{U_i\cap U_j} = \pi_j|_{U_i\cap U_j}$. By Morphism of locally ringed spaces glue 8.3.2, there exists a unique $\pi \in \operatorname{Mor}(X,Y)$ such that $\pi|_{U_i} = \pi_i$. Hence $\rho(\pi) = \varphi$ by the injectivity of β, which proves that ρ is surjective. □

Remark This is even true in the category of locally ringed spaces (see Stacks Project [8] Lemma 01I1).

Remark In particular, we have a bijection

$$\operatorname{Mor}(X,\operatorname{Spec}\Gamma(X,\mathscr{O}_X)) \xrightarrow{\sim} \operatorname{Hom}(\Gamma(X,\mathscr{O}_X),\Gamma(X,\mathscr{O}_X)),$$

hence there is a canonical morphism from a scheme to Spec of its ring of global sections. (**Warning:** Even if X is a finite type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.) The canonical morphism $X \to \operatorname{Spec}\Gamma(X,\mathscr{O}_X)$ is an isomorphism if and only if X is affine, and in this case it is the isomorphism hinted at in Remark after Definition 5.3.4.

✍ **Exercise 8.3** If $S_\bullet$ is a finitely generated graded A-algebra, describe a natural "structure morphism" $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

Proof Say $\operatorname{Proj} S_\bullet = \bigcup_i D_+(f_i)$, by Proposition 5.5.9, we have $D_+(f_i) \cong \operatorname{Spec}((S_\bullet)_{f_i})_0$. Define $(\pi_i)^\sharp_{\operatorname{Spec} A} : A \to ((S_\bullet)_{f_i})_0$ by setting

$$A = S_0 \hookrightarrow S_\bullet \xrightarrow{\text{localization}} (S_\bullet)_{f_i} \longrightarrow ((S_\bullet)_{f_i})_0.$$

the image of $(\pi_i)^\sharp_{\operatorname{Spec} A}$ in $((S_\bullet)_{f_i})_0$. By Proposition 8.3.7, $(\pi_i)^\sharp_{\operatorname{Spec} A}$ induces a morphism of schemes $(\pi_i,\pi_i^\sharp) : D_+(f_i) \to \operatorname{Spec} A$. We want to show that $\pi_i|_{D_+(f_i)\cap D_+(f_j)} = \pi_j|_{D_+(f_i)\cap D_+(f_j)}$ as morphisms of schemes. It suffice to show that $(\pi_i|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A} = (\pi_j|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A}$. In fact, by diagram (8.6), $(\pi_i|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A}$ is given by

$$A \xrightarrow{(\pi_i)^\sharp_{\operatorname{Spec} A}} ((S_\bullet)_{f_i})_0 \xrightarrow{\text{localization}} ((S_\bullet)_{f_i})_{\frac{f_j^{\deg f_i}}{f_i^{\deg f_j}}} \cong ((S_\bullet)_{f_if_j})_0,$$

and $(\pi_i|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A}$ is given by

$$A \xrightarrow{(\pi_j)^\sharp_{\operatorname{Spec} A}} ((S_\bullet)_{f_j})_0 \xrightarrow{\text{localization}} ((S_\bullet)_{f_j})_{\frac{f_i^{\deg f_j}}{f_j^{\deg f_i}}} \cong ((S_\bullet)_{f_if_j})_0.$$

Hence $(\pi_i|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A}$ and $(\pi_j|_{D_+(f_i)\cap D_+(f_j)})^\sharp_{\operatorname{Spec} A}$ give the same ring map, and therefore

$$\pi_i|_{D_+(f_i)\cap D_+(f_j)} = \pi_j|_{D_+(f_i)\cap D_+(f_j)}$$

as morphisms of schemes. By Morphism of locally ringed spaces glue 8.3.2, there is a unique morphism of locally ringed spaces $\pi : \operatorname{Proj} S_\bullet \to \operatorname{Spec} A$ such that $\pi|_{D_+(f_i)} = \pi_i$. □

Remark From Proposition 8.3.7, it is one small step to some products of schemes exists: if A and B are rings, then $\operatorname{Spec} A \times \operatorname{Spec} B = \operatorname{Spec}(A \otimes_{\mathbb{Z}} B)$; and if A and B are C-algebras, then $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_C B)$. But we are in no hurry, so we wait until Chapter 11 to discuss this properly.

Remark **Side fact for experts: Γ and Spec are adjoints.** We have a contravariant functor Spec from rings to locally ringed spaces, and a contravariant functor Γ from locally ringed spaces to ring. In fact $(\Gamma, \operatorname{Spec})$ is an adjoint pair! Thus we could have defined Spec by requiring it to be right-adjoint to Γ. In fact, if we used ringed spaces rather than locally ringed spaces, Γ again has a right adjoint.

Corollary 8.3.1

Spec *is a contravariant functor from rings to locally ringed spaces,* Γ *is a contravariant functor from locally ring spaces to rings, then* $(\Gamma, \operatorname{Spec})$ *is adjoint pair.*

Proof Denote $\mathscr{C}$ be the category of locally ringed spaces. In fact, $\Gamma : \mathscr{C}^{\mathrm{op}} \to \mathbf{Rings}$ and $\operatorname{Spec} : \mathbf{Rings} \to \mathscr{C}^{\mathrm{op}}$. Let $X \in \mathscr{C}$ and $A \in \mathbf{Rings}$, by Stacks Project [8] Lemma 01I1, we have a bijection

$$\operatorname{Hom}_{\mathbf{Rings}}(\Gamma(X, \mathscr{O}_X), A) \xrightarrow{\sim} \operatorname{Mor}_{\mathscr{C}^{\mathrm{op}}}(X, \operatorname{Spec} A),$$

which implies that $(\Gamma, \operatorname{Spec})$ is adjoint pair. □

Our ability to easily describe morphisms to affine schemes will allow us to revisit our earlier discussion of schemes over a given field or ring (§6.3.2).

8.3.3 The category of complex schemes (or more generally the category of k-schemes where k is a field, or more generally the category of A-schemes where A is a ring, or more generally the category of S-scheme where S is a scheme)

Definition 8.3.4 (The category of S-schemes $\mathbf{Sch}_S$)

*The **category of** S**-schemes** $\mathbf{Sch}_S$ (where S is a scheme) defined as follows. The objects S-schemes are morphisms of the form*

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

*(The morphism to S is called the **structure morphism**. A motivation for this terminology is the fact that if $S = \operatorname{Spec} A$, the structure morphism gives the functions on each open set of X the structure of an A-algebra, cf. §6.3.2) The morphisms in the category of S-schemes are defined to be commutative diagrams*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \xrightarrow{=} & S \end{array}$$

which is more conveniently written as a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

When there is no confusion (if the base scheme is clear), simply the top row of the diagram is given. In the case where $S = \operatorname{Spec} A$, where A is a ring, we get the notion of an A-scheme, which is the same as the same definition as in Definition 6.3.3, but in a more satisfactory form.

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

✍ **Exercise 8.4** Show that this definition of A-scheme given in Definition 8.3.4 agrees with the earlier definition with the earlier definition of Definition 6.3.3.

Proof Let X be an A-scheme, then we get a morphism of schemes $\pi : X \to \operatorname{Spec} A$, we want to show that for all open sets $U \subseteq X$, $\Gamma(U, \mathscr{O}_X)$ is A-algebra. By the definition of morphism of schemes, we have ring morphism $\pi^\sharp_{\operatorname{Spec} A} : A \to \Gamma(X, \mathscr{O}_X)$. Hence for all open subset $U \subseteq X$, we have ring map

$$A \longrightarrow \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(U, \mathscr{O}_X).$$

It follows that $\Gamma(U, \mathscr{O}_X)$ is an A-algebra, and therefore X is A-scheme as Definition 6.3.3.

Conversely, if X is an A-scheme as Definition 6.3.3, then for all open $U \subseteq X$, $\Gamma(U, \mathscr{O}_X)$ is A-algebra. In particular, $\Gamma(X, \mathscr{O}_X)$ is an A-algebra, hence there is a ring map

$$A \longrightarrow \Gamma(X, \mathscr{O}_X),$$

by Proposition 8.3.7, there exists a unique morphism of schemes $X \to \operatorname{Spec} A$, and therefore X is an A-scheme as in Definition 8.3.4. □

Proposition 8.3.8

$\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.) If A is any ring (for example, a field k), then $\operatorname{Spec} A$ is the final object in the category of A-schemes.

Proof Note that $\mathbb{Z}$ is the initial object of **Rings**, by Proposition 8.3.7, $\operatorname{Spec} \mathbb{Z}$ is the final object of **Sch**.

Suppose A is any ring, and X is an A-scheme. Let $\pi : X \to \operatorname{Spec} A$ be any morphism of schemes from X to $\operatorname{Spec} A$, we want to show that π must be structure morphism. Say $\varphi : X \to \operatorname{Spec} A$ be structure morphism. We have the following commutative diagram.

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & \operatorname{Spec} A \arrow[dl, "\mathrm{id}_{\operatorname{Spec} A}"] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Then we have $\mathrm{id}_{\operatorname{Spec} A} \circ \pi = \pi = \varphi$, which implies that $\operatorname{Spec} A$ is the final object in the category $\mathbf{Sch}_{\operatorname{Spec} A}$. □

8.3.4 Coproducts of schemes

Proposition 8.3.9

Suppose X and Y are schemes. Then their disjoint union $X\coprod Y$ (Definition 5.3.6) is the coproduct of X and Y in the category of schemes, justifying the use of the coproduct symbol $\coprod$.

Proof Let W be any scheme, with morphisms $f_1 : X \to W$ and $f_2 : Y \to W$. Say $Z = X\coprod Y$. Consider the following diagram.

$$\begin{tikzcd} & Y \arrow[d, "i_Y"'] \arrow[ddr, "f_2"] & \\ X \arrow[r, "i_X"] \arrow[drr, "f_1"'] & Z \arrow[dr, dashed] & \\ & & W \end{tikzcd} \tag{8.7}$$

We want to show that there exists unique morphism from Z to W such that above diagram commutes. Define $\varphi : Z \to X$ as follow. As morphism between topological spaces, set

$$\varphi(z) = \begin{cases} f_1(z) & \text{if } z \in X \\ f_2(z) & \text{if } z \in Y. \end{cases}$$

Clearly, as morphism of topological spaces, diagram (8.7) commutes. As morphism of sheaves, consider $\varphi^\sharp : \mathscr{O}_W \to \varphi_*\mathscr{O}_Z$, for all open $U \subseteq W$, we have

$$\mathscr{O}_Z(\varphi^{-1}(U)) = \mathscr{O}_Z(f_1^{-1}(U)\coprod f_2^{-1}(U)) = \mathscr{O}_X(f_1^{-1}(U)) \times \mathscr{O}_Y(f_2^{-1}(U)).$$

Define $\varphi^\sharp_U : \mathscr{O}_W(U) \to \mathscr{O}_Z(\varphi^{-1}(U))$ by setting $s \mapsto ((f_1)^\sharp_U(s), (f_2)^\sharp_U(s))$. It is easy to show that $\varphi^\sharp$ is a morphism of sheaves. We next want to show that $\varphi^\sharp_p : \mathscr{O}_{W,\varphi(p)} \to \mathscr{O}_{Z,p}$ is a local ring map for all $p \in Z$. If $p \in X$, then $\mathscr{O}_{Z,p} \cong \mathscr{O}_{X,p}$, since $f_1 : X \to W$ is a local ring map, so is $\varphi^\sharp_p$. Similarly, if $p \in Y$, we have $\varphi^\sharp_p$ is a local ring map. Hence, $\varphi_W \to Z$ is a morphism of schemes.

Now we shall to show that diagram (8.7) commutes. As morphism of topological spaces, it is clear. We want to show that $f_1^\sharp = (\varphi \circ i_X)^\sharp = i_X^\sharp \circ h^\sharp$. Let $U \subseteq W$ open, then

$$(\varphi \circ i_X)^\sharp_U = (i_X)^\sharp_{\varphi^{-1}(U)} \circ \varphi^\sharp_U : \mathscr{O}_W(U) \to \mathscr{O}_X(i_X^{-1} \circ \varphi^{-1}(U)) = \mathscr{O}_X(f^{-1}(U)).$$

Pick $s \in \mathscr{O}_W(U)$, then

$$(\varphi \circ i_X)^\sharp_U(s) = (i_X)^\sharp_{\varphi^{-1}(U)} \circ \varphi^\sharp_U(s) = (i_X)^\sharp_{\varphi^{-1}(U)}((f_1)^\sharp_U(s), (f_2)^\sharp_U(s)) = (f_1)^\sharp_U(s),$$

which implies that $f_1^\sharp = (\varphi \circ i_X)^\sharp$. Similarly we have $f_2^\sharp = (\varphi \circ i_Y)^\sharp$.

By the definition of φ, clearly, φ is unique. Hence $X\coprod Y$ is the coproduct of X and Y in the category of schemes. □

8.3.5 Morphisms from (some) affine schemes

Morphisms from affine schemes are not quite as simple as morphisms to affine schemes, but some cases are worth pointing out.

Proposition 8.3.10

Suppose p is a point of a scheme X.

(a) *There exists a canonical (choice-free) morphism* $\operatorname{Spec} \mathscr{O}_{X,p} \to X$.

(b) There exists a canonical morphism $\operatorname{Spec}\kappa(p)\to X$*. (This is often written* $p\to X$*; one gives* p *the obvious interpretation as a scheme,* $\operatorname{Spec}\kappa(p)$*.)*

Proof

(a) Let $p\in X$, then exists affine open $\operatorname{Spec}A\hookrightarrow X$ such that $p\in\operatorname{Spec}A$. In fact, we have a canonical ring homomorphism $A\to A_p$, by Proposition 8.3.6, we get a morphism of affine schemes, $\operatorname{Spec}A_p\to\operatorname{Spec}A$. Note that $\mathscr{O}_{X,p}\cong A_p$, then we have a morphism of schemes

$$\operatorname{Spec}\mathscr{O}_{X,p}\longrightarrow\operatorname{Spec}A\hookrightarrow X.$$

To show that the map is canonical, suppose $V=\operatorname{Spec}B$ is another affine neighborhood containing p so that there exists some prime ideal $\mathfrak{q}\subseteq$ with $\mathscr{O}_{X,p}=B_{\mathfrak{q}}$. Say $U=\operatorname{Spec}A$, and $\mathfrak{p}\in\operatorname{Spec}A$ such that $\mathscr{O}_{X,p}=A_{\mathfrak{p}}$. Since $p\in U\cap V$, by Proposition 6.3.1, there exists a distinguished open subscheme $W\subseteq U\cap V$ of the form $\operatorname{Spec}A_f\hookrightarrow U$ and $\operatorname{Spec}B_g\hookrightarrow V$. Since $p=[\mathfrak{p}]\in\operatorname{Spec}A_f$, $f\notin\mathfrak{p}$, i.e., f is invertible in $A_{\mathfrak{p}}$. By the universal property of localization, we get a morphism $A_f\to A_{\mathfrak{p}}$. Similarly, we get a morphism $B_g\to B_{\mathfrak{q}}$. In fact, we have $\mathscr{O}_{X,p}\cong A_{\mathfrak{p}}\cong B_{\mathfrak{q}}$ and $A_f=B_g$, applying functor Spec, we see that $\operatorname{Spec}\mathscr{O}_{X,p}\to U$ and $\operatorname{Spec}\mathscr{O}_{X,p}\to V$ must factor through $W\subseteq U\cap V$ and thus $\operatorname{Spec}\mathscr{O}_{X,p}\to X$ is independent of the choice of affine neighborhood.

(b) Suppose $U=\operatorname{Spec}A$ is an affine neighborhood of p then exists $[\mathfrak{p}]\in\operatorname{Spec}A$ such that $p=[\mathfrak{p}]$. Thus $\kappa(p)=A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Note that we have ring homomorphism $A_{\mathfrak{p}}\to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, by Proposition 8.3.6, we obtain a morphism of schemes $\operatorname{Spec}\kappa(p)\to\operatorname{Spec}\mathscr{O}_{X,p}$. By part (a), we get a canonical morphism $\operatorname{Spec}\kappa(p)\to\operatorname{Spec}\mathscr{O}_{X,p}\to X$.

□

Proposition 8.3.11 (Morphisms from Spec of a local ring to X)

Suppose X *is a scheme, and* $(A,\mathfrak{m})$ *is a local ring. Suppose we have a scheme morphism* $\pi:\operatorname{Spec}A\to X$ *sending* $[\mathfrak{m}]$ *to* p*. Then*

(a) any open set containing p *contains the image of* π*;*

(b) there is a bijection between $\operatorname{Mor}(\operatorname{Spec}A,X)$ *and*

$$\{(p\in X,\ \textit{some local homomorphism } \mathscr{O}_{X,p}\to A)\}.$$

Proof

(a) Let U be any open set which contains p, we want to show that $\pi(\operatorname{Spec}A)\subseteq U$. Since $\pi([\mathfrak{m}])=p\in U$, we have $[\mathfrak{m}]\in\pi^{-1}(U)$. It suffices to show that $\pi^{-1}(U)=\operatorname{Spec}A$. If not, $\operatorname{Spec}A\setminus\pi^{-1}(U)$ is closed in $\operatorname{Spec}A$. Let $\mathfrak{p}\in\operatorname{Spec}A\setminus\pi^{-1}(U)$, since $(A,\mathfrak{m})$ is a local ring, $\mathfrak{p}\subseteq\mathfrak{m}$, and therefore $[\mathfrak{m}]\in\overline{\{\mathfrak{p}\}}\subseteq\operatorname{Spec}A\setminus\pi^{-1}(U)$, contradicts to the fact that $[\mathfrak{m}]\in\pi^{-1}(U)$. Hence $\pi^{-1}(U)=\operatorname{Spec}A$, which implies that $\pi(\operatorname{Spec}A)\subseteq U$.

(b) By part (a), let $\pi\in\operatorname{Mor}(\operatorname{Spec}A,X)$, π actually factors as $\operatorname{Spec}A\to U\hookrightarrow X$ where U is an open neighborhood of $\pi([\mathfrak{m}])=p$. Say $\pi=i_U\circ f_U$, where $i_U:U\hookrightarrow X$ and $f_U:\operatorname{Spec}A\to U$. Thus a morphism $\pi:\operatorname{Spec}A\to X$ mapping $[\mathfrak{m}]$ to p corresponds to a equivalence class of morphisms $f_U:\operatorname{Spec}A\to U$ mapping $[\mathfrak{m}]$ to p, where U is an affine open neighborhood of p. More precisely, we have

$$\pi\longleftrightarrow(f_U:\operatorname{Spec}A\to U)_U$$

where $(f_U:\operatorname{Spec}A\to U)_U$ is equivalence class which defined by that $f_U\sim f_V$ if $\pi=i_U\circ f_U$ and

$\pi = i_V \circ f_V$. Note that such f_U corresponds to a homomorphism $f_U^\sharp : \mathscr{O}_X(U) \to A$ and $(f_U^\sharp)^{-1}(\mathfrak{m}) = p$, say $U = \operatorname{Spec} B$, we have $f_U^\sharp(B - p) \subseteq A - \mathfrak{m}$, by the universal property of localization, we get a homomorphism of rings $\mathscr{O}_{X,p} = B_p \to A$. Hence we have a bijection

$$\operatorname{Mor}(\operatorname{Spec} A, X) \longleftrightarrow \{(p \in X, \text{local homomorphism } \mathscr{O}_{X,p} \to A)\}.$$

$\square$

Proposition 8.3.10 and Proposition 8.3.11 lead us to the notion of field-valued points, and more generally ring-valued points, and more generally still, scheme-valued points.

8.3.6 Definition:The functor of points, and scheme-valued points (and ring-valued points, and field-valued points) of a scheme

Definition 8.3.5 (Scheme-valued points)

*If Z is a scheme, then Z-**valued points** of a scheme X, denoted $X(Z)$, are defined to be maps $Z \to X$, i.e., $X(Z) = \operatorname{Mor}(Z, X)$. If A is a ring, then A-**valued points** of a scheme X, denoted $X(A)$, are defined to be the $(\operatorname{Spec} A)$-valued points of the scheme. (The most common case of this is when A is a field.)*

If you are working over a base scheme B — for example, complex algebraic geometers will consider only schemes and morphisms over $B = \operatorname{Spec} C$ — then in the above definition, there is an implicit structure map $Z \to B$ (or $\operatorname{Spec} A \to B$ in the case of $X(A)$). For example, for a complex geometer, if X is a scheme over $\mathbb{C}$, the $\mathbb{C}(t)$-valued points of X correspond to commutative diagrams of the form

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}(t) \arrow[rr] \arrow[dr, "\xi"'] & & X \arrow[dl, "\pi"] \\ & \operatorname{Spec} \mathbb{C} & \end{tikzcd}$$

where $\pi : X \to \operatorname{Spec} \mathbb{C}$ is the structure map for X, and ξ corresponds to the obvious inclusion of rings $\mathbb{C} \to \mathbb{C}(t)$.

Remark **Warning:** A k-valued point of a k-scheme X is sometimes called a “rational point” of X, which is dangerous, as for most of the world, “rational” refers to $\mathbb{Q}$. We will use the safer phrase “k-valued point” of X. Another safe choice is “k-rational point” of X or k-point.

The terminology “Z-valued point” (and A-valued point) is unfortunate, because we earlier defined the notion of points of a scheme, and Z-valued points (and A-valued points) are not (necessarily) points! But these usages are well-established in the literature.

Proposition 8.3.12

(a) A morphism of schemes $X \to Y$ induces a map of Z-valued points $X(Z) \to Y(Z)$.

(b) Morphisms of schemes $X \to Y$ are determined by their induced maps of Z-valued points, as Z varies over all schemes.

Proof

(a) Let $\pi : X \to Y$ be a morphism of schemes, pick $\{Z \to X\} \in X(Z)$, then $Z \to X \to Y$ belong to $Y(Z)$, i.e., $\pi : X \to Y$ induced map $\pi \circ \square : X(Z) \to Y(Z)$.

(b) By part (a), any morphisms of schemes $X \to Y$ gives a natural transformation $\tau : \operatorname{Mor}(\square, X) \to$

$\mathrm{Mor}(\square, Y)$. Then by Yoneda's Lemma 2.2.1, we have a bijection

$$y : \mathrm{Nat}(\mathrm{Mor}(\square, X), \mathrm{Mor}(\square, Y)) \longrightarrow \mathrm{Mor}(X, Y)$$
$$\tau \longmapsto \tau_X(\mathrm{id}_X),$$

as we desired.

$\square$

Example 8.3 **Morphisms of schemes $X \to Y$ are not determined by their "underlying" map of points.** For example, let k be a field with $\mathrm{char}(k) = p$, let $X = Y = \operatorname{Spec} k[x]$. Consider Frobenius endomorphism $k[x] \to k[x]$ which maps x to x^p and identity $\mathrm{id}_{k[x]}$. They induce the same morphisms as morphisms of topological space, but induced morphisms are not the same as morphisms of schemes.

Furthermore, we will see that "products of Z-valued points" behave as you might hope (Chapter 11). A related reason this language is suggestive: the notation $X(Z)$ suggests the interpretation of X as a (contravariant) functor h_X from schemes to sets — the **functor of (scheme-valued) points** of the scheme X (cf. Example 2.19). More precisely:

Definition 8.3.6 (Functor of points)

Given a scheme X we can define a functor

$$h_X : \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Sets}, \qquad T \longmapsto \mathrm{Mor}(T, X).$$

*h_X is called the **functor of points of** X.*

Here is a low-brow reason A-valued points are a useful notion:

Proposition 8.3.13

The A-valued points of an affine scheme $\operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ (where $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ are relations) are precisely the solutions to the equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$$

in the ring A.

Proof Say $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. The A-valued points of an affine scheme X is $\mathrm{Mor}(\operatorname{Spec} A, X)$, by Proposition 8.3.7, it can be seen as $\mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r), A)$. Let

$$\varphi : \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \longrightarrow A$$

be a ring map, then φ determined by $\varphi(x_i) = a_i$. Note that $\varphi(f_j) = f_j(a_1, \ldots, a_n) = 0$, we know that $\mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r), A)$ is one-to-one corresponding to a solution of

$$f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$$

in the ring A, then we done. $\square$

Example 8.4 For example, the rational solutions to $x^2 + y^2 = 16$ are precisely the $\mathbb{Q}$-valued points of $\operatorname{Spec} \mathbb{Z}[x, y]/(x^2 + y^2 - 16)$. The integral solutions are precisely the $\mathbb{Z}$-valued points. So A-valued points of an affine scheme (finite type over $\mathbb{Z}$) can be interpreted simply. In the special case where A is local, A-valued points of a general scheme have a good interpretation too (Proposition 8.3.10 and Proposition 8.3.11).

Proposition 8.3.14

(a) *Suppose B is a ring. If X is a B-scheme, and $f_0, \ldots, f_n$ are $n+1$ functions on X with no common zeros, then $[f_0 : \cdots : f_n]$ gives a morphism of B-schemes $X \to \mathbb{P}^n_B$.*

(b) Suppose g is a nowhere vanishing function on X, and f_i are as in part (a). Then the morphisms $[f_0 : \cdots : f_n]$ and $[gf_0 : \cdots : gf_n]$ to $\mathbb{P}^n_B$ are the same.

Proof

(a) Let $\mathbb{P}^n_B = \bigcup_{i=0}^n D_+(x_i)$ where $D_+(x_i) \cong \operatorname{Spec}(B[x_0, \ldots, x_n]_{x_i})_0$. Pick $p \in X$, since $f_0, \ldots, f_n$ have no common zeros, there must have $p \in X_{f_i}$ for some i (X_{f_i} defined in Definition 7.2.4), hence $X = \bigcup\limits_{i=0}^{n} X_{f_i}$. Define $\pi_i^\sharp : (B[x_0, \ldots, x_n]_{x_i})_0 \to \Gamma(X_{f_i}, \mathscr{O}_X)$ by setting $x_{j/i} \mapsto \frac{f_j}{f_i}$, hence we get a morphism of schemes, $\pi_i : X_{f_i} \to D_+(x_i)$. We next show that $\pi_i|_{X_{f_i} \cap X_{f_j}} = \pi_j|_{X_{f_i} \cap X_{f_j}}$. It suffices to show that $\pi_i|^\sharp_{X_{f_i} \cap X_{f_j}} = \pi_j|^\sharp_{X_{f_i} \cap X_{f_j}}$. Note that $\pi_i|^\sharp_{X_{f_i} \cap X_{f_j}} : (B[x_0, \ldots, x_n]_{x_i x_j})_0 \to \Gamma(X_{f_i} \cap X_{f_j}, \mathscr{O}_X)$ maps $\frac{x_k x_l}{x_i x_j}$ to $\frac{f_k f_l}{f_i f_j}$, similar to $\pi_j|^\sharp_{X_{f_i} \cap X_{f_j}}$, we have $\pi_i|^\sharp_{X_{f_i} \cap X_{f_j}} = \pi_j|^\sharp_{X_{f_i} \cap X_{f_j}}$. Hence $\pi_i|_{X_{f_i} \cap X_{f_j}} = \pi_j|_{X_{f_i} \cap X_{f_j}}$. Say $\iota_k : D_+(x_i) \hookrightarrow \mathbb{P}^n_B$ be open immersion, then

$$\iota_i \circ \pi_i|_{X_{f_i} \cap X_{f_j}} = (\iota_i \circ \pi_i)|_{X_{f_i} \cap X_{f_j}} = (\iota_j \circ \pi_j)|_{X_{f_i} \cap X_{f_j}} = \iota_j \circ \pi_j|_{X_{f_i} \cap X_{f_j}},$$

by Morphisms of locally ringed spaces glue 8.3.2, we get a morphism of schemes $\pi : X \to \mathbb{P}^n_B$ such that $\pi|_{X_{f_i}} = \pi_i$. Hence π is of the form $[f_0 : f_1 : \cdots : f_n]$. Since all ring map is also B-algebra map, $\pi : X \to \mathbb{P}^n_B$ is a morphism of B-schemes.

(b) Since g is a nowhere vanishing function on X, we have $X_{f_i} = X_{gf_i}$. Say $\varphi = [gf_0 : \cdots : gf_n]$, consider $\varphi|_{X_{gf_i}}$, it induces a ring map $(B[x_0, \ldots, x_n]_{x_i})_0 \to \Gamma(X_{gf_i}, \mathscr{O}_X)$ mapping $x_{j/i} \mapsto \frac{gf_j}{gf_i} = \frac{f_j}{f_i}$. It is same as $\pi|_{X_{gf_i}} = \pi|_{X_{f_i}}$, hence $\pi = \varphi$.

$\square$

Example 8.5 Consider the $n+1$ functions $x_0, \ldots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^{n+1} \setminus \{0\}$. Hence they determine a morphism $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

You might hope that Proposition 8.3.14 (a) gives all morphisms to projective space (over B). But this isn't the case. Indeed, even the identity morphism $X = \mathbb{P}^1_k \to \mathbb{P}^1_k$ isn't of this form, as the source $\mathbb{P}^1$ has no nonconstant global functions with which to build this map. (There are similar examples with an affine source.) However, there is a correct generalization (characterizing all maps from schemes to projective schemes) in Chapter 16. This result roughly states that this works, so long as the f_i are not quite functions, but sections of a line bundle. Our desire to understand maps to projective schemes in a clean way will be one important motivation for understanding line bundles.

We will see more ways to describe maps to projective space in the next section. A different description directly generalizing Proposition 8.3.14 (a) will be given in Chapter 16, which will turn out to be a "universal" description.

Incidentally, before Grothendieck, it was considered a real problem to figure out the right way to interpret points of projective space with "coordinates" in a ring. These difficulties were due to a lack of functorial reasoning. And the clues to the right answer already existed (the same problems arise for maps from a manifold to $\mathbb{RP}^n$) — if you ask such a geometric question (for projective space is geometric), the answer is necessarily geometric, not purely algebraic!

8.3.7 Visualizing morphisms: Picturing maps of schemes when nilpotents are present

You now know how to visualize the points of schemes (§4.3), and nilpotents (§5.2 and §7.6). The following imprecise exercise will give you some sense of how to visualize maps of schemes when nilpotents are involved.

Exercise 8.5 Suppose $a \in \mathbb{C}$. Consider the map of rings $\mathbb{C}[x] \to \mathbb{C}[\varepsilon]/(\varepsilon^2)$ given by $x \mapsto a\varepsilon$. Recall that $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ may be pictured as a point with a tangent vector (§5.2). How would you picture this map if $a \neq 0$? How does your picture change if $a = 0$? (The tangent vector should be "crushed" in this case.)

Solution *Say $\pi^\sharp : \mathbb{C}[x] \to \mathbb{C}[\varepsilon]/(\varepsilon^2)$ given by $x \mapsto a\varepsilon$. It gives a morphism of schemes $\pi : \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} \mathbb{C}[x]$. In fact, $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ has only one closed point $[(\varepsilon)]$ and $\pi([(\varepsilon)]) = (\pi^\sharp)^{-1}([(\varepsilon)]) = [(x)]$. By Proposition 8.2.6, π induces a map of stalks $\pi^\sharp_{[(\varepsilon)]} : \mathbb{C}[x]_{(x)} \to (\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)}$. Since $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ is a local ring, $(\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)} \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$. The maximal ideal in $\mathbb{C}[x]_{(x)}$ is $(x)\mathbb{C}[x]_{(x)}$, say $\mathfrak{m}_{(x)} = (x)\mathbb{C}[x]_{(x)}$, then $\pi^\sharp_{[\varepsilon]}\big|_{\mathfrak{m}_{(x)}} : \mathfrak{m}_{(x)} \to (\varepsilon)(\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)}$. It induce a map of cotangent space $\widetilde{\pi^\sharp_{[(\varepsilon)]}} : \mathfrak{m}_{(x)}/\mathfrak{m}^2_{(x)} \to \mathfrak{m}_{(\varepsilon)}/\mathfrak{m}^2_{(\varepsilon)}$, where $\mathfrak{m}_{(\varepsilon)} = (\varepsilon)(\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)}$. In fact*

$$\mathfrak{m}_{(x)}/\mathfrak{m}^2_{(x)} = (x)\mathbb{C}[x]_{(x)}/(x^2)\mathbb{C}[x]_{(x)} \cong \mathbb{C}x$$

and

$$\mathfrak{m}_{(\varepsilon)}/\mathfrak{m}^2_{(\varepsilon)} = \frac{(\varepsilon)(\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)}}{(\varepsilon^2)(\mathbb{C}[\varepsilon]/(\varepsilon^2))_{(\varepsilon)}} \cong \mathbb{C}\varepsilon,$$

we have $\widetilde{\pi^\sharp_{[(\varepsilon)]}} : \mathbb{C}x \to \mathbb{C}\varepsilon$ maps x to $a\varepsilon$. If $a \neq 0$, $\widetilde{\pi^\sharp_{[(\varepsilon)]}}$ is an isomorphism, which implies that the cotangent space is isomorphic, take dual, their tangent space at origin are isomorphic. If $a = 0$, $\widetilde{\pi^\sharp_{[(\varepsilon)]}} = 0$, it follows that tangent vector should be "crushed" in this case.

8.3.8 ⋆⋆ Analytification of complex algebraic varieties

Any discussion of analytification is only for readers who are familiar with the notion of complex analytic varieties, or willing to develop it on their own in parallel with our development of schemes.

8.4 Maps of graded rings and maps of projective schemes

8.4.1 Graded ring morphisms and their induced scheme maps

As maps of rings correspond to maps of affine schemes in the opposite direction, maps of graded rings (over a base ring A) sometimes give maps of projective schemes in the opposite direction. This is an imperfect generalization: not every map of graded rings gives a map of projective schemes; not every map of projective scheme comes from a map of graded rings; and different maps of graded rings can yield the same map of schemes.

Definition 8.4.1 (Homomorphism of graded ring)

We say a ring map $\varphi : S_\bullet \to R_\bullet$ is a homomorphism of $\mathbb{Z}^{\geq 0}$-graded rings, if there exists an $r \geq 1$ such that for every $d \geq 0$, we have $\varphi(S_d) \subseteq R_{rd}$.

Proposition 8.4.1

Suppose that $\varphi : S_\bullet \to R_\bullet$ is a homomorphism of $\mathbb{Z}^{\geq 0}$-graded rings. This induces a morphism of schemes

$$(\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+)) \longrightarrow \operatorname{Proj} S_\bullet.$$

In particular, if

$$V(\varphi(S_+)) = \varnothing, \tag{8.8}$$

then we have a morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$*. If* φ *is furthermore a morphism of* A*-algebras, then the induced morphism* $(\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+)) \to \operatorname{Proj} S_\bullet$ *is a morphism of* A*-schemes.*

Proof Suppose $\operatorname{Proj} S_\bullet = \bigcup_{f\in S_+} \operatorname{Spec}((S_\bullet)_f)_0$. We claim that $(\operatorname{Proj} R_\bullet)\setminus V(\varphi(S_+)) = \bigcup_{f\in S_+} \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0$. Pick $[\mathfrak{q}] \in (\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+))$, then $\mathfrak{q}$ is a homogeneous ideal with $\mathfrak{q} \not\supseteq R_+$ and $\mathfrak{q} \not\supseteq \varphi(S_+)$. Since φ is a morphism of graded ring, we have $\varphi^{-1}(\mathfrak{q})$ is a homogenous ideal with $\varphi^{-1}(\mathfrak{q}) \supseteq S_+$. Hence there exists $f \in S_+$ such that $f \notin \varphi^{-1}(\mathfrak{q})$, and therefore $\varphi(f) \notin \mathfrak{q}$, it follows that $\mathfrak{q} \in \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0$. Conversely, pick $[\mathfrak{q}] \in \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0$ for some $f \in S_+$, then $\varphi(f) \notin \mathfrak{q}$, so that $\mathfrak{q} \not\supseteq \varphi(S_+)$, which implies that $[\mathfrak{q}] \in (\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+))$. Hence $(\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+)) = \bigcup_{f\in S_+} \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0$. Define $\pi_f^\sharp : ((S_\bullet)_f)_0 \to ((R_\bullet)_{\varphi(f)})_0$ by setting

$$\frac{h}{f^n} \longmapsto \frac{\varphi(h)}{\varphi(f)^n},$$

by definition of morphism of graded ring, $\pi_f^\sharp$ is well-defined. Hence this gives a morphism of affine schemes $\pi_f : \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0 \to \operatorname{Spec}((S_\bullet)_f)_0$. Say $D_{R_\bullet,f} = \operatorname{Spec}((R_\bullet)_{\varphi(f)})_0$ and $D_{S_\bullet,f} = \operatorname{Spec}((S_\bullet)_f)_0$, let $\iota_f : D_{S_\bullet,f} \hookrightarrow \operatorname{Proj} S_\bullet$ be open immersion. We want to show that $(\iota_f\circ\pi_f)|_{D_{R_\bullet,f}\cap D_{R_\bullet,g}} = \iota_f\circ\pi_f|_{D_{R_\bullet,f}\cap D_{R_\bullet,g}} = \iota_g \circ \pi_g|_{D_{R_\bullet,f}\cap D_{R_\bullet,g}} = (\iota_g \circ \pi_g)|_{D_{R_\bullet,f}\cap D_{R_\bullet,g}}$. It suffices to check $\pi_f|^\sharp_{D_{R_\bullet,f}\cap D_{R_\bullet,g}} = \pi_g|^\sharp_{D_{R_\bullet,f}\cap D_{R_\bullet,g}}$ as ring maps. It is clear. By Morphisms of locally ringed spaces glue 8.3.2, we get a morphism of schemes

$$\pi : (\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+)) \longrightarrow \operatorname{Proj} S_\bullet.$$

In particular, if $V(\varphi(S_+)) = \varnothing$, then we have a morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$.

If φ is a morphism of A-algebras, above ring maps which induced by φ are all morphism of A-algebras, and therefore the induced morphism $(\operatorname{Proj} R_\bullet) \setminus V(\varphi(S_+)) \to \operatorname{Proj} S_\bullet$ is a morphism of A-schemes. □

Example 8.6 Let's see Proposition 8.4.1 in action. We will scheme-theoretically interpret the map of complex projective manifolds $\mathbb{CP}^1$ to $\mathbb{CP}^2$ given by

$$\begin{aligned} \mathbb{CP}^1 &\longrightarrow \mathbb{CP}^2 \\ [s:t] &\longrightarrow [s^{20} : s^9t^{11} : t^{20}] \end{aligned}$$

Notice first that this is well-defined: $[\lambda s, \lambda t]$ is sent to the same point of $\mathbb{CP}^2$ as $[s,t]$. The reason for it to be well-defined is that the three polynomials s^{20}, s^9t^{11}, and t^{20} are all homogeneous of degree 20.

Algebraically, this corresponds to a map of graded rings in the opposite direction

$$\varphi : \mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[s,t]$$

given by $x \mapsto s^{20}$, $y \mapsto s^9t^{11}$, $z \mapsto t^{20}$.

Let $S_\bullet = \mathbb{C}[x,y,z]$ and $R_\bullet = \mathbb{C}[s,t]$, then $S_+ = (x,y,z)$, and therefore $\varphi(S_+) = (s^{20}, s^9t^{11}, t^{20})$. Hence $V(\varphi(S_+)) = \varnothing$. By Proposition 8.4.1, we have $\operatorname{Proj}\mathbb{C}[s,t] \to \operatorname{Proj}\mathbb{C}[x,y,z]$ which given by $[s:t] \to [s^{20} : s^9t^{11} : t^{20}]$.

Example 8.7 Notice that there is no map of complex manifolds $\mathbb{CP}^2 \to \mathbb{CP}^1$ given by $[x:y:z] \mapsto [x:y]$, because the map is not defined when $x = y = 0$. This corresponds to the fact that the map of graded rings $\mathbb{C}[s,t] \to \mathbb{C}[x,y,z]$ given by $s \mapsto x$ and $t \mapsto y$, doesn't satisfy hypothesis (8.8).

Proposition 8.4.2

If $\varphi : S_\bullet \to R_\bullet$ *satisfies* $\sqrt{(\varphi(S_+))} = R_+$*, then hypothesis (8.8) is satisfied.*

Proof Note that

$$V(\sqrt{(\varphi(S_+))}) = V((\varphi(S_+))) = V(\varphi(S_+)) = V(R_+) = \varnothing,$$

as we desired. $\square$

Remark This algebraic formulation of the more geometric hypothesis can some times be easier to verify.

Exercise 8.6 This exercise shows that different maps of graded rings can given the same map of schemes. Let $R_\bullet = k[x,y,z]/(xz,yz,z^2)$ and $S_\bullet = k[a,b,c]/(ac,bc,c^2)$, where every variable has degree 1. Show that $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet \cong \mathbb{P}^1_k$. Show that the maps $S_\bullet \to R_\bullet$ given by $(a,b,c) \mapsto (x,y,z)$ and $(a,b,c) \mapsto (x,y,0)$ given the same isomorphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$. (The real reason is that all of these constructions are insensitive to what happens in a finite number of degrees. This will be made precise in a number of ways later.)

Proof We claim that $R_\bullet \cong k[x,y]$. In fact we have $k[x,y] \cong k[x,y,z]/(z)$. We want to show that $R_\bullet \cong k[x,y,z]/(z)$. Define $k[x,y,z] \to k[x,y,z]/(xz,yz,z^2)$ by setting $(x,y,z) \mapsto (x,y,z)$. Clearly, this map is a surjective. Also $\operatorname{Ker}(k[x,y,z] \to k[x,y,z]/(xz,yz,z^2)) = (z)$, by the Fundamental Theorem of Isomorphism, we have $k[x,y,z]/(z) \cong R_\bullet \cong k[x,y]$. Similarly, we have $k[a,b,c]/(c) \cong S_\bullet \cong k[a,b]$. Hence $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet \cong \mathbb{P}^1_k$.

Say $\varphi^\sharp : S_\bullet \to R_\bullet$ given by $(a,b,c) \mapsto (x,y,z)$ and $\psi^\sharp : S_\bullet \to R_\bullet$ given by $(a,b,c) \mapsto (x,y,0)$. As in the proof of Proposition 8.4.1, we may define isomorphism $\varphi^\sharp_f : ((S_\bullet)_f)_0 \to ((R_\bullet)_{\varphi(f)})_0$ by setting

$$\frac{h(a,b,c)}{f(a,b,c)^n} \longmapsto \frac{h(x,y,z)}{f(x,y,z)^n}$$

and define isomorphism $\psi^\sharp_f : ((S_\bullet)_f)_0 \to ((R_\bullet)_{\psi(f)})_0$ by setting

$$\frac{h(a,b,c)}{f(a,b,c)^n} \longmapsto \frac{h(x,y,0)}{f(x,y,0)^n}.$$

We claim that $((R_\bullet)_{\varphi(f)})_0 \cong ((R_\bullet)_{\psi(f)})_0$. Note that $R_\bullet \cong k[x,y,z]/(z)$, it suffices to show that

$$((k[x,y,z]/(z))_{\varphi(f)})_0 \cong ((k[x,y,z]/(z))_{\psi(f)})_0.$$

Consider $\frac{h(x,y)}{f(x,y)^n} \mapsto \frac{h(x,y)}{f(x,y)^n}$, this morphism is clearly isomorphism. Hence we have a commutative diagram.

$$\begin{tikzcd}
((S_\bullet)_f)_0 \arrow[r, "\varphi^\sharp_f"] \arrow[d, "\sim"'] & ((R_\bullet)_{\varphi(f)})_0 \arrow[d, "\sim"] \\
((S_\bullet)_f)_0 \arrow[r, "\psi^\sharp_f"] & ((R_\bullet)_{\psi(f)})_0
\end{tikzcd}$$

It follows $\varphi^\sharp$ and $\psi^\sharp$ induce same isomorphisms of affine subschemes as in the proof of Proposition 8.4.1, and therefore $\varphi^\sharp$ and $\psi^\sharp$ give the same isomorphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$. $\square$

Remark In Chapter 16, we will show that not every morphism of schemes $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$ comes from a map of graded rings $S_\bullet \to R_\bullet$, even in quite reasonable circumstances.

8.4.2 Veronese subrings

Here is a useful construction.

Definition 8.4.2 (Veronese subring)

Suppose $S_\bullet$ is a finitely generated graded ring. Define the n-th ***Veronese subring*** *of $S_\bullet$ by $S_{n\bullet} = \bigoplus_{j=0}^{\infty} S_{nj}$. (The "old degree" n is "new degree" 1.)*

The geometric interpretation of Veronese subring is the important **Veronese embedding**, discussed in

Chapter 10.

Proposition 8.4.3

The map of graded rings $S_{n\bullet} \hookrightarrow S_\bullet$ induces an isomorphism $\operatorname{Proj} S_\bullet \xrightarrow{\sim} \operatorname{Proj} S_{n\bullet}$.

Proof We claim that $\operatorname{Proj} S_\bullet = \bigcup_{g \in S_+} D_+(g) = \bigcup_{\substack{f \in S_m \\ n \mid m}} D_+(f)$. Pick $[\mathfrak{p}] \in \bigcup_{g \in S_+} D_+(g)$, then $[\mathfrak{p}] \in D_+(g)$ for some $g \in S_+$. Hence $g \notin \mathfrak{p}$, say $g = g_0 + g_1 + \cdots + g_r$ where g_i is homogeneous of degree i, then exists $g_i \notin \mathfrak{p}$, and therefore $g_i^n \notin \mathfrak{p}$. It follows that $[\mathfrak{p}] \in D_+(g_i^n)$. Note that $\deg g_i^n = ni$, $g_i^n \in S_{ni}$, hence $[\mathfrak{p}] \in \bigcup_{\substack{f \in S_m \\ n \mid m}} D_+(f)$. Consequently, we have $\operatorname{Proj} S_\bullet = \bigcup_{g \in S_+} D_+(g) = \bigcup_{\substack{f \in S_m \\ n \mid m}} D_+(f)$. In fact, $\operatorname{Proj} S_{n\bullet} = \bigcup_{f \in S_{n+}} D_+(f)$. Say $i : S_{n\bullet} \hookrightarrow S_\bullet$. Define ring map $i_f^\sharp : ((S_{n\bullet})_f)_0 \to ((S_\bullet)_f)_0$ by setting $i_f^\sharp = \mathrm{id}$ (since f is homogeneous with $n \mid \deg f$, $i_f^\sharp$ is well-defined), this gives an isomorphism of affine schemes $D_+(f) \xrightarrow{\sim} D_+(f)$, where left $D_+(f)$ is open subscheme of $\operatorname{Proj} S_\bullet$ and right $D_+(f)$ is open subscheme of $\operatorname{Proj} S_{n\bullet}$. Hence $i : S_{n\bullet} \hookrightarrow S_\bullet$ induces an isomorphism $\operatorname{Proj} S_\bullet \xrightarrow{\sim} \operatorname{Proj} S_{n\bullet}$. ☐

Proposition 8.4.4

If $S_\bullet$ is generated in degree 1*, then $S_{n\bullet}$ is also generated in degree* 1.

Proof It suffices to show that each S_{nk} is generated by S_n. Since $S_\bullet$ is generated in degree 1, S_1 generates S_k for all k. In particular, S_1 generates S_{nk} and S_1 generates S_n. In other words, we have surjective

$$\begin{tikzcd} S_1^{\otimes nk} = (S_1^{\otimes n})^{\otimes k} \arrow[r, two heads] \arrow[dr, two heads] & (S_n)^{\otimes k} \arrow[d, two heads] \\ & S_{nk}, \end{tikzcd}$$

where surjectivity of $(S_n)^{\otimes k} \twoheadrightarrow S_{nk}$ is given by $S_1^{\otimes nk} \twoheadrightarrow S_{nk}$ and $S_1^{\otimes nk} \twoheadrightarrow (S_n)^{\otimes k}$, it follows that S_{nk} is generated by S_n, i.e., $S_{n\bullet}$ is generated in degree 1. ☐

Proposition 8.4.5

If $R_\bullet$ and $S_\bullet$ are the same finitely generated graded rings except in a finite number of nonzero degrees; more precisely, there exists a finite set D such that for all $d \notin D$, $S_d = R_d$. Then $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet$.

Proof Take $n > \max D$, then for all i we have $S_{ni} = R_{ni}$, hence $S_{n\bullet} = R_{n\bullet}$ for all i, hence $R_{n\bullet} = S_{n\bullet}$. By Proposition 8.4.3, we have isomorphism $\operatorname{Proj} S_\bullet \cong \operatorname{Proj} S_{n\bullet}$ and $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} R_{n\bullet}$, and therefore $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet$. ☐

Proposition 8.4.6

Suppose $S_\bullet$ is generated over S_0 by $f_1, \ldots, f_n$. There exists a d such that $S_{d\bullet}$ is finitely generated in "new" degree 1 *(= "old" degree d).*

Proof Suppose there are generators $f_1, \ldots, f_n$ of degrees $d_1, \ldots, d_n$ respectively. We claim that any monomial $f_1^{a_1} \cdots f_n^{a_n}$ of degree at least $nd_1 \cdots d_n$ has $a_i \geq (\prod_j d_j)/d_i$ for some i. If not, then $a_i < (\prod_j d_j)/d_i$ for all i, hence $a_1 d_1 + \cdots + a_n d_n < n \prod_j d_j$, a contradiction. Say $m = nd_1 \cdots d_n$. We shall to show that the m-th Veronese subring is generated by elements in S_m. It suffices to show that any monomial in S_{km} can be written as a monomial in elements of S_m. Prove by induction on k. If $k = 1$, then $S_{km} = S_m$, so this case is trivial.

Suppose we have shown this to hold for some $k_0 > 1$ and let $f_1^{a_1} \cdots f_n^{a_n}$ be a monomial in $S_{(k_0+1)m}$. By above discussion, there exists $a_i > (\prod_j d_j)/d_i$ for some i, then

$$f_1^{a_1} \cdots f_n^{a_n} = f_i^{(\prod_j d_j)/d_i} \cdot f_1^{a_1} \cdots f_i^{a_i - (\prod_j d_j)/d_i} \cdots f_n^{a_n},$$

where $\deg\left(f_1^{a_1} f_2^{a_2} \cdots f_i^{a_i - (\prod_j d_j)/d_i} \cdots f_n^{a_n}\right) = (k_0+1)nd_1 \cdots d_n - d_1 \cdots d_n = ((k_0+1)n-1)d_1 \cdots d_n$. Consider $f_1^{a_1} \cdots f_i^{a_i - (\prod_j d_j)/d_i} \cdots f_n^{a_n}$, repeat above process $n-1$ times, we get a factorization of $f_1^{a_1} \cdots f_n^{a_n}$,

$$f_1^{a_1} \cdots f_n^{a_n} = (f_1^{b_1} \cdots f_n^{b_n}) \cdot (f_1^{c_1} \cdots f_n^{c_n}),$$

where $\deg(f_1^{b_1} \cdots f_n^{b_n}) = nd_1 \cdots d_n$ and $\deg(f_1^{c_1} \cdots f_n^{c_n}) = k_0 n d_1 \cdots d_n$. By the induction hypothesis, $f_1^{c_1} \cdots f_n^{c_n}$ can be written as a monomial in elements of S_m. Hence $S_{nd_1 \cdots d_n \bullet}$ is finitely generated in $S_{nd_1 \cdots d_n}$, as we desired. □

Proposition 8.4.6, in combination with Proposition 8.4.3, shows that there is little harm in assuming the finitely generated graded rings are generated in degree 1, as after a regrading (or more precisely, keeping only terms of degree a multiple of d, then dividing the degree by d), this is indeed the case. This is handy, as it means that, using Proposition 8.4.3, we can assume that any finitely generated graded ring is generated in degree 1. Chapter 10 will later imply as a consequence that we can embed every Proj in some projective space.

Proposition 8.4.7 (Less important)

Suppose $S_\bullet$ is a finitely generated graded ring. Then $S_{n\bullet}$ is a finitely generated graded ring.

Proof By Proposition 8.4.6 and Proposition 8.4.3, we may assume that $S_\bullet$ is generated in degree 1. By Proposition 5.5.3, $S_\bullet$ is a finitely generated S_0-algebra, i.e., there is a surjective $S_0[x_0, \ldots, x_n] \twoheadrightarrow S_\bullet$, where $\deg x_i = 1$. Clearly, we have $S_\bullet \twoheadrightarrow S_{n\bullet}$, hence

$$S_0[x_0, \ldots, x_n] \longrightarrow\!\!\!\!\!\rightarrow S_\bullet \longrightarrow\!\!\!\!\!\rightarrow S_{n\bullet},$$

i.e., $S_{n\bullet}$ is a finitely generated graded ring. □

8.5 Rational maps from reduced schemes

Informally speaking, a "rational map" is "a morphism defined almost everywhere", much as a rational function (Definition 7.6.10, Proposition 6.2.9, Definition 6.2.3) is a name for a function defined almost everywhere. We will later see that in good situations, just as with rational functions, where a rational map is defined (the Reduced-to-Separated Theorem), and has a largest "domain of definition". ***For this section only, we assume X to be reduced.*** A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

8.5.1 Definition: Rational map

Definition 8.5.1 (Rational map of reduced schemes)

*A **rational map** π **of reduced schemes**, from X to Y, denoted $\pi : X \dashrightarrow Y$, is the data of a morphism $\alpha : U \to Y$ from a dense open set $U \subseteq X$, with the equivalence relation $(\alpha : U \to Y) \sim (\beta : V \to Y)$ if there is a dense open set $Z \subseteq U \cap V$ such that $\alpha|_Z = \beta|_Z$.*

Remark

(1) In Chapter 12, we will improve this to: if $\alpha|_{U \cap V} = \beta|_{U \cap V}$ in good circumstances — when Y is separated.

(2) People often use the word "map" for "morphism", which is quite reasonable, except that a rational map need not be a map. So to avoid confusion, when one means "rational map", one should never just say "map".

We will also discuss rational maps of S-schemes for a scheme S.

Definition 8.5.2 (Rational maps of S-schemes)

A rational map π of S-schemes, from $X \to Y$, denoted $\pi : X \dashrightarrow Y$, is the data of a morphism of S-schemes $\alpha : U \to Y$ from a dense open set $U \subseteq X$, with the equivalence relation $(\alpha : U \to Y) \sim (\beta : V \to Y)$ if there is a dense open set $Z \subseteq U \cap V$ such that $\alpha|_Z = \beta|_Z$.

Remark ⋆ **Rational maps more generally.** Just as with rational functions, Definition 8.5.1 can be extended to where X is not reduced, as is (using the same name, "rational map"), or in a version that imposes some control over what happens over the nonreduced locus (pseudo-morphisms, see Stacks Project [8] Remark 01RX). We will see in Chapter 12 that rational maps from reduced schemes to separated schemes behave particularly well, which is why they are usually considered in this context. the reason for the definition of pseudo-morphisms is to extend these results to when X is nonreduced. We will not use the notion of pseudo-morphism.

Example 8.8 An obvious example of a rational map is a morphism. Another important example is the projection $\mathbb{P}^n_A \dashrightarrow \mathbb{P}^{n-1}_A$ given by $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}]$. (Pick the dense open subset $D_+(x_n) \hookrightarrow \mathbb{P}^n_A$, then $D_+(x_n) = \bigcup_{i=0}^{n-1} D_+(x_i x_n)$, define $D_+(x_i x_n) \to \operatorname{Spec}(k[x_0, \ldots, x_{n-1}]_{x_i})_0$ by $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}]$, we glue these morphisms together, then obtain a morphism of schemes $D_+(x_n) \to \mathbb{P}^{n-1}_A$. Hence projection is rational map.)

✍ **Exercise 8.7** Suppose X is an integral (irreducible and reduced) scheme. Describe how rational maps $X \dashrightarrow \mathbb{A}^1$ should be identified with rational functions on X.

Proof Let $\pi : X \dashrightarrow \mathbb{A}^1$ be a rational map, pick $(\alpha : U \to \mathbb{A}^1)$ be a representative morphism, then we get a ring map $\alpha^\sharp : k[t] \to \Gamma(U, \mathscr{O}_X)$, which is determined by $t \mapsto \alpha^\sharp(t)$. Define φ from rational maps $X \dashrightarrow \mathbb{A}^1$ to rational functions on X, by setting $\varphi(\alpha) = \alpha^\sharp(t)$. We next show φ is well defined. If $(\beta : V \to \mathbb{A}^1)$ be another representative, then $(\alpha : U \to \mathbb{A}^1) \sim (\beta : V \to \mathbb{A}^1)$, i.e., there exists a dense open set $Z \subseteq U \cap V$ such that $\alpha|_Z = \beta|_Z$, hence $(\alpha|_Z)^\sharp(t) = (\beta|_Z)^\sharp(t)$, which implies that $\alpha^\sharp(t)|_Z = \beta^\sharp(t)|_Z$. It follows that φ is a well-defined.

Let (U, f) be a rational function on X, where U is a dense open subset of X. Define ring map $\alpha^\sharp : k[t] \to \Gamma(U, \mathscr{O}_X)$ by setting $t \mapsto \alpha^\sharp(t)$. It induces a morphism of scheme $\alpha : U \to \operatorname{Spec} k[t] = \mathbb{A}^1$. Let $(U, f) \sim (V, g)$, then exists dense open subset $Z \subseteq U \cap V$ such that $f|_Z = g|_Z$. By above discussion $g|_Z$ induces a morphism of schemes $\beta : V \to \mathbb{A}^1$. We want to show that $(\alpha : U \to \mathbb{A}^1) \sim (\beta : V \to \mathbb{A}^1)$. Restrict α and β to Z, then they induced by the same ring maps, hence $(\alpha : U \to \mathbb{A}^1) \sim (\beta : V \to \mathbb{A}^1)$. Define ψ from rational functions on X to rational maps $X \dashrightarrow \mathbb{A}^1$ by setting $(U, f) \to (\alpha : U \to \operatorname{Spec} k[t])$ where $\alpha^\sharp(t) = f$.

By the construction of φ and ψ, there is a bijection between rational maps $X \dashrightarrow \mathbb{A}^1$ and rational functions on X, as we desired. □

Motivated by Exercise 8.7, we can define rational functions on schemes.

Definition 8.5.3 (Rational function)

Let X be a reduced scheme. A rational function on X is rational map $X \dashrightarrow A^1$.

Definition 8.5.4 (Dominant)

A rational map $\pi : X \dashrightarrow Y$ is ***dominant*** *(or in some sources, dominating) if for some (and hence every) representative $U \to Y$, the image is dense in Y. A morphism is a* ***dominant morphism*** *(or dominating morphism) if it is dominant as a rational map.*

Remark **Why every image of representative $U \to Y$ is dense in Y?** Topology fact: if $U \subseteq U'$ are dense and $f(U)$ is dense then $f(U')$ must be dense. Let $(\alpha : U \to \mathbb{A}^1)$ and $(\beta : K \to \mathbb{A}^1)$ be two representative of rational map $X \dashrightarrow Y$, also let $\alpha(U)$ dense in Y. Then there exists dense open subset $Z \subseteq U \cap K$ such that $\alpha|_Z = \beta|_Z$. We next show that $\alpha(Z)$ is dense. If suffices to show that $\alpha(Z) \cap V \neq \varnothing$ for all open subset $V \subseteq Y$. Since $\alpha(U)$ is dense in Y, for all open subset $V \subseteq Y$, $\alpha(U) \cap V \neq \varnothing$, hence $U \cap \alpha^{-1}(V) \neq \varnothing$. Since Z is dense in X, $Z \cap (U \cap \alpha^{-1}(V)) = Z \cap \alpha^{-1}(V) \neq \varnothing$ for all open subset $V \subseteq Y$, hence $\alpha(Z) \cap V \neq \varnothing$ for all open subset $V \subseteq Y$. It follows that $\alpha(Z)$ is dense in Y. Note that $Z \subseteq K$, by the topology fact and $\alpha(Z) = \beta(Z)$, $\beta(K)$ is dense in Y.

Proposition 8.5.1

Suppose X, Y are reduced scheme. A rational map $\pi : X \dashrightarrow Y$ of irreducible schemes is dominant if and only if π sends the generic point of X to the generic point of Y. (In fact, by Proposition 6.2.7, X and Y are integral schemes.)

Proof Since X and Y are irreducible schemes, we may assume that $X = \overline{\{\xi\}}$ and $Y = \overline{\{\eta\}}$, where ξ and η are generic point. If $\pi : X \dashrightarrow Y$ is dominant. Let $\alpha : U \to Y$ be a representative of $\pi : X \dashrightarrow Y$. Since ξ is generic point of X, by Proposition 4.6.10, we have $\xi \in U$. Consider $\overline{\alpha(\xi)}$, we have $\overline{\alpha(\xi)} \subseteq \overline{\alpha(U)} = Y$, since Y is irreducible, $\overline{\alpha(\xi)} = Y$. By Proposition 6.1.2, we have $\alpha(\xi) = \eta$. It follows that π sends the generic point of X to the generic point of Y. Conversely, if $\pi(\xi) = \eta$, pick $\alpha : U \to Y$ be a representative of π, then $\alpha(\xi) = \eta$. Hence $\overline{\alpha(\xi)} = \overline{\{\eta\}} = Y$. Note that $\alpha(U) \supseteq \alpha(\xi)$, we have $\overline{\alpha(U)} \supseteq Y$, i.e., $\overline{\alpha(U)} = Y$. Hence $\pi : X \dashrightarrow Y$ is dominant. □

Remark A little thought will convince you that you can compose (in a well-defined way) a dominant map $\pi : X \dashrightarrow Y$ from an integral scheme X to an integral scheme Y with a rational map $\phi : Y \dashrightarrow Z$. Furthermore, the composition $\rho \circ \pi$ will be dominant if ρ is dominant. Integral schemes and dominant rational maps between them form a category which is geometrically interesting.

Proposition 8.5.2

Dominant rational maps of integral schemes give morphisms of function fields in the opposite direction.

Proof Let $\pi : X \dashrightarrow Y$ be a dominant rational map, say $(\alpha : U \to Y)$ be a representative of π. Suppose ξ is the generic point of X and η is the generic point of Y, by Proposition 8.5.1, $\alpha(\xi) = \eta$. From $\alpha : U \to Y$, we get a morphism of sheaves $\alpha^\sharp : \mathscr{O}_Y \to \alpha_*(\mathscr{O}_X|_U)$. By Proposition 8.2.6, it induce a map of stalks $\alpha^\sharp_\xi : \mathscr{O}_{Y,\eta} \to \mathscr{O}_{X,\xi}$. By Proposition 6.2.9, we have $\mathscr{O}_{Y,\eta} \cong K(Y)$ and $\mathscr{O}_{X,\xi} \cong K(X)$, and therefore we get a morphism of function fields $\alpha^\sharp_\xi : K(Y) \to K(X)$, as we desired. □

In "suitably classical situations" (integral finite type k-schemes — and in particular, irreducible varieties, to be defined in Chapter 12) — this is reversible: dominant rational maps correspond to inclusions of function fields in the opposite direction. We make this precise in below. But it is not true that morphisms of function fileds *always* give dominant rational maps, or even rational maps. For example:

Example 8.9 Spec $k[x]$ and Spec $k(x)$ have the same function field $k(x)$, but there is no corresponding rational map $\operatorname{Spec} k[x] \dashrightarrow \operatorname{Spec} k(x)$ of k-schemes. Reason: such a rational map would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $\operatorname{Spec} k[x, 1/f(x)]$, to $\operatorname{Spec} k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ (sending k identically to k and x to x) for any one $f(x)$.

Next proposition gives a evidence that the topologically-defined notion of dominance is simultaneously algebraic.

Proposition 8.5.3 (Unimportant)

If $\varphi : A \to B$ is a ring morphism, then the corresponding morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant if and only if φ has kernel contained in the nilradical of A.

Proof Say $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ induced by ring map $\varphi : A \to B$. It suffices to show two things: (i) π is dominant if and only if $\varphi^{-1}(\mathfrak{N}(B)) \subseteq \mathfrak{N}(A)$; (ii) $\varphi^{-1}(\mathfrak{N}(B)) \subseteq \mathfrak{N}(A)$ if and only if $\operatorname{Ker} \varphi \subseteq \mathfrak{N}(A)$.

(i) π is dominant iff for all $D(f) \neq \varnothing$ (which implies that $f \notin \mathfrak{N}(A)$), we have $D(f) \cap \pi(\operatorname{Spec} B) \neq \varnothing$ iff exists $f \notin \varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} B$ iff $\varphi(f) \notin \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Spec} B$ and some $f \notin \mathfrak{N}(A)$ iff $\varphi(f) \notin \mathfrak{N}(B)$ for some $f \notin \mathfrak{N}(A)$ iff $f \notin \pi(\mathfrak{N}(B))$ for some $f \notin \mathfrak{N}(A)$ iff $\varphi^{-1}(\mathfrak{N}(B)) \subseteq \mathfrak{N}(A)$.

(ii) Pick $f \in \operatorname{Ker} \varphi$, we want to show that $f \in \mathfrak{N}(A)$. Since $f \in \operatorname{Ker} \varphi$, $\varphi(f) = 0$, and therefore $\varphi(f) \in \mathfrak{N}(B)$, i.e., $f \in \varphi^{-1}(\mathfrak{N}(B)) \subseteq \mathfrak{N}(A)$.
Conversely, if $\operatorname{Ker} \varphi \subseteq \mathfrak{N}(A)$, we want to show that $\varphi^{-1}(\mathfrak{N}(B)) \subseteq \mathfrak{N}(A)$. Pick $f \in \varphi^{-1}(\mathfrak{N}(B))$, then $\varphi(f) \in \mathfrak{N}(B)$. Hence $\varphi(f)^n = \varphi(f^n) = 0$ for some n, and therefore $f^n \in \operatorname{Ker} \varphi \subseteq \mathfrak{N}(A)$, thus $f \in \mathfrak{N}(A)$.

$\square$

Definition 8.5.5 (Birational)

*(i) A rational map $\pi : X \dashrightarrow Y$ is said to be **birational** if it is dominant, and there is another rational map $\psi : Y \dashrightarrow X$ that is also dominant, such that $\pi \circ \psi$ is (in the same equivalence class as) the identity (as rational map) on Y, and $\psi \circ \pi$ is (in the same equivalence class as) the identity (as rational map) on X.*

*(ii) A morphism is a **birational morphism** if it is birational as a rational map. (Note that the "inverse" to a birational morphism may be only a rational map, not a morphism.)*

*(iii) We say X and Y are **birational** (to each other) if there exists a birational map $X \dashrightarrow Y$.*

Remark

(i) Birational map is the notion of isomorphism in the category of integral schemes and dominant rational maps. (In the differentiable category, this is not particularly interesting.)

(ii) If X and Y are irreducible, then birational maps induce isomorphisms of function fields. The fact that maps of function fields correspond to rational maps in the opposite direction for integral finite type k-schemes, to be proved in Proposition 8.5.5, shows that a map between integral finite type k-schemes that induces an isomorphism of function fields is birational.

Definition 8.5.6 (Rational)

*An integral finite type k-scheme is said to be **rational** if it is birational to $\mathbb{A}^n_k$ for some n.*

Proposition 8.5.4

Suppose X and Y are reduced schemes. Then X and Y are birational if and only if there is a dense open subscheme U of X and a dense open subscheme V of Y such that $U \cong V$.

Remark

(a) We have "integral" hypotheses can be dropped with sufficient care and modified definitions.

(b) Proposition 8.5.4 tells you how to think of birational maps. Just as a rational map is a "mostly-defined function", two birational reduced schemes are "mostly isomorphic". For example, a reduced finite type k-scheme (such as an affine k-variety) is rational if and only if it has a dense open subscheme isomorphic to an open subscheme of $\mathbb{A}^n_k$.

Proof The "if" direction is immediate, so we prove the "only if" direction. We basically follow our nose, and use what we are given.

f	$g \circ f$	$g \circ f$ $= \mathrm{id}$ as:		g	$f \circ g$	$f \circ g$ $= \mathrm{id}$ as:
rational map	rational map	rational map	$X \overset{\sim}{\dashleftarrow\dashrightarrow} Y$	rational map	rational map	rational map
morphism	″	″	$X_1 \quad Y_1$	morphism	″	″
″	morphism	″	X_2			
″	″	morphism	$X_3 \leftarrow Y_2$	″	morphism	″
			$X_4 \overset{\sim}{\leftrightarrow} Y_3$	″	″	morphism

We have inverse rational maps $F : X \dashrightarrow Y$ and $G : Y \dashrightarrow X$. Choose representative morphisms $f : X_1 \to Y$ (where $X_1 \subseteq X$) and $g : Y_1 \to X$ (where $Y_1 \subseteq Y$) for the rational maps F and G respectively. If $X_2 = (f|_{X_1})^{-1}(Y_1) \subseteq X_1$, then $g \circ f$ is a morphism from X_2 to X that is the identity as a rational map. Thus there is a dense open subset $X_3 \subseteq X_2$ such that the morphism $g \circ f : X_3 \to X$ is the identity morphism on X_3 (or more precisely, it is the open embedding $X_3 \hookrightarrow X$).

Similarly, let $Y_2 = (g|_{Y_1})^{-1}(X_3) \subseteq Y_1$, so $f \circ g$ is a morphism from $Y_2 \to Y$ that is the identity as a rational map. Then let $Y_3 \subseteq Y_2$ be a dense open subset such that $f \circ g : Y_3 \to Y$ is the inclusion (the "identity").

Finally, if $X_4 = (f|_{X_3})^{-1}(Y_3) \subseteq X_3$, then $(g \circ f)|_{X_4}$ is the identity morphism on X_4 (by way of Y_3), and $(f \circ g)|_{Y_3}$ is the identity morphism on Y_3 (by way of X_4), so we have found our isomorphism of open sets that is a "representative" for our birational map. □

8.5.2 Rational maps of irreducible varieties

Proposition 8.5.5

Suppose X is an integral k-scheme and Y is an integral finite type k-scheme, and we are given an extension of function fields $\varphi^\sharp : K(Y) \hookrightarrow K(X)$ preserving k. Then there exists a dominant rational map of k-schemes $\varphi : X \dashrightarrow Y$ inducing $\varphi^\sharp$.

Proof By replacing Y with an open subset, we may assume that Y is affine, say $\operatorname{Spec} B$, where B is generated over k by finitely many elements $y_1, \ldots, y_n$. Since we only need to define φ on an open subset of X, we may similarly assume that $X = \operatorname{Spec} A$ is affine. Then $\varphi^\sharp$ gives an inclusion $\varphi^\sharp : B \hookrightarrow K(A)$. Write $\varphi^\sharp(y_i)$ as f_i/g_i ($f_i, g_i \in A$), and let $g := \prod_{i=1}^{n} g_i$. Therefore $\varphi : \operatorname{Spec} A_g \to \operatorname{Spec} B$ induces $\varphi^\sharp$. The morphism φ is dominant because the inverse image of the zero ideal under the inclusion $B \hookrightarrow A_g$ is the zero ideal, so φ takes the generic point of X to the generic point of Y, by Proposition 8.5.1, φ is dominant. $\square$

Definition 8.5.7 (Finitely generated (field extension))

Let K be a finitely generated field extension of k, if there is a finite "generating set" $x_1, \ldots, x_n$ in K such that every element of K can be written as a rational function function in $x_1, \ldots, x_n$ with coefficients in k.

Proposition 8.5.6

Let K be a finitely generated field extension of k, then there exists an irreducible affine k-variety with function field K.

Proof Consider the map $\varphi : k[t_1, \ldots, t_n] \to K$ given by $t_i \mapsto x_i$. Since the image of $k[t_1, \ldots, t_n]$ is the subring of field K, by the Fundamental theorem of isomorphism, $\varphi(k[t_1, \cdots, t_n]) \cong k[t_1, \ldots, t_n]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. Consider $K(k[t_1, \ldots, t_n]/\mathfrak{p}) \to K$, then φ induce a homomorphism $\varphi^\dagger : K(k[t_1, \cdots, t_n]/\mathfrak{p}) \to K$ given by $\overline{t_i} \mapsto x_i$ where $\overline{t_i}$ is the image of t_i in $K(k[t_1, \cdots, t_n]/\mathfrak{p})$. Clearly, $\varphi^\dagger$ is a surjective, also $K(k[t_1, \cdots, t_n]/\mathfrak{p}) \hookrightarrow K$, we have $K(k[t_1, \cdots, t_n]/\mathfrak{p}) \cong K$. It follows that $\operatorname{Spec} k[t_1, \ldots, t_n]/\mathfrak{p}$ is an irreducible affine k-variety with function field K. $\square$

Remark Interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the one-point image (say p be the image of $[(0)]$ in $\operatorname{Spec} k[t_1, \ldots, t_n]$, then $\overline{\{p\}} = \operatorname{Spec}(k[t_1, \ldots, t_n]/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$, also $K(k[t_1, \ldots, t_n]/\mathfrak{p}) \cong K$).

Proposition 8.5.7

The following two categories are equivalent.

(a) the category with objects "integral affine k-varieties", and morphisms "dominant rational maps defined over k"; and

(b) the opposite category with objects "finitely generated field extensions of k", and morphisms "inclusions extending the identity on k".

Proof Denote $\mathscr{C}$ be the category with objects "integral affine k-varieties", and morphisms "dominant rational maps defined over k"; and denote $\mathscr{D}$ be the category with objects "finitely generated field extensions of k", and morphisms "inclusions extending the identity on k". Define functor $F : \mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ as follow. Let $X \in \operatorname{obj}(\mathscr{C})$, define $F(X) = K(X)$. Let $X \dashrightarrow Y$ be a dominant rational map defined over k. By Proposition 8.5.2, it induces a morphism of function fields $K(Y) \to K(X)$, hence define $F(X \dashrightarrow Y)$ by setting $K(Y) \to K(X)$.

Then we defined a functor $F : \mathscr{C} \to \mathscr{D}^{\mathrm{op}}$.

Define functor $G : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}$ as follow. Let $K \in \mathrm{obj}(\mathscr{D}^{\mathrm{op}})$, by Proposition 8.5.6, $K \cong K(X)$ where X is an integral k-variety, hence set $G(K(X)) = X$. Let $K(Y) \to K(X)$ be an extension of function fields preserving k, by Proposition 8.5.5, there exists a dominant rational map of k-schemes $X \dashrightarrow Y$ inducing $K(Y) \to K(X)$, so we define $G(K(Y) \to K(X))$ by setting $X \dashrightarrow Y$. Then we defined a functor $G : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}$.

We next show that $F \circ G \cong \mathrm{id}_{\mathscr{D}^{\mathrm{op}}}$ and $G \circ F \cong \mathrm{id}_{\mathscr{C}}$. Let $K(Y) \xrightarrow{\varphi} K(X)$ in $\mathscr{D}^{\mathrm{op}}$. Then

$$F \circ G(K(Y) \xrightarrow{\varphi} K(X)) = F(X \dashrightarrow Y) = K(Y) \to K(X).$$

Hence the following diagram commutes.

$$\begin{tikzcd} F \circ G(K(Y)) \arrow[r] \arrow[d] & F \circ G(K(X)) \arrow[d] \\ K(Y) \arrow[r] & K(X) \end{tikzcd}$$

Let $X \overset{\pi}{\dashrightarrow} Y$ in $\mathscr{C}$. We may assume that $X = \operatorname{Spec} A$, then $G \circ F(X) = G(K(X)) = \operatorname{Spec} A'$ where $K(A') = K(X)$. Hence there exists a birational map such that

$$X \dashleftarrow\dashrightarrow \operatorname{Spec} A'.$$

We may assume that $A = A'$. Consider $G \circ F(\pi)$, then

$$G \circ F(\pi) = G(K(Y) \to K(X)) = \operatorname{Spec} A \dashrightarrow \operatorname{Spec} B = \pi$$

where $Y = \operatorname{Spec} B$. Hence the following diagram commutes.

$$\begin{tikzcd} G \circ F(X) = \operatorname{Spec} A \arrow[r, dashed, "G \circ F(\pi)"] \arrow[d, dashed, leftrightarrow, "\sim"'] & G \circ F(Y) = \operatorname{Spec} B \arrow[d, dashed, leftrightarrow, "\sim"] \\ X \arrow[r, dashed, "\pi"] & Y \end{tikzcd}$$

By above discussion, $\mathscr{C}$ and $\mathscr{D}$ are equivalent. □

Remark Once we define varieties in general in Chapter 12, you can add in: (c) the category with objects "integral k-varieties", and morphisms "dominant rational maps defined over k".

Remark In particular, an integral affine k-variety X is rational if its function field $K(X)$ is a purely transcendental extension of k, i.e., $K(X) \cong k(x_1, \ldots, x_n)$ for some n. (This needs to be said more precisely: the map $k \hookrightarrow K(X)$ induced by $X \to \operatorname{Spec} k$ should agree with the "obvious" map $k \hookrightarrow k(x_1, \ldots, x_n)$ under this isomorphism.)

8.5.3 More examples of rational maps

A recurring theme in these examples is that domains of definition of rational maps to projective schemes extend over regular codimension one points. We will make this precise in the Curve-to-Projective Extension Theorem, when we discuss curves.

The classical formula for Pythagorean triples

The first example is the classical formula for Pythagorean triples, and its derivation by "stereographic projection". Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$ (Figure 8.1). One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic C (now interpreted as $\operatorname{Spec} \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$) to $\mathbb{A}^1_{\mathbb{Q}}$, given by

$(x, y) \mapsto y/(x-1)$. (Something subtle just happened: we were talking about $\mathbb{Q}$-points on a circle, and ended up with a rational map of schemes.)

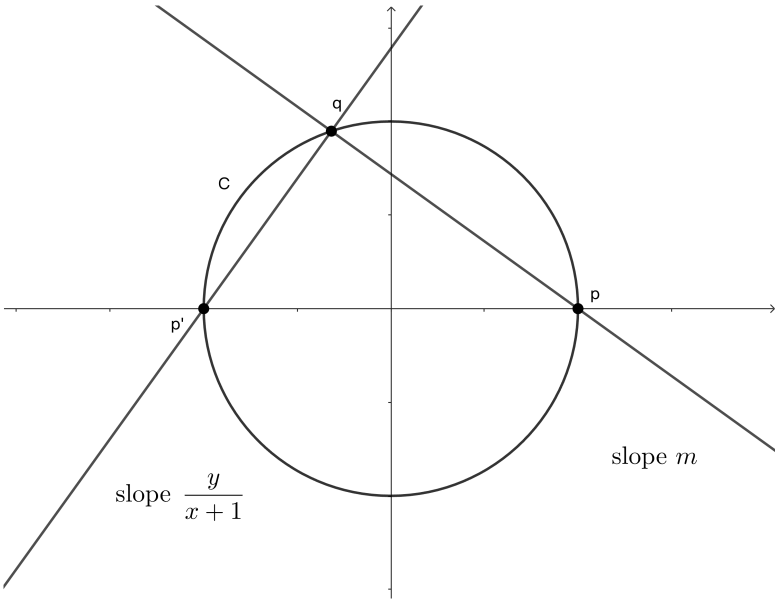


Figure 8.1: Finding primitive Pythagorean triples using geometry

Conversely, given a line of slope m through p, where m is rational, we can recover q by solving the equations

$$\begin{cases} m(x-1) - y &= 0, \\ x^2 + y^2 - 1 &= 0. \end{cases}$$

We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} & x^2 + (m(x-1))^2 &&= 1 \\ \Longrightarrow & (m^2+1)x^2 + (-2m^2)x + (m^2-1) &&= 0 \\ \Longrightarrow & (x-1)((m^2+1)x - (m^2-1)) &&= 0 \end{aligned}$$

The other solution is $x = \frac{m^2-1}{m^2+1}$, which gives $y = \frac{-2m}{m^2+1}$. Thus we get a birational map between the conic C and $\mathbb{A}^1_{\mathbb{Q}}$ with coordinate m, given by $f : (x, y) \mapsto \frac{y}{x-1}$ which is defined for $x \neq 1$, and with inverse rational map given by $m \mapsto \left(\frac{m^2-1}{m^2+1}, \frac{-2m}{m^2+1}\right)$ which is defined away from $m^2 + 1 = 0$.

We can extend to a rational map $C \dashrightarrow \mathbb{P}^1_{\mathbb{Q}}$ via the "inclusion" $\mathbb{A}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ which we later call an open embedding. Then f is given by $(x, y) \mapsto [y : x-1]$. We then have an interesting question: *what is the domain of definition of f?* It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e., everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1 : -y]$ where $(x, y) \neq (-1, 0)$ agrees with f on their common domains of definition, as $[x+1 : -y] = [y, x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of the Curve-to-Projective Extension Theorem.

✍ **Exercise 8.8** Use the above to find a "formula" yielding all Pythagorean triples.

Proof $(0, 0, 0)$ is trivial. Consider equation $x^2 + y^2 = z^2$ with $z \neq 0$ and $\gcd(x, y, z) = 1$. Then we get an equation $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1$. By previous paragraph we know that we must have $\left(\frac{x}{z}, \frac{y}{z}\right) = (1, 0)$ or

$\left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{m^2-1}{m^2+1}, \frac{-2m}{m^2+1}\right)$. Hence $x = z = 1, y = 0$ or $x = m^2 - 1, y = -2m, z = m^2 + 1$ for some m. $\square$

Exercise 8.9 Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}_k^2$ is isomorphic to $\mathbb{P}_k^1$ for any field k of characteristic not 2.

Proof In fact, $x^2 + y^2 = z^2$ in $\mathbb{P}_k^2$ can be written as $\operatorname{Proj}\left(k[x, y, z]/(x^2 + y^2 - z^2)\right)$. Define $\varphi^\sharp$: $k[x, y, z]/(x^2 + y^2 - z^2) \to k[u, v]$ by setting

$$x \mapsto u^2 - v^2,\ y \mapsto -2uv,\ z \mapsto u^2 + v^2.$$

Note that $\varphi^\sharp(\left(k[x, y, z]/(x^2 + y^2 - z^2)\right)_+) = \varphi^\sharp(x, y, z) = (u^2 - v^2, -2uv, u^2 + v^2) = (u^2, uv, v^2)$, hence

$$V(\varphi^\sharp(\left(k[x, y, z]/(x^2 + y^2 - z^2)\right)_+)) = V(u^2, uv, v^2) = \varnothing.$$

By Proposition 8.4.1, $\varphi^\sharp$ induces a morphism of schemes $\varphi : \mathbb{P}_k^1 \to \operatorname{Proj}(k[x, y, z]/(x^2 + y^2 - z^2))$ which given by

$$[a, b] \longmapsto [a^2 - b^2 : -2ab : a^2 + b^2].$$

We next show that φ is an isomorphism. We claim that $\operatorname{Proj}(k[x, y, z]/(x^2+y^2-z^2)) = D_+(x-z) \cup D_+(x+z)$. By Proposition 5.5.8, it suffices to show that $V(x - z, x + z) = \varnothing$. If not, there exists homogeneous prime $[\mathfrak{p}] \in V(x - z, x + z)$, i.e., $x - z, x + z \in \mathfrak{p}$. Hence $x, z \in \mathfrak{p}$. Note that in $\operatorname{Proj}(k[x, y, z]/(x^2 + y^2 - z^2))$, we have $y^2 = z^2 - x^2$, hence $y^2 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime ideal, we have $y \in \mathfrak{p}$. It follows that $\mathfrak{p} = (k[x, y, z]/(x^2 + y^2 - z^2))_+$, contradicts to the fact that $\mathfrak{p} \neq (k[x, y, z]/(x^2 + y^2 - z^2))_+$. Hence $\operatorname{Proj}(k[x, y, z]/(x^2 + y^2 - z^2)) = D_+(x - z) \cup D_+(x + z)$. In fact, we have $\mathbb{P}_k^1 = D_+(u) \cup D_+(v)$. Consider $\varphi_{x-z}^\sharp$ and $\varphi_{x+z}^\sharp$. Note that

$$\begin{aligned}
((k[x, y, z]/(x^2 + y^2 - z^2))_{x-z})_0 &\cong k\left[\frac{x}{x-z}, \frac{y}{x-z}, \frac{z}{x-z}\right] \Big/ \left(\frac{x^2}{(x-z)^2} + \frac{y^2}{(x-z)^2} - \frac{z^2}{(x-z)^2}\right) \\
&\cong k\left[\frac{x}{x-z}, \frac{y}{x-z}\right] \Big/ \left(\frac{y^2}{(x-z)^2} + \frac{2x}{x-z} - 1\right) \\
&\cong k\left[\frac{y}{x-z}\right]
\end{aligned}$$

and

$$(k[u, v]_v)_0 \cong k\left[\frac{u}{v}\right],$$

we know that $\varphi_{x-z}^\sharp : ((k[x, y, z]/(x^+y^2-z^2))_{x-z})_0 \to (k[u, v]_v)_0$ which given by $\frac{y}{x-z} \mapsto \frac{u}{v}$ is an isomorphism. It induces an isomorphism of affine schemes $D_+(x-z) \xrightarrow{\sim} D_+(v)$. Similarly, we have $D_+(x+z) \xrightarrow{\sim} D_+(u)$. Hence

$$\mathbb{P}_k^1 \xrightarrow{\sim} \operatorname{Proj}(k[x, y, z]/(x^2 + y^2 - z^2)),$$

as we desired. $\square$

Remark In fact, any conic in $\mathbb{P}_k^2$ with a k-valued point (i.e., a point with residue field k) of rank 3 (after base change to $\overline{k}$, so "rank" makes sense, see Exercise 6.7) is isomorphic to $\mathbb{P}_k^1$. (The hypothesis of having a k-valued point is certainly necessary: $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ is a conic that is not isomorphic to $\mathbb{P}_k^1$.)

Exercise 8.10 Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}_{\mathbb{Q}}^1$, mimicking what worked with the conic. (In Chapter 20, we will see that these points basically form a group, and that this is a degenerate elliptic curve.)

Proof Denote $C : y^2 = x^3 + x^2$. Let rational point $p = (0, 0)$, q be another rational point on curve C. Then pq is a line of rational slope. This gives a rational map from the curve C to $\mathbb{A}_{\mathbb{Q}}^1$ given by $(a, b) \mapsto \frac{b}{a}$. Conversely, given a line of slope t through p, where t is rational, we can recover q by solving the following equations

$$\begin{cases} y - tx & = 0 \\ x^3 + x^2 - y^2 & = 0 \end{cases} \implies \begin{cases} x & = t^2 - 1 \\ y & = t(t^2 - 1). \end{cases}$$

Hence we get a birational map between the curve C and $\mathbb{A}^1_{\mathbb{Q}}$ with coordinate t, given by $(x, y) \mapsto \frac{y}{x}$ which defined for $x \neq 0$, and with inverse rational map given by $t \mapsto (t^2 - 1, t^3 - t)$. By above discussion all rational solutions to $y^2 = x^3 + x^2$ have the form $(t^2 - 1, t^3 - t)$. □

You will obtain a rational map to $\mathbb{P}^1_{\mathbb{Q}}$ that is not defined over the "node" $x = y = 0$, and cannot be extended over this codimension 1 set. This is an example of the limits of our future result, the Curve-to-Projective Extension Theorem, showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be regular.

✍ **Exercise 8.11** Use a similar idea to find a birational map form the quadric surface $Q = \{x^2+y^2 = w^2+z^2\} \subseteq \mathbb{P}^3_{\mathbb{Q}}$ to $\mathbb{P}^2_{\mathbb{Q}}$. Use this to find all rational points on Q.

Proof Consider $x^2 + y^2 = w^2 + z^2$, then $(x - w)(x + w) = (z - y)(z + y)$. Define a rational map from Q to $\mathbb{P}^2_{\mathbb{Q}}$ given by $[x : y : z : w] \mapsto [z - y : x - w : z + y]$ which defined for $x - w \neq 0$. Conversely, give the inverse rational map by setting $[u : v : s] \mapsto [v^2 + us : v(s - u) : v(u + s) : us - v^2]$ which defined for $v \neq 0$. Hence we get a morphism of affine schemes $\operatorname{Spec}((\mathbb{Q}[x, y, z, w]/(x^2 + y^2 - z^2 - w^2))_{x-w})_0 \to \operatorname{Spec}(\mathbb{Q}[u, v, s]_v)_0$ given by

$$[a : b : c : d] \mapsto [c - b : a - d : c + b].$$

It induce a ring map $(\mathbb{Q}[u, v, s]_v)_0 \to ((\mathbb{Q}[x, y, z, w]/(x^2 + y^2 - z^2 - w^2))_{x-w})_0$ given by

$$\frac{u}{v} \longmapsto \frac{z - y}{x - w}, \quad \frac{s}{v} \longmapsto \frac{z + y}{x - w}.$$

Note that

$$((\mathbb{Q}[x, y, z, w]/(x^2 + y^2 - z^2 - w^2))_{x-w})_0 \cong \mathbb{Q}\left[\frac{y}{x - w}, \frac{z}{x - w}\right],$$

above ring map is an isomorphism. Hence we get a birational map $Q \overset{\sim}{\dashrightarrow} \mathbb{P}^2_{\mathbb{Q}}$. We next use this map to find all rational points on Q. Let $[x : y : z : w] \in Q$. If $x - w \neq 0$, then $[x : y : z : w]$ must be the form $[v^2 + us : v(s - u) : v(s + u) : us - v^2]$ for some u, v, s. If $x = w$, then $y = z$ or $y = -z$, hence solution of Q have the form $[x : y : y : x]$ or $[x : y : -y : x]$. □

Remark This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.

Definition 8.5.8 (The Cremona transformation)

The Cremona transformation is a rational map $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ *given by*

$$[x, y, z] \mapsto \left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right] = [yz : xz : xy].$$

The domain of Cremona transformation is $\mathbb{P}^2_k \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. *(Since if* $yz = xz = xy = 0$, *we get* $[1 : 0 : 0]$ *or* $[0 : 1 : 0]$ *or* $[0 : 0 : 1]$. *Now,* $xyz \neq 0$, *we get the domain.)*

Remark You will observe that you can extend it over "codimension 1 sets" (ignoring the fact that we don't yet know what codimension means). This again foreshadows the Curve-to-Projective Extension Theorem.

⋆ Complex curves that are not rational (fun but inessential)

We now describe two examples of curves C that do not admit a nonconstant rational map from $\mathbb{P}^1_{\mathbb{C}}$. (Admittedly, we do not yet know what "curve" means, but no matter.) Both proofs are by Fermat's method of infinite descent. These results can be interpreted as the fact that these curves have no "nontrivial" $\mathbb{C}(t)$-valued points, where by this, we mean that any $\mathbb{C}(t)$-valued point is secretly a $\mathbb{C}$-valued point. You may notice that if you consider the same examples with $\mathbb{C}(t)$ replaced by $\mathbb{Q}$ (and where C is a curve over $\mathbb{Q}$ rather then $\mathbb{C}$), you get two fundamental questions in number theory and geometry. The analog of Exercise 8.14 is the question of rational points on elliptic curves, and you may realize the analog of Exercise 8.12 is even more famous. Also, the arithmetic analog of Exercise 8.14 (a) is the "Four Squares Theorem" (there are not four integer squares in arithmetic progression), first stated by Fermat. These examples will give you a glimpse of how and why facts over number fields are often paralleled by facts over function fields of curves. This parallelism is a recurring deep theme in the subject.

Exercise 8.12 If $n > 2$, show that $\mathbb{P}^1_{\mathbb{C}}$ has no dominant rational maps to the "Fermat curve" $x^n + y^n = z^n$ in $\mathbb{P}^2_{\mathbb{C}}$.

Proof Suppose that there exists a dominant rational map from $\mathbb{P}^1_{\mathbb{C}}$ to $x^n + y^n = z^n$ in $\mathbb{P}^2_{\mathbb{C}}$. We may assume representative $U \to \operatorname{Proj}(\mathbb{C}[x, y, z]/(x^n + y^n - z^n))$ given by $t \mapsto [f(t) : g(t) : h(t)]$ where $f(t)^n + g(t)^n = h(t)^n$. Since this rational map is dominant, the image of rational map is nonconstant (if not, the image is single point, contradicts to the fact that rational map is dominant). By clearing denominators, we may assume that $f(t), g(t), h(t)$ are relatively prime polynomials in $\mathbb{C}[t]$ and $(f(t), g(t), h(t))$ is the solution of $x^n + y^n = z^n$ with minimal degree, i.e., $\max\{\deg f(t), \deg g(t), \deg h(t)\}$ is minimal. Since $\mathbb{C}[t]$ is a UFD, we may assume that $h(t)^n - g(t)^n = \prod_{i=0}^{n-1} (h(t) - \zeta^i g(t)) = f(t)^n$, where $\zeta = e^{\frac{2\pi i}{n}}$. We claim that each factor $h(t) - \zeta^i g(t)$ is an n-th power in $\mathbb{C}[t]$. We first show each $h(t) - \zeta^i g(t)$ are pairwise coprime. Assume there exists an irreducible polynomial $p(t)$ dividing both $h(t) - \zeta^i g(t)$ and $h(t) - \zeta^j g(t)$, then $p(t)$ divides the linear combination $\zeta^j (h(t) - \zeta^i g(t)) - \zeta^i (h(t) - \zeta^j g(t)) = (\zeta^j - \zeta^i)h(t)$ and $(h(t) - \zeta^i g(t)) - (h(t) - \zeta^j g(t)) = (\zeta^j - \zeta^i)g(t)$. Since $\zeta^i \neq \zeta^j$, we have $p(t) \mid h(t)$ and $p(t) \mid g(t)$. Note that $f(t)^n = h(t)^n - g(t)^n$, we have $p(t)^n \mid f(t)^n$, since $p(t)$ is irreducible polynomial, $p(t) \mid f(t)$. We now show the claim. Let p be any irreducible factor of $h(t) - \zeta^k g(t)$ with multiplicity m_k. By pairwise coprimality, p does not divide any other $h(t) - \zeta^l g(t)$ $(l \neq k)$. Hence in $f(t)^n$, the multiplicity of p is a multiple of n, i.e., $m_k = n \cdot e_k$ for all k. Hence, each $h(t) - \zeta^i g(t)$ is an n-th power $h(t) - \zeta^i g(t) = k_i(t)^n$, where $k_i(t) \in \mathbb{C}[t]$. Pick $\alpha = -\zeta^{-1} - 1$, $\beta = -\zeta^{-1}$, then we have

$$(h(t) - g(t)) + \alpha(h(t) - \zeta g(t)) = \beta(h(t) - \zeta^2 g(t)),$$

and therefore we get

$$k_0(t)^n + \alpha k_1(t)^n = \beta k_2(t)^n.$$

Then $(k_0(t), \alpha^{\frac{1}{n}} k_1(t), \beta^{\frac{1}{n}} k_2(t))$ is another solution of $x^n + y^n = z^n$. Note that

$$\deg k_0(t), \deg \alpha^{\frac{1}{n}} k_1(t), \deg \beta^{\frac{1}{n}} k_2(t) < \max\{\deg f(t), \deg g(t), \deg h(t)\},$$

we have

$$\max\{\deg k_0(t), \deg \alpha^{\frac{1}{n}} k_1(t), \deg \beta^{\frac{1}{n}} k_2(t)\} < \max\{\deg f(t), \deg g(t), \deg h(t)\},$$

this contradicts to the fact that $\max\{\deg f(t), \deg g(t), \deg h(t)\}$ is minimal. Hence $\mathbb{P}^1_{\mathbb{C}}$ has no dominant rational maps to $x^n + y^n = z^n$ in $\mathbb{P}^2_{\mathbb{C}}$. □

Exercise 8.13 Give two smooth complex curves X and Y such that no nonempty open subset of X is isomorphic to a nonempty open subset of Y.

Proof Pick $X = \mathbb{P}^1_{\mathbb{C}}$ and $Y = \mathrm{Proj}(\mathbb{C}[x,y,z]/(x^n + y^n - z^n))$, by Exercise 8.12, we done. $\square$

Exercise 8.14 Suppose a, b and c are distinct complex numbers. By the following steps, show that if $x(t)$ and $y(t)$ are two rational functions of t such that

$$y(t)^2 = (x(t) - a)(x(t) - b)(x(t) - c), \tag{8.9}$$

then $x(t)$ and $y(t)$ are constants ($x(t), y(t) \in \mathbb{C}$). (Here $\mathbb{C}$ may be replaced by any field K of characteristic not 2; slight extra care is needed if K is not algebraically closed.)

Proof We first show the following fact: **Suppose $P, Q \in \mathbb{C}[t]$ are relatively prime polynomials such that four linear combinations of them are perfect squares, no two of which are constant multiples of each other. Then P and Q are constant.**

Let $S_i = a_i P + b_i Q$ as above hypothesis for all $i = 1, 2, 3, 4$. Let $P' = S_1$ and $Q' = S_2$, then P' and Q' are coprime perfect squares. Since $\gcd(P, Q) = 1$, we may write $S_3 = cP' + dQ'$ and $S_4 = eP' + fQ'$ where $c, d, e, f \in \mathbb{C}$. Consider $\frac{S_3}{c}$ and $\frac{S_4}{e}$, they are also perfect squares and not constant multiples of each other. Hence we may assume that the perfect squares are $P, Q, P - Q, P - \lambda Q$ for some $\lambda \in \mathbb{C}$. Say $P = u^2$ and $Q = v^2$, then

$$P - Q = u^2 - v^2 = (u - v)(u + v)$$

and

$$P - \lambda Q = (u - \sqrt{\lambda} v)(u + \sqrt{\lambda} v)$$

are perfect squares. Let $d = \gcd(u - v, u + v)$, then $d \mid ((u - v) + (u + v)) = 2u$ and $d \mid ((u + v) - (u - v)) = 2v$, since $\gcd(u, v) = 1$, we have $d \mid 2$, it follows that $\gcd(u - v, u + v) = 1$. Similarly, we have $\gcd(u - \sqrt{\lambda} v, u + \sqrt{\lambda} v) = 1$. Since $P - Q$ and $P - \lambda Q$ are perfect squares, $u - v, u + v, u - \sqrt{\lambda} v, u + \sqrt{\lambda} v$ must be perfect squares and no two of which are constant multiples of each other.

Now we show the fact. If P and Q are not both constant and suppose $\max\{\deg P, \deg Q\}$ be the minimal. Say $P = u^2$ and $Q = v^2$, by above discussion, $u - v, u + v, u - \lambda v, u + \lambda v$ are perfect squares with no two of which are constant multiples of each other. Note that

$$0 < \max\{\deg u, \deg v\} < \max\{\deg P, \deg Q\},$$

this contradicts to the fact that $\max\{\deg P, \deg Q\}$ is minimal. Hence P and Q are constant.

We next show the Exercise 8.14. Suppose $(x(t), y(t)) = \left(\frac{p(t)}{q(t)}, \frac{r(t)}{s(t)}\right)$ is a solution of equation (8.9), where $p, q, r, s \in \mathbb{C}[t]$, also we may assume that p/q and r/s are in lowest term, i.e., $\gcd(p, q) = \gcd(r, s) = 1$. Clear denominators in (8.9), we have

$$r^2 q^3 = s^2 (p - aq)(p - bq)(p - cq).$$

Since $\gcd(r, s) = 1$, form $s^2 \mid r^2 q^3$, we get $s^2 \mid q^3$. Since $\gcd(p, q) = 1$, we have $q \nmid p - aq, p - bq, p - cq$, hence $q^3 \nmid (p - aq)(p - bq)(p - cq)$. Note that $q^3 \mid s^2 (p - aq)(p - bq)(p - cq)$, we get $q^3 \mid s^2$. Hence we may assume that $s^2 = \delta q^3$ for some $\delta \in \mathbb{C}$. Then we have

$$r^2 = \delta (p - aq)(p - bq)(p - cq).$$

Since $\gcd(p, q) = 1$, we know that $p - aq, p - bq, p - cq$ are pairwise coprime, hence they are all perfect squares with no two of which are constant multiples of each other. Note that $s^2 = \delta q^3$, we know that q must be perfect square. Hence $q, p - aq, p - bq, p - cq$ are all perfect squares with no two of which are constant multiples of each other. Apply above fact, we know that p and q are constants. So s and r are also constants, which implies that $x(t)$ and $y(t)$ are constants. $\square$

A much better geometric approach to Exercise 8.12 and 8.14 is given in Chapter 22.

8.6 ⋆ Representable functors and group schemes

8.6.1 Maps to $\mathbb{A}^1$ correspond to functions

If X is a scheme, there is a bijection between the maps $X \to \mathbb{A}^1$ and global sections of the structure sheaf: by Proposition 8.3.7, maps $\pi : X \to \mathbb{A}^1_{\mathbb{Z}}$ correspond to maps of rings $\pi^\sharp : \mathbb{Z}[t] \to \Gamma(X, \mathscr{O}_X)$, and $\pi^\sharp(t)$ is a function on X; this is reversible.

This map is very natural in an informal sense: you can even picture this map to $\mathbb{A}^1$ as being given by the function. (By analogy, a function on a manifold is a map to $\mathbb{R}$.) But it is natural in a more precise sense: this bijection is functorial in X. We will ponder this example at length, and see that it leads us to two important sophisticated notions: representable functors and group schemes.

Proposition 8.6.1

Suppose X is a $\mathbb{C}$-scheme. Then there is a natural bijection between maps $X \to \mathbb{A}^1_{\mathbb{C}}$ in the category of $\mathbb{C}$-schemes and functions on X. (Here the base ring $\mathbb{C}$ can be replaced by any ring A.)

Proof Give a morphism of schemes $X \to \mathbb{A}^1_{\mathbb{C}}$, we get a ring map $\mathbb{C}[t] \to \Gamma(X, \mathscr{O}_X)$. This ring map is given by $t \mapsto f(t)$, and $f(t)$ is a function on X. Conversely, let $f(t) \in \Gamma(X, \mathscr{O}_X)$, define a ring map $k[t] \to X$ by setting $t \mapsto f(t)$. By Proposition 8.3.7, we get a morphism of schemes $X \to \mathbb{A}^1_{\mathbb{C}}$. Clearly, above process reversible, hence there is a natural bijection between maps $X \to \mathbb{A}^1_{\mathbb{C}}$ in the category of $\mathbb{C}$-schemes and functions on X. □

This interpretation can be extended to rational maps, as follows.

Proposition 8.6.2 (Unimportant)

Rational functions on an integral scheme as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$.

Proof Let X be an integral scheme. Say $\eta \in X$ be the generic point of X. Then $K(X) = \mathscr{O}_{X,\eta}$. Let $f \in K(X)$, then $f \in \Gamma(U, \mathscr{O}_X)$ for some open subset $U \subseteq X$. Define the ring map $\mathbb{Z}[t] \to \Gamma(U, \mathscr{O}_X)$ by setting $t \mapsto f$. By Proposition 8.3.7, we get a morphism of schemes $U \to \operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}$. Since X is irreducible, U is dense in X, hence we get a rational map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$. Let $V \to \mathbb{A}^1_{\mathbb{Z}}$ be a representative of rational map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$, by Proposition 8.3.7, we get a ring map $\mathbb{Z}[t] \to \Gamma(V, \mathscr{O}_X)$ which defined by $t \mapsto g$. Since $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ is rational map, we can shrink U and V to a smaller open subset Z such that $(U \to \mathbb{A}^1_{\mathbb{Z}})|_Z = (V \to \mathbb{A}^1_{\mathbb{Z}})|_Z$. Hence the image of f in $\mathscr{O}_{X,\eta}$ is same as the image of g in $\mathscr{O}_{X,\eta}$.

Conversely, let $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ be an rational map, then for dense open subset $U \subseteq X$ we have a morphism of schemes $U \to \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$. By Proposition 8.3.7, we have a ring map $\mathbb{Z}[t] \to \Gamma(U, \mathscr{O}_X)$. This ring map is given by $t \mapsto f$. Consider the image of f in $\mathscr{O}_{X,\eta} = K(X)$, also denote it f, then f is a rational function on X. From f, apply Proposition 8.3.7, we may recover the rational map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$. □

Remark If you wish, you can extend your argument to rational maps on locally Noetherian schemes, see Definition 7.6.10.

8.6.2 Representable functors

We restate the bijection of §8.6.1 as follows.

Definition 8.6.1 (Representable functors)

We have two different contravariant functors from **Sch** *to* **Sets***: maps to* $\mathbb{A}^1$ *(i.e.,* $H : X \mapsto \operatorname{Mor}(X, \mathbb{A}^1_{\mathbb{Z}})$*), and functions on* X *(i.e.,* $F : X \mapsto \Gamma(X, \mathscr{O}_X)$*). The "naturality" of the bijection — the functoriality in* X *— is precisely the statement that the bijection gives a natural isomorphism of functors: given any* $\pi : X \to X'$*, the diagram*

$$\begin{tikzcd} H(X') \arrow[r, "H(\pi)"] \arrow[d] & H(X) \arrow[d] \\ F(X') \arrow[r, "F(\pi)"] & F(X) \end{tikzcd}$$

(where the vertical maps are the bijections given in §8.6.1) commutes.

More generally, if Y *is an element of a category* $\mathscr{C}$ *(we care about the special case* $\mathscr{C} = \mathbf{Sch}$*), recall the contravariant functor* $h_Y : \mathscr{C} \to \mathbf{Sets}$ *defined by* $h_Y(X) = \operatorname{Mor}(X, Y)$*. We say a contravariant functor from* $\mathscr{C}$ *to* **Sets** *is* ***represented by*** Y *if it is naturally isomorphic to the functor* h_Y*. We say it is* ***representable*** *if it is represented by some* Y*.*

Remark The bijection of §8.6.1 may now be restated as: **the global section functor is represented by** $\mathbb{A}^1$**.**

Proposition 8.6.3 (Representing objects are unique up to unique isomorphism)

If a contravariant functor F *is represented by* Y *and by* Z*, then we have a unique isomorphism* $Y \xrightarrow{\sim} Z$ *induced by the natural isomorphism of functors* $h_Y \xrightarrow{\sim} h_Z$*.*

Proof Since F is represented by Y and Z, F is naturally isomorphic to the functor h_Y and h_Z, and therefore h_Y is naturally isomorphic to h_Z. By Yoneda's Lemma 2.2.1, there is a bijection

$$\operatorname{Nat}(h_Y, h_Z) \xleftrightarrow{\sim} \operatorname{Mor}(Y, Z),$$

since $h_Y \xrightarrow{\sim} h_Z$, we have a unique isomorphism $Y \xrightarrow{\sim} Z$. □

Remark You have implicitly seen this notion before: **you can interpret the existence of products and fibered products in a category as examples of representable functors.**

Let $Y \times Z$ be a product, then for any object X there exists unique morphism such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & Y \times Z \arrow[r, "\mathrm{pr}_Y"'] \arrow[d, "\mathrm{pr}_Z"] & Y \\ & Z & \end{tikzcd}$$

Define the contravariant functor F by setting

$$F(X) := h_Y(X) \times h_Z(X),$$

we shall to show that F is represented by $Y \times X$. Define the natural transformation $\tau : h_{Y \times Z} \to F$ by setting

$$\tau_X : \operatorname{Mor}(X, Y \times Z) \longrightarrow \operatorname{Mor}(X, Y) \times \operatorname{Mor}(X, Z), \quad \varphi \longmapsto (\mathrm{pr}_Y \circ \varphi, \mathrm{pr}_X \circ \varphi).$$

We want to show that τ_X is a bijection. For surjectivity, let $(f, g) \in \operatorname{Mor}(X, Y) \times \operatorname{Mor}(X, Z)$, by the universal property of product $Y \times Z$, there exists unique φ such that $\mathrm{pr}_Y \circ \varphi = f$ and $\mathrm{pr}_X \circ \varphi = g$, hence the surjectivity holds. For injectivity, if $\tau_X(\varphi) = \tau_X(\varphi')$, by the universal property again, we get $\varphi = \varphi'$. It follows that F is represented by $Y \times X$.

You may wish to work out how a natural isomorphism $h_{Y\times Z} \xrightarrow{\sim} h_Y \times h_Z$ induces the projection maps $Y \times Z \to Y$ and $Y \times Z \to Z$: Say $\tau : h_{Y\times Z} \xrightarrow{\sim} h_Y \times h_Z$ be the natural isomorphism. Consider

$$\tau_{Y\times Z} : \operatorname{Mor}(Y \times Z, Y \times Z) \to \operatorname{Mor}(Y \times Z, Y) \times \operatorname{Mor}(Y \times Z, Z),$$

say $\tau_{Y\times Z}(\mathrm{id}_{Y\times Z}) = (p, q)$, by the universal property of product, p, q are the projections map, i.e., $p = \mathrm{pr}_Y$ and $q = \mathrm{pr}_Z$.

Exercise 8.15 Suppose F is the contravariant functor $\mathbf{Sch} \to \mathbf{Sets}$ defined by $F(X) = \{\text{Grothendieck}\}$ for all schemes X. Show that F is representable.

Proof To show that F is representable, it suffices to show that there exists $Y \in \mathbf{Sch}$ such that there is a natural isomorphism $\tau : h_Y \xrightarrow{\sim} F$. Let $Y = \operatorname{Spec} \mathbb{Z}$, by Proposition 8.3.8, Y is the final object in the category of $\mathbf{Sch}$, hence $h_Y(X) = \operatorname{Mor}(X, Y)$ is a single point set, say $h_Y(X) = \{*_X\}$. Define $\tau_X : h_Y(X) \to F(X)$ by setting $*_X \mapsto$ Grothendieck. It is easy to see that τ_X is a bijection for each $X \in \mathbf{Sch}$, and therefore τ is a natural isomorphism, i.e., F is representable. □

Exercise 8.16 In this exercise, $\mathbb{Z}$ may be replaced by any ring.

(a) (Affine n-space represents the functor of n functions.) Show that the contravariant functor from $\mathbb{Z}$-schemes to $\mathbf{Sets}$

$$X \longmapsto \{(f_1, \dots, f_n) : f_i \in \Gamma(X, \mathscr{O}_X)\}$$

is represented by $\mathbb{A}^n_{\mathbb{Z}}$. Show that $\mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \cong \mathbb{A}^2_{\mathbb{Z}}$, i.e., that $\mathbb{A}^2$ satisfies the functorial description of $\mathbb{A}^1 \times \mathbb{A}^1$. (You will undoubtedly be able to immediately show that $\prod \mathbb{A}^{m_i}_{\mathbb{Z}} \xleftrightarrow{\sim} \mathbb{A}^{\sum m_i}_{\mathbb{Z}}$.)

(b) (The functor of invertible functions is representable.) Show that the contravariant functor from $\mathbb{Z}$-schemes to $\mathbf{Sets}$ taking X to invertible functions on X is representable by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

Proof

(a) By Proposition 8.3.7, there is a bijection between $\operatorname{Mor}(X, \mathbb{A}^n_{\mathbb{Z}})$ and $\operatorname{Hom}(\mathbb{Z}[x_1, \dots, x_n], \Gamma(X, \mathscr{O}_X))$. Say F is the contravariant functor by setting

$$X \longmapsto \{(f_1, \dots, f_n) : f_i \in \Gamma(X, \mathscr{O}_X)\}.$$

Define the natural transformation $\tau : h_{\mathbb{A}^n_{\mathbb{Z}}} \to F$ as follow: define $\tau_X : \operatorname{Mor}(X, \mathbb{A}^n_{\mathbb{Z}}) \to F(X)$ by setting

$$\pi \longleftrightarrow \pi^\sharp \longmapsto (\pi^\sharp(x_1), \dots, \pi^\sharp(x_n)).$$

It is easy to check that τ_X is bijection. Hence, F is represented by $\mathbb{A}^n_{\mathbb{Z}}$. In fact, we have

$$\mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \cong \operatorname{Spec} \mathbb{Z}[x] \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[y] \cong \operatorname{Spec}(\mathbb{Z}[x] \otimes \mathbb{Z}[y]) \cong \operatorname{Spec} \mathbb{Z}[x, y] = \mathbb{A}^2_{\mathbb{Z}}.$$

By the induction, we also have

$$\prod \mathbb{A}^{m_i}_{\mathbb{Z}} \xleftrightarrow{\sim} \mathbb{A}^{\sum m_i}_{\mathbb{Z}}.$$

(b) Let G be the contravariant functor from $\mathbb{Z}$-schemes to $\mathbf{Sets}$ taking X to invertible functions on X, i.e., $G(X) = \Gamma(X, \mathscr{O}_X)^\times$. Define the natural transformation $\tau : h_{\operatorname{Spec} \mathbb{Z}[t, t^{-1}]} \to G$ as follow: define $\tau_X : \operatorname{Mor}(X, \operatorname{Spec} \mathbb{Z}[t, t^{-1}]) \to \Gamma(X, \mathscr{O}_X)^\times$ by setting

$$\pi \longmapsto \pi^\sharp(t).$$

It is easy to check that τ_X is a bijection. Hence G is representable by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

□

Definition 8.6.2 (Multiplicative group)

The scheme $\operatorname{Spec}\mathbb{Z}[t,t^{-1}]$ *is called the* ***multiplicative group*** *(or multiplicative group scheme)* $\mathbb{G}_m$. *"*$\mathbb{G}_m$ *over a field* k*" ("the multiplicative group over* k*") means* $\operatorname{Spec} k[t,t^{-1}]$, *with the same group operations.*
Better: *it represents the group of invertible functions in the category of* k*-schemes.*
We can similarly define $\mathbb{G}_m$ *over an arbitrary ring or even arbitrary scheme.*

✍ **Exercise 8.17** (Less important exercise.) Fix a ring A. Consider the functor H from the category of locally ringed spaces to **Sets** given by $H(X) = \{A \to \Gamma(X, \mathscr{O}_X)\}$. Show that this functor is representable (by $\operatorname{Spec} A$). This gives another (admittedly odd) motivation for the definition of $\operatorname{Spec} A$, closed related to the fact that Γ and Spec are adjoints.

Proof By Stacks Project [8]Lemma 01I1, there is a bijection between $\operatorname{Mor}(X, \operatorname{Spec} A)$ and $\operatorname{Hom}(A, \Gamma(X, \mathscr{O}_X))$ for all locally ringed space X. It follows that there exits a natural isomorphism $h_{\operatorname{Spec} A} \xrightarrow{\sim} H$, i.e., H is represented by $\operatorname{Spec} A$. □

8.6.3 ⋆⋆ Group schemes (or more generally, group objects in a category)

8.7 ⋆⋆ The Grassmannian: First construction

Chapter 9 Useful classes of morphisms of schemes

We now define an excessive number of types of morphisms. Some (often finiteness properties) are useful because most "reasonable" morphism has such properties, and they will be used in proofs in obvious ways. Others correspond to geometrically meaningful behavior, and you should have a picture of what each means.

Change of perspective: morphisms are more fundamental than objects. One of Grothendieck's lesson is that things that we often think of as properties of objects are better understood as properties of morphisms. One way of turning properties of objects into properties of morphisms is as follows.

Definition 9.0.1 (Morphism has property $\mathscr{P}$)

*If $\mathscr{P}$ is a property of schemes, we often (but not always) say that a **morphism** $\pi : X \to Y$ **has** $\mathscr{P}$ if for every affine open subset $U \subseteq Y$, $\pi^{-1}(U)$ has $\mathscr{P}$.*

We will see this for $\mathscr{P}$ = quasicompact, quasiseperated, affine, and more. (As you might hope, in good circumstance, $\mathscr{P}$ will satisfy the hypotheses of the Affine Communication Lemma 6.3.1, so we don't have to check every affine open subset.) Informally, you can think of such a morphism as one where all the fibers have $\mathscr{P}$. (You can quickly define the fiber of a morphism as a topological space, but once we define fibered product, we will define the **scheme-theoretic fiber**, and then this discussion will make sense.) But it means more than that: it means that "being $\mathscr{P}$" is really not just fiber-by-fiber, but behaves well as the fiber varies. (For comparison, "submersion of manifolds" means more than that the fibers are smooth.)

9.1 "Reasonable" classes of morphisms (such as open embeddings)

You will notice that almost all classes of morphisms that are useful have many properties in common, which follow from the following three basic properties.

Definition 9.1.1 ("Reasonable" class of morphisms of schemes)

*We call any class of morphisms satisfying these properties a **"reasonable" class of morphisms of schemes**. (To avoid any stupidities, we assume that the class includes all isomorphisms; but this requirement doesn't deserve a number.)*

*(i) The class is **preserved by composition**: if $\pi : X \to Y$ and $\rho : Y \to Z$ are both in this class, then so is $\rho \circ \pi$.*

*(ii) The class is **preserved by "base change" (or "pullback" or "fibered product")**. Precisely: if $\pi : X \to Y$ is in this class, then for any $Y' \to Y$, the induced map $X \times_Y Y' \to Y'$ is also in this class. (Implicit in this statement is that the fibered product $X \times_Y Y'$ exists; but we will soon see that all fibered products of schemes exist, Chapter 11.)*

*(iii) The class is **local on the target**. In other words,*

(a) if $\pi : X \to Y$ is in the class, then for any open subset V of Y, the restricted morphism $\pi^{-1}(V) \to V$ is in the class;

(b) for a morphism $\pi : X \to Y$, if there is an open cover $\{V_i\}$ of Y for which each restricted morphism $\pi^{-1}(V_i) \to V_i$ is in the class, then π is in the class.

In particular, as schemes are built out of affine schemes, properties are often easy to verify on any affine cover (the properties are "affine-local" on the target), as described in Definition 9.0.1.

(Stalk-local properties are automatically local on the target.)

Properties (i) and (ii) imply a useful additional property — (iv) "reasonable" classes of morphisms are **preserved by product**:

Proposition 9.1.1

Suppose $\mathscr{P}$ is a property of morphisms preserved by composition and base change ((i) and (ii) in Definition 9.1.1), and $X \to Y$ and $X' \to Y'$ are two morphisms of S-schemes with property $\mathscr{P}$. Assume that $X \times_S X'$ and $Y \times_S Y'$ exist (which is indeed true, Chapter 11). Then $X \times_S X' \to Y \times_S Y'$ has property $\mathscr{P}$ as well.

Proof By the universal property of fiber product,

$$\begin{tikzcd}
X \times_S X' \arrow[dd] \arrow[dr, dashed] \arrow[drr, bend left] & & \\
& X \times_S Y' \arrow[r] \arrow[d] & X \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times_S Y' \arrow[dd] \arrow[dr, dashed] \arrow[drr, bend left] & & \\
& Y \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Y \arrow[r] & S
\end{tikzcd}$$

we have a decomposition of $X \times_S X' \to Y \times_S Y'$,

$$X \times_S X' \longrightarrow X \times_S Y' \longrightarrow Y \times_S Y'.$$

In fact, we have canonical isomorphism

$$X \times_S X' \cong X \times_S (Y' \times_{Y'} X'), \quad X \times_S Y' \cong (X \times_Y Y) \times_S Y'.$$

Consider following diagram.

$$\begin{tikzcd}
& X \times_S X' \arrow[rr] \arrow[dl] & & X \times_S Y' \arrow[rr] \arrow[dl] \arrow[dr] & & Y \times_S Y' \arrow[dr] & \\
X' \arrow[rr] & & Y' & & X \arrow[rr] & & Y
\end{tikzcd}$$

Since $X' \to Y'$ has property $\mathscr{P}$ and $\mathscr{P}$ preserved by base change, $(X \times_S Y') \times_{Y'} X' \to X \times_S Y'$ has property $\mathscr{P}$, i.e., $X \times_S X' \to X \times_S Y'$ has property $\mathscr{P}$. Similarly, $X \times_S Y' \to Y \times_S Y'$ has property $\mathscr{P}$. Since $\mathscr{P}$ preserved by composition, $X \times_S X' \to Y \times_S Y'$ has property $\mathscr{P}$. □

(v) Another extremely important consequence of (i) and (ii) is the **Cancellation Theorem** for such morphism, see Chapter 12.

When you learn of a new class of morphisms, you should immediately ask whether these properties (i)-(iii) (and hence (iv) and (v)) hold. The answer will almost always be "yes"; and if it is "no", then you should exercise caution, and possibly figure out how to make the definition better.

To prepare you to think about these properties for schemes, you may want to verify that the analogous properties to (i)-(iii) holds in the category of topological spaces with the classes "injections", "surjections", "open embeddings" (open embedding of topological space = isomorphism with an open subset), and "closed embeddings" (closed embedding of topological spaces = isomorphism with a closed subset).

Proposition 9.1.2

Isomorphism of schemes are "reasonable" in this sense, i.e., that they satisfy the three properties (i)-(iii) in Definition 9.1.1.

Proof (i) is clearly. Let $\pi : X \xrightarrow{\sim} Y$ be an isomorphism of schemes. For any $Y' \to Y$, we have $X \times_Y Y' \cong Y \times_Y Y' \cong Y'$ is an isomorphism of schemes. Hence (ii) holds. Let V be any open subset of Y, since $\pi : X \to Y$ is an isomorphism, $\pi^{-1}(V) \to V$ is also an isomorphism. Let $\{V_i \hookrightarrow Y\}$ be an open cover of Y for which each restricted morphism $\pi^{-1}(V_i) \to V_i$ is isomorphism. For all $p \in X$, we have an isomorphism of stalks $\mathscr{O}_{Y,\pi(p)} \xrightarrow{\sim} \mathscr{O}_{X,p}$. Hence $\pi : X \xrightarrow{\sim} Y$ is isomorphism, i.e., (iii) holds. □

We are now ready to introduce a fundamental class of morphisms of schemes.

Definition 9.1.2 (Open embedding, open subscheme)

*A morphism $\pi : X \to Y$ of schemes is an **open embedding** (or **open immersion**) if it is an open embedding as ringed spaces (Definition 8.2.2). In other words, a morphism $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of schemes is an open embedding if π factors as*

$$(X, \mathscr{O}_X) \xrightarrow[\sim]{\rho} (U, \mathscr{O}_Y|_U) \xhookrightarrow{\tau} (Y, \mathscr{O}_Y)$$

where ρ is an isomorphism, and $\tau : U \hookrightarrow Y$ is an inclusion of an open set.
*If X is actually a subset of Y (and π is the inclusion, i.e., ρ is the identity), then we say $(X, \mathscr{O}_X)$ is an **open subscheme** of $(Y, \mathscr{O}_Y)$.*

Remark

(1) It is immediate that isomorphisms are open embeddings. The symbol $\hookrightarrow$ is often used to indicate that a morphism is an open embedding (or more generally, a locally closed embedding).

(2) The difference between open embeddings and open subschemes is a bit confusing, and not too important: at the level of sets, open subschemes are subsets, while open embeddings are bijections onto subsets.

"Open subschemes" are scheme-theoretic analogs of open subsets. ("Closed subschemes" are scheme-theoretic analogs of closed subsets, but they have a surprisingly different flavor, as we will see in Chapter 10.)

The next two Propositions verify that the class of open embeddings is "reasonable" in the sense described above (Definition 9.1.1).

Proposition 9.1.3

The class of open embedding is preserved by composition and local on the target.

Proof We first show that the class of open embedding is local on the target. Suppose $\pi : X \to Y$ is an open embedding and $V \subseteq Y$ is an open subset. We want to show that $\pi^{-1}(V) \to V$ is an open embedding. Since $\pi : X \to Y$ is an open embedding, π factors as

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (U, \mathscr{O}_Y|_U) \hookrightarrow (Y, \mathscr{O}_Y),$$

where U is an open subset of Y. Hence the restricted morphism factor as

$$(\pi^{-1}(V), \mathscr{O}_X|_{\pi^{-1}(V)}) \xrightarrow{\sim} (U \cap V, \mathscr{O}_Y|_{U \cap V}) \hookrightarrow (V, \mathscr{O}_Y|_V).$$

It follows that $\pi^{-1}(V) \to V$ is an open embedding.

Let V_i is an open cover of Y with each restricted morphism $\mathrm{res}_i : \pi^{-1}(V_i) \to V_i$ is open embedding, we want to show that $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an open embedding. Since res_i is an open embedding, res_i factors as

$$(\pi^{-1}(V_i), \mathscr{O}_X|_{\pi^{-1}(V_i)}) \xrightarrow{\sim} (\widetilde{V}_i, \mathscr{O}_Y|_{\widetilde{V}_i}) \hookrightarrow (V_i, \mathscr{O}_Y|_{V_i}).$$

By Gluing sheaves 3.5.3 and Morphism of locally ringed spaces glue 8.3.2, $\mathscr{O}_X|_{\pi^{-1}(V_i)}$ glue to $\mathscr{O}_X$ and $\mathscr{O}_Y|_{\widetilde{V}_i}$

glue to $\mathscr{O}_Y|_{\operatorname{Im}\pi}$ where $\operatorname{Im}\pi = \bigcup_i \widetilde{V}_i$, and we have

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (\operatorname{Im}\pi, \mathscr{O}_Y|_{\operatorname{Im}\pi}) \hookrightarrow (Y, \mathscr{O}_Y).$$

Note that $\operatorname{Im}\pi$ is an open subset of Y, we know that $\pi : X \to Y$ is an open embedding. Hence the class of open embedding is local on the target.

Let $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ are open embeddings, then π_1 factors as

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (U, \mathscr{O}_Y|_U) \hookrightarrow (Y, \mathscr{O}_Y).$$

Since the class of open embedding is local one the target, the restriction morphism $\pi_2|_U$ is also open embedding, so $\pi_2|_U$ factors as

$$(U, \mathscr{O}_Y|_U) \xrightarrow{\sim} (V, \mathscr{O}_Z|_V) \hookrightarrow (Z, \mathscr{O}_Z).$$

Then we have

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (U, \mathscr{O}_Y|_U) \xrightarrow{\sim} (V, \mathscr{O}_Z|_V) \hookrightarrow (Z, \mathscr{O}_Z),$$

it follows that $\pi_2 \circ \pi_1$ is an open embedding, i.e., the class of open embedding is preserved by composition. □

Proposition 9.1.4 (Fibered products with open embeddings exist)

The class of open embeddings is preserved by "base change". More specifically: suppose $i : U \to Z$ is an open embedding, and "$\rho : Y \to Z$" is any morphism. Then $U \times_Z Y$ exists and $U \times_Z Y \to Y$ is an open embedding. In particular, if $U \hookrightarrow Z$ and $V \hookrightarrow Z$ are open embeddings, $U \times_Z V \cong U \cap V$: "intersection of open embeddings is fiber product".

Proof We claim that $(\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)})$ is the fiber product. Without loss of generality, we may assume that U is open subset of Z. Consider the following diagram,

$$\begin{tikzcd}
(\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)}) \arrow[r, hook] \arrow[d, "\rho|_{\rho^{-1}(U)}"'] & (Y, \mathscr{O}_Y) \arrow[d, "\rho"] \\
(U, \mathscr{O}_Z|_U) \arrow[r, "i"'] & (Z, \mathscr{O}_Z)
\end{tikzcd} \tag{9.1}$$

where $(\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)})$ is open subscheme of $(Y, \mathscr{O}_Y)$. Clearly diagram (9.1) commutes. Let $(W, \mathscr{O}_W)$ be any scheme with $i \circ \alpha = \rho \circ \beta$, we want to show that there exists unique morphism from $(W, \mathscr{O}_W)$ to $(\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)})$ such that the following diagram commutes.

$$\begin{tikzcd}
(W, \mathscr{O}_W) \arrow[dr, dashed] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& (\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)}) \arrow[r, hook] \arrow[d, "\rho|_{\rho^{-1}(U)}"] & (Y, \mathscr{O}_Y) \arrow[d, "\rho"] \\
& (U, \mathscr{O}_Z|_U) \arrow[r, "i"'] & (Z, \mathscr{O}_Z)
\end{tikzcd}$$

Since $\rho^{-1}(U)$ is subscheme of Y, $W \to \rho^{-1}(U)$ must be β. Next we check $\beta : W \to \rho^{-1}(U)$ makes above diagram commute. Since $\rho \circ \beta = i \circ \alpha$ and $\beta(W) \subseteq \rho^{-1}(U)$, we have $\alpha = \rho|_{\rho^{-1}(U)} \circ \beta$. Hence $(\rho^{-1}(U), \mathscr{O}_Y|_{\rho^{-1}(U)}) = U \times_Z Y$, by the construction of $U \times_Z Y$, $U \times_Z Y \to Y$ is an open embedding.

In particular, if $U \hookrightarrow Z$ and $V \hookrightarrow Z$ are open embeddings, without loss of generality, we may assume $U \subseteq Z$ and $V \subseteq Z$. Say $i_1 : U \hookrightarrow Z$ and $i_2 : V \hookrightarrow Z$, by above discussion, we have

$$U \times_Z V = (i_2^{-1}(U), \mathscr{O}_Z|_{i_2^{-1}(U)}) = (U \cap V, \mathscr{O}_Z|_{U \cap V}),$$

as we desired. □

Corollary 9.1.1

The class of open embedding is "reasonable".

Proof By Proposition 9.1.3 and Proposition 9.1.4, we get immediately. □

Proposition 9.1.5

Open embeddings of schemes are monomorphisms (in the category of schemes).

Proof By Proposition 8.2.2, we done. □

Exercise 9.1 Suppose $\pi : X \to Y$ is an open embedding. If Y is locally Noetherian, then X is too. If Y is Noetherian, then X is too. However, if Y is quasicompact, X need not be.

Proof If Y is locally Noetherian, we may assume that $Y = \bigcup_i \operatorname{Spec} A_i$ where each A_i is Noetherian. Since $\pi : X \hookrightarrow Y$ is an open embedding, by Corollary 9.1.1, $\pi^{-1}(\operatorname{Spec} A_i) \hookrightarrow \operatorname{Spec} A_i$ is an open embedding. Hence $\pi^{-1}(\operatorname{Spec} A_i)$ can be seen as open subscheme of $\operatorname{Spec} A_i$, and therefore $\pi^{-1}(\operatorname{Spec} A_i)$ is the union of distinguished open subset of $\operatorname{Spec} A_i$. Since $\operatorname{Spec} A_i$ is Noetherian, each distinguished open subset is also Noetherian, hence $X = \bigcup_i \pi^{-1}(\operatorname{Spec} A_i)$ is locally Noetherian.

If Y is Noetherian, we may assume that $Y = \bigcup\limits_{i=1}^{n} \operatorname{Spec} A_i$, by above discussion $X = \bigcup\limits_{i=1}^{n} \operatorname{Spec} A_i$ is locally Noetherian, since this is a finite cover, X is Noetherian.

Let Y be $\operatorname{Spec} k[x_1, x_2, x_3, \dots]$, then Y is quasicompact. Say $\mathfrak{m} = (x_1, x_2, \dots)$, then $U = Y \setminus V(\mathfrak{m})$ is a nonquasicompact open subset of Y. Let $X = U$, then X is not quasicompact. □

Definition 9.1.3 (Local on the source, affine-local on the source, and affine-local on the target)

We call a property $\mathscr{P}$ of morphisms is ***local on the source****, if satisfies:*

(a) if $\pi : X \to Y$ has property $\mathscr{P}$, and $i : U \hookrightarrow X$ is an open subset, then $\pi \circ i : U \to Y$ has $\mathscr{P}$;

(b) for a morphism $\pi : X \to Y$, if for any open cover $\{\iota_i : U_i \hookrightarrow X\}$ of X each $\pi \circ \iota_i : U_i \to Y$ has property $\mathscr{P}$, then $\pi : X \to Y$ has property $\mathscr{P}$.

We call a property $\mathscr{P}$ of morphisms is ***affine-local on the source****, if satisfies:*

(a) if $\pi : X \to Y$ has $\mathscr{P}$, and $i : \operatorname{Spec} A \hookrightarrow X$ is an affine open subset, then $\pi \circ i : \operatorname{Spec} A \to Y$ has $\mathscr{P}$;

(b) for a morphism $\pi : X \to Y$, if for any affine open cover $\{\iota_i : \operatorname{Spec} A_i \hookrightarrow X\}$ of X each $\pi \circ \iota_i : \operatorname{Spec} A_i \to Y$ has property $\mathscr{P}$, then $\pi : X \to Y$ has property $\mathscr{P}$.

We call a property $\mathscr{P}$ of morphisms is ***affine-local on the target****, if satisfies:*

(a) if $\pi : X \to Y$ has property $\mathscr{P}$, then for any affine open subset $\operatorname{Spec} B$ of Y, the restricted morphism $\pi^{-1}(\operatorname{Spec} B) \to \operatorname{Spec} B$ has property $\mathscr{P}$.

(b) for a morphism $\pi : X \to Y$, if for any affine open cover $\{\operatorname{Spec} B_i \hookrightarrow Y\}$ of Y each restricted morphism $\pi^{-1}(\operatorname{Spec} B_i) \to \operatorname{Spec} B_i$ has property $\mathscr{P}$, then $\pi : X \to Y$ has property $\mathscr{P}$.

Remark The use of "affine-local" rather than "local" is to emphasize that the criterion on affine schemes is simple to describe. (Clearly the Affine Communication Lemma 6.3.1 will be very handy.)

Exercise 9.2 The notion of "open embedding" is not local on the source.

Proof Consider the projection $\operatorname{pr} : \mathbb{A}_k^1 \coprod \mathbb{A}_k^1 \to \mathbb{A}_k^1$ given by $\mathfrak{p} \times \{i\} \mapsto \mathfrak{p}$. In fact, $\mathbb{A}_k^1 \coprod \mathbb{A}_k^1 = (\operatorname{Spec} k[x] \times \{1\}) \cup (\operatorname{Spec} k[y] \times \{2\})$ is an open cover. The restriction maps $\operatorname{pr}|_{\operatorname{Spec} k[x] \times \{1\}}$ and $\operatorname{pr}|_{\operatorname{Spec} k[x] \times \{2\}}$ are clearly isomorphism, and therefore is open embedding. But pr is not open embedding, since pr is not injection. Hence "open embedding" is not local on the source. □

9.2 Another algebraic interlude: Lying Over and Nakayama

To set up our discussion in the next section on integral morphisms, we develop some algebraic preliminaries. A clever trick we use can also be used to show Nakayama's Lemma, so we discuss this as well.

9.2.1 Integral

Definition 9.2.1 (Integral)

Suppose $\varphi : B \to A$ is a ring morphism. We say $x \in A$ is ***integral*** *over B if x satisfies some monic polynomial*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

where the coefficients lie in $\varphi(B)$.

A ring morphism $\varphi : B \to A$ is ***integral*** *if every element of A is integral over $\varphi(B)$. An integral ring morphism φ is an* ***integral extension*** *if φ is an inclusion of rings.*

Remark You should think of integral morphisms and integral extensions as ring-theoretic generalizations of the notion of algebraic extensions of fields.

Proposition 9.2.1

If $\varphi : B \to A$ is a ring morphism, $(b_1, \ldots, b_n) = 1$ in B, and $B_{b_i} \to A_{\varphi(b_i)}$ is integral for all i, then φ is integral.

Proof Replace B by $\varphi(B)$ to reduce to the case where B is a subring of A. Suppose $x \in A$, then $x \in A_{b_i}$ for all i. Since $B_{b_i} \to A_{b_i}$ is integral, there exists an equations

$$x^{m(i)} + k_{i,n-1}x^{m(i)-1} + \cdots + k_{i,0} = 0.$$

Clear the denominator, we have

$$b_i^{t(i)}x^{m(i)} + b_i^{t(i)}k_{i,n-1}x^{m(i)-1} + \cdots + b_i^{t(i)}k_{i,0} = 0,$$

where $b_i^{t(i)}k_{i,j} \in B$. Hence $b_i^{t(i)}x^{m(i)} \in B + Bx + \cdots + Bx^{m(i)-1}$. By Proposition 4.5.2 and Proposition 4.5.4, we have $\operatorname{Spec} B = \bigcup_{i=1}^n D(b_i) = \bigcup_{i=1}^n D(b_i^t)$, where $t = \max_i t(i)$. Hence $(b_1^t, b_2^t, \ldots, b_n^t) = 1$. Let $s(i) = t - t(i)$, then

$$b_i^{s(i)}x^{s(i)} \cdot b_i^{t(i)}x^{m(i)} = b_i^t x^t \in B + Bx + \cdots + Bx^{m(i)-1},$$

and therefore

$$\sum_{i=1}^n b_i^t x^t = x^t \in B + Bx + \cdots + Bx^{t-1}.$$

It follows that

$$x^t + k_{t-1}x^{t-1} + \cdots + k_0 = 0$$

for some $k_j \in B$. Hence $\varphi : B \to A$ is integral. □

Proposition 9.2.2

(a) *The property of a morphism $\varphi : B \to A$ being integral is always preserved by localization and quotient of B, and quotient of A, but not localization of A. More precisely: suppose φ is integral, the induced maps $T^{-1}B \to \varphi(T)^{-1}A$, $B/J \to A/\varphi(J)A$, and $B \to A/I$ are integral, where T*

is a multiplicative subset of B, J is an ideal of B, and I is an ideal of A. But $B \to S^{-1}A$ need not be integral, where S is a multiplicative subset of A.

(b) The property of φ being an integral extension is preserved by localization of B, but not localization or quotient of A.

(c) In fact the property of φ being an integral extension (as opposed to integral morphism) is not preserved by taking quotients of B either. But it is in some cases. Suppose $\varphi : B \to A$ is an integral extension, and prime ideal $J \subseteq B$ is the restriction of a prime ideal $I \subseteq A$. The induced map $B/J \to A/JA$ is an integral extension.

Proof

(a) Let T be a multiplicative subset of B. Suppose $\frac{a}{\varphi(t)} \in \varphi(T)^{-1}A$. Since $\varphi : B \to A$ is integral, we have an equation

$$a^n + k_{n-1}a^{n-1} + \cdots + k_0 = 0,$$

where $k_i \in B$. Dividing the equation by $\varphi(t)^n$ given us

$$\frac{a^n}{\varphi(t)^n} + \frac{k_{n-1}}{\varphi(t)} \cdot \frac{a^{n-1}}{\varphi(t)^{n-1}} + \cdots + \frac{k_0}{\varphi(t)^n} = 0,$$

it follows that $T^{-1}B \to \varphi(T)^{-1}A$ is integral.

Next, let J be an ideal of B. Suppose $\overline{a} \in A/\varphi(J)A$. Since $\varphi : B \to A$ is integral, we have equation

$$a^n + \varphi(b_{n-1})a^{n-1} + \cdots + \varphi(b_0) = 0,$$

where $b_i \in B$. Hence, all $b_i \in J$ if and only if $\overline{a}^n = 0$ (i.e., $a \in \varphi(J)$). It follows that integrality is well-defined and preserved on the quotient of B.

Let I be an ideal of A. Suppose $\overline{a} \in A/I$. Since $\varphi : B \to A$ is integral, we have an equation

$$a^n + \varphi(b_{n-1})a^{n-1} + \cdots + \varphi(b_0) = 0,$$

where each $b_i \in B$. Take quotient, we have

$$\overline{a}^n + \overline{\varphi(b_{n-1})}\,\overline{a}^{n-1} + \cdots + \overline{\varphi(b_0)} = 0.$$

Hence $B \to A/I$ is integral.

We now give an counterexample to show that $B \to S^{-1}A$ need not be integral. Consider $\mathrm{id} : k[t] \to k[t]$, clearly it is integral. We want to show that $k[t] \to k[t]_{(t)}$ is not. Suppose $k[t] \to k[t]_{(t)}$ is integral. Consider $\frac{1}{t} \in k[t]_{(t)}$, then we have an equation

$$\frac{1}{t^n} + k_{n-1}(t)\frac{1}{t^{n-1}} + \cdots + k_0(t) = 0,$$

where $k_i(t) \in k[t]$. Multiplying the equation by t^n, we have

$$1 + k_{n-1}(t)t + \cdots + k_0(t)t^n = 0.$$

This is impossible. Hence $k[t] \to k[t]_{(t)}$ is not integral.

(b) Let $i : B \to A$ be an integral extension. Let T be a multiplicative subset of B. By part (a), $T^{-1}B \to T^{-1}A$ is integral. Note that $T^{-1}B$ is the subring of $T^{-1}A$, this map is an inclusion. Hence $T^{-1}B \to T^{-1}A$ is an integral extension.

We next give two counterexamples to show that the property of morphisms being an integral extension is not preserved by localization of A and quotient of A. Consider $\mathrm{id} : k[t] \to k[t]$, clearly, $\mathrm{id} : k[t] \to k[t]$ is an integral extension. By part (a), we know that induced map $k[t] \to k[t]_{(t)}$ is not integral. Now consider $k[t] \to k[t]/(t)$. In fact, $k[t]/(t) \cong k$, map $k[t] \to k$ is clearly integral. But $k[t] \to k$ is not inclusion (since $k[t] \to k[t] \to k[t]/(t)$ is not injection), and therefore is not integral extension. Hence the property of morphisms being an integral extension is not preserved by localization or quotient of A.

(c) We first give a counterexample to show that the property of morphisms being an integral extension is not preserved by taking quotients of B either. Let $B = k[x,y]/(y^2)$ and $A = k[x,y,z]/(z^2, xz - y)$, then we have an obvious inclusion $B \hookrightarrow A$. It is easy to see that $B \hookrightarrow A$ is integral, hence $B \hookrightarrow A$ is integral extension. Consider $B/(x) \to A/(x)$. In fact, $B/(x) \cong k[y]/(y^2)$ and $A/(x) \cong k[x,y,z]/(z^2, xz - y, x) \cong k[z]/(z^2)$, and $B/(x) \to A/(x)$ is given by $a + by \mapsto a + bxz = a$. Map $B/(x) \to A/(x)$ is not injection.
Now suppose $\varphi : B \to A$ is an integral extension, and $J \subseteq B$ is the restriction of an ideal $I \subseteq A$. We want to show that $B/J \to A/JA$ is an integral extension. Since $\varphi : B \to A$ is an integral extension, by part (a), we know that $B/J \to A/JA$ is integral. Since $J = I \cap B$, $B/J \to A/I$ factors through A/JA, i.e., we have $B/J \to A/JA \to A/I$. Since $\varphi : B \to A$ is injective, B can be seen as the subring of A, hence $B/J \to A/I$ is injective. By Atiyah-Macdonald [1] Corollary 5.9, we have $JA = I$. Hence $A/JA \cong A/I$, and therefore $B/J \to A/JA$ is injective.

□

The following lemma uses a useful but sneaky trick.

Lemma 9.2.1

Suppose $\varphi : B \to A$ is a ring morphism. Then $a \in A$ is integral over B if and only if it is contained in a subalgebra of A that is a finitely generated B-module.

Proof If $a \in A$ is integral over B, then a satisfies a monic polynomial equation of degree n. Hence the B-submodule of A generated by $1, a, \ldots, a^{n-1}$ is closed under multiplication, and hence a subalgebra of A.

Conversely, assume that a is contained in a subalgebra A' of A that is a finitely generated B-module. Choose a finite generating set $m_1, \ldots, m_n$ of A' (as a B-module). Then $am_i = \sum b_{ij} m_j$ where $b_{ij} m_j := \varphi(b_{ij}) m_j$, for some $b_{ij} \in B$. Thus

$$(a\,\mathrm{id}_{n\times n} - (b_{ij})_{ij}) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{9.2}$$

where id_n is the $n \times n$ identity matrix in A. We can't invert the matrix $(a\,\mathrm{id}_{n\times n} - (b_{ij})_{ij})$, but we almost can. Recall that an $n \times n$ matrix M has an adjugate matrix $\mathrm{adj}(M)$ such that $\mathrm{adj}(M)M = \det(M)\,\mathrm{id}_n$. Multiplying (9.2) by $\mathrm{adj}(a\,\mathrm{id}_n - (b_{ij})_{ij})$, we get

$$\det(a\,\mathrm{id}_n - (b_{ij})_{ij}) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

So $\det(a\,\mathrm{id}_n - (b_{ij})_{ij})$ annihilates the generating elements m_i, and hence every element of A', i.e., $\det(a\,\mathrm{id}_n - (b_{ij})) = 0$. Expanding the determinant yields an integral equation for a with coefficients in B. □

Corollary 9.2.1 (Finite implies integral)

*If A is a **finite B-algebra** (a finitely generated B-module), then φ is an integral ring morphism.*

Proof Apply Lemma 9.2.1, we done. □

Remark The converse is false: integral does not imply finite, as $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ is an integral ring morphism, but $\overline{\mathbb{Q}}$ is not a finite $\mathbb{Q}$-module. (A field extension is integral if it is algebraic.)

Proposition 9.2.3

If $C \to B$ and $B \to A$ are both integral ring morphisms, then their composition $C \to A$ is integral ring morphism.

Proof Let $a \in A$, since $B \to A$ is integral, by Lemma 9.2.1 a is contained in a subalgebra R of A that is a finitely generated B-module. We may assume R is generated by $b_1, \ldots, b_n$. Since $C \to B$ is integral, by Lemma 9.2.1, each b_i is contained in a subalgebra R_i of B that is finitely generated C-module. Hence R is a finitely generated C-module. By Lemma 9.2.1 again, a is integral over C, and therefore $C \to A$ is integral. □

Proposition 9.2.4

Suppose $\varphi : B \to A$ is a ring morphism. The elements of A integral over B form a subalgebra of A, i.e.,

$$\widetilde{A} = \{a \in A : a \text{ is integral over } B\}$$

is an B-subalgebra of A.

Proof Suppose $a_1, a_2 \in A$ are integral over B, by Lemma 9.2.1, there exists R_1, R_2 be two subalgebras of A such that $a_i \in R_i$ and R_i is finitely generated B-module. Consider R_1R_2, then $a_1a_2 \in R_1R_2$ and $a_1 + a_2 \in R_1R_2$, hence a_1a_2 and $a_1 + a_2$ are integral over B. It follows that the elements of A integral over B form a subalgebra of A. □

Remark **Transcendence theory.** These ideas lead to the main facts about transcendence theory we will need for a discussion of dimension of varieties, see Chapter 13.

9.2.2 The Lying Over and Going-Up Theorems

The Lying Over Theorem is a useful property of integral extensions.

Theorem 9.2.1 (The Lying Over Theorem)

Suppose $\varphi : B \to A$ is an integral extension. Then for any prime ideal $\mathfrak{q} \subseteq B$, there is a prime ideal $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.

Remark To clear on how weak the hypotheses are: B need not be Noetherian, and A need not be finitely generated over B.

Theorem 9.2.2 (The Lying Over Theorem (Geometric version))

If $B \to A$ is an integral extension, then $\operatorname{Spec} A \to \operatorname{Spec} B$ *is surjective.*

Remark Although the Lying Over Theorem 9.2.1 is a theorem in algebra, the name can be interpreted geometrically: the theorem asserts that the corresponding morphism of schemes is surjective. (A map of schemes is **surjective** if the underlying map of sets is surjective.) Translation: "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{p}$ "upstairs", see Figure 9.1. (For this reason, it is often said that $\mathfrak{p}$ "lies over" $\mathfrak{q}$ if $\mathfrak{p} \cap B = \mathfrak{q}$.)

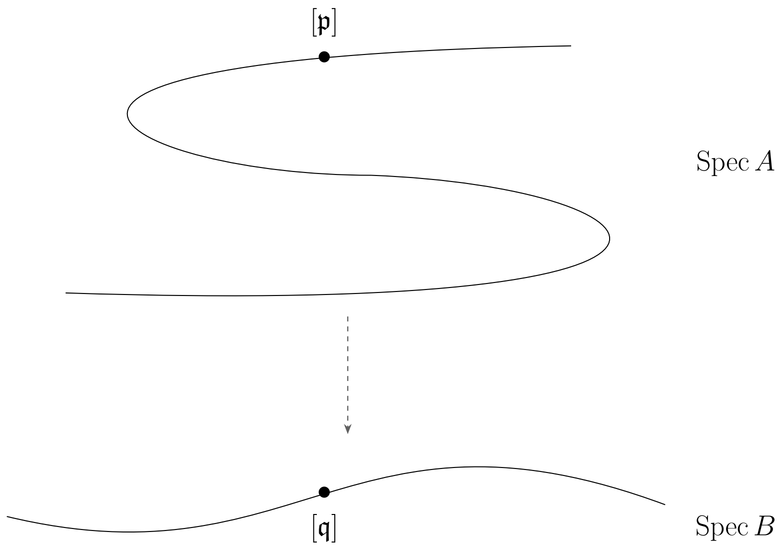


Figure 9.1: A picture of the Lying Over Theorem 9.2.1: if $\varphi : B \to A$ is an integral extension, then $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective

The following lemmas set up the proof.

Lemma 9.2.2 (The special case of the Lying Over Theorem 9.2.1 where A is a field)

Suppose A is a field. Let $B \subseteq A$ is a subring with A integral over B, then B is a field.

Proof Let $x \in B$, $x \neq 0$. Then $x^{-1} \in A$, since A is a field. Note that A integral over B, we have an equation

$$x^{-n} + b_{n-1}x^{-(n-1)} + \cdots + b_1 x^{-1} + b_0 = 0,$$

where $b_i \in B$. It follows that

$$x^{-1} = -(b_{n-1} + \cdots + b_1 x^{n-2} + b_0 x^{n-1}) \in A,$$

hence A is a field. □

Proof of the Lying Over Theorem 9.2.1:

Proof We first make a reduction: by localizing at $\mathfrak{q}$ and Proposition 9.2.2 (b), we can assume that $(B, \mathfrak{q})$ is a local ring. Then let $\mathfrak{p}$ be any maximal ideal of A. Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, two heads] & A/\mathfrak{p} & \text{field} \\ B \arrow[u, hook] \arrow[r, two heads] & B/(\mathfrak{p} \cap B) \arrow[u, hook] & \end{tikzcd}$$

The right vertical arrow is an integral extension by Proposition 9.2.2 (c). By Lemma 9.2.2, $B/(\mathfrak{p} \cap B)$ is a field too, so $\mathfrak{p} \cap B$ is a maximal ideal, hence it is $\mathfrak{q}$. □

Theorem 9.2.3 (The Going-Up Theorem)

Suppose $\varphi : B \to A$ is an integral ring morphism (not necessarily an integral extension). If $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ for all $1 \leq i \leq m$ (and $1 \leq m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ such that each $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ for all $1 \leq i \leq n$.

Proof

$$\begin{array}{ccc} A & & \operatorname{Spec} A \\ \uparrow & & \downarrow \\ B & & \operatorname{Spec} B \end{array} \qquad \begin{array}{ccccccccccc} \mathfrak{p}_1 & \subseteq & \cdots & \subseteq & \mathfrak{p}_m & \subseteq & \mathfrak{p}_{m+1} & \subseteq & \cdots & \subseteq & \mathfrak{p}_n \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ \mathfrak{q}_1 & \subseteq & \cdots & \subseteq & \mathfrak{q}_m & \subseteq & \mathfrak{q}_{m+1} & \subseteq & \cdots & \subseteq & \mathfrak{q}_n \end{array}$$

By induction we reduce immediately to the case $m = 1$, $n = 2$. Let $\overline{B} = B/\mathfrak{q}_1$ and $\overline{A} = A/\mathfrak{p}_1$. Since $\varphi : B \to A$ is integral, by Proposition 9.2.2 (a), $\overline{B} \to \overline{A}$ is integral. Hence, by the Lying Over Theorem 9.2.1, there exists a prime ideal $\overline{\mathfrak{p}_2}$ of $\overline{A}$ such that $\overline{\mathfrak{p}_2} \cap \overline{B} = \overline{\mathfrak{q}_2}$, the image of $\mathfrak{q}_2$ in $\overline{B}$. Lift back $\overline{\mathfrak{p}_2}$ to A and we have a prime ideal $\mathfrak{p}_2$ with the required properties. $\square$

There are analogus "Going-Down" results (requiring quite different hypotheses); see Chapter 13 and Chapter 25.

9.2.3 Nakayama's Lemma

The trick in the proof of Lemma 9.2.1 can be used to quickly prove Nakayama's Lemma, which we will use repeatedly in the future. This name is used for several different but related results, which we discuss here. (A geometric interpretation will be given in Chapter 15). We may as well prove it while the trick is fresh in our minds.

Lemma 9.2.3 (Nakayama's Lemma version 1)

Suppose A is a ring, I is an ideal of A, and M is a finitely-generated A-module, such that $M = IM$. Then there exists an $a \in A$ with $a \equiv 1 \pmod I$ with $aM = 0$. (Equivalently, there is some $i \in I$ for which multiplication by i induces the identity on M: $im = m$ for all $m \in M$.)

Proof Say M is generated by $m_1, \dots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{id}_n - Z) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0 \tag{9.3}$$

where $Z = (a_{ij})$. Multiplying both sides of (9.3) on the left by $\operatorname{adj}(\mathrm{id}_n - Z)$, we obtain

$$\det(\mathrm{id}_n - Z) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0.$$

Expand out $\det(\mathrm{id}_n - Z)$, as Z has entires in I, we get something that is $1 \pmod I$. $\square$

Definition 9.2.2 (Jacobson radical)

Let A be a ring. The Jacobson radical of A is $\operatorname{rad}(A) = \bigcap_{\mathfrak{m} \in \operatorname{Max} A} \mathfrak{m}$ the intersection of all the maximal ideals of R.

Here is why you care. Suppose I is contained in all maximal ideals of A. (I contained in Jacobson radical, see Definition 9.2.2.) Then any $a \equiv 1 \pmod I$ is invertible. (We are not using Nakayama yet!) Reason: otherwise $(a \neq A)$, so the ideal (a) is contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. As a is invertible, we have the following.

Lemma 9.2.4 (Nakayama's Lemma version 2)

Suppose A is a ring, I is an ideal of A contained in all maximal ideals, and M is a finitely generated A-module. (The most interesting case is when A is a local ring, and I is the maximal ideal.) Suppose $M = IM$. Then $M = 0$.

Proof By Nakayama's Lemma 9.2.3, there exists an $a \in A$ with $a \equiv 1 \pmod I$ with $aM = 0$. By the previous discussion, we know that a is invertible, and therefore $M = 0$. □

Lemma 9.2.5 (Nakayama's Lemma version 3)

Suppose A is a ring, and I is an ideal of A contained in all maximal ideals. Suppose M is a finitely generated A-module, and $N \subseteq M$ is a submodule. If $N/IN \to M/IM$ is surjective, then $M = N$.

Proof We claim that $N + IM = M$. Say $\varphi : N/IN \to M/IM$, and let $\varphi(n + IN) = m + IM$, then $n + IM = m + IM$, hence $m - n \in IM$. It follows that $M \subseteq N + IM$. Since $IM \subseteq M$ and $N \subseteq M$, we have $M = N + IM$. Consider the quotient module M/N, we have $M/N \cong (N + IM)/N \cong I(M/N)$. By Nakayama's Lemma 9.2.4, we done. □

Lemma 9.2.6 (Nakayama's Lemma version 4: generators of M/mM lift to generators of M)

Suppose $(A, \mathfrak{m})$ is a local ring. Suppose M is a finitely generated A-module, and $f_1, \ldots, f_n \in M$, with (the images of) $f_1, \ldots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \ldots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

Proof Let $N = (f_1, \ldots, f_n)$. Since the image of $f_1, \ldots, f_n$ generate M, map $N \to M/\mathfrak{m}M$ is surjective. Note that $N \to M/\mathfrak{m}M$ factors through $N/\mathfrak{m}N$, i.e., $N \to N/\mathfrak{m}N \to M/\mathfrak{m}M$. Hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is surjective. Apply Nakayama's Lemma 9.2.5, we have $M = (f_1, \ldots, f_n)$. □

Remark **Small remark:** In the proof of Lemma 9.2.6 we use the following fact:

Lemma

Let $f : M \to M'$ and $g : M \to N$ are A-module homomorphism, g is surjective and $\operatorname{Ker} g \subseteq \operatorname{Ker} f$, then exists unique homomorphism $h : N \to M'$ such that

$$f = hg.$$

Also $\operatorname{Ker} h = g(\operatorname{Ker} f)$, $\operatorname{Im} h = \operatorname{Im} f$. So h is injective iff $\operatorname{Ker} g = \operatorname{Ker} f$; h is surjective iff f is surjective.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, two heads, "g"'] & M' \\ & N \arrow[u, dashed, "\exists! h"'] \end{tikzcd}$$

Definition 9.2.3 (Faithful module)

*A B-module N is said to be **faithful** if the only element of B acting on N by the identity is I, i.e., if $b \cdot n = n$ for all $n \in N$ then $b = 1$ (or equivalently, if the only element of B acting as the 0-map on N is 0, i.e., $\operatorname{Ann}(N) = 0$).*

Lemma 9.2.7 (Generalizing Lemma)

Suppose S is a subring of a ring A, and $r \in A$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Then r is integral over S.

Proof Since M is finitely generated as an S-module, we may assume that $M = (m_1, \ldots, m_n)$. Since $S[r]$-module M is finitely generated as an S-module, suppose $rm_i = \sum_j a_{ij} m_j$ where $a_{ij} \in S$, then we have

$$(r \operatorname{id}_n - (a_{ij})_{ij}) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0. \tag{9.4}$$

Say $Z = (a_{ij})_{ij}$. Multiplying both sides of (9.4) on the left by $\operatorname{adj}(r \operatorname{id}_n - Z)$, we obtain

$$\det(r \operatorname{id}_n - Z) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0.$$

Since M is faithful $S[r]$-module, $\det(r \operatorname{id}_n - Z) = 0$. Expanding the determinant yields an integral equation for r with coefficients in S. □

Proposition 9.2.5

Suppose B is a subring of a ring A. The following are equivalent:

(i) $x \in A$ is integral over B;

(ii) $B[x]$ is a finitely generated B-module;

(iii) $B[x]$ is contained in a subring S of A such that C is a finitely generated B-module.

(iv) There exists a faithful $B[x]$-module M which is finitely generated as an B-module.

Proof (i) $\Rightarrow$ (ii): Since $x \in A$ is integral over B, we have an equation of x,

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0 = 0,$$

where $b_i \in B$. It follows that

$$x^{n+r} = -(b_{n-1}x^{n+r-1} + \cdots + b_0 x^r)$$

for all $r \geq 0$. Hence, by induction, all positive powers of x lie in the B-module generated by $1, x, \ldots, x^{n-1}$. Hence $B[x]$ is generated (as an B-module) by $1, x, \ldots, x^{n-1}$.

(ii) $\Rightarrow$ (iii): Take $C = B[x]$.

(iii) $\Rightarrow$ (iv): Take $M = C$, which is a faithful $A[x]$-module (pick $y \in A[x]$, let $yC = 0$, since $1 \in C$, we have $y \cdot 1 = y = 0$, it follows that M is faithful $A[x]$-module).

(iv) $\Rightarrow$ (i): Lemma 9.2.7. □

Definition 9.2.4 (Integral closure)

Let $B \to A$ be a ring map. The ring $\widetilde{A} \subseteq A$ of elements integral over B, see Proposition 9.2.4, is called the ***integral closure*** *of B in A. If $B \subseteq A$, we say that B is* ***integrally closed*** *in A if $B = \widetilde{A}$.*

Proposition 9.2.6

Suppose A is an integral domain, and $\widetilde{A}$ is the integral closure of A in $K(A)$. Then $\widetilde{A}$ is integrally closed in $K(\widetilde{A}) = K(A)$.

Proof We first show that $K(\widetilde{A}) = K(A)$. Since $\widetilde{A} \subseteq K(A)$ and $K(\widetilde{A})$ is the minimal field which contains $\widetilde{A}$, we have $K(\widetilde{A}) \subseteq K(A)$. Since $A \subseteq \widetilde{A}$, we have $K(A) \subseteq K(\widetilde{A})$, hence $K(A) = K(\widetilde{A})$.

We next show that $\widetilde{A}$ is integrally closed in $K(A)$. Let $x \in K(A)$ and x is integral over $\widetilde{A}$, then we have an equation of x

$$x^n + \widetilde{a_{n-1}}x^{n-1} + \cdots + \widetilde{a}_0 = 0, \tag{9.5}$$

where $\widetilde{a}_i \in \widetilde{A}$. Say $R = A[\widetilde{a}_0, \ldots, \widetilde{a_{n-1}}]$, since each $\widetilde{a}_i$ is integral over A, by the induction and Proposition 9.2.5 (ii), $A[\widetilde{a}_0, \ldots, \widetilde{a_{n-1}}]$ is a finitely generated A-module. From (9.5), we know that x is integral over R, by Proposition 9.2.5, $R[x]$ is a finitely generated R-module. Since R is finitely generated A-module, $R[x]$ is a finitely generated A-module. Let $M = R[x]$, then M can be seen as $A[x]$-module and also is a finitely generated A-module. Clearly, M is faithful, by Generalizing Lemma 9.2.7, x is integral over A. It follows that $\widetilde{A}$ is integrally closed in $K(\widetilde{A}) = K(A)$. □

9.3 A gazillion finiteness conditions on morphisms

By the end of this section, you will have seen the following types of morphisms: quasi-compact, quasi-separated, affine, finite, integral, closed, (locally) of finite type, quasi-finite — and possibly, (locally) of finite presentation.

9.3.1 Quasi-compact and quasi-separated morphisms.

Definition 9.3.1 (Quasi-compact morphism)

*A morphism $\pi : X \to Y$ of schemes is **quasi-compact** if for every affine open subset U of Y, $\pi^{-1}(U)$ is quasi-compact. (Equivalently, the preimage of any quasi-compact open subset is quasi-compact. This is the right definition in other parts of geometry.)*

Remark We will like this notion because:

(i) finite sets have advantages over infinite sets (e.g., a finite set of integers has a maximum; also, things can be proved inductively),

(ii) most reasonable schemes will be quasi-compact.

Along with quasi-compactness comes the weird notion of quasi-separatedness.

Definition 9.3.2 (Quasi-separated morphism)

*A morphism $\pi : X \to Y$ is **quasi-separated** if for every affine open subset U of Y, $\pi^{-1}(U)$ is a quasi-separated scheme (Definition 6.1.2). (Equivalently, the preimage of any quasi-separated open subset is quasi-separated, although we won't worry about proving this. This the definition that extends to other parts of geometry.)*

Note *This will be a useful hypothesis in theorems, usually in conjunction with quasi-compactness. (For this reason, "quasi-compact and quasi-separated" is often abbreviated as **qcqs**.)*

Various interesting kinds of morphisms (locally Noetherian source, affine, separated) are quasi-separated, and having the word "quasi-separated" will allow us to state theorems more succinctly.

Remark We will give an equivalent definition of quasi-separateness in Chapter 12 ("quasi-separated=quasi-compact diagonal"), which will be much simpler to use.

Proposition 9.3.1

The composition of two quasi-compact morphisms is quasi-compact.

Proof Let $\pi : X \to Y$ and $\rho : Y \to Z$ be two quasi-compact morphisms. Pick $U \subseteq Z$ be any open affine subset U of Z. We want to show that $\pi^{-1} \circ \rho^{-1}(U)$ is quasi-compact. Since ρ is quasi-compact, $\rho^{-1}(U)$ is quasi-compact. Since π is quasi-compact, the preimage of any quasi-compact open subset is quasi-compact. So $\pi^{-1} \circ \rho^{-1}(U)$ is quasi-compact. It follows that the composition of two quasi-compact morphisms is quasi-compact. □

Remark It is also true — but not easy — that the composition of two quasi-separated morphisms is quasi-separated. You are free to show this directly, but will in any case follow easily once we understand it in a more sophisticated way, see Chapter 12.

Following Grothendieck's philosophy of thinking that the important notions are properties of morphisms, not of objects, we can restate the definition of a quasi-compact (resp., quasi-separated) scheme as a scheme that is quasi-compact (resp., quasi-separated) over the final object $\operatorname{Spec} \mathbb{Z}$ in the category of schemes (Proposition 8.3.8).

Proposition 9.3.2

(a) Any morphism from a Noetherian scheme is quasi-compact.

(b) Any morphism from a quasi-separated scheme is quasi-separated.

Proof

(a) Let X be a Noetherian scheme, and $\pi : X \to Y$ be a morphism from X to any scheme Y. We want to show that π is quasi-compact. Let U be any affine open subset of Y, then $\pi^{-1}(U)$ is open in X. Since X is Noetherian, by Proposition 4.6.19, $\pi^{-1}(U)$ is quasi-compact. It follows that $\pi : X \to Y$ is quasi-compact.

(b) Let X be a quasi-separated scheme, and $\pi : X \to Y$ be a morphism from X to any scheme Y. Let U be any affine open subset of Y, then $\pi^{-1}(U)$ is open. Let V_1, V_2 be any two affine open subsets of $\pi^{-1}(U)$, then V_1, V_2 are also affine open subsets of X. Since X is quasi-separated, by Proposition 6.1.6, $V_1 \cap V_2$ is a finite union of affine open subsets. By Proposition 6.1.6 again, X is quasi-separated.

□

Remark **Caution.** Two parts of the Proposition 9.3.2 may lead you to suspect that any morphism $\pi : X \to Y$ with quasi-compact source and target is necessarily quasi-compact. **This is false**, and you may verify that the following is a counterexample.

Let Z be the nonquasi-separated scheme constructed in Example 6.1 , and let $X = \operatorname{Spec} k[x_1, x_2, \dots]$ as in Example 6.1. The obvious open embedding $\pi : X \to Z$ (identifying X with one of the two pieces glued together to get Z) is not quasi-compact (remark after Proposition 4.6.4, pick $\operatorname{Spec} X/\{[\mathfrak{m}]\} \subseteq Z$, the inverse image of $\operatorname{Spec} X/\{[\mathfrak{m}]\}$ is also $\operatorname{Spec} X/\{[\mathfrak{m}]\}$, but $\operatorname{Spec} X/\{[\mathfrak{m}]\}$ is not quasi-separated). (But once you see the Cancellation Theorem, you will quickly see that any morphism from a quasi-compact source to a quasi-separated target is necessaily quasi-compact.)

Corollary 9.3.1

Any morphism from a locally Noetherian scheme is quasi-separated. Thus we working only with locally Noetherian schemes may take quasi-separateness as a standing hypothesis.

Proof By Proposition 6.3.3, we know that any locally Noetherian schemes are quasi-separated, by Proposition 9.3.1 (b), locally Noetherian scheme is quasi-separated. $\square$

Proposition 9.3.3

(a) ***(Quasi-compactness is affine-local on the target.)*** *A morphism $\pi : X \to Y$ is quasi-compact if there is a cover of Y by affine open sets U_i such that $\pi^{-1}(U_i)$ is quasi-compact.*

(b) ***(Quasi-separateness is affine-local on the target.)*** *A morphism $\pi : X \to Y$ is quasi-separated if there is a cover of Y by affine open sets U_i such that $\pi^{-1}(U_i)$ is quasi-separated.*

Proof

(a) Let $\pi : X \to Y$ be a morphism of schemes. Consider the property "preimage of affine open subset of Y is quasi-compact", we want to show this property is affine-local property (Definition 6.3.1). It suffices to show two things:

(i) Let $\operatorname{Spec} A \hookrightarrow Y$ be an affine open subset with $\pi^{-1}(\operatorname{Spec} A)$ is quasi-compact, then $\pi^{-1}(\operatorname{Spec} A_f)$ is quasi-compact for any $f \in A$;

(ii) Let $(f_1, \ldots, f_n) = A$ with $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is quasi-compact for all i, then $\pi^{-1}(\operatorname{Spec} A)$ is quasi-compact.

We first show (i). Pick $f \in A$. In fact, $\pi^{-1}(\operatorname{Spec} A_f) = \pi^{-1}(\operatorname{Spec} A) \setminus V(\pi^\sharp f)$. Since $\pi^{-1}(\operatorname{Spec} A)$ is quasi-compact, it can be covered by finite numbers of affine open subset, say $\pi^{-1}(\operatorname{Spec} A) = \bigcup_{i=1}^n \operatorname{Spec} B_i$. Then $\operatorname{Spec} B_i \setminus V(\pi^\sharp f)$ is also affine. Hence

$$\pi^{-1}(\operatorname{Spec} A_f) = \pi^{-1}(\operatorname{Spec} A) \setminus V(\pi^\sharp f) = \bigcup_{i=1}^n (\operatorname{Spec} B_i \setminus V(\pi^\sharp f)).$$

It follows that $\pi^{-1}(\operatorname{Spec} A_f)$ is quasi-compact.

Next we show (ii). Since $(f_1, \ldots, f_n) = A$, we have $\operatorname{Spec} A = \bigcup_{i=1}^n \operatorname{Spec} A_{f_i}$. Hence

$$\pi^{-1}(\operatorname{Spec} A) = \pi^{-1}\left(\bigcup_{i=1}^n \operatorname{Spec} A_{f_i}\right) = \bigcup_{i=1}^n \pi^{-1}(\operatorname{Spec} A_{f_i}).$$

Since each $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is quasi-compact, we know that $\pi^{-1}(\operatorname{Spec} A)$ is quasi-compact. Hence "preimage of affine open subset of Y is quasi-compact" is affine-local property.

Let $\{U_i \hookrightarrow Y\}$ is an open cover of Y such that $\pi^{-1}(U_i)$ is quasi-compact, apply Affine Communication Lemma 6.3.1, the preimage of every affine open subset of Y is quasi-compact, i.e., $\pi : X \to Y$ is quasi-compact.

(b) Let $\pi : X \to Y$ be a morphism of schemes. Consider the property "preimage of affine open subset of Y is quasi-separated", we want to show this property is affine-local property. It suffices to show two things:

(i) Let $\operatorname{Spec} A \hookrightarrow Y$ be an affine open subset with $\pi^{-1}(\operatorname{Spec} A)$ is quasi-separated, then $\pi^{-1}(\operatorname{Spec} A_f)$ is quasi-compact for any $f \in A$;

(ii) Let $(f_1, \ldots, f_n) = A$ with $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is quasi-compact for all i, then $\pi^{-1}(\operatorname{Spec} A)$ is quasi-separated.

We first show (i). Pick $f \in A$. Since $\pi^{-1}(\operatorname{Spec} A)$ is quasi-separated, by Corollary 6.1.3, there exists an affine cover $\{\operatorname{Spec} B_i \hookrightarrow \pi^{-1}(\operatorname{Spec} A)\}$ such that any two of which have intersection covered by a finite number of affine open subsets. By part (a), we know that

$$\pi^{-1}(\operatorname{Spec} A_f) = \pi^{-1}(\operatorname{Spec} A) \setminus V(\pi^\sharp f) = \bigcup_i (\operatorname{Spec} B_i \setminus V(\pi^\sharp f)) = \bigcup_i (\operatorname{Spec}(B_i)_{\pi^\sharp f}).$$

Consider $(\operatorname{Spec} B_i \setminus V(\pi^\sharp f)) \cap (\operatorname{Spec} B_j \setminus V(\pi^\sharp f))$, since $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$ is the union of finite number of affine open subsets,

$$(\operatorname{Spec} B_i \setminus V(\pi^\sharp f)) \cap (\operatorname{Spec} B_j \setminus V(\pi^\sharp f)) = (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) \setminus V(\pi^\sharp f)$$

is the union of finite number of distinguished open subsets. By Corollary 6.1.3 again, $\pi^{-1}(\operatorname{Spec} A)$ is quasi-separated.

We next show (ii). Let $\operatorname{Spec} B, \operatorname{Spec} C$ be any two affine open subsets of $\pi^{-1}(\operatorname{Spec} A)$. We want to show that $\operatorname{Spec} B \cap \operatorname{Spec} C$ is a finite union of affine open subsets. In fact,

$$\operatorname{Spec} B \cap \operatorname{Spec} C = \bigcup_{i=1}^{n} (\operatorname{Spec} B \cap \operatorname{Spec} C \cap \pi^{-1}(\operatorname{Spec} A_{f_i})).$$

Since each $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is quasi-separated, by Proposition 6.1.6, we know that

$$\operatorname{Spec} B \cap \operatorname{Spec} C \cap \pi^{-1}(\operatorname{Spec} A_{f_i}) = (\operatorname{Spec} B \cap \pi^{-1}(\operatorname{Spec} A_{f_i})) \cap (\operatorname{Spec} C \cap \pi^{-1}(\operatorname{Spec} A_{f_i}))$$

is finite union of affine open subsets. Hence $\operatorname{Spec} B \cap \operatorname{Spec} C$ is a finite union of affine open subsets, by Proposition 6.1.6, $\pi^{-1}(\operatorname{Spec} A)$ is quasi-separated. Hence "preimage of affine open subset of Y is quasi-separated"" is an affine-local property.

Let $\{U_i \hookrightarrow Y\}$ is an open cover of Y such that $\pi^{-1}(U_i)$ is quasi-separated, apply Affine Communication Lemma 6.3.1, the preimage of every affine open subset of Y is quasi-separated, i.e., $\pi : X \to Y$ is quasi-separated.

□

9.3.2 Affine morphisms

Definition 9.3.3 (Affine morphism)

A morphism $\pi : X \to Y$ is ***affine*** *if for every affine open set U of Y, $\pi^{-1}(U)$ (interpreted as an open subscheme of X) is an affine scheme.*

Proposition 9.3.4

The composition of two affine morphisms is affine.

Proof Clearly! □

Proposition 9.3.5

Affine morphisms are quasi-compact and quasi-separated.

Proof Let $\pi : X \to Y$ be an affine morphism, then for every affine open set U of Y, $\pi^{-1}(U)$ is an affine scheme, and therefore is quasi-compact. By Corollary 6.1.2, affine scheme is quasi-separated. Hence affine morphisms are quasi-compact and quasi-separated. □

Proposition 9.3.6 (The property of "affineness" of a morphism is affine-local on the target)

A morphism $\pi : X \to Y$ is affine if there is a cover of Y by affine open sets U such that $\pi^{-1}(U)$ is affine.

Proof We want to show that the condition "π is affine over" is an affine-local property. We check our two criteria. First, suppose $\pi : X \to Y$ is affine over $\operatorname{Spec} B$, i.e., $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. Then for any $s \in B$,

$$\pi^{-1}(\operatorname{Spec} B_s) = \pi^{-1}(\operatorname{Spec} B \setminus V(s)) = \pi^{-1}(\operatorname{Spec} B) \setminus \pi^{-1}(V(s)) = \operatorname{Spec} A \setminus V(\pi^\sharp s) = \operatorname{Spec} A_{\pi^\sharp s}.$$

Second, suppose we are given $\pi : X \to \operatorname{Spec} B$ and $(s_1, \dots, s_n) = B$ with $X_{\pi^\sharp s_i} := \operatorname{Spec} A_i$. (Recall from Definition 7.2.4 that $X_{\pi^\sharp s_i}$ is the open subset of X where $\pi^\sharp s_i$ doesn't vanish.) We wish to show that X is affine too. Let $A = \Gamma(X, \mathscr{O}_X)$. Then $X \to \operatorname{Spec} B$ factors through the tautological map $\alpha : X \to \operatorname{Spec} A$ (arising from the (iso)morphism $A \to \Gamma(X, \mathscr{O}_X)$, Proposition 8.3.7).

$$\begin{tikzcd} X = \bigcup_{i=1}^n X_{\pi^\sharp s_i} \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & \operatorname{Spec} A \arrow[dl, "\beta"] \\ & \operatorname{Spec} B = \bigcup_{i=1}^n D(s_i) & \end{tikzcd}$$

We want to show that α is an isomorphism. Now $\beta^{-1}(D(s_i)) = D(\beta^\sharp s_i) \cong \operatorname{Spec} A_{\beta^\sharp s_i}$ (the preimage of a distinguished open set is a distinguished open set), and $\pi^{-1}(D(s_i)) = \operatorname{Spec} A_i$. By Proposition 9.3.3, X is quasi-compact and quasi-separated, so the hypotheses of the Qcqs Lemma 7.2.3 are satisfied. Hence by the Qcqs Lemma 7.2.3, we have an induced isomorphism of $A_{\beta^\sharp s_i} = \Gamma(X, \mathscr{O}_X)_{\beta^\sharp s_i} \cong \Gamma(X_{\beta^\sharp s_i}, \mathscr{O}_X) = A_i$. Thus α induces an isomorphism $\operatorname{Spec} A_i \xrightarrow{\sim} \operatorname{Spec} A_{\beta^\sharp s_i}$ (an isomorphism of rings induces an isomorphism of affine schemes, Proposition 5.3.1). Thus α is an isomorphism over each $\operatorname{Spec} A_{\beta^\sharp s_i}$, which cover $\operatorname{Spec} A$, and hence α is an isomorphism. Therefore $X \to \operatorname{Spec} A$ is an isomorphism, so X is affine as desired. $\square$

The affine-locality of affine morphisms (Proposition 9.3.6) has some nonobvious consequence, as shown in the next proposition.

Proposition 9.3.7 (Useful)

Suppose Z is a closed subset of an affine scheme $\operatorname{Spec} A$ locally cut out by one equation. (In other words, $\operatorname{Spec} A$ can be covered by smaller open sets, and on each such set Z is cut out by one equation.) Then the complement Y of Z is affine. (This is clear if Z is globally cut out by one function f, even set-theoretically; then $Y = \operatorname{Spec} A_f$. However, Z is not always of this form, see Chapter 20.)

Proof We may assume that $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$, say $U_i = D(f_i)$. By the condition, we know that $U_i \cap Z = V(g_i)$. Consider the inclusion $\iota : Y \to \operatorname{Spec} A$, we have

$$\iota^{-1}(U_i) = U_i \cap (\operatorname{Spec} A \setminus Z) = U_i \setminus Z = U_i \setminus (U_i \cap Z) = U_i \setminus V(g_i) = U_i \cap D(g_i) = D(f_i g_i).$$

Since $\operatorname{Spec} A$ is covered by U_i and $\iota^{-1}(U_i)$ is affine, by Proposition 9.3.6, the morphism $\iota : Y \to \operatorname{Spec} A$ is affine, and therefore Y is affine. $\square$

9.3.3 Finite and integral morphisms

Before defining finite and integral morphisms, we give an example to keep in mind:

Example 9.1 If L/K is a field extension, then $\operatorname{Spec} L \to \operatorname{Spec} K$

(i) is always affine;
(ii) is integral if L/K is algebraic;
(iii) is finite if L/K is finite.

Finite morphisms

Definition 9.3.4 (Finite algebra)

If we have a ring morphism $B \to A$ such that A is a finitely generated B-module then we say that A is a ***finite*** *B-algebra.*

Remark This is stronger than being a finitely generated B-algebra. (The similarity of the terminology "finite" and "finitely generated" B-algebra is unfortunate.)

Definition 9.3.5 (Finite morphism)

We say a morphism $\pi : X \to Y$ is ***finite*** *if for every affine open set $\operatorname{Spec} B$ of Y, $\pi^{-1}(\operatorname{Spec} B)$ is the spectrum of a finite B-algebra.*

Proposition 9.3.8

Finite morphisms are affine.

Proof By definition, it is clear. □

Proposition 9.3.9 (The property of finiteness is affine-local on the target)

A morphism $\pi : X \to Y$ is finite if there is a cover of Y by affine open sets $\operatorname{Spec} A$ such that $\pi^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

Proof Let $\pi : X \to Y$ be a morphism of schemes. We want to show that the property "the preimage of affine open sets of Y is the spectrum of a finite algebra" is an affine local property. If suffices to show the following two things:

(i) Let $\operatorname{Spec} A$ be an affine open subsets of Y and $\pi^{-1}(\operatorname{Spec} A)$ is the spectrum of finite A-algebra, than $\pi^{-1}(\operatorname{Spec} A_f)$ is the spectrum of a finite A_f-algebra for any $f \in A$.

(ii) Let $A = (f_1, \ldots, f_n)$ with each $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is the spectrum of finite A_{f_i}-algebra, then $\pi^{-1}(\operatorname{Spec} A)$ is the spectrum of finite A-algebra.

To show (i), we may assume $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ where B is a finite A-algebra. In fact,

$$\pi^{-1}(\operatorname{Spec} A_f) = \pi^{-1}(\operatorname{Spec} A) \setminus V(\pi^\sharp f) = \operatorname{Spec} B_{\pi^\sharp f}.$$

From $\pi|_{\operatorname{Spec} B} : \operatorname{Spec} B \to \operatorname{Spec} A$, we get a ring morphism $\pi|^\sharp_{\operatorname{Spec} B} : A \to B$, it induces a map $A_f \to B_{\pi^\sharp f}$, hence $B_{\pi^\sharp f}$ is an A_f-algebra. Since B is a finite A-algebra, we know that $B_{\pi^\sharp f}$ is a finite A_f-algebra.

To show (ii), we may assume $\pi^{-1}(\operatorname{Spec} A_{f_i}) = \operatorname{Spec} B_i$ where B_i is a finite A_{f_i}-algebra. Say $Z = \pi^{-1}(\operatorname{Spec} A)$. As the proof in Proposition 9.3.6, we know that $Z = \operatorname{Spec} \Gamma(Z, \mathscr{O}_X)$ and $\Gamma(Z, \mathscr{O}_X)_{\pi^\sharp f_i} = B_i$, hence $\Gamma(Z, \mathscr{O}_X)_{\pi^\sharp f_i}$ is a finite A_{f_i}-algebra, and therefore is a finite A-algebra, by Proposition 6.3.2, $\Gamma(Z, \mathscr{O}_X)$ is a finite A-algebra, as we desired.

Hence the property "the preimage of affine open sets of Y is the spectrum of a finite algebra" is an affine-local property. Apply Affine Communication Lemma 6.3.1, we know that $\pi : X \to Y$ is finite. □

The following four example will give you some feeling for finite morphisms. In each example, you will notice two things. In each case, you will notice two things. In each case, the maps are always finite-to-one (as maps of sets). We will verify this in general. You will also notice that the morphisms are **closed** as maps of topological spaces, i.e., the images of closed sets are closed. We will show that finite morphisms are always closed (and give a second proof in Chapter 10). Intuitively, you should think of finite as being closed plus finite fibers, although this isn't quite true. We will make this precise in Chapter 29.

Example 9.2 **Branched covers.** Consider the morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[u]$ given by $u \mapsto p(t)$ where $p(t) \in k[t]$ is a degree n polynomial (see Figure 9.2). This is finite: $k[t]$ is generated as a $k[u]$-module by $1, t, t^2, \ldots, t^{n-1}$ (we may assume $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, let $u = p(t)$, then $t^n \in k[u] + k[u]t + \cdots + k[u]t^{n-1}$, hence $k[t]$ is a finitely generated $k[u]$-module).

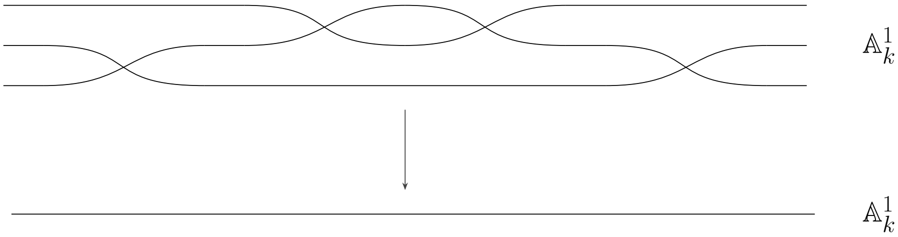


Figure 9.2: The "branched cover" $\mathbb{A}^1_k \to \mathbb{A}^1_k$ of the "u-line" by the "t-line" given by $u \mapsto p(t)$ is finite

Example 9.3 **Closed embeddings** (to be defined soon, in Chapter 10). If I is an ideal of a ring A, consider the morphism $\operatorname{Spec} A/I \to \operatorname{Spec} A$ given by the obvious map $A \to A \setminus I$ (see Figure 9.3 for an example, with $A = k[t]$, $I = (t)$). This is a finite morphism ($A \setminus I$ is generated as an A-module by the element $1 \in A/I$).

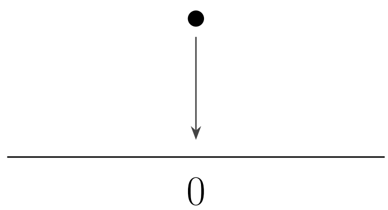


Figure 9.3: The "closed embedding" $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ given by $t \mapsto 0$ is finite

Example 9.4 **Normalization (to be defined in Chapter 11).**

Consider the morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ corresponding to $k[x,y]/(y^2-x^2-x^3) \to k[t]$ given by $x \mapsto t^2 - 1$, $y \mapsto t^3 - t$, see figure 9.4. This is a finite morphism, as $k[t]$ is generated as a $(k[x,y]/(y^2 - x^2 - x^3))$-module by 1 and t (since $t^2 = x + 1$, $t^3 = y + t$). The figure suggests that this is an isomorphism away from the "node" of the target. We can verify this, by checking that it induces an isomorphism between $D(t^2 - 1)$ in the source and $D(x)$ in the target. (Note that under the morphism $x \mapsto t^2 - 1, y \mapsto t^3 - t^2$ we have isomorphism $(k[x,y]/(y^2 - x^2 - x^3))_x \cong k[t^2 - 1, t^3 - t]_{t^2-1} \cong k[t]_{t^2-1}$). We will meet this example again repeatedly!

Proposition 9.3.10 (Finite morphism to $\operatorname{Spec} k$. Important!)

If $X \to \operatorname{Spec} k$ is a finite morphism, then X is a finite union of points with the discrete topology, each point with residue field a finite extension of k, see Figure 9.5.

Remark An example is

$$\operatorname{Spec}(\mathbb{F}_8 \times \mathbb{F}_4[x,y]/(x^2, y^4) \times \mathbb{F}_4[t]/(t^9) \times \mathbb{F}_2) \longrightarrow \operatorname{Spec} \mathbb{F}_2.$$

Proof Suppose $\pi : X \to \operatorname{Spec} k$ is a finite morphism, then $\pi^{-1}(\operatorname{Spec} k) = X$ is the spectrum of finite k-algebra. We may assume that $X = \operatorname{Spec} A$, where A is a finite k-algebra. Pick $[\mathfrak{p}] \in \operatorname{Spec} A$, then $A/\mathfrak{p}$ is an integral domain and also is a finite k-algebra, by Proposition 4.2.1, we know that $A/\mathfrak{p}$ is a field, and therefore $\mathfrak{p}$ is maximal ideal. Hence $\operatorname{Spec} A = \operatorname{Max} A$. Since A is a finitely generated k-vector space, A is a finite dimension k-vector space, and therefore A is Noetherian ring. Note that $\operatorname{Spec} A = \operatorname{Max} A$, A is Artin ring.

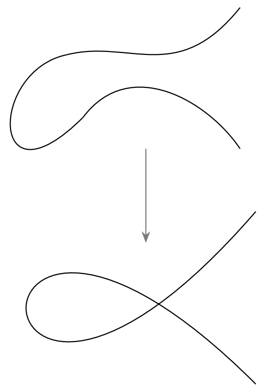


Figure 9.4: The "normalization" of the nodal cubic $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ given by $(x,y) \mapsto (t^2-1, t^3-t^2)$ is finite

Figure 9.5: A picture of a finite morphism to $\operatorname{Spec} k$. Bigger fields are depicted as bigger points.

By Atiyah-MacDonald [1] Proposition 8.3, we know that A has only a finite number of maximal ideal. Hence $\operatorname{Spec} A$ has finite closed points, i.e. $\operatorname{Spec} A$ is finite, and therefore discrete. Let $[\mathfrak{m}] \in \operatorname{Spec} A$, then the residue field is $\kappa([\mathfrak{m}]) \cong K(A/\mathfrak{m}) \cong A/\mathfrak{m}$. Since A is finite dimensional k-vector space, the residue field $A/\mathfrak{p}$ is a finite extension of k. $\square$

Proposition 9.3.11

The composition of two finite morphisms is also finite.

Proof Let $\pi : X \to Y$ and $\psi : Y \to Z$ be two finite morphisms. Let $\{\operatorname{Spec} C_i \hookrightarrow Z\}$ be a cover of Z. Since $\psi : Y \to Z$ is finite, we know that each $\psi^{-1}(\operatorname{Spec} C_i)$ is the spectrum of finite C_i-algebra. We may assume $\psi^{-1}(\operatorname{Spec} C_i) = \operatorname{Spec} B_i$ where B_i is finite C_i-algebra. Since $\{\psi^{-1}(\operatorname{Spec} C_i) \hookrightarrow Y\}$ is an open cover of Y, we know that $\{\pi^{-1} \circ \psi^{-1}(\operatorname{Spec} C_i) \hookrightarrow X\}$ is an open cover of X, since π is finite morphism, each $\pi^{-1} \circ \psi^{-1}(\operatorname{Spec} C_i)$ is the spectrum of finite B_i-algebra, say $\pi^{-1} \circ \psi^{-1}(\operatorname{Spec} C_i) = \operatorname{Spec} A_i$ where A_i is finite B_i-algebra. Each A_i can be seen as finitely generated B_i-module, since B_i can be seen as finitely generated C_i-module, we know that A_i is a finitely generated C_i-module. Hence A_i is also a finite C_i-algebra. By Proposition 9.3.9, $\psi \circ \pi : X \to Y$ is finite morphism. $\square$

Proposition 9.3.12 (Finite morphisms to $\operatorname{Spec} A$ are projective)

If R is an A-algebra, define a graded ring $S_\bullet$ by $S_0 = A$, and $S_n = R$ for $n > 0$. Then there is an isomorphism $\operatorname{Proj} S_\bullet \xrightarrow{\sim} \operatorname{Spec} R$. Suppose $\pi : X \to \operatorname{Spec} A$ is a finite morphism, then X is a projective

A-scheme (Definition 5.5.11).

Proof In fact, we have $\operatorname{Proj} S_\bullet = \bigcup_{f\in S_1} D_+(f)$ and $\operatorname{Spec} R = \bigcup_{f\in R} D(f)$. By Proposition 5.5.9, we know that $D_+(f) \cong \operatorname{Spec}((S_\bullet)_f)_0$ and $D(f) \cong \operatorname{Spec} R_f$. Define a ring map $R_f \to ((S_\bullet)_f)_0$ by setting

$$\frac{r}{f^k} \longmapsto \frac{r}{f^k},$$

where r can be seen as the element belong to $S_k = R$. Hence this ring map is well-defined, and clearly is an isomorphism. By Proposition 5.3.1, we obtain an isomorphism of schemes $\pi_f : D_+(f) \xrightarrow{\sim} D(f)$. Clearly π_f "agree on the overlaps", i.e. $\pi_f|_{D_+(f)\cap D_+(g)} = \pi_g|_{D_+(f)\cap D_+(g)}$, by Morphism of locally ringed spaces glue 8.3.2, there is an isomorphism $\operatorname{Proj} S_\bullet \xrightarrow{\sim} \operatorname{Spec} R$.

Suppose $\pi : X \to \operatorname{Spec} A$ is a finite morphism, then $X = \operatorname{Spec} R$ for some R where R is a finite A-algebra. By above discussion we know that $X \cong \operatorname{Proj} S_\bullet$, and $S_\bullet$ is a finitely generated graded ring over A, and hence that X is a projective A-scheme. □

Remark Hence finite morphisms are affine (by definition) and projective. The converse is also true, see Chapter 19.

Proposition 9.3.13 (Important)

The finite morphisms have finite fibers.

Remark This is a useful proposition, because you will have to figure out how to get at points in a fiber of a morphism: given $\pi : X \to Y$, and $q \in Y$, what are the points of $\pi^{-1}(q)$? This will be easier to do once we discuss fibers in greater detail, but it will be enlightening to do it now.

Proof We reduce the problem via the following steps.

$$\begin{array}{ccccc}
 & X \xrightarrow{\ \pi\ } Y & \ni & q & \\
 & \uparrow \qquad\quad \uparrow & & & \\
\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A & \operatorname{Spec} A \xrightarrow{\text{finite}} \operatorname{Spec} B & \ni & q = [\mathfrak{q}] & \\
A' = A/\pi^\sharp(\mathfrak{q})A & \operatorname{Spec} A' \xrightarrow{\text{finite}} \operatorname{Spec} B' & \ni & [(0)] & B' = B/\mathfrak{q} \\
A'' = \pi^\sharp(B' \setminus \{(0)\})^{-1}A' & \operatorname{Spec} A'' \xrightarrow{\text{finite}} \operatorname{Spec} B'' & \ni & [(0)B'_{(0)}] & B'' = B'_{(0)}
\end{array}$$

Suppose the morphism of schemes $\pi : X \to Y$ is finite and $q \in Y$, then q contained in some affine open subsets of Y, say $q \in \operatorname{Spec} B$. Since $\pi : X \to Y$ is finite, we may assume $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ where A is a finite B-algebra. Hence we may reduce the problem by setting $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $q = [\mathfrak{q}] \in \operatorname{Spec} B$. Clearly, the morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is finite.

Now we throw out everything in B outside $\overline{\{q\}}$ by modding out by $\mathfrak{q}$. From finite ring map $B \to A$, it induces a finite ring map $B/\mathfrak{q} \to A/\pi^\sharp(\mathfrak{q})A$, i.e., $A/\pi^\sharp(\mathfrak{q})A$ is a finite $B/\mathfrak{q}$-algebra. Then we obtain a finite morphism of schemes $\operatorname{Spec} A/\pi^\sharp(\mathfrak{q})A \to \operatorname{Spec} B/\mathfrak{q}$, and $q \in \operatorname{Spec} B$ is corresponding to $[(0)] \in \operatorname{Spec} B/\mathfrak{q}$. So we have reduced to the case where Y is the Spec of an integral domain B, and $[\mathfrak{q}] = [(0)]$ is the generic point.

Next, we can throw out the rest of the points of B by localizing at (0). From ring map $\pi^\sharp : B \to A$, we obtain ring map $B_{(0)} \to \pi^\sharp(B \setminus \{0\})^{-1}A$. Since A is finitely generated B-module, $(B \setminus \{0\})^{-1}A$ is finitely generated $B_{(0)}$-module, and therefore $(B \setminus \{0\})^{-1}A$ is finite $B_{(0)}$-algebra. Hence morphism of schemes

$\operatorname{Spec}(\pi^\sharp(B \setminus \{0\})^{-1}A) \to \operatorname{Spec} B_{(0)}$ is finite. Note that B is an integral domain, $B_{(0)}$ is a field. Also q is corresponding to $[(0)B_{(0)}] \in \operatorname{Spec} B_{(0)}$. So we may reduce to the case that $Y = \operatorname{Spec} k$ and $q = [(0)] \in Y$.

By Proposition 9.3.10, $\pi^{-1}(q)$ is a finite union of points, i.e. fiber is finite. □

Proposition 9.3.12 and Proposition 9.3.13 show that finite morphisms are projective and have finite fibers. The converse is also true, see Chapter 19.

There is more to finiteness than finite fibers, and three examples to keep in mind are Example 9.5 , Example 9.6 , and figure in Chapter 20 (a variant of Figure 9.4 showing a morphism that is affine, closed, one-to-one on points and even a monomorphism, but not finite).

Example 9.5 The open embedding $\mathbb{A}^2_k \setminus \{(0,0)\} \to \mathbb{A}^2_k$ has finite fibers, but is not affine (as $\mathbb{A}^2_k \setminus \{(0,0)\}$ isn't affine, §5.4.1) and hence not finite.

Example 9.6 The open embedding $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}}$ has finite fibers and is affine, but is not finite.

Proof Note that $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \cong \operatorname{Spec} \mathbb{C}[x]_x$, but $\mathbb{C}[x]_x$ is not finitely generated $\mathbb{C}[x]$-module, and therefore $\mathbb{C}[x]_x$ is not finite $\mathbb{C}[x]$-algebra. Hence open embedding $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}}$ is not finite. □

Integral morphisms

Definition 9.3.6 (Integral)

A morphism $\pi : X \to Y$ of schemes in ***integral*** *if π is affine, and for every affine open subset $\operatorname{Spec} B \subseteq Y$, with $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$, the induced map $B \to A$ is an integral ring morphism.*

Proposition 9.3.14

The property of integrality of morphisms is affine-local on the target.

Proof By Proposition 9.2.1 and Proposition 9.2.2, and Affine Communication Lemma 6.3.1. □

Proposition 9.3.15

The composition of two integral morphisms is also integral.

Proof By Proposition 9.2.3. □

Integral morphisms are mostly useful because:

Proposition 9.3.16

Finite morphisms are integral. But the converse implication doesn't hold.

Proof First part by Corollary 9.2.1. Consider $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$, since $\mathbb{Q} \to \overline{\mathbb{Q}}$ is integral, we know that $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ is integral morphism. But $\overline{\mathbb{Q}}$ is not a finite $\mathbb{Q}$-module, and therefore $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ is not finite, it follows that the converse implication doesn't hold. □

Proposition 9.3.17

Integral morphisms are closed, i.e., the image of closed subsets are closed.

Proof Reduce to the affine case. If $\pi^\sharp : B \to A$ is a ring map, inducing the integral morphism $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$. Suppose $I \subseteq A$ be an ideal of A, then it cuts out a closed set $V(I)$ of $\operatorname{Spec} A$. Say $J = (\pi^\sharp)^{-1}(I)$, by Proposition 9.2.2, induced ring map $B/J \to A/I$ is integral. By The Lying Over Theorem 9.2.1, for any prime ideal $\mathfrak{q} \subseteq B/J$ there exists a prime ideal $\mathfrak{p} \supseteq A/I$ such that $\mathfrak{p} \cap B = \mathfrak{q}$. It follows that $\pi(V(I)) = V(J)$, i.e. the image of $V(I)$ is closed. □

Corollary 9.3.2

Finite morphisms are closed. (A second proof will be given in §10.3.2.)

Proof Finite implies integral (Proposition 9.3.16), apply Proposition 9.3.17. $\square$

Proposition 9.3.18 (Unimportant)

Suppose $B \to A$ is integral. Then for any ring morphism $B \to C$, the induced map $C \to A \otimes_B C$ is integral.

Proof Suppose $\varphi : B \to A$ is integral and $\psi : B \to C$ be any ring morphism from B. Let $S = \{\sum_{i=1}^n a_i \otimes c_i \in A \otimes_B C : \sum_{i=1}^n a_i \otimes c_i \text{ is integral over } C\}$, by Proposition 9.2.4, we know that S is an C-subalgebra of $A \otimes_B C$. In fact, $A \otimes_B C$ is generated by $\{a \otimes 1 : a \in A\} \cup \{1 \otimes c : c \in C\}$. Clearly, $\{1 \otimes c : c \in C\} \subseteq S$. It suffices to show that $\{a \otimes 1 : a \in A\} \subseteq S$. Pick $a \otimes 1$. Since A is integral over B, a satisfies some monic polynomial,

$$p(x) = x^n + \varphi(b_{n-1})x^{n-1} + \cdots + \varphi(b_1)x + \varphi(b_0) = 0,$$

where $b_i \in B$. Induced map $\tau : C \to A \otimes_B C$ is given by $c \mapsto 1 \otimes c$. Let

$$\begin{aligned} q(x) =& x^n + \tau \circ \psi(b_{n-1})x^{n-1} + \cdots + \tau \circ \psi(b_1)x + \tau \circ \psi(b_0) \\ =& x^n + (1 \otimes \psi(b_{n-1}))x^{n-1} + \cdots + (1 \otimes \psi(b_1))x + 1 \otimes \psi(b_0) \end{aligned}$$

we want to check that $q(a \otimes 1) = 0$. Consider $q(a \otimes 1)$, we have

$$\begin{aligned} q(a \otimes 1) =& a^n \otimes 1 + a^{n-1} \otimes \psi(b_{n-1}) + \cdots + a \otimes \psi(b_1) + 1 \otimes \psi(b_0) \\ =& a^n \otimes 1 + \varphi(b_{n-1})a^{n-1} \otimes 1 + \cdots + \varphi(b_1)a \otimes 1 + \varphi(b_0) \otimes 1 \\ =& p(a) \otimes 1 \\ =& 0. \end{aligned}$$

It follows that $a \otimes 1$ is integral over C, hence $\{a \otimes 1 : a \in A\} \cup \{1 \otimes c : c \in C\} \subseteq S$, and therefore $S = A \otimes_B C$, which implies that induced map $C \to A \otimes_B C$ is integral. $\square$

Remark Once we know what "base change" is, this will imply that the property of integrality of a morphisms is preserved by base change.

Fibers of integral morphisms

Unlike finite morphisms (Proposition 9.3.13), integral morphisms don't always have finite fibers. However, once we make sense of fibers as topological spaces (or even schemes) in Chapter 11, you can check that the fibers have the property that no point is in the closure of any other point.

Example 9.7 **Example of integral morphisms don't have finite fibers.** Consider finite field $\mathbb{F}_2$, and ring $\mathbb{F}_2[x]/(x^2) = \{0, 1, x, x+1\}$. Then $\mathbb{F}_2 \to \mathbb{F}_2[x]/(x^2)$ is integral. Note that

$$p(t) = t^2(t-1)(t+1)$$

killing all of $\mathbb{F}_2[x]/(x^2)$, $p(t)$ also kills the element of $\prod_{i \in \mathbb{N}} \mathbb{F}_2[x]/(x^2)$. Hence $\mathbb{F}_2 \to \mathbb{F}_2[x]/(x^2)$ is integral. So we obtain an integral morphism of schemes

$$\pi : \operatorname{Spec} \prod_{i \in \mathbb{N}} \mathbb{F}_2[x]/(x^2) \longrightarrow \operatorname{Spec} \mathbb{F}_2,$$

the fiber $\pi^{-1}([(0)])$ is infinite.

9.3.4 Morphisms (locally) of finite type

Locally of finite type, finite type

Definition 9.3.7 (Locally of finite type)

A morphism $\pi : X \to Y$ is ***locally of finite type*** *if for every affine open set $\operatorname{Spec} B$ of Y, and every affine open subset $\operatorname{Spec} A$ of $\pi^{-1}(\operatorname{Spec} B)$, the induced morphism $B \to A$ express A as a finitely generated B-algebra. By the affine-locality of finite typeness of B-schemes (Proposition 6.3.2), this is equivalent to: $\pi^{-1}(\operatorname{Spec} B)$ can be covered by affine open subsets $\operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra.*

Definition 9.3.8 (Finite type)

A morphism π is ***of finite type*** *if it is a locally of finite type and quasi-compact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $\pi^{-1}(\operatorname{Spec} B)$ can be covered with a finite number of open sets $\operatorname{Spec} A_i$ such that the induced morphism $B \to A_i$ expresses as A_i as a finitely generated B-algebra.*

Remark **Linguistic curiosity.** It is a common practice to name properties as follows: $\mathscr{P}$ equals locally $\mathscr{P}$ plus quasi-compact. Two exceptions are "ringed space" (§8.3) and "finite presentation".

Example 9.8 The "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n+1$ open sets of the form $\operatorname{Spec} A[x_1, \ldots, x_n]$.

Our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") from Definition 6.3.4 is now a special case of this more general notion: the phrase "a scheme X is of finite type over k" means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism" Definition 8.3.4) that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

Proposition 9.3.19 (The notions "locally finite type" and "finite type" are affine-local on the target)

A morphism $\pi : X \to Y$ is locally of finite type if there is a cover of Y by affine open sets $\operatorname{Spec} B_i$ such that $\pi^{-1}(\operatorname{Spec} B_i)$ is locally finite type over B_i (Definition 6.3.4). Hence the notions of "locally finite type" and "finite type" are both affine-local on the target.

Proof Suppose $\pi : X \to Y$ be a morphism. We shall to show that the property "preimage of affine open subset of Y is locally finite type" is affine-local property. It suffices to show the following two things:

(i) Let $\operatorname{Spec} B$ be an affine open subset of Y, with $\pi^{-1}(\operatorname{Spec} B)$ is locally finite type over B, then $\pi^{-1}(\operatorname{Spec} B_f)$ is locally finite type over B_f for all $f \in B$.

(ii) If $B = (f_1, \ldots, f_n)$ and $\pi^{-1}(\operatorname{Spec} B_{f_i})$ is locally finite type over B_{f_i} for all i, then $\pi^{-1}(\operatorname{Spec} B)$ is locally finite type over B.

We first show (i). Since $\pi^{-1}(\operatorname{Spec} B)$ is locally finite type over B, we may assume $\pi^{-1}(\operatorname{Spec} B) = \bigcup_i \operatorname{Spec} A_i$ where each A_i is finitely generated B-algebra. Localizing each A_i at $\pi^\sharp(f)$, we know that $(A_i)_{\pi^\sharp(f)}$ is finitely generated B_f-algebra. Note that

$$\begin{aligned}\pi^{-1}(\operatorname{Spec} B_f) &= \pi^{-1}(\operatorname{Spec} B) \setminus \pi^{-1}(V(f)) = \pi^{-1}(\operatorname{Spec} B) \setminus V(\pi^\sharp(f)) \\ &= \bigcup_i (\operatorname{Spec} A_i \setminus V(\pi^\sharp(f))) \\ &= \bigcup_i \operatorname{Spec}(A_i)_{\pi^\sharp(f)},\end{aligned}$$

it follows that $\pi^{-1}(\operatorname{Spec} B_f)$ is locally finite type over B_f.

We next show (ii). Suppose $B = (f_1, \ldots, f_n)$ and $\pi^{-1}(\operatorname{Spec} B_{f_i})$ is locally finite type over B_{f_i} for all i. We may assume $\pi^{-1}(\operatorname{Spec} B_{f_i}) = \bigcup_j \operatorname{Spec} A_{ij}$ where A_{ij} is finitely generated B_{f_i}-algebra. Hence A_{ij} is of the form $B_{f_i}[t_1, \ldots, t_m]/I_{ij}$ where $B_{f_i}[t_1, \ldots, t_m]$ is polynomial ring. Since localization commutes with quotient, we know that $A_{ij} \cong (C_{ij})_{f_i}$, hence $(C_{ij})_{f_i}$ is finitely generated B_{f_i}-algebra, and therefore is finitely generated B-algebra. Note that

$$\begin{aligned}\pi^{-1}(\operatorname{Spec} B) =&\pi^{-1}\left(\bigcup_{i=1}^{n} \operatorname{Spec} B_{f_i}\right) = \bigcup_{i=1}^{n} \pi^{-1}(\operatorname{Spec} B_{f_i}) \\ =&\bigcup_{i=1}^{n}\bigcup_{j} \operatorname{Spec} A_{ij} \cong \bigcup_{i=1}^{n}\bigcup_{j} \operatorname{Spec}(C_{ij})_{f_i}\end{aligned}$$

and each $(C_{ij})_{f_i}$ is finitely generated B-algebra, we know that $\pi^{-1}(\operatorname{Spec} B)$ is locally finite type over B.

Hence the property "preimage of affine open subset of Y is locally finite type" is affine-local property. Apply Affine Communication Lemma 6.3.1, the notions "locally finite type" is affine-local on the target. □

Proposition 9.3.20 (Finite = Integral + Finite Type)

(a) Finite morphisms are of finite type.

(b) A morphism is finite if and only if it is integral and of finite type.

Proof

(a) Let $\pi : X \to Y$ be a finite morphism, then for any affine open subset $\operatorname{Spec} B \hookrightarrow Y$ preimage $\pi^{-1}(\operatorname{Spec} B)$ is the spectrum of a finite B-algebra, say $\operatorname{Spec} A$ where A is a finite B-algebra. Hence A is finitely generated B-module, and therefore is finitely generated B-algebra. It follows that $\pi : X \to Y$ is of finite type.

(b) If $\pi : X \to Y$ is a finite morphism, by Proposition 9.3.16 and part (a), we done.
Conversely, if $\pi : X \to Y$ is integral and of finite type, we want to show that $\pi : X \to Y$ is finite. Let $\operatorname{Spec} B \hookrightarrow Y$ be any affine open subset of Y, since $\pi : X \to Y$ is integral, $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ where induced ring map $B \to A$ is integral. Since $\pi : X \to Y$ is also of finite type, A is finitely generated B-algebra. Then there is an evaluation homomorphism $\mathrm{val} : B[t_1, \ldots, t_n] \twoheadrightarrow A$ given by $t_i \mapsto a_i$. Hence $A \cong B[t_1, \ldots, t_n]/\operatorname{Ker} \mathrm{val}$. Since A is integral over B, each a_i satisfies polynomial equation

$$x^{m_i} + \pi^\sharp(b_{i,n-1})x^{m_i-1} + \cdots + \pi^\sharp(b_{i,1})x + \pi^\sharp(b_{i,0}) = 0,$$

where $b_{i,j} \in B$. Then we have

$$a_i^{m_i} = -\pi^\sharp(b_{i,n-1})a_i^{m_i-1} - \cdots - \pi^\sharp(b_{i,1})a_i - \pi^\sharp(b_{i,0}).$$

It follows that

$$\begin{aligned}A \cong&B[t_1, \ldots, t_n]/\operatorname{Ker} \mathrm{val} \\ \cong&Ba_1 + \cdots + Ba_1^{m_1-1} + Ba_2 + \cdots + Ba_2^{m_2-1} + \cdots + Ba_n^{m_n-1},\end{aligned}$$

i.e., A is finitely generated B-module, and therefore is finite B-algebra. Hence $\pi : X \to Y$ is finite.

□

Proposition 9.3.21 (Important!)

(a) Every open embedding is locally of finite type, and hence that every quasi-compact open embedding is of finite type. Every open embedding into a locally Noetherian scheme is of finite type.

(b) The composition of two morphisms locally of finite type is locally of finite type. The composition of two morphisms of finite type is of finite type.

(c) *Suppose $\pi : X \to Y$ is locally of finite type, and Y is locally Noetherian. Then X is also locally Noetherian. If $\pi : X \to Y$ is a morphism of finite type, and Y is Noetherian, then X is Noetherian.*

Proof

(a) Let $\iota : U \hookrightarrow Y$ be any open embedding. For any affine open subset $\operatorname{Spec} B$ of Y, $\iota^{-1}(\operatorname{Spec} B) = \operatorname{Spec} B \cap U$, hence $\iota^{-1}(\operatorname{Spec} B)$ is an open subset of $\operatorname{Spec} B$, and therefore it is covered by some distinguished subset $\operatorname{Spec} B_f$. Consider the canonical morphism $B \to B_f$, clearly, $B_f = B[f^{-1}]$ is finitely generated B-algebra. Hence open embedding $\iota : U \hookrightarrow Y$ is locally of finite type. Hence every quasi-compact open embedding is of finite type.

If Y is a locally Noetherian scheme, then B is a Noetherian ring. Since $U \cap \operatorname{Spec} B$ is open subset of $\operatorname{Spec} B$, by Proposition 4.6.19, $U \cap \operatorname{Spec} B$ is quasi-compact. So apply above discussion, we know that Every open embedding into a locally Noetherian scheme is of finite type.

(b) Let $\pi : X \to Y$ and $\psi : Y \to Z$ be two morphisms locally of finite type, we want to show that $\psi \circ \pi : X \to Z$ locally of finite type. Let $\operatorname{Spec} C$ be any affine open subset of Z, then $\psi^{-1}(\operatorname{Spec} C)$ is covered by affine open subsets $\operatorname{Spec} B_i$ where each B_i is a finitely generated C-algebra. Since $\pi : X \to Y$ locally of finite type, we know that each $\pi^{-1}(\operatorname{Spec} B_i)$ can be covered by $\operatorname{Spec} A_{ij}$ where A_{ij} is finitely generated B_i algebra, and therefore is finitely generated C-algebra. It follows that $(\psi \circ \pi)^{-1}(\operatorname{Spec} C)$ can be cover by the spectrums of C-algebra, i.e., $\psi \circ \pi$ locally of finite type.

Hence as the composition of two quasi-compact morphisms is quasi-compact, Proposition 9.3.1, the composition of two morphisms of finite type is of finite type.

(c) Since Y is locally Noetherian, we may assume $Y = \bigcup_{i=1}^{n} \operatorname{Spec} B_i$ where each B_i is Noetherian, then X is covered by $\pi^{-1}(\operatorname{Spec} B_i)$. Since $\pi : X \to Y$ is locally of finite type, we may assume $\pi^{-1}(\operatorname{Spec} B_i) = \bigcup_{j=1}^{m_i} \operatorname{Spec} A_{ij}$ where each A_{ij} is finitely generated B_i-algebra. Hence $A_{ij} \cong B_i[t_1, \ldots, t_l]/I_{ij}$, since B_i is Noetherian, by Hilbert's Basis Theorem, we know that A_{ij} is Noetherian. Thus X is covered by the spectrum of Noetherian ring which implies that X is locally Noetherian.

Similarly, if Y is Noetherian, from the quasi-compactness of Y, apply above discussion, we know that X can be covered by finite numbers of spectrums of Noetherian ring, it follows that X is Noetherian.

□

Absolute Frobenius morphism

Definition 9.3.9 (Absolute Frobenius morphism)

Suppose X is a scheme over $\mathbb{F}_p$ where p is a prime number. Define an endomorphism $F : X \to X$ such that for each affine open subset $\operatorname{Spec} A \subseteq X$, F corresponds to the map $A \to A$ given by $f \mapsto f^p$ for all $f \in A$. The morphism F is called the ***absolute Frobenius morphism****.*

Proposition 9.3.22

Suppose X is a scheme over $\mathbb{F}_p$ where p is a prime number. If X is locally of finite type over $\mathbb{F}_p$, then absolute Frobenius morphism is finite.

Proof Let $\operatorname{Spec} A$ be any affine open subset of X. We first show that F is an identity as map of topological spaces. Pick $[\mathfrak{p}] \in \operatorname{Spec} A$, then

$$F([\mathfrak{p}]) = (F^\sharp)^{-1}(\mathfrak{p}) = \{f \in A : f^p \in \mathfrak{p}\} = [\mathfrak{p}].$$

It follows that F is an identity as map of topological spaces.

Since X is locally of finite type over $\mathbb{F}_p$, X can be covered by affine open subsets $\operatorname{Spec} A_i$ such that each A_i is a finitely generated $\mathbb{F}_p$-algebra. Each $\operatorname{Spec} A_i$ is quasi-compact and A_i is obviously finitely generated A_i-algebra, by Proposition 9.3.19, F is of finite type. To show that F is finite, it suffices to show that F is integral. By Proposition 9.3.14, it suffices to show that each A_i is integral over $F^\sharp(A_i)$. Since A_i is finitely generated $\mathbb{F}_p$-algebra, suppose $a_1, \ldots, a_m$ be the generators of A_i. Note that each a_i agree with the equation

$$x^p - a_i^p = 0,$$

hence A_i is integral over $F^\sharp(A_i)$, which implies that $F : X \to X$ is integral. Apply Proposition 9.3.20, absolute Frobenius morphisms is finite. □

Quasi-finite morphisms

Definition 9.3.10 (Quasi-finite (less important))

A morphism $\pi : X \to Y$ is quasi-finite if it is of finite type, and for all $q \in Y$, $\pi^{-1}(q)$ is a finite set.

Remark The main point of this notion is the "finite fiber" part; the "finite type" hypothesis will ensure that this notion is "preserved by fibered product", Chapter 11.

Proposition 9.3.23

Finite morphisms are quasi-finite.

Proof By Proposition 9.3.20, finite morphisms are of finite type. By Proposition 9.3.13, finite morphisms have finite fibers. Hence finite morphisms are of finite type. □

Example 9.9 **Example of quasi-finite morphisms which are not finite.** The open embedding $\mathbb{A}_k^2 \setminus \{(0,0)\} \to \mathbb{A}_k^2$ has finite fibers, and of finite type, hence is quasi-finite. But not finite, see Example 9.5 .

However, in section §9.4, we will soon see that quasi-finite morphism to $\operatorname{Spec} k$ are finite.

Example 9.10 **A morphism with finite fibers that is not quasi-finite.** A key example of a morphism with finite fibers that is not quasi-finite is $\operatorname{Spec} \mathbb{C}(t) \to \operatorname{Spec} \mathbb{C}$ ($\mathbb{C}(t)$ is function field). Another is $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ (not of finite type, see Chapter 11).

How to picture quasi-finite morphisms?

Proposition 9.3.24

If $\pi : X \to Y$ is a finite morphism of locally Noetherian schemes, then for any quasi-compact open subset $U \subseteq X$, the induced morphism $\pi|_U : U \to Y$ is quasi-finite.

Proof It suffices to show that $\pi|_U : U \to Y$ is of finite type and finite fibers. We first show that $\pi|_U : U \to Y$ is of finite type. Let $\operatorname{Spec} B \subseteq Y$ be an affine open subset, since $\pi : X \to Y$ is a finite morphism, we may assume that $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ where A is finite B-algebra. Consider $(\pi|_U)^{-1}(\operatorname{Spec} B)$, we have $(\pi|_U)^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A \cap U$. Since X is locally Noetherian, A is Noetherian, by Proposition 4.6.18, $\operatorname{Spec} A$ is Noetherian space, by Proposition 4.6.19, $U \cap \operatorname{Spec} A$ is quasi-compact. Since $U \cap \operatorname{Spec} A$ is open subset of $\operatorname{Spec} A$, $U \cap \operatorname{Spec} A$ covered by finite numbers of distinguished open subsets, say

$$U \cap \operatorname{Spec} A = \bigcup_{i=1}^{n} \operatorname{Spec} A_{f_i}.$$

Since A is finite B-algebra, we know that A_{f_i} is finitely generated B-algebra. Hence $\pi|_U$ is of finite type.

We next show that $\pi|_U$ has finite fibers. Let $p \in Y$, since $\pi : X \to Y$ is a finite morphism, by Proposition 9.3.13, we know that $\pi^{-1}(p)$ is finite. Since $(\pi|_U)^{-1}(p) = U \cap \pi^{-1}(p)$, $(\pi|_U)^{-1}(p)$ is finite. Hence $\pi|_U : U \to Y$ is quasi-finite. □

In fact, every reasonable quasi-finite morphism arises in this way. (This simple-sounding statement is in fact a deep and important result — a form of Zariski's Main Theorem.) Thus the right way to visualize quasi-finiteness is as a finite map with some (closed locus of) points removed.

9.3.5 ⋆⋆ (Locally) finitely presented morphisms

The following variant of "locally of finite type" is useful in non-Noetherian situations.

9.4 Images of morphisms: Chevalley's Theorem and elimination theory

In this section, we will answer a question that you may have wondered about long before hearing the phrase "algebraic geometry". If you have a number of polynomial equations in a number of variables with indeterminate coefficients, you would reasonably ask what conditions there are on the coefficients for a (common) solution to exist. Give the algebraic nature of the problem, you might hope that the answer should be purely algebraic in nature — it shouldn't be "random", or involve bizarre functions like exponentials or cosines. You should expect the answer to be given by "algebraic conditions". This is indeed the case, and it can be profitably interpreted as a question about images of maps of varieties or schemes, in which guise it is answered by **Chevalley's Theorem**. As a consequence we get an immediate proof of the Nullstellensatz 4.2.2.

In certain cases, the image is nicer still. For example, the image of a finite morphism is always closed (Corollary 9.3.2). We will prove a classical result, the Fundamental Theorem of Elimination Theory, which essentially generalizes this to maps from projective space.

In a different direction, in the distant future we will see that in certain good circumstances ("flat" plus a bit more, see Chapter 25), morphisms are open (the image of open subsets is open); one example is $\mathbb{A}^n_B \to \operatorname{Spec} B$.

9.4.1 Chevalley's Theorem

If $\pi : X \to Y$ is a morphism of schemes, the notion of the image of π as sets is clear: we just take the points in Y that are the image of points in X. We know that the image can be open (open embeddings), and we have seen examples where it is closed, and more generally, locally closed. But it can be weirder still: consider the morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $(x, y) \mapsto (x, xy)$. The image is the plane, with the y-axis removed, but origin put back in (Figure 9.6). We make a definition to capture this phenomenon.

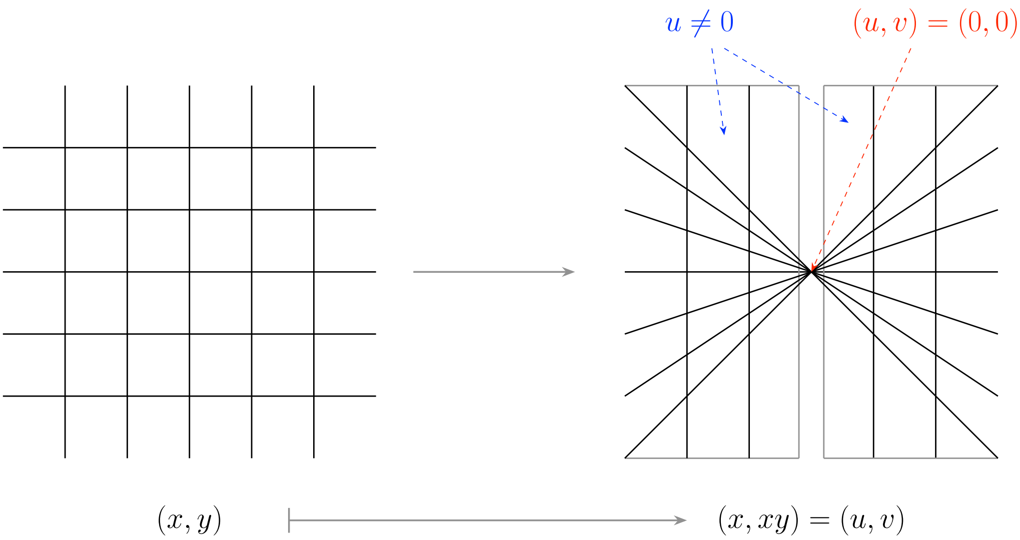


Figure 9.6: The image of $(x, y) \mapsto (x, xy) = (u, v)$

Definition 9.4.1 (Constructible subset)

*A **constructible subset** of a Noetherian topological space is a subset which belongs to the smallest family of subsets such that*

(i) every open set is in the family,

(ii) a finite intersection of family members is in the family,

(iii) the complement of a family member is also in the family.

Example 9.11 The image of $(x, y) \mapsto (x, xy)$ is constructible.

Proof Say $\pi : \mathbb{A}_k^2 \to \mathbb{A}_k^2$ given by $(x, y) \mapsto (x, xy)$, then as set

$$\begin{aligned}\pi(\mathbb{A}_k^2) &= \left\{(u, v) \in \mathbb{A}_k^2 : \text{exists } (a, b) \in \mathbb{A}_k^2, \text{ such that } (u, v) = (a, ab)\right\} \\ &= (\mathbb{A}_k^2 \setminus V(x)) \cup \{(0, 0)\} \\ &= (V(x) \cap (\mathbb{A}_k^2 \setminus \{(0, 0)\}))^c.\end{aligned}$$

Since $\mathbb{A}_k^2 \setminus \{(0, 0)\}$ is open and $V(x)$ is closed, $\pi(\mathbb{A}_k^2)$ is constructible subset. □

Definition 9.4.2 (Locally closed)

*A subset of a topological space X is **locally closed** if it is the intersection of an open subset and a closed subset. Equivalently, it is an open subset of a closed subset, or a closed subset of an open subset.*

Remark We will later have trouble extending this to open and closed and locally closed subschemes, see Chapter 10.

Proposition 9.4.1 (Constructible subsets are finite disjoint unions of locally closed subsets)

A subset of a Noetherian topological space X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Proof Let $Z = \coprod_{i=1}^{n} (U_i \cap C_i)$ where each U_i is open and C_i is closed, then

$$Z^c = \bigcap_{i=1}^{n} (U_i \cap C_i)^c.$$

From U_i open and C_i closed, we know that each U_i and C_i are constructible, hence $U_i \cap C_i$ is constructible, and therefore $(U_i \cap C_i)^c$ is constructible. Since Z^c is the finite intersection of $(U_i \cap C_i)^c$, we know that Z^c is constructible, and therefore Z is constructible.

Conversely, let $\mathscr{G}$ be a collection of the finite disjoint union of locally closed subsets. We want to show that $\mathscr{G}$ holds the following conditions:

(i) every open sets of X in $\mathscr{G}$,

(ii) a finite intersection of members of $\mathscr{G}$ is in $\mathscr{G}$,

(iii) the complement of member of $\mathscr{G}$ is also in $\mathscr{G}$.

Let U be an open subset of X, then $U = U \cap X$, since X is closed, U is locally closed subset.

Let $\coprod_{i=1}^{n} (U_i \cap C_i)$, $\coprod_{j=1}^{m} (V_i \cap D_i) \in \mathscr{G}$, where U_i, V_i are open subsets and C_i, D_i are closed subsets. Consider $\left(\coprod_{i=1}^{n} (U_i \cap C_i)\right) \cap \left(\coprod_{j=1}^{m} (V_j \cap D_j)\right)$, then

$$\left(\coprod_{i=1}^{n}(U_i \cap C_i)\right) \cap \left(\coprod_{j=1}^{m}(V_j \cap D_j)\right) = \coprod_{i=1}^{n}\coprod_{j=1}^{m}(U_i \cap V_j) \cap (C_i \cap D_j).$$

Hence $\left(\coprod_{i=1}^{n} (U_i \cap C_i)\right) \cap \left(\coprod_{j=1}^{m} (V_j \cap D_j)\right) \in \mathscr{G}$.

Let $\coprod_{i=1}^{n} (U_i \cap C_i) \in \mathscr{G}$. Consider $\left(\coprod_{i=1}^{n} (U_i \cap C_i)\right)^c$, then

$$\left(\coprod_{i=1}^{n}(U_i \cap C_i)\right)^c = \bigcap_{i=1}^{n}(U_i \cap C_i)^c.$$

So, it suffices to show that $(U_i \cap C_i)^c \in \mathscr{G}$. Note that

$$(U_i \cap C_i)^c = U_i^c \cup C_i^c = U_i^c \coprod (C_i^c \setminus U_i^c) = (X \cap U_i^c) \coprod ((C_i^c \cap U_i) \cap X)$$

where U_i^c is closed and $C_i^c \cap U_i$ is open, hence $(U_i \cap C_i)^c$ belong to $\mathscr{G}$.

We say $\mathscr{F}$ be the smallest family of subsets of X such (i), (ii) and (iii) holds, i.e., $\mathscr{F}$ is the collection of constructible subsets. By above discussion, we know that $\mathscr{F} \subseteq \mathscr{G}$. Pick $A \in \mathscr{F}$, then A is the finite disjoint union of locally closed subsets. $\square$

Remark Important remark: the only reason for the hypothesis of the topological space in question being Noetherian is because this is the only setting in which we have defined constructible sets. An extension of the notion of constructibility to more general topological spaces is mentioned in Chapter 10.

Corollary 9.4.1

If $X \to Y$ is a continuous map of Noetherian topological spaces, then the preimage of a constructible set is a constructible set.

Proof Since $X \to Y$ is continuous, preimage of open set is open and preimage of closed set is closed. Hence preimage of locally closed subset is locally closed subset. By Proposition 9.4.1, constructible set is the finite disjoint union of locally closed subsets, and therefore the preimage of constructible set is constructible. $\square$

✍ **Exercise 9.3** Show that the generic point of $\mathbb{A}_k^1$ does not form a constructible subset of $\mathbb{A}_k^1$ (where k is a field).

Proof The generic point of $\mathbb{A}_k^1$ is $\eta = [(0)]$. If $\{\eta\}$ is a constructible subset, by Proposition 9.4.1, $\{\eta\}$ must be locally closed subset. So we may assume that $\eta = U \cap C$ where U is open and C is closed. Note that the closed subset which contains η is only $\mathbb{A}_k^1$, hence $\eta = U$. By Proposition 4.6.10, η is contained in any open subset of

$\mathbb{A}_k^1$, but $\{\eta\}$ is not open (since closed subset of $\mathbb{A}_k^1$ is $\mathbb{A}_k^1$ and the set which constructed by finite closed points), a contradiction. □

Proposition 9.4.2

(a) A constructible subset of a Noetherian scheme is closed if and only if it is "stable under specialization". More precisely, if Z is a constructible subset of a Noetherian scheme X, then Z is closed if and only if for every pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$, if $y_2 \in Z$, then $y_1 \in Z$.

(b) A constructible subset of a Noetherian scheme is open if and only if it is "stable under generization". More precisely, if Z is a constructible subset of a Noetherian scheme X, then Z is open if and only if for every pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$, if $y_1 \in Z$, then $y_2 \in Z$.

Proof Let X be a Noetherian scheme, Z be a constructible subset of X.

(a) If Z is closed. Consider pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$. Let $y_2 \in Z$, then $\overline{\{y_2\}} \subseteq Z$, and therefore $y_1 \in Z$.

Conversely, if Z is "stable under specialization". Since Z is a constructible subset, we may assume $Z = \coprod_{i=1}^n U_i \cap Z_i$ where $U_i \subseteq X$ open and $Z_i \subseteq X$ closed. By Proposition 4.6.12, each Z_i is the finite union of irreducible closed subsets, none contained in any other, say $Z_i = \bigcup_{j=1}^{m_i} Z_{ij}$. So

$$Z = \coprod_{i=1}^n (U_i \cap Z_i) = \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} (U_i \cap Z_{ij}). \tag{9.6}$$

We next show that $Z_{ij} \subseteq Z$. By Proposition 6.1.2, we may assume $Z_{ij} = \overline{\{y_{ij}\}}$. It suffices to show that each y_{ij} is contained in Z. Since $U_i \cap Z_{ij}$ is open subset of Z_{ij}, by Proposition 4.6.10, $y_{ij} \in U_i \cap Z_{ij}$, and therefore $y_{ij} \in Z$. Since Z is "stable under specialization", we know that $Z_{ij} \subseteq Z$, hence $\bigcup_{i=1}^n \bigcup_{j=1}^{m_i} Z_{ij} \subseteq Z$. From (9.6), $Z \subseteq \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} Z_{ij}$, hence $Z = \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} Z_{ij}$, and therefore Z is closed.

(b) If Z is open. Consider pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$. Let $y_1 \in Z$. If $y_2 \notin Z$, then $y_2 \in X \setminus Z$, since $X \setminus Z$ is closed, $\overline{\{y_2\}} \subseteq X \setminus Z$, but $y_1 \notin X \setminus Z$, a contradiction.

Conversely, if Z is "stable under generization". Consider $X \setminus Z$, we claim that $X \setminus Z$ is "stable under specialization". For every pair of points y_1 and y_2 with $y_1 \in \overline{\{y_2\}}$, by the condition, we know that "if $y_1 \in Z$, then $y_2 \in Z$", i.e., "if $y_1 \notin X \setminus Z$, then $y_2 \notin X \setminus Z$", i.e., "if $y_2 \in X \setminus Z$, then $y_1 \in X \setminus Z$". Hence $X \setminus Z$ is "stable under specialization", by part (a), $X \setminus Z$ is closed, and therefore Z is open.

□

The image of a morphism of schemes can be stranger than a constructible set. Indeed if S is any subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $\pi : X \to Y$ be the natural map. This is quite pathological, but in any reasonable situation, the image is essentially no worse than what arose in the previous example of $(x, y) \mapsto (x, xy)$. This is made precise by Chevalley's Theorem.

Theorem 9.4.1 (Chevalley's Theorem)

If $\pi : X \to Y$ is a finite type morphism of Noetherian schemes, then the image of any constructible set is constructible. In particular, the image of π is constructible.

Proof Since Y is Noetherian scheme, suppose $Y = \bigcup_{i=1}^n \operatorname{Spec} B_i$, then $X = \bigcup_{i=1}^n \pi^{-1}(\operatorname{Spec} B_i)$. Since

$\pi : X \to Y$ is a finite type morphism, each $\pi^{-1}(\operatorname{Spec} B_i) = \bigcup_{j=1}^{m_i} \operatorname{Spec} A_{ij}$ where A_{ij} is finitely generated B_i-algebra. So $X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \operatorname{Spec} A_{ij}$, hence

$$\pi(X) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi(\operatorname{Spec} A_{ij}).$$

By the definition of constructible subset, constructible preserves finite union of closed subset, so we can reduce to the case that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ where A is finitely generated B-algebra and A, B are Noetherian rings. Since A is finitely generated B-algebra, A can be written as $B[t_1, \ldots, t_n]/I$. Note that $B \to B[t_1, \ldots, t_n]/I$ factors through ring map $B \to B[t_1, \ldots, t_n]$,

$$\begin{tikzcd} B \arrow[rr, "\pi^\sharp"] \arrow[dr] & & B[t_1, \ldots, t_n]/I \\ & B[t_1, \ldots, t_n] \arrow[ur, two heads] & \end{tikzcd}$$

it induces morphisms of affine schemes.

$$\begin{tikzcd} X = \operatorname{Spec} B[t_1, \ldots, t_n]/I \arrow[rr, "\pi"] \arrow[dr, "\iota"'] & & Y = \operatorname{Spec} B \\ & \operatorname{Spec} B[t_1, \ldots, t_n] \arrow[ur, "\psi"'] & \end{tikzcd}$$

We next show that if $\operatorname{Spec} B[t_1, \ldots, t_n] \to Y$ holds Chevalley's Theorem then π also holds. Since ι is integral morphism, by Proposition 9.3.17, the image of ι is closed. Hence $\psi(\iota(X)) = \pi(X)$ is constructible. So we may reduce to the case that $X = \operatorname{Spec} B[t_1, \ldots, t_n]$ and $Y = \operatorname{Spec} B$ where B is Noetherian. If we show that $\operatorname{Spec} B[t] \to \operatorname{Spec} B$ where B is Noetherian holds, by the induction, $X = \operatorname{Spec} B[t_1, \ldots, t_n]$ and $Y = \operatorname{Spec} B$ where B is Noetherian holds. Hence we may reduce to the case that $X = \operatorname{Spec} B[t]$ and $Y = \operatorname{Spec} B$ where B is Noetherian.

We have simplified the task to showing that given $\pi : X = \operatorname{Spec} B[t] \to \operatorname{Spec} B = Y$, the image of any constructible subset of X is constructible in Y. Because constructible sets are finite unions of locally closed subsets (Proposition 9.4.1), we need only show that the image of any locally closed subset Z of X is constructible in Y.

(**An important special case.**) We begin with the case when Z is a closed subset of $X = \mathbb{A}^1_B = \operatorname{Spec} B[t]$, say

$$Z = V(f_1(t), \ldots, f_r(t)), \qquad \text{where } f_i(t) = a_{id_i} t^{d_i} + \cdots + a_{i1} t + a_{i0},$$

with $a_{ij} \in B$, so $f_i(t) \in B[t]$ and $d_i = \deg f_i(t)$.

Let $q \in Y$, consider $\pi^{-1}(q)$. In fact,

$$\pi^{-1}(q) = X \times_Y \operatorname{Spec} \kappa(q) \cong \operatorname{Spec}(B[t] \otimes_B \kappa(q)) \cong \operatorname{Spec}(\kappa(q)[t]).$$

Let $Y_{\vec{e}}$ indexed by $\vec{e} = (e_1, \ldots, e_r)$, where $e_i \in \mathbb{Z}$, $-1 \le e_i \le d_i$, and $Y_{\vec{e}}$ is the locus where $f_1(t), \ldots, f_r(t)$ have "true degree" $e_1, \ldots, e_r$ respectively. More precisely:

$$\begin{aligned} Y_{\vec{e}} &= \left\{ q \in Y : {\scriptstyle \forall 1, \ldots, r, \text{ the coefficient of } t^{e_i} \text{ in } f_i(t) \in \kappa(q)[t], \atop \text{and the higher coefficients are zero.}} \right\} \\ &= \{ q \in Y : a_{ie_i} \neq 0 \in \kappa(q), a_{ij} = 0 \in \kappa(q) \text{ for } j > e_i \} \end{aligned}$$

(The zero polynomial has "true degree" -1 under this convention.)

More precisely still, we define $Y_{\vec{e}} := \operatorname{Spec} B_{\vec{e}}$, where

$$B_{\vec{e}} := \frac{B_{\prod_i a_{ie_i}}}{(a_{ij})_{j>e_i}}, \qquad \text{so } Y_{\vec{e}} = \left(\bigcap_i D(a_{ie_i})\right) \cap \left(\bigcap_{\substack{i \\ j>e_i}} V(a_{ij})\right).$$

Since each point in Y unique determine index $\vec{e}$. So $Y = \operatorname{Spec} B$ is the finite disjoint union of locally closed subsets $Y_{\vec{e}}$, i.e., $Y = \coprod_{\vec{e}} Y_{\vec{e}}$.

If $\vec{e} = (-1, \ldots, -1)$, then

$$\pi^{-1}(Y_{\vec{e}}) \cap Z \xrightarrow{\pi} Y_{\vec{e}} \tag{9.7}$$

is $\operatorname{Spec} B_{(-1,\ldots,-1)}[t] \to \operatorname{Spec} B_{(-1,\ldots,-1)}$, since corresponding ring map $B_{(-1,\ldots,-1)} \to B_{(-1,\ldots,-1)}[t]$ is integral extension, by The Lying Over Theorem 9.2.2, (9.7) is surjective.

If $\vec{e} \neq (-1, \ldots, -1)$, (9.7) is a finite morphism corresponding to the map of rings

$$B_{\vec{e}} \longrightarrow \frac{B_{\vec{e}}}{\left(t^{e_1} + \frac{a_{1(e_1-1)}}{a_{1e_1}} t^{e_1-1} + \cdots + \frac{a_{10}}{a_{1e_1}}, \ldots, t^{e_r} + \frac{a_{r(e_r-1)}}{a_{re_r}} t^{e_r-1} + \cdots + \frac{a_{r0}}{a_{re_r}}\right)}$$

(The ring on the right is generated as a $B_{\vec{e}}$-module by $1, t, \ldots, t^{\max(e_i)-1}$.) Because the images of finite morphisms are closed (Proposition 9.3.17 and Proposition 9.3.20), $\pi(Z) \cap Y_{\vec{e}}$ (the image of (9.7)) is a locally closed subset of Y, so

$$\pi(Z) = \pi(Z) \cap Y = \coprod_{\vec{e}} (\pi(Z) \cap Y_{\vec{e}})$$

is indeed constructible when Z is closed. We have completed our important special case.

Remark

Now we proved the image of closed subset is constructible. We want to show that the image of locally closed subset is constructible. Suppose $Z \subseteq X$ is a locally closed subset, then $Z = \overline{Z} \setminus \delta Z$, where $\delta Z = \overline{Z} \setminus Z$. $\overline{Z}$ and δZ are both closed subsets of X (since $Z = U \cap \overline{Z}$ where U is open, hence $\delta Z = \overline{Z} \cap U^c$ is closed). We hope $\pi(Z) = \pi(\overline{Z}) \setminus \pi(\delta Z)$, then, by $\pi(\overline{Z})$ and $\pi(\delta Z)$ are constructible, $\pi(Z)$ is construction. This seems to complete the proof, but unfortunately this is **false**! In fact, $\pi(Z) \neq \pi(\overline{Z}) \setminus \pi(\delta Z)$. Let's take a look at what went wrong. Pick $q \in Y$, such that $\pi^{-1}(q) \subseteq \overline{Z}$. If $\pi^{-1}(q) \cap \delta Z \neq \varnothing$ and $\pi^{-1}(q) \cap Z \neq \varnothing$, then $q \in \pi(Z)$ and $q \in \pi(\delta Z)$, this is where the problem arises. So we need to define something to capture the situation where the fibers fall completely into a closed set, this is why we will define **fiber locus**.

Note that the locus of points $q \in Y$ where the entire fiber $\pi^{-1}(q)$ is contained in Z (Z is closed), $\{q \in Y : \pi^{-1}(q) \subseteq Z\}$, is a closed subset of Y: it is $\operatorname{Spec} B_{(-1,\ldots,-1)} = \operatorname{Spec} B/(a_{ij})$. We call this $FL(Z) \subseteq Y$, the "fibral locus for Z", only for the rest of this proof.

Finally, if Z is locally closed, then $\overline{Z} \subseteq X$ is closed, so $\pi(\overline{Z})$ is a constructible subset of $Y = \operatorname{Spec} B$ (apply above consequence). Define $\delta Z := \overline{Z} \setminus Z$, a closed subset of X (since $Z = U \cap \overline{Z}$ where U is open, hence $\delta Z = \overline{Z} \cap U^c$ is closed); Since $\overline{Z}$ and δZ are both closed, by above discussion (fibral locus for closed subset is closed), we know that $FL(\overline{Z})$ and $FL(\delta Z)$ are both closed subsets of Y. The point q in $FL(\delta Z)$ means the fiber $\pi^{-1}(q)$ contained in δZ, so $\pi^{-1}(q) \cap Z = \varnothing$, and therefore $FL(\delta Z) \cap \pi(Z) = \varnothing$. Hence we can cut $FL(\delta Z)$ out of Y, i.e., $Y \setminus FL(\delta Z)$. The point q in $FL(\overline{Z}) \setminus FL(\delta Z)$ means the fiber $\pi^{-1}(q) \subseteq \overline{Z}$ and $\pi^{-1}(q) \nsubseteq \delta Z$, so $\pi^{-1}(q) \cap \pi(Z) \neq \varnothing$, and therefore $q \in \pi(Z)$, i.e., $FL(\overline{Z}) \setminus FL(\delta Z) \subseteq \pi(Z)$. Note that $FL(\overline{Z}) \setminus FL(\delta Z)$ is locally closed, so we can cut $FL(\overline{Z}) \setminus FL(\delta Z)$ out of $Y \setminus FL(\delta Z)$, i.e., $Y \setminus FL(\overline{Z})$. It suffices to prove the result for the open subset $Y' := Y \setminus (FL(\overline{Z}) \cup FL(\delta Z)) \subseteq Y$. Let $\pi' : X' = \pi^{-1}(Y') \to Y'$ be the restriction of π above Y', and let $Z' = Z \cap X'$. Say $\delta Z' = \overline{Z'} \setminus Z' \subseteq X'$, where $\overline{Z'}$ is the closure of Z' in X'. In fact, $Z' \to Y'$ is quasi-finite. Since π is finite type, $Z' \to Y'$ is also finite type. Pick $q \in Y'$, then $\pi'^{-1}(q) = \operatorname{Spec} \kappa(q)[t] \cap Z'$. Since $\kappa(q)[t]$ is PID, any ideal in $\kappa(q)[t]$ is principal. Since $Z' \cap \operatorname{Spec} \kappa(q)[t]$ is closed in $\operatorname{Spec} \kappa(q)[t]$, then $Z' \cap \operatorname{Spec} \kappa(q)[t] = V(g(t))$. Since $\kappa(q)[t]$ is PID, we may assume

$$g(t) = p_1(t)^{k_1} \cdots p_m(t)^{k_m},$$

so $Z' \cap \operatorname{Spec} \kappa(q)[t]$ is finite, i.e., $X' \to Y'$ has finite fibers. Hence $X' \to Y'$ is quasi-finite.

We next show that the closed subset $\delta Z' \subseteq X'$ does not meet any generic fiber of π', i.e., the preimage of any generic point of Y'. Since Y' is open subscheme of Noetherian scheme Y, we may assume that $Y' = \bigcup \operatorname{Spec} B_f$. If there exists a generic point of Y', say η, such that $\pi'^{-1}(\eta) \cap \delta Z' \neq \varnothing$, then $\eta = [\mathfrak{q}] \in \operatorname{Spec} B_f$ for some f. Since η is generic point of Y', $\operatorname{cl}_{\operatorname{Spec} B_f}(\eta) = \operatorname{Spec} B_f$, so $\pi'^{-1}(\eta) = \operatorname{Spec} K(B_f/\mathfrak{q})[t]$. Since $K(B_f/\mathfrak{q})[t]$ is PID, by Proposition 6.2.8 and Proposition 6.2.7, $\pi'^{-1}(\eta)$ is irreducible. We claim that $\pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z')$ is proper closed in $\pi'^{-1}(\eta)$. If $\pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z') = \pi'^{-1}(\eta)$, then $\operatorname{cl}_{X'}(Z') \subseteq \pi^{-1}(\eta)$, it follows that $\eta \in FL(\operatorname{cl}_{X'}(Z'))$. Since $\operatorname{cl}_{X'}(Z') \subseteq \operatorname{cl}_X(Z)$, $FL(\operatorname{cl}_{X'}(Z')) \subseteq FL(\operatorname{cl}_X(Z))$, so $\eta \in FL(\operatorname{cl}_X(Z))$. But $Y' = Y \setminus (FL(\operatorname{cl}_X(Z)) \cap FL(\delta Z))$, $\eta \notin FL(\operatorname{cl}_X(Z))$, a contradiction. Hence $\pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z')$ is proper closed in $\pi'^{-1}(\eta)$. Since $\pi'^{-1}(\eta)$ is irreducible, the generic point of $\pi'^{-1}(\eta)$ does not belong to $\pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z')$, in the other words, $\pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z')$ is the finite union of closed point (finiteness because $Z' \to Y'$ is quasi-finite). Since $\pi'^{-1}(\eta) \cap Z' \subseteq \pi'^{-1}(\eta) \cap \operatorname{cl}_{X'}(Z')$, $\pi'^{-1}(\eta) \cap Z'$ is also the finite union of closed point, and therefore

$$\pi'^{-1}(\eta) \cap Z' = \operatorname{cl}_{\pi'^{-1}(\eta)}(\pi'^{-1}(\eta) \cap Z') = \pi'^{-1}(\eta) \cap Z'.$$

Hence

$$\begin{aligned} \delta Z' \cap \pi'^{-1}(\eta) =& (\operatorname{cl}_{X'}(Z') \setminus Z') \cap \pi'^{-1}(\eta) \\ =& (\operatorname{cl}_{X'}(Z') \cap \pi'^{-1}(\eta)) \setminus (Z' \cap \pi'^{-1}(\eta)) \\ =& \varnothing, \end{aligned}$$

a contradiction. Hence the closed subset $\delta Z' \subseteq X'$, does not meet any generic fiber of π', i.e., $\pi'(\delta Z')$ does not contain any generic point of Y'. So $\pi'(\delta Z')$ not dense in Y', it follows that $Y' \setminus \operatorname{cl}_{Y'}(\pi'(\delta Z')) \neq \varnothing$. We may assume $Y' = \bigcup_{i=1}^m Y_i$ where each Y_i is irreducible component of Y'. Clearly $\pi'(\delta Z') \cap Y_i$ not dense in Y_i, so there exists open subset W_i of Y_i such that $W_i \cap (\pi'(\delta Z') \cap Y_i) = \varnothing$. Let $V_i = W_i \setminus \bigcup_{j \neq i} cl_{Y'}(Y_j)$, then V_i is open in Y_i. Since Y_i is irreducible, V_i is dense in Y_i. Let $V = \bigcup_{i=1}^m V_i$, then open subset V is dense in Y' and $V \cap \pi'(\delta Z') = \varnothing$.

We finish the proof of Chevalley's Theorem by using Noetherian induction 4.6.1. Define the property $\mathscr{P}$ as follow: let C is a closed subset of Noetherian scheme Y', we say $\mathscr{P}$ holds for C if $\pi'(Z' \cap \pi'^{-1}(C))$ is constructible in Y'. Suppose $\mathscr{P}$ holds for every proper closed subset of C, we want to show that $\mathscr{P}$ holds for C. By our previous discussion, there exists a dense open subset $V \subseteq Y'$ such that $V \cap \pi'(\delta Z') = \varnothing$. Note that

$$C = (C \cap V) \coprod (C \cap (Y' \setminus V)),$$

we have

$$\begin{aligned} \pi'(Z' \cap \pi'^{-1}(C)) =& \pi'(Z' \cap (\pi'^{-1}(C \cap V) \coprod \pi'^{-1}(C \cap (Y' \setminus V)))) \\ =& \pi'(Z' \cap \pi'^{-1}(C \cap V)) \cup \pi'(Z' \cap \pi'^{-1}(C \cap (Y' \setminus V))). \end{aligned}$$

Consider $\pi'(Z' \cap \pi^{-1}(C \cap V))$, since $\pi^{-1}(V) \cap \delta Z' = \varnothing$, we have

$$\pi'(Z' \cap \pi^{-1}(C \cap V)) = \pi'(\operatorname{cl}_{X'}(Z') \cap \pi^{-1}(C \cap V)).$$

Since $\mathrm{cl}_{X'}(Z') \cap \pi^{-1}(C \cap V)$ is closed in $\pi^{-1}(C \cap V)$, $\pi'(\mathrm{cl}_{X'}(Z') \cap \pi^{-1}(C \cap V))$ is constructible in $C \cap V$, and therefore is constructible in Y'. Hence $\pi'(Z' \cap \pi'^{-1}(C \cap V))$ is constructible in Y'.

Consider $\pi'(Z' \cap \pi^{-1}(C \cap (Y' \setminus V)))$, since $C \cap (Y' \setminus V)$ is proper closed in C, by the induction hypothesis, $\pi'(Z' \cap \pi^{-1}(C \cap (Y' \setminus V)))$ is constructible in Y'.

Hence $\pi'(Z' \cap \pi'^{-1}(C))$ is constructible in Y'. By the Noetherian induction 4.6.1, $\mathscr{P}$ holds for Y', i.e., $\pi'(Z' \cap \pi'^{-1}(Y')) = \pi'(Z')$ is constructible in Y'. Note that

$$\pi(Z) = \pi'(Z) \cup (FL(\mathrm{cl}_X(Z)) \setminus FL(\delta Z)),$$

we know that $\pi(Z)$ is constructible. We finished the proof of Chevalley's Theorem. □

Remark For the minority who might care: see Chapter 11 for an extension to locally finitely presented morphisms.

We next give some important consequences that may seem surprising, in that they are algebraic corollaries of a seemingly quite geometric and topological theorem. The first is a proof of the Nullstellensatz. Recall the Nullstellensatz 4.2.2:

Theorem 9.4.2 (Hilbert's Nullstellensatz)

If k is any field, every maximal ideal of $k[x_1, \ldots, x_n]$ has residue field $k[x_1, \ldots, x_n]/\mathfrak{m}$ a finite extension of k.

Proof We wish to show that if K is a field extension of k that is finitely generated as a k-algebra, say by $x_1, \ldots, x_n$, then it is a finite extension of fields. It suffices to show that each x_i is algebraic over k. But if x_i is not algebraic over k, then we have an inclusion of rings $k[x_i] \to K$, corresponding to a dominant morphism $\pi : \operatorname{Spec} K \to \mathbb{A}^1_k$ of finite type k-schemes. Of course $\operatorname{Spec} K$ is a single point, so the image of π is one point. By Chevalley's Theorem 9.4.1 and Exercise 9.3, the image of π is not the generic point of $\mathbb{A}^1_k$. Consider $\mathrm{cl}_{\mathbb{A}^1_k}(\mathrm{Im}(\pi))$, let $y \in \mathrm{cl}_{\mathbb{A}^1_k}(\mathrm{Im}(\pi))$, since $\mathrm{Im}(\pi)$ is constructible, by Proposition 9.4.2, we know that $y \in \mathrm{Im}(\pi)$, i.e., so $y = \mathrm{Im}(\pi)$. It follows that $\mathrm{Im}(\pi)$ is closed point, and therefore π is not dominant, a contradiction. □

A similar ideal can be used in the following Corollaries of Chevalley's Theorem 9.4.1.

Corollary 9.4.2 (Quasi-finite morphisms to a filed are finite)

Suppose $\pi : X \to \operatorname{Spec} k$ is a quasi-finite morphism. Then π is finite.

Proof Since $\pi : X \to \operatorname{Spec} k$ is a quasi-finite morphism, $\pi : X \to \operatorname{Spec} k$ is of finite type and has finite fibers. Since $\pi : X \to \operatorname{Spec} k$ is of finite type, suppose $\pi^{-1}(\operatorname{Spec} k) = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$ where each A_i is a finitely generated k-algebra. Restrict π to $\operatorname{Spec}_{A_i}$, then we get a morphism $\pi|_{\operatorname{Spec} A_i} : \operatorname{Spec} A_i \to \operatorname{Spec} k$. Clearly, $\pi|_{\operatorname{Spec} A_i}$ is of finite type. We want to show that $\pi|_{\operatorname{Spec} A_i}$ is integral. It suffices to show that corresponding ring map $k \to A_i$ is integral. Suppose A_i contains an element x that is not algebraic over k, then we have an inclusion $k[x] \hookrightarrow A_i$, and A_i is finitely generated $k[x]$-algebra. Since A_i is finitely generated k-algebra, by Hilbert's Basis Theorem, A_i is Noetherian. Consider the corresponding dominant morphism $\varphi : \operatorname{Spec} A_i \to \operatorname{Spec} k[x]$, φ is a finite type morphism of Noetherian schemes, by Chevalley's Theorem 9.4.1, $\varphi(\operatorname{Spec} A_i)$ is constructible. By the discussion in the proof of Hilbert's Nullstellensatz 9.4.2, we know that $\varphi(\operatorname{Spec} A_i)$ is closed point, and therefore φ is not dominant, a contradiction. Hence $\pi|^{\sharp}_{\operatorname{Spec} A_i} : k \to A_i$ is integral, by Proposition 9.3.20, $\pi|_{\operatorname{Spec} A_i} \operatorname{Spec} A_i \to \operatorname{Spec} k$ is integral for all i. By Proposition 9.3.10, $\operatorname{Spec} A_i$ is a finite union of points, each point with residue field a finite extension of k. So $\pi^{-1}(\operatorname{Spec} k)$ is a finite union of points, and therefore $\pi^{-1}(\operatorname{Spec} k)$ can be written as disjoint union of affine scheme, by Proposition 5.3.5 (a), $\pi^{-1}(\operatorname{Spec} k)$ is an affine scheme, say $\pi^{-1}(\operatorname{Spec} k) = \operatorname{Spec} A$, where A is finite product of residue field. Since each residue filed is a finite extension of k, A is finite generated k-vector space, i.e., A is finite k-algebra. Consequently, $\pi : X \to \operatorname{Spec} k$ is finite. $\square$

Corollary 9.4.3

Suppose X is a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, and p is a closed point of X. Then the residue field $\kappa(p)$ is a finite field.

Proof Consider the structure morphism $\pi : X \to \operatorname{Spec}\mathbb{Z}$, since X is of finite type over $\operatorname{Spec}\mathbb{Z}$, $\pi^{-1}(\operatorname{Spec}\mathbb{Z})$ can be covered by finite number of open subscheme $\operatorname{Spec} A_i$ where A_i is finitely generated $\mathbb{Z}$-algebra. Since $\mathbb{Z}$ is Noetherian, by Hilbert's Basis Theorem, A_i is Noetherian, and therefore X is Noetherian scheme. Since p is a closed point of X, p must contained in $\operatorname{Spec} A \subseteq X$ for some A where A is Noetherian ring, say $p = [\mathfrak{m}] \in \operatorname{Spec} A$ where $\mathfrak{m}$ is maximal ideal of A, hence $\kappa(p) = A/\mathfrak{m}$. Consider the composition $\mathbb{Z} \to A \to A/\mathfrak{m}$. Since $\mathbb{Z}$ is PID, say $\operatorname{Ker}(\varphi : \mathbb{Z} \to A/\mathfrak{m}) = (p)$. Since $\operatorname{Ker}(\varphi : \mathbb{Z} \to A/\mathfrak{m}) = \varphi^{-1}(0)$ and (0) is prime ideal of $A/\mathfrak{m}$, $\operatorname{Ker}(\varphi : \mathbb{Z} \to A/\mathfrak{m})$ must be prime ideal, so p is prime. We obtain a field extension $\mathbb{F}_p \hookrightarrow A/\mathfrak{m}$. Since A is finitely generated $\mathbb{Z}$-algebra, $A/\mathfrak{m}$ is finitely generated $\mathbb{Z}$-algebra, and therefore $A/\mathfrak{m}$ is finitely generated $\mathbb{F}_p$-algebra. If $A/\mathfrak{m}$ not a finite field, then there exists $x \in A/\mathfrak{m}$ not algebraic over $\mathbb{F}_p$. Hence we obtain a ring map $\mathbb{F}_p[x] \hookrightarrow A/\mathfrak{m}$. Clearly, $A/\mathfrak{m}$ is finitely generated $\mathbb{F}_p[x]$-algebra, the corresponding dominant morphism $\psi : \operatorname{Spec} A/\mathfrak{m} \to \operatorname{Spec}\mathbb{F}_p[x]$ is of finite type. Note that $A/\mathfrak{m}$ and $\mathbb{F}_p[x]$ is Noetherian, apply Chevalley's Theorem 9.4.1, $\psi(\operatorname{Spec} A/\mathfrak{m})$ is constructible in $\operatorname{Spec}\mathbb{F}_p[x]$. Since $\operatorname{Spec} A/\mathfrak{m}$ is a single point, $\psi(\operatorname{Spec} A/\mathfrak{m})$ is also a single point, by Exercise 9.3, $\psi(\operatorname{Spec} A/\mathfrak{m})$ is not the generic point of $\operatorname{Spec}\mathbb{F}_p[x]$. It follows that $\psi(\operatorname{Spec} A/\mathfrak{m})$ is closed point (use Proposition 9.4.2 or see the proof of Hilbert's Nullstellensatz 9.4.2), hence ψ is not dominant, a contradiction. Hence $A/\mathfrak{m}$ is finite field. $\square$

Corollary 9.4.4 (For maps of varieties, surjective can be checked on closed points)

A morphism of affine k-varieties $\pi : X \to Y$ is surjective if and only if it is surjective on closed points (i.e., if every closed point of Y is surjective if and only if it is surjective on closed point of X).

Proof If $\pi : X \to Y$ is surjective, obviously, it is surjective on closed points.

Conversely, if $\pi : X \to Y$ is surjective on closed points, we want to show that $\pi : X \to Y$ is surjective. If π is not surjective, there exists $p \notin \operatorname{Im}(\pi)$. By Chevalley's Theorem 9.4.1, $\operatorname{Im}(\pi)$ is constructible, and therefore the complement of $\operatorname{Im}(\pi)$ is also constructible, say $p \in U \cap C$ where U is open and C is closed. By Proposition 4.6.7, the set of closed points of C is dense, hence $U \cap C$ contains a closed point of Y. Since $\pi : X \to Y$ is surjective on closed point, $\operatorname{Im}(\pi)$ must contains all closed point of Y, a contradiction. $\square$

Remark Once we define varieties in general, in Chapter 12, we will see that our argument works without change with the adjective "affine" removed.

9.4.2 Elimination of quantifiers

A basic sort of question that arises in any number of contexts is when a system of equations has a solution. Suppose for example you have some polynomials in variables $x_1, \ldots, x_n$ over an algebraically closed field $\overline{k}$, some of which you set to be zero, and some of which you set to be nonzero. (This question is of fundamental interest even before you know any scheme theory!) Then there is an algebraic condition on the coefficients which will tell you if there is a solution. Define the **Zariski topology on $\overline{k}^n$** in the obvious way: closed subsets are cut out by equations. (A mild generalization of this appears in Chapter 13.)

Proposition 9.4.3 (Elimination of quantifiers, over an algebraically closed field)

Fix an algebraically closed field $\overline{k}$. Suppose

$$f_1, \dots, f_p, g_1, \dots, g_q \in \overline{k}[W_1, \dots, W_m, X_1, \dots, X_n]$$

are given. There is a (Zariski-)constructible subset Y of $\overline{k}^m$ such that

$$f_1(w_1, \dots, w_m, X_1, \dots, X_n) = \cdots = f_p(w_1, \dots, w_m, X_1, \dots, X_n) = 0 \tag{9.8}$$

and

$$g_1(w_1, \dots, w_m, X_1, \dots, X_n) \neq 0 \quad \cdots \quad g_q(w_1, \dots, w_m, X_1, \dots, X_n) \neq 0 \tag{9.9}$$

has a solution $(X_1, \dots, X_n) = (x_1, \dots, x_n) \in \overline{k}^n$ if and only if $(w_1, \dots, w_n) \in Y$.

Proof Let X be the locally closed subset of $\mathbb{A}^{m+n}$ cut out by the equalities and inequalities (9.8) and (9.9), i.e.,

$$X = V(f_1, \dots, f_p) \cap \left(\bigcap_{j=1}^{q} D(g_j) \right).$$

Consider the projection $\mathbb{A}^{m+n} \to \mathbb{A}^m$, it is given by the ring map $\overline{k}[W_1, \dots, W_m] \to \overline{k}[W_1, \dots, W_m, X_1, \dots, X_n]$, restrict the projection to X, i.e., $\pi : X \to \mathbb{A}^m$.

Remark

If Z is a finite type scheme over $\overline{k}$, and the closed points are denoted Z^{cl} ("cl" is for either "closed" or "classical"), then under the inclusion of topological spaces $Z^{\mathrm{cl}} \hookrightarrow Z$, the Zariski topology on Z, the Zariski topology on Z induces the Zariski topology on Z^{cl}.

By Hilbert's Nullstellensatz 4.7.1 and Proposition 6.3.8, we can identify $(\mathbb{A}^m_{\overline{k}})^{\mathrm{cl}}$ with $\overline{k}^m$. Consider the diagram

$$\begin{tikzcd}
X^{\mathrm{cl}} \arrow[r, hook] \arrow[d, "\pi^{\mathrm{cl}}"'] & X \arrow[r, hook, "\text{loc.cl.}"] \arrow[d, "\pi"] & \mathbb{A}^{m+n} \arrow[dl] \\
\overline{k}^m \arrow[r, hook] & \mathbb{A}^m &
\end{tikzcd}$$

where X^{cl} is the closed points of X, and $\pi^{\mathrm{cl}} : X^{\mathrm{cl}} \to \overline{k}^m$ is given by restrict π to X^{cl}, i.e.,

$$(w_1, \dots, w_m, x_1, \dots, x_n) \mapsto (w_1, \dots, w_m).$$

Let $Y = \operatorname{Im} \pi^{\mathrm{cl}}$. Clearly, $\pi : X \to \mathbb{A}^m$ is a finite type morphism of Noetherian schemes, by Chevalley's Theorem 9.4.1, $\operatorname{Im} \pi$ is constructible, and hence so is $(\operatorname{Im} \pi) \cap \overline{k}^m$. It remains to show that $(\operatorname{Im} \pi) \cap \overline{k}^m = Y$.

Let $\pi^{\mathrm{cl}}(p) \in Y$, then $\pi^{\mathrm{cl}}(p) \in \operatorname{Im} \pi$, so $\pi^{\mathrm{cl}}(p) \in (\operatorname{Im} \pi) \cap \overline{k}^m$. Hence $Y \subseteq (\operatorname{Im} \pi) \cap \overline{k}^m$.

Conversely, let $q \in (\operatorname{Im} \pi) \cap \overline{k}^m$, then q is closed point in $\mathbb{A}^m$. Hence $\pi^{-1}(q)$ is a closed subset of X. By Proposition 6.3.8, X^{cl} is dense in X, so $X^{\mathrm{cl}} \cap \pi^{-1}(q) \neq \varnothing$. Hence there exists $p \in X^{\mathrm{cl}}$ such that $\pi^{\mathrm{cl}}(p) = q$, it follows that $q \in \operatorname{Im} \pi^{\mathrm{cl}}$. Hence $Y = (\operatorname{Im} \pi) \cap \overline{k}^m$, and therefore Y is constructible subset. □

Remark This is called "elimination of quantifiers" because it gets rid of the quantifier "there exists a solution". The analogous statement for real numbers, where inequalities are also allowed, is a special case of Tarski's celebrated theorem of elimination of quantifiers for real closed fields.

9.4.3 The Fundamental Theorem of Elimination Theory

In the case of projective space (and later, projective morphisms), one can do better than Chevalley (at least in describing image of closed sets).

Theorem 9.4.3 (Fundamental Theorem of Elimination Theory)

*The morphism $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ is **closed** (sends closed sets to closed sets).*

Note that no Noetherian hypotheses are needed.

A great deal of classical algebra and geometry is contained in this theorem as special cases. Here are some example.

Example 9.12 Let $A = k[a, b, c, \ldots, i]$, and consider the closed subset of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0, dx + ey + fz = 0, gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a nontrivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 0.$$

Thus the ideal of the determinant is embedded in elimination theory.

Example 9.13 Let $A = k[a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n]$. Now consider the closed subset of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0x^m + a_1x^{m-1}y + \cdots + a_my^m = 0$ and $b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n = 0$. Then there is a polynomial in the coefficients $a_0, \ldots, b_n$ (an element of A) which vanishes if and only if these two polynomials have a common nonzero root — this polynomial is called the **resultant**.

More generally, given a number of homogeneous equations in $n + 1$ variables with indeterminate coefficients, Theorem 9.4.3 implies that one can write down equations in the coefficients that precisely determine when the equations have a nontrivial solution.

Proof of the Fundamental Theorem of Elimination Theory 9.4.3.

Proof Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed subset. We wish to show that $\pi(Z)$ is closed.

By the definition of the Zariski topology on $\operatorname{Proj} A[x_0, \ldots, x_n]$ (Definition 5.5.6), Z is cut out (set-theoretically) by some homogeneous elements $f_1, f_2, \cdots \in A[x_0, \ldots, x_n]$. We wish to show that the points $p \in \operatorname{Spec} A$ that are in $\pi(Z)$ form a closed subset. Unpacked, this means that $Z = V_+(\{f_i\}_{i \in I})$ where f_i is homogeneous elements of $A[x_0, \ldots, x_n]$. We want to show that $\pi(Z)$ is closed in $\operatorname{Spec} A$. In fact,

$$\begin{aligned}
\pi(Z) &= \left\{ \pi(\mathfrak{p}) : \mathfrak{p} \in \bigcap_{i \in I} V_+(f_i) \text{ in } \operatorname{Proj} A[x_0, \ldots, x_n] \right\} \\
&= \left\{ \mathfrak{p}_0 \in \operatorname{Spec} A : \bigcap_{i \in I} \pi|^{-1}_{V_+(f_i)}(\mathfrak{p}_0) \neq \varnothing \right\} \\
&= \left\{ \mathfrak{p}_0 \in \operatorname{Spec} A : \bigcap_{i \in I} (V_+(f_i) \cap \pi^{-1}(\mathfrak{p}_0)) \neq \varnothing \right\}.
\end{aligned}$$

Define the ring map $\varphi_{\mathfrak{p}_0} : A[x_0, \ldots, x_n] \to \kappa(\mathfrak{p}_0)[x_0, \ldots, x_n]$ induced by $A \to \kappa(\mathfrak{p}_0)$ (for each prime ideal $\mathfrak{p}_0 \in \operatorname{Spec} A$), then we have $V_+(f_i) \cap \pi^{-1}(\mathfrak{p}_0) = V_+(\varphi_{\mathfrak{p}_0}(f_i)) \subseteq \operatorname{Proj} \kappa(\mathfrak{p}_0)[x_0, \ldots, x_n]$. Hence,

$$\pi(Z) = \left\{ \mathfrak{p}_0 \in \operatorname{Spec} A : \bigcap_{i \in I} V_+(\varphi_{\mathfrak{p}_0}(f_i)) \neq \varnothing \text{ in } \mathbb{P}^n_{\kappa(\mathfrak{p}_0)} \right\}.$$

So, equivalently, we want to show that those $p \in \operatorname{Spec} A$ for which $f_1, f_2, \ldots$ have a common zero in

$\operatorname{Proj}\kappa(p)[x_0,\ldots,x_n]$ form a closed subset of $\operatorname{Spec} A$, i.e.,

$$\pi(Z) = \left\{ p \in \operatorname{Spec} A : \bigcap_{i \in I} V_+(\varphi_p(f_i)) \neq \varnothing \text{ in } \mathbb{P}^n_{\kappa(p)} \right\} \tag{9.10}$$

form a closed subset of $\operatorname{Spec} A$.

Remark

To motivate our argument, we consider a related question. Suppose that $S_\bullet := k[x_0,\ldots,x_n]$, and that $g_1, g_2, \cdots \in k[x_0,\ldots,x_n]$ are homogeneous polynomials. How can we tell if $g_1, g_2, \ldots$ have a common zero in $\operatorname{Proj} S_\bullet = \mathbb{P}^n_k$?

They would have a common zero if and only if in

$$\mathbb{A}^{n+1}_k = \operatorname{Spec} S_\bullet,$$

they cut out (set-theoretically) more than the origin, i.e., if set-theoretically

$$V(g_1, g_2, \ldots) \nsubseteq V(x_0,\ldots,x_n).$$

But the inclusion-reversing bijection between closed subsets and radical ideals (Theorem 4.7.1), this is true if and only if

$$\sqrt{(g_1, g_2, \ldots)} \nsupseteq \sqrt{(x_0,\ldots,x_n)}.$$

Since $(x_0,\ldots,x_n)$ is maximal ideal, so this is true if and only if

$$\sqrt{(g_1, g_2, \ldots)} \nsupseteq (x_0,\ldots,x_n).$$

This is true if and only if not all of the generators $x_0,\ldots,x_n$ of the ideal $(x_0,\ldots,x_n)$ are in $\sqrt{(g_1, g_2, \ldots)}$, which is true if and only if there is some i such that no power of x_i is in $(g_1, g_2, \ldots)$. This is true if and only if

$$(x_0,\ldots,x_n)^N \nsubseteq (g_1, g_2, \ldots) \quad \text{for all } N.$$

This is equivalent to

$$S_N \nsubseteq (g_1, g_2, \ldots) \quad \text{for all } N,$$

which may be rewritten as

$$S_N \nsubseteq g_1 S_{N-\deg g_1} \oplus g_2 S_{N-\deg g_2} \oplus \cdots$$

for all N. In other words, this is equivalent to the statement that the k-linear map

$$S_{N-\deg g_1} \oplus S_{N-\deg g_2} \oplus \cdots \longrightarrow S_N$$

is not surjective. This map is given by a matrix with $\dim S_N$ rows. (It may have an infinite number of columns, but this will not bother us.) To check that this linear map is not surjective, we need only check that all the "maximal" ($\dim S_N \times \dim S_N$) determinants are zero. (Of cause, we need to check this for all N.) Thus the condition that $g_1, g_2, \ldots$ have common zeros in $\mathbb{P}^n_k$ is the same as checking some (admittedly infinite) number of equations. In other words, it is a Zariski-closed condition on the coefficients of the polynomials $g_1, g_2, \ldots$.

Say $S_\bullet = A[x_0,\ldots,x_n]$. $f_1, f_2, \ldots$ have a common zero in $\operatorname{Proj} S_\bullet$ if and only if $\varphi_p(f_1), \varphi_p(f_2), \ldots$ have common zeros in $\operatorname{Proj}\kappa(p)[x_0,\ldots,x_n]$ for some $p \in \operatorname{Spec} A$, by above remark, if and only if there exists

$p \in \operatorname{Spec} A$ such that the $\kappa(p)$-linear map

$$\alpha_{N,p} : \bigoplus_{i\geq 1} \kappa(p)[x_0, \dots, x_n]_{N-\deg \varphi_p(f_i)} \longrightarrow \kappa(p)[x_0, \dots, x_n]_N \tag{9.11}$$

given by

$$(k_i)_{i\geq 1} \longmapsto \sum_{i\geq 1} k_i \varphi_p(f_i).$$

is not surjective, for all N. By above remark, (9.11) gives a matrix with coefficient in $\kappa(p)$, say $M_{N,p}$, and $\alpha_{N,p}$ is surjective if and only if all $\dim \kappa(p)[x_0, \dots, x_n]_N \times \dim \kappa(p)[x_0, \dots, x_n]_N$ minor of $M_{N,p}$ are zero. Say $\dim \kappa(p)[x_0, \dots, x_n]_N = r$.

Obviously, $\deg f_i = \deg \varphi_p(f_i)$, so $N - \deg f_i = N - \deg \varphi_p(f_i)$. Consider the following diagram,

$$\begin{tikzcd}
\bigoplus_{i\geq 1} S_{N-\deg f_i} \arrow[r, "\alpha_N"] \arrow[d, "\text{induced by } \varphi_p"'] & S_N \arrow[d, "\varphi_p"] \\
\bigoplus_{i\geq 1} \kappa(p)[x_0, \dots, x_n]_{N-\deg \varphi_p(f_i)} \arrow[r, "\alpha_{N,p}"] & \kappa(p)[x_0, \dots, x_n]_N
\end{tikzcd}$$

we can lift $\alpha_{N,p}$ to a morphism from $\bigoplus_{i\geq 1} S_{N-\deg f_i}$ to S_N via φ_p. The A-module map α_N gives a matrix with coefficient in A, say M_N. Let $M_N = (m_{ij})$, define $\widetilde{\varphi_p}(M_N) = (\varphi_p(m_{ij}))_{ij}$, by the construction of (9.11), we know that $\widetilde{\varphi_p}(M_N) = M_{N,p}$, and the $r \times r$ minor of M_N in $\kappa(p)$ is the $r \times r$ minor of $M_{N,p}$. Hence, $\varphi_p(f_1), \varphi_p(f_2), \dots$ have common zeros in $\operatorname{Proj} \kappa(p)[x_0, \dots, x_n]$ for some $p \in \operatorname{Spec} A$ if and only if all $r \times r$ minor of M_N in $\kappa(p)$ are zero for some p.

Note that $\dim \kappa(p)[x_0, \dots, x_n]_N = \operatorname{rank}(A[x_0, \dots, x_n]_N)$, let I_N be an ideal which generated by $r \times r$ minor of M_N, then all $r \times r$ minor of M_N in $\kappa(p)$ are zero if and only if $p \in V(I_N)$. Hence, $\varphi_p(f_1), \varphi_p(f_2), \dots$ have common zeros in $\operatorname{Proj} \kappa(p)[x_0, \dots, x_n]$ for some $p \in \operatorname{Spec} A$ if and only if $p \in V(I_N)$ for all N.

By (9.10), we have

$$\begin{aligned}
\pi(Z) &= \left\{ p \in \operatorname{Spec} A : \bigcap_{i\in I} V_+(\varphi_p(f_i)) \neq \varnothing \text{ in } \mathbb{P}^n_{\kappa(p)} \right\} \\
&= \bigcap_{N\geq 0} V(I_N),
\end{aligned}$$

hence $\pi(Z)$ is closed. □

Remark Notice that projectivity was crucial to the proof: we used graded rings in an essential way. Notice also that the proof is essentially just linear algebra.

Chapter 10 Closed embeddings and related notions

The scheme-theoretic analog of closed subsets has a surprisingly different flavor from the analog of open sets (open subschemes/embeddings). However, just as open subschemes (the scheme-theoretic version of open set) are locally modeled on open sets $U \subseteq Y$, the analog of closed subsets also has a local model. This was foreshadowed by our understanding of closed subsets of $\operatorname{Spec} B$ as roughly corresponding to ideals. If $I \subseteq B$ is an ideal, then $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ is a morphism of schemes, and we have checked that on the level of topological spaces, this describes $\operatorname{Spec} B/I$ as a closed subset of $\operatorname{Spec} B$, with the subspace topology (Proposition 4.4.7). This morphism is our "local model" of a closed subscheme/embedding.

10.1 Closed embeddings and closed subschemes

10.1.1 Definition of closed embeddings and closed subschemes

Definition 10.1.1 (Closed embedding, closed subscheme)

A morphism $\pi : X \to Y$ is a ***closed embedding*** *(or* ***closed immersion****) if it is an affine morphism, and for every affine open subset $\operatorname{Spec} B \subseteq Y$, with $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, the map $B \to A$ is surjective (i.e., of the form $B \to B/I$, our desired local model). The symbol $\hookrightarrow$ often is used to indicate that a morphism is a closed embedding (or more generally, a locally closed embedding, §10.2).*

If X is a subset of Y (and π on the level of sets is the inclusion), we say that X is a ***closed subscheme*** *of Y. A closed embedding is the same thing as an isomorphism with a closed subscheme.*

Remark "Embedding" is preferable to "immersion", because the differential geometric notion of immersion is closer to what algebraic geometers call unramified, which we will define in Chapter 22.

Proposition 10.1.1

A closed embedding $\pi : X \hookrightarrow Y$ identifies the topological space of X with a closed subset of the topological space of Y, i.e., X is homeomorphic to a closed subset of Y.

Proof Let $Y = \bigcup_i \operatorname{Spec} B_i$, then $X = \bigcup_i \pi^{-1}(\operatorname{Spec} B_i)$. Since $\pi : X \to Y$ is closed embedding, we may assume $\pi^{-1}(\operatorname{Spec} B_i) \cong \operatorname{Spec} A_i$ where $\pi|^{\sharp}_{\operatorname{Spec} A_i} : B_i \to A_i$ is surjective. Hence, by the Fundamental Theorem of Isomorphism for rings, we have $A_i \cong B_i/\operatorname{Ker}(\pi|^{\sharp}_{\operatorname{Spec} A_i})$. Say $I_i = \operatorname{Ker}(\pi|^{\sharp}_{\operatorname{Spec} A_i})$, then as topological space

$$\pi : X \xrightarrow{\sim} \bigcup_i \operatorname{Spec} B_i/I_i.$$

By Proposition 4.4.7, each $\operatorname{Spec} B_i/I_i$ is closed in $\operatorname{Spec} B_i$. We want to show that $\pi(X) = \bigcup_i \operatorname{Spec} B_i/I_i$ is closed in Y. Consider $Y \setminus \pi(X)$,

$$\begin{aligned} Y \setminus \pi(X) &= \left(\bigcup_i \operatorname{Spec} B_i\right) \setminus \pi(X) \\ &= \bigcup_i (\operatorname{Spec} B_i \setminus (\operatorname{Spec} B_i \cap \pi(X))). \end{aligned}$$

It suffices to show that each Spec $B_i \cap \pi(X)$ is closed in Spec B_i. Note that

$$\begin{aligned}\operatorname{Spec} B_i \cap \pi(X) &= \operatorname{Spec} B_i \cap \bigcup_i \operatorname{Spec} B_i/I_i \\ &= \operatorname{Spec} B_i/I_i \cup \left(\bigcup_{j \neq i} (\operatorname{Spec} B_j/I_j \cap \operatorname{Spec} B_i)\right),\end{aligned}$$

consider Spec $B_j/I_j \cap$ Spec B_i, we claim that

$$(\operatorname{Spec} B_j/I_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) = (\operatorname{Spec} B_i/I_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j).$$

By Proposition 6.3.1, Spec $B_i \cap$ Spec B_j can be covered by simultaneously distinguished open subsets of Spec B_i and Spec B_j, say

$$\operatorname{Spec} B_i \cap \operatorname{Spec} B_j = \bigcup_k \operatorname{Spec}(B_i)_{f_k} \cong \bigcup_k \operatorname{Spec}(B_j)_{g_k},$$

where $\operatorname{Spec}(B_i)_{f_k} \cong \operatorname{Spec}(B_j)_{g_k}$. Hence, we have

$$(\operatorname{Spec} B_j/I_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) = \bigcup_k (\operatorname{Spec} B_j/I_j \cap \operatorname{Spec}(B_j)_{g_k})$$

and

$$(\operatorname{Spec} B_i/I_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) = \bigcup_k (\operatorname{Spec} B_i/I_i \cap \operatorname{Spec}(B_i)_{f_k}).$$

It suffices to show that $\operatorname{Spec} B_j/I_j \cap \operatorname{Spec}(B_j)_{g_k} \cong \operatorname{Spec} B_i/I_i \cap \operatorname{Spec}(B_i)_{f_k}$ for all k. Note that $\operatorname{Spec} B_j/I_j \cap \operatorname{Spec}(B_j)_{g_k} \cong \operatorname{Spec}(B_j/I_j)_{\overline{g_k}}$ and $\operatorname{Spec} B_i/I_i \cap \operatorname{Spec}(B_i)_{f_k} \cong \operatorname{Spec}(B_i/I_i)_{\overline{f_k}}$. Since π is closed embedding, we have $\pi^{-1}(\operatorname{Spec}(B_i)_{f_k}) \cong \pi^{-1}(\operatorname{Spec} B_i/I_i)_{\pi^\sharp(f_k)} \cong \operatorname{Spec}(B_i/I_i)_{\overline{f_k}}$ and $\pi^{-1}(\operatorname{Spec}(B_j)_{g_k}) \cong \operatorname{Spec}(B_j/I_j)_{\overline{g_k}}$. Since $\operatorname{Spec}(B_i)_{f_k} \cong \operatorname{Spec}(B_j)_{g_k}$, we have $\pi^{-1}(\operatorname{Spec}(B_i)_{f_k}) \cong \pi^{-1}(\operatorname{Spec}(B_j)_{g_k})$, and therefore $\operatorname{Spec}(B_i/I_i)_{\overline{f_k}} \cong \operatorname{Spec}(B_j/I_j)_{\overline{g_k}}$, as we desired.

Hence we have proved

$$(\operatorname{Spec} B_j/I_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) = (\operatorname{Spec} B_i/I_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j).$$

So

$$\begin{aligned}\operatorname{Spec} B_i \cap \pi(X) &= \operatorname{Spec} B_i/I_i \cup \left(\bigcup_{j \neq i} (\operatorname{Spec} B_j/I_j \cap \operatorname{Spec} B_i)\right) \\ &= \operatorname{Spec} B_i/I_i \cup \left(\bigcup_{j \neq i} (\operatorname{Spec} B_i/I_i \cap \operatorname{Spec} B_j)\right) \\ &= \operatorname{Spec} B_i/I_i,\end{aligned}$$

i.e., Spec $B_i \cap \pi(X)$ is closed in Spec B_i, and therefore $Y \setminus \pi(X)$ is open, we done! □

Remark **Caution:** The closed embeddings $\operatorname{Spec} k[x]/(x) \hookrightarrow \operatorname{Spec} k[x]$ and $\operatorname{Spec} k[x]/(x^2) \hookrightarrow \operatorname{Spec} k[x]$ show that the closed subset does not determine the closed subscheme. The "infinitesimal" information, or "fuzz", is lost.

Proposition 10.1.2

Closed embeddings are finite morphisms, hence of finite type.

Proof Let $\pi : X \hookrightarrow Y$ be a closed embedding, say $Y = \bigcup_i \operatorname{Spec} B_i$, as in the proof of Proposition 10.1.1, we may assume

$$X = \bigcup_i \operatorname{Spec} B_i/I_i,$$

where I_i is an ideal of B_i, and $\pi^{-1}(\operatorname{Spec} B_i) \cong \operatorname{Spec} B_i/I_i$. Clearly, B_i/I_i is a finitely generated B_i-module, and therefore is finite B_i-algebra. By Proposition 9.3.9, closed embedding $\pi : X \hookrightarrow Y$ is finite morphism, hence of finite type. □

Proposition 10.1.3

The composition of two closed embeddings is a closed embedding.

Proof Let $\pi : X \hookrightarrow Y$ and $\psi : Y \hookrightarrow Z$ be two closed embeddings, we want to show that $\psi \circ \pi : X \to Z$ is closed embedding. Let $\operatorname{Spec} C \subseteq Z$, since ψ is closed embedding, we have $\psi^{-1}(\operatorname{Spec} C) \cong \operatorname{Spec} B$ where $C \to B$ is surjective. Since π is closed embedding, $\pi^{-1}(\psi^{-1}(\operatorname{Spec} C)) \cong \pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, where $B \to A$ is surjective. Hence the composition $C \to B \to A$ is surjective, and therefore $\psi \circ \pi : X \to Z$ is closed embedding. □

Proposition 10.1.4 (The property of being a closed embedding is affine-local on the target)

A morphism $\pi : X \to Y$ is closed embedding if there is a cover of Y by affine pen sets $\operatorname{Spec} B_i$ such that $\pi^{-1}(\operatorname{Spec} B_i) \cong \operatorname{Spec} A_i$ where $B_i \to A_i$ is surjective.

Proof Let $\pi : X \to Y$ be a morphism of schemes. By hypothesis and Proposition 9.3.6, $\pi : X \to Y$ is affine. We want to show that the property of being a closed embedding is affine local property. It suffices to show the following two things:

(i) let $\operatorname{Spec} B$ be an affine open subsets of Y and $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X)$ where $B \to \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X)$ is surjective, then $\pi^{-1}(\operatorname{Spec} B_f) \cong \operatorname{Spec} \Gamma(\pi^{-1}(\operatorname{Spec} B_f), \mathscr{O}_X)$ where $B_f \to \Gamma(\pi^{-1}(\operatorname{Spec} B_f), \mathscr{O}_X)$ is surjective, for all $f \in B$;

(ii) if $B = (f_1, \ldots, f_n)$ and $\pi^{-1}(\operatorname{Spec} B_{f_i}) \cong \operatorname{Spec} A_i$ where $B_{f_i} \to A_i$ is surjective, for all i, then $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ where $B \to A$ is surjective, for some A.

To show (i). Since $\pi : X \to Y$ is affine, we know that $\pi^{-1}(\operatorname{Spec} B_f) \cong \operatorname{Spec} \Gamma(\pi^{-1}(\operatorname{Spec} B_{f_i}), \mathscr{O}_X)$ for all $f \in B$. It suffices to show that $B_f \to \Gamma(\pi^{-1}(\operatorname{Spec} B_f), \mathscr{O}_X)$ is surjective, for all $f \in B$. In fact,

$$\begin{aligned}
\pi^{-1}(\operatorname{Spec} B_f) =& \pi^{-1}(\operatorname{Spec} B \setminus V(f)) = \pi^{-1}(\operatorname{Spec} B) \setminus \pi^{-1}(V(f)) \\
=& \operatorname{Spec} \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X) \setminus V(\pi^\sharp f) \\
=& \operatorname{Spec} \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X)_{\pi^\sharp f}.
\end{aligned}$$

Hence, it suffices to show that $B_f \to \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X)_{\pi^\sharp f}$ is surjective, this is obvious, since $B \to \Gamma(\pi^{-1}(\operatorname{Spec} B), \mathscr{O}_X)$ is surjective.

To show (ii). By the proof of Proposition 9.3.6, we know that there is a ring A such that $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ and $A_{\pi^\sharp f_i} \cong A_i$. By the hypothesis, $B_{f_i} \to A_{\pi^\sharp f_i}$ is surjective for all i, we want to show that $B \to A$ is surjective. Note that $B_{f_i} = \Gamma(\operatorname{Spec} B_{f_i}, \mathscr{O}_Y)$, $B = \Gamma(\operatorname{Spec} B, \mathscr{O}_Y)$, $A_{\pi^\sharp f_i} = \Gamma(\operatorname{Spec} A_{\pi^\sharp f_i}, \mathscr{O}_X)$ and $A = \Gamma(\operatorname{Spec} A, \mathscr{O}_X)$, consider the following commutative diagram (definition of sheaves),

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\operatorname{Spec} B, \mathscr{O}_Y) \arrow[r, "\beta"] \arrow[d] & \prod_{i=1}^n \Gamma(\operatorname{Spec} B_{f_i}, \mathscr{O}_Y) \arrow[d, two heads, "\gamma"] \\
0 \arrow[r] & \Gamma(\operatorname{Spec} A, \mathscr{O}_X) \arrow[r, "\alpha"] & \prod_{i=1}^n \Gamma(\operatorname{Spec} A_{\pi^\sharp f_i}, \mathscr{O}_X)
\end{tikzcd}$$

where α, β induced by restriction and γ induced by $\pi|^\sharp_{\operatorname{Spec} A_{\pi^\sharp f_i}} : B_{f_i} \to A_{\pi^\sharp f_i}$. Let $f \in A$, then $\alpha(f) =$

$(f, \dots, f)$. By the construction of γ, $\gamma((\pi|^\sharp_{\operatorname{Spec} A_{\pi^\sharp f_1}})^{-1}(f), \dots, (\pi|^\sharp_{\operatorname{Spec} A_{\pi^\sharp f_n}})^{-1}(f)) = (f, \dots, f)$, and each $(\pi|^\sharp_{\operatorname{Spec} A_{\pi^\sharp f_i}})^{-1}(f) \in B$, so $(\pi|^\sharp_{\operatorname{Spec} A_{\pi^\sharp f_i}})^{-1}(f) = (\pi^\sharp)^{-1}(f)$. Hence $\beta((\pi^\sharp)^{-1}(f)) = ((\pi^\sharp)^{-1}(f), \dots, (\pi^\sharp)^{-1}(f))$, it follows that $B \to A$ is surjective.

By (i) and (ii), the property of being a closed embedding is affine local property, by Affine Communication Lemma 6.3.1, $\pi : X \hookrightarrow Y$ is closed embedding. $\square$

In particular, if $B \to A$ is a surjection of rings, then the induced morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is a closed embedding.

10.1.2 Important: Closed subschemes correspond to quasi-coherent sheaves of ideals

Definition 10.1.2 (Ideal sheaf)

*A closed subscheme $\pi : X \hookrightarrow Y$ gives a surjection of $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$. We define the **ideal sheaf** (or sheaf of ideals) on Y, denoted $\mathscr{I}_{X/Y}$, to be the kernel of $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$, i.e.,*

$$\mathscr{I}_{X/Y} = \operatorname{Ker}(\mathscr{O}_Y \to \pi_*\mathscr{O}_X).$$

Remark On each open subset $U \subseteq Y$, it gives an ideal $\mathscr{I}_{X/Y}(U)$ of the ring $\mathscr{O}_Y(U)$; on the affine open subset $\operatorname{Spec} B \subseteq Y$, it gives the ideal I of B determining the surjection $B \to A = B/I$ in the definition of "closed subscheme" (Definition 10.1.1).

Proposition 10.1.5

$\pi : X \hookrightarrow Y$ is a closed embedding. Then $\pi_*\mathscr{O}_X$ is a quasi-coherent sheaf on Y.

Proof $\pi : X \hookrightarrow Y$ induces a morphism of sheaves $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, then $\pi_*\mathscr{O}_X$ can be seen as $\mathscr{O}_Y$-module. Let $\operatorname{Spec} B \subseteq Y$ be an affine open subset, since $\pi : X \hookrightarrow Y$ is a closed embedding, we may assume $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} B/I$ where I is an ideal of B. B/I can be seen as B-module, we claim that $(\pi_*\mathscr{O}_X)|_{\operatorname{Spec} B} \cong \widetilde{B/I}$. It suffices to check on the distinguished base. Let $\operatorname{Spec} B_f \hookrightarrow \operatorname{Spec} B$ be any distinguished open subset of $\operatorname{Spec} B$, then

$$\begin{aligned}\Gamma(\operatorname{Spec} B_f, (\pi_*\mathscr{O}_X)|_{\operatorname{Spec} B}) &= \Gamma(\pi^{-1}(\operatorname{Spec} B_f), \mathscr{O}_X) = \Gamma(\operatorname{Spec}(B/I)_{\pi^\sharp f}, \mathscr{O}_X) \\ &= (B/I)_{\pi^\sharp f} \cong B_f \otimes_B (B/I) \\ &= \widetilde{B/I}(\operatorname{Spec} B_f).\end{aligned}$$

Hence for any affine open subset $\operatorname{Spec} B \hookrightarrow Y$ we have $(\pi_*\mathscr{O}_X)|_{\operatorname{Spec} B} \cong \widetilde{B/I}$, and therefore $\pi_*\mathscr{O}_X$ is a quasi-coherent sheaf on Y. $\square$

Quasi-coherent sheaves form an abelian category, so $\mathscr{I}_{X/Y} = \operatorname{Ker}(\mathscr{O}_Y \to \pi_*\mathscr{O}_X)$ is also a quasi-coherent sheaf on Y. Thus every closed subscheme of Y yields a quasi-coherent sheaf on Y, i.e., ideal sheaf on Y.

Definition 10.1.3 (Closed subscheme exact sequence)

We call the exact sequence (of quasi-coherent sheaves on Y)

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_Y \longrightarrow \pi_*\mathscr{O}_X \longrightarrow 0 \tag{10.1}$$

*the **closed subscheme exact sequence** for $\pi : X \hookrightarrow Y$.*

We may recover X (and its map π to Y) from the ideal sheaf $\mathscr{I}_{X/Y}$ via (10.1):

Proposition 10.1.6 (Important)

Every quasi-coherent sheaf of ideals on Y produces a closed subscheme on Y.

Proof Suppose $Y = \bigcup_i \operatorname{Spec} B_i$. Let $\mathscr{I}$ be an ideal sheaf on Y, then $I(B_i) := \mathscr{I}(\operatorname{Spec} B_i)$ is an ideal of B_i. Consider quotient map $B_i \to B_i/I(B_i)$, it gives a closed subscheme $\operatorname{Spec} B_i/I(B_i)$ of $\operatorname{Spec} B_i$. We want to glue $\{\operatorname{Spec} B_i/I(B_i)\}$ to a closed subscheme $X \hookrightarrow Y$.

For any $\operatorname{Spec} B_i$ and $\operatorname{Spec} B_j$, by Proposition 6.3.1, we may assume that $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$ is covered by $\operatorname{Spec}(B_i)_{f_t} \cong \operatorname{Spec}(B_j)_{g_t}$. Hence

$$(B_i)_{f_t} \cong (B_j)_{g_t},$$

and therefore we have $I((B_i)_{f_t}) \cong I((B_j)_{g_t})$. Since $\mathscr{I}$ is a quasi-coherent sheaf, by Theorem 7.2.2, we have $I((B_i)_{f_t}) \cong I(B_i)_{f_t}$ and $I((B_j)_{g_t}) \cong I(B_j)_{g_t}$. Hence

$$(B_i)_{f_t}/I(B_i)_{f_t} \cong (B_i/I(B_i))_{f_t} \cong (B_j/I(B_j))_{g_t} \cong (B_j)_{g_t}/I(B_j)_{g_t},$$

sine each $\operatorname{Spec} B_i/I(B_i) \hookrightarrow \operatorname{Spec} B_i$ is closed subscheme, we have

$$\operatorname{Spec}(B_i)_{f_t} \cap \operatorname{Spec} B_i/I(B_i) = \operatorname{Spec}(B_i/I(B_i))_{f_t} \cong \operatorname{Spec}(B_j/I(B_j))_{g_t} = \operatorname{Spec}(B_j)_{g_t} \cap \operatorname{Spec} B_j/I(B_j),$$

it follows that we have an isomorphism

$$f_{ij} : \operatorname{Spec} B_i/I(B_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j) \xrightarrow{\sim} \operatorname{Spec} B_j/I(B_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j).$$

We next show the morphisms $\{f_{ij}\}_{ij}$ agree on triple intersections (Theorem 5.4.1). Say

$$X_{ij} = \operatorname{Spec} B_i/I(B_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j),$$

then

$$\begin{aligned} X_{ij} \cap X_{ik} &= \operatorname{Spec} B_i/I(B_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k) \\ X_{ji} \cap X_{jk} &= \operatorname{Spec} B_j/I(B_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k) \\ X_{kj} \cap X_{ki} &= \operatorname{Spec} B_k/I(B_k) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k). \end{aligned}$$

Consider $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k$, since $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j = \bigcup_t \operatorname{Spec}(B_i)_{f_t}$, we have

$$\begin{aligned} \operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k &= \left(\bigcup_t \operatorname{Spec}(B_i)_{f_t}\right) \cap \operatorname{Spec} B_k \\ &= \bigcup_t (\operatorname{Spec}(B_i)_{f_t} \cap \operatorname{Spec} B_k). \end{aligned}$$

By Proposition 6.3.1, each $\operatorname{Spec}(B_i)_{f_t} \cap \operatorname{Spec} B_k$ can be covered by simultaneously distinguished open subset of $\operatorname{Spec}(B_i)_{f_t}$ and $\operatorname{Spec} B_k$. Since distinguished open subset of $\operatorname{Spec}(B_i)_{f_t}$ is also the distinguished open subset of $\operatorname{Spec} B_i$ and $\operatorname{Spec} B_j$, $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k$, since $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j = \bigcup_t \operatorname{Spec}(B_i)_{f_t}$ can be covered by simultaneously distinguished open subset of $\operatorname{Spec} B_i$, $\operatorname{Spec} B_j$, and $\operatorname{Spec} B_k$. Similar to the previous process, we have

$$f_{ij}|_{X_{ij} \cap X_{ik}} : \operatorname{Spec} B_i/I(B_i) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k) \xrightarrow{\sim} \operatorname{Spec} B_j/I(B_j) \cap (\operatorname{Spec} B_i \cap \operatorname{Spec} B_j \cap \operatorname{Spec} B_k).$$

Hence,

$$f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}.$$

By Theorem 5.4.1, we can glue $\{\operatorname{Spec} B_i/I(B_i)\}$ together, say X. By the construction of X, clearly, X is a closed subscheme on Y. □

Example 10.1 **(Unimportant example: a sheaf of ideals (as $\mathscr{O}$-modules) that is not quasi-coherent.)** Let $X = \operatorname{Spec} k[x]_{(x)}$, the "germ of the affine line at the origin", which has two points, the closed point and the generic point η. Define $\mathscr{I}(X) = \{0\} \subseteq \mathscr{O}_X(X) = k[x]_{(x)}$, and $\mathscr{I}(\eta) = k(x) = \mathscr{O}_X(\eta)$ ($\{\eta\} = D(x)$ is open in X). Show that this sheaf of ideals does not correspond to a closed subscheme.

Proof We show that $\mathscr{I}$ is not quasi-coherent sheaf, and therefore not correspond to a closed subscheme.

If $\mathscr{I}$ is a quasi-coherent sheaf, then $\mathscr{I} \cong \widetilde{I}$ where $I \subseteq A$ is an ideal. By our definition, $\mathscr{I}(X) = \{(0)\}$, hence $\widetilde{I}(X) = I = (0)$. We now check $\widetilde{(0)}(\eta)$,

$$\widetilde{(0)}(\eta) = (k[x]_{(x)})_x \otimes_{k[x]_{(x)}} (0) = k(x) \otimes_{k[x]_{(x)}} (0) = 0 \neq k(x),$$

it follows that $\mathscr{I}$ is not quasi-coherent sheaf, and therefore not correspond to a closed subscheme. □

In the literature, the usual definition of a closed embedding is:

Definition 10.1.4 (Closed embedding)

*A **closed embedding** is a morphism $\pi : X \to Y$ such that π induces a homeomorphism of the underlying topological space of X onto a closed subset of the topological space of Y, and the induced map $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ of sheaves on Y is surjective. (By "surjective" we mean that the ring morphism on stalks is surjective.)*

Remark We not use Definition 10.1.4 later.

We next show Definition 10.1.4 agrees with Definition 10.1.1.

Lemma 10.1.1

Let $\pi : X \to Y$ be a morphism of schemes. If π is a homeomorphism onto a closed subset of Y, then π is affine.

Proof Let $y \in Y$ be a point. If $y \notin \pi(X)$, then there exists an affine neighborhood of y which is disjoint from $\pi(X)$. If $y \in \pi(X)$, since π is homeomorphism, let $x \in X$ be the unique point of X mapping to y. Let $V \subseteq Y$ be an affine open neighborhood of y. Let $U \subseteq X$ be an affine open neighborhood of x which maps into V. Since $\pi(U) \subseteq V \cap \pi(X)$ is open in the induced topology by our assumption on π we may choose a $h \in \Gamma(V, \mathscr{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap \pi(X) \subseteq \pi(U)$. Denote $h' \in \Gamma(U, \mathscr{O}_X)$ the restriction of $\pi^\sharp(h)$ to U. Then we see that $D(h') \subseteq U$ is equal to $\pi^{-1}(D(h))$. In other words, every point of Y has an open neighborhood whose inverse image is affine. Thus π is affine. □

Proposition 10.1.7

Definition 10.1.4 agrees with Definition 10.1.1.

Proof To show that Definition 10.1.4 implies Definition 10.1.1. Let $\pi : X \to Y$ be a morphism such that π induces a homeomorphism of the underlying topological space of X onto a closed subset of the topological space Y, and the induced map $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ of sheaves on Y is surjective. By hypothesis, apply Lemma 10.1.1, we know that π is affine. Let $\operatorname{Spec} B$ be any affine open subset of Y, we may assume that $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, since $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is surjective,

$$\mathscr{O}_Y(\operatorname{Spec} B) = B \longrightarrow A = \pi_*\mathscr{O}_X(\operatorname{Spec} B)$$

is surjective. It follows that Definition 10.1.4 implies Definition 10.1.1.

We next show that Definition 10.1.1 implies Definition 10.1.4. By Proposition 10.1.1, X is homeomorphic to a closed subset of Y. It suffices to show that the induced map $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ of sheaves on Y is surjective. By Proposition 3.5.1, it suffices to show that $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ on a base $\operatorname{Spec} B_f$ is surjective for all $f \in B$. By the hypothesis, we may assume that $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ where $\operatorname{Spec} B \subseteq Y$ is open and $B \twoheadrightarrow A$. Hence

$$\mathscr{O}_Y(\operatorname{Spec} B_f) = B_f \longrightarrow \pi_* \mathscr{O}_X(\operatorname{Spec} B_f) = \mathscr{O}_X(\pi^{-1}(\operatorname{Spec} B_f)) = A_{\pi^\sharp f}.$$

Since $B \twoheadrightarrow A$ is surjective, $B_f \to A_{\pi^\sharp f}$ is surjective. It follows that $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$ on a base $\operatorname{Spec} B_f$ is surjective for all $f \in B$, then we done! □

We have now defined the analog of open subsets and closed subsets in the land of schemes. Their definition is slightly less "symmetric" than in the classical topological setting: the "complement" of a closed subscheme is a unique open subscheme, but there are many "complementary" closed subschemes to a given open subscheme in general. (We will soon define one that is "best", that has a reduced structure.)

10.1.3 Examples and properties

Using the one-to-one corresponding between quasi-coherent sheaves of ideals and closed subschemes (§10.1.2) we can define scheme theoretic

Definition 10.1.5 (Finite (scheme-theoretic) union of closed subschemes)

Let Z be a scheme. Let $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathscr{I}_{X/Z} \hookrightarrow \mathscr{O}_Z$ and $\mathscr{I}_{Y/Z} \hookrightarrow \mathscr{O}_Z$. The scheme theoretic union of X and Y is the closed subscheme of Z cut out by $\mathscr{I}_{X/Z} \cap \mathscr{I}_{Y/Z}$.

Definition 10.1.6 (Arbitrary (scheme-theoretic) intersection of closed subschemes)

Let Z be a scheme. Let $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathscr{I}_{X/Z} \hookrightarrow \mathscr{O}_Z$ and $\mathscr{I}_{Y/Z} \hookrightarrow \mathscr{O}_Z$. The scheme theoretic intersection of X and Y is the closed subscheme of Z cut out by $\mathscr{I}_{X/Z} + \mathscr{I}_{Y/Z}$.

✍ **Exercise 10.1** Describe the scheme-theoretic intersection of $V(y - x^2)$ and $V(y)$ in $\mathbb{A}^2$. See Figure 5.5 for a picture. (For example, explain informally how this corresponds to two curves meeting at a single point with multiplicity 2 — notice how the 2 is visible in your answer. Alternatively, what is the non-reducedness telling you — both its "size" and its "direction"?) Describe their scheme-theoretic union.

Solution *Let $Z = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$, $X = V(y - x^2) = \operatorname{Spec} k[x,y]/(y - x^2)$, and $Y = V(y) = \operatorname{Spec} k[x,y]/(y)$. Then the corresponding ideal sheaves of X and Y are $\widetilde{(y - x^2)}$ and $\widetilde{(y)}$, hence the corresponding ideal sheaves of $X \cap Y$ is $\widetilde{(y - x^2)} + \widetilde{(y)} = \widetilde{(y - x^2) + (y)} = \widetilde{(y, x^2)}$, and therefore $X \cap Y = \operatorname{Spec} k[x,y]/(y, x^2) \cong \operatorname{Spec} k[x]/(x^2)$. $\operatorname{Spec} k[x]/(x^2)$ is non-reduced scheme, its Krull dimension is $\dim_k \operatorname{Spec} k[x]/(x^2) = 2$, it means two curves meeting at a single point have "thickness". See Figure 5.3, $X \cap Y$ remembers the derivative only in the x direction.*

By Definition 10.1.5, the corresponding ideal sheaves of $X \cup Y$ is $\widetilde{(y - x^2)} \cap \widetilde{(y)} = \widetilde{(y - x^2) \cap (y)} = \widetilde{(y(y - x^2))}$, and therefore $X \cup Y = \operatorname{Spec} k[x,y]/(y(y - x^2)) = V(y(y - x^2))$.

Proposition 10.1.8

(a) The underlying set of a finite union of closed subschemes is the finite union of the underlying sets.
(b) The underlying set of intersection of closed subschemes is the intersection of the underlying sets.

Proof

(a) Let X be a scheme, and let $Z_1, \ldots, Z_n$ be a finite collection of closed subschemes of X. Then the corresponding ideal sheaf of $\bigcup_{j=1}^{n} Z_j$ is $\bigcap_{j=1}^{n} \mathscr{I}_{Z_j/X}$. Suppose $X = \bigcup_i \operatorname{Spec} A_i$, by Proposition 10.1.6,

$$\bigcup_{j=1}^{n} Z_j = \bigcup_i \operatorname{Spec}\left(A_i \Big/ \left(\bigcap_{j=1}^{n} \mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) \right) \right).$$

Note that

$$\begin{aligned} \bigcup_{j=1}^{n} \operatorname{Spec} A_i/\mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) &= \bigcup_{j=1}^{n} V_{\operatorname{Spec} A_i}(\mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i)) \\ &= V_{\operatorname{Spec} A_i}\left(\bigcap_{j=1}^{n} \mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) \right) \\ &= \operatorname{Spec}\left(A_i \Big/ \left(\bigcap_{j=1}^{n} \mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) \right) \right), \end{aligned}$$

it follows that a finite union of closed subschemes is the finite union of the underlying sets.

(b) Let X be a scheme, and let $\{Z_j\}_{j\in\mathscr{J}}$ be a collection of closed subschemes of X. Then the corresponding ideal sheaf of $\bigcap_{j\in\mathscr{J}}$ is $\sum_{j\in\mathscr{J}} \mathscr{I}_{Z_j/X}$. Suppose $X = \bigcup_i \operatorname{Spec} A_i$, by Proposition 10.1.6,

$$\begin{aligned} \bigcap_{j\in\mathscr{J}} Z_j &= \bigcup_i \operatorname{Spec}\left(A_i \Big/ \left(\sum_{j\in\mathscr{J}} \mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) \right) \right) \\ &= \bigcup_i V_{\operatorname{Spec} A_i}\left(\sum_{j\in\mathscr{J}} \mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i) \right) \\ &= \bigcup_i \bigcap_{j\in\mathscr{J}} V_{\operatorname{Spec} A_i}(\mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i)) \\ &= \bigcap_{j\in\mathscr{J}} \bigcup_i V_{\operatorname{Spec} A_i}(\mathscr{I}_{Z_j/X}(\operatorname{Spec} A_i)), \end{aligned}$$

it follows that the underlying set of intersection of closed subschemes is the intersection of the underlying sets.

□

Exercise 10.2 Describe the scheme-theoretic intersection of $V(y^2 - x^2)$ and $V(y)$ in $\mathbb{A}^2$. Draw a picture.

Solution *The scheme-theoretic intersection is* $V(y^2 - x^2) \cap V(y) = V(y, x^2) \cong \operatorname{Spec} k[x]/(x^2)$.

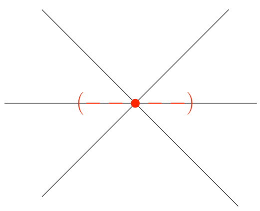


Figure 10.1: Exercise 10.2

Example 10.2 In general, if X, Y, and Z are closed subschemes of W, then $(X\cap Z)\cup(Y\cap Z) \neq (X\cup Y)\cap Z$. Here is an example. Consider $X = V(y-x)$, $Y = V(y+x)$, $Z = V(y)$, and $W = \mathbb{A}^2$. Then

$$(X\cap Z)\cup(Y\cap Z) = \{(0,0)\}\cup\{(0,0)\} = \{(0,0)\}$$

and

$$(X\cup Y)\cap Z = V(y^2 - x^2)\cap V(y) = V(x^2, y),$$

hence

$$(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z.$$

Remark In particular, not all properties of intersection and union carry over from sets to schemes.

Proposition 10.1.9

Closed embeddings are monomorphisms.

Proof Same as Proposition 8.2.2. □

In many cases, loci which are a priori closed sets often have enriched versions as closed subschemes. The remaining notions are examples of this. (As described in §7.6.14, we often indicate that we are considering scheme-theoretic enrichments by adding the adjective scheme-theoretic — for example, say "scheme-theoretic intersection", "scheme-theoretic union", and so forth.)

Definition 10.1.7 (Vanishing scheme)

*Suppose $Y = \bigcup_i \operatorname{Spec} B_i$ is scheme, and $s \in \Gamma(Y, \mathscr{O}_Y)$. Define the closed subscheme **cut out by** s as follow:*

$$V(s) := \bigcup_i \operatorname{Spec} B_i/(s_{B_i}),$$

*where $s_{B_i} = s|_{\operatorname{Spec} B_i}$ is the restriction of s to $\operatorname{Spec} B_i$. We call this the **vanishing scheme** $V(s)$ of s. If S is a set of functions, define $V(S)$ be the scheme-theoretic intersection*

$$V(S) = \bigcap_{s \in S} V(s).$$

*We call this the **vanishing scheme** $V(S)$ of S.*

Proposition 10.1.10

Suppose Y is a scheme, and $s \in \Gamma(Y, \mathscr{O}_Y)$. If u is an invertible function, then $V(s) = V(su)$.

Proof We may assume that $Y = \operatorname{Spec} B$, then $V(s) = \operatorname{Spec} B/(s)$ and $V(su) = \operatorname{Spec} B/(su)$. Since $u \in B$ is invertible, we have $B/(s) \cong B/(su)$, hence $V(s) = V(su)$. □

Definition 10.1.8 (Scheme-theoretic support)

*If M is an A-module, and $m \in M$, we call the closed subscheme $\operatorname{Spec}(A/\operatorname{Ann} m)$ of $\operatorname{Spec} A$ the **scheme-theoretic support** of m.*

Remark This is often described as the "smallest closed subscheme on which m is defined", since $\operatorname{Spec}(A/\operatorname{Ann} m)$ is the smallest closed subscheme such that m is not zero divisor on global section.

Proposition 10.1.11

The underlying set of the scheme-theoretic support of m is the usual "set-theoretic" support, $\operatorname{Supp} m$.

Proof Let M is an A-module, and $m \in M$, we want to show that as set

$$\operatorname{Spec}(A/\operatorname{Ann} m) = \{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0 \text{ in } M_{\mathfrak{p}}\}.$$

If $m_{\mathfrak{p}} \neq 0$ in $M_{\mathfrak{p}}$, then for all $s \in A - \mathfrak{p}$ we have $sm \neq 0$, hence $(A - \mathfrak{p}) \cap \operatorname{Ann} m = \varnothing$, and therefore $\mathfrak{p} \supseteq \operatorname{Ann} m$, vice versa. Hence

$$\operatorname{Spec}(A/\operatorname{Ann} m) = \{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0 \text{ in } M_{\mathfrak{p}}\}.$$

□

Definition 10.1.9 (Scheme-theoretic support of a finitely generated module) *Define the* ***scheme-theoretic support of a finitely generated*** A***-module*** M *to be the scheme-theoretic intersection of all closed subschemes* $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ *for which* M *is an* (A/I)*-module. More precisely,*

$$\operatorname{Supp} M := \bigcap_{I \subseteq \operatorname{Ann} M} \operatorname{Spec} A/I = \operatorname{Spec}(A/\operatorname{Ann}_A M).$$

Proposition 10.1.12 *The scheme-theoretic support of a finitely generated* A*-module* M *is the scheme-theoretic union of the scheme-theoretic supports of any finite generating set of* M*.*

Proof Let M be a finitely generated A-module which generated by $m_1, \ldots, m_n$. We want to show that

$$\operatorname{Supp} M = \bigcup_{i=1}^{n} \operatorname{Spec}(A/\operatorname{Ann} m_i)$$

where the union is scheme-theoretic union. The corresponding ideal sheaf of $\operatorname{Supp} M$ is $\sum\limits_{I \subseteq \operatorname{Ann} M} \widetilde{I} = \widetilde{\sum\limits_{I \subseteq \operatorname{Ann} M} I}$. Consider $\sum\limits_{I \subseteq \operatorname{Ann} M} I$. Note that

$$\sum_{I \subseteq \operatorname{Ann} M} I = \operatorname{Ann} M = \bigcap_{i=1}^{n} \operatorname{Ann} m_i,$$

we have

$$\sum_{I \subseteq \operatorname{Ann} M} \widetilde{I} = \widetilde{\bigcap_{i=1}^{n} \operatorname{Ann} m_i} = \bigcap_{i=1}^{n} \widetilde{\operatorname{Ann} m_i}.$$

Since $\bigcup\limits_{i=1}^{n} \operatorname{Spec}(A/\operatorname{Ann} m_i)$ correspond to the ideal sheaf $\bigcap\limits_{i=1}^{n} \widetilde{\operatorname{Ann} m_i}$, we have

$$\operatorname{Supp} M = \bigcup_{i=1}^{n} \operatorname{Spec}(A/\operatorname{Ann} m_i).$$

□

We next define the **scheme-theoretic support of finite type quasi-coherent sheaf**. Suppose $X = \bigcup_i \operatorname{Spec} A_i$ is a scheme and $\mathscr{F}$ is a finite type quasi-coherent sheaf on X. Since $\mathscr{F}$ is quasi-coherent sheaf, we may assume that $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$, where M is finitely generated A-module. Define annihilate sheaf $\mathscr{A}\mathrm{nn}(\mathscr{F})$ as follow:

$$\mathscr{A}\mathrm{nn}(\mathscr{F})(\operatorname{Spec} A) := \operatorname{Ann}_A M.$$

Then $\mathscr{A}\mathrm{nn}(\mathscr{F})$ is a quasi-coherent sheaf on X. By Proposition 10.1.6, $\mathscr{A}\mathrm{nn}(\mathscr{F})$ corresponds to a closed subscheme on X, say $\operatorname{Supp} \mathscr{F}$. If $X = \bigcup_i \operatorname{Spec} A_i$, then

$$\operatorname{Supp} \mathscr{F} = \bigcup_i \operatorname{Spec}(A_i/\mathscr{A}\mathrm{nn}(\mathscr{F})(\operatorname{Spec} A_i)) = \bigcup_i \operatorname{Supp} M_i,$$

where $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$.

Definition 10.1.10 (Scheme-theoretic support of finite type quasi-coherent sheaf)

Suppose $X = \bigcup_i \operatorname{Spec} A_i$ is a scheme and $\mathscr{F}$ is a finite type quasi-coherent sheaf on X. Say $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$, define the ***scheme-theoretic support of finite type quasi-coherent sheaf*** *$\mathscr{F}$ be the scheme-theoretic union of the scheme-theoretic support of finitely generated A-module $\operatorname{Supp} M_i$, i.e.,*

$$\operatorname{Supp} \mathscr{F} := \bigcup_i \operatorname{Supp} M_i.$$

10.2 Locally closed embeddings and locally closed subschemes

Now that we have defined analogs of open and closed subsets, it is natural to define the analog of locally closed subsets. Recall that locally closed subsets are intersections of open subsets and closed subsets. Hence they are closed subsets of open subsets, or equivalently open subsets of closed subsets. The analog of these equivalences will be a little problematic in the land of schemes. (Similarly, the notion of subgroup and quotient group are not quite complementary. Rhetorical question: should "subquotient" of a group G be defined as a subgroup of a quotient of G, or a quotient of a subgroup of G, or are these the same? We will confront the same issue when defining locally closed embeddings.)

Definition 10.2.1 (Locally closed embedding, Locally closed immersion)

We say a morphism $\pi : X \to Y$ is a ***locally closed embedding*** *(or* ***locally closed immersion****) if π can be factored into*

$$X \overset{\rho}{\underset{\text{closed}}{\hookrightarrow}} Z \overset{\tau}{\underset{\text{open}}{\hookrightarrow}} Y$$

where (as indicated) ρ is a closed embedding and τ is an open embedding.
If X is a subset of Y (and π on the level of sets is the inclusion), we say X is a ***locally closed subscheme*** *of Y.*

Remark The symbol $\hookrightarrow$ is often used to indicate that a morphism is a locally closed embedding. Because closed embeddings and open embeddings are monomorphisms (Proposition 10.1.9 and Proposition 8.2.2 respectively), and monomorphisms are preserved by composition, locally closed embeddings are also monomorphisms.

Remark **Warning:** the naked term **subscheme** is often used to mean locally closed subscheme, but we will avoid this unhappy usage.

Example 10.3 The morphism $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ given by $(x, y) \mapsto (t, 0)$ is a locally closed embedding (Figure 10.2), since $\operatorname{Spec} k[t, t^{-1}] \cong D(x) \cap V(y)$.

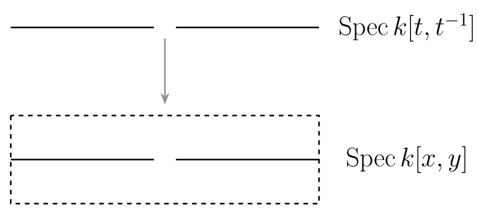


Figure 10.2: The locally closed embedding $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ ($t \mapsto (t, 0) = (x, y)$, i.e., $(x, y) \to (t, 0)$)

Remark **Warning:** The term immersion is often used instead of locally closed embedding or locally closed immersion, but this is unwise terminology, for reasons that already arose for closed embedding in Definition 10.1.1: The differential geometric notion of immersion is closer to what algebraic geometers call unramified, which we will define in Chapter 22. Also, the naked term embedding should be avoided, because it is needlessly imprecise.

Proposition 10.2.1

Locally closed embedding are locally of finite type.

Proof Let $\pi : X \hookrightarrow Y$ be a locally closed embedding, then π can be factored into

$$X \xhookrightarrow{\rho} Z \xhookrightarrow{\tau} Y,$$

where ρ is closed embedding and τ is an open embedding. We may assume that $Y = \operatorname{Spec} B$, since τ is open embedding, suppose that $Z = \bigcup_i \operatorname{Spec} B_{f_i}$. Since ρ is closed embedding, we may assume that $X = \bigcup_i \operatorname{Spec} B_{f_i}/I_i$. Note that each B_{f_i}/I_i is finitely generated B-algebra, π is locally of finite type. $\square$

Proposition 10.2.2

Suppose $\pi : X \to Y$ is a locally closed embedding whose image is a closed subset of Y. Then π is a closed embedding.

Proof Since $\pi : X \to Y$ is a locally closed embedding, π can be factored into $X \hookrightarrow U \hookrightarrow Y$ where U is open subscheme of Y. Note that $\pi(X)$ is closed in Y, Y can be covered by $Y \setminus \pi(X)$ and U. Consider $\pi^{-1}(Y \setminus \pi(X))$ and $\pi^{-1}(U)$. In fact, $\pi^{-1}(Y \setminus \pi(X)) = \varnothing$, and therefore $\varnothing \hookrightarrow Y \setminus \pi(X)$ is closed embedding. By our assumption, $\pi^{-1}(U) \hookrightarrow U$ is closed embedding. By Proposition 10.1.4, $\pi : X \to Y$ is closed embedding. $\square$

At this point, you could define the intersection of two locally closed embeddings in a scheme X (which will also be a locally closed embedding in X). But it would be awkward, as you would have to show that your construction is independent of the factorizations of each locally closed embedding into a closed embedding and an open embedding. Instead, we wait until Chapter 11, when recognizing the intersection as a fibered product will make this easier.

Clearly an open subscheme U of a closed subscheme V of X can be interpreted as a closed subscheme of an open subscheme: as the topology on V is induced from the topology on X, the underlying set of U is the intersection of some open subset U' on X with V. We can take $V' = V \cap U'$, and then $V' \to U'$ is a closed embedding, and $U' \to X$ is an open embedding.

It is not clear that a closed subscheme V' of an open subscheme U' can be expressed as an open subscheme of a closed subscheme V. In the category of topological spaces, we would take V as the closure of V', so we are now motivated to define the analogous construction, which will give us an excuse to introduce several related ideas, in §10.4. We will then resolve this issue in goof cases (e.g., if X is Noetherian).

We formalize our discussion in a proposition.

Proposition 10.2.3

Suppose $V \to X$ is a morphism. Consider three conditions:

(i) V is the intersection of an open subscheme of X and a closed subscheme of X.

(ii) V is an open subscheme of a closed subscheme of X, i.e., it factors into an open embedding followed by a closed embedding.

(iii) V is a closed subscheme of an open subscheme of X, i.e., V is locally closed embedding.

Then (i) and (ii) are equivalent, and both imply (iii).

Proof Consider the following diagram.

$$\begin{tikzcd} V \arrow[r, hook, "\text{open emb.}"] \arrow[d, hook, "\text{closed emb.}"'] & K \arrow[d, hook, "\text{closed emb.}"] \\ U \arrow[r, hook, "\text{open emb.}"'] & X \end{tikzcd}$$

We define the intersection of an open subscheme of X and a closed subscheme of X be the fiber product of open subscheme and closed subscheme. More precisely, if $K \hookrightarrow X$ is closed subscheme of X and $U \hookrightarrow X$ is open subscheme of X, define

$$U \cap K := U \times_X K.$$

By Proposition 9.1.4, fiber product exists, and therefore well-defined.

(i) $\Rightarrow$ (ii): By Proposition 9.1.4, $U \times_X K \to K$ is an open embedding, hence $U \cap K$ is an open subscheme of closed subscheme K of X.

(ii) $\Rightarrow$ (i): Since the class of open embeddings is preserved by base change (Proposition 9.1.4), $U \times_X K$ exists.

(i) $\Rightarrow$ (iii): Say $\rho : K \hookrightarrow X$ be closed embedding, by Proposition 9.1.4, we know that $U \times_X K = (\rho^{-1}(U), \mathscr{O}_K|_{\rho^{-1}(U)})$. We want to show that $\rho|_{\rho^{-1}(U)} : U \times_X K \to U$ is closed embedding. Let $\operatorname{Spec} B \subseteq U$ be any affine open subset of U, then $\operatorname{Spec} B$ is also affine open subset of X. Since ρ is closed embedding,

$$\rho|_{\rho^{-1}(U)}^{-1}(\operatorname{Spec} B) = \rho^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$$

where $B \twoheadrightarrow A$. Hence $\rho|_{\rho^{-1}(U)} : U \times_X K \to U$ is closed embedding, and therefore $U \times_X K$ is a closed subscheme of open subscheme U of X.

(ii) $\Rightarrow$ (iii): Since (i) $\Leftrightarrow$ (ii), we done! $\square$

Remark (iii) does not always imply (i) and (ii), see the pathological example Stacks Project[8] Example 01QW.

Proposition 10.2.4

The composition of two locally closed embeddings is a locally closed embedding.

Proof Let $X \hookrightarrow Y$ and $Y \hookrightarrow Z$ be two locally closed embeddings. Then we have

$$X \xhookrightarrow{\text{closed}} U \xhookrightarrow{\text{open}} Y \xhookrightarrow{\text{closed}} V \xhookrightarrow{\text{open}} Z.$$

By Proposition 10.2.3, we have the following commutative diagram.

$$\begin{tikzcd} & & Y \arrow[dr, hook, "\text{closed}"] & & \\ X \arrow[r, hook, "\text{closed}"] & U \arrow[ur, hook, "\text{open}"] \arrow[dr, hook, "\text{closed}"'] & & V \arrow[r, hook, "\text{open}"] & Z \\ & & W \arrow[ur, hook, "\text{open}"'] & & \end{tikzcd}$$

Since the composition of closed embeddings is closed embedding (Proposition 10.1.3) and the composition of open embeddings is open embeddings (Proposition 9.1.1), we have

$$X \xhookrightarrow{\text{closed}} W \xhookrightarrow{\text{open}} Z,$$

i.e., $X \to Z$ is locally closed embedding. $\square$

Remark **(Unimportant but fun remark.)** It may feel odd that in the definition of a locally closed embedding, we had to make a choice (as a composition of a closed embedding followed by an open embedding, rather than vice versa), but this type of issue comes up earlier: a subquotient of a group can be defined as the quotient of a subgroup, or a subgroup of a quotient. Which is the right definition? Or are they the same? (The subquotient of a subquotient should certainly be a subquotient, hence define subquotient is a quotient object of subobject.)

10.3 Important examples from projective geometry

We now interpret closed embeddings in terms of graded rings. Don't worry; most of the annoying foundational discussion of graded rings is complete, and we now just take advantage of our earlier work.

10.3.1 Example: Closed embeddings in projective space $\mathbb{P}^n_A$

Recall the definition of projective space $\mathbb{P}^n_A$ given in Definition 5.4.1 (and the terminology defined there). Any homogeneous polynomial f in $x_0, \ldots, x_n$ defines a closed subscheme. (Thus even if f doesn't make sense as a function, its vanishing scheme still makes sense.) On the open set U_i, the closed subscheme is $V(f(x_{0/i}, \ldots, x_{n/i}))$, which we privately think of as $V(f(x_0, \ldots, x_n)/x_i^{\deg f})$. On the overlap

$$U_i \cap U_j = \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1),$$

these functions on U_i and U_j don't exactly agree, but they agree up to a nonvanishing scalar, and hence cut out the same closed subscheme of $U_i \cap U_j$ (Definition 10.1.7):

$$f(x_{0/i}, \ldots, x_{n/i}) = x_{j/i}^{\deg f} f(x_{0/j}, \ldots, x_{n/j}).$$

Similarly, a collection of homogeneous polynomials in $A[x_0, \ldots, x_n]$ cuts out a closed subscheme of $\mathbb{P}^n_A$. (Chapter 16 will show that all closed subschemes of $\mathbb{P}^n_A$ are of the form.)

Definition 10.3.1 (Hypersurface, degree of hypersurface)

As usual, let k be a field. A closed subscheme of $\mathbb{P}^n_k$ cut out by a single (nonzero, homogeneous) equation is called a ***hypersurface*** *(in $\mathbb{P}^n_k$). (Be careful: the word "hypersurface" can be used to mean slightly different things in algebraic geometry, for entirely reasonable reasons, see for Chapter 13.)*

The ***degree of a hypersuface*** *is the degree of the polynomial. (Implicit in this is that this notion can be determined from the subscheme itself. You may have the tools to prove this now, but we won't formally prove it until Chapter 19.)*

A hypersurface of degree 1 (resp., degree $2, 3, \ldots$) is called a ***hyperplane*** *(resp.,* ***quadric, cubic, quartic, quintic, sextic, septic, octic, ... hypersurface****). If $n = 2$, hypersufaces are called* ***curves****; a degree 1 hypersurface is called a* ***line****, and a degree 2 hypersurface is called a* ***conic curve****, or a* ***conic*** *for short. If $n = 3$, a hypersurface is called a* ***surface****. (In Chapter 13, we will justify the terms curve and surface.)*

Remark Of cause, a hypersurface is not cut out by a single global function on $\mathbb{P}^n_k$: there are no nonconstant global functions (Proposition 5.4.2).

Exercise 10.3

(a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible closed subscheme in $\mathbb{P}^3_k$. In fact it is a curve, a notion we will define once we know what dimension is. This curve is called the **twisted cubic**. (The twisted cubic is a good nontrivial example of many things, so you should make friends with it as soon as possible. It implicitly appeared earlier in Exercise 4.18.)

(b) Show that the twisted cubic is isomorphic to $\mathbb{P}^1_k$.

Proof

(a) It suffices to show that $\operatorname{Proj} k[x,y,z,w]/(wz-xy, x^2-wy, y^2-xz)$ is irreducible closed subscheme. Note that

$$\begin{aligned}
&\operatorname{Proj} \frac{k[x,y,z,w]}{(wz-xy,x^2-wy,y^2-xz)} \\
&= \operatorname{Spec} \frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)} \cup \operatorname{Spec} \frac{k\left[\frac{x}{y},\frac{z}{y},\frac{w}{y}\right]}{\left(\frac{w}{y}\frac{z}{y}-\frac{x}{y},\frac{x^2}{y^2}-\frac{w}{y},1-\frac{x}{y}\frac{z}{y}\right)} \\
&\cup \operatorname{Spec} \frac{k\left[\frac{x}{z},\frac{y}{z},\frac{w}{z}\right]}{\left(\frac{w}{z}-\frac{x}{z}\frac{y}{z},\frac{x^2}{z^2}-\frac{w}{z}\frac{y}{z},\frac{y^2}{z^2}-\frac{x}{z}\right)} \cup \operatorname{Spec} \frac{k\left[\frac{x}{w},\frac{y}{w},\frac{z}{w}\right]}{\left(\frac{z}{w}-\frac{x}{w}\frac{y}{w},\frac{x^2}{w^2}-\frac{y}{w},\frac{y^2}{w^2}-\frac{x}{w}\frac{z}{w}\right)},
\end{aligned}$$

we know that $\operatorname{Proj} \frac{k[x,y,z,w]}{(wz-xy,x^2-wy,y^2-xz)}$ is a closed subscheme of $\mathbb{P}^3_k$. Since

$$\operatorname{Spec} \frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)} = \operatorname{Proj} \frac{k[x,y,z,w]}{(wz-xy,x^2-wy,y^2-xz)} \cap \operatorname{Spec} k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right],$$

we know that

$$\operatorname{cl}_{\mathbb{P}^3_k}\left(\operatorname{Spec} \frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)}\right) = \operatorname{Proj} \frac{k[x,y,z,w]}{(wz-xy,x^2-wy,y^2-xz)}.$$

Since the closure of irreducible subset is also irreducible (Proposition 4.6.1 (b)), it suffices to show that $\operatorname{Spec} \frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)}$ is irreducible, by Proposition 4.6.2, it suffices to show that $\frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)}$ is integral domain. Say $a=\frac{y}{x}, b=\frac{z}{x}, c=\frac{w}{x}$, then

$$\frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)} = k[a,b,c]/(bc-a,1-ac,a^2-b).$$

Note that

$$\frac{k[a,b,c]}{(bc-a,1-ac,a^2-b)} \cong \frac{k[a,a^{-1},b]}{(a^2-b)} \cong k[a,a^{-1}],$$

and $k[a,a^{-1}]$ is an integral domain, $\frac{k\left[\frac{y}{x},\frac{z}{x},\frac{w}{x}\right]}{\left(\frac{w}{x}\frac{z}{x}-\frac{y}{x},1-\frac{w}{x}\frac{y}{x},\frac{y^2}{x^2}-\frac{z}{x}\right)}$ is integral domain, as we desired.

(b) Say $C = \operatorname{Proj} \frac{k[w,x,y,z]}{(wz-xy,x^2-wy,y^2-xz)}$, then there is a morphism

$$\begin{aligned}
\varphi : \mathbb{P}^1_k &\longrightarrow C \subseteq \mathbb{P}^3_k \\
[s:t] &\longmapsto [s^3 : s^2t : st^2 : t^3].
\end{aligned}$$

In fact, $C = (D_+(w)\cap C)\cup(D_+(z)\cap C)$. On $D_+(w)$ we have

$$\begin{aligned}
D_+(w)\cap C &\longrightarrow \mathbb{P}^1_k \\
[w:x:y:z] &\longmapsto [w:x]
\end{aligned}$$

is isomorphism. On $D_+(z)$ we have

$$\begin{aligned}
D_+(z)\cap C &\longrightarrow \mathbb{P}^1_k \\
[w:x:y:z] &\longmapsto [y:z]
\end{aligned}$$

is isomorphism. On the overlap $D_+(w)\cap D_+(z)$, we have $wz = xy$, so they glue together, and therefore we have $C \xrightarrow{\sim} \mathbb{P}^1_k$.

□

We now extend this discussion to projective schemes in general.

Proposition 10.3.1

Suppose that $S_\bullet \twoheadrightarrow R_\bullet$ is a surjection of graded rings. Then the domain of the induced morphism (Proposition 8.4.1) is $\operatorname{Proj} R_\bullet$, and that the induced morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$ is a closed embedding.

Proof Say $\pi^\sharp : S_\bullet \twoheadrightarrow R_\bullet$ be a surjection of graded rings, by Proposition 8.4.1, it suffices to show that $V(\pi^\sharp(S_+)) = \varnothing$. Since $S_\bullet \twoheadrightarrow R_\bullet$ is surjective, $\pi^\sharp(S_+) = R_+$, hence $V(\pi^\sharp(S_+)) = \varnothing$. By Proposition 8.4.1, $S_\bullet \to R_\bullet$ induces a morphism of projective schemes

$$\pi : \operatorname{Proj} R_\bullet \longrightarrow \operatorname{Proj} S_\bullet. \tag{10.2}$$

We next show that induced map is closed embedding. Since $\pi^\sharp(S_+) = R_+$, we may assume that $\operatorname{Proj} R_\bullet = \bigcup\limits_{\pi^\sharp(f)\in\pi^\sharp(S_+)} \operatorname{Spec}((R_\bullet)_{\pi^\sharp(f)})_0$. On the other hand, $\operatorname{Proj} S_\bullet = \bigcup\limits_{f\in S_+} \operatorname{Spec}((S_\bullet)_f)_0$. By Proposition 8.4.1, we know that (10.2) is glued by

$$\pi|_{D_+(\pi^\sharp(f))} : \operatorname{Spec}((R_\bullet)_{\pi^\sharp(f)})_0 \longrightarrow \operatorname{Spec}((S_\bullet)_f)_0,$$

hence $\pi^{-1}(\operatorname{Spec}((S_\bullet)_f)_0) = \operatorname{Spec}((R_\bullet)_{\pi^\sharp(f)})_0$. Since $S_\bullet \twoheadrightarrow R_\bullet$ is surjective, $((S_\bullet)_f)_0 \twoheadrightarrow ((R_\bullet)_{\pi^\sharp(f)})_0$ is surjective, and therefore $\pi : \operatorname{Proj} R_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$ is closed embedding. □

In Chapter 16 we will show the converse to Proposition 10.3.1: every closed embedding $X \hookrightarrow \operatorname{Proj} S_\bullet$ into a projective A-scheme ($S_\bullet$ is a finitely generated graded A-algebra) is of the form $\operatorname{Proj}(S_\bullet/I)$, where I is a homogeneous ideal, of "projective functions" vanishing on X.

Proposition 10.3.2

An injective linear map of k-vector spaces $V \hookrightarrow W$ induces a closed embedding $\mathbb{P}V \hookrightarrow \mathbb{P}W$. (This is another justification for the definition of $\mathbb{P}V$ in Definition 5.5.12 in terms of the dual of V.)

Proof By Definition 5.5.12, we have $\mathbb{P}V = \operatorname{Proj}(\operatorname{Sym}^\bullet V^\vee)$ and $\mathbb{P}W = \operatorname{Proj}(\operatorname{Sym}^\bullet W^\vee)$. By Proposition 10.3.1, it suffices to show that $\operatorname{Sym}^\bullet W^\vee \to \operatorname{Sym}^\bullet V^\vee$ is surjective. Say $\varphi : V \to W$ is injective k-linear map, it induces a dual k-linear map

$$\varphi^\vee : W^\vee \longrightarrow V^\vee, \qquad f \longmapsto f \circ \varphi,$$

we want to show that $\varphi^\vee$ is surjective. Let $V = \operatorname{Span}_k\{v_1, \dots, v_n\}$, since $\varphi : V \to W$ is injective, $\varphi(V)$ can be seen as subspace of W. Then $\varphi(V) = \operatorname{Span}_k\{\varphi(v_1), \dots, \varphi(v_n)\}$, by the Basis Extension Theorem, we have $W = \operatorname{Span}_k\{\varphi(v_1), \dots, \varphi(v_n), w_{n+1}, \dots, w_m\}$. Let $V^\vee = \operatorname{Span}_k\{x_1, \dots, x_n\}$ where $x_i(v_j) = \delta_{ij}$. Let $g = \sum\limits_{i=1}^n a_i x_i \in V^\vee$, define $f \in W^\vee$ as follow:

$$f(\varphi(v_i)) = a_i, \qquad f(w_j) = 0,$$

where $1 \le i \le n$ and $n+1 \le j \le m$. Hence $\varphi^\vee(f) = g$, it follows that $\varphi^\vee$ is surjective. Hence $\operatorname{Sym}^\bullet W^\vee \to \operatorname{Sym}^\bullet V^\vee$ is surjective, as we desired. □

Definition 10.3.2 (Linear space)

*The closed subscheme defined in Proposition 10.3.2 is called a **linear space**. Once we know about dimension, we will call this closed subscheme a linear space of dimension $\dim V - 1 = \dim \mathbb{P}V$. More explicitly, a linear space of dimension n in $\mathbb{P}^N$ is any closed subscheme cut out by $N - n$ k-linearly independent homogeneous linear polynomials in $x_0, \dots, x_N$. A linear space of dimension 1 (resp., 2, n, $\dim \mathbb{P}V - 1$) is called a **line** (resp., **plane**, n**-plane**, **hyperplane**).*

Remark If the linear map in the previous linear polynomials in Proposition 10.3.2 is not injective, then the hypothesis (8.8) in Proposition 8.4.1 fails.

✍ **Exercise 10.4** **(A special case of Bézout's Theorem.)** Suppose $X \subseteq \mathbb{P}_k^n$ is a degree d hypersurface cut out by $f = 0$, and l is a line not contained in X. A very special case of Bézout's Theorem implies that X and l meet with multiplicity d, "counted correctly". Make sense of this, by restricting the homogeneous degree d polynomial f to the line l, and using the fact that a degree d polynomial in $k[x]$ has d roots, counted properly. (If it makes you feel better, assume $k = \overline{k}$.)

Solution *Say* $\mathbb{P}_k^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. *Let* $X = V(f)$ *where* $f \in k[x_0, \ldots, x_n]$ *and* $\deg f = d$. *Suppose line* l *is generated by two points* $[a_0 : \cdots : a_n]$ *and* $[b_0 : \cdots : b_n]$, *then*

$$l = \{[sa_0 + tb_0 : \cdots : sa_n + tb_n] : s \neq 0 \text{ or } t \neq 0\}.$$

Define $\varphi : \mathbb{P}^1 \to l \subseteq \mathbb{P}^n$ *by setting*

$$\varphi([s : t]) = [sa_0 + tb_0 : \cdots : sa_n + tb_n].$$

Hence we may restrict X *to* l *by using* $f \circ \varphi$:

$$g(s,t) := f(\varphi([s : t])) = f(sa_0 + tb_0 : \cdots : sa_n + tb_n).$$

Since f *is homogeneous polynomial of degree* d, g *is homogeneous polynomial of degree* d. *Since* $l \nsubseteq X$, g *is not zero polynomial. In fact, geometric points of* $X \cap l$ *is one-to-one corresponding to points of* $g(s,t) = 0$. *In* $k[s,t]$, *we may factors* $g(s,t)$ *as*

$$g(s,t) = c \prod_i (\alpha_i s - \beta_i t)^{m_i}$$

where $\sum_i m_i = d$. *Each linear factor* $\alpha_i s - \beta_i t$ *cuts out a point* $[\beta_i : \alpha_i] \in \mathbb{P}^1$, *this point is corresponding to a point in* $X \cap l$. *The exponent* m_i *is exactly the intersection multiplicity of* X *and* l *at point* $\varphi([\beta_i : \alpha_i])$. *Also the sum of intersection multiplicities is* d.

✍ **Exercise 10.5** Show that the map of graded rings $k[w,x,y,z] \to k[s,t]$ given by $(w,x,y,z) \mapsto (s^3, s^2t, st^2, t^3)$ induces a closed embedding $\mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^3$, which yields an isomorphism of $\mathbb{P}_k^1$ with the twisted cubic (defined in Exercise 10.3 — in fact, this will solve Exercise 10.3 (b)). Doing this in a hands-on way will set you up well for the general Veronese construction of §10.3.3.

Proof Let $S_\bullet = k[w,x,y,z]$ and $R_\bullet = k[s,t]$, say $\varphi^\sharp : S_\bullet \to R_\bullet$ which defined by

$$(w,x,y,z) \longmapsto (s^3, s^2t, st^2, t^3).$$

Consider $i^\sharp : R_{3\bullet} \hookrightarrow R_\bullet$, by the construction of $\varphi^\sharp$, we have $\varphi^\sharp = i^\sharp \circ \varphi'^\sharp$, where $\varphi'^\sharp : S_\bullet \to R_{3\bullet}$ defined by $\varphi^\sharp$. Clearly, $\varphi'^\sharp : S_\bullet \to R_{3\bullet}$ is a surjection of graded rings, by Proposition 10.3.1, it induces a closed embedding $\varphi' : \operatorname{Proj} R_{3\bullet} \hookrightarrow \operatorname{Proj} S_\bullet$. By Proposition 8.4.3, $R_{3\bullet} \hookrightarrow R_\bullet$ induces an isomorphism $\operatorname{Proj} R_\bullet \xrightarrow{\sim} \operatorname{Proj} R_{3\bullet}$. Hence we get a closed embedding $\mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^3$. □

10.3.2 A particularly nice case: when $S_\bullet$ is generated in degree 1

Suppose $S_\bullet$ is a finitely generated graded ring generated in degree 1 (Definition 5.5.4). Then S_1 is a finitely generated S_0-module, and the irrelevant ideal S_+ is generated in degree 1 (Proposition 5.5.3 (a)).

Proposition 10.3.3

If $S_\bullet$ *is generated (as an* A*-algebra) in degree* 1 *by* $n+1$ *elements* $x_0, \ldots, x_n$, *then* $\operatorname{Proj} S_\bullet$ *may be described as a closed subscheme of* $\mathbb{P}_A^n$ *as follows. Consider* $A^{\oplus(n+1)}$ *as a free module with generators*

$t_0, \dots, t_n$ associated to $x_0, \dots, x_n$. The surjection of

$$\begin{aligned} \operatorname{Sym}^\bullet(A^{\oplus(n+1)}) = A[t_0, t_1, \dots, t_n] &\twoheadrightarrow S_\bullet \\ t_i &\longmapsto x_i \end{aligned}$$

implies $S_\bullet = A[t_0, t_1, \dots, t_n]/I$, where I is a homogeneous ideal.
In particular, $\operatorname{Proj} S_\bullet$ can always be interpreted as a closed subscheme of some $\mathbb{P}^n_A$ if $S_\bullet$ is finitely generated in degree 1. Then using Proposition 8.4.3 and Proposition 8.4.6, we can remove the hypothesis of generation in degree 1.

This is analogous to the fact that if R is a finitely generated A-algebra, then choosing n generators of R as an algebra is the same as describing $\operatorname{Spec} R$ as a closed subscheme of $\mathbb{A}^n_A$. In the affine case this is "choosing coordinates"; in the projective case this is "choosing projective coordinates".

Recall (Proposition 5.4.3) that if k is algebraically closed, then we can interpret the closed points of $\mathbb{P}^n$ as the lines through the origin in $(n+1)$-space. The following proposition states this more generally.

Proposition 10.3.4

Suppose $S_\bullet$ is a finitely generated graded ring over an algebraically closed field k, generated in degree 1 by $x_0, \dots, x_n$, inducing closed embeddings $\operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ and $\operatorname{Spec} S_\bullet \hookrightarrow \mathbb{A}^{n+1}$. Then there is a bijection between the closed points of $\operatorname{Proj} S_\bullet$ and the "lines through the origin" in $\operatorname{Spec} S_\bullet \subseteq \mathbb{A}^{n+1}$.

Proof Since $S_\bullet$ is a finitely generated graded ring over k, by Proposition 5.5.3, $S_\bullet$ is a finitely generated k-algebra, then we have surjective

$$\begin{aligned} k[t_0, t_1, \dots, t_n] &\twoheadrightarrow S_\bullet \\ t_i &\longmapsto x_i. \end{aligned}$$

By the Fundamental Theorem of Isomorphism of rings, $S_\bullet \cong k[t_0, \dots, t_n]/I$, where I is a homogeneous ideal. Then $\operatorname{Proj} S_\bullet \cong \operatorname{Proj} k[t_0, \dots, t_n]/I$ and $\operatorname{Spec} S_\bullet \cong \operatorname{Spec} k[t_0, \dots, t_n]/I$. Pick closed point $[a_0 : \dots : a_n] \in \operatorname{Proj} S_\bullet$, since I is homogeneous ideal, $(\lambda a_0, \dots, \lambda a_n) \in \operatorname{Spec} S_\bullet$ for all $\lambda \in k$, this gives a line through the origin, say

$$L := \{(\lambda a_0, \dots, \lambda a_n) : \lambda \in k\} \subseteq \operatorname{Spec} S_\bullet.$$

Conversely, let L be a line through the origin in $\operatorname{Spec} S_\bullet$, we may assume this line through the point $(a_0, \dots, a_n) \in \operatorname{Spec} S_\bullet$, then

$$L = \{(\lambda a_0, \dots, \lambda a_n) : \lambda \in k\}.$$

This gives a projective coordinates $[a_0 : \dots : a_n]$ in $\operatorname{Proj} S_\bullet$. By our discussion, this correspondence is a bijection. $\square$

A second proof that finite morphisms are closed

This interpretation of $\operatorname{Proj} S_\bullet$ as a closed subscheme of projective space (when it is generated in degree 1) yields the following second proof of the fact (shown in Corollary 9.3.2) that finite morphisms are closed.

Proposition

Finite morphisms are closed.

Proof Suppose $\pi : X \to Y$ is a finite morphism. The question is local on the target, so it suffices to consider the affine case $Y = \operatorname{Spec} B$, by the finiteness, we may assume that $X = \operatorname{Spec} A$ where A is finite B-algebra. It suffices to show that $\pi(X)$ is closed. Then by Proposition 9.3.12, X can be seen as a projective B-scheme, by Proposition 10.3.3, X can be described as a closed subscheme of some $\mathbb{P}^n_B$, and hence by the Fundamental Theorem of Elimination Theory 9.4.3

$$X \xhookrightarrow{\text{closed}} \mathbb{P}^n_B \xrightarrow{\text{closed}} \operatorname{Spec} B,$$

$\pi(X)$ is closed. □

10.3.3 Important classical construction: The Veronese embedding

Suppose $S_\bullet = k[x, y]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^1_k$. Then $S_{2\bullet} = k[x^2, xy, y^2] \subseteq k[x, y]$ (see Definition 8.4.2). We identify this subring as follows.

Exercise 10.6 Let $u = x^2$, $v = xy$, $w = y^2$. Show that $S_{2\bullet} \cong k[u, v, w]/(uw - v^2)$, by mapping u, v, w to x^2, xy, y^2, respectively.

Proof Define $\varphi : k[u, v, w] \to k[x^2, xy, y^2]$ by setting

$$(u, v, w) \longmapsto (x^2, xy, y^2),$$

then φ is clearly a surjective. Consider $\operatorname{Ker} \varphi$, we claim that $\operatorname{Ker} \varphi = (uw - v^2)$. Note that $k[u, v, w]$ can be seen as $k[u, w][v]$, since $v^2 - uw$ is monic polynomial in $k[u, w][v]$, for all $f(u, v, w) \in k[u, w][v]$ we have

$$f(u, v, w) = (v^2 - uw)g(u, v, w) + r(u, v, w),$$

where $g(u, v, w), r(u, v, w) \in k[u, w][v]$ and $\deg_v r(u, v, w) < 2$. We may assume that

$$r(u, v, w) = a(u, w)v + b(u, w),$$

where $a(u, w), b(u, w) \in k[u, w]$. Now let $f(u, v, w) \in \operatorname{Ker} \varphi$, then

$$\varphi(f(u, v, w)) = r(x^2, xy, y^2) = a(x^2, y^2)xy + b(x^2, y^2) = 0.$$

Hence $a(x^2, y^2) = b(x^2, y^2) = 0$, since $k[u, w] \cong k[x^2, y^2]$ (given by $(u, w) \mapsto (x^2, y^2)$), $a(u, w) = b(u, w) = 0$, it follows that $\operatorname{Ker} \varphi = (uw - v^2)$. By the Fundamental Theorem of Isomorphism of rings, we have

$$S_{2\bullet} \cong k[u, v, w]/(uw - v^2).$$

□

We have a graded ring generated by three element in degree 1. Thus we think of it as sitting "in" $\mathbb{P}^2$, via the construction of Proposition 10.3.3. This can be interpreted as "$\mathbb{P}^1$ as a conic in $\mathbb{P}^2$".

Lemma 10.3.1

$\mathbb{P}^1$ can be seen as a conic in $\mathbb{P}^2$.

Proof Say $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$. By Proposition 8.4.3, we know that $\operatorname{Proj} S_\bullet \xrightarrow{\sim} \operatorname{Proj} S_{2\bullet}$, by Exercise 10.6, we have $\operatorname{Proj} S_{2\bullet} \cong \operatorname{Proj} k[u, v, w]/(uw - v^2)$, where $u = x^2, v = xy, w = y^2$. Hence

$$\mathbb{P}^1 \cong \operatorname{Proj} S_{2\bullet} \cong \operatorname{Proj} k[u, v, w]/(uw - v^2) \subseteq \mathbb{P}^2.$$

□

Thus if k is algebraically closed of characteristic not 2, using the fact that we can diagonalize quadratics

(Exercise 6.7), the conics in $\mathbb{P}^2$, up to change of coordinates, come in only a few flavors: sums of three (and no fewer) squares (e.g., our conic of Exercise 10.6), sums of two (and no fewer) squares (e.g., $y^2 - x^2 = 0$, the union of two lines), a single (nonzero) square (e.g., $x^2 = 0$, which looks set-theoretically like a line, and is non-reduced), and 0 (perhaps not a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three non-zero squares (and no fewer) is isomorphic to $\mathbb{P}^1$. (See Exercise 8.9 for a closely related fact.)

We now soup up this example.

Proposition 10.3.5

Take $S_\bullet = k[x,y]$. Then $\operatorname{Proj} S_{d\bullet}$ is given by the equations that

$$\begin{bmatrix} t_0 & t_1 & \cdots & t_{d-1} \\ t_1 & t_2 & \cdots & t_d \end{bmatrix}$$

*is rank 1 (i.e., that all the 2×2 minors vanish). This is called the **degree d rational normal curve** "in" $\mathbb{P}^d$.*

Remark We did the twisted cubic case $d = 3$ in Exercise 10.3 and Exercise 10.5.

Proof In fact, $S_{d\bullet} = k[x^d, x^{d-1}y, \ldots, y^d]$. Consider the ring homomorphism

$$\begin{aligned} \varphi : \qquad k[t_0, t_1, \ldots, t_d] &\twoheadrightarrow S_{d\bullet} = k[x^d, x^{d-1}y, \ldots, y^d] \\ (t_0, t_1, \ldots, t_d) &\longmapsto (x^d, x^{d-1}y, \ldots, y^d) \end{aligned}$$

it is clearly surjective. By Exercise 4.18, the kernel of φ is given by the equations (all 2×2 minors vanish) that

$$\begin{bmatrix} t_0 & t_1 & \cdots & t_{d-1} \\ t_1 & t_2 & \cdots & t_d \end{bmatrix}$$

is rank 1. □

More generally:

Definition 10.3.3 (d-ulpe Veronese embedding)

*If $S_\bullet = k[x_0, \ldots, x_n]$, then $v_d : \operatorname{Proj} S_{d\bullet} \to \mathbb{P}^{N-1}$ (where N is the dimension of the vector space of homogeneous degree d polynomials in $x_0, \ldots, x_n$) is called the **d-uple embedding or the (d-uple) Veronese embedding**.*

Proposition 10.3.6

(d-uple) Veronese embedding $v_d : \operatorname{Proj} S_{d\bullet} \to \mathbb{P}^{N-1}$ is closed embedding, where $S_\bullet = k[x_0, \ldots, x_n]$.

Proof By Proposition 8.4.4, $\operatorname{Proj} S_{d\bullet}$ is generated in degree 1, by Proposition 10.3.3, $v_d : \operatorname{Proj} S_{d\bullet} \hookrightarrow \mathbb{P}^{N-1}$ is closed embedding. □

Proposition 10.3.7

In Definition 10.3.3, $N = \binom{n+d}{d}$.

Proof By Definition 10.3.3, N is the dimension of the vector space of homogeneous degree d polynomials in $x_0, \ldots, x_n$. Hence N is the number of non-negative integer solution to equation

$$a_0 + a_1 + \cdots + a_n = d.$$

This is a classical "stars and bars" combinatorial problem. Imagine placing d identical stars (representing units of degree) and n identical bars (separating $n+1$ variables). Total positions are $d+n$. Choose n positions for bars, we get $N = \binom{n+d}{d}$. $\square$

✍ **Exercise 10.7** (**Unimportant exercise.**) Find six linearly independent quadratic equations vanishing on the **Veronese surface** $\operatorname{Proj} S_{2\bullet}$ where $S_\bullet = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$. (You needn't show that these equations generate all the equations cutting out the Veronese surface, although this is in fact true.)

Proof The ring homomorphism $k[t_0, \ldots, t_5] \to S_{2d}$ is given by

$$(t_0, \ldots, t_5) \longmapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$

Then six linearly independent quadratic equations are all 2×2 minors of matrix

$$\begin{bmatrix} t_0 & t_1 & t_2 \\ t_1 & t_3 & t_4 \\ t_2 & t_4 & t_5 \end{bmatrix}.$$

$\square$

10.3.4 Rulings on the quadric surface

We return to rulings on the quadric surface, which first appeared in the Example 5.7 .

✍ **Exercise 10.8** (**Useful geometric exercise: The rulings on the quadric surface** $wz = xy$**.**) This exercise is about the lines on the quadric surface X given by $wz - xy = 0$ in $\mathbb{P}_k^3$ (where the projective coordinates on $\mathbb{P}_k^3$ are ordered w, x, y, z). This construction arises all over the place in nature.

(a) Suppose a_0 and b_0 are given elements of k, not both zero. Make sense of the statement: as $[c : d]$ varies in $\mathbb{P}^1$, $[a_0c : b_0c : a_0d : b_0d]$ is a line in the quadric surface. (This describes "a family of lines parametrized by $\mathbb{P}^1$", although we can't yet make this precise.) Find another family of lines. These are the two **rulings** of the smooth quadric surface.

(b) Show that through every k-valued point of the quadric surface X, these passes one line from each ruling.

(c) Show there are no other lines. (There are many ways of proceeding. At risk of predisposing you to one approach, here is a germ of an idea. Suppose L is a line on the quadric surface, and $[1 : x : y : z]$ and $[1 : x' : y' : z']$ are distinct points on it. Because they are both on the quadric, $z = xy$ and $z' = x'y'$. Because all of L is on the quadric, $(1+t)(z+tz') - (x+tx')(y+ty') = 0$ for all t. After some algebraic manipulation, this translates into $(x - x')(y - y') = 0$. How can this be made watertight? Another possible approach uses Bézout's Theorem, in the form of Exercise 10.4.)

Proof

(a) Fix points $[a_0 : b_0] \in \mathbb{P}^1$, let $[c, d]$ varies in $\mathbb{P}^1$, then we get a line through points $[a_0 : b_0 : 0 : 0]$ and $[0 : 0 : a_0 : b_0]$, say

$$L_{[a_0:b_0]} := \{[a_0c : b_0c : a_0d : b_0d] \in \mathbb{P}^3 : [c : d] \in \mathbb{P}^1\}.$$

Clearly, line $L_{[a_0:b_0]}$ is contained in $\operatorname{Proj} k[w, x, y, z]/(wz - xy)$. Hence collection

$$\mathscr{L} = \{L_{[a_0,b_0]} : [a_0 : b_0] \in \mathbb{P}^1\}$$

is a family of lines which contained in quadric surface.

Now, we fix points $[c_0 : d_0] \in \mathbb{P}^1$, let $[a : b]$ varies in $\mathbb{P}^1$, then we get a line through points $[c_0 : 0 : d_0 : 0]$ and $[0 : c_0 : 0 : d_0]$, say

$$M_{[c_0:d_0]} := \{[c_0a : c_0b : d_0a : d_0b] \in \mathbb{P}^3 : [a : b] \in \mathbb{P}^1\}.$$

$M_{[c_0:d_0]}$ is contained in quadric surface. And collection

$$\mathscr{M} = \{M_{[c_0:d_0]} : [c_0 : b_0] \in \mathbb{P}^1\}$$

is another family of lines which contained in quadric surface.

(b) Pick k-valued points $p = [p_0 : p_1 : p_2 : p_3]$ of X, where $p_0p_3 = p_1p_2$.

Small remark

What is the k-valued point of X?

k-valued points of the quadric surface $X = \operatorname{Proj} k[w, x, y, z]/(wz - xy)$ is defined by $X(k) = \operatorname{Mor}(\operatorname{Spec} k, X)$ (Definition 8.3.5). Say $p = \pi([(0)])$, by Proposition 8.3.1, π induces an inclusion $\kappa(p) \hookrightarrow k$, note that $k \subseteq \kappa(p)$, $\kappa(p) = k$, it follows that p is closed point of X.

We may assume $[p_0 : p_1] \in \mathbb{P}^1$, let $[a_0 : b_0] = [p_0 : p_1]$. We need to find $[c : d] \in \mathbb{P}^1$ such that

$$[p_0 : p_1 : p_2 : p_3] = [a_0c : b_0c : a_0d : b_0d].$$

Note that $c = \frac{p_0}{p_0} = \frac{p_1}{p_1} = 1$ and $p_0p_3 = p_1p_2$, take

$$d = \frac{p_2}{p_0} = \frac{p_3}{p_1}.$$

It follows that $p \in L_{[p_0:p_1]}$. Since $[p_0 : p_1] \in \mathbb{P}^1$, we may assume $p_0 \neq 0$, then $[p_0 : p_2] \in \mathbb{P}^1$. Let $[c_0 : d_0] = [p_0 : p_2]$. We want to find $[a : b] \in \mathbb{P}^1$ such that

$$[p_0 : p_1 : p_2 : p_3] = [c_0a : c_0b : d_0a : d_0b].$$

Note that $a = \frac{p_0}{p_0} = \frac{p_2}{p_2} = 1$ and $p_0p_3 = p_1p_2$, take

$$b = \frac{p_1}{p_0} = \frac{p_3}{p_2}.$$

Then $p \in M_{[p_0:p_2]}$. By uniqueness of $[c : d]$ and $[a : b]$, $L_{[p_0:p_1]}$ and $M_{[p_0:p_2]}$ are the unique lines passing through p in each ruling family.

(c) Let L be a line on the quadric surface, and $P = [1 : x : y : z]$ and $Q = [1 : x' : y' : z']$ are distinct points on it where $z = xy$ and $z' = x'y'$, then

$$L = \{[s + t : xs + x't : ys + y't : zs + z't] : [s : t] \in \mathbb{P}^1\}.$$

Let $s = 1$, since $L \subseteq X$, we have equation

$$(1 + t)(z + z't) - (x + x't)(y + y't) = 0,$$

by some calculations,

$$(x' - x)(y' - y)t = 0, \quad \forall t \in k,$$

hence $(x' - x)(y' - y) = 0$. It follows that $x' = x$ or $y' = y$. Suppose that $x' = x$, by part (b), $P, Q \in L_{[1,x]}$, i.e., $L = L_{[1,x]}$.

□

Remark Hence by Exercise 6.7, if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines. (In Chapter 11, we will recognize this quadric as $\mathbb{P}^1 \times \mathbb{P}^1$.)

Remark The quadric surface sketched in figure 10.3 is actually $a^2 + b^2 = c^2 + 1$. **Why this is an affine chart for S?**

This is a linear algebra problem. We need to transform the quadratic form $wz - xy$ into its canonical form.

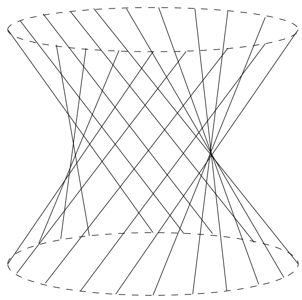


Figure 10.3: One of the two rulings on the quadric surface $S := V(wz - xy) \subseteq \mathbb{P}^3$. One ruling contains the line $V(w,x)$ and the other contains the line $V(w,y)$.

Let

$$\begin{cases} a' = & \frac{w+z}{2} \\ b' = & \frac{w-z}{2} \\ c' = & \frac{x+y}{2} \\ d' = & \frac{x-y}{2} \end{cases},$$

then we have $a'^2 + b'^2 - c'^2 - d'^2 = 0$. On the chart $D_+(d')$, we have $a^2 + b^2 = c^2 + 1$ where $a = \frac{a'}{d'}$, $b = \frac{b'}{d'}$ and $c = \frac{c'}{d'}$.

Remark **Side remark: Rulings on quadric hypersurfaces.** The exercise of these two rulings is the first chapter of a number of important and beautiful stories. The second chapter is often the following. If k is an algebraically closed field, then a maximal rank (Exercise 6.7) quadric hypersurface X of dimension m contains no linear spaces of dimension greater than $m/2$. (We will see in Chapter 14 that the maximal rank quadric hypersurfaces are the "smooth" quadrics.) If $m = 2a+1$, then X contains an irreducible $\binom{a+2}{2}$-dimensional family of a-planes. If $m = 2a$, then X contains two irreducible $\binom{a+2}{2}$-dimensional families of a-planes, and furthermore two a-planes Λ and Λ' are in the same family if and only if $\dim(\Lambda \cap \Lambda') \equiv a \pmod 2$. These families of linear spaces are also called **rulings**.

10.3.5 Affine and projective cones (Figure 10.5)

Definition 10.3.4 (Affine cone)

*If $S_\bullet$ is a finitely generated graded ring, then the **affine cone** of* $\operatorname{Proj} S_\bullet$ *is* $\operatorname{Spec} S_\bullet$.

Figure 10.4: The cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$.

Remark **Caution:** this terminology is not ideal, as this construction depends on $S_\bullet$, not just on $\operatorname{Proj} S_\bullet$. As motivation, consider the graded ring $S_\bullet = \mathbb{C}[x,y,z]/(z^2 - x^2 - y^2)$. Figure 10.4 is a sketch of $\operatorname{Spec} S_\bullet$. (Here we draw the "real picture" of $z^2 = x^2 + y^2$ in $\mathbb{R}^3$.) It is a cone in the traditional sense; the origin $(0, 0, 0)$ is the "cone point".

This gives a useful way of picture Proj (even over arbitrary rings, not just $\mathbb{C}$). Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} S_\bullet$. The following proposition makes that precise.

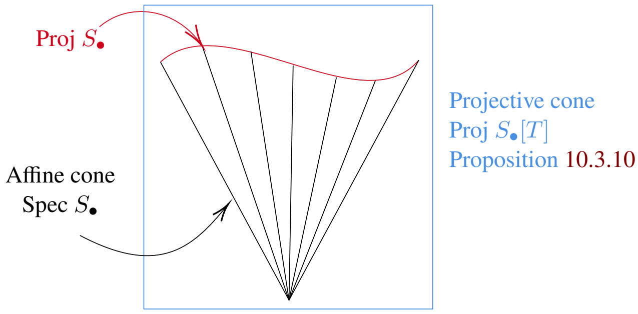


Figure 10.5: The affine and projective cones of $\operatorname{Proj} S_\bullet$

Proposition 10.3.8

If $\operatorname{Proj} S_\bullet$ *is a projective scheme over a field* k*, then there is a natural morphism* $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$.

Proof In fact, $\operatorname{Spec} S_\bullet \setminus V(S_+) = \bigcup\limits_{f \in S_+} \operatorname{Spec}(S_\bullet)_f$ and $\operatorname{Proj} S_\bullet = \bigcup\limits_{f \in S_+} \operatorname{Spec}((S_\bullet)_f)_0$. From inclusion $\iota_f^\sharp : ((S_\bullet)_f)_0 \hookrightarrow (S_\bullet)_f$, by Proposition 8.3.6, we have a morphism of affine schemes

$$\iota_f : \operatorname{Spec}(S_\bullet)_f \longrightarrow \operatorname{Spec}((S_\bullet)_f)_0.$$

We now check ι_f agree on the overlaps, i.e., $\iota_f|_{D(f) \cap D(g)} = \iota_g|_{D(f) \cap D(g)}$ where $f, g \in S_+$. It is clear, since they are both given by inclusion

$$((S_\bullet)_{fg})_0 \longrightarrow (S_\bullet)_{fg},$$

by Morphism of locally ringed spaces glue 8.3.2, they glue together, then we get a natural morphism

$$\operatorname{Spec} S_\bullet \setminus V(S_+) \longrightarrow \operatorname{Proj} S_\bullet.$$

$\square$

Remark Why $V(S_+)$ is a single point, and should reasonably be called the origin?

Since $S_0 = k$ is a field, $S_+ = \bigoplus\limits_{i \geq 1} S_i$ is maximal ideal of $S_\bullet$, $V(S_+)$ is a single point. This point is called "the origin" because it is the location where all elements of positive degree, i.e., elements in S_+, vanish.

This readily generalizes to the following proposition, which again motivates the terminology "irrelevant".

Proposition 10.3.9

If $S_\bullet$ *is a finitely generated graded ring over a base ring* A*, then there is a natural morphism*

$$\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet.$$

Proof Same as Proposition 10.3.8. $\square$

In fact, it can be made precise that $\operatorname{Proj} S_\bullet$ is the quotient (by the multiplicative group of scalars) of the affine cone minus the origin.

Definition 10.3.5 (Projective cone)

*The **projective cone** of* $\operatorname{Proj} S_\bullet$ *is* $\operatorname{Proj} S_\bullet[T]$*, where* T *is a new variable of degree* 1*. The projective cone is sometimes called the **projective completion** of* $\operatorname{Spec} S_\bullet$*. (Note: this depends on* $S_\bullet$*, not just on* $\operatorname{Proj} S_\bullet$*.)*

Example 10.4 The cone corresponding to the conic $\operatorname{Proj} k[x,y,z]/(z^2-x^2-y^2)$ is $\operatorname{Proj} k[x,y,z,T]/(z^2-x^2-y^2)$.

Proposition 10.3.10 (Less important)

The "projective cone" $\operatorname{Proj} S_\bullet[T]$ *of* $\operatorname{Proj} S_\bullet$ *has a closed subscheme isomorphic to* $\operatorname{Proj} S_\bullet$ *(informally, corresponding to* $T=0$*), whose complement (the distinguished open set* $D_+(T) \hookrightarrow \operatorname{Proj} S_\bullet[T]$*) is isomorphic to the affine cone* $\operatorname{Spec} S_\bullet$*.*

Proof Clearly, $S_\bullet \cong S_\bullet[T]/(T)$, hence $\operatorname{Proj} S_\bullet \cong \operatorname{Proj} S_\bullet[T]/(T)$. Note that we have surjective $S_\bullet[T] \twoheadrightarrow S_\bullet[T]/(T)$, by Proposition 10.3.1, it induces closed embedding $\operatorname{Proj} S_\bullet \hookrightarrow \operatorname{Proj} S_\bullet[T]$. In fact, distinguished open set $D_+(T) \hookrightarrow \operatorname{Proj} S_\bullet[T]$ is $D_+(T) \cong \operatorname{Spec}((S_\bullet[T])_T)_0$. Define $\varphi : S_\bullet \to (S_\bullet[T]_T)_0$ by setting

$$f_d \longmapsto \frac{f_d}{T^d},$$

where $f_d \in S_d$. It is easy to check that φ gives an isomorphism of rings. Hence $\operatorname{Spec} S_\bullet \cong D_+(T)$, as we desired. $\square$

This construction can be usefully pictured as the affine cone union some points "at infinity", and the points at infinity form the Proj. (The reader may wish to start with Figure 10.4, and try to visualize the conic curve "at infinity", and then compare this visualization to Figure 5.10.)

We have thus completely described the algebraic analog of the classical picture of §5.5.1.

Definition 10.3.6 (Weighted projective space)

If we put a nonstandard weighting on the variables of $k[x_0,\dots,x_n]$ *— say we give* $\deg x_i = d_i$ *— then* $\operatorname{Proj} k[x_0,\dots,x_n]$ *is called **weighted projective space*** $\mathbb{P}(d_0,d_1,\dots,d_n)$*.*

Exercise 10.9

(a) Show that $\mathbb{P}(m,n)$ is isomorphic to $\mathbb{P}^1$.

(b) Show that $\mathbb{P}(1,1,2) \cong \operatorname{Proj} k[u,v,w,z]/(uw-z^2)$.

Proof

(a) Suppose $R_\bullet = k[t_0,t_1]$ where $\deg t_i = 1$. Let $x \in R_m$ and $y \in R_n$, say $S_\bullet = k[x,y]$, then $\mathbb{P}(m,n) = \operatorname{Proj} S_\bullet$. Denote $\gcd(m,n)=d$, then $m=dm'$ and $n=dn'$ where $\gcd(m',n')=1$. Consider $S_{m'n'\bullet}$, note that $x^{n'}, y^{m'} \in S_{m'n'}$, we want to show that $S_{m'n'\bullet} \cong k[x^{n'},y^{m'}]$. It suffices to show that any monomial in $S_{m'n'\bullet}$ is generated by $x^{n'}$ and $y^{m'}$. Suppose $x^a y^b \in S_{km'n'}$, then we have $am+bn = km'n'$. Since $\gcd(m',n')=1$, $m'|b$ and $n'|a$, say $a=n'a'$ and $b=m'b'$, and therefore

$$x^a y^b = (x^{n'})^{a'}(y^{m'})^{b'}.$$

It follows that $x^a y^b$ is generated by $x^{n'}$ and $y^{m'}$. Hence $S_{m'n'\bullet} \cong k[x^{n'},y^{m'}]$. Since $k[x^{n'},y^{m'}]$ is finitely generated in degree 1, we have $k[x^{n'},y^{m'}] \cong k[t_0,t_1]$, by Proposition 8.4.3, we have

$$\mathbb{P}(m,n) = \operatorname{Proj} S_\bullet \cong \operatorname{Proj} S_{m'n'\bullet} \cong \operatorname{Proj} R_\bullet = \mathbb{P}^1.$$

(b) Suppose $\mathbb{P}(1,1,2) = \operatorname{Proj} S_\bullet := \operatorname{Proj} k[r,s,t]$ where $\deg r = \deg s = 1$ and $\deg t = 2$. Consider Veronese subring $S_{2\bullet} = k[r^2,t,s^2,rs]$, then $S_{2\bullet}$ is finitely generated in degree 1. Define homomorphism of graded rings $\varphi : k[u,v,w,z] \to S_{2\bullet}$ by setting

$$(u,v,w,z) \longmapsto (r^2, t, s^2, rs).$$

Clearly, φ is surjective and $\operatorname{Ker}\varphi = (z^2 - uw)$. By the Fundamental Theorem of Isomorphism, we have $S_{2\bullet} \cong k[u,v,w,z]/(vw - z^2)$, and therefore

$$\mathbb{P}(1,1,2) = \operatorname{Proj} S_\bullet \cong \operatorname{Proj} S_{2\bullet} \cong \operatorname{Proj} k[u,v,w,z]/(uw - z^2).$$

□

Remark $\mathbb{P}(1,1,2)$ is a projective cone over a conic curve $\operatorname{Proj} k[u,v,w]/(uw - v^2)$. Over a field of characteristic not 2, it is isomorphic to the traditional cone $x^2 + y^2 = z^2$ in $\mathbb{P}^3$, see Figure 10.4.

Proposition 10.3.11

$\mathbb{P}(1,1,n)$ is isomorphic to the projective cone over the degree n rational normal curve.

Proof Let $\mathbb{P}(1,1,n) = \operatorname{Proj} S_\bullet := \operatorname{Proj} k[x,y,z]$ where $\deg x = \deg y = 1$ and $\deg z = n$. Consider Veronese subring $S_{n\bullet} = k[x^n, x^{n-1}y, \ldots, y^n, z]$, $S_{n\bullet}$ is finitely genenrated in degree 1, by Proposition 8.4.3, we have

$$\mathbb{P}(1,1,n) = \operatorname{Proj} S_\bullet \cong \operatorname{Proj} S_{n\bullet}.$$

Note that $S_{n\bullet} = k[x^n, x^{n-1}y, \ldots, y^n][z]$, say $R_\bullet = k[x,y]$, then $S_{n\bullet} = R_{n\bullet}[z]$. Hence $\mathbb{P}(1,1,n) \cong \operatorname{Proj} R_{n\bullet}[z]$, it follows that $\mathbb{P}(1,1,n)$ is isomorphic to the projective cone over the degree n rational normal curve (Proposition 10.3.5). □

10.4 The (closed sub)scheme-theoretic image

We now define a series of notions that are all of the form "the smallest closed subscheme such that something or other is true". One example will be th enotion of scheme-theoretic closure of a locally closed embedding, which will allow us to interpret locally closed embeddings in three equivalent ways (open subscheme intersect closed subscheme; open subscheme of closed subscheme; and closed subscheme of open subscheme — cf. Proposition 10.2.3).

10.4.1 Scheme-theoretic image

We start with the notion of scheme-theoretic image. Set-theoretic images are badly behaved in general (§9.4.1), and even with reasonable hypotheses such as those in Chevalley's Theorem 9.4.1, things can be confusing. For example, there is no reasonable way to impose a scheme structure on the image of $\mathbb{A}_k^2 \to \mathbb{A}_k^2$ given by $(x,y) \mapsto (x, xy)$ (the set-theoretic image of this map is $D(u) \cup \{(0,0)\}$). It will be useful (e.g., Proposition 10.4.3) to define a notion of a closed subscheme of the target that "best approximates" the image. This will incorporate the notion that the image of something with non-reduced structure ("fuzz") can also have non-reduced structure. As usual, we will need to impose reasonable hypotheses to make this notion behave well (see Theorem 10.4.1 and Corollary 10.4.1).

Definition 10.4.1 (Scheme-theoretic image)

Suppose $i : Z \hookrightarrow Y$ is a closed subscheme, giving an exact sequence

$$0 \longrightarrow \mathscr{I}_{Z/Y} \longrightarrow \mathscr{O}_Y \longrightarrow i_*\mathscr{O}_Z \longrightarrow 0.$$

*We say that **the image of** $\pi : X \to Y$ **lies in** Z if the composition $\mathscr{I}_{Z/Y} \to \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is zero.*
*If the image of π lies in some subschemes Z_j (as j runs over some index set), it clearly lies in their intersection (cf. Proposition 10.1.8 on intersections of closed subschemes). We then define the **scheme-theoretic image of** π, a closed subscheme of Y, as the "smallest closed subscheme containing the image", i.e., the intersection of all closed subschemes containing the image. In particular (and in our first example), if Y is affine, the scheme-theoretic image is cut out by functions on Y that are 0 when pulled back to X.*

Remark Why we define "the image of $\pi : X \to Y$ lies in Z" this way?

Informally, locally, functions vanishing on Z pull back to the zero function on X.

Note *Other reasonable names for scheme-theoretic image are the **closed subscheme-theoretic image**, or **image closed subscheme**. They have the advantage that they remind the reader that the scheme-theoretic image is a closed subscheme, but we won't use these phrases.*

Example 10.5 Consider $\pi : \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} k[x] = \mathbb{A}_k^1$ given by $x \mapsto \varepsilon$. Then the scheme-theoretic image of π is given by $\operatorname{Spec} k[x]/(x^2)$ (the polynomials pulling back to 0 are precisely (x^2)). Thus the image of the fuzzy point still has some fuzz.

Example 10.6 Consider $\pi : \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} k[x] = \mathbb{A}_k^1$ given by $x \mapsto 0$. Then the scheme-theoretic image is given by $\operatorname{Spec} k[x]/(x)$: the image is reduced (Definition 4.2.3). In this picture, the fuzz is "collapsed" by π.

Example 10.7 Consider $\pi : \operatorname{Spec} k[t, t^{-1}] = \mathbb{A} - \{0\} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ given by $u \mapsto t$. Any function $g(u)$ which pulls back to 0 as a function of t must be the zero-function. Thus the scheme-theoretic image is everything. the set-theoretic image, on the other hand, is the distinguished open set $\mathbb{A}^1 \setminus \{0\}$. Thus in not -too-pathological cases, the underlying set of the scheme-theoretic image is not the set-theoretic image. But the situation isn't terrible: the underlying set of the scheme-theoretic image must be closed, and indeed it is the closure of the set-theoretic image. We might imagine that in reasonable cases this will be true, and in even nicer cases, the underlying set of the scheme-theoretic image will be the set-theoretic image. We will later see that this is indeed the case.

But sadly pathologies can sometimes happen in, well, pathological situations.

Example 10.8 (Figure 10.6) Let $X = \coprod \operatorname{Spec} k[\varepsilon_n]/((\varepsilon_n)^n)$ (a scheme which appeared in the hint to Example 6.4) and $Y = \operatorname{Spec} k[x]$, and define $X \to Y$ by $x \mapsto \varepsilon_n$ on the n-th component of X (in fact, $\Gamma(X, \mathscr{O}_X) = \prod_n k[\varepsilon_n]/((\varepsilon_n)^n)$). If a function $g(x)$ on Y pulls back to 0 on X, then its Taylor extension is 0 to order n (by examining the pullback to the n-th component of X) for all n, so $g(x)$ must be 0. (This argument will be vastly generalized in Chapter 14). Thus the scheme-theoretic image is $V(0)$ on Y, i.e., Y itself, while the set-theoretic image is easily seen to be just the origin (the closed point 0). (This morphism implicitly arises in Caution/Example 10.10 .)

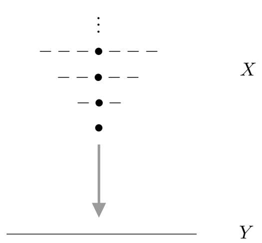


Figure 10.6: Yuck (Example 10.8)

10.4.2 Criteria for computing scheme-theoretic images affine-locally

Example 10.8 clearly is weird though, and we can show that in "reasonable circumstances" such pathology doesn't occur.

In the special case when the target Y is affine (pay attention to how this argument works in Example 10.5 through 10.7), say $Y = \operatorname{Spec} B$, almost by definition the scheme-theoretic image of $\pi : X \to Y$ is cut out by the ideal $I \subseteq B$ of functions on Y which pull back to zero on X.

It would be great to use this to compute the scheme-theoretic image affine by affine by affine (affine-locally). On the affine open set $\operatorname{Spec} B \subseteq Y$, define the ideal $I(B) \subseteq B$ of functions which pull back to 0 on X. Formally, $I(B) := \operatorname{Ker}(B \to \Gamma(\operatorname{Spec} B, \pi_* \mathscr{O}_X))$. If for each such B, and each $g \in B$, the map $\varphi : I(B)_g \to I(B_g)$ is an isomorphism, then we will have defined the scheme-theoretic image as a closed subscheme (see Proposition 10.1.6, i.e., show that the sheaf $\mathscr{I}(\operatorname{Spec} B) := I(B)$ form a quasi-coherent sheaf, Theorem 7.2.2). Injectivity of φ is straightforward: let $\varphi(\frac{r}{g^n}) = \frac{r}{g^n} = 0$ where $r \in I(B)$, then there exists some m such that $g^m r = 0$ in B, since $r \in I(B)$, we have $\frac{r}{g^n} = 0$ in $I(B)_g$, it follows that φ is injective.

So the question is the surjectivity of φ, which translates to: given a function $\frac{r}{g^n}$ on $D(g)$ that pulls back to zero on $\pi^{-1}(D(g))$, i.e., $\pi^\sharp(\frac{r}{g^n}) = \frac{\pi^\sharp(r)}{\pi^\sharp(g)^n} = 0$ in $\pi^{-1}(D(g))$, we want to show that $\frac{r}{g^n}$ in $I(B)_g$, i.e., show that $rg^m \in I(B)$ for some m, i.e., show that $\pi^\sharp(rg^m) = 0$ on $\pi^{-1}(\operatorname{Spec} B)$.

(i) We first show that the answer is yes if $\pi^{-1}(\operatorname{Spec} B)$ is reduced:
In this case we can take advantage of Proposition 6.2.1, that a function on a reduced scheme is zero if it has value zero at every point. We may take $m = 1$ (as $\pi^\sharp(r)$ vanishes on $\pi^{-1}(D(g))$ and g vanishes on $V(g)$, so $\pi^\sharp(rg)$ vanishes on $\pi^{-1}(\operatorname{Spec} B) = \pi^{-1}(D(g)) \cup \pi^{-1}(V(g))$.)

(ii) The answer is also yes if $\pi^{-1}(\operatorname{Spec} B)$ is affine, say $\operatorname{Spec} A$:
if $r' = \pi^\sharp(r)$ and $g' = \pi^\sharp(g)$ in A, then if $r'/g'^n = 0$ on $D(g')$, then there is an m such that $r'(g')^m = 0$ (as the statement $r'/g'^n = 0$ in $D(g')$ means precisely this fact — the functions on $D(g')$ are $A_{g'}$).

(iii) More generally, the answer is yes if $\pi^{-1}(\operatorname{Spec} B)$ is quasi-compact:
Cover $\pi^{-1}(\operatorname{Spec} B)$ with finitely many affine open sets U_i. For each one there will be some m_i such that $\pi|_{U_i}^\sharp(rg^{m_i}) = 0$ on U_i. Then let $m = \max\{m_i\}$, we have $\pi^\sharp(rg^m) = 0$ on $\pi^{-1}(\operatorname{Spec} B)$. (We see again that quasi-compactness is our friend!)

In conclusion, we have proved the following (subtle) theorem.

Theorem 10.4.1

Suppose $\pi : X \to Y$ is a morphism of schemes. If X is reduced or π is quasi-compact, then the scheme-theoretic image of π may be computed affine-locally: on $\operatorname{Spec} A \subseteq Y$, the scheme-theoretic image of X is cut out by the functions (elements of A) that pull back to the function 0 (on $\pi^{-1}(\operatorname{Spec} A)$).

Corollary 10.4.1

Suppose $\pi : X \to Y$ is a morphismof schemes. If X is reduced or π is quasi-compact, then the closure of the set-theoretic image of π is the underlying set of the scheme-theoretic image.

(Example 10.8 above shows how we need these hypotheses.)

Remark In particular, if the set-theoretic image is closed (e.g., if π is finite (Corollary 9.3.2) or projective), the set-theoretic image is the underlying set of the scheme-theoretic image, as promised in Example 10.7

Proof of Corollary 10.4.1

Proof By definition of scheme-theoretic image (Definition 10.4.1), we know that the set-theoretic image is in the underlying set of the scheme-theoretic image, i.e., $\pi(X) \subseteq \pi(X)_{\mathbf{Sch}}$ where $\pi(X)_{\mathbf{Sch}}$ means scheme-theoretic image of X. The underlying set of the scheme-theoretic image is closed, so the closure of the set-theoretic image is contained in the underlying set of the scheme-theoretic image, i.e., $\mathrm{cl}_Y(\pi(X)) \subseteq \pi(X)_{\mathbf{Sch}}$. On the other hand, if U is the complement of the closure of the set-theoretic image, $\pi^{-1}(U) = \varnothing$. Under these hypotheses, the scheme theoretic image can be computed locally (Theorem 10.4.1), so the scheme-theoretic image is the empty set on U, i.e., $\pi(X)_{\mathbf{Sch}} \cap U = \varnothing$. Hence $\pi(X)_{\mathbf{Sch}} \subseteq \mathrm{cl}_Y(\pi(X))$, we done. □

We conclude with a few stray remarks.

Proposition 10.4.1

If X is reduced, then the scheme-theoretic image of $\pi : X \to Y$ is also reduced.

Proof Since the scheme-theoretic image of π may be computed affine-locally (Theorem 10.4.1), we may assume that $Y = \operatorname{Spec} B$, then the scheme-theoretic image of π is $\mathrm{cl}_Y(\pi(X)) = \operatorname{Spec} B/I$, where $I = \operatorname{Ker}(B \to \Gamma(\operatorname{Spec} B, \pi_*\mathscr{O}_X))$. Since X is reduced, by Proposition 6.2.3, $\Gamma(\operatorname{Spec} B, \pi_*\mathscr{O}_X)$ is reduced. By the Fundamental Theorem of Isomorphism, we have

$$B/I \cong \operatorname{Im}(B \to \Gamma(\operatorname{Spec} B, \pi_*\mathscr{O}_X)) \subseteq \Gamma(\operatorname{Spec} B, \pi_*\mathscr{O}_X),$$

since $\Gamma(\operatorname{Spec} B, \pi_*\mathscr{O}_X)$ is reduced, B/I is reduced, by Proposition 6.2.2, the scheme-theoretic image $\mathrm{cl}_Y(\pi(X))$ is reduced. □

More generally, you might expect there to be non unnecessary non-reduced structure on the image not forced by non-reduced structure on the source. The following makes this precise in the locally Noetherian case (when we can talk about associated points.)

Proposition 10.4.2 (Unimportant)

If $\pi : X \to Y$ is a quasi-compact morphism of locally Noetherian schemes, then the associated points of the image subscheme are a subset of the image of the associated points of X.

Proof Since the scheme-theoretic image of π may be computed affine-locally (Theorem 10.4.1), we may assueme that $Y = \operatorname{Spec} B$ where B is Noetherian ring. Since π is quasi-compact, we know that $X = \pi^{-1}(\operatorname{Spec} B)$ is quasi-compact, since X is locally Noetherian, X is Noetherian. Hence we may assume that $X = \operatorname{Spec} A$ where A is Noetherian, and therefore we reduced to the case that $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an affine morphism of Noetherian scheme.

By Theorem 10.4.1, the scheme-theoretic image of π is $\mathrm{cl}_Y(\pi(X)) = \operatorname{Spec} B/I$ where $I = \operatorname{Ker}(B \to A)$. We want to show that $\operatorname{Ass}_{B/I}(B/I) \subseteq \pi(\operatorname{Ass}_A(A))$. Let $\mathfrak{p} = \operatorname{Ann}_{B/I}(b) \in \operatorname{Ass}_{B/I}(B/I)$. By the Fundamental Theorem of Isomorphism, we have $B/I \cong \pi^\sharp(B)$, then $\pi^\sharp : B \to A$ factors as

$$\begin{tikzcd} B \arrow[rr, "\pi^\sharp"] \arrow[dr, two heads, "\mathrm{pr}"'] & & A \\ & B/I. \arrow[ur, hook, "i"'] & \end{tikzcd}$$

We may look B/I as a subring of A, then i is inclusion. Hence $\mathfrak{p} = \operatorname{Ann}_A(b) \cap B/I$. Inclusion $i : B/I \hookrightarrow A$ induces ring map $B/I \to A/\operatorname{Ann}_A(b)$, consider $\operatorname{Ker}(B/I \to A/\operatorname{Ann}_A(b))$, note that $\operatorname{Ker}(B/I \to A/\operatorname{Ann}_A(b)) = B/I \cap \operatorname{Ann}_A(b) = \mathfrak{p}$, then we get an injective

$$(B/I)/\mathfrak{p} \longhookrightarrow A/\operatorname{Ann}_A(b),$$

where $(B/I)/\mathfrak{p}$ is integral domain. We claim that there is a prime ideal of $A/\operatorname{Ann}_A(b)$ such that $(0) =$

$\mathfrak{q} \cap (B/I)/\mathfrak{p}$. Let

$$\Sigma = \{\text{ideal } J \subseteq A/\operatorname{Ann}_A(b) : J \cap (B/I)/\mathfrak{p} = (0)\}$$

ordered by the relation $\subseteq$. Since A is Noetherian, $A/\operatorname{Ann}_A(b)$ is Noetherian, hence Σ has maximal element, say $\mathfrak{q}$. We want to show that $\mathfrak{q}$ is prime. Suppose $ab \in \mathfrak{q}$ with $0 \neq a \notin \mathfrak{q}$ and $0 \neq b \notin \mathfrak{q}$, Consider $\mathfrak{q}+(a)$ and $\mathfrak{q}+(b)$, since $\mathfrak{q} \subsetneq \mathfrak{q}+(a), \mathfrak{q}+(b)$, $(\mathfrak{q}+(a)) \cap (B/I)/\mathfrak{p} \neq 0$ and $(\mathfrak{q}+(b)) \cap (B/I)/\mathfrak{p} \neq 0$. Let $r \in (\mathfrak{q}+(a)) \cap (B/I)/\mathfrak{p}$ and $r' \in (\mathfrak{q}+(b)) \cap (B/I)/\mathfrak{p}$. Since $ab \in \mathfrak{q}$, we have

$$rr' \in (\mathfrak{q}+(a))(\mathfrak{q}+(b)) \cap (B/I)/\mathfrak{p} = \mathfrak{q} \cap (B/I)/\mathfrak{p} = 0.$$

Since $(B/I)/\mathfrak{p}$ is integral domain, we have $r = 0$ or $r' = 0$, a contradiction. Hence $\mathfrak{q}$ is prime. We now lift $\mathfrak{q}$ to A, also say $\mathfrak{q}$, then $\mathfrak{q} \supseteq \operatorname{Ann}_A(b)$ and $\mathfrak{p} = \mathfrak{q} \cap B/I$. Since $\mathfrak{q} \in \operatorname{Spec} A$, $\mathfrak{q} \in \operatorname{Ass}_A(\operatorname{Ann}_A(b))$, by Proposition 7.6.7, we have $\operatorname{Ass}_A(\operatorname{Ann}_A(b)) \subseteq \operatorname{Ass}_A(A)$, hence $\mathfrak{q} \in \operatorname{Ass}_A(A)$.

Note that

$$\pi([\mathfrak{q}]) = (\pi^\sharp)^{-1}(\mathfrak{q}) = \operatorname{pr}^{-1}(\mathfrak{q} \cap B/I) = \operatorname{pr}^{-1}(\mathfrak{p}),$$

also note that point $[\operatorname{pr}^{-1}(\mathfrak{p})]$ can be seen as point $[\mathfrak{p}]$ in $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$, we proved that $\operatorname{Ass}_{B/I}(B/I) \subseteq \pi(\operatorname{Ass}_A(A))$. □

Remark Intuitively, this proposition shows that a quasi-compact morphism "preserves" associated points: the geometric features of the scheme-theoretic image (described by its associated points) are entirely determined by those of the source scheme.

Example 10.9 (Example of Proposition 10.4.2 not work) Consider $\pi : \coprod_{a\in\mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t-a) \to \operatorname{Spec} \mathbb{C}[t]$, by Definition 7.6.5, $\operatorname{Ass}(X) = \{((0), a) : a \in \mathbb{C}\}$. Since $\coprod_{a\in\mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t-a)$ is reduced, by Corollary 10.4.1, the scheme-theoretic image of π is $\operatorname{Spec} \mathbb{C}[t]/(0) = \operatorname{Spec} \mathbb{C}[t]$, and $\operatorname{Ass}(\operatorname{Spec} \mathbb{C}[t]) = \{[(0)]\}$. Now consider $\pi(\operatorname{Ass}(X))$, in fact, $\pi(\operatorname{Ass}(X)) = \{[(t-a)] \in \operatorname{Spec} \mathbb{C}[t] : a \in \mathbb{C}\}$. But $\operatorname{Ass}(\operatorname{Spec} \mathbb{C}[t]) \nsubseteq \pi(\operatorname{Ass}(X))$. Hence this example shows that can go wrong if you give up quasi-compactness — note that reducedness of the source doesn't help.

10.4.3 Scheme-theoretic closure of a locally closed subscheme

Definition 10.4.2 (Scheme-theoretic closure of locally closed embedding)

*We define the **scheme-theoretic closure** of a locally closed embedding $\pi : X \hookrightarrow Y$ as the scheme-theoretic image of π.*

Remark A shorter phrase for this is **schematic closure**, although this more elegant nomenclature has not caught on.

Proposition 10.4.3

If a locally closed embedding $V \hookrightarrow X$ is quasi-compact (e.g., if V is Noetherian, Proposition 9.3.2 (a)), or if V is reduced, then (iii) implies (i) and (ii) in Proposition 10.2.3. Thus in this fortunate situation, a locally closed embedding can be thought of in three different ways, whichever is convenient.

Proof Let Z be the scheme-theoretic image of $\pi : V \hookrightarrow X$, by Corollary 10.4.1, we have $Z = \operatorname{cl}_X(\pi(V))$. Since $\pi : V \hookrightarrow X$ is locally closed embedding, π factors as

$$V \xhookrightarrow[\text{closed}]{\tau} U \xhookrightarrow[\text{open}]{i} X,$$

where $i : U \hookrightarrow X$ is inclusion. Since $V \hookrightarrow U$ is closed embedding, we have $\pi(V) = U \cap \operatorname{cl}_X(\pi(V))$, which implies that $\pi(V)$ is open in $\mathrm{cl}_X(\pi(V))$. Since $Z = \mathrm{cl}_X(\pi(V))$ is a scheme, $\pi(V)$ is an open subscheme of Z, hence $V \hookrightarrow Z = \mathrm{cl}_X(\pi(V))$ is open embedding, i.e., (iii) implies (ii). By Proposition 10.2.3, (i) $\Leftrightarrow$ (ii), we done. $\square$

Proposition 10.4.4 (Unimportant, but useful for intuition)

If $\pi : X \hookrightarrow Y$ is a locally closed embedding into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic closure are (naturally in bijection with) the associated points of X.

Informally, the only non-reduced structure on the scheme-theoretic closure is that "forced by" non-reduced structure on X.

Proof We first show that locally closed embedding $\pi : X \hookrightarrow Y$ is quasi-compact. We may assume that π is inclusion. Since $\pi : X \hookrightarrow Y$ is locally closed embedding, π factors as

$$X \overset{\tau}{\hookrightarrow} U \overset{i}{\hookrightarrow} Y,$$

where τ is closed embedding and i is open embedding. Let $\operatorname{Spec} B \subseteq Y$ be an affine open subscheme of Y, then $\pi^{-1}(\operatorname{Spec} B) = X \cap \operatorname{Spec} B$. Since Y is locally Noetherian scheme, $\operatorname{Spec} B$ is Noetherian, by Proposition 4.6.19, $X \cap \operatorname{Spec} B$ is quasi-compact. Hence $\pi : X \hookrightarrow Y$ is quasi-compact. By Proposition 10.4.2, we have $\mathrm{Ass}(\mathrm{cl}_Y(\pi(X))) \subseteq \pi(\mathrm{Ass}(X))$, i.e., $\mathrm{Ass}(\mathrm{cl}_Y(X)) \supseteq \mathrm{Ass}(X)$. Since $X \subseteq \mathrm{cl}_Y(X)$, by Proposition 7.6.7, $\mathrm{Ass}(X) \subseteq \mathrm{Ass}(\mathrm{cl}_Y(X))$. Hence $\mathrm{Ass}(\mathrm{cl}_Y(X)) = \mathrm{Ass}(X)$, we done. $\square$

10.4.4 The (reduced) subscheme structure on a closed subset

Suppose X^{set} is a closed subset of a scheme Y. Then we can define a canonical scheme structure X on X^{set} that is reduced. We could describe it as being cut out by those functions whose values are zero at all the points of X^{set}. On the affine open set $\operatorname{Spec} B$ of Y, if the set X^{set} corresponds to the radical ideal $I = I(X^{\mathrm{set}} \cap \operatorname{Spec} B)$ (recall the $I(\cdot)$ function from §4.7), the scheme X corresponds to $\operatorname{Spec} B/I$. We check that this behaves well with respect to any distinguished inclusion $\operatorname{Spec} B_f \hookrightarrow \operatorname{Spec} B$:

Lemma 10.4.1

Let X^{set} is a closed subset of a scheme Y, $\operatorname{Spec} B \subseteq Y$ be an affine open subset of Y, let $I = I(X^{set} \cap \operatorname{Spec} B)$, then for any distinguished open subset $\operatorname{Spec} B_f \hookrightarrow \operatorname{Spec} B$, there is an isomorphism

$$\operatorname{Spec}(B/I)_f \cong \operatorname{Spec} B_f/I',$$

where $I' = I(X^{set} \cap \operatorname{Spec} B_f)$.

Proof It suffices to show that $(B/I)_f \cong B_f/I'$. Since localization commutes with quotient, we have $(B/I)_f \cong B_f/IB_f$, hence it suffices to show that $I' = IB_f$.

Let $\frac{g}{f^n} \in IB_f$, where $g \in I$. Since $g \in I$, for all $p \in X^{\mathrm{set}} \cap \operatorname{Spec} B$, $g(p) = 0$, in particular, $g(p) = 0$ for all $p \in X^{\mathrm{set}} \cap \operatorname{Spec} B_f$. Since $p \in \operatorname{Spec} B_f$, we have $f \notin \operatorname{Spec} B_f$, i.e., $f(p) \neq 0$, hence $\frac{g}{f^n} \in I'$, i.e., $IB_f \subseteq I'$.

Conversely, let $\frac{g}{f^n} \in I'$. To show $\frac{g}{f^n} \in IB_f$, it suffices to show that $f^m g \in I = \bigcap_{[\mathfrak{p}] \in X^{\mathrm{set}} \cap \operatorname{Spec} B} \mathfrak{p}$ for some m. If $[\mathfrak{p}] \in X^{\mathrm{set}} \cap \operatorname{Spec} B_f$, then $g([\mathfrak{p}]) = 0$ and $f([\mathfrak{p}]) = 0$, hence $f^m g([\mathfrak{p}]) = 0$ for all m. If $[\mathfrak{p}] \in X^{\mathrm{set}} \cap (\operatorname{Spec} B - \operatorname{Spec} B_f)$, then $f([\mathfrak{p}]) = 0$. Pick $m = 1$, we have $fg(p) = 0$ for all $p \in X^{\mathrm{set}} \cap \operatorname{Spec} B$, hence $I' \subseteq IB_f$.

By above discussion, we know that $I' = IB_f$, and therefore $\operatorname{Spec}(B/I)_f \cong \operatorname{Spec} B_f/I'$. $\square$

By Lemma 10.4.1 and Theorem 7.2.2, $\mathscr{I}(\operatorname{Spec} B) = I(X^{\text{set}} \cap \operatorname{Spec} B)$ gives an ideal sheaf. We give the first definition of the reduced subscheme structure on the closed subset X^{set}.

Definition 10.4.3 (The reduced subscheme structure on the closed subset)

*Let X^{set} be a closed subset of a scheme Y. We define **the reduced subscheme structure** on X^{set} be the closed subscheme which produced by ideal sheaf*

$$\mathscr{I}(\operatorname{Spec} B) = I(X^{set} \cap \operatorname{Spec} B)$$

for any $\operatorname{Spec} B \subseteq Y$.

We also consider this construction as an example of a scheme-theoretic image in the following crazy way: let W be the scheme that is a disjoint union of all the points of X^{set}, where the point corresponding to p in X^{set} is Spec of the residue field of $\mathscr{O}_{Y,p}$, i.e., $W = \coprod_{p \in X^{\text{set}}} \operatorname{Spec} \kappa(p)$. Let $\rho : W \to Y$ be the "canonical" map sending "p to p", and giving an isomorphism on residue fields. Then the scheme structure on X is the scheme-theoretic image of ρ. Then we give the second definition of the reduced subscheme structure on the closed subset X^{set}.

Definition 10.4.4 (The reduced subscheme structure on the closed subset)

*Let X^{set} be a closed subset of a scheme Y. Suppose $W = \coprod_{p \in X^{set}} \operatorname{Spec} \kappa(p)$, define $\rho : W \to Y$ by setting $p \mapsto p$. We define **the reduced subscheme structure** X on X^{set} by setting X is the scheme-theoretic image of ρ.*

Finally, we give the third definition:

Definition 10.4.5 (The reduced subscheme structure on the closed subset)

*We define **the reduced subscheme structure** X on X^{set} by setting X is the smallest closed subscheme whose underlying set contains X^{set}.*

We next show that Definition 10.4.3, 10.4.4, and 10.4.5 are equivalent.

Proposition 10.4.5

Definition 10.4.3, 10.4.4, and 10.4.5 are equivalent.

Proof

(1) Definition 10.4.3 implies Definition 10.4.4.

It suffices to show that X is the scheme-theoretic image of ρ. Suppose $Y = \bigcup_i \operatorname{Spec} B_i$, then the scheme-theoretic image of ρ is $\bigcup_i \operatorname{Spec} B_i/I_i$, where $I_i = \operatorname{Ker}(B_i \to \Gamma(\operatorname{Spec} B_i, \rho_* \mathscr{O}_W))$. It suffices to show that $I_i = I(X^{\text{set}} \cap \operatorname{Spec} B_i)$. Consider $\Gamma(\operatorname{Spec} B_i, \rho_* \mathscr{O}_W)$, note that $\rho^{-1}(\operatorname{Spec} B_i) = \coprod_{p \in \operatorname{Spec} B_i \cap X^{\text{set}}} \operatorname{Spec} \kappa(p)$, we have

$$\Gamma(\operatorname{Spec} B_i, \rho_* \mathscr{O}_W) = \mathscr{O}_W(\rho^{-1}(\operatorname{Spec} B_i)) = \prod_{p \in \operatorname{Spec} B \cap X^{\text{set}}} \kappa(p).$$

We next consider $\operatorname{Ker}(B_i \to \prod_{p \in \operatorname{Spec} B \cap X^{\text{set}}} \kappa(p))$, $B_i \to \prod_{p \in \operatorname{Spec} B \cap X^{\text{set}}} \kappa(p)$ is given by $f \mapsto (f(p))_p$, hence $\operatorname{Ker}(B_i \to \prod_{p \in \operatorname{Spec} B \cap X^{\text{set}}} \kappa(p)) = I(X^{\text{set}} \cap \operatorname{Spec} B_i)$.

(2) Definition 10.4.4 implies Definition 10.4.5.

Note that $W = \coprod_{p \in X^{\text{set}}} \operatorname{Spec} \kappa(p)$ is reduced, by Corollary 10.4.1, the closure of the set-theoretic image of ρ is the underlying set of the scheme-theoretic image, i.e., $X = X^{\text{set}}$. It is clearly the smallest closed subscheme whose underlying set contains X^{set}.

(3) Definition 10.4.5 implies Definition 10.4.3.

Let X be the smallest closed subscheme whose underlying set contains X^{set}. For any $\operatorname{Spec} B$, functions on $X \cap \operatorname{Spec} B$ must vanish on $X^{\text{set}} \cap \operatorname{Spec} B$, hence X must produced by ideal sheaf $\mathscr{I}(\operatorname{Spec} B) = I(X^{\text{set}} \cap \operatorname{Spec} B)$, we done.

□

Definition 10.4.6 (The reduced subscheme structure on the closed subset)

The construction as Definition 10.4.3 or 10.4.4 or 10.4.5 is called the (induced) reduced subscheme structure on the closed subset X^{set}.

Proposition 10.4.6

Let Y be a scheme. Let $X^{set} \subseteq Y$ be a closed subset. Then there exists unique reduced subscheme structure X on the closed subset X^{set} with the following properties:

(a) the underlying topological space of X is equal to X^{set};

(b) X is reduced.

Proof We use the first definition, i.e., Definition 10.4.3. For any affine open $\operatorname{Spec} B \subseteq Y$, we have $X \cap \operatorname{Spec} B = \operatorname{Spec} B/I$, where $I = I(X^{\text{set}} \cap \operatorname{Spec} B)$. We want to show that B/I is reduced. Consider $\mathfrak{N}(B/I)$, let $f + I \in \mathfrak{N}(B/I)$, then $f^n \in I$ for some n. Since I is radical ideal, we have $f \in I$. Hence $\mathfrak{N}(B/I) = 0$, B/I is reduced. By Proposition 6.2.2, $X \cap \operatorname{Spec} B$ is reduced. Hence X is reduced. By the construction of X, the underlying closed subset of X must be X^{set}.

We next show the uniqueness. Let $X' \subseteq Y$ be a second reduced subscheme structure on the closed subset X^{set}. By the definition, for any affine open $\operatorname{Spec} B \subseteq Y$, we have $X' \cap \operatorname{Spec} B = \operatorname{Spec} B/I'$ for some ideal $I' \subseteq B$. By Proposition 6.2.2, B/I' is reduced, and therefore I' is radical. Since $V(I') = X^{\text{set}} \cap \operatorname{Spec} B = V(I)$, by Hilbert's Nullstellensatz 4.7.1, $I = I'$. Hence X and X' are defined by the same ideal sheaf and hence are equal. □

10.4.5 Reduced version of a scheme

Consider the case where X^{set} is all of Y, we obtain a reduced closed subscheme which is called the reduction of Y:

Definition 10.4.7

Let Y be a scheme, by Proposition 10.4.6, we obtain a reduced closed subscheme $Y^{\text{red}} \to Y$, called the ***reduction*** *of Y. On the affine open subset $\operatorname{Spec} B \hookrightarrow Y$, $Y^{\text{red}} \hookrightarrow Y$ corresponds to (the quotient by) the nilradical $\mathfrak{N}(B) \subseteq B$.*

Remark The reduction of a scheme is the "reduced version" of the scheme, and informally corresponds to "shearing off the fuzz".

Remark **An alternative equivalent definition of the reduction:** on the affine open subset $\operatorname{Spec} B \hookrightarrow Y$, the reduction of Y corresponds to the nilradical $\mathfrak{N}(B) \subseteq B$. By Proposition 7.2.2, for any $f \in B$, $\mathfrak{N}(B)_f = \mathfrak{N}(B_f)$, by Theorem 7.2.2, $\mathscr{I}(\operatorname{Spec} B) := \mathfrak{N}(B)$ form a quasi-coherent sheaf. By Proposition 10.1.6 this defines a closed subscheme, which we call the reduction.

Example 10.10 (Caution/Example) It is not true that for *every open subset* $U \subseteq Y$, $\Gamma(U, \mathscr{O}_{Y^{\text{red}}})$ is $\Gamma(U, \mathscr{O}_Y)$ modulo its nilpotents. For example, on $Y = \coprod \operatorname{Spec} k[x]/(x^n)$, $\Gamma(Y, \mathscr{O}_Y) = \prod k[x]/(x^n)$, function $(x, \dots)$ is not nilpotent. $Y^{\text{red}} = \coprod \operatorname{Spec}(k[x]/(x^n))/\mathfrak{N}(k[x]/(x^n)) = \coprod \operatorname{Spec} k$, hence $(x, \dots)$ is 0 on Y^{red}, as it is "locally nilpotent". This may remind you of Example 10.8 .

10.5 Slicing by effective Cartier divisors, regular sequences and regular embeddings

We now introduce regular embeddings, an important class of locally closed embeddings, an important class of locally closed embeddings. Locally closed embeddings of regular schemes in regular schemes are one important example of regular embeddings (Chapter 13). Effective Cartier divisors, basically the codimension 1 case, will turn out to be repeatedly useful as well, with repeated "slicing by effective Cartier divisors" providing the inductive step in many arguments. (When we say **slicing** in many future contexts, we will mean slicing by effective Cartier divisors.)

10.5.1 Locally principle closed subschemes, and effective Cartier divisors

Definition 10.5.1 (Locally principal closed subscheme)

*A closed subscheme is **locally principal** if on each open set in a small enough open cover it is cut out by a single equation (i.e., by a principal ideal, hence the terminology). More specifically, a locally principal closed subscheme is a closed subscheme $\pi : X \hookrightarrow Y$ for which there is an open cover $\{U_i\}$ of Y for which for each i, $\pi^{-1}(U_i) \hookrightarrow U_i$ the closed subscheme $V(s_i)$ of U_i for some $s_i \in \Gamma(U_i, \mathscr{O}_X)$. (This open cover can clearly chosen to be by affine open subsets.)*

Example 10.11 Hypersurfaces in $\mathbb{P}^n_k$ (Definition 10.3.1) are locally principal: each homogeneous polynomial in $n+1$ variables defines a locally principal closed subscheme of $\mathbb{P}^n_A$. (Warnings: unlike "lcoal principality", "principality" is not an affine-local condition, see Chapter 20! Also, the example of a projective hypersurface, §10.3.1, shows that a locally principal closed subscheme need not be cut out by a globally-defined function. Finally, we unfortunately use the phrase "locally principal" in a different way in Chapter 16.)

Definition 10.5.2 (Effective Cartier divisor)

*If the ideal sheaf is locally generated near every point by a function that is not a zero-divisor, we call the closed subscheme an effective Cartier divisor. More precisely: if $\pi : X \to Y$ is a closed embedding, and there is a cover Y by affine open subsets $\operatorname{Spec} A_i \subseteq Y$, and there exists non-zerodivisors $t_i \in A_i$ with $V(t_i) = X|_{\operatorname{Spec} A_i}$ (scheme-theoretically — i.e., the ideal sheaf of X over $\operatorname{Spec} A_i$ is generated by t_i), then we say that X is **effective Cartier divisor on** Y. (We will not explain the origin of the phrase, as it is not relevant for this point of view.) We will often use the evocative phrase **slicing by an effective Cartier divisor** when we mean "restrict to an effective Cartier divisor".*

The notion of an effective Cartier divisor is central. The notion of a locally principal closed subscheme is much less so.

Proposition 10.5.1

Suppose $t \in A$ is a non-zerodivisor. Then t is a non-zerodivisor in $A_{\mathfrak{p}}$ for each prime $\mathfrak{p}$.

Proof Suppose t is a zerodivisor in $A_{\mathfrak{p}}$, then there exists $0 \neq \frac{a}{b} \in A_{\mathfrak{p}}$ such that $\frac{a}{b} \cdot \frac{t}{1} = \frac{at}{b} = 0$, hence there exists $s \in A - \mathfrak{p}$ such that $sat = 0$ in A. Since t is a non-zerodivisor in A, $sa = 0$. Hence $\frac{a}{b} = 0$ in $A_{\mathfrak{p}}$, a contradiction! $\square$

Remark Caution. If D is an effective Cartier divisor on an affine scheme $\operatorname{Spec} A$, it is not necessarily true that $D = V(t)$ for some $t \in A$ (see Chapter 16 — Chapter 20 gives a different flavor of example). In other words, the condition of a closed subscheme being an effective Cartier divisor can be verified on an affine cover, but cannot be checked on an arbitrary affine cover — it is not an affine-local condition in this obvious a way.

Proposition 10.5.2

Suppose X is a locally Noetherian scheme.

(a) If $t \in \Gamma(X, \mathscr{O}_X)$ is a function on X, then t (or more precisely the closed subscheme $V(t)$) is an effective Cartier divisor if and only if it doesn't vanish on any associated point of X.

(b) A locally principal closed subscheme of X is an effective Cartier divisor if and only if it doesn't contain any associated points of X.

Proof

(a) If $V(t)$ is an effective Cartier divisor. Suppose $X = \bigcup_i \operatorname{Spec} A_i$ where each A_i is Noetherian. Since $V(t)$ is an effective Cartier divisor, $V(t) = \bigcup_i \operatorname{Spec} A_i/(t_i)$ where $t_i = t|_{\operatorname{Spec} A_i}$ is non-zerodivisor in A_i. Suppose exists $p \in \operatorname{Ass}(X)$ such that $t(p) = 0$. Since X is covered by $\{\operatorname{Spec} A_i\}$, $p \in \operatorname{Spec} A_i$ for some i. Since $p \in \operatorname{Ass}(X)$, $\mathfrak{m}_p = pA_{ip} = \operatorname{Ann}(f)$ for some $f \in A_{ip}$ and the image of t in $A_{ip}/\operatorname{Ann}(f)$ is 0. Hence $\frac{t_i}{1} \in \operatorname{Ann}(f)$, this contradicts to the fact that t_i is non-zerodivisor.
Conversely, if t doesn't vanish on any associated point of X. Suppose $X = \bigcup_i \operatorname{Spec} A_i$ where A_i is Noetherian, then the vanish scheme can be written as $V(t) = \bigcup_i \operatorname{Spec} A_i/(t_i)$ where $t_i = t|_{\operatorname{Spec} A_i}$. Since t doesn't vanish on any associated point of X, t_i does't vanish on any associated point of $\operatorname{Spec} A_i$, note that A_i is Noetherian, by Proposition 7.6.11, t_i is not zerodivisor on B_i for all i. Hence vanish scheme $V(t)$ is effective Cartier divisor.

(b) Let locally principal closed subscheme D be an effective Cartier divisor on X. Suppose $X = \bigcup_i \operatorname{Spec} A_i$ where each A_i is Noetherian, then D can be written as $D = \bigcup_i \operatorname{Spec} A_i/(t_i)$ where t_i is non-zerodivisor in A_i, hence t_i doesn't vanishes at any associated points of A_i by Proposition 7.6.11, i.e., $\frac{t_i}{1} \notin p(A_i)_p$ for all $p \in \operatorname{Ass}(A_i)$, i.e., $t_i \notin p$ for all $p \in \operatorname{Ass}(A_i)$. Hence $\operatorname{Spec} A_i/(t_i)$ doesn't contain any any associated points of $\operatorname{Spec} A_i$, and therefore D doesn't contain any any associated points of X.
Conversely, if locally principal closed subscheme D doesn't contain any associated points of X. By Definition 10.5.1, there is an open cover $\{\operatorname{Spec} A_i \hookrightarrow X\}_i$ such that $D = \bigcup_i \operatorname{Spec} A_i/(s_i)$ where $s_i \in A_i$. By the hypothesis, each $\operatorname{Spec} A_i/(s_i)$ doesn't contain any associated points of $\operatorname{Spec} A_i$, hence s_i doesn't vanishes at any associated points of A_i, note that A_i is Noetherian, by Proposition 7.6.11, s_i is non-zerodivisor on A_i for all i. It follows that D is effective Cartier divisor on X.

$\square$

Proposition 10.5.3

Suppose $V(t) = V(t') \hookrightarrow \operatorname{Spec} A$ is an effective Cartier divisor, with t and t' non-zerodivisors in A. Then t is an invertible function times t'.

Proof Since $V(t) = V(t')$ as closed subscheme, they must have the same ideal sheaf, i.e., $(t) = (t')$. Hence we may assume

$$t = at', \quad t' = bt, \tag{10.3}$$

where $a, b \in A$. From (10.3), we have

$$t = at' = abt \quad \Longrightarrow \quad (1 - ab)t = 0.$$

Since t is non-zerodivisor in A, $1 - ab = 0$, i.e., a, b are invertible functions, as we desired. □

The idea of effective Cartier divisors lead us to the notion of regular sequences. (We will close the loop in Exercise 9.5.H, where we will interpret effective Cartier divisors on any reasonable scheme as regular embedding of codimension 1.)

10.5.2 Regular sequences

The notion of "regular sequence", like so much else, is due to Serre. The definition of regular sequence is the algebraic version of the following geometric idea: locally, we take an effective Cartier divisor (a non-zerodivior); then an effective Cartier divisor on that; then an effective Cartier divisor on that; and so on, a finite number of times. A little care is necessary; for example, we might want this to be independent of the order of the equations imposed, and this is true only when we say this in the right way.

We make the definition of regular sequence for a ring A, and more generally for an A-module M.

Definition 10.5.3 (M-regular sequence)

*If M is an A-module, a sequence $x_1, \ldots, x_r \in A$ is called an M-**regular sequence** (or a **regular sequence for** M) if the following two conditions are satisfied.*

(i) For each i, x_i is not a zerodivisor for $M/(x_1, \ldots, x_{i-1})M$. (The case $i = 1$ should be interpreted as: "x_1 is not a zerodivisor of M.")

(ii) We have a proper inclusion $(x_1, \ldots, x_r)M \subsetneq M$.

In the case most relevant to us, when $M = A$, this should be seen as a reasonable approximation of a "complete intersection", and indeed we will use this as the definition (10.5.3).

*We say r is the **length of the regular sequence** $x_1, \ldots, x_r \in A$. An A-regular sequence is just called a **regular sequence**.*

Proposition 10.5.4

If M is an A-module, then an M-regular sequence continues to satisfy condition (i) of the Definition 10.5.3 of regular sequence upon any localization. More precisely, let $x_1, \ldots, x_r \in A$ be an M-regular sequence, then $\frac{x_i}{1}$ is not a zerodivisor for $S^{-1}M / \left(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1}\right) S^{-1}M$ for each i.

Proof Suppose exits $\frac{x_i}{1}$ is a zerodivisor for $S^{-1}M / \left(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1}\right) S^{-1}M$, then there exists $\frac{f}{g} \in S^{-1}M$ where $g \in S$ such that

$$\frac{x_i}{1} \cdot \frac{f}{g} \in \left(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1}\right) S^{-1}M.$$

We may assume that

$$\frac{x_i}{1} \cdot \frac{f}{g} = \sum_{j=1}^{i-1} \frac{m_j x_j}{s_j}, \tag{10.4}$$

pick $s = \prod_{k=1}^{i-1} s_k$, we may write (10.4) as

$$\frac{x_i f}{g} = \frac{m}{s}$$

where $m \in (x_1, \ldots, x_{i-1})M$. Then there exists $t \in S$ such that $t(x_i f s - mg) = 0$, i.e., $x_i stf = tgm$. Note that $tgm \in (x_1, \ldots, x_{i-1})M$, $x_i stf \in (x_1, \ldots, x_{i-1})M$. But this is impossible, since x_i is not zerodivisor for $M/(x_1, \ldots, x_{i-1})M$. Hence M-regular sequence continues to satisfy condition (i) of the Definition 10.5.3 of regular sequence upon any localization. □

Remark Once you know what flatness is, you will see that your argument shows that condition (i) is preserved by any flat base change.

Lemma 10.5.1

Suppose $x_1, x_2, \ldots, x_n$ is an M-regular sequence. If $x_1 m \in (x_2, \ldots, x_k)M$, then $m \in (x_2, \ldots, x_k)M$.

Proof We prove this by induction on n. If $n = 1$ and $x_1 m \in (0)$, since x_1 is not a zerodivisor of M, $m = 0$. Suppose consequence holds for the case $n-1$, i.e., if $x_1, \ldots, x_{n-1}$ is M-regular sequence and $x_1 m \in (x_2, \ldots, x_{n-1})M$ then $m \in (x_2, \ldots, x_{n-1})M$. We now check the case for n, suppose $x_1, x_2, \ldots, x_n$ is M-regular sequence and $x_1 m \in (x_2, \ldots, x_n)M$, want to show that $m \in (x_2, \ldots, x_n)M$. Define homomorphism $\times x_1 : M/(x_2, \ldots, x_n)M \to M/(x_2, \ldots, x_n)M$ by setting $a \mapsto ax_1$. It suffices to show that $\times x_1$ is injective. Let $ax_1 \equiv 0 \pmod{(x_2, \ldots, x_n)M}$, then

$$ax_1 = m_2 x_2 + \cdots + m_n x_n \tag{10.5}$$

for $m_i \in M$. Then $m_n x_n \in (x_1, \ldots, x_{n-1})M$. Since x_n is not a zerodivsor of $M/(x_1, \ldots, x_{n-1})M$, $m_n \in (x_1, \ldots, x_{n-1})M$. Hence exists $c_1, \ldots, c_{n-1} \in M$ such that

$$m_n = c_1 x_1 + \cdots + c_{n-1} x_{n-1}.$$

So by (10.6), we have

$$ax_1 = m_2 x_2 + \cdots + m_{n-1} x_{n-1} + x_n(c_1 x_1 + \cdots + c_{n-1} x_{n-1}),$$

rearranging terms, we have

$$x_1(a - c_1 x_n) = x_2(m_2 + c_2 x_n) + \cdots + x_{n-1}(m_{n-1} + c_{n-1} x_n) \in (x_2, \ldots, x_{n-1})M.$$

Since $x_1, x_2, \ldots, x_{n-1}$ is M-regular sequence, note that $x_1(a - c_1 x_n) \in (x_2, \ldots, x_{n-1})M$, by induction hypothesis, we know that $a - c_1 x_n \in (x_2, \ldots, x_{n-1})M$. Hence $a \in (x_2, \ldots, x_n)M$, it follows that $a \equiv 0 \pmod{(x_2, \ldots, x_n)M}$, i.e., $\times x_1$ is injective, we finished the proof. □

Proposition 10.5.5

If $x_1, \ldots, x_n$ is an M-regular sequence, and $a_1, \ldots, a_n \in \mathbb{Z}^+$, then $x_1^{a_1}, \ldots, x_n^{a_n}$ is a regular sequence.

Proof We first show that if $x_1, \ldots, x_n$ is an M-regular sequence then $x_1^N, x_2, \ldots, x_n$ is an M-regular sequence. We prove it by induction on N. If $N = 1$, then $x_1, \ldots, x_n$ is an M-regular sequence. Suppose $x_1^i, x_2, \ldots, x_n$ is an M-regular sequence for any $i < N$, we want to show that $x_1^N, x_2, \ldots, x_n$ is an M-regular sequence. Suppose $x_1^N, x_2, \ldots, x_n$ is not an M-regular sequence. Then there exists minimal $k \le n$ such that $x_1^N, x_2, \ldots, x_k$ is not an M-regular sequence. Without loss of generally, we may assume $k = n$, i.e., $x_1^N, x_2, \ldots, x_n$ is not an M-regular sequence and $x_1^N, x_2, \ldots, x_{n-1}$ is an M-regular sequence. Since x_n is not an M-regular sequence, there exists $m \in M \setminus (x_1^N, x_2, \ldots, x_{n-1})M$ such that $x_n m \equiv 0 \pmod{(x_1^N, x_2, \ldots, x_{n-1})M}$. Since x_n is not a zerodivisor of $M/(x_1^{N-1}, x_2, \ldots, x_{n-1})M$, m must belong to $(x_1^{N-1}, x_2, \ldots, x_{n-1})M$, hence we may assume $m = m_1x_1^{N-1} + \sum_{i=2}^{n-1} m_ix_i$, then we have

$$x_nm = m_1x_1^{N-1}x_n + \sum_{i=2}^{n-1} m_ix_ix_n \in (x_1^N, x_2, \ldots, x_{n-1})M. \tag{10.6}$$

On the other hand, since $x_nm \in (x_1^N, x_2, \ldots, x_{n-1})M$, we may assume that

$$x_nm = k_1x_1^N + \sum_{i=2}^{n-1} k_ix_i. \tag{10.7}$$

Let (10.6) $-$ (10.7), rearranging terms, we have

$$x_1^{N-1}(m_1x_n - k_1x_1) + \sum_{i=2}^{n-1} x_i(m_ix_n - k_i) = 0.$$

Hence $x^{N-1}(m_1x_n - k_1x_1) \in (x_2, \ldots, x_{n-1})M$, by Lemma 10.5.1, we know that $m_1x_n - k_1x_1 \in (x_2, \ldots, x_{n-1})M$, and therefore $m_1x_n \in (x_1, x_2, \ldots, x_{n-1})M$. Since x_n is not a zerodivisor of $M/(x_1, \ldots, x_{n-1})M$, $m_1 \in (x_1, \ldots, x_{n-1})M$, say $m_1 = \sum_{i=1}^{n-1} m_{1,i}x_i$. Then

$$\begin{aligned} m = &\left(\sum_{i=1}^{n-1} m_{1,i}x_i\right) x_1^{N-1} + \sum_{i=2}^{n-1} m_ix_i \\ =&m_{1,1}x_1^N + \sum_{i=2}^{n-1} x_i(m_{1,i}x_1^{N-1} + m_i) \in (x_1^N, x_2, \ldots, x_{n-1})M, \end{aligned}$$

that contradicts to the fact that $m \in M \setminus (x_1^N, x_2, \ldots, x_{n-1})M$. Hence $x_1^N, x_2, \ldots, x_n$ is an M-regular sequence.

Now we prove this proposition, if $x_1, \ldots, x_n$ is an M-regular sequence and $a_1, \ldots, a_n \in \mathbb{Z}^+$, then $x_1^{a_1}, x_2, \ldots, x_n$ is an M-regular sequence, by our above discussion. Hence $x_2, \ldots, x_n$ is an $M/(x_1^{a_1}M)$-regular sequence, by above discussion again, $x_2^{a_2}, x_3, \ldots, x_n$ is an $M/(x_1^{a_1}M)$-regular sequence. Repeat this process, we have x_n is an $M/(x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}})M$-regular sequence, hence x_n is not a zerodivisor of $M/(x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}})M$, and therefore $x_n^{a_n}$ is an $M/(x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}})M$-regular sequence. Thus $x_1^{a_1}, \ldots, x_n^{a_n}$ is an M-regular sequence. □

Example 10.12 (The regularity of a sequence depends on its order.) We now give an example showing that the order of a regular sequence matters. Suppose $A = k[x, y, z]/(x-1)z$, so $X = \operatorname{Spec} A$ is the union of the $z = 0$ plane and the $x = 1$ plane — X is reduced and has two components (see Figure 10.7). You can

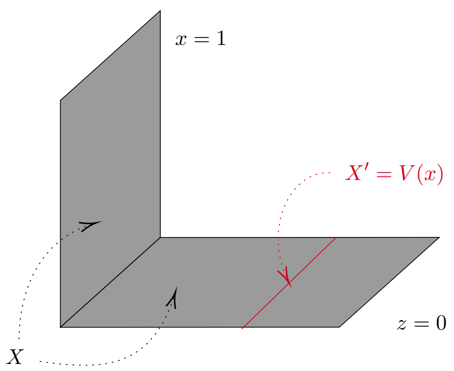


Figure 10.7: Order matters in a regular sequence (in the "non-local" situation), for unsurprising reasons

readily verify that x is a non-zerodivisor of A ($x = 0$ misses one component of X, and doesn't vanish entirely on the other, note that $\operatorname{Ass}(X) = \{[(z)], [(x-1)]\}$, recall Proposition 7.6.11 and Proposition 7.6.22), and that the corresponding effective Cartier divisor $X' = \operatorname{Spec} k[x,y,z]/(x,z)$ (on X) is integral. Then $(x-1)y$ gives an effective Cartier divisor on X' (it doesn't vanish entirely on X'), so $x, (x-1)y$ is a regular sequence for A. However, $(x-1)y$ is not a non-zerodivisor of A, as it does vanish entirely on $x = 1$. Thus $(x-1)y, x$ is not a regular sequence. The reason that reordering the regular sequence $x, (x-1)y$ ruins regularity is clear: there is a locus on which $(x-1)y$ isn't effective Cartier divisor, but it disappears if we enforce $x = 0$ first. This problem is one of "nonlocality" — "near" $x = y = z = 0$ there is no problem. This may motivate the fact that in the (Noetherian) local situation, this problem disappears. We now make this precise.

Theorem 10.5.1

Suppose $(A, \mathfrak{m})$ is a Noetherian local ring, and M is a finitely generated A-module. Then any M-regular sequence $(x_1, \ldots, x_r)$ in $\mathfrak{m}$ remains a regular sequence upon any reordering.

Remark In [10], Dieudonné gives a half-page example showing that Noetherian hypotheses are necessary in Theorem 10.5.1. Also, in Chapter 25, we give a different, somehow simpler proof in the case where A contains a field.

Proof Before proving Theorem 10.5.1, we prove the first nontrivial case, when $r = 2$. This discussion is secretly a baby case of the Koszul complex (which is briefly mentioned in Chapter 25).

Suppose $x, y \in \mathfrak{m}$, and x, y is an M-regular sequence. Translation: $x \in \mathfrak{m}$ is a non-zerodivisor on M, and $y \in \mathfrak{m}$ is a non-zerodivisor on M/xM.

Consider the double complex

$$\begin{tikzcd} M \arrow[r, "\times(-x)"] & M \\ M \arrow[u, "\times y"] \arrow[r, "\times x"] & M \arrow[u, "\times y"'] \end{tikzcd} \tag{10.8}$$

where the bottom left is considered to be in position $(0, 0)$. (The only reason for the minus sign in the top row is solely our arbitrary preference for anti-commuting rather than commuting squares in §2.6.1, but it really doesn't matter.)

We compute the cohomology of the total complex using a (simple) spectral sequence, beginning with the rightward orientation. On the first page, we have

$$\begin{tikzcd} \operatorname{Ker}(M \xrightarrow{\times x} M) & M/xM \\ \operatorname{Ker}(M \xrightarrow{\times x} M) \arrow[u, "\times y"] & M/xM \arrow[u, "\times y"'] \end{tikzcd}$$

The entries $\operatorname{Ker}(M \xrightarrow{\times x} M)$ in the first column are 0, as x is a non-zerodivisor on M. Taking homology in the vertical direction to obtain the second page, we find

$$\begin{tikzcd}
0 & & \\
0 & M/(x,y)M \arrow[ul] & \\
0 & 0 & \\
& & 0 \arrow[uul]
\end{tikzcd} \tag{10.9}$$

using the fact that y is a non-zerodivisor on M/xM. The sequence clearly converges here. Thus the original double complex (10.8) only has nonzero cohomology in degree 2, where it is $M/(x,y)M$.

Now we run the spectral sequence on (10.8) using the upward orientation. The first page of the sequence is:

$$\begin{tikzcd}
M/yM \arrow[r, "\times(-x)"] & M/yM \\
\operatorname{Ker}(M \xrightarrow{\times y} M) \arrow[r, "\times x"] & \operatorname{Ker}(M \xrightarrow{\times y} M)
\end{tikzcd} \tag{10.10}$$

And the second page of the sequence is:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & \\
0 \arrow[drr] & 0 \arrow[drr] & A \arrow[drr] & B \arrow[drr] & & \\
& & C \arrow[drr] & D \arrow[drr] & 0 & 0 \\
& & & & 0 & 0
\end{tikzcd}$$

where $A = \operatorname{Ker}(M/yM \xrightarrow{\times(-x)} M/yM)$, $B = \operatorname{Coker}(M/yM \xrightarrow{\times(-x)} M/yM)$, $C = \operatorname{Ker}(\operatorname{Ker}(M \xrightarrow{\times y} M) \to \operatorname{Ker}(M \xrightarrow{\times y} M))$, and $D = \operatorname{Coker}(\operatorname{Ker}(M \xrightarrow{\times y} M) \to \operatorname{Ker}(M \xrightarrow{\times y} M))$. Since $H^0(E^\bullet) = H^1(E^\bullet) = 0$, the top row in (10.10) is injective and the bottom row in (10.10) is isomorphism. Since the top row in (10.10) is injective, x is a non-zerodivisor on M/yM. Since A is Noetherian and M is finitely generated A-module, we have M is Noetherian (M is the quotient of A^n, the quotient of Noetherian ring is also Noetherian), and therefore $\operatorname{Ker}(M \xrightarrow{\times y} M)$ is finitely generated A-module. As $x \in \mathfrak{m}$, by version 2 of Nakayama's Lemma 9.2.4, this implies that $\operatorname{Ker}(M \xrightarrow{\times y} M) = 0$, so y is non-zerodivisor on M. Thus we have shown that y, x is a regular sequence on M — the $r = 2$ case of Theorem 10.5.1.

We next prove Theorem 10.5.1. First we show that if $x_1, \ldots, x_i, \ldots, x_j, \ldots, x_r \in \mathfrak{m}$ is M-regular sequence, then $x_1, \ldots, x_j, \ldots, x_i, \ldots, x_r$ is M-regular sequence, i.e., regular sequences are invariant under transposition. By our above discussion, we know that $x_1, \ldots, x_i, \ldots, x_j, x_{j-1}, x_{j+1}, \ldots, x_r$ is regular sequence, repeat this process, we have

$$\begin{array}{cccccc}
x_1 & \cdots & x_j & x_i & \cdots & x_r \\
 & & & & & \\
 & & i\text{-th} & (i+1)\text{-th} & &
\end{array}$$

is regular sequence. Now we perform transpositions of x_i with its right neighbor until x_i reaches the original position of x_j, then we have

$$\begin{array}{ccccccc} x_1 & \cdots & x_j & \cdots & x_i & \cdots & x_r \\ & & & & & & \\ & & i\text{-th} & & j\text{-th} & & \end{array}$$

is regular sequence. Since every permutation is a composition of transpositions, we have thus completed the proof of Theorem 10.5.1. $\square$

10.5.3 Regular embeddings

Definition 10.5.4 (Regular embedding (of codimension r))

*Suppose $\pi : X \hookrightarrow Y$ is a locally closed embedding. We say that π is **a regular embedding (of condimension r) at a point** $p \in X$ if in the local ring $\mathscr{O}_{Y,p}$, the ideal of X is generated by a regular sequence (of length r). We say that π is a **regular embedding (of codimension r)** if it is a regular embedding (of codimension r) at all $p \in X$.*

Remark Another reasonable name for a regular embedding might be "local complete intersection". Unfortunately, "local complete intersection morphism", or "lci morphism", is already used for a related notion, see Stacks Project[8] Section 068E.

Our terminology uses the word "codimension", which we have not defined as a word on its own. The reason for using this word will become clearer once you meet Krull's Principal Ideal Theorem (Chapter 13) and Krull's Height Theorem (Chapter 13).

Lemma 10.5.2

If I and J are ideals of a Noetherian ring A, and $\mathfrak{p} \subseteq A$ is a prime ideal such that $I_\mathfrak{p} = J_\mathfrak{p}$, then there exists $a \in A \setminus \mathfrak{p}$ such that $I_a = J_a$ in A_a.

Proof Since A is Noetherian, I, J are finitely generated, say $I = (f_1, \dots, f_n)$. Since $I_\mathfrak{p} = J_\mathfrak{p}$, we have $\frac{f_i}{1} \in I_\mathfrak{p} = J_\mathfrak{p}$. Hence exists $\frac{g_i}{s_i} \in J_\mathfrak{p}$ such that $\frac{g_i}{s_i} = \frac{f_i}{1}$, hence exists $t_i \in A \setminus \mathfrak{p}$ such that $t_i(s_i f_i - g_i) = 0$. Let $t = \prod_{i=1}^n t_i$ and $s = \prod_{i=1}^n s_i$, then $stf_i = g_i t \prod_{j \neq i} s_j \in J$ for all i. Say $b = st$, then we have $bI \subseteq J$. Since $s, t \in A - \mathfrak{p}$, $b \in A - \mathfrak{p}$. Localize A at b, we have $(bI)_b = I_b$, and therefore $((bI)_b)_\mathfrak{p} = I_\mathfrak{p} = J_\mathfrak{p} = (J_b)_\mathfrak{p}$. Hence we may reduce to the case that $I \subseteq J$.

Let $I \subseteq J$, denote $K = J/I$. Since $I_\mathfrak{p} = J_\mathfrak{p}$, we have $K_\mathfrak{p} = (J/I)_\mathfrak{p} \cong J_\mathfrak{p}/I_\mathfrak{p} = 0$. We want to show that $K_a = 0$ for some $a \in A \setminus \mathfrak{p}$. Since K is finitely generated, we may assume that $K = (k_1, \dots, k_m)$. Since $K_\mathfrak{p} = 0$, for each k_i, there exists $a_i \in A \setminus \mathfrak{p}$ such that $a_i k_i = 0$. Let $a = \prod a_i$, then $ak_i = 0$ for all i, hence $aK = 0$. Now we localize aK at a, we have $(aK)_a = K_a = 0$, i.e., $I_a = J_a$. $\square$

Proposition 10.5.6 (The condition of locally closed embedding being regular embedding is open)

If a locally closed embedding $\pi : X \hookrightarrow Y$ of locally Noetherian schemes is a regular embedding at p, then it is a regular embedding in some open neighborhood of p in X.

Proof Reduce to the case where π is a closed embedding, and then where Y (hence X) is affine, say $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} B/I$, and $p = [\mathfrak{p}]$, where B is Noetherian. Since $\pi : X \hookrightarrow Y$ is a regular embedding at p, there are $f_1, \dots, f_r \in B$ such that in $\mathscr{O}_{Y,p} = B_\mathfrak{p}$, the image of the f_i are a $B_\mathfrak{p}$-regular sequence generating

$I_\mathfrak{p} = IB_\mathfrak{p}$.

We first show there exists an open neighborhood U of p such that $(f_{1,q},\dots,f_{r,q}) = I_\mathfrak{q} = IB_\mathfrak{q}$ for all $q = [\mathfrak{q}] \in U$ where $f_{i,q} = f_i/1$ is the image of f_i in $\mathscr{O}_{Y,q}$, i.e., $(f_1,\dots,f_r) = I$ "in an open neighborhood of p". Suppose for any open neighborhood U of p exists $q = [\mathfrak{q}] \in U$ such that $(f_{1,q},\dots,f_{r,q}) \neq IB_\mathfrak{q}$. Since $(f_{1,p},\dots,f_{r,p}) = I_\mathfrak{p}$, by Lemma 10.5.2, there exists $g \in B \setminus \mathfrak{p}$ such that $(f_{1,g},\dots,f_{r,g}) = I_g = IB_g$. Hence $p \in D(g)$, by our assumption, there exists $q = [\mathfrak{q}] \in D(g)$ such that $(f_{1,q},\dots,f_{r,q}) \neq IB_\mathfrak{q}$. Since $g \notin \mathfrak{q}$, we may localize B_g at $[\mathfrak{q}]$. Note that $(f_{1,g},\dots,f_{r,g}) = IB_g$, we have

$$(f_{1,g},\dots,f_{r,g})_\mathfrak{q} = (f_{1,q},\dots,f_{r,q}) = (IB_g)_\mathfrak{q} = IB_\mathfrak{q},$$

a contradiction! Hence there exists an open neighborhood U of p such that $(f_1,\dots,f_r) = I$ in U.

We next show that there exists an open neighborhood $V \subseteq U$ of p such that $(f_{1,q},\dots,f_{r,q})$ is a $B_\mathfrak{q}$-regular sequence for all $q = [\mathfrak{q}] \in V$. Define the homomorphism as follow,

$$\begin{aligned} \varphi_i : B/(f_1,\dots,f_{i-1}) &\longrightarrow B/(f_1,\dots,f_{i-1}) \\ b &\longmapsto f_i b, \end{aligned}$$

denote $J_i := \operatorname{Ker}\varphi_i$, then $(f_{1,q},\dots,f_{r,q})$ is a $B_\mathfrak{q}$-regular sequence if and only if $(J_i)_\mathfrak{q} = 0$ for all i. Since $(f_{1,p},\dots,f_{r,p})$ is a $B_\mathfrak{p}$-regular sequence, $(J_i)_\mathfrak{p} = 0$ for all i, by Lemma 10.5.2, there exists $h \in B \setminus \mathfrak{p}$ such that $(J_i)_h = 0$ in B_h for all i. Consider $D(h) \cap U$, it is an open neighborhood of p. Note that for all $[\mathfrak{q}] \in D(h) \cap U$, $h \notin \mathfrak{q}$, hence we may localize $(J_i)_h$ at point $[\mathfrak{p}]$, then we have $((J_i)_h)_\mathfrak{q} = (J_i)_\mathfrak{q} = 0$ for all i and $[\mathfrak{q}] \in D(h) \cap U$. It follows that for all $q = [\mathfrak{q}] \in D(h) \cap U$, $(f_{1,q},\dots,f_{r,q})$ is a $B_\mathfrak{q}$-regular sequence. Let $V = D(h) \cap U$, we done! □

Corollary 10.5.1

If X is locally of finite type over a field (e.g., a variety), then to check that a closed embedding π is a regular embedding it suffices to check at closed points of X (Proposition 6.3.8).

Proof Let X be a locally of finite type over a field, then X is a locally Noetherian scheme. Suppose $\pi : X \hookrightarrow Y$ is a closed embedding of locally Noetherian schemes. By Proposition 10.5.6, set

$$Z = \{p \in X : \pi : X \hookrightarrow Y \text{ is a regular embedding at } p\}$$

is open. If we check $\pi : X \hookrightarrow Y$ is a regular embedding at all closed points of X, say C be the set of all closed points of X, then $C \subseteq Z$. By Proposition 6.3.8, we know that C is dense. We claim that $Z = X$ as topological space. If not $X \setminus Z$ is a non-empty closed subset, since C is dense, $C \cap (X \setminus Z) \neq \varnothing$, hence exists closed point not belong to Z, a contradiction! □

Remark In Chapter 14, we will show that not all closed embeddings are regular embeddings.

Proposition 10.5.7

A closed embedding $X \hookrightarrow Y$ of locally Noetherian schemes is a regular embedding of codimension 1 if and only if X is an effective Cartier divisor on Y.

Proof If X is an effective Cartier divisor, then exists a cover of Y, say $\{\operatorname{Spec} B_i \hookrightarrow Y\}_i$, such that $X = \operatorname{Spec} B_i/(t_i)$, where t_i is not a zerodivisor of B_i. Pick p be any point of X, then exists affine piece $\operatorname{Spec} B_i \ni p = [\mathfrak{p}]$, we want to show that $t_{i,\mathfrak{p}} = \frac{t_i}{1}$ is a $(B_i)_\mathfrak{p}$-regular sequence, i.e., $\frac{t_i}{1}$ is not a zerodivisor of $(B_i)_\mathfrak{p}$. Since t_i is not a zerodivisor of B_i, by Proposition 10.5.4, $\frac{t_i}{1}$ is not a zerodivisor of $(B_i)_\mathfrak{p}$. Hence $\pi : X \hookrightarrow Y$ is a regular embedding of codimension 1.

If closed embedding $X \hookrightarrow Y$ is a regular embedding of codimension 1, we may reduce to the case that $Y = \operatorname{Spec} B$ where B is a Noetherian ring. Since $\pi : X \hookrightarrow Y$ is closed embedding, we may assume that $X = \operatorname{Spec} B/I$. Then for all $p = [\mathfrak{p}] \in X$, $I_\mathfrak{p}$ is generated by one element which is not a zerodivisor of $B_\mathfrak{p}$. Say $I_\mathfrak{p} = (f_\mathfrak{p})B_\mathfrak{p}$, then $I_\mathfrak{p} = (f)_\mathfrak{p}$, by Lemma 10.5.2, there exists $h_p \in B \setminus \mathfrak{p}$ such that $I_{h_p} = (f)_{h_p} = (f_{h_p})$. Consider $\operatorname{Ann}_{B_{h_p}}(f_{h_p})$, note that $\operatorname{Ann}_{B_{h_p}}(f_{h_p})_\mathfrak{p} = \operatorname{Ann}_{B_\mathfrak{p}}(f_\mathfrak{p}) = 0$, by Lemma 10.5.2, there exists $l_p \in B \setminus \mathfrak{p}$ such that $\operatorname{Ann}_{B_{h_p}}(f_{h_p})_{l_p} = \operatorname{Ann}_{B_{h_p l_p}}(f_{h_p l_p}) = 0$, say $s_p = h_p l_p$, then f_{s_p} is not a zerodivisor of B_{s_p} and $I_{s_p} = (f_{s_p})$. Hence $X = \bigcup_{p \in X} \operatorname{Spec} B_{s_p}/(f_{s_p})$, where f_{s_p} is not a zerodivisor of B_{s_p}, i.e., X is an effective Cartier divisor on Y. □

Remark (**Unimportant remark.**) The Noetherian hypotheses can be replaced by requiring $\mathscr{O}_Y$ to be coherent, and essentially the same argument applies. It is interesting to note that "effective Cartier divisor" implies "regular embedding of codimension 1" always, but that the converse argument requires Noetherian(-like) assumption.

Definition 10.5.5 (Codimension r complete intersection)

*A **codimension r complete intersection** in a scheme Y is closed subscheme X that can be written as the scheme-theoretic intersection of r effective Cartier divisors $D_1, \ldots, D_r$ such that at every point $p \in X$, the equations corresponding to $D_1, \ldots, D_r$ form a regular sequence. The phrase **complete intersection** means "codimension r complete intersection for some r".*

Chapter 11 Fibered products of schemes, and base change

Fibered products have an unexpectedly central place in algebraic geometry. Experience will gradually tech you why they play such an outsized role. Until you have acquired that experience, you should pay closer attention to them than you think they might deserve.

11.1 They exist

Before we get to products, we note that coproducts exists in the category of schemes: just as with the category of sets (Exercise 2.16), coproduct is disjoint union.

We will now construct the fibered product in the category of schemes.

Theorem 11.1.1 (Fibered products exist)

Suppose $\alpha : X \to Z$ and $\beta : Y \to Z$ are morphisms of schemes. Then the fibered product

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & Z \end{tikzcd}$$

exists in the category of schemes.

Note *If A is a ring, people often sloppily write $\times_A$ for $\times_{\operatorname{Spec} A}$. If B is an A-algebra, and X is an A-scheme, people often write X_B or $X \times_A B$ for $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$.*

Remark **Warning: products of schemes aren't products of sets.** Before showing existence, here is a warning: the product of schemes isn't a product of sets (and more generally for fibered products). We have made a big deal about schemes being sets, endowed with a topology, upon which we have a structure sheaf. So you might think that we will construct the product in this order: take the product set, endow it with the product topology, and then figure out what the structure sheaf must be. But we won't, because scheme products behave oddly on the level of sets. You may have checked (Exercise 8.16 (a)) that the product of two affine lines over your favorite algebraically closed field $\overline{k}$ is the affine plane: $\mathbb{A}^1_{\overline{k}} \times_{\overline{k}} \mathbb{A}^1_{\overline{k}} \cong \mathbb{A}^2_{\overline{k}}$. But the underlying set of the latter is not the underlying set of the former — we get additional points corresponding to curves in $\mathbb{A}^2$ that are not lines parallel to the axes!

Remark On the other hand, W-valued points (where W is a scheme, Definition 8.3.5) do behave well under (fibered) products. This is just the universal property definition of fibered product: a W-valued point of a scheme X is defined as an element of $\operatorname{Hom}(W, X)$, and the fibered product is defined (via Yoneda Lemma, see §2.2.1) by

$$\operatorname{Hom}(W, X \times_Z Y) = \operatorname{Hom}(W, X) \times_{\operatorname{Hom}(W,Z)} \operatorname{Hom}(W, Y). \tag{11.1}$$

This is one justification for making the definition of scheme-valued point. For this reason, those classical people preferring to think only about varieties over an algebraically closed field $\overline{k}$ (or more generally, finite type scheme over $\overline{k}$), and preferring to understand them through their closed points — or equivalently, the $\overline{k}$-valued points, by the Nullstellensatz (Proposition 6.3.8) — needn't worry: the closed points of the product of two finite type $\overline{k}$-schemes over $\overline{k}$ are (naturally identified with) the product of the closed points of the factors. This will follow from the fact that the product is also finite type over $\overline{k}$, which we will verify in §11.2. This is one of the reasons that varieties over algebraically closed fields can be easier to work with. But over a non-algebraically closed field, things become even more interesting; An example in §11.2 is a first glimpse.

Remark **Fancy side.** You may feel that (i) "products of topological spaces are products on the underlying sets" is natural, while (ii) "products of schemes are not necessarily products on the underlying sets" is weird. But really (i) is the lucky consequence of the fact that the underlying set of a topological space can be interpreted as set of p-valued points, where p is a point, so it is best seen as a consequence of above paragraph, which is the "more correct" — i.e., more general — fact.

Remark **Warning on Noetherianness.** The fibered product of Noetherian schemes need not be Noetherian. You will later be able to verify that an example in §11.2, i.e., that $A := \overline{\mathbb{q}} \otimes_{\mathbb{q}} \overline{\mathbb{q}}$ is not Noetherian, as follows. By Chapter 13, $\dim A = 0$. A Noetherian dimension 0 scheme has a finite number of points. But by §11.2, $\operatorname{Spec} A$ has an infinite number of points.

On the other hand, the fibered product of finite type k-schemes over finite type k-schemes is a finite type k-scheme, so this pathology does not arise for varieties.

11.1.1 Philosophy behind the proof of Theorem 11.1.1

The proof of Theorem 11.1.1 can be confusing. The following comments may help a little.

We already basically know existence of fibered products in two cases: the case where X, Y, and Z are affine (stated explicitly below), and case where $\beta : Y \to Z$ is an open embedding (Proposition 9.1.4).

Proposition 11.1.1

Let $C \to A$ and $C \to B$ be two ring maps, then

$$\operatorname{Spec}(A \otimes_C B) \cong \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B.$$

Hence the fibered product of affine schemes exists (in the category of schemes).

Proof Consider the following commutative diagram (Exercise 2.17).

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \otimes_C B \end{tikzcd} \tag{11.2}$$

By Proposition 8.3.7, we have the following diagram.

$$\begin{tikzcd} \operatorname{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C \end{tikzcd} \tag{11.3}$$

Since each morphisms induced by ring homomorphism, by the commutative of (11.2), diagram (11.3) is commutative.

We next check $\operatorname{Spec}(A \otimes_C B)$ agree with the universal property of fibered product. Let W be any scheme with maps to $\operatorname{Spec} A$ and $\operatorname{Spec} B$ whose compositions to $\operatorname{Spec} C$ agree commute, we want to check that these maps factor through some unique $W \to \operatorname{Spec} A \times_{\operatorname{Spec} Z} \operatorname{Spec} B$.

We first consider the case that $W = \operatorname{Spec} E$. By the universal property of fibered coproduct. There exists an unique morphism from $A \otimes_C B$ to E such that the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & A \otimes_C B \arrow[dr, dashed] & \\
 & & E
\end{tikzcd}$$

By Proposition 8.3.7, we have the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Spec} E \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \operatorname{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
 & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

Now let W be any scheme, suppose $W = \bigcup_i \operatorname{Spec} E_i := \bigcup_i U_i$. By our above discussion, there exists unique morphism $\theta_i : \operatorname{Spec} E_i \to \operatorname{Spec}(A \otimes_C B)$ such that the following diagram commutes.

$$\begin{tikzcd}
U_i \arrow[dr, dashed, "\theta_i"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \operatorname{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
 & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

By the gluability axiom, $\theta_i : U_i \to \operatorname{Spec}(A \otimes_C B)$ glue to unique $\theta : W \to \operatorname{Spec}(A \otimes_C B)$ such that the following diagram commutes.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\theta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \operatorname{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
 & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

By the definition of fibered product, we have

$$\operatorname{Spec}(A \otimes_C B) \cong \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B.$$

□

Remark This generalizes the fact that the product of affine lines exist, Exercise 8.16 (a).

The main theme of the proof of Theorem 11.1.1 is that because schemes are built by gluing affine schemes along open subsets, these two special cases will be all that we need. The argument will repeatedly use the same ideas — roughly, that schemes glue (Theorem 5.4.1), and that morphisms of schemes glue (Proposition 8.3.2). This is a sign that something more structural is going on; §11.1.3 describes this for experts.

11.1.2 Proof of Theorem 11.1.1

Proof The key idea is this: we cut everything up into affine open sets, do fibered products there, and show that everything glues nicely. The conceptually difficult part of the proof comes from the gluing, and the realization that we have to check almost nothing. We divide the proof up into a number of bite-sized pieces.

Step 1: fibered products of affine with "almost-affine" over affine. We begin by combining the affine case with the open embedding case as follow. Suppose X and Z are affine, and $\beta : Y \to Z$ factor as

$$Y \xhookrightarrow{\iota} Y' \longrightarrow Z$$

where ι is an open embedding and Y' is affine. Then $X \times_Z Y$ exists. This is because if the two small squares of

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook, "\text{open}"'] & Y \arrow[d, hook, "\text{open}"] \\ W' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

are Cartesian diagrams, then the "outside rectangle" is also a Cartesian diagram. (This was Exercise 2.13, although you should be able to see this on the spot.) It will be important to remember (from Important Proposition 9.1.4) that "open embeddings" are "preserved by fibered product": the fact that $Y \to Y'$ is an open embedding implies that $W \to W'$ is an open embedding.

Key Step 2: fibered product of affine with arbitrary over affine exists. We now come to the key part of the argument: if X and Z are affine, and Y is arbitrary. This is confusing when you first see it, so we first deal with a special case, when Y is the union of two affine open sets $Y_1 \cup Y_2$. Let $Y_{12} = Y_1 \cap Y_2$.

Now for $i = 1$ and 2, $X \times_Z Y_i$ exists by the affine cases, Proposition 11.1.1. Call this W_i. Also, $X \times_Z Y_{12}$ exists by **Step 1** (call it W_{12}), and comes with canonical open embeddings into W_1 and W_2 (by construction of fibered products with open embeddings, see the last sentence of **Step 1**). Thus we can glue W_1 to W_2 along W_{12}; call this resulting scheme W.

We check that the result is the fibered product by verifying that it satisfies the universal property. Suppose we have maps $\alpha'' : V \to X$, $\beta'' : V \to Y$ that compose (with α and β respectively) to the same map $V \to Z$. We need to construct a unique map $\gamma : V \to W$, so that $\alpha' \circ \gamma = \beta''$ and $\beta' \circ \gamma = \alpha''$.

$$\begin{tikzcd} V \arrow[dr, dashed, "\exists!\gamma?"] \arrow[drr, bend left, "\beta''"] \arrow[ddr, bend right, "\alpha''"'] & & \\ & W \arrow[r, "\alpha'"'] \arrow[d, "\beta'"] & Y \arrow[d, "\beta"'] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd} \tag{11.4}$$

For $i = 1$ and 2, defined $V_i := (\beta'')^{-1}(Y_i)$. Define $V_{12} := (\beta'')^{-1}(Y_{12}) = V_1 \cap V_2$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are as desired (by the universal property of the fibered product $X \times_Z Y_i = W_i$), hence a unique map $\gamma_i : V_i \to W$. Similarly, there is a unique map $\gamma_{12} : V_{12} \to W$ such that the composed maps $V_{12} \to X$ and $V_{12} \to Y$ are as desired. But the restriction of γ_i to V_{12} is one such map, so it must be γ_{12}. Thus the maps γ_1 and γ_2 agree on V_{12}, and glue together to a unique map $\gamma : V \to W$. We have shown existence and uniqueness of desired γ.

We have thus shown that if Y is the union of two affine open sets, and X and Z are affine, then $X \times_Z Y$ exists.

We now tackle the general case. We now cover Y with open sets Y_i, as i runs over some index set (not necessarily finite!). As before, we define W_i and W_{ij}. We need to check the cocycle condition. Say

$f_{ij} : W_{ij} \xrightarrow{\sim} W_{ji}$, it suffices to show that $f_{ik}|_{W_{ij}\cap W_{ik}} = f_{jk}|_{W_{ji}\cap W_{jk}} \circ f_{ij}|_{W_{ij}\cap W_{ik}}$. In fact, we have

$$\begin{aligned} W_{ij} \cap W_{ik} =& \alpha_i'^{-1}(Y_i \cap Y_j) \cap \alpha_i'^{-1}(Y_i \cap Y_k) \\ =& \alpha_i'^{-1}(Y_i \cap Y_j \cap Y_k) \\ =& \alpha'^{-1}(Y_i \cap Y_j \cap Y_k) \\ \cong& W_{ji} \cap W_{jk} \cong W_{ki} \cap W_{kj}. \end{aligned}$$

Then we have the triple intersection cocycle conditions. By Theorem 5.4.1, we can glue these together to produce a scheme W along with open sets we identify with W_i.

As in the two-affine case, we show that W is the fibered product by showing that it satisfies the universal property. Suppose we have maps $\alpha'' : V \to X$, $\beta'' : V \to Y$ that compose to the same map $V \to Z$. We construct a unique map $\gamma : V \to W$, so that $\alpha' \circ \gamma = \beta''$ and $\beta' \circ \gamma = \alpha''$. Define $V_i = (\beta'')^{-1}(Y_i)$ and $V_{ij} := (\beta'')^{-1}(Y_{ij}) = V_i \cap V_j$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are as desired, hence a unique map $\gamma_i : V_i \to W$. Similarly, there is a unique map $\gamma_{ij} : V_{ij} \to W$ such that the composed maps $V_{ij} \to X$ and $V_{ij} \to Y$ are as desired. But the restriction of γ_i to V_{ij} is one such map, so it must be γ_{ij}. Thus the maps γ_i and γ_j agree on V_{ij}. Thus the γ_i glue together to a unique map $\gamma : V \to W$. We have shown existence and uniqueness of the desired γ, completing this step.

Step 3: Z affine, X and Y arbitrary. We next show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just follow **Step 2**, with the roles of X and Y reversed, using the fact that by the previous step, we can assume that the fibered product of an affine scheme with an arbitrary scheme over an affine scheme exists.

Step 4: Z "almost-affine", X and Y arbitrary. This is akin to **Step 1**. Let $Z \hookrightarrow Z'$ be an open embedding into an affine scheme. Then $X \times_Z Y$ satisfies the universal property of $X \times_{Z'} Y$, by Exercise 2.20 and that open embeddings are monomorphisms.

Step 5: the general case. We employ the same trick yet again. Suppose $\alpha : X \to Z$, $\beta : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subschemes Z_i, and let $X_i = \alpha^{-1}(Z_i)$ and $Y_i = \beta^{-1}(Z_i)$. Define $Z_{ij} := Z_i \cap Z_j$, $X_{ij} := \alpha^{-1}(Z_{ij})$, and $Y_{ij} := \beta^{-1}(Z_{ij})$. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i (**Step 3**), and $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ exists for all i, j (**Step 4**), and for each i and j, W_{ij} comes with a canonical open embedding into both W_i and W_j (see the last sentence in **Step 1**). Since $X_i = \alpha^{-1}(Z_i)$, by Proposition 9.1.4, $X_i = X \times_Z Z_i$. Note that we have the tower of cartesian diagrams

$$\begin{array}{ccc} W_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Z_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z, \end{array}$$

by Exercise 2.13, we have $W_i = X \times_Z Y_i$. Similarly, we identify W_{ij} with $X \times_Z Y_{ij}$.

We then proceed exactly as in **Step 2**: the W_i's can be glued together along the W_{ij} (the cocycle condition can be readily checked to be satisfied), and W can be checked to satisfy the universal property of $X \times_Z Y$ (again, exactly as in **Step 2**). $\square$

11.1.3 ⋆⋆ Describing the existence of fibered products using the fancy language of representable functors

The proof above can be described more cleanly in the language of representable functors (§8.6). This will be enlightening only after you have absorbed the above argument and meditated on it for a long time. It may be most useful to shed light on representable functors, rather than on the existence of the fibered product.

11.2 Computing fibered products in practice

Before giving some examples, we first see how to compute fibered products in practice. There are four types of morphisms §11.2.1-§11.2.4 that it is particularly easy to take fibered products with, and all morphisms can be built from these atomic components. More precisely, §11.2.1 will imply that we can compute fibered products locally on the source and target. Thus to understand fibered products in general, it suffices to understand them on the level of affine sets, i.e., to be able to compute $A \otimes_B C$ given ring maps $B \to A$ and $B \to C$. Any map $B \to A$ (and similarly $B \to C$) may be expressed as $B \to B[t_1, \dots]/I$, so if we know how to base change by "adding variables" §11.2.2 and "taking quotients" §11.2.3, we can "compute" any fibered product (at least in theory). The fourth type of morphism §11.2.4, corresponding to localization, is useful to understand explicitly as well.

11.2.1 Base change by open embeddings.

We have already done this (Proposition 9.1.4), and we used it in the proof hat fibered products of schemes exist.

11.2.2 Adding an extra variable.

Lemma 11.2.1

$A \otimes_B B[t] \cong A[t]$, so the following is a Cartesian diagram.

$$\begin{tikzcd} \operatorname{Spec} A[t] \arrow[r] \arrow[d] & \operatorname{Spec} B[t] \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

Proof Since tensor product is fibered coproduct, we have the following Cartesian diagram.

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "\varphi"'] & B[t] \arrow[d, hook] \\ A \arrow[r, hook] & A \otimes_B B[t] \end{tikzcd}$$

Consider ring homomorphism $i_A : A \hookrightarrow A[t]$ and $\widetilde{\varphi} : B[t] \to A[t]$ which induced by $\varphi : B \to A$, consider the following diagram.

$$\begin{tikzcd}
B \arrow[r, hook, "i_B"] \arrow[d, "\varphi"'] & B[t] \arrow[d, hook] \arrow[ddr, bend left, "\widetilde{\varphi}"] & \\
A \arrow[r, hook] \arrow[drr, bend right, "i_A"'] & A \otimes_B B[t] \arrow[dr, dashed, "\theta"] & \\
& & A[t]
\end{tikzcd} \tag{11.5}$$

Clearly, $i_A \circ \varphi = \widetilde{\varphi} \circ i_B$, by the universal property of fibered coproduct, there exists a unique morphism of rings $\theta : A \otimes_B B[t] \to A[t]$ such that diagram (11.5) commutes.

We next show that $A[t]$ is fibered coproduct. Consider the following diagram,

$$\begin{tikzcd}
B \arrow[r, hook, "i_B"] \arrow[d, "\varphi"'] & B[t] \arrow[d, "\widetilde{\varphi}"'] \arrow[ddr, bend left, "\psi"] & \\
A \arrow[r, hook, "i_A"] \arrow[drr, bend right, "\eta"'] & A[t] \arrow[dr, dashed, "\theta'"] & \\
& & C
\end{tikzcd} \tag{11.6}$$

where $\eta \circ \varphi = \psi \circ i_B$. We want to show that there exists a unique morphism $A[t] \to C$ such that diagram (11.6) commutes. Define $\theta' : A[t] \to C$ by setting

$$\sum_{i=0}^{n} a_i t^i \longmapsto \sum_{i=0}^{n} \eta(a_i)\psi(t)^i.$$

Let $a \in A$, then

$$\theta' \circ i_A(a) = \theta'(a) = \eta(a),$$

i.e., $\eta = \theta' \circ i_A$. Let $\sum\limits_{i=0}^{n} b_i t^i \in B[t]$, then by $\eta \circ \varphi = \psi \circ i_B$ we have

$$\theta' \circ \widetilde{\varphi}\left(\sum_{i=0}^{n} b_i t^i\right) = \theta'\left(\sum_{i=0}^{n} \varphi(b_i) t^i\right) = \sum_{i=0}^{n} (\eta \circ \varphi)(b_i)\psi(t)^i = \sum_{i=0}^{n} (\psi \circ i_B)(b_i)\psi(t)^i = \psi\left(\sum_{i=0}^{n} b_i t^i\right),$$

i.e., $\psi = \theta' \circ \widetilde{\varphi}$. By the definition of θ', uniqueness is obviously. Hence $A[t]$ is fibered coproduct. Let $C = A \otimes_B B[t]$, then there exists a unique morphism $A[t] \to A \otimes_B B[t]$. Hence

$$A \otimes_B B[t] \cong A[t].$$

By Proposition 8.3.7, we have the following Cartesian diagram.

$$\begin{tikzcd}
\operatorname{Spec} A[t] \arrow[r] \arrow[d] & \operatorname{Spec} B[t] \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} B
\end{tikzcd}$$

□

Definition 11.2.1 (Affine space over an arbitrary scheme)

If X is any scheme, we define $\mathbb{A}^n_X$ as $X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. Clearly, $\mathbb{A}^n_{\operatorname{Spec} A}$ is canonically the same as $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$.

11.2.3 Base change by closed embeddings.

Definition 11.2.2 (Extension ideal)

Suppose $\varphi : B \to A$ is a ring homomorphism, and $I \subseteq B$ is an ideal. Let $I^e := (\varphi(i))_{i \in I} = \varphi(I)A \subseteq A$ be the ***extension of*** I ***to*** A.

Lemma 11.2.2

Suppose $\varphi : B \to A$ is a ring homomorphism, and $I \subseteq B$ is an ideal. There is a natural isomorphism $A/I^e \xleftrightarrow{\sim} A \otimes_B (B/I)$.

Proof Consider the following exact sequence.

$$0 \longrightarrow I \longrightarrow B \longrightarrow B/I \longrightarrow 0$$

By the right exactness of $\square \otimes_B A$ (Exercise 2.37), the sequence

$$0 \longrightarrow I \otimes_B A \longrightarrow B \otimes_B A \longrightarrow B/I \otimes_B A \longrightarrow 0$$

is exact, hence

$$A \otimes_B (B/I) \xleftrightarrow{\sim} (A \otimes_B B)/(A \otimes_B I).$$

We claim that $A/I^e \cong (A \otimes_B B)/(A \otimes_B I)$. Define $\theta : A \to (A \otimes_B B)/(A \otimes_B I)$ by setting

$$a \longmapsto a \otimes 1.$$

Since $A \hookrightarrow A \otimes_B B$, θ is injective. Let $a \otimes b \in (A \otimes_B B)/(A \otimes_B I)$, then

$$a \otimes b = \varphi(b)a \otimes 1,$$

hence $\theta(\varphi(b)a) = a \otimes b$, i.e., θ is surjective. Hence $A/I^e \cong (A \otimes_B B)/(A \otimes_B I)$, and therefore

$$A/I^e \xleftrightarrow{\sim} A \otimes_B (B/I).$$

$\square$

Proposition 11.2.1 (Closed embeddings are preserved by base change)

The fibered product with a closed subscheme is a closed subscheme of the fibered product. More precisely, if $X \hookrightarrow Z$ is a closed embedding, then for any $Y \to Z$, induced map $X \times_Z Y \hookrightarrow Y$ is closed embedding.

Proof By Lemma 11.2.2 and Proposition 11.1.1, we get this consequence immediately. $\square$

Proposition 11.2.2

The intersection of two closed embeddings into X is their fibered product over X.

Proof It suffices to check this in the affine case. We may assume that $X = \operatorname{Spec} B$, and $Y = \operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ and $Z = \operatorname{Spec} B/J \hookrightarrow \operatorname{Spec} B$ be two closed embedding. Their fibered product over X is $\operatorname{Spec} B/I \times_B \operatorname{Spec} B/J$. By Proposition 11.1.1, we know that $\operatorname{Spec} B/I \times_B \operatorname{Spec} B/J \cong \operatorname{Spec}(B/I \otimes_B B/J)$. We claim that $B/I \otimes_B B/J \cong B/(I+J)$. Consider the following exact sequence.

$$0 \longrightarrow I \longrightarrow B \longrightarrow B/I \longrightarrow 0$$

By the right exactness of $\square \otimes_B (B/J)$ (Exercise 2.37), the sequence

$$0 \longrightarrow I \otimes_B (B/J) \longrightarrow B \otimes_B (B/J) \longrightarrow (B/I) \otimes_B (B/J) \longrightarrow 0$$

is exact. Note that $B \otimes_B (B/J) \cong B/J$, then we have exact sequence

$$0 \longrightarrow I \otimes_B (B/J) \longrightarrow B/J \longrightarrow (B/I) \otimes_B (B/J) \longrightarrow 0.$$

We now compute $I \otimes_B (B/J)$, claim that $I \otimes_B (B/J) \cong I/IJ$. Define $\varphi : I \otimes_B (B/J) \to I/IJ$ by setting

$$i \otimes b \longmapsto \overline{ib},$$

clearly this morphism is well-defined. We next define the inverse of φ, define $\psi : I/IJ \to I \otimes_B (B/J)$ by setting

$$\bar{i} \longmapsto i \otimes \bar{1}.$$

Let $i \in IJ$, then $i = \sum_k i_k j_k$ where $i_k \in I$ and $j_k \in J$, then $i \otimes \bar{1} = \sum_k i_k \otimes j_k = 0$. Hence ψ is well-defined, and therefore $I/IJ \cong I \otimes_B (B/J)$, then we get an exact sequence

$$0 \longrightarrow I/IJ \longrightarrow B/J \longrightarrow (B/I) \otimes_B (B/J) \longrightarrow 0,$$

i.e., $(B/I) \otimes_B (B/J) \cong (B/J)/\operatorname{Im}(I/IJ \to B/J)$. In fact, $\operatorname{Im}(I/IJ \to B/J) = (I+J)/J$, hence $(B/I) \otimes_B (B/J) \cong (B/J)/((I+J)/J)$, by the 3-rd Fundamental Theorem of Isomorphism, we have

$$(B/I) \otimes_B (B/J) \cong (B/J)/((I+J)/J).$$

Hence

$$Y \cap Z = \operatorname{Spec}(B/(I+J)) \cong \operatorname{Spec}(B/I \otimes_B B/J) \cong \operatorname{Spec} B/I \times_B \operatorname{Spec} B/J.$$

□

Proposition 11.2.3 (Locally closed embeddings are preserved by base change)

If $\pi : X \hookrightarrow Y$ is a locally closed embedding, then for any $Y' \to Y$, the induced map $X \times_Y Y' \to Y'$ is also a locally closed embedding.

Proof Since $\pi : X \hookrightarrow Y$ is a locally closed embedding, π can be factored into

$$X \hookrightarrow Z \hookrightarrow Y,$$

where $X \hookrightarrow Z$ is closed embedding and $Z \hookrightarrow Y$ is open embedding. Since open embedding is preserved by base change (Proposition 9.1.4), $Z \times_Y Y' \to Y'$ is an open embedding. By Proposition 11.2.1, $X \times_Z (Z \times_Y Y') \to Z \times_Y Y'$ is a closed embedding. Note that we have the following commutative diagram,

$$\begin{tikzcd}
X \times_Y Y' \arrow[dr, dashed, shift left] \arrow[drr, bend left] \arrow[dddr, bend right] & & \\
& X \times_Z (Z \times_Y Y') \arrow[r] \arrow[d, hook] \arrow[ul, dashed, shift left] & X \arrow[d, hook] \\
& Z \times_Y Y' \arrow[r] \arrow[d, hook] & Z \arrow[d, hook] \\
& Y' \arrow[r] & Y
\end{tikzcd}$$

by the universal property of $X \times_Y Y'$ and $X \times_Z (Z \times_Y Y')$, there is an isomorphism

$$X \times_Z (Z \times_Y Y') \cong X \times_Y Y'.$$

Hence $X \times_Y Y' \to Y'$ can be factored into

$$X \times_Y Y' \xhookrightarrow{\text{closed}} Z \times_Y Y' \xhookrightarrow{\text{open}} Y',$$

i.e., $X \times_Y Y' \hookrightarrow Y'$ is a locally closed embedding. □

Definition 11.2.3 (Intersection of n locally closed embeddings)

We define the ***intersection of*** n ***locally closed embeddings*** $X_i \hookrightarrow Z$ *($1 \le i \le n$) by the fibered product of the X_i over Z, i.e., $X_1 \times_Z \cdots \times_Z X_n$.*

Proposition 11.2.4

The intersection of a finite number of locally closed embeddings is also a locally closed embedding.

Proof It suffices to show the case that $n = 2$, by Proposition 11.2.3, $X_1 \times_Z X_2 \hookrightarrow X_2$ is locally closed embedding, by Proposition 10.2.4, $X_1 \times_Z X_2 \hookrightarrow X_2 \hookrightarrow Z$ is locally closed embedding. By induction on n, we done. □

Proposition 11.2.5

Suppose $I \subseteq A[x_1, \dots, x_m]$ and $J \subseteq A[y_1, \dots, y_n]$ are ideals. Then there is an isomorphism

$$A[x_1, \dots, x_m]/I \otimes_A A[y_1, \dots, y_n]/J \xleftrightarrow{\sim} A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J).$$

Proof By Lemma 11.2.1, we have

$$\begin{aligned} A[x_1, \dots, x_m, y_1, \dots, y_n] &\cong A[x_1, \dots, x_m, y_1, \dots, y_{n-1}] \otimes_A A[y_n] \\ &\cong A[x_1, \dots, x_m, y_1, \dots, y_{n-2}] \otimes_A A[y_{n-1}, y_n] \\ &\cong \cdots \cong A[x_1, \dots, x_m] \otimes_A A[y_1, \dots, y_n]. \end{aligned}$$

Clearly, we have

$$\begin{aligned} A[x_1, \dots, x_m, y_1, \dots, y_n]/(J) &\cong (A[x_1, \dots, x_m] \otimes_A A[y_1, \dots, y_n])/(1 \otimes J) \\ &\cong A[x_1, \dots, x_m] \otimes_A (A[y_1, \dots, y_n]/J). \end{aligned}$$

Consider the ring map $A[x_1, \dots, x_m] \to A[x_1, \dots, x_m, y_1, \dots, y_n]/(J)$, by Lemma 11.2.2, we have

$$(A[x_1, \dots, x_m, y_1, \dots, y_n]/(J))/I^e \cong (A[x_1, \dots, x_m, y_1, \dots, y_n]/(J)) \otimes_{A[x_1, \dots, x_m]} (A[x_1, \dots, x_m]/I).$$

Note that

$$(A[x_1, \dots, x_m, y_1, \dots, y_n]/(J))/I^e \cong A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J),$$

hence

$$\begin{aligned} A[x_1, \dots, x_m, y_1, \dots, y_n]/(I, J) &\cong (A[y_1, \dots, y_n]/J) \otimes_A A[x_1, \dots, x_m] \otimes_{A[x_1, \dots, x_m]} (A[x_1, \dots, x_m]/I) \\ &\cong (A[x_1, \dots, x_m]/I) \otimes_A (A[y_1, \dots, y_n]/J), \end{aligned}$$

we are done! □

As an application, we can compute tensor products of finitely generated k-algebras over k.

Example 11.1 For example we have

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Proposition 11.2.6

Suppose X and Y are locally of finite type A-schemes, then $X \times_A Y$ is also locally of finite type over A. Also the same thing with "locally" removed from both the hypothesis and conclusion.

Proof Since X and Y are locally of finite type, we may assume $X = \bigcup_i \operatorname{Spec} B_i$ and $Y = \bigcup_j \operatorname{Spec} C_j$, where B_i and C_j are finitely generated A-algebras. Note that $X \times_A Y = \bigcup_{i,j} (\operatorname{Spec} B_i \times_A \operatorname{Spec} C_j)$, it suffices to show that each $\operatorname{Spec} B_i \times_A \operatorname{Spec} C_i$ is locally of finite type A-scheme. By Proposition 11.1.1, $\operatorname{Spec} B_i \times_A \operatorname{Spec} C_j \cong \operatorname{Spec}(B_i \otimes_A C_j)$, since B_i and C_j are finitely generated A-algebras, $B_i \otimes_A C_j$ is finitely generated A-algebra, hence $\operatorname{Spec} B_i \times_A \operatorname{Spec} C_j$ is locally of finite type A-scheme, as we desired. □

Example 11.2 We can use these ideas to compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong & \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) \\
\cong & (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2+1) && \text{by Lemma 11.2.2} \\
\cong & \mathbb{C}[x]/(x^2+1) && \text{by Lemma 11.2.1} \\
\cong & \mathbb{C}[x]/((x-i)(x+i)) \\
\cong & \mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i) && \text{by the Chinese Remainder Theorem 5.4.2} \\
\cong & \mathbb{C} \times \mathbb{C}.
\end{aligned}$$

Thus $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \cong \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C}$. This example is the first example of many different behaviors. Notice for example that two points somehow correspond to the Galois group of $\mathbb{C}$ over $\mathbb{R}$; for one of them, x (the "i" in one of the copies of $\mathbb{C}$) equals i (the "i" in the other copy of $\mathbb{C}$), and in the other, $x = -i$.

Remark Here is a clue that there is something deep going on behind Example 11.2 . If L/K is a (finite) Galois extension with Galois group G, then $L \otimes_K L$ is isomorphic to L^G (the product of $|G|$ copies of L). This is turns out to be a restatement of the classical form of linear independence of characters! In the language of schemes, $\operatorname{Spec} L \times_K \operatorname{Spec} L$ is a union of a number of copies of $\operatorname{Spec} L$ that naturally form a torsor over the Galois group G; but we will not define torsor here.

✍ **Exercise 11.1** ⋆ **Hard but fascinating exercise for those familiar with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.** Show that the points of $\operatorname{Spec}(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

Proof Let $K/\mathbb{Q}$ be an finite Galois extension with Galois group $G = \operatorname{Gal}(K/\mathbb{Q})$. Consider the $K \otimes_{\mathbb{Q}} K$, we claim that $K \otimes_{\mathbb{Q}} K \cong \prod_{\sigma \in G} K$. Since $K/\mathbb{Q}$ is finite Galois extension, suppose $[K : \mathbb{Q}] = n$. Since $K/\mathbb{Q}$ is finite Galois extension, $K/\mathbb{Q}$ is finite separated, then there exists a primitive element of $K/\mathbb{Q}$, say α, with minimal polynomial $f(x) \in \mathbb{Q}[x]$ of degree n. Since $K/\mathbb{Q}$ is a Galois extension, $f(x)$ splits completely in K, i.e.,

$$f(x) = \prod_{\sigma \in G} (x - \sigma(\alpha)).$$

Consider the isomorphism $K \cong \mathbb{Q}[x]/(f(x))$. Then

$$\begin{aligned}
K \otimes_{\mathbb{Q}} K &\cong K \otimes_{\mathbb{Q}} \mathbb{Q}[x]/(f(x)) \\
&\cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[x] \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(f(x))) \\
&\cong K[x] \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(f(x)) \\
&\cong K[x]/(f(x)).
\end{aligned}$$

by the Chinese Remainder Theorem, since $f(x)$ splits into distinct linear factors in K, we have

$$K[x]/(f(x)) \cong \prod_{\sigma \in G} K[x]/(x - \sigma(\alpha)) \cong \prod_{\sigma \in G} K.$$

Hence we have $K \otimes_{\mathbb{Q}} K \cong \prod_{\sigma \in G} K$, and therefore $\operatorname{Spec} K \otimes_{\mathbb{Q}} K \cong \operatorname{Spec} \prod_{\sigma \in G} K$. Note that the prime ideal of $\prod_{\sigma \in G} K$ is

$$\mathfrak{p}_\sigma = \{(x_\tau)_{\tau \in G} : x_\tau = K, \text{ if } \tau \neq \sigma, x_\sigma = (0)\}.$$

Then we get a one-to-one corresponding between $\operatorname{Spec} K \otimes_{\mathbb{Q}} K$ and $\operatorname{Gal}(K/\mathbb{Q})$.

Now consider $\overline{\mathbb{Q}}/\mathbb{Q}$. It is easy to check that $\overline{\mathbb{Q}} = \operatorname{colim}_{i \in I} K_i$, where K_i is finite Galois extension of $\mathbb{Q}$

and I is a filtered index set. Since tensor product is left adjoint functor, it preserves colimit, so we have

$$\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \operatorname{colim}_{i,j \in I}(K_i \otimes_{\mathbb{Q}} K_j).$$

Since I is filtered, note that the following diagram,

$$\begin{tikzcd}
\operatorname{colim}_{i,j\in I} K_i \otimes_{\mathbb{Q}} K_j \arrow[rrr, dashed] & & & \operatorname{colim}_{i\in I} K_i \otimes_{\mathbb{Q}} K_i \\
& K_i \otimes_{\mathbb{Q}} K_j \arrow[ul] \arrow[r, hook, "\exists l \geq i,j"] \arrow[d, hook] & K_l \otimes_{\mathbb{Q}} K_l \arrow[ur] \arrow[d, hook] & \\
& K_s \otimes_{\mathbb{Q}} K_t \arrow[uul] \arrow[r, hook, "\exists m \geq s,t,l"] & K_m \otimes_{\mathbb{Q}} K_m \arrow[uur] &
\end{tikzcd}$$

and diagram

$$\begin{tikzcd}
\operatorname{colim}_{i\in I} K_i \otimes_{\mathbb{Q}} K_i \arrow[rr, dashed] & & \operatorname{colim}_{i,j\in I} K_i \otimes_{\mathbb{Q}} K_j \\
& K_i \otimes_{\mathbb{Q}} K_i \arrow[ul] \arrow[ur] \arrow[d, hook] & \\
& K_s \otimes_{\mathbb{Q}} K_s \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

by the universal property of colimit, we have

$$\operatorname{colim}_{i,j\in I}(K_i \otimes_{\mathbb{Q}} K_j) \cong \operatorname{colim}_{i\in I}(K_i \otimes_{\mathbb{Q}} K_i)$$

The Spec is a contravariant functor, so we have

$$\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \lim_{i\in I} \operatorname{Spec}(K_i \otimes_{\mathbb{Q}} K_i).$$

By our above discussion, we have a bijection $\operatorname{Spec} K \otimes_{\mathbb{Q}} K \overset{\sim}{\longleftrightarrow} \operatorname{Gal}(K/\mathbb{Q})$, so it suffices to consider $\lim_{i\in I} \operatorname{Gal}(K_i/\mathbb{Q})$. Consider the following universal problem,

$$\begin{tikzcd}
G \arrow[rr, dashed, "\exists!?"] \arrow[dr, dashed, "\varphi_j"] \arrow[ddr, bend right, "\varphi_i"'] & & \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[dl, dashed, "\operatorname{res}_{\overline{\mathbb{Q}},K_j}"'] \arrow[ddl, bend left, "\operatorname{res}_{\overline{\mathbb{Q}},K_i}"] \\
& \operatorname{Gal}(K_j/\mathbb{Q}) \arrow[d, dashed, "\operatorname{res}_{K_j,K_i}"] & \\
& \operatorname{Gal}(K_i/\mathbb{Q}) &
\end{tikzcd}$$

where G is any group with $\varphi_i = \operatorname{res}_{K_j,K_i} \circ \varphi_j$ and $\operatorname{res}_{K_j,K_i}(\sigma) = \sigma|_{K_i}$. Define $\theta : G \to \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as following. Pick $q \in \overline{\mathbb{Q}}$, then q is algebraic over $\mathbb{Q}$, so it must belong to some K_i, define $\theta(g)(q) = \varphi_i(g)(q)$, since each $\varphi_i(g)$ belong to $\operatorname{Gal}(K_i, \mathbb{Q})$, we know that $\theta(g) \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Uniqueness is clear, and diagram commutes is a straightforward check. Hence

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cong \lim_{i\in I} \operatorname{Gal}(K_i/\mathbb{Q}),$$

and therefore there is a bijection

$$\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \lim_{i\in I} \operatorname{Spec}(K_i \otimes_{\mathbb{Q}} K_i) \overset{\sim}{\longleftrightarrow} \lim_{i\in I} \operatorname{Gal}(K_i/\mathbb{Q}) = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

as we desired. □

At this point, we can compute any $A \otimes_B C$ (where A and C are B-algebras): any map of rings $\varphi : B \to A$ can be interpreted by adding variables (perhaps infinitely many) to B, and then imposing relations. But in practice §11.2.4 is useful, as we will see in examples.

11.2.4 Base change of affine schemes by localization.

Proposition 11.2.7 (Localizations are preserved by base change)

Suppose $\varphi : B \to A$ is a ring morphism, and $S \subseteq B$ is a multiplicative subset of B, which implies that $\varphi(S)$ is a multiplicative subset of A. Then there is a natural isomorphism

$$\varphi(S)^{-1}A \xleftrightarrow{\sim} A \otimes_B (S^{-1}B).$$

Proof Define $\tau : A \otimes_B (S^{-1}B) \to \varphi(S)^{-1}A$ by setting

$$a \otimes \frac{b}{s} \longmapsto \frac{a\varphi(b)}{\varphi(s)}.$$

Let $\frac{a}{\varphi(s)} \in \varphi(S)^{-1}A$, then $\tau\left(a \otimes \frac{1}{s}\right) = \frac{a}{\varphi(s)}$, it follows that τ is surjective. Let $\tau\left(a \otimes \frac{b}{s}\right) = 0$ in $\varphi(S)^{-1}A$, then exists $\varphi(t) \in \varphi(S)$ such that $a\varphi(bt) = 0$. Note that

$$a \otimes \frac{b}{s} = a\varphi(bt) \otimes \frac{1}{st} = 0 \otimes \frac{1}{st} = 0,$$

we know that τ is injective. Then we have an isomorphism

$$\varphi(S)^{-1}A \xleftrightarrow{\sim} A \otimes_B (S^{-1}B).$$

□

Remark Informal translation: "the fibered product with a localization is the localization of the fibered product in the obvious way." We say that "localizations are preserved by base change". This is handy if the localization is of the form $B \hookrightarrow B_f$ (corresponding to taking distinguished open sets) or (if B is an integral domain) $B \hookrightarrow K(B)$ (corresponding to taking the generic point), and various things in between.

11.2.5 Examples

Lemma 11.2.3 (The three important types of monomorphisms of schemes)

The following are monomorphisms: open embeddings, closed embeddings, and localization of affine schemes.

Proof We have already showed that open embeddings and closed embeddings are monomorphisms in Proposition 9.1.5 and Proposition 10.1.9. It suffices to show that localization of affine schemes are monomorphisms. Suppose $\iota : \operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, we want to show that ι is a monomorphism. Let $\mu_1 : Z \to \operatorname{Spec} A_f$ and $\mu_2 : Z \to \operatorname{Spec} A_f$ such that $\iota \circ \mu_1 = \iota \circ \mu_2$. Then $\iota^\sharp : A \to A_f, \mu_1^\sharp : A_f \to \Gamma(Z, \mathscr{O}_Z)$ and $\mu_2^\sharp : A_f \to \Gamma(Z, \mathscr{O}_Z)$. Since $\iota \circ \mu_1 = \iota \circ \mu_2$, we have $\mu_1^\sharp \circ \iota^\sharp = \mu_2^\sharp \circ \iota^\sharp$, by the universal property of localization, $\mu_1^\sharp = \mu_2^\sharp$, and therefore $\mu_1 = \mu_2$, which implies that ι is a monomorphism. □

Remark As monomorphisms are preserved by composition compositions of the above are also monomorphisms — for example, locally closed embeddings, or maps from "Spec of stalks at points of X" to X.

Remark **Caution:** if p is a point of a scheme X, the natural morphism $\operatorname{Spec} \mathscr{O}_{X,p} \to X$, cf. Proposition 8.3.10, is a monomorphism but is not in general an open embedding (since it is the composition of open embedding and localization of affine scheme).

✍ **Exercise 11.2** Recall that $\mathbb{A}_A^n \cong \operatorname{Spec} A \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^n$ (Definition 11.2.1 and Lemma 11.2.1). Prove similarly that $\mathbb{P}_A^n \cong \mathbb{P}_{\mathbb{Z}}^n \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$. Thus affine space and projective space are pulled back from their "universal manifestation" over the final object $\operatorname{Spec} \mathbb{Z}$.

Proof Note that

$$\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n] = \bigcup_{i=0}^{n} \operatorname{Spec} \mathbb{Z}[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$$

and each $\operatorname{Spec} \mathbb{Z}[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$ is isomorphic to $\mathbb{A}^n_{\mathbb{Z}}$, hence by the proof of Theorem 11.1.1 (**Step 2**, they glue together) and Lemma 11.2.1 we have

$$\begin{aligned}
\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A &\cong \bigcup_{i=0}^{n} (\operatorname{Spec} \mathbb{Z}[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A) \\
&\cong \bigcup_{i=0}^{n} \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \\
&\cong \mathbb{P}^n_A,
\end{aligned}$$

as we desired. □

Extending the base field.

One special case of base change is called **extending the base field**.

Definition 11.2.4 (Extending the base field)

*If X is a k-scheme, and l is a field extension of k (often l is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ (sometimes informally written $X \times_k l$ or X_l) is an l-scheme. This special case of base change is called **extending the base field**.*

Often properties of X can be checked by verifying them instead on X_l. This is the subject of descent — certain properties "descend" from X_l to X. We have already seen that the property of being the Spec of a normal integral domain descends in this way (Exercise 6.10). Proposition 11.2.8 and Proposition 11.2.9 give other examples of properties which descend: the property of two morphisms being equal, and the property of a morphism being a closed embedding, both descend in this way. Those interested in schemes over non-algebraically closed fields will use this repeatedly, to reduce results to the algebraically closed case.

Proposition 11.2.8

Suppose $\pi : X \to Y$ and $\rho : X \to Y$ are morphisms of k-schemes, l/k is a field extension, and $\pi_l : X \times_{\operatorname{Spec} k} \operatorname{Spec} l \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} l$ and $\rho_l : X \times_{\operatorname{Spec} k} \operatorname{Spec} l \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} l$ are the induced maps of l-schemes. If $\pi_l = \rho_l$, then $\pi = \rho$.

Remark In the proof, we use the fact that $X \times_{\operatorname{Spec} k} \operatorname{Spec} l \to X$ is surjective, we will soon prove in §11.4.

Proof Consider the following diagram.

$$\begin{tikzcd}
X \times_{\operatorname{Spec} k} \operatorname{Spec} l \arrow[dr, dashed, shift left, "\pi_l"] \arrow[dr, dashed, shift right, "\rho_l"'] \arrow[drr, bend left] \arrow[dd, "\iota_X"'] & & \\
& Y \times_{\operatorname{Spec} k} \operatorname{Spec} l \arrow[r] \arrow[d, "\iota_Y"] & \operatorname{Spec} l \arrow[d] \\
X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\rho"'] & Y \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Then we get $\pi_l : X \times_{\operatorname{Spec} k} \operatorname{Spec} l \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} l$ and $\rho_l : X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ which are the induced maps of l-schemes, by the universal property of fibered product. Suppose $\pi_l = \rho_l$.

We first show that $\pi = \rho$ on the level of sets. Let $p \in X$, we want to show that $\pi(p) = \rho(p)$. Since

$X \times_{\operatorname{Spec} k} \operatorname{Spec} l \to X$ is surjective, there exists $q \in X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ such that $\iota_X(q) = p$, then we have

$$\pi(p) = \pi \circ \iota_X(q) = \iota_Y \circ \pi_l(q) = \iota_Y \circ \rho_l(q) = \rho \circ \iota_X(q) = \rho(p),$$

as we desired.

We next show that $\pi = \rho$ on the level of scheme. We may reduced the case where X and Y are affine. Suppose $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, by Proposition 11.1.1, we have $X \times_{\operatorname{Spec} k} \operatorname{Spec} l \cong \operatorname{Spec}(A \otimes_k l)$ and $Y \times_{\operatorname{Spec} k} \operatorname{Spec} l \cong \operatorname{Spec}(B \otimes_k l)$. Since $\pi_l = \rho_l$, we have $\pi_l^\sharp = \rho_l^\sharp$. Consider the following diagram.

$$\begin{tikzcd} B \arrow[r, shift left, "\pi^\sharp"] \arrow[r, shift right, "\rho^\sharp"'] \arrow[d, "\iota_Y^\sharp"'] & A \arrow[d, "\iota_X^\sharp"] \\ B \otimes_k l \arrow[r, shift left, "\pi_l^\sharp"] \arrow[r, shift right, "\rho_l^\sharp"'] & A \otimes_k l \end{tikzcd}$$

we want to show that $\pi^\sharp = \rho^\sharp$. Let $b \in B$, then we have

$$\iota_X^\sharp \circ \pi^\sharp(b) = \pi_l^\sharp \circ \iota_Y^\sharp(b) = \rho_l^\sharp \circ \iota_Y^\sharp(b) = \iota_X^\sharp \circ \rho^\sharp(b),$$

since $\iota_X^\sharp$ is injective, we have $\pi^\sharp(b) = \rho^\sharp(b)$, i.e., $\pi^\sharp = \rho^\sharp$. We are done! $\square$

Proposition 11.2.9

Suppose $\pi : X \to Y$ is a(n affine) morphism over k, and l/k is a field extension. Then π is a closed embedding if and only if $\pi \times_k l : X \times_k l \to Y \times_k l$ is closed embedding.

Remark The affine hypothesis is not necessary for this result, but it makes the proof easier, and the this is the situation in which we will most need it.

Proof If $\pi : X \to Y$ is a closed embedding, we may suppose that $X = \operatorname{Spec} B/I$ and $Y = \operatorname{Spec} B$. By Proposition 11.1.1, we know that $X \times_k l \cong \operatorname{Spec}(B/I \otimes_k l)$ and $Y \times_k l \cong \operatorname{Spec}(B \otimes_k l)$. Clearly $B \otimes_k l \to B/I \otimes_k l$ is surjective, and therefore $\pi \times_k l : X \times_k l \to Y \times_k l$ is closed embedding.

Conversely, if $\pi \times_k l : X \times_k l \to Y \times_k l$ is a closed embedding,, we want to show that $\pi : X \to Y$ is closed embedding. Let $\operatorname{Spec} B \subseteq Y$ be an affine open subset of Y, since $\pi : X \to Y$ is affine, we may assume $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. Since $\pi \times_k l$ is closed embedding, by Proposition 11.1.1, we know that $B \otimes_k l \to A \otimes_k l$ is surjective. Consider the following exact sequence.

$$B \longrightarrow A \longrightarrow \operatorname{Coker}(B \to A) \longrightarrow 0$$

By Exercise 2.37, we know that $\square \otimes_k l$ is a right-exact covariant functor, so we have the exact sequence

$$B \otimes_k l \longrightarrow A \otimes_k l \longrightarrow \operatorname{Coker}(B \to A) \otimes_k l \longrightarrow 0.$$

Since $B \otimes_k l \to A \otimes_k l$ is surjective, we know that $\operatorname{Coker}(B \to A) \otimes_k l = 0$, then $\operatorname{Coker}(B \to A) = 0$, i.e., $B \to A$ is surjective. It follows that $\pi : X \to Y$ is closed embedding. $\square$

Exercise 11.3 Seemingly pathological behavior in nonpathological circumstances Suppose k is a field, and $A = k(x) \otimes_k k(y)$. Show that A is a (nonzero) localization of $k[x, y]$, hence an integral domain. Thus (0) is the unique minimal prime ideal of A. Show that the remaining prime ideals of A correspond to ideals $(f(x, y)) \subseteq k[x, y]$, where $f(x, y)$ is an irreducible polynomial in $k[x, y]$ containing both the variables x and y.

Proof Say $S_x = k[x] \setminus (0)$ and $S_y = k[y] \setminus (0)$, then we have

$$k(x) \otimes_k k(y) = S_x^{-1}k[x] \otimes_k S_y^{-1}k[y] \cong (S_x \otimes S_y)^{-1}(k[x] \otimes_k k[y]),$$

Let $S = \{p(x)q(y) : p(x) \in S_x \text{ and } q(y) \in S_y\}$, by Lemma 11.2.1, we have $k[x] \otimes_k k[y] \cong k[x, y]$, and therefore we have

$$S^{-1}k[x,y] \cong (S_x \otimes S_y)^{-1}(k[x] \otimes_k k[y]).$$

Since $k[x,y]$ is integral domain, we know that $S^{-1}k[x,y]$ is integral domain, thus (0) is the unique minimal prime ideal of A. Let $\mathfrak{p}$ be an prime ideal of A, by our isomorphism, $\mathfrak{p}$ corresponds to the prime ideal of $k[x,y]$ which don't meet S, i.e., $(f(x,y)) \subseteq k[x,y]$, where $f(x,y)$ is an irreducible polynomial in $k[x,y]$ containing both the variables x and y. □

This example will come up again in remarks of Chapter 13 and Chapter 25. The idea in the solution to Exercise 11.3 also yields the following.

Exercise 11.4 Unimportant but fun exercise Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. This scheme doesn't come up in nature, but it is certainly neat!

Proof Note that

$$\mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C} \cong (\mathbb{Q}(t)_{(0)} \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{Q}[t]_{(0)} \otimes_{\mathbb{Q}[t]} \mathbb{C}[t] \cong S^{-1}C[t]$$

where $S = \mathbb{Q}[t] \setminus (0)$, hence $\operatorname{Spec}(\mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}) \cong \operatorname{Spec} S^{-1}\mathbb{C}[t]$. The prime ideal of $S^{-1}\mathbb{C}[t]$ is the prime ideal of $\mathbb{C}[t]$ which do not meet $\mathbb{Q}[t] \setminus (0)$. Let $\mathfrak{p} = (t-a)$ be the prime ideal of $\mathbb{C}[t]$ which do not meet $\mathbb{Q}[t] \setminus (0)$, then for all $f(t) \in \mathbb{Q}[t] \setminus (0)$, $f(a) \neq 0$, i.e., a is transcendental complex numbers. □

11.3 Interpretations: Pulling back families, and fibers of morphisms

11.3.1 Pulling back families

Before making any definitions, we give a motivating informal example. Consider the "family of curves"

$$y^2 = x^3 + tx$$

in the xy-plane parametrized by t. Translation: consider

$$\operatorname{Spec} k[x,y,t]/(y^2 - x^3 - tx) \longrightarrow \operatorname{Spec} k[t].$$

If we "pull back this family" to the uv-plane via $uv = t$, we get the family

$$y^2 = x^3 + uvx.$$

If instead we set t to 3, we get the curve

$$y^2 = x^3 + 3x,$$

which we interpret as the fiber of the original family above $t = 3$. You may have noticed the fibered products underlying in there constructions:

$$\begin{tikzcd} \operatorname{Spec} k[x,y,u,v]/(y^2-x^3-uvx) \arrow[r] \arrow[d] & \operatorname{Spec} k[x,y,t]/(y^2-x^3-tx) \arrow[d] \\ \operatorname{Spec} k[u,v] \arrow[r, "uv \leftarrow t"] & \operatorname{Spec} k[t] \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{Spec} k[x,y]/(y^2-x^3-3x) \arrow[r] \arrow[d] & \operatorname{Spec} k[x,y,t]/(y^2-x^3-tx) \arrow[d] \\ \operatorname{Spec} k \arrow[r, "3 \leftarrow t"] & \operatorname{Spec} k[t] \end{tikzcd}$$

We now formalize this.

Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a **base scheme** (or just plain **base**) Z." Then if we have another another morphism $\psi : X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family" (see Figure 11.1).

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\text{pulled back family}"'] & Y \arrow[d, "\text{family}"] \\ X \arrow[r, "\psi"] & Z \end{tikzcd}$$

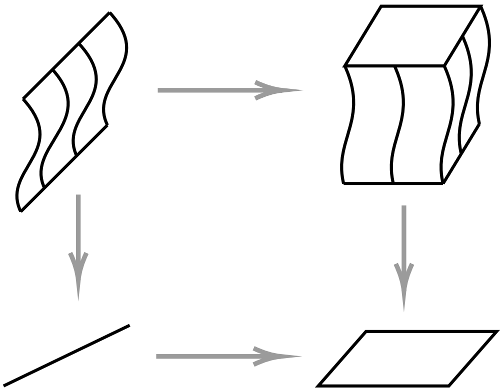


Figure 11.1: A picture of a pulled back family

Remark **Informal remark:** imagine Z as a line (the parameter axis), and Y as a family of curves floating above Z. When we map another space X to Z via ψ, we pull Y back along ψ to above X, forming the new family $X \times_Z Y$. This is analogous to changing the coordinate system or the parameterization method.

We sometimes say that $X \times_Z Y$ is the **scheme-theoretic preimage** (or scheme-theoretic pullback, or scheme-theoretic inverse image, or inverse image scheme) of Y. (Our fourth-coming discussion of fibers may give some motivation for this.) For this reason, fibered product is often called **base change** or **change of base** or **pullback**. In addition to the various names for a Cartesian diagram given in §2.2.4, in algebraic geometry it is often called a **base change diagram** or a **pullback diagram**, and $X \times_Z Y \to X$ is called the **pullback** of $Y \to Z$ by ψ, and $X \times_Z Y$ is called the **pullback** of Y by ψ. One often uses the phrase "over X" or "above X" when discussing $X \times_Z Y$, especially if X is a locally closed subscheme of Z.

Remark **Random side remark:** scheme theoretic preimage always makes sense, while the notion of scheme-theoretic image in somehow problematic, as discussed in §10.4.1.

11.3.2 Fibers of morphisms

Recall Proposition 9.3.13, that finite morphisms have finite fibers, you will not find the following discussion surprising. A special case of pull pack is the notion of a fiber of a morphism. We motivate this with the notion of fiber in the category of topological spaces.

Proposition 11.3.1

If $Y \to Z$ is a continuous map of topological spaces, and X is a point p of Z, then the fiber of Y over p (the set-theoretic fiber, with the induced topology) is naturally identified with $X \times_Z Y$.

Proof Say the continuous map $\pi : Y \to Z$, we want to show that $\pi^{-1}(p)$ is the fibered product of $\{p\}$ and Y

over Z. Consider the following diagram,

$$
\begin{tikzcd}
W \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed] \arrow[ddr, bend right, "\beta"'] & & \\
& \pi^{-1}(p) \arrow[r, hook, "i_1"'] \arrow[d, "\pi|_{\{p\}}"] & Y \arrow[d, "\pi"] \\
& \{p\} \arrow[r, hook, "i_2"'] & Z
\end{tikzcd}
\tag{11.7}
$$

where $W \in \operatorname{obj}(\mathbf{Top})$ be any topological space. We want to show that there exists unique continuous map $\theta : W \to \pi^{-1}(p)$ such that diagram (11.7) commutes. Define $\theta : W \to \pi^{-1}(p)$ by setting $\theta = \pi^{-1} \circ \beta$. Pick $q \in W$, by $\pi \circ \alpha = i_2 \circ \beta$, we have

$$i_1 \circ \theta(q) = i_1 \circ \pi^{-1} \circ \beta(q) = \pi^{-1} \circ \pi \circ \alpha(q) = \alpha(q),$$

i.e., $i_1 \circ \theta = \alpha$. By the definition of θ, θ must be unique, so $\pi^{-1}(p) \cong \{p\} \times_Z Y$. □

More generally, we have:

Corollary 11.3.1

For any $\pi : X \to Z$, the fiber of $X \times_Z Y \to X$ over a point p of X is naturally identified with the fiber of $Y \to Z$ over $\pi(p)$.

Proof Consider the follows diagram.

$$
\begin{tikzcd}
& & \psi^{-1}(\pi(p)) \arrow[dr] \arrow[dd, "\psi"] \arrow[dll, dashed, shift left] & \\
\widetilde{\psi}^{-1}(p) \arrow[urr, dashed, shift left] \arrow[dr, hook] \arrow[dd, "\widetilde{\psi}"] & & & Y \arrow[dd, "\psi"] \\
& X \times_Z Y \arrow[urr] \arrow[dd, "\widetilde{\psi}"] & \{\pi(p)\} \arrow[dr, hook] & \\
\{p\} \arrow[urr, "\pi"] \arrow[dr, hook] & & & Z \\
& X \arrow[urr, "\pi"'] & &
\end{tikzcd}
\tag{11.8}
$$

By Proposition 11.3.1, we know that $\widetilde{\psi}^{-1}(p)$ and $\psi^{-1}(\pi(p))$ are fibered products, by the universal property of fibered product, we give dashed arrows in diagram (11.8), and therefore $\widetilde{\psi}^{-1}(p) \cong \psi^{-1}(\pi(p))$, i.e., the fiber of $X \times_Z Y \to X$ over a point p of X is naturally identified with the fiber of $Y \to Z$ over $\pi(p)$. □

Motivated by topology, we return to the category of schemes.

Definition 11.3.1 (Scheme-theoretic fiber, generic fiber)

Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). More precisely, if p is a point with residue field K, consider the map $\operatorname{Spec} K \to Z$ sending $\operatorname{Spec} K$ to p, with the natural isomorphism of residue fields. Then if $g : Y \to Z$ is any morphism, the scheme-theoretic preimage of p is called the ***scheme-theoretic fiber*** *of g above p, and is denoted $g^{-1}(p) := \operatorname{Spec} K \times_Z Y$.*
If Z is irreducible, the scheme-theoretic fiber above the generic point of Z is called the ***generic fiber*** *(of g).*

Remark In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism $\operatorname{Spec} K \to Z$ corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. This is the composition of localization and closed embedding, and thus can be computed by the tricks above. (Note that $p \to Z$ is a monomorphism, by Lemma 11.2.3.)

Proposition 11.3.2

The underlying topological space of the scheme-theoretic fiber of $X \to Y$ above a point p is naturally identified with the topological fiber $X \to Y$ above p.

Proof Say $\pi : X \to Y$. Let $p \in Y$, the scheme-theoretic fiber of π above p is $\operatorname{Spec}\kappa(p) \times_Y X$. As set, by Exercise 2.10,

$$\operatorname{Spec}\kappa(p) \times_Y X = \{(p, x) \in \operatorname{Spec}\kappa(p) \times X : \pi(x) = p\}.$$

Hence, as set, we have $\operatorname{Spec}\kappa(p) \times_Y X = \pi^{-1}(\operatorname{Spec}\kappa(p))$. Consider the following diagram (as topological spaces),

$$\begin{tikzcd}
\operatorname{Spec}\kappa(p) \times_Y X \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\mathrm{pr}_X"] \arrow[ddr, bend right] & & \\
& \pi^{-1}(\operatorname{Spec}\kappa(p)) \arrow[r, hook, "i"'] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\
& \operatorname{Spec}\kappa(p) \arrow[r, hook] & Y
\end{tikzcd}$$

dashed arrow given by the universal property of $\pi^{-1}(\operatorname{Spec}\kappa(p))$ (as topological space), and $\theta : \operatorname{Spec}\kappa(p) \times_Y X \to \pi^{-1}(\operatorname{Spec}\kappa(p))$ is defined by $(p, x) \mapsto x$. Let V be an open subset of $\pi^{-1}(\operatorname{Spec}\kappa(p))$, its also an open subset of X. Say $\mathrm{pr}_X : \operatorname{Spec}\kappa(p) \times_Y X \to X$ and $i : \pi^{-1}(\operatorname{Spec}\kappa(p)) \hookrightarrow X$, by $i \circ \theta = \mathrm{pr}_X$, we have $\mathrm{pr}_X^{-1}(V) = \theta^{-1}(V)$ is an open subset of $\operatorname{Spec}\kappa(p) \times_Y X$, and therefore θ is continuous. We next show that θ is open, pick $U \subseteq \operatorname{Spec}\kappa(p) \times_Y X$ be an open subset, since $\operatorname{Spec}\kappa(p)$ only has two open set $\varnothing$ and $\operatorname{Spec}\kappa(p)$, hence the open subset of $\operatorname{Spec}\kappa(p) \times_Y X$ looks like $\mathrm{pr}_X^{-1}(V)$ where V is any open subset of X. Hence, by $i \circ \theta = \mathrm{pr}_X$, we know that θ is open. Since θ is continuous, open, and bijective, θ is homeomorphism. It follows that the underlying topological space of the scheme-theoretic fiber of $X \to Y$ above a point p is naturally identified with the topological fiber $X \to Y$ above p. $\square$

Proposition 11.3.3

Suppose that $\pi : Y \to Z$ and $\tau : X \to Z$ are morphisms, and $p \in X$ is a point. Then the fiber of $X \times_Z Y \to X$ over p is (isomorphic to) the base change to p of the fiber of $\pi : Y \to Z$ over $\tau(p)$.

Proof Consider the following diagram.

$$\begin{tikzcd}
\psi^{-1}(p) \arrow[rrrr] \arrow[dr, dashed, shift left] \arrow[dddr] & & & & X\times_Z Y \arrow[dd, "\psi"] \arrow[dr] & \\
& p\times_{\tau(p)}(\tau(p)\times_Z Y) \arrow[dd] \arrow[dr] \arrow[ul, dashed, shift left] & & & & Y \arrow[dd, "\pi"] \\
& & \tau(p)\times_Z Y \arrow[uurrr] \arrow[dd] & & X \arrow[dr, "\tau"] & \\
& p \arrow[uurrr, hook] \arrow[dr, "\tau"] & & & & Z \\
& & \tau(p) \arrow[urrr, hook] & & &
\end{tikzcd}$$

Each square is Cartesian square, and by the universal property of fibered product, we have

$$\psi^{-1}(p) \cong p \times_{\tau(p)} (\tau(p) \times_Z Y),$$

as we desired. □

Example 11.3 (Enlightening in several ways) Consider the projection of the parabola $y^2 = x$ to the x-axis over $\mathbb{Q}$, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens. (you should look at Figure 11.2, which is a flipped version of the parabola of Figure 4.8, and figure out how to edit it to reflect what we glean here.) Writing $\mathbb{Q}[y]$ as $\mathbb{Q}[x,y]/(y^2 - x)$ helps us interpret the morphism conveniently.

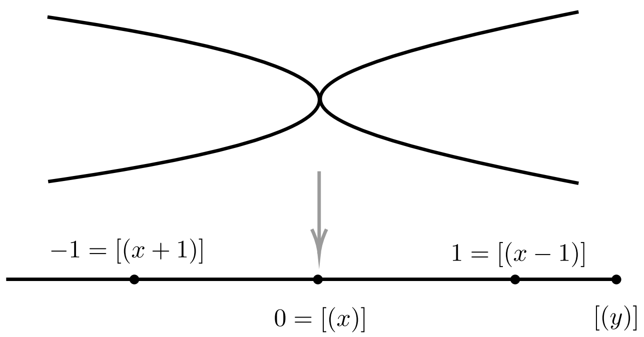


Figure 11.2: The map $\mathbb{C} \to \mathbb{C}$ given by $y \mapsto y^2$ (cf. Figure 4.8)

(i) Then the preimage of 1 is two points:

$$\begin{aligned}
\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x, x - 1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y^2 - 1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y-1) \coprod \operatorname{Spec} \mathbb{Q}[y]/(y+1).
\end{aligned}$$

(ii) The preimage of 0 is one nonreduced point:

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x, x) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2).$$

(iii) The preimage of -1 is one reduced point, but of "size 2 over the base field".

$$\begin{aligned}\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x+1) &\cong \operatorname{Spec}\mathbb{Q}[y]/(y^2+1)\\ &\cong \operatorname{Spec}\mathbb{Q}[i]\\ &= \operatorname{Spec}\mathbb{Q}(i).\end{aligned}$$

(iv) The preimage of the generic point is again one reduced point, but of "size 2 over the residue field", as we verify now.

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}[x]}\mathbb{Q}(x)\cong \operatorname{Spec}\mathbb{Q}[y]\otimes_{\mathbb{Q}[y^2]}\mathbb{Q}(y^2)$$

i.e., (informally) the Spec of the ring of polynomials in y divided by polynomials in y^2. A little thought shows you that in this ring you may invert any polynomial in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)}=\frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}[x]}\mathbb{Q}(x)\cong\mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$ (note that $\mathbb{Q}(x)=\mathbb{Q}(y^2)$).

(You might want to work through the preimages of other points of $\mathbb{A}^1_{\mathbb{Q}}$, such as $[(x^3-4)]$ and $[(x^2-2)]$.)

Notice the following interesting fact: in each of the four cases, the number of preimage can be interpreted as 2, where you count to two in several ways: you can count points (as in the case of the preimage of 1); you can get nonreduced behavior (as in the case of the preimage of 0); or you can have a field extension of degree 2 (as in the case of the preimage of -1 or the generic point). In each case, the fiber is an affine scheme whose dimension as a vector space over the residue field of the point is 2. Number theoretic readers may have seen this behavior before. We will discuss this example again in Chapter 17. This is going to be symptomatic of a very important kind of morphism (a finite flat morphism, see Chapter 25).

Try to draw a picture of this morphism if you can, so you can develop a pictorial shorthand for what is going on. A good first approximation is the parabola of Figure 11.2, but you will want to somehow depict the peculiarities of (iii) and (iv).

✍ **Exercise 11.5 (Important for those with more arithmetic background)** What is the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i]\to\operatorname{Spec}\mathbb{Z}$ over the prime (p)? Your answer will depend on p, and there are four cases, corresponding to the four cases of Example 11.3 .

Proof

(i) If (p) is generic point, i.e., $p=0$, the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i]\to\operatorname{Spec}\mathbb{Z}$ over (0) is

$$\operatorname{Spec}\mathbb{Z}_{(0)}/(0)\mathbb{Z}_{(0)}\times_{\mathbb{Z}}\operatorname{Spec}\mathbb{Z}[i]\cong\operatorname{Spec}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}[i])\cong\operatorname{Spec}\mathbb{Q}(i).$$

Hence the preimage of the generic point is $\operatorname{Spec}\mathbb{Q}(i)$, which is a reduced point of "size 2 over the residue field".

(ii) If $p=2$, the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i]\to\operatorname{Spec}\mathbb{Z}$ over (2) is

$$\begin{aligned}\operatorname{Spec}\mathbb{Z}_{(2)}/(2)\mathbb{Z}_{(2)}\times_{\mathbb{Z}}\operatorname{Spec}\mathbb{Z}[i] &\cong \operatorname{Spec}\mathbb{F}_2\otimes_{\mathbb{Z}}\mathbb{Z}[i]\\ &\cong\operatorname{Spec}\mathbb{F}_2[x]/(x^2+1)\\ &\cong\operatorname{Spec}\mathbb{F}_2[x]/(x+1)^2.\end{aligned}$$

Hence the preimage of (2) is nonreduced point.

(iii) If $p\equiv 1 \pmod 4$, assume that $p=4k+1$, by Euler's criterion, there exists an integer b such that

$b^2 \equiv -1 \pmod p$. Hence the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec} mbz$ over (p) is

$$\operatorname{Spec}\mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[i] \cong \operatorname{Spec}\mathbb{F}_p \coprod \operatorname{Spec}\mathbb{F}_p.$$

Then the preimage of (p) is two points.

(iv) If $p \equiv 3 \pmod 4$, by Euler's criterion, there are not integer b such that $b^2 \equiv -1 \pmod p$. Hence x^2+1 is irreducible over $\mathbb{F}_2[x]$, and therefore the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ over (p) is

$$\operatorname{Spec}\mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[i] \cong \operatorname{Spec}\mathbb{F}_p[x]/(x^2+1) \cong \operatorname{Spec}\mathbb{F}_p(i).$$

Hence the preimage of (p) is a reduced point of "size 2 over the residue field".

$\square$

Remark We used following fact:

Lemma (Euler's criterion)

Let p be an odd prime and a be an integer coprime to p. Then

$$a^{\frac{p-1}{2}} \equiv 1 \pmod p$$

if and only if there is an integer x such that $x^2 \equiv a \pmod p$.

Exercise 11.6 (This exercise will give you practice in computing a fibered product over something that is not a field.) Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $u \mapsto t^2$, where $\operatorname{char} k \neq 2$. Show that $X \times_Y X$ has two irreducible components. (What happens if $\operatorname{char} k = 2$? See Exercise in §11.5. A for a clue.)

Proof Clearly, $X \times_Y X = \operatorname{Spec}(k[t] \otimes_{k[u]} k[t])$. We claim the $k[t] \otimes_{k[u]} k[t] \cong k[x,y]/(x^2-y^2)$. Define $\theta : k[x,y] \to k[t] \otimes_{k[u]} k[t]$ by setting

$$x \longmapsto t \otimes 1, \qquad y \longmapsto 1 \otimes t.$$

Then θ is a surjective. We next calculate $\operatorname{Ker}\theta$. We claim that $\operatorname{Ker}\theta = (x^2-y^2)$. Let $f \in (x^2-y^2)$, since $\theta(x^2-y^2) = t^2 \otimes 1 - 1 \otimes t^2 = u \otimes 1 - 1 \otimes u = 0$, we have $(x^2-y^2) \subseteq \operatorname{Ker}\theta$. Conversely, let $f \in \operatorname{Ker}\theta$. Define $\psi : k[t] \otimes_{k[u]} k[t] \to k[x,y]/(x^2-y^2)$, by setting

$$t \otimes 1 \longmapsto x, \qquad 1 \otimes t \longmapsto y.$$

Then $\operatorname{Ker}(\psi \circ \theta) = (x^2-y^2)$. Since $\theta(f) = 0$, we have $\psi \circ \theta(f) = 0$, which implies that $f \in \operatorname{Ker}(\psi \circ \theta) = (x^2-y^2)$. Hence $\operatorname{Ker}\theta = (x^2-y^2)$. By the Fundamental Theorem of Isomorphism of rings, we have

$$k[t] \otimes_{k[u]} k[t] \cong k[x,y]/(x^2-y^2).$$

Hence

$$\begin{aligned} X \times_Y X &\cong \operatorname{Spec} k[x,y]/(x^2-y^2) \\ &\cong \operatorname{Spec}(k[x,y]/(x-y) \times k[x,y]/(x+y)) \\ &\cong \operatorname{Spec} k[x] \coprod \operatorname{Spec} k[x] \\ &\cong \mathbb{A}_k^1 \coprod \mathbb{A}_k^1. \end{aligned}$$

If $\operatorname{char} k = 2$, we have

$$X \times_Y X \cong \operatorname{Spec} k[x,y]/(x^2-y^2) \cong \operatorname{Spec} k[x,y]/(y-x)^2,$$

which is nonreduced. $\square$

11.3.3 A first view of a blow-up

Exercise 11.7 (Important concrete exercise) (The discussion here immediately generalizes to $\mathbb{A}^n_A$.) Define a closed subscheme $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ of $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$ as follows (see Figure 11.3). If the coordinates on $\mathbb{A}^2_k$ are x, y, and the projective coordinate on $\mathbb{P}^1_k$ are u, v, this subscheme is cut out in $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$ by the single equation $xv = yu$. (You may wish to interpret $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ as follows. The $\mathbb{P}^1_k$ parametrizes lines through the origin. The blow-up corresponds to ordered pairs of (point p, line l) such that $(0,0)$ and p both lie on l.)

(i) Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$.

(ii) Show that the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$ is an isomorphism away from $(0,0) \in \mathbb{A}^2_k$.

(iii) Show that the fiber of $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$ over $(0,0)$ is an effective Cartier divisor (Definition 10.5.2, a closed subscheme that is locally cut out by a single equation, which is not a zero divisor). It is called the **exceptional divisor**.

We will discuss blow-ups in Chapter 23.

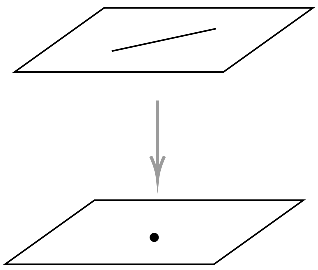


Figure 11.3: A first example of a blow-up

Proof

(i) Consider $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$, we have

$$\begin{aligned}
\mathbb{A}^2_k \times_k \mathbb{P}^1_k &= \operatorname{Spec} k[x,y] \times_{\operatorname{Spec} k} \operatorname{Proj} k[u,v] \\
&\cong \left(\operatorname{Spec} k[x,y] \times_{\operatorname{Spec} k} \operatorname{Spec} k\left[\frac{u}{v}\right]\right) \cup \left(\operatorname{Spec} k[x,y] \times_{\operatorname{Spec} k} \operatorname{Spec} k\left[\frac{v}{u}\right]\right) \\
&\cong \operatorname{Spec}\left(k[x,y] \otimes_k k\left[\frac{u}{v}\right]\right) \cup \operatorname{Spec}\left(k[x,y] \otimes_k k\left[\frac{v}{u}\right]\right) \\
&\cong \operatorname{Spec} k\left[x, y, \frac{u}{v}\right] \cup \operatorname{Spec} k\left[x, y, \frac{v}{u}\right].
\end{aligned}$$

Since $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is cut out by $xv = yu$, we know that

$$\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \cong \operatorname{Spec} \frac{k[x,y,\frac{u}{v}]}{(x - \frac{u}{v} y)} \cup \operatorname{Spec} \frac{k[x,y,\frac{v}{u}]}{(\frac{v}{u} x - y)} \cong \operatorname{Spec} k\left[y, \frac{u}{v}\right] \cup \operatorname{Spec} k\left[x, \frac{v}{u}\right]. \tag{11.9}$$

Let $P = [a:b]$ be a closed point of $\mathbb{P}^1_k$, we may assume that $b \neq 0$, then P is in the affine piece $\operatorname{Spec} k\left[\frac{u}{v}\right]$. The residue field of P is

$$\kappa(P) = \mathscr{O}_{\mathbb{P}^1, P}/\mathfrak{m}_P \cong k\left[\frac{u}{v}\right] \Big/ \left(\frac{u}{v} - \frac{a}{b}\right).$$

Hence the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ is

$$\begin{aligned}
\operatorname{Spec} \kappa(P) \times_{\mathbb{P}^1_k} \mathrm{Bl}_{(0,0)} \mathbb{A}^2_k &\cong \operatorname{Spec} \kappa(P) \times_{k[\frac{u}{v}]} \operatorname{Spec} k\left[y, \frac{u}{v}\right] \\
&\cong \operatorname{Spec}\left(\kappa(P) \otimes_{k[\frac{u}{v}]} k\left[y, \frac{u}{v}\right]\right) \\
&\cong \operatorname{Spec} k\left[y, \frac{u}{v}\right] \Big/ \left(\frac{u}{v} - \frac{a}{b}\right) \\
&\cong \operatorname{Spec} k[y] \cong \mathbb{A}^1_k.
\end{aligned}$$

(ii) Note that $\mathbb{A}_k^2 \setminus \{(0,0)\} = \operatorname{Spec} k[x,y]_x \cup \operatorname{Spec} k[x,y]_y$, we claim that $\operatorname{Spec} k\left[x,y,\frac{u}{v}\right] / \left(x - \frac{u}{v}y\right) \cong \operatorname{Spec} k[x,y]_y$. Say $t = \frac{u}{v}$, define $\varphi : k[x,y]_y \to k[x,y,t]/(x-ty)$, by setting

$$x \longmapsto ty, \qquad y \longmapsto y.$$

Define $\psi : k[x,y,t]/(x-ty) \to k[x,y]_y$ by setting

$$y \longmapsto y, \qquad t \longmapsto \frac{x}{y}.$$

It is easy to see that $\psi \circ \varphi = \mathrm{id}_{k[x,y]_y}$ and $\varphi \circ \psi = \mathrm{id}_{k[x,y,t]/(x-ty)}$. Hence $\operatorname{Spec} k\left[x,y,\frac{u}{v}\right] / \left(x - \frac{u}{v}y\right) \cong \operatorname{Spec} k[x,y]_y$. Similarly, we have $\operatorname{Spec} k\left[x,y,\frac{v}{u}\right] / \left(\frac{v}{u}x - y\right) \cong \operatorname{Spec} k[x,y]_x$. Since they can glue together, we have $\mathrm{Bl}_{(0,0)} \mathbb{A}_k^2 \cong \mathbb{A}_k^2 \setminus \{(0,0)\}$

(iii) By (11.9), the fiber of $\mathrm{Bl}_{(0,0)} \mathbb{A}_k^2 \to \mathbb{A}_k^2$ over $(0,0)$ is

$$\begin{aligned}
\operatorname{Spec} \kappa((x,y)) \times_{\mathbb{A}_k^2} \mathrm{Bl}_{(0,0)} \mathbb{A}_k^2 &\cong \left(\operatorname{Spec} k \times_{k[x,y]} \operatorname{Spec} k\left[y, \frac{u}{v}\right]\right) \cup \left(\operatorname{Spec} k \times_{k[x,y]} \operatorname{Spec} k\left[x, \frac{v}{u}\right]\right) \\
&\cong \operatorname{Spec}\left(k[x,y]/(x,y) \otimes_{k[x,y]} k\left[x,y,t\right]/(x-yt)\right) \\
&\quad \cup \operatorname{Spec}\left(k[x,y]/(x,y) \otimes_{k[x,y]} k\left[x,y,s\right]/(sx-y)\right) \\
&\cong \operatorname{Spec}(k\left[x,y,t\right]/(x-yt))/(x,y)^e \cup \operatorname{Spec}(k\left[x,y,s\right]/(sx-y))/(x,y)^e \\
&\cong \operatorname{Spec} k[y,t]/(y) \cup \operatorname{Spec} k[x,s]/(x) \\
&\cong \mathbb{P}_k^1,
\end{aligned}$$

where $t = \frac{u}{v}$ and $s = \frac{v}{u}$, and they glue together via $t = \frac{1}{s}$. Hence the fiber of $\mathrm{Bl}_{(0,0)} \mathbb{A}_k^2 \to \mathbb{A}_k^2$ over $(0,0)$ is an effective Cartier divisor.

$\square$

We haven't yet discussed regularity, but here is a hang-waving argument suggesting that the $\mathrm{Bl}_{(0,0)} \mathbb{A}_k^2$ is "smooth": the preimage above either standard open set $U_i \subseteq \mathbb{P}^1$ is isomorphic to $\mathbb{A}^2$. Thus "the blow-up is a surgery that takes the smooth surface $\mathbb{A}_k^2$, cuts out a point, and glues back in a $\mathbb{P}^1$, in such a way that the outcome is a another smooth surface."

11.3.4 General fibers, generic fibers, generically finite morphisms

The phrase "generic fiber" and "general fiber" parallel the phrases "generic point" and "general point" (Definition 4.6.6).

Definition 11.3.2 (General fiber)

*Suppose $\pi : X \to Y$ is a morphism of schemes. When one says the **general fiber** of π has a certain property, this means that there exists a dense open subset $U \subseteq Y$ such that the fibers above any point in U have the property.*

Remark When one says the generic fiber of $\pi : X \to Y$ (Definition 11.3.1), this implicitly means that Y is irreducible, and the phrase refers to the fiber over the generic point. **General fiber and generic fiber are not the same thing!** Clearly if something holds for the general fiber, then it holds for the generic fiber, but the converse is not always true. However, in good circumstances, it can be — properties of the generic fiber extend to an honest open neighborhood.

Example 11.4 For example, if Y is irreducible and Noetherian, and π is finite type, then if the generic fiber of π is empty (resp., nonempty), then the general fiber is empty (resp., nonempty), by Chevalley's Theorem 9.4.1.

Proof By Chevalley's Theorem 9.4.1, we know that $\pi(X)$ is constructible set (Definition 9.4.1). Let η be the generic point of Y. Since the generic fiber of π is empty, we have $\eta \notin \pi(X)$, and therefore $\eta \in Y \setminus \pi(X)$. By Definition 9.4.1, $Y \setminus \pi(X)$ is constructible subset. By Proposition 9.4.1, there exists a locally closed subset $U \cap C$ such that $\eta \in U \cap C$ where U is a dense open subset of Y and C is closed. Since $\overline{\{\eta\}} = Y$, we may replace C by Y, then we have $\eta \in U \cap Y = U \subseteq Y \setminus \pi(X)$. Hence the fiber above any point in U is empty, as we desired. □

Definition 11.3.3 (Generically finite)

*If $\pi : X \to Y$ is finite type, we say π is **generically finite** is π is finite after base change to the generic point of each irreducible component (or equivalently, by Corollary 9.4.2, if the preimage of the generic point of each irreducible component of Y is a finite set.)*

Remark The notion of generic finiteness can be defined in more general circumstances, see Stacks Project [8] Remark 073A.

Proposition 11.3.4 ("Generically finite" usually means "Generally finite")

Suppose $\pi : X \to Y$ is an affine, finite type, generically finite morphism of locally Noetherian schemes, and Y is reduced. Then there is an open neighborhood of each generic point of Y over which π is actually finite. (The hypotheses can be weakened considerably, see Stacks Project [8] Lemma 02NW.)

Proof Since $\pi : X \to Y$ is affine, we may reduced to the case that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. Since Y is reduced, B is reduced, i.e., $\mathfrak{N}(B) = (0)$. Since Y is irreducible, by Proposition 4.6.2 (ii), $\mathfrak{N}(B)$ is prime, and therefore B is integral domain. Since $\pi : X \to Y$ is of finite type, A is a finitely generated B-algebra, so we may write $A = B[x_1, \ldots, x_n]/I$ where I is an ideal of polynomial ring $B[x_1, \ldots, x_n]$. Since π is generically finite, the induced map $\operatorname{Spec} A \otimes_B K(B) \to \operatorname{Spec} K(B)$ is finite, hence $A \otimes_B K(B)$ is a finite $K(B)$ algebra. By Corollary 9.2.1, $K(B) \to A \otimes_B K(B)$ is integral, so there are monic polynomials $f_i(t) \in K(B)[t]$ such that $f_i(x_i) = 0$ in $A \otimes_B K(B)$. Let b be the product of the (finite number of) denominators appearing in the coefficients in the $f_i(x)$, then $f_i(t) \in B_b[t]$ is monic and $f_i(x_i) = 0$ in $A \otimes_B B_b \cong A_b$. Hence for each $f_i(x_i)$ there exists s_i such that $s_i f_i(x_i) = 0$ in A. Let $s = \prod_i s_i$, localize A_b at s, we have $f_i(x_i) = 0$ in A_{bs}. Hence $B_{bs} \to A \otimes_B B_{bs} \cong A_{bs}$ is integral, and therefore $\operatorname{Spec} A_{bs} \to \operatorname{Spec} B_{bs}$ is integral. Since π is of finite type, we know that $\operatorname{Spec} A_{bs} \to \operatorname{Spec} B_{bs}$ is of finite type. By Proposition 9.3.20, $\operatorname{Spec} A_{bs} \to \operatorname{Spec} B_{bs}$ is finite. □

11.3.5 ⋆⋆ Finitely presented families (morphisms) are locally pullbacks of particularly nice families

11.4 Properties preserved by base change

All "reasonable" properties of morphisms are preserved under base change (cf. Definition 9.1.1 (ii)). We discuss this, and in §11.5.1 we will explain how to fix those that don't fit this pattern.

We have already shown that the notion of "open embedding" is preserved by base change (Proposition 9.1.4). We did this by explicitly describing what the fibered product of an open embedding is: if $Y \hookrightarrow Z$ is an open embedding, and $\psi : X \to Z$ is any morphism, then we check that the open subscheme $\psi^{-1}(Y)$ of X satisfies the universal property of fibered products.

We have also shown that the notion of "closed embedding" is preserved by base change (Proposition

11.2.1). In other words, given a Cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, hook, "\text{cl. emb.}"] \\ X \arrow[r] & Z \end{tikzcd}$$

where $Y \hookrightarrow Z$ is a closed embedding, $W \to X$ is as well.

Proposition 11.4.1

The closed embeddings and locally closed embeddings are both "reasonable" class of morphisms of schemes (Definition 9.1.1).

Proof The composition of closed embeddings is also closed embedding (Proposition 10.1.3), i.e., closed embedding is preserved by composition. By Proposition 11.2.1, closed embeddings are preserved by base change. By Proposition 10.1.4, the property of being a closed embedding is affine-local on the target. Hence closed embeddings is reasonable class of morphisms of schemes.

By Proposition 10.2.4, the composition of locally closed embeddings is also a locally closed embedding. Since open embeddings and closed embeddings local on the target (Proposition 9.1.3), locally closed embeddings is local one the target. Consider the following cartesian diagrams

$$\begin{tikzcd} (X \times_Z U) \times_U Y \arrow[r] \arrow[d, hook, "\text{cl. emb.}"'] & Y \arrow[d, hook, "\text{cl. emb.}"] \\ X \times_Z U \arrow[r] \arrow[d, hook, "\text{op. emb.}"'] & U \arrow[d, hook, "\text{op. emb.}"] \\ X \arrow[r] & Z \end{tikzcd}$$

where $X \times_Z U \hookrightarrow X$ is open embedding given by Proposition 9.1.4 and $(X \times_Z U) \times_U Y \hookrightarrow X \times_Z U$ is closed embedding given by Proposition 11.2.1. Hence $(X \times_Z U) \times_U Y \hookrightarrow X$ is locally closed embedding. By Exercise 2.13, we have $(X \times_Z U) \times_U Y \cong X \times_Z Y$, so $X \times_Z Y \hookrightarrow X$ is locally closed embedding, i.e., locally closed embedding is preserved by base change. Hence locally closed embeddings is reasonable class of morphisms of schemes. □

Corollary 11.4.1

Suppose $X \to Z$ and $Y \to Z$ are both locally closed embeddings, then $X \times_Z Y \to Z$ is a locally closed embedding.

Proof In the proof of Proposition 11.4.1, we know that locally closed embedding is preserved by base change. So $X \times_Z Y \hookrightarrow X$ and $X \times_Z Y \hookrightarrow Y$ are both locally closed embeddings. The composition of locally closed embeddings is locally closed embedding. Hence $X \times_Z Y \hookrightarrow Z$ is a locally closed embedding. □

Apply Corollary 11.4.1, we may define the scheme-theoretic intersection of two locally closed embeddings.

Definition 11.4.1 (Scheme-theoretic intersection of two locally closed embeddings)

Let $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ be two locally closed embeddings, we defined the scheme-theoretic intersection of two locally closed embedding by setting $X \times_Z Y \hookrightarrow Z$.

Proposition 11.4.2

The locally principal closed subschemes (Definition 10.5.1) pull back to locally principal closed subschemes.

Proof Let $\pi : X \hookrightarrow Z$ be a locally principal closed subschemes, and $\psi : Y \to Z$ be any morphism. We want to show that $X \times_Z Y \to Y$ is locally principal closed subscheme. Since $X \hookrightarrow Z$ is a locally principal closed subscheme, there exists an open cover of Z, say V_i such that $\pi^{-1}(V_i)$ is cut out by s_i where $s_i \in \Gamma(V_i, \mathscr{O}_Z)$. Say $V_i = \bigcup_j \operatorname{Spec} B_{ij}$, then $\pi^{-1}(V_i) = \bigcup_j \operatorname{Spec} B_{ij}/(s_i|_{B_{ij}})$. Say $\psi^{-1}(\operatorname{Spec} B_{ij}) = \bigcup_k \operatorname{Spec} A_{ijk}$, then $\psi^{-1}(V_i) = \bigcup_j \bigcup_k \operatorname{Spec} A_{ijk}$, and therefore $Y = \bigcup_{i,j,k} \operatorname{Spec} A_{ijk}$. We next calculate $X \times_Z Y$,

$$\begin{aligned} X \times_Z Y &= \bigcup_{i,j,k} \operatorname{Spec} B_{ij}/(s_i|_{B_{ij}}) \times_{\operatorname{Spec} B_{ij}} \operatorname{Spec} A_{ijk} \\ &= \bigcup_{i,j,k} \operatorname{Spec} A_{ijk} \otimes_{B_{ij}} B_{ij}/(s_i|_{B_{ij}}) \\ &\cong \bigcup_{i,j,k} \operatorname{Spec} A_{ijk}/(s_i|_{B_{ij}})^e, \end{aligned}$$

the last $\cong$ given by Lemma 11.2.2. By Definition 10.5.1, $X \times_Z Y$ is a locally principal closed subscheme. □

Similarly, other important properties are preserved by base changes.

Proposition 11.4.3

The following properties of morphisms are preserved by base change.

(a) quasi-compact
(b) affine morphism
(c) finite
(d) integral
(e) locally of finite type
(f) finite type
⋆ (g) locally of finite presentation

Proof Direct check. □

Corollary 11.4.2

The following properties of morphisms are all "reasonable" class of morphisms of schemes (Definition 9.1.1).

(a) quasi-compact
(b) affine morphism
(c) finite
(d) integral
(e) locally of finite type
(f) finite type
⋆ (g) locally of finite presentation

Proof Direct check. □

Proposition 11.4.4

(a) The notion of "quasifinite morphism" (finite type + finite fibers, Definition 9.3.10) is preserved by base change.

(b) The notion of "quasifinite morphism" is "reasonable" class of morphisms of schemes (Definition 9.1.1).

Proof

(a) Let $\pi : X \to Z$ be a quasi-finite morphism, and $\varphi : Y \to Z$ be any morphism of schemes. By Proposition 11.4.3, $X \times_Z Y \to Y$ is of finite type. It suffices to check that $X \times_Z Y \to Y$ is finite fibers. Let p be a point in Y, we want to show that $(X \times_Z Y) \times_Y \operatorname{Spec} \kappa(p)$ is a finite set. By Proposition 11.3.3, we know that

$$(X \times_Z Y) \times_Y \operatorname{Spec} \kappa(p) \cong \operatorname{Spec} \kappa(p) \times_{\operatorname{Spec} \kappa(\varphi(p))} (\operatorname{Spec} \kappa(\varphi(p)) \times_Z X).$$

Since $X \to Z$ is quasi-finite morphism, $\operatorname{Spec} \kappa(\varphi(p)) \times_Z X$ is finite set, so $\operatorname{Spec} \kappa(p) \times_{\operatorname{Spec} \kappa(\varphi(p))} (\operatorname{Spec} \kappa(\varphi(p)) \times_Z X)$ is also finite set, i.e., $X \times_Z Y \to Y$ is finite fibers.

(b) By part (a), we know that quasi-finite morphism is preserved by base change.

We first show that quasi-finite morphism is preserved by composition. Let $\pi : X \to Y$ and $\varphi : Y \to Z$ be two quasi-finite morphism, since finite type is preserved by composition, it suffices to show that $\pi^{-1} \circ \varphi^{-1}(p)$ is finite set, for all $p \in Z$. Since $\varphi^{-1}(p)$ is finite, we may assume $\varphi^{-1}(p) = \{q_1, \ldots, q_r\}$. Since π is finite fibers, so $\pi^{-1}(q_i)$ is finite for all i, and therefore $\pi^{-1}(\varphi^{-1}(p))$ is a finite set. Hence $\varphi \circ \pi$ is a quasi-finite morphism, i.e., quasi-finite is preserved by composition.

We next show that quasi-finiteness is local on the target. Recall Definition 9.1.1, property (iii) (a) is straightforward. We now check (iii) (b). Let $\pi : X \to Z$ be a quasi-finite morphism, and $\{V_i \hookrightarrow Y\}_i$ is an affine open cover of Y for which each restricted morphism $\pi_i : \pi^{-1}(V_i) \to V_i$ is quasi-finite, we need to show that π is quasi-finite. By Corollary 11.4.2, finite type is local on the target, so it suffices to show that π is finite fibers. We only need to consider this issue at the set level. Pick $p \in Z$, then p in some V_i, so $\pi^{-1}(p) \subseteq \pi^{-1}(V_i)$. Note that $\pi_i^{-1}(p) = \pi^{-1}(p) \cap \pi^{-1}(V_i) = \pi^{-1}(p)$, since π_i is finite fibers morphism, we know that $\pi^{-1}(p)$ is finite set, and therefore π is a quasi-finite morphism. Hence quasi-finiteness is local on the target.

□

Remark The notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 11.1.

Proposition 11.4.5

*(a) The surjectivity is preserved by base change. (**Surjectivity has its usual meaning: surjective as a map of sets.**)*

(b) Surjective morphisms form a "reasonable" class of morphisms of schemes (Definition 9.1.1).

Proof

(a) Let $\pi : X \to S$ be a surjective morphism, and $\varphi : Y \to S$ be any morphism. We want to show that $\psi : X \times_S Y \to Y$ is a surjective morphism. Let $p \in Y$, by Proposition 11.3.3, we want to show that

$$\psi^{-1}(p) \cong (X \times_S Y) \times_Y \operatorname{Spec} \kappa(p) \cong \operatorname{Spec} \kappa(p) \times_{\operatorname{Spec} \kappa(q)} (\operatorname{Spec} \kappa(q) \times_S X)$$

is not empty, where $q = \varphi(p)$. Since $\pi : X \to S$ is surjective, there exits $s \in X$ such that $\pi(s) = q$.

Consider the following diagram,

$$\begin{tikzcd}
\operatorname{Spec}\kappa(p) \times_{\operatorname{Spec}\kappa(q)} \operatorname{Spec}\kappa(s) \arrow[rr] \arrow[dr, hook] \arrow[ddr] & & \operatorname{Spec}\kappa(s) \arrow[dr, hook] \arrow[ddr] & \\
& \psi^{-1}(p) \arrow[rr, "\pi"] \arrow[d] & & \operatorname{Spec}\kappa(q) \times_S X \arrow[d, "\pi"] \\
& \operatorname{Spec}\kappa(p) \arrow[rr] & & \operatorname{Spec}\kappa(q)
\end{tikzcd}$$

it suffices to show that $\operatorname{Spec}\kappa(p) \times_{\operatorname{Spec}\kappa(q)} \operatorname{Spec}\kappa(s)$ is non-empty. Let $k_1 = \operatorname{Spec}\kappa(p)$, $k_2 = \operatorname{Spec}\kappa(s)$, and $k_3 = \operatorname{Spec}\kappa(q)$, then

$$\operatorname{Spec}\kappa(p) \times_{\operatorname{Spec}\kappa(q)} \operatorname{Spec}\kappa(s) \cong \operatorname{Spec} k_1 \otimes_{k_3} k_2.$$

Note that $k_1 \otimes_{k_3} k_2 \neq 0$, by Zorn's Lemma $k_1 \otimes_{k_3} k_2$ must have a maximal ideal, so $\operatorname{Spec} k_1 \otimes_{k_3} k_2$ is not empty, and therefore $\psi^{-1}(p)$ is not empty, as we desired.

(b) Clearly, surjectivity is preserved by composition and local on the target. By part (a), we know that surjective morphisms form a "reasonable" class of morphisms of schemes.

□

Remark On the other hand, injectivity is not preserved by base change — witness the bijection $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, which loses injectivity upon base change by $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ (see Example 11.2). This can be rectified (see §11.5).

Proposition 11.4.6 (cf. Proposition 11.2.6)

Suppose X and Y are integral finite type $\overline{k}$-schemes. Then $X \times_{\overline{k}} Y$ is an integral finite type $\overline{k}$-scheme.

Remark Once we define "variety", this will become the important fact that the product of irreducible varieties over an algebraically closed field is an irreducible variety. The fact that the base field $\overline{k}$ is algebraically closed is important, see §11.5.

Proof Reduce to the case where X and Y are both affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with A and B are integral finitely generated $\overline{k}$-algebra. Since A and B are finitely generated, we know that $A \otimes_{\overline{k}} B$ is finitely generated $\overline{k}$-algebra. So we only need to show that $A \otimes_{\overline{k}} B$ is an integral domain.

Suppose $(\sum a_i \otimes b_i)(\sum a'_j \otimes b'_j) = 0$ in $A \otimes_{\overline{k}} B$ with $a_i, a'_j \in A$, $b_i, b'_j \in B$, where both $\{b_i\}_i$ and $\{b'_j\}_j$ are linearly independent over $\overline{k}$, and a_1 and a'_1 are nonzero (these condition also implies that $\sum a_i \otimes b_i \neq 0$ and $\sum a'_j \otimes b'_j$). Since A is integral domain and a_1, a'_1 are nonzero, we have $a_1 a'_1 \neq 0$, so $D(a_1 a'_1) \subseteq \operatorname{Spec} A$ is nonempty. By the Weak Nullstellensatz 4.2.1, there is a maximal ideal $\mathfrak{m} \subseteq A$ in $D(a_1 a'_1)$ with $A/\mathfrak{m} = \overline{k}$. Define $\varphi : A \otimes_{\overline{k}} B \to \overline{k} \otimes_{\overline{k}} B \cong B$ by setting

$$\sum_i a_i \otimes b_i \longmapsto \sum_i \overline{a_i} b_i.$$

It is easy to check that φ is a well-defined ring homomorphism. Then we have

$$\varphi\left(\left(\sum a_i \otimes b_i\right)\left(\sum a'_j \otimes b'_j\right)\right) = \varphi\left(\sum a_i \otimes b_i\right) \cdot \varphi\left(\sum a'_j \otimes b'_j\right) = \left(\sum \overline{a_i} b_i\right)\left(\sum \overline{a'_j} b'_j\right) = 0,$$

it contradicts to the fact that B is integral domain. So $A \otimes_{\overline{k}} B$ is integral domain, combine that $A \otimes_{\overline{k}} B$ is finitely generated $\overline{k}$-algebra, we know that $X \times_{\overline{k}} Y$ is an integral finite type $\overline{k}$-scheme. □

11.5 ⋆ Properties not preserved by base change, and how to fix them

We saw in the previous section that many useful properties of morphisms are preserved by base change. Indeed, "preserved by base change" is one of the properties any "reasonable" class of morphisms should have (Definition 9.1.1). But some seemingly reasonable notions are not preserved by base change, and in this section we discuss how to modify them appropriately so that they are. We do this not for our own amusement, but because the resulting notions come up in nature, and perhaps in retrospect are the notions we should have started with.

The "universal" patch to this problem is as follows.

Definition 11.5.1 (Universally $\mathscr{P}$ morphism)

*If $\mathscr{P}$ is some property of schemes, then a morphism of schemes is said to be **universally** $\mathscr{P}$ if it remains $\mathscr{P}$ under any base change.*

Then the class of universally P morphism is preserved by base change.

An important example is the notion of **universally closed morphisms**. Another example is that of **universally injective morphisms**, which turns out to generalize (and "geometrize") purely inseparable field extensions.

One problem with "universally P" morphisms is that it is a priori hard to determine whether a given morphism is universally P — you seem to have to check every single base change of the morphism. Finding other equivalent criteria is thus essential.

11.5.1 Geometric fiber

There are some notions that you should reasonably expect to be preserved by pullback based on your geometric intuition. Given a family in the topological category, fibers pull back in reasonable ways. So for example, any pullback of family in which all the fibers are irreducible will also have this property; ditto for connected. Unfortunately, both of these fail in algebraic geometry, as Example 11.2 shows:

$$\begin{tikzcd} \operatorname{Spec}\mathbb{C}\coprod\operatorname{Spec}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{C} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}\mathbb{R} \end{tikzcd}$$

The family on the right (the vertical map) has irreducible and connected fibers (the fiber is $\operatorname{Spec}\mathbb{C}$, it is single point, so it is irreducible and connected), and the one on the left doesn't (not connected, since $\operatorname{Spec}\mathbb{C}\coprod\operatorname{Spec}\mathbb{C}$ is two not connected points). The same example shows that the notion of "integral fibers" doesn't behave well under pullback. And we used it in §11.4 to show that injectiveity isn't preserved by base change.

Exercise 11.8 Suppose k is a field of characteristic p, so $k(u)/k(u^p)$ is a purely inseparable extension. By considering $k(u)\otimes_{k(u^p)}k(u)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (We will soon see that this happens only in characteristic p, in the presence of inseparability.)

Proof Consider $\pi:\operatorname{Spec}k(u)\to\operatorname{Spec}k(u^p)$, let $q\in\operatorname{Spec}k(u^p)$, then $\kappa(q)=k(u^p)$. So the fiber over q is

$$\begin{aligned}\operatorname{Spec}k(u^p)\times_{\operatorname{Spec}k(u^p)}\operatorname{Spec}k(u) &\cong \operatorname{Spec}k(u^p)\otimes_{k(u^p)}k(u)\\ &\cong \operatorname{Spec}k(u).\end{aligned}$$

Since $k(u)$ is a field, $k(u)$ is reduced, and therefore $\operatorname{Spec}k(u)$ is reduced, i.e., $\pi:\operatorname{Spec}k(u)\to\operatorname{Spec}k(u^p)$ has reduced fiber.

Consider $k(u) \otimes_{k(u^p)} k(u)$, we have

$$\begin{aligned} k(u) \otimes_{k(u^p)} k(u) &\cong k(u) \otimes_{k(u^p)} k(u^p)[x]/(x^p - u^p) \\ &\cong k(u) \otimes_{k(u^p)} (k(u^p)[x] \otimes_{k(u^p)[x]} k(u^p)[x]/(x^p - u^p)) \\ &\cong k(u)[x] \otimes_{k(u^p)[x]} k(u^p)[x]/(x^p - u^p) \\ &\cong k(u)[x]/(x^p - u^p). \end{aligned}$$

Pick $m \in \operatorname{Spec} k(u)$, then $\kappa(m) = k(u)$, so we have

$$\begin{aligned} \operatorname{Spec} \kappa(m) \times_{k(u)} (\operatorname{Spec} k(u) \times_{k(u^p)} \operatorname{Spec} k(u)) &\cong \operatorname{Spec} k(u) \times_{\operatorname{Spec} k(u)} \operatorname{Spec} k(u) \otimes_{k(u^p)} k(u) \\ &\cong \operatorname{Spec} k(u) \times_{k(u)} \operatorname{Spec} k(u)[x]/(x^p - u^p) \\ &\cong \operatorname{Spec} k(u) \otimes_{k(u)} k(u)[x]/(x^p - u^p) \\ &\cong \operatorname{Spec} k(u)[x]/(x^p - u^p). \end{aligned}$$

Hence $\operatorname{Spec} k(u) \times_{\operatorname{Spec} k(u^p)} \operatorname{Spec} k(u) \to \operatorname{Spec} k(u)$ do not have reduced fiber, i.e., the notion of "reduced fibers" does not necessarily behave will under pullback. □

We rectify this problem as follows.

Definition 11.5.2 (Geometric point)

*A **geometric point** of a scheme X is defined to be a morphism $\operatorname{Spec} k \to X$ where k is an algebraically closed field.*

Awkwardly, this is now the third kind of "point" of scheme! There are just plain points, which are elements of underlying set; there are Z-valued points (Z is a scheme), which are maps $Z \to X$, Definition 8.3.5; and there are geometric points. Geometric points are clearly a flavor of a scheme-valued point, but they are also an enriched version of a (plan) point: they are the data of a point with an inclusion of the residue field of the point in an algbraically closed field.

Definition 11.5.3 (Geometric fiber)

*A **geometric fiber** of a morphism $X \to Y$ is defined to be the fiber over a geometric point of Y, i.e., the fibered product with the geometric point $\operatorname{Spec} k \to Y$.*

*A morphism has **connected** (resp., **irreducible**, **integral**, **reduced**) **geometric fibers** if all its geometric fibers are connected (resp., irreducible, integral, reduced). One usually says that the morphism has **geometrically connected** (resp., **geometrically irreducible, geometrically integral, geometrically reduced**) fibers.*

*A k-scheme X is **geometrically connected** (resp., **geometrically irreducible**, **geometrically integral**, **geometrically reduced**) if the structure morphism $X \to \operatorname{Spec} k$ has geometrically connected (resp., irreducible, integral, reduced) fibers.*

We will soon see that to check any of these conditions, we need only base change to $\overline{k}$.

Remark **Warning:** in some sources, in the definition of "geometric point", "algebraically closed" is replaced by "separably closed".

✍ **Exercise 11.9 (Exercise for the arithmetically-minded)** Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points. (Cf. Example 11.2 .) Thus $\operatorname{Spec} \mathbb{C}$ is a geometrically reduced but not geometrically irreducible $\mathbb{R}$-scheme.

Proof The geometric point of $\operatorname{Spec} \mathbb{R}$ is morphism $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{R}$, where k is algebraically closed field.

We shall to show that $\operatorname{Spec} k \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ consist of two reduced points. Consider $k \otimes_{\mathbb{R}} \mathbb{C}$,

$$\begin{aligned} k \otimes_{\mathbb{R}} \mathbb{C} \cong& k \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2+1) \\ \cong& k \otimes_{\mathbb{R}} (\mathbb{R}[x] \otimes_{\mathbb{R}[x]} \mathbb{R}[x]/(x^2+1)) \\ \cong& k[x] \otimes_{\mathbb{R}[x]} \mathbb{R}[x]/(x^2+1) \\ \cong& k[x]/(x^2+1) \\ \cong& k[x]/(x+i) \times k[x]/(x-i) \\ \cong& k \times k \end{aligned}$$

i.e., $\operatorname{Spec} k \times_{\mathbb{R}} \times \operatorname{Spec} \mathbb{C} \cong \operatorname{Spec} k \coprod \operatorname{Spec} k$, i.e., geometric fibers consist of two reduced points. Thus $\operatorname{Spec} \mathbb{C}$ is a geometrically reduced but not geometrically irreducible $\mathbb{R}$-scheme. □

Example 11.5 (k-scheme is reduced but not geometrically reduced) Let $k = \mathbb{F}_p(t)$, where p is a prime and t is an indeterminate. Consider the k-scheme $X = \operatorname{Spec} k[x]/(x^p - t)$, $x^p - t$ is irreducible on $k[x]$, so $k[x]/(x^p - t)$ is a field, and therefore X is reduced scheme.

Consider $X \times_k \overline{k} = \operatorname{Spec}(\overline{k} \otimes_k k[x]/(x^p - t))$,

$$\begin{aligned} \overline{k} \otimes_k k[x]/(x^p - t) \cong& \overline{k} \otimes_k (k[x] \otimes_{k[x]} k[x]/(x^p - t)) \\ \cong& \overline{k}[x] \otimes_{k[x]} k[x]/(x^p - t) \\ \cong& \overline{k}[x]/(x^p - t). \end{aligned}$$

Since $\overline{k}$ is algebraically closed, $x^p - t$ split on $\overline{k}[x]$, i.e., there exists $\alpha \in \overline{k}$ such that $\alpha^p = t$, and therefore $x^p - t = (x - \alpha)^p$, so $X \times_k \overline{k}$ is non-reduced scheme, i.e., X is not geometrically reduced.

Example 11.6 (k-scheme is connected but not geometrically connected) Consider $\mathbb{R}$-scheme $X = \operatorname{Spec} \mathbb{C}$. Since X is a single point, X is connected. Consider $X \times_{\mathbb{R}} \mathbb{C} = \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$, by Example 11.2 , we have $X \times_{\mathbb{R}} \mathbb{C} = \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C}$, which implies that $X \times_{\mathbb{R}} \mathbb{C}$ is not geometrically connected.

Example 11.7 (k-scheme is integral but not geometrically integral) Let $k = \mathbb{F}_p(t)$, where p is a prime and t is an indeterminate. Consider the k-scheme $X = \operatorname{Spec} k[x]/(x^p - t)$, by Example 11.5 , we know that $k[x]/(x^p - t)$ is a field, so $k[x]/(x^p - t)$ is an integral domain, and therefore X is an integral k-scheme.

Consider $X \times_k \overline{k}$, by Example 11.5 , we already know that

$$\overline{k} \otimes_k k[x]/(x^p - t) \cong \overline{k}[x]/(x^p - t) \cong \overline{k}[x]/((x-\alpha)^p),$$

where $\alpha \in \overline{k}$ and $\alpha^p = t$. Note that $\overline{k} \otimes_k k[x]/(x^p - t)$ not integral domain, so $X \times_k \overline{k}$ is not integral. It follows that X is not geometrically integral.

✍ **Exercise 11.10 (To convince geometers why geometric fibers are meaningful)** Recall Example 11.3 , the projection of the parabola $y^2 = x$ to the x-axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fiber "over $0 = [(x)]$". (Note that $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Q}[x]$ and $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}[x]$, both corresponding to ring maps with $x \mapsto 0$, are both geometric points "above 0".)

Proof Say $X = \operatorname{Spec} \mathbb{Q}[x,y]/(x - y^2)$. Let $\operatorname{Spec} \overline{k} \to \operatorname{Spec} \mathbb{Q}[x]$, where $\overline{k}$ is an algebraically closed field. Consider $\overline{k} \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x,y]/(x - y^2)$,

$$\begin{aligned} \overline{k} \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x,y]/(x-y^2) \cong& \overline{k} \otimes_{\mathbb{Q}[x]} (\mathbb{Q}[x][y] \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x,y]/(x-y^2)) \\ \cong& \overline{k}[y] \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x,y]/(x-y^2) \\ \cong& \overline{k}[y]/(y^2 - a), \end{aligned}$$

where a is the image of x under the ring map $\mathbb{Q}[x] \to \overline{k}$.

If $a = 0$, then $\overline{k} \times_{\mathbb{Q}[x]} X = \operatorname{Spec} \overline{k}[y]/(y^2)$, i.e., a non-reduced point.

If $a \neq 0$, since $\overline{k}$ is algebraically closed, exits $b \in \overline{k}$ such that $b^2 = a$, by Chinese Remainder Theorem 5.4.2, then

$$\overline{k} \otimes_{\mathbb{Q}[x]} X \cong \overline{k}[y]/(y-b) \times \overline{k}[y]/(y+b) \cong \overline{k} \times \overline{k},$$

i.e., two points. $\square$

Checking whether a k-scheme is geometrically connected etc. seems annoying: you need to check every single algebraically closed field containing k. However, in each of these four cases, the failure of nice behavior of geometric fibers can already be detected after a finite extension of fields. For example, $\operatorname{Spec} \mathbb{Q}(i) \to \operatorname{Spec} \mathbb{Q}$ is not geometrically connected, and in fact you only need to base change by $\operatorname{Spec} \mathbb{Q}(i)$ to see this (let $\overline{k}$ be an algebraically closed field with filed extension $\mathbb{Q} \to \overline{k}$, note that $\overline{k} \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \overline{k}[x]/(x^2+1) \cong \overline{k} \times \overline{k}$). We make this precise as follows.

Note *Suppose X is a k-scheme. If K/k is a field extension, define $X_k = X \times_k \operatorname{Spec} K$. Consider the following twelve statements of the form "X_k is [property] for all fields [condition]":*

- $(C_K), (C_{K=\overline{K}}), (C_{\overline{k}}), (C_{k^s})$,
- $(I_K), (I_{K=\overline{K}}), (I_{\overline{k}}), (I_{k^s})$,
- $(R_K), (R_{K=\overline{K}}), (R_{\overline{k}}), (R_{k^p})$.

Here C means "connected", I means "irreducible", and R means "reduced". Also, k^s is the separable closure of k, and k^p is the perfect closure (see Stacks Project [8] Lemma 046W and Definition 046X, and purely inseparable extension, see Patrick Morandi [11] page 39-49) of k. If $\operatorname{char} k = 0$, then $\overline{k} = k^s$ and $k = k^p$, so life is simpler.

Thus (C_K) means "X_k is connected for all fields K", $(I_{\overline{k}})$ means "$X_{\overline{k}}$ is irreducible", and (R_{k^p}) means "X_{k^p} is reduced". Geometrically connected (resp., geometrically irreducible, geometrically reduced) translates to $(C_{K=\overline{K}})$ (resp., $(I_{K=\overline{K}})$, $(R_{K=\overline{K}})$).

Trivially (C_K) implies $(C_{K=\overline{K}})$ implies $(C_{\overline{k}})$, and (C_K) implies (C_{k^s}), and similarly (with appropriate changes) with "connected" replaced by "irreducible" and "reduced". See the following diagram,

$$\begin{tikzcd}
(C_K) \arrow[r, shift right] \arrow[d] \arrow[dr] & (C_{K=\overline{K}}) \arrow[l, dotted, shift right] \arrow[dl] \\
(C_{\overline{k}}) \arrow[r, dotted] & (C_{k^s})
\end{tikzcd}$$

connectedness conditions that will later turn out to be equivalent (solid arrows are trivial, dotted arrows are Proposition 11.5.1).

Proposition 11.5.1

(a) Suppose that E/F is a field extension, and A is an F-algebra. Then A is an F-subalgebra of $A \otimes_F E$.

(b) $(C_{K=\overline{K}})$ implies (C_K) and $(C_{\overline{k}})$ implies (C_{k^s}).

(c) $(I_{K=\overline{K}})$ implies (I_K) and $(I_{\overline{k}})$ implies (I_{k^s}).

(d) $(R_{K=\overline{K}})$ implies (R_K) and $(R_{\overline{k}})$ implies (R_{k^p}).

Proof

(a) It suffices to show that $A \to A \otimes_F E$ is injective. $\varphi : A \to A \otimes_F E$ is given by

$$a \longmapsto a \otimes 1.$$

Let $\varphi(a) = a \otimes 1 = 0$, we want to show that $a = 0$. If $a \neq 0$, since A can be seen as F-vector space (note that F is a field), there exists a F-linear map $f : A \to F$ such that $f(a) \neq 0$. Define $g : A \times E \to E$

by setting $g(x,e) = f(x)e$, g is an F-bilinear map, by the universal property of tensor product, there exits unique F-linear map $h : A \otimes_F E \to E$ such that $h \circ t = g$, where $t : A \times E \to A \otimes_F E$. In fact, $h : A \otimes_F E \to E$ is given by $h(x \otimes e) = g(x,e) = f(x)e$. Consider $h(a \otimes 1)$, since $a \otimes 1 = 0$, $h(a \otimes 1) = 0$. On the other hand, $h(a \otimes 1) = f(a) \cdot 1 = f(a)$, so $f(a) = 0$, a contradiction! Hence $a = 0$, and therefore $\varphi : A \to A \otimes_F E$ is an injective. It follows that A is an F-subalgebra of $A \otimes_F E$.

(b) If $(C_{K=\overline{K}})$ holds, let K/k be any field extension, we want to show that X_K is connected. Note that

$$X_{\overline{K}} = X \times_k \operatorname{Spec} \overline{K} = X \times_k (\operatorname{Spec} K \times_K \operatorname{Spec} \overline{K}) = X_K \times_K \operatorname{Spec} \overline{K},$$

consider the following diagram.

$$\begin{tikzcd} X_{\overline{K}} \arrow[r, two heads] \arrow[d] & X_K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \overline{K} \arrow[r, two heads] & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Since $K \to \overline{K}$ is an integral extension, by the Lying Over Theorem 9.2.2, $\operatorname{Spec} \overline{K} \twoheadrightarrow \operatorname{Spec} K$ is surjective. By Proposition 11.4.5, surjectivity is preserved by base change, we know that $X_{\overline{K}} \twoheadrightarrow X_K$ is surjective. By hypothesis, $X_{\overline{K}}$ is connected, note that $X_{\overline{K}} \to X_K$ is continuous and surjective as map of topological spaces, X_K is connected.

If $(C_{\overline{k}})$ holds, we want to show that X_{k^s} is connected. Note that

$$X_{\overline{k}} = X \times_k \overline{k} = X \times_k (k^s \times_{k^s} \overline{k}) = X_{k^s} \times_{k^s} \overline{k},$$

consider the following diagram.

$$\begin{tikzcd} X_{\overline{k}} \arrow[r, two heads] \arrow[d] & X_{k^s} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \overline{k} \arrow[r, two heads] & \operatorname{Spec} k^s \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Since $\overline{k}/k^s$ is purely inseparable extension (and therefore algebraic extension, see Patrick Morandi [11]), $k^s \to \overline{k}$ is an integral extension, by the Lying Over Theorem 9.2.2, $\operatorname{Spec} \overline{k} \twoheadrightarrow \operatorname{Spec} k^s$ is surjective. Surjectivity is preserved by base change (Proposition 11.4.5), we know that $X_{\overline{k}} \to X_{k^s}$ is surjective. By hypothesis, $X_{\overline{k}}$ is connected, since $X_{\overline{k}} \twoheadrightarrow X_{k^s}$ is continuous and surjective as map of topological spaces, X_{k^s} is connected.

(c) As part (b), we already know that $X_{\overline{K}} \to X_K$ and $X_{\overline{k}} \to X_{k^s}$ are surjective and continuous as maps of topological spaces, note that irreducibility is preserved under the continuous surjective map, so $(I_{K=\overline{K}})$ implies (I_K) and $(I_{\overline{k}})$ implies (I_{k^s}).

(d) Since reducedness is an affine-local property (Example 6.5), we may assume that $X = \operatorname{Spec} A$, where A is a k-algebra.

If $(R_{K=\overline{K}})$ holds, let K/k be any field extension, we want to show that X_K is reduced. Note that $K \otimes_k A \hookrightarrow \overline{K} \otimes_K (K \otimes_k A) = \overline{K} \otimes_k A$ (by part (a)), since $\overline{K} \otimes_k A$ is reduced ring, $K \otimes_k A$ is reduced ring, and therefore $X_K = \operatorname{Spec} A \otimes_k K$ is reduced.

If $(R_{\overline{k}})$ holds, note that $k^p \hookrightarrow \overline{k}$, similar to the case that "$(R_{K=\overline{K}})$ implies (R_K)", we have (R_{k^p}).

□

Thus for example a k-scheme is geometrically integral if and only if it remains integral under any field extension. Hence "geometrically integral" means "universally, integral fibers".

Proposition 11.5.2 (Harder fact)

(a) $(C_K) \Leftrightarrow (C_{K=\overline{K}}) \Leftrightarrow (C_{\overline{k}}) \Leftrightarrow (C_{k^s})$.
(b) $(I_K) \Leftrightarrow (I_{K=\overline{K}}) \Leftrightarrow (I_{\overline{k}}) \Leftrightarrow (I_{k^s})$.
(c) $(R_K) \Leftrightarrow (R_{K=\overline{K}}) \Leftrightarrow (R_{\overline{k}}) \Leftrightarrow (R_{k^p})$.

We defer the explanation to the end of this section, in a double-starred discussion (§11.5.3).

The following exercise may help even the geometrically-minded reader appreciate the utility of these notions. (There is nothing important about the dimension 2 and the degree 4 in this exercise!)

Exercise 11.11 ⋆ Recall that "degree d hypersurfaces in $\mathbb{P}^n$ are parametrized by $\mathbb{P}^{\binom{n+d}{d}-1}$", the quartic curves in $\mathbb{P}^2_k$ are parametrized by a $\mathbb{P}^{14}_k$. (This will be made much more precise in Chapter 26.) Show that the points of $\mathbb{P}^{14}_k$ corresponding to geometrically integral curves form an open subset. Explain the necessity of the modifier "geometrically" (even if k is algebraically closed).

11.5.2 Universally injective (= radical) morphisms

As remark in §11.4, injectivity is not preserved by base change. A batter notion is that of **universally injective morphisms**:

Definition 11.5.4 (Universally injective (radical))

*Let $\pi : X \to S$ be a morphism of schemes. We say that π is **universally injective** (or **radical**) if and only if for any morphism of schemes $S' \to S$ the base change $\pi_{S'} : X_{S'} \to S'$ is injective (on underlying topological spaces).*

Remark In general, if $\pi : X \to S$ is a morphism, we say π is radical if π is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x)/\kappa(\pi(x))$ is purely inseparable, see Stacks Project [8] Definition 01S3. We will prove this is same as Definition 11.5.4 in Proposition 11.5.4.

Example 11.8 As a first example: all locally closed embeddings are universally injective (as they are injective, and remain locally closed embeddings upon any base change). If you wish, you can show more generally that all monomorphisms are universally injective. (Hint: show that monomorphisms are injective, and that the class of monomorphisms is preseved by base change.)

Suppose you want to determine whether a given morphism is universally injective. A map of sets is injective if and only if each fiber contains at most one point. Now "the fiber of a base change is the base change of the fiber" (Proposition 11.3.3). Also the underlying set of the scheme-theoretic fiber is the fiber of the map on underlying sets (Proposition 11.3.1). Hence the question of universal injectivity turns into one about field theory. We make this precise in Proposition 11.5.4, after first making the field theory question precise in Proposition 11.5.3. En route, we will see why universal injectivity is the algebro-geometric generalization of the notion of purely inseparable extensions of fields.

Proposition 11.5.3

Suppose E/F is an extension of fields, inducing $\varphi : \operatorname{Spec} E \to \operatorname{Spec} F$.

(a) If E contains an element x transcendental over F, then φ is not universally injective.
(b) If $E \setminus F$ contains an element x algebraic over F, and separable over F, then φ is not universally injective.
*(c) Suppose E contains no elements of the above forms, i.e., E/F is a **purely inseparable extension***

(all elements of E are algebraic over F, and have some p^Nth power in F). If E'/F is any other field extension, then $\operatorname{Spec} E \otimes_F E'$ *contains precisely one point.*

Proof

(a) Let y be an transcendental element over F. Consider the following Cartesian diagrams.

$$\begin{tikzcd} \operatorname{Spec} E \otimes_F F(y) \arrow[r, two heads] \arrow[d] & \operatorname{Spec} F(x) \otimes_F F(y) \arrow[r] \arrow[d] & \operatorname{Spec} F(y) \arrow[d] \\ \operatorname{Spec} E \arrow[r, two heads] & \operatorname{Spec} F(x) \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

Since $\operatorname{Spec} E \to \operatorname{Spec} F(x)$ is a morphism from a point to a point, so it is surjective. Surjectivity is preserved by base change (Proposition 11.4.5), we know that $\operatorname{Spec} E \otimes_F F(y) \twoheadrightarrow \operatorname{Spec} F(x) \otimes_F F(y)$ is surjective. Consider $\operatorname{Spec} F(x) \otimes_F F(y)$, by Exercise 11.3, the prime ideal of $F(x) \otimes_F F(y)$ correspond to ideals $(f(x,y)) \subseteq F[x,y]$, where $f(x,y)$ is an irreducible polynomial in $F[x,y]$ containing both variables x and y. Since x is transcendental over F, $\operatorname{Spec} F(x) \otimes_F F(y)$ is not single point, by the surjectivity of $\operatorname{Spec} E \otimes_F F(y) \twoheadrightarrow \operatorname{Spec} F(x) \otimes_F F(y)$, we know that $\operatorname{Spec} E \otimes_F F(y)$ not single point. Since $\operatorname{Spec} E$ is a single point, $\operatorname{Spec} E \otimes_F F(y) \to \operatorname{Spec} E$ not injective. Hence φ is not universally injective.

(b) Since x is algebraic over F, let $\operatorname{Irr}(t,F) \in F[t]$ be a minimal polynomial of x, hence $F(x) \cong F[t]/(\operatorname{Irr}(t,F))$. Consider the following Cartesian diagrams.

$$\begin{tikzcd} \operatorname{Spec} E \otimes_F \overline{F} \arrow[r] \arrow[d] & \operatorname{Spec} F[t]/(\operatorname{Irr}(t,F)) \otimes_F \overline{F} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{F} \arrow[d] \\ \operatorname{Spec} E \arrow[r] & \operatorname{Spec} F[t]/(\operatorname{Irr}(t,F)) \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

We may assume that $\deg \operatorname{Irr}(t,F) = n$, since x is separable over F, we may assume $\operatorname{Irr}(t,F) = \prod_{i=1}^{n}(t-x_i)$ in $\overline{F}$. Consider $F[t]/(\operatorname{Irr}(t,F)) \otimes_F \overline{F}$,

$$\begin{aligned} F[t]/(\operatorname{Irr}(t,F)) \otimes_F \overline{F} &\cong (F[t]/(\operatorname{Irr}(t,F)) \otimes_{F[t]} F[t]) \otimes_F \overline{F} \\ &\cong F[t]/(\operatorname{Irr}(t,F)) \otimes_{F[t]} \overline{F}[t] \\ &\cong \overline{F}[t]/(\operatorname{Irr}(t,F)) \\ &\cong \prod_{i=1}^{n} \overline{F}[t]/(t-x_i) \cong \prod_{i=1}^{n} \overline{F}, \end{aligned}$$

hence $\operatorname{Spec} F[t]/(\operatorname{Irr}(t,F)) \otimes_F \overline{F} \cong \coprod_{i=1}^{n} \operatorname{Spec} \overline{F}$, and therefore $\operatorname{Spec} F[t]/(\operatorname{Irr}(t,F))$ have n point. Note that $\operatorname{Spec} E \to \operatorname{Spec} F[t]/(\operatorname{Irr}(t,F))$ is a morphism from a point to a point, so it is surjective. Surjectivity is preserved by base change (Proposition 11.4.5), we know that $\operatorname{Spec} E \otimes_F \overline{F} \to \operatorname{Spec} F[t]/(\operatorname{Irr}(t,F)) \otimes_F \overline{F}$ is surjective. Hence $\operatorname{Spec} E \otimes_F \overline{F}$ at least n points, so $\operatorname{Spec} E \otimes_F \overline{F} \to \operatorname{Spec} E$ is not injective, and therefore φ is not universally injective.

(c) Suppose $\operatorname{char} F = p > 0$. Let $x \in E \otimes_F E'$, then x is of the form

$$x = \sum_{i=1}^{n} a_i \otimes b_i,$$

where $a_i \in E$ and $b_i \in E'$. Since E/F is purely inseparable extension, $a_i^{p^{m_i}} \in F$. Let $m =$

$\max\{m_1, \ldots, m_n\}$. Consider x^{p^m},

$$x^{p^m} = \sum_{i=1}^{n} (a_i \otimes b_i)^{p^m} = \sum_{i=1}^{n} a_i^{p^m} \otimes b_i^{p^m} = \sum_{i=1}^{n} (1 \otimes a_i^{p^m} b_i^{p^m}) = 1 \otimes \sum_{i=1}^{n} a_i^{p^m} b_i^{p^m}.$$

Note that $1 \otimes E' \cong E'$, x^{p^m} can be seen as an element belong to E'. Since E' is a field, x^{p^m} must be 0 or unit. Hence $E \otimes_F E'$ is a local ring. Let $\mathfrak{p} \in \operatorname{Spec} E \otimes_F E'$, note that $0 \in \mathfrak{p}$, every nilpotent of $E \otimes_F E'$ must belong to $\mathfrak{p}$, so $\mathfrak{p} = \mathfrak{m}$. Hence $\operatorname{Spec} E \otimes_F E' = \{[\mathfrak{m}]\}$, as we desired.

□

Proposition 11.5.4

Suppose $\pi : X \to Y$ is a morphism of schemes. The following are equivalent.

(i) π is universally injective.

(ii) The morphism π is injective, and for each $p \in X$, the field extension $\kappa(p)/\kappa(\pi(p))$ is purely inseparable.

Proof (i) $\Rightarrow$ (ii): Note that $X \times_Y Y \cong X$, since $\pi : X \to Y$ is universally injective, $\pi : X \to Y$ must be injective.

If there exists $p \in X$ such that the $\kappa(p)/\kappa(\pi(p))$ is not purely inseparable. Hence $\kappa(p)$ contains an element x transcendental over F or $\kappa(p) \setminus \kappa(\pi(p))$ contains an element x algebraic and separable over $\kappa(\pi(p))$, by Proposition 11.5.3, we know that π is not universally injective, a contradiction!

(ii) $\Rightarrow$ (i): Let $Z \to Y$ be any morphism of schemes, we want to show that $\pi_Z : X \times_Y Z \to Z$ is injective. Let $p_1', p_2' \in X \times_Y Z$, with $\pi_Z(p_1') = \pi_Z(p_2') = q' \in Z$. Consider the following Cartesian diagram.

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "\pi_Z"'] & X \arrow[d, "\pi"] \\
Z \arrow[r] & Y
\end{tikzcd}$$

Let p_1, p_2 be the images of p_1', p_2' in X respectively, and q be the image of q' in Y. So we have $\pi(p_1) = \pi(p_2) = q$, by the Cartesian diagram. Since π is injective, we have $p := p_1 = p_2$, and therefore $p_1, p_2 \in \kappa(p) \times_X (X \times_Y Z) \cong \kappa(p) \times_Y Z$. Since $\pi_Z(p_1') = \pi_Z(p_2') = q'$, we have $p_1, p_2 \in \kappa(q') \times_Z (\kappa(p) \times_Y Z) \cong \kappa(q') \times_Y \kappa(p)$. Note that

$$\begin{tikzcd}
\kappa(p) \times_{\kappa(q)} \kappa(q') \arrow[dr, dashed, shift left] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& \kappa(p) \times_Y \kappa(q') \arrow[ul, dashed, shift left] \arrow[r] \arrow[d] & \operatorname{Spec} \kappa(p) \arrow[d, hook, "\pi"'] \arrow[ddr] & \\
& \operatorname{Spec} \kappa(q') \arrow[r, hook] \arrow[drr] & Y & \\
& & & \operatorname{Spec} \kappa(q) \arrow[ul, hook]
\end{tikzcd}$$

where dashed arrows given by the universal property of fibered product, we have

$$p_1, p_2 \in \kappa(p) \times_Y \kappa(q') \cong \kappa(p) \times_{\kappa(q)} \kappa(q').$$

Since $\kappa(p)/\kappa(q)$ is purely inseparable, by Proposition 11.5.3, $\kappa(p) \times_{\kappa(q)} \kappa(q') = \operatorname{Spec} \kappa(p) \otimes_{\kappa(q)} \kappa(q')$ is a single point. So $p_1 = p_2$, i.e. π_Z is injective, and therefore π is universally injective. □

You may already see that the class of universally injective morphisms is also preserved by composition, and local on the target, and hence form a “reasonable” class (Definition 9.1.1). In any case, we will see this in a different way in Chapter 12.

11.5.3 ⋆⋆ Proof of Harder fact 11.5.2

We will use one fact we will not prove until much later. Recall that a map of topological spaces is an **open map** if the image of every open set is an open set.

Fact, we will proved in Chapter 25:

Theorem 11.5.1

*Suppose X is a k-scheme. Then $X \to \operatorname{Spec} k$ is **universally open**, i.e., remains open after any base change.*

11.6 Products of projective schemes: The Segre embedding

We next describe

11.7 Normalization

Chapter 12 Separated and proper morphisms, and varieties

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