第1章 整除 参考答案

计算证明

1. 计算下面整数对的最大公因子和最小公倍数.

（1）$(202, 282)$ 2，28482

（2）$(-666, 1410)$ 6，-156510

（3）$(30, 105, 360)$ 15，2520

（4）$(8n^2+28n+12, 12n^2+30n+12)$

$(8n^2+28n+12, 12n^2+30n+12) = (4(2n+1)(n+3), 6(2n+1)(n+2)) = (4,6)\cdot((2n+1)(n+3), (2n+1)(n+2))$ =2(2n+1)

$[8n^2+28n+12, 12n^2+30n+12] = 12(2n+1)(n+2)(n+3)$

2. 求下面整数的标准分解式.

（1）69 $69 = 3^1\times 23^1$

（2）200 $200 = 2^3\times 5^2$

（3）3288 $3288 = 2^3\times 3^1\times 137^1$

（4）22345680 $22345680 = 2^4\times 3^1\times 5^1\times 7^1\times 47^1\times 283^1$

3. 若 $a\in\mathbb{Z}^+\cup\{0\}$， a^4-3a^2+9 是质数还是合数?

解 $a^4-3a^2+9 = (a^2-3a+3)(a^2+3a+3) = [(a-1)(a-2)+1][(a+1)(a+2)+1]$

当 $a=0$， $a^4-3a^2+9=9\rightarrow$合数

当 $a=1$， $a^4-3a^2+9=7\rightarrow$质数

当 $a=2$， $a^4-3a^2+9=13\rightarrow$质数

当 $a>2$， $(a+1)(a+2)+1>(a-1)(a-2)+1>1\rightarrow$合数

综上所述，当 $a=1$ 或 $a=2$ 时为质数， 当 $a=0$或 $a>2$ 时为合数.

4. 若 $m-p\mid mn+pq$,求证 $m-p\mid mq+np$.

证明 易知 $m-p\mid(m-p)(n-q)$ ， 又 $m-p\mid mn+pq$ ， 而 $-(m-p)(n-q)+(mn+pq)=mq+np$ ， 则 $m-p\mid mq+np$ ， 证毕.

5. 设 $3 \mid a^2+b^2$，求证： $3 \mid a$ 且 $3 \mid b$.

证明 设 $a=3m+p,\ b=3n+q,\ (m,n\in\mathbb{Z},\ p,q\in\{-1,0,1\})$，

则 $a^2+b^2=(3m+p)^2+(3n+q)^2=3(3m^2+3n^2+2mp+2nq)+p^2+q^2$.

由 $3 \mid a^2+b^2$，得 $3 \mid p^2+q^2$，而 $p^2+q^2=\begin{cases}0, & p=q=0\\ 1, & p+q=\pm1\\ 2, & p=\pm1, q=\pm1\end{cases}$，故 $p=q=0$，$3 \mid a$ 且 $3 \mid b$，证毕.

6. 设 $a=qn-t$，若 $a \mid pm$，已知 $p-q=t$ 且 $(a,n+1)=1$，求证： $a \mid tm$.

证明 由 $a \mid pm$ 知 $a \mid pmn$，又 $a \mid am$，由整系数线性组合得 $a \mid pmn-am$，而 $a=qn-t$，则 $a \mid (p-q)m(n+1)$，又有 $p-q=t$ 且 $(a,n+1)=1$，得 $a \mid tm$，证毕.

7. 求证任意 n 个连续的正整数整数乘积都被 $n!$ 整除.（写出严谨的证明过程）

证明 数学归纳法.

记任意 n 个连续的正整数分别为 $a_0,a_1,\cdots,a_{n-1}$. 其中 $a_i=m+i\ (m\in\mathbb{N})$，乘积记为 T_m.

当 $m=1$ 时，$T_1=n!$，$n! \mid T_1$ 成立 .

假设当 $m=k\ (k\in\mathbb{N})$ 时，$n! \mid T_k$，即 $n! \mid k(k+1)\cdots(k+n-1)$ 成立 .

那么当 $m=k+1$ 时，有 $T_{k+1}=\prod_{i=0}^{n-1}a_i=(k+1)(k+2)\cdots(k+n)=k(k+1)\cdots(k+n-1)+n(k+1)\cdots(k+n-1)$.

只需证 $n! \mid n(k+1)\cdots(k+n-1)$，即$(n-1)! \mid (k+1)\cdots(k+n-1)$.

取 n 为 $n-1$，m 为 k，重复上述证明过程.

易知经过有限步后，$n=1$，而 $\forall m\in\mathbb{N},\ 1 \mid m$，假设成立，证毕.

8. 求证 $12 \mid n^4+2n^3+11n^2+10n,\ n\in\mathbb{Z}$.

证明 由$n^4+2n^3+11n^2+10n=(n-1)n(n+1)(n+2)+12n(n+1)$，只需证 $12 \mid (n-1)n(n+1)(n+2)$，易知 $[1,2,3,4] \mid (n-1)n(n+1)(n+2)$ （严谨证明可用**数学归纳法**），证毕.

9. 证明 n 的标准分解式中次数都是偶数当且仅当 n 是完全平方数 .

证明 充分性. 设 $n=m^2$，由算术基本定义有 $\begin{cases}n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}, & p_i\text{是素数且}p_i<p_j(0<i<j\le s)\\ m=q_1^{\beta_1}\cdots q_t^{\beta_t}, & q_i\text{是素数且}q_i<q_j(0<i<j\le t)\end{cases}$，由算术分解的唯一性可得 $\begin{cases}s=t\\ p_i=q_i, & 0<i\le s\\ \alpha_i=2\beta_i, & 0<i\le s\end{cases}$. 必要性. 设 $n=p_1^{2\alpha_1}\cdots p_s^{2\alpha_s}$, p_i 是素数且 $p_i<p_j(0<i<j\le s)$，取 $m=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ 即可满足 $n=m^2$. 证毕.

10. 证明 $\sqrt{5}$ 是无理数. *并将其表示为简单连分数的形式.（*标注表示不作强制要求）

证明 反证法. 反设$\sqrt{5}=\dfrac{p}{q}$，其中 $p,q\in\mathbb{Z}^+$, $(p,q)=1$. 得 $p^2=5q^2$，则 $5 \mid p$.可设 $p=5m$，代入得$q^2=5m^2$，同理 $5 \mid q$. 则 $(p,q)=5\ne1$矛盾，故反设不成立，证毕.

对无理数$\alpha=\sqrt{5}$，有：

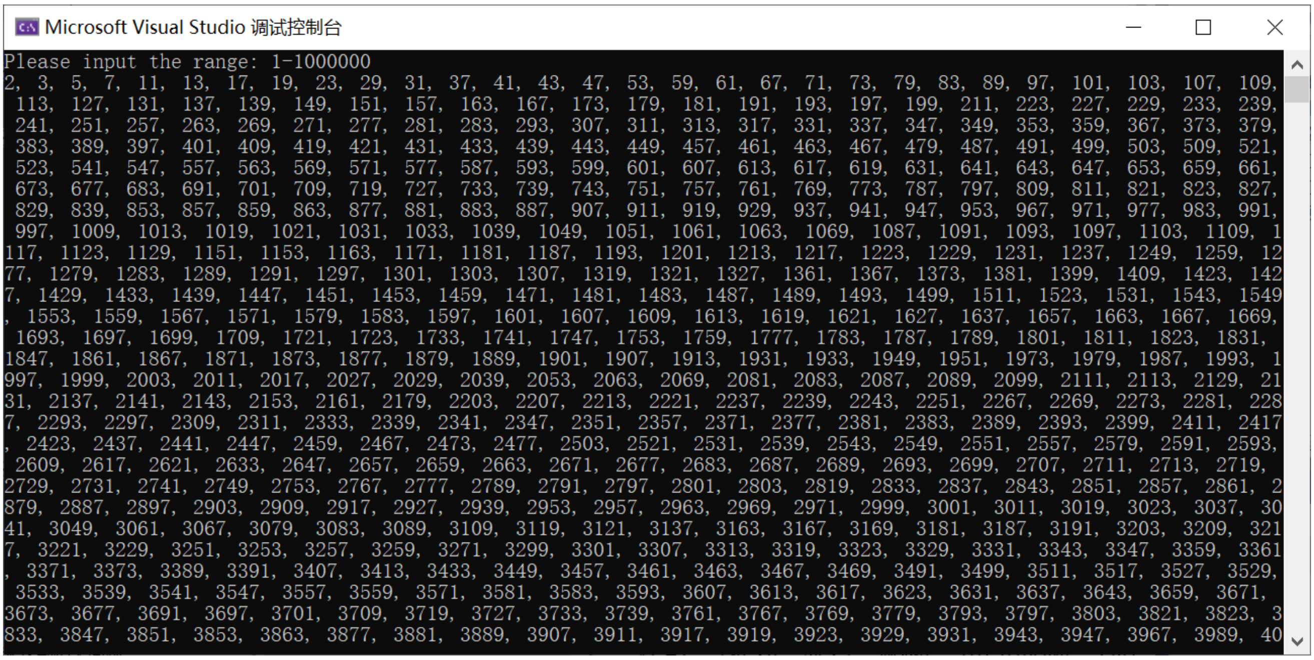
$$a_0=[\sqrt{5}]=2,\ \alpha_1=\frac{1}{\sqrt{5}-2}=\sqrt{5}+2$$

$$a_1=[\sqrt{5}+2]=4,\ \alpha_2=\frac{1}{\sqrt{5}+2-4}=\sqrt{5}+2=\alpha_1$$

故 $\sqrt{5}=[2;\overline{4}]$.

编程练习（基于C/C++）

1. 编写程序使用Eratosthenes筛法打印1 000 000内所有素数及个数，效果如图所示．（*思考：a.对比筛法与普通算法的性能差异；b.递归调用该算法求更大范围素数进行优化；c.求更大的素数（如2^{512}数量级）该方法是否适用？会引入哪些新的问题？）


Microsoft Visual Studio 调试控制台

```
Please input the range: 1-1000000
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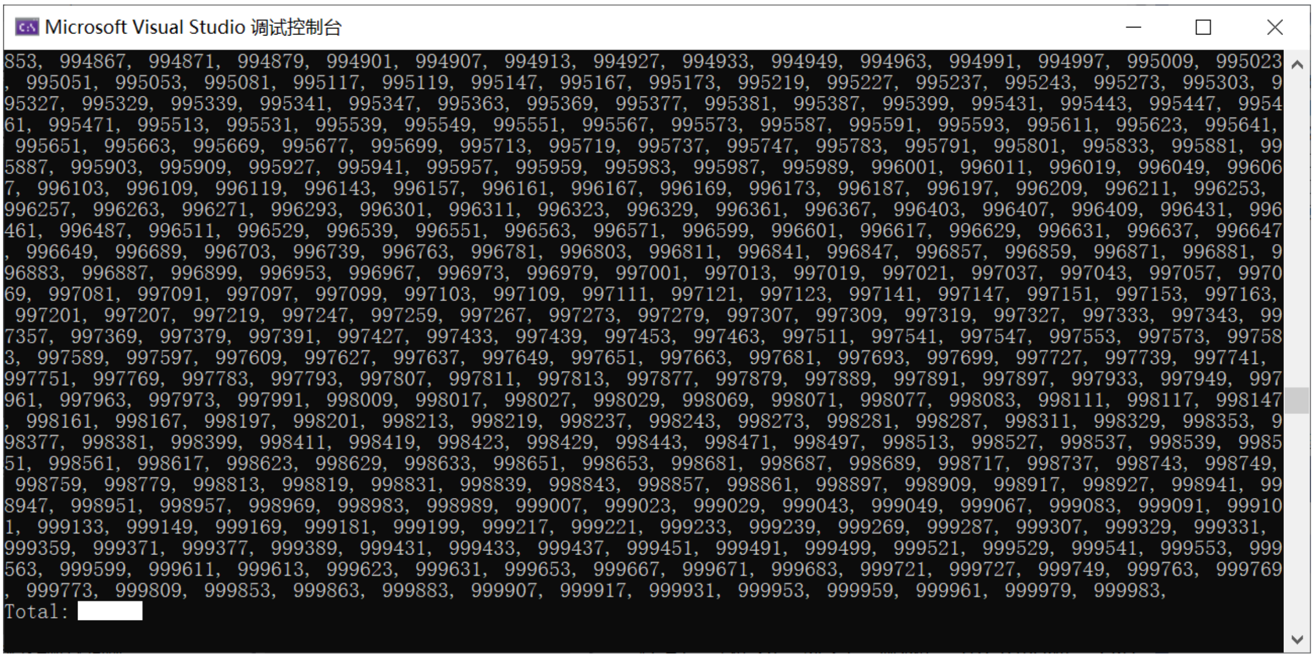
Microsoft Visual Studio 调试控制台

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, 999773, 999809, 999853, 999863, 999883, 999907, 999917, 999931, 999953, 999959, 999961, 999979, 999983,
Total:
```

Total: 78498

```cpp
#include<iostream>
#include<math.h>
using namespace std;

bool is_prime(int n)
{
    if (n < 2)return false;
    bool flag = true;
    for (int i = 2; i < n; i++)
    {
        if (n % i == 0)
        {
            flag = false;
            break;
        }
    }
    return flag;
}

//based on Eratosthenes once
int find_prime(int range)
{
    int count = 0;
    bool* num = new bool[range + 1];
    for (int i = 1; i < range + 1; i++)
        num[i] = true;
    for (int i = 2; i < sqrt(range); i++)
    {
        if(is_prime(i))
        {
            for (int temp = i + i; temp <= range; temp += i)
                num[temp] = false;
        }
    }
    for (int i = 2; i < range + 1; i++)
    {
        if (num[i])
        {
            count++;
            cout << i << ", ";
        }
    }
    cout << endl;
    return count;
}

int main()
{
    int n;
    cout << "Please input the range: 1-";
    cin >> n;
    int count = find_prime(n);
    cout << "Total: " << count << endl;
    system("pause");
}
```

2. 编写程序计算最大公因数和最小公倍数，效果如图所示.

Microsoft Visual Studio 调试控制台

```
a=9876
b=6789
gcd(a,b)=3
lcm(a,b)=22349388
```

```cpp
#include<iostream>
using namespace std;

int Euclid(int a, int b)
{

    if (a < 0)a = -a;
    if (b < 0)b = -b;
    int r = a;
    while (r != 0)
    {
        r = a % b;
        a = b;
        b = r;
    }
    return a;
}

int main()
{
    int a, b;
    cout << "a=";
    cin >> a;
    cout << "b=";
    cin >> b;
    int gcd = Euclid(a, b);
    int lcm = a * b / gcd;
    cout << "gcd(a,b)=" << gcd << endl;
    cout << "lcm(a,b)=" << lcm << endl;
}
```

3. 编写程序实现算术基本定理，效果如下所示.

Microsoft Visual Studio 调试控制台

```
Please input n(n>0): 888
888=2^3*3^1*37^1
```

```cpp
#include<iostream>
#include<map>
using namespace std;
void num_decompose(int n)
{
    cout << n << "=";
    map<int, int>nums;
    while (n!=1)
    {
        int i = 2;
```

```cpp
        while (i <= n)
        {
            if (n % i == 0)
            {
                auto iter = nums.find(i);
                iter == nums.end() ? nums[i] = 1 : iter->second++;
                n /= i;
                break;
            }
            i++;
        }
    }

    for (auto iter = nums.begin(); iter != nums.end(); )
    {
        cout << iter->first << "^" << iter->second;
        if (++iter == nums.end())
            cout << endl;
        else
            cout << "*";
    }
}

int main()
{
    int n;
    cout << "Please input n(n>0): ";
    cin >> n;
    if (n == 1)
        cout << "1=1" << endl;
    else
        num_decompose(n);
    return 0;
}
```