



MSML610: Advanced Machine Learning

Probabilistic Reinforcement Learning

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References:

- AIMA Chap 17: Making complex decisions
- AIMA Chap 22: Reinforcement Learning

Sequential Decision Problems

- **Sequential Decision Problems**
 - Utilities Over Time
 - Algorithms for MDPs
- Reinforcement Learning

Sequential Decision Problems

- **Agents need to make decisions in the real world:**
 - In a stochastic environment (randomness, unpredictability)
 - E.g., weather conditions affecting a delivery route
 - Where utility depends on a sequence of decisions
 - E.g., planning a multi-step journey where each step influences the next
- **What is involved**
 - Uncertainty
 - Represent the lack of certainty in outcomes, modeled using probabilities
 - E.g., weather forecasts often include uncertainty (70% chance of rain)
 - Utility functions
 - Measure the desirability of outcomes by quantifying preferences
 - E.g., assign higher values to outcomes with more profit and lower risk
 - Rewards
 - Yielded by the environment as feedback for actions taken
 - E.g., receive points in a game for completing a level
 - Sensing
 - Gather information about the environment using sensors
 - E.g., a robot using a camera to detect obstacles in its path
 - Search and planning
 - Find a sequence of actions to achieve a goal
 - E.g., a GPS system planning the shortest route to a destination

Markov Decision Process

- MDPs are a formal model for sequential decision making
- Assumptions
 - Fully observable but stochastic environment
 - Sensors give agent the complete state of the environment $\forall t$
 - Next state is not completely determined by current state and agent's action
 - Initial state s_0
 - An agent takes action $a \in \text{Actions}(s)$ in each state s
 - Transition model
 - $\text{Pr}(s'|s, a)$ is probability of reaching state s' , if action a is done in state s
 - Markov assumption: probability depends on (s, a) , not on history
 - Reward function
 - For every transition $s \rightarrow s'$ via a the agent receives a reward $R(s, a, s')$
 - Total reward depends on sequence of states and actions, i.e., environment history
 - Goal states
- E.g., a robot navigating a slippery surface
 - It knows its exact location and the map (fully observable)
 - Its wheels may slip unpredictably (stochastic outcome)



MDP: Solution

- The solution of an MDP is a **policy** “in state s take action a ”

$$\pi(s) : s \rightarrow a \in \text{Actions}(s)$$

- Any execution of the policy leads to a different **environment history** because of the stochastic nature of the environment
- Environment history is a sequence of states and actions
 $(s_0, a_0) \rightarrow (s_1, a_1) \rightarrow \dots \rightarrow (s_i, a_i) \rightarrow \dots$
- A policy is measured by the **expected utility** of the environment history

$$U(\pi(s)) = \mathbb{E}[f(s_0, \pi(s), R(s, a, s'))]$$

- The **optimal policy** $\pi^*(s)$ yields the highest expected utility

$$\pi^*(s) = \operatorname{argmax}_{\pi} \mathbb{E}[f(s_0, \pi(s), R(s, a, s'))]$$

- Note: the optimal policy is a function of the reward function
- MDP is often solved with **dynamic programming**
 1. Break the problem in smaller pieces recursively
 2. Solve the sub-problems
 3. Remember solutions of the pieces

MDP: 4x3 Environment Example

- **Environment**

- A 4×3 grid world
- Fully observable: the agent always knows its location
- Non-deterministic: actions are not reliable
 - $\Pr(\text{intended action}) = 0.8$
 - $\Pr(\text{move right/left angle}) = 0.1$

- **Agent**

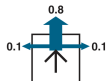
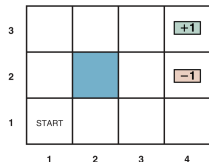
- Begin at the START cell
- Choose actions Up, Down, Left, Right at each step
- Aim to reach goal states marked +1 or -1

- **Transition Model**

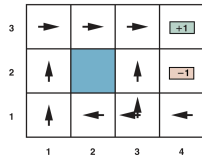
- Result of each action in each state $\Pr(s'|s, a)$

- **Utility Function**

- The reward for each state transition $s \rightarrow s'$ via action a is $R(s, a, s')$
 - -0.04 for all transitions (to encourage reaching terminal states swiftly)
 - +1 or -1 upon reaching terminal states
- Total utility is the sum of all received rewards



Valid actions



Example of optimal policy

Utilities Over Time

- Sequential Decision Problems
 - **Utilities Over Time**
 - Algorithms for MDPs
- Reinforcement Learning

Utility Function

- The **utility function** for environment histories (finite or infinite) is expressed as:

$$U_h([s_0, a_0, s_1, a_1, \dots, s_n, \dots])$$

- A **finite horizon** indicates a fixed time N after which nothing matters:

$$U_h([s_0, a_0, s_1, \dots, s_N, \dots s_{N+k}]) = U_h([s_0, a_0, s_1, \dots, s_N]) \quad \forall k > 0$$

- Actions are chosen based on the current state and remaining steps
- Lead to non-stationary policies
- **Infinite horizon**
 - No fixed time limit, i.e., the process continues indefinitely
 - Utility is often defined using a discount factor $\gamma < 1$ for convergence
 - The optimal policy can be **stationary**
 - Policies do not depend on the specific time step
 - Same action is chosen whenever the agent visits the same state

Additive Rewards

- **Additive rewards:**

- Rewards for each transition $s_i \xrightarrow{a_i} s_{i+1}$ are summed:

$$U_h([s_0, a_0, s_1, a_1, \dots]) = \sum_{i=0} R(s_i, a_i, s_{i+1})$$

- **Additive discounted rewards:**

- Include a discount factor $\gamma \in [0, 1]$:

$$\begin{aligned} U_h([s_0, a_0, s_1, a_1, \dots]) &= R(s_0, a_0, s_1) + \gamma R(s_1, a_1, s_2) + \gamma^2 R(s_2, a_2, s_3) + \dots \\ &= \sum_{i=0} \gamma^i R(s_i, a_i, s_{i+1}) \end{aligned}$$

where:

- $\gamma = 1$: purely additive rewards
- $\gamma \rightarrow 0$: future rewards are negligible
- $\gamma \rightarrow 1$: future rewards significant

- **Pros of discounted rewards:**

- Reflect human tendency to prioritize near-term rewards
- In economics, early rewards can be reinvested, compounding further rewards
- Support infinite horizons, preventing unbounded rewards

Expected Utility of a Policy

- We said that a policy leads to different environment histories
- The expected **utility of executing policy** π from state s :

$$U^\pi(s) = \mathbb{E}\left[\sum_{i=0}^{\infty} \gamma^i R(S_i, \pi(S_i), S_{i+1})\right]$$

where the expectation $\mathbb{E}[\cdot]$ is:

- Over state sequences determined by $s, \pi(s)$
- The environment's transition model $\Pr(s'|s, a)$
- The agent should choose the **optimal policy**:

$$\pi_s^* = \operatorname{argmax}_{\pi} U^\pi(s)$$

- With discounted utilities and infinite horizons, the optimal policy is independent of the starting state: $\pi_s^* = \pi^*$
- This is not true for finite-horizon policies or other reward combinations

Principle of Maximum Expected Utility (MEU)

- MEU posits: *“A rational agent should choose the action that maximizes its expected utility based on its beliefs”*

- **Formal Definition**

- Possible actions: $a \in A$
- Possible outcomes: s'
- Probability distribution: $\Pr(s'|a)$ for each action
- Utility function: assign a numerical value $U(s')$ to each outcome
- The expected utility of action a is (recursive):

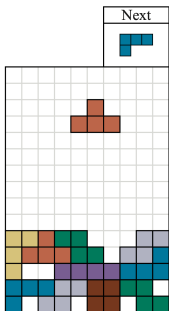
$$EU(a) = \mathbb{E}[U(a)] = \sum_{s'} U(s') \Pr(s'|a)$$

- Choose the action $a^* = \operatorname{argmax}_{a \in A} \sum_{s'} U(s') \Pr(s'|a)$

- **Example:**

- E.g., an agent must choose between:
 - Action A: 80% chance of reward 10; 20% chance of reward 0
 - Action B: 100% chance of reward 6
- By MEU, choose Action A, since $EU(A) = 0.8 \cdot 10 + 0.2 \cdot 0 = 8 > EU(B) = 1.0 \cdot 6 = 6$

MDP: Tetris Example



- **States S**
 - Current board configuration and falling piece
- **Actions A**
 - Valid final placements of the piece
 - Rotation (0–3 positions)
 - Horizontal movement (left, right)
 - Hard drop (instant placement)
- **Transition Model $T(s, a, s')$**
 - Deterministic or stochastic based on next piece modeling
 - Piece generation often random (uniform or “bag” system)
- **Reward $R(s, a, s')$**
 - +1 for each cleared line
 - Negative reward for new block addition or height increase
 - Game over may have large negative reward
- **Discount Factor γ**
 - Close to 1 for valuing long-term survival and line-clearing

Utility of a State

- The utility of a state s , $U(s)$, reflects the long-term desirability of a state under optimal behavior
 - I.e., $U(s) = U^{\pi^*}(s)$, the expected sum of discounted rewards under an optimal policy from s
 - To remove the dependency from the policy, use the optimal policy
 - Calculated based on the expected rewards and the discount factor
- **Example:**
 - In a 4x3 environment, the utility of a state is:
 - Higher closer to the +1 state, as fewer steps are needed to reach it
 - Lower for the one close to the -1 state, since the agent needs to go around it
 - E.g., if the agent is two steps away from the +1 state, the utility will be higher compared to being four steps away
 - This assumes certain reward (e.g., $\gamma = 1$ and $r = -0.04$ for non-terminal transitions)

3	0.8516	0.9078	0.9578	+1
2	0.8016		0.7003	-1
1	0.7453	0.6953	0.6514	0.4279
	1	2	3	4

Bellman Equation

- The **utility of a state** s is the expected reward for the next transition plus the discounted utility of the next state, assuming the agent chooses the optimal action:

$$U(s) = \max_{a \in A(s)} \sum_{s'} \Pr(s'|s, a) [R(s, a, s') + \gamma U(s')]$$

where:

- $A(s)$: set of actions available in state s
- $\Pr(s'|s, a)$: probability of transitioning to state s' from state s by action a
- $R(s, a, s')$: reward after transitioning from state s to s' using a
- γ : discount factor, where $0 \leq \gamma < 1$
- Writing Bellman equations for all states gives a system of equations
 - Each state has its own equation based on its possible actions and transitions
 - Each equation is recursive: utility of s depends on utilities of its successor states
- Under certain conditions (e.g., finite state/action spaces, $\gamma < 1$):
 - This system has a unique solution
 - The utility function $U(s)$ is well-defined
 - E.g., in a grid world with a finite number of cells and actions

Bellman Equation: Intuition

- The **Bellman Equation**: says

"Utility of a state = Best immediate action + Future potential"

- Balances short-term gain and long-term value when outcomes are partly under the control of a decision-maker and partly random
- E.g., to find the fastest path to the goal in a maze, the Bellman equation prescribes:
 - *"Your current position is only as valuable as the best path out of it"*
 - Best path combines current proximity (reward now) and future position quality (reward later)
 - Value backs up from future to present, similar to tracing a route from finish to start
- E.g., in a chess game, the optimal strategy involves making the best move at each turn while considering future moves and potential outcomes

Q-Function

- Aka “Action-utility function”
- The Q-function $Q(s, a)$ is the expected utility of taking an *action* in a *given state*
 - Gives the expected value of choosing action a in state s , and then acting optimally afterward
- Utility of actions $Q(s, a)$ is the “dual” view of utility of states $U(s)$
 - Express the utility of a state in terms of utility of actions:

$$U(s) = \max_a Q(s, a)$$

- Bellman equation for Q-functions

$$Q(s, a) = \sum_{s'} \Pr(s'|s, a) [R(s, a, s') + \gamma \max_{a'} Q(s', a')]$$

- An optimal policy picks the “best” action:

$$\pi^*(s) = \operatorname{argmax}_a Q(s, a)$$

Shaping Theorem

- For discounted sums of rewards, the **scale of utilities** is arbitrary:
 - An affine transformation $U'(s) = m \cdot U(s) + b$ does not change the optimal policy $\pi^*(s)$
 - What matters for decision-making is the relative ordering of utilities (which is preserved)
- More generally, a **potential-based reward shaping**, i.e., using a function $\Phi(s)$ of the state s doesn't change the optimal policy:

$$R'(s, a, s') = R(s, a, s') + \gamma\Phi(s') - \Phi(s)$$

- It ensures the difference in value between states remains consistent
- **Pros**
 - Speed: Can significantly speed up learning by guiding the agent
 - E.g., adding a potential function that increases with proximity to a goal can encourage faster convergence
 - E.g., animal trainers provide a small treat to the animal for each step in the target sequence
 - Safety: Prevents misleading the agent into a suboptimal policy
 - E.g., an agent might prioritize short-term rewards over long-term gains

Representing MDP

- The transition model $\Pr(s'|s, a)$ and the reward function $R(s, a, a')$ can be represented with:
 - Three-dimensional tables of size $|S|^2 \cdot |A|$
 - For sparse MDPs (i.e., each s transitions to only a few states s'), the table size is $O(|S| \cdot |A|)$
- MDPs can be represented using **Dynamic Decision Networks (DDNs)**:
 - DDNs are a type of probabilistic graphical model extending Bayesian networks for sequential decision problems
 - DDNs offer a factored representation, compactly encoding state variables and dependencies
 - They are more scalable and expressive than atomic (flat) representations
 - E.g., in a large MDP with many states, a DDN can efficiently represent the problem without explicitly listing every possible state transition

Dynamic Decision Networks: Tetris Example

- DDN models Tetris in terms of time slices with the game's state, actions, and rewards

- State variables:**

- $Board_t$: grid configuration at time t
 - $Piece_t$: current piece falling
 - $NextPiece_t$: upcoming piece

- Decision variable:**

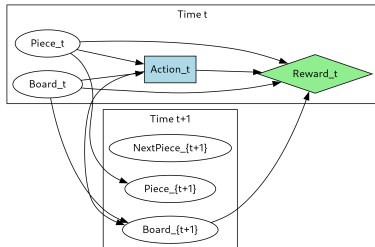
- $Action_t$: placement of $Piece_t$ (rotation and position)

- Chance nodes (transition):**

- $Board_{t+1}$: board after action
 - $Piece_{t+1}$: next piece, depending on $NextPiece_t$ or random selection

- Utility node:**

- $Reward_t$: derived from $Board_{t+1}$ (e.g., lines cleared, holes created)



Algorithms for MDPs

- Sequential Decision Problems
 - Utilities Over Time
 - **Algorithms for MDPs**
- Reinforcement Learning

Value Iteration (1/2)

- **Value iteration** solves MDPs using 2 steps:
 - Compute optimal utility for each state $U(s)$
 - Extract optimal policy π^* from utilities $U(s)$
- **Step 1:** compute optimal utility for each state
 - There are n possible states, so n Bellman equations, one per state

$$U(s) = \max_{a \in A(s)} \sum_{s'} \Pr(s'|s, a) [R(s, a, s') + \gamma U(s')]$$

- Each equation relates the utility of a state to the utilities of its successors
- The state utilities $U(s)$ are n unknowns
- Solve these equations n equations with n unknowns simultaneously
 - Problem: equations are non-linear due to max operator
 - Solution: use an iterative approach

Value Iteration (2/2)

- **Solve system of Bellman equations**

- Start with arbitrary values for utilities $U(s) = 0$
- Perform Bellman updates:

$$U_{i+1}(s) \leftarrow \max_a \sum_{s'} \Pr(s'|s, a)[R(s, a, s') + \gamma U_i(s')]$$

- Calculate the right-hand side and plug it into the left-hand side
- No strict update order required for convergence, but intelligent ordering can improve speed, especially in large or structured MDPs
- Repeat until equilibrium or close to convergence $\|U_{i+1} - U_i\| < \epsilon$
- Guaranteed to converge to the unique fixed point (optimal policy) for additive discounted rewards and $\gamma < 1$
- **Step 2:** compute optimal policy
 - Derive optimal policy by choosing action a that maximizes expected utility for each state s :

$$\pi^*(s) = \operatorname{argmax}_a \sum_{s'} \Pr(s'|s, a)[R(s, a, s') + \gamma U(s')]$$

Policy Iteration

- **Policy iteration** solves MDPs by iteratively improving a policy
 - Alternates between evaluating the current policy and improving it
 - Uses the simplified Bellman equation with a fixed action per state
- **Algorithm steps**
 - Start with an initial (random) policy π
 - Policy Evaluation: compute $U^\pi(s)$ by solving:

$$U^\pi(s) = \sum_{s'} \Pr(s'|s, \pi(s)) [R(s, \pi(s), s') + \gamma U^\pi(s')]$$

- Policy Improvement: for each state, find:

$$\pi'(s) = \operatorname{argmax}_a \sum_{s'} \Pr(s'|s, a) [R(s, a, s') + \gamma U^\pi(s')]$$

- Repeat until policy is unchanged or close to convergence
- **Convergence Guarantee**
 - Each iteration strictly improves or maintains policy performance
 - Guaranteed to terminate with an optimal policy for finite MDPs
- **Efficiency Considerations**
 - Policy evaluation involves solving linear equations
 - Typically converges in fewer iterations than value iteration

Off-Line vs On-Line Solution of MDPS

- **Offline methods** (e.g., value iteration, policy iteration) precompute full solutions
 - Pros:
 - Compute the entire optimal policy $\pi^* \forall s$ before taking any action
 - Cons:
 - Assumes full knowledge of transition probabilities $\Pr(s'|s, a)$ and reward function $R(s, a, s')$
 - Not feasible for large MDPs (e.g., Tetris with 10^{62} states)
- **Online methods** compute actions at runtime, using only reachable parts of the state space
 - Interleave planning and acting
 - Agent explores the environment and updates estimates (e.g., Q-learning)
 - Pros:
 - Focuses computation only on relevant parts of the state space
 - Scales to large problems with appropriate heuristics and approximations
 - Allows adaptive, real-time decision-making
 - No need for full model of the MDP
 - Cons
 - Requires fast and accurate state evaluation functions
 - May require significant computation at each decision point
 - Needs exploration and careful tradeoff with exploitation
 - Sensitive to model accuracy and search depth

The n -Bandit Problem

- A simplified reinforcement learning scenario
 - There are n different actions (arms)
 - Each arm a_i yields a reward drawn from an unknown probability distribution R_i
 - At each timestep t , agent selects an arm a_t and receives reward $r_t \sim R_{a_t}$
 - No state transitions: the environment is static and memoryless
 - Goal: maximize total reward over a sequence of pulls
- **Exploration vs. Exploitation**
 - Exploration: try different arms to learn their rewards
 - Exploitation: choose the best-known arm to maximize immediate reward
- **Applications**
 - Online advertising (choosing ads to show)
 - Clinical trials (testing treatments)
 - A/B testing in web development



Partially Observable MDPs (POMDPs)

- **Motivation**

- Traditional MDPs assume full observability of the environment
- The agent knows in which state it is in
- In real-world situations, agents often lack precise knowledge of the current state
- POMDPs (read “pom-dee-pees”) extend MDPs to handle uncertainty in state perception

- **Definition**

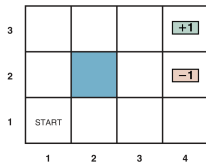
- A POMDP is defined by:
 - States S
 - Actions A
 - Transition model $\Pr(s'|s, a)$
 - Reward function $R(s, a, s')$
 - Sensor model $\Pr(e|s)$: probability of observing evidence e in state s

- **Belief States**

- A belief state $b(s)$ is a probability distribution over possible actual states s (i.e., the probability of being in s)
- The agent maintains $b(s)$ as its internal representation of the environment
- Optimal policies depend on belief states: $\pi^*(b)$

POMDP: 4x3 World with Noisy Four-Bit Sensor

- The world is the 4x3 grid with partial and probabilistic information about the environment
- Use a noisy four-bit sensor, instead of knowing where the agent is
 - Detect obstacles in four directions: North, East, South, West
 - Produces a four-bit string (e.g., 1010), each bit indicating presence (1) or absence (0) of a wall in one direction
- **Error Model**
 - Each bit is correct with probability $1 - \epsilon$, incorrect with probability ϵ
 - Errors are assumed to be independent across bits
 - Example: true config is 1100, observed is 1110
- **Localization Rule**
 - Helps infer the robot's position by comparing sensor output with map-based expectations (integrated into belief state updates)
 - Localization is achievable with high error rate by aggregating observations over time
 - E.g., if the robot believes to be in (3, 2), moves left



Belief State Transitions and Value of Information

- **Belief Update**

- After action a and observation e , belief state b is updated:

$$b'(s') = \alpha \Pr(e|s') \sum_s \Pr(s'|s, a) b(s)$$

where α normalizes the distribution

- Same equation as the filtering task to calculate the new belief state $b'(s)$ from the previous belief state $b(s)$ and the new evidence e

- **Belief space**

- Everything (policy, transition and reward models) is now function of belief state
- It can't be function of the actual state the agent is in, since the agent doesn't know the actual state
- Intermediate belief states have lower utility due to uncertainty
- Information-gathering actions can improve future decision quality

- **Transition and Reward Models in Belief Space**

- Transition: $\Pr(b'|b, a)$ defined using:

$$\Pr(b'|b, a) = \sum_e \Pr(b'|e, a, b) \Pr(e|a, b)$$

- Expected reward in belief state:

Solving POMDPs

- **Observable MDP over Belief Space**

- A POMDP on an actual state space can be converted into an MDP on the belief space

- **Value Iteration for POMDPs**

- Maintains a set of conditional plans p with associated utility vectors α_p
- Expected utility of a plan in belief state b is $b \cdot \alpha_p$
- Optimal utility is piecewise linear and convex over belief space

- **Recursive Plan Evaluation**

$$\alpha_p(s) = \sum_{s'} \Pr(s'|s, a) \left[R(s, a, s') + \gamma \sum_e \Pr(e|s') \alpha_{p.e}(s') \right]$$

- **Challenges**

- Number of plans grows exponentially with depth
- Even small problems generate many plans (e.g., 2^{255} plans for a two-state POMDP at depth 8)
- Approximation Techniques

Reinforcement Learning

- Sequential Decision Problems
- **Reinforcement Learning**
 - Passive Reinforcement Learning
 - Active Reinforcement Learning
 - Generalization in Reinforcement Learning
 - Policy Search

Problem with Supervised Learning

- **In supervised learning**
 - An agent learns by observing examples of input / outputs
 - It's hard to find labeled data for all situations
- E.g., apply supervised learning to play chess
 - Take a board position as input \underline{x} and return a move m
 - Build a DB of grandmaster games with positions and winner (assuming moves by winner are good)
 - Problems
 - In a new game, positions differ from DB, as we have few examples compared to possible positions (10^{40})
 - The agent doesn't understand the game's goal (i.e., checkmate) or valid moves of each piece
- *"The AI revolution will not be supervised"* (Yann LeCun)

Reinforcement Learning

- **Reinforcement Learning (RL) Paradigm**

- Agent learns from direct interaction with the environment
- Periodically receives reward signals indicating success or failure (“reinforcements”)
- Learns a policy to maximize cumulative future rewards
- Goal: maximize expected sum of rewards

- **RL vs supervised learning**

- Providing a reward signal to the agent is easier than providing inputs / outputs
- RL is active since the agent explores the environment and learn from actions and consequences

- **RL vs MDP**

- The goal of both is to maximize the expected sum of rewards
- In RL the agent:
 - Doesn't know the transition model or the reward function (doesn't know the rules)
 - Needs to act to learn more

Sparse vs Immediate Rewards

- Sparse rewards = in the vast majority of states the agent is not given informative reward
 - E.g., win/lose at the end of a chess game
 - The agent must explore many states to find the few that provide rewards
 - Often requires more sophisticated exploration strategies
- Immediate / intermediate rewards help guide learning
 - E.g.,
 - In tennis, you can get rewards for every point scored
 - Learning to crawl, any forward motion is a reward
 - In a video game, collecting coins or power-ups can serve as intermediate rewards
 - Provides continuous feedback to the agent

Applications of Reinforcement Learning

- **Games and Simulations**

- RL has achieved superhuman performance in games like Go, Chess, and Dota2
- Algorithms learn strategies through self-play and reward-driven improvement

- **Robotics**

- RL enables learning of complex control policies for walking, grasping, and manipulation
- Applications include robotic arms, quadrupeds, and autonomous drones

- **Autonomous Vehicles**

- RL used for decision-making and control in self-driving cars
- Handles tasks like lane merging, navigation, and obstacle avoidance

- **Recommendation Systems**

- Adaptive recommendation based on user interactions (e.g., Netflix, YouTube) to optimize long-term engagement and satisfaction

- **Finance and Trading**

- Portfolio management and trading strategies learned through market simulations
- Agents aim to maximize returns under uncertainty and risk constraints

- **Healthcare**

- Personalized treatment policies learned from patient data

Model-Based Reinforcement Learning

- **Definition**

- Learns an explicit model of the environment's dynamics and uses it to make a decision about how to act
- **Transition model**: estimates $\Pr(s'|s, a)$, i.e., probability of reaching state s' from s after action a
- **Reward model**: estimates $R(s, a)$, i.e., expected reward after taking action a in state s
- Intuition: learn to drive by studying the manual and physics

- **Learning Process**

- Collects experience tuples (s, a, r, s')
- Updates the model of the environment (transition and reward)
- Plans using the model to improve policy (e.g., via value iteration or policy iteration)
- Dyna-Q algorithm: combines model-free updates with simulated planning steps

- **Advantages**

- Efficient sample usage: fewer real-world interactions required
- Enables planning by simulating outcomes

- **Disadvantages**

- Learning an accurate model is challenging
- Errors in the model can propagate and lead to poor decisions

Model-Free Reinforcement Learning

- **Definition**

- Learns directly from interactions with the environment without building a model of dynamics
- Agent observes (s, a, r, s') and updates value or policy estimates based on observed outcomes
- No attempt to predict $P(s'|s, a)$ or $R(s, a)$
- Intuition: learn to drive by trial and error

- **Learning Process**

- Value-based methods: Learn state or state-action values (e.g., $Q(s, a)$)
 - E.g., Q-learning
- Policy-based methods: Learn the policy directly
 - E.g., REINFORCE, actor-critic

- **Advantages**

- Simpler to implement when environment model is unknown or too complex
- Robust to model inaccuracies since no model is used

- **Disadvantages**

- Requires more environment interactions (sample inefficient)
- Harder to incorporate planning or long-term reasoning

Model-Based vs Model-Free Reinforcement Learning

- **Core Distinction**

- Model-Based RL: Learns a model of environment dynamics $P(s'|s, a)$ and $R(s, a)$ and uses it for planning
- Model-Free RL: Learns value functions $Q(s, a)$ or policies $\pi(a|s)$ directly from experience

- **Sample Efficiency**

- Model-Based: Generally more sample efficient due to simulated planning
- Model-Free: Typically needs more environment interactions

- **Computation**

- Model-Based: Higher planning overhead; simulations required
- Model-Free: Simpler computations per step; often more scalable

- **Flexibility and Robustness**

- Model-Based: Sensitive to model inaccuracies
- Model-Free: More robust to model errors (since it doesn't learn one)

- **Typical Use Cases**

- Model-Based: Robotics, planning tasks, known environments
- Model-Free: Games, large-scale unknown or stochastic environments

- **Examples**

- Model-Based: Dyna-Q, PILCO
- Model-Free: Q-learning, Deep Q Networks (DQN), REINFORCE

Active vs Passive Reinforcement Learning

- **Basic Distinction**

- Passive RL: Learns value of a fixed policy; does not choose actions
- Active RL: Learns both the value function and the optimal policy through exploration

- **Policy Handling**

- Passive: Follows a given policy $\pi(s)$ and estimates $V^\pi(s)$ or $Q^\pi(s, a)$
- Active: Improves policy over time, aiming for $\pi^*(s)$ that maximizes reward

- **Exploration**

- Passive: No exploration — strictly evaluates the given policy
- Active: Explores actions to improve the policy (e.g., ϵ -greedy, softmax)

- **Learning Goal**

- Passive: Accurate value function for a known policy
- Active: Optimal policy and value function via interaction

- **Algorithms**

- Passive: Temporal Difference Learning (TD), Adaptive Dynamic Programming for a fixed policy
- Active: Q-learning, SARSA, policy iteration methods

- **Use Cases**

- Passive: Evaluation of policies from human demonstrations or expert systems
- Active: Autonomous agents discovering optimal strategies from scratch

Passive Reinforcement Learning

- Sequential Decision Problems
- Reinforcement Learning
 - **Passive Reinforcement Learning**
 - Active Reinforcement Learning
 - Generalization in Reinforcement Learning
 - Policy Search

Passive Learning Agent

- Consider a fully observable environment with a small number of actions and states
- **The agent:**
 - Has a fixed policy $\pi(s)$ to determine its action
 - Needs to learn $U^\pi(s)$, the expected discounted reward if policy π is executed starting in state s
 - Doesn't know the transition model $\Pr(s'|s, a)$ and the reward function $R(s, a, s')$
- The agent executes a set of trials using the policy π :
 - Starts from an initial state and experiences state transitions until reaching terminal states
 - Stores actions and rewards at each state $(s_0, a_0, r_1, s_1, \dots, s_n)$
 - Estimates:

$$U^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(S_t, \pi(S_t), S_{t+1})\right]$$

- Direct utility estimation
 - For each state s , average the returns from all episodes in which s was visited:

Adaptive Dynamic Programming

- **Objective**

- Learn utility estimates $U^\pi(s)$ for a fixed policy π using an estimated model of the environment

- **Key Components**

- Model learning: Estimate transition probabilities $\Pr(s'|s, a)$ and reward function $R(s, a)$ from experience
- Utility update: Solve the Bellman equations for the fixed policy:

$$U^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s'} \Pr(s'|s, \pi(s)) U^\pi(s')$$

- **Learning Process**

- Collect transitions $(s, \pi(s), r, s')$ during execution
- Update model estimates:
 - $\Pr(s'|s, a) \approx$ empirical frequency
 - $R(s, a) \approx$ average observed reward
- Use dynamic programming to compute $U^\pi(s)$

- **Advantages**

- More sample-efficient than direct utility estimation
- Leverages structure of the MDP to generalize better

- **Limitations**

- Requires accurate model estimation
- Computational cost of solving Bellman equations repeatedly

Temporal-Difference Learning

- **Objective**

- Estimate utility values $U^\pi(s)$ for a fixed policy π using experience without a model

- **Key Idea**

- Combine benefits of Monte Carlo methods and Dynamic Programming
- Update estimates after every transition using bootstrapping

- **TD(0) Update Rule**

- When a transition occurs from state s to state s' via action $\pi(s)$, we apply the update:

$$U^\pi(s) \leftarrow U^\pi(s) + \alpha[r + \gamma U^\pi(s') - U^\pi(s)]$$

where:

- s is the current state
- r is the immediate reward
- s' is the next state
- α is the learning rate
- γ is the discount factor

- **Characteristics**

- Online and incremental: updates occur after each step
- Does not require knowledge of model $P(s'|s, a)$ or $R(s, a)$

- **Advantages**

- More efficient and lower variance than Monte Carlo methods

Active Reinforcement Learning

- Sequential Decision Problems
- Reinforcement Learning
 - Passive Reinforcement Learning
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Active Reinforcement Learning

- Passive RL assumes agent has a fixed policy and passively receives reward signals
 - In many real-world cases, agent needs to decide what actions to take and rewards must be actively sought or queried
- Active RL includes cost-sensitive decisions about when to query for rewards
 - Useful when querying is expensive or limited (e.g., human feedback)
- Key problem: balancing cost of querying against benefit of accurate reward
- Formal model:
 - Agent observes state s and selects action a
 - Decides whether to query for reward r
 - Cost c incurred if query is made
- Objective:
 - Maximize cumulative reward minus query costs
 - $\sum (r_t - c_t)$ where $c_t = c$ if query made, 0 otherwise
- Optimal policy needs to learn both:
 - What actions to take
 - When it is worth querying for reward
- Applications:
 - Robotics with costly sensors

Greedy Agent in Reinforcement Learning

- A greedy agent always selects the action with the highest estimated value based on current knowledge or Q-values

$$a = \operatorname{argmax}_a Q(s, a)$$

for state s

- No exploration: purely exploits known information \$\$
- An agent must make a tradeoff between
 - Exploitation of current best action to maximize its short-term reward
 - Exploration of unknown states to gain information that can lead to a change in policy (and greater rewards in the future)
 - E.g., in life you need to decide continuing a comfortable existence, or try something unknown in the hopes of a better life
- Goal: efficient learning with minimal queries to maximize information gain per unit cost
- Strategies include:
 - Random follow greedy policy or explore
 - Cost-aware exploration: modify exploration bonus based on query cost
 - Confidence-based querying: only query when uncertain about reward

Safe Exploration in Reinforcement Learning

- In idealized settings, agents can explore freely and learn from negative outcomes (e.g., losing in chess or simulations)
 - E.g., a self-driving car in simulation can crash without consequences
- In the real world, exploration has risks:
 - Irreversible actions may lead to states that cannot be recovered from
 - Agents can enter “absorbing states” where no further rewards or actions are possible
 - E.g., a crash that destroys a self-driving car permanently limits its future learning
- Safer Policy Approaches
 - **Bayesian Reinforcement Learning:** Maintain a probability distribution over possible models
 - Compute a policy that maximizes expected utility across all plausible models
 - In complex cases, leads to an “exploration POMDP” which is computationally intractable but conceptually useful
 - **Robust Control Theory:** Optimize for the worst-case scenario among all plausible models
 - Resulting policies are conservative but safe
 - E.g., agent avoids any action that could possibly lead to death
 - Impose constraints to prevent the agent from taking dangerous actions
 - E.g., safety controllers can intervene in risky states for autonomous helicopters

Temporal-Difference Q-Learning

- Q-learning is a model-free reinforcement learning algorithm
 - Learns the value of taking an action in a given state, denoted $Q(s, a)$
 - Does not require a model of the environment
- Temporal-difference (TD) learning updates estimates based on other learned estimates
 - Unlike Monte Carlo methods, it updates after every step using bootstrapping

- **Q-learning update rule:**

$$Q(s, a) \leftarrow Q(s, a) + \alpha[r + \gamma \max_{a'} Q(s', a') - Q(s, a)]$$

- α : learning rate
 - r : reward received after action a
 - γ : discount factor for future rewards
 - s' : next state
 - a' : next action
- The update aims to reduce “the TD error” $r + \gamma \max_{a'} Q(s', a') - Q(s, a)$, i.e., the difference between current estimate and observed return

Generalization in Reinforcement Learning

- Sequential Decision Problems
- Reinforcement Learning
 - Passive Reinforcement Learning
 - Active Reinforcement Learning
 - **Generalization in Reinforcement Learning**
 - Policy Search

Generalization in Reinforcement Learning (1/2)

- Tabular representations become infeasible for large state spaces
 - Real-world problems often have millions or more distinct states
 - Example: Backgammon has $\sim 10^{20}$ states, but successful agents visit only a small fraction
- Function approximation enables scalability and generalization
 - Replace large tables with parameterized functions: $\hat{U}_\theta(s)$ or $\hat{Q}_\theta(s, a)$
 - Linear example: $\hat{U}_\theta(s) = \theta_1 f_1(s) + \dots + \theta_n f_n(s)$
- Benefit: Generalizes from visited states to unvisited ones
 - Allows efficient learning with fewer examples
- Temporal-Difference (TD) and Q-learning adapt to function approximation
 - TD update:

$$\theta_i \leftarrow \theta_i + \alpha [r + \gamma \hat{U}_\theta(s') - \hat{U}_\theta(s)] \frac{\partial \hat{U}_\theta(s)}{\partial \theta_i}$$

- Q-learning update:

$$\theta_i \leftarrow \theta_i + \alpha [r + \gamma \max_{a'} \hat{Q}_\theta(s', a') - \hat{Q}_\theta(s, a)] \frac{\partial \hat{Q}_\theta(s, a)}{\partial \theta_i}$$

- Issues and solutions:
 - **Divergence:** parameters can grow uncontrollably
 - **Catastrophic forgetting:** important knowledge can be lost
 - **Solution:** experience replay reuses old data to stabilize learning

Policy Search

- Sequential Decision Problems
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 - Passive Reinforcement Learning
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Policy Search in Reinforcement Learning

- A policy $\pi(s)$ maps states to actions
 - Use a parameterized representation with fewer parameters than states (e.g., linear, deep neural network): $\pi_\theta(s)$
 - Directly optimizes parameters θ of the policy $\pi_\theta(s)$ rather than value functions
 - Pick the value with highest predicted value

$$\pi_\theta(s) = \operatorname{argmax}_a \hat{Q}_\theta(s, a)$$

- Useful in high-dimensional or continuous action spaces
- Even if learning a function replaces the Q-function, it is not an approximation of Q-function (i.e., Q_learning)
 - Seek a function that gives good performance and might differ from the optimal Q-function Q^*
- To avoid jittery policy for discrete actions, use stochastic policies for smoother optimization:
 - E.g., softmax over Q-values

$$\pi_\theta(s, a) = \frac{e^{\beta \hat{Q}_\theta(s, a)}}{\sum_{a'} e^{\beta \hat{Q}_\theta(s, a')}}$$

where β controls exploration vs exploitation

- If everything is continuous and differentiable, use gradient descent to find