Hadamard Transform :→

⇒ The 1-D, forward Hadamard Kernel is given by the relation

$$g(x,u) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

i.e.

$$g(x,u) = \frac{1}{N}(-1)^{\sum\limits_{i=0}^{n-1} b_i(x) b_i(u)}$$

where summation in the exponent is performed in modulo 2 arithmetic and $b_k(z)$ is the kth bit in the binary representation of z.

⇒ The 1-D Hadamard transform is given by

$$H(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

where $N=2^n$, and u has values in the range $0, 1, 2, \ldots, N-1$

⇒ The Hadamard Kernel forms a matrix having orthogonal rows and columns

⇒ The inverse Hadamard Kernel is given by

$$h(x,u) = (-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

⇒ The inverse Hadamard Transform

$$f(x) = \sum_{u=0}^{N-1} H(u)(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

for $x = 0, 1, 2, \ldots, N-1$

The 2-D Kernels are given by

$$g(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

and

$$h(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

⇒ The 2-D Hadamard tranform pair

$$H(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

and

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} H(u,v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

⇒ Hadamard Kernels are separable and symmetric. Hence

$$g(x,y,u,v) = g_1(x,u)\,g_1(y,v)$$

$$= h_1(x,u)\,h_1(y,v)$$

$$= \left[\frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1} b_i(x)\,b_i(u)}\right]\left[\frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1} b_i(y)\,b_i(v)}\right]$$

⇒ As 2-D Hadamard Kernels are separable the 2-D tranform may be obtained by successive applications of 1-D Hadamard tranform algorithm

For $N=8$, the Hadamard Kernel is given by Eq^n. (1). Here the constant term $1/N$ has been omitted for simplicity

$u \backslash x$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	-1	1	-1	1	-1	1	-1
2	1	1	-1	-1	1	1	-1	-1
3	1	-1	-1	1	1	-1	-1	1
4	1	1	1	1	-1	-1	-1	-1
5	1	-1	1	-1	-1	+1	-1	+1
6	1	1	-1	-1	-1	-1	+1	+1
7	1	-1	-1	1	-1	+1	+1	-1

⇒ The Hadamard matrix of lowest order ($N=2$) is

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

⇒ Letting H_N represent the matrix of order N, the recursive relationship is

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

where H_{2N} is the Hadamard matrix of order $2N$ and $N=2^n$ is assumed

The transformation matrix is given by

$$A = \frac{1}{\sqrt{N}} H_N$$

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 \end{bmatrix}$$

and

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 & 1 & -1 & -1 & +1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & +1 & -1 & +1 \\ 1 & 1 & -1 & -1 & -1 & -1 & +1 & +1 \\ 1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix}$$

ALTERNATIVE :→ (A.K. Jain)

⇒ The Hadamard transform matrices, H_n are $N \times N$ matrices where

$$N = 2^n, \; n = 1, 2, 3$$

These can be generated by the core matrix

$$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the Kronecker product recursion

$$H_n = H_{n-1} \otimes H_1 = H_1 \otimes H_{n-1}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

For Eg :→, for n = 3

$$H_3 = H_1 \otimes H_2$$

$$H_2 = H_1 \otimes H_1$$

Hadamard Transform

Definition :→

$$X(b) = \left(\frac{1}{2}\right)^{n/2} \cdot \sum_{a=0}^{N-1} x(a)(-1)^{\sum_{k=0}^{n-1} a(k)b(k)}$$

with

$a \leftrightarrow a(n-1)\ldots\ldots a(1)a(0)$

$b \leftrightarrow b(n-1)\ldots\ldots b(1)b(0)$

$a(k), b(k) = 0, 1$

Explication :→

A Hadamard matrix contains only the values +1 and −1 in such a way that all rows (columns) are mutually orthogonal. The Kronecker product of two Hadamard matrices is again a Hadamard matrix. The smallest Hadamard matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

A Hadamard tranform is discrete by definition.

A Hadamard matrix may be ordered, since all rows (columns) are orthogonal. However there are three important orderings.

Natural Ordering :→

The natural order follows after repeatedly applying the definition given above

$$
\begin{array}{c}
 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}
\begin{array}{c}
0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5
\end{array}
\qquad
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
$$

0 1

Sequential Ordering

This ordering is based on the number of sign changes in a row analog to the sines and cosines in the Fourier transform. The transformed data is ordered in increasing frequency.

$$
\begin{array}{c}
\text{row} \\ 0 \\ 4 \\ 6 \\ 2 \\ 3 \\ 7 \\ 5 \\ 1
\end{array}
\begin{array}{c}
 \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
\end{array}
\begin{array}{c}
\text{sign changes} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
$$

Bit Reversed Ordering

When ordered this way the Hadamard transform is identical to the discrete generalised Walsh transform of order 2. (see Walsh transform for details)

$$
\begin{array}{c}
\text{row} \\ 0 \\ 4 \\ 2 \\ 6 \\ 1 \\ 5 \\ 3 \\ 7
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}
$$

The natural ordering can be transformed into sequential ordering by using Gray codes and bit reversing.

Natural decimal	binary	Gray		Bit Reversed	Sequential decimal
0	000	000	$\longrightarrow$	000	0
1	001	001	$\longrightarrow$	100	4
2	010	011	$\longrightarrow$	110	6
3	011	010	$\longrightarrow$	010	2
4	100	110	$\longrightarrow$	011	3
5	101	111	$\longrightarrow$	111	7
6	110	101	$\longrightarrow$	101	5
7	111	100	$\longrightarrow$	001	1

THE HAAR TRANSFORM

The haar functions $h_K(x)$ are defined on a continuous interval, $x \in [0, 1]$, and for $K = 0, \ldots N-1$, where $N = 2^n$. The integer K can be uniquely decomposed as

$$\boxed{K = 2^p + q - 1}$$

where $0 \le p \le n-1$; $q = 0, 1$ for $p = 0$ and $1 \le q \le 2^p$ for $p \ne 0$. For Eg:→ when $N=4$, we have

K	0	1	2	3
p	0	0	1	1
q	0	1	1	2

$N = 2^n$

$N = 4$

$2^n = 4$

$2^n = 2^2$

$\therefore n = 2$

Representing K by (p, q), the Haar functions are defined as

$$h_0(x) \equiv h_{0,0}(x) = \frac{1}{\sqrt{N}}, \quad \underbrace{x \in [0,1]}$$

$$h_k(x) = h_{p,q}(x) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2}, & \frac{q-1}{2^p} \le x < \frac{q-\frac{1}{2}}{2^p} \\ -2^{p/2}, & \frac{q-\frac{1}{2}}{2^p} \le x < \frac{q}{2^p} \\ 0 & \text{otherwise for } x \in [0,1] \end{cases}$$

The Haar transform is obtained by letting x take discrete values at m/N, $m = 0, 1, \dots, N-1$. For $N=8$, the Haar transform is given by

$$H_r = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & & & & \end{bmatrix}$$

$$\begin{array}{c c} & \text{Sequency} \\ H_r = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{matrix} \end{array}$$

Although some work has been done for using the Haar transform in image data compression problems, its full potential in feature extraction and image analysis problems has not been determined.

Properties of the Haar Transform :→

① The Haar transform is real and orthogonal. Therefore,

$$H_r = H_r^*$$
$$H_r^{-1} = H_r^T$$

② The basis vectors of the Haar ~~transform~~ matrix are sequency ordered.

③ The Haar transform has poor energy compaction for images.

Orthogonal and Unitary Matrices

An orthogonal matrix is such that its inverse is equal to its transpose i.e. A is orthogonal if

$$A^{-1} = A^T$$

$$A^T A = A A^T = I$$

A matrix is called unitary if its inverse is equal to its conjugate transpose, i.e.,

$$A^{-1} = A^{*T}$$

or

$$A A^{*T} = A^{*T} A = I$$

A real orthogonal matrix is also unitary, but a unitary matrix need not be orthogonal. The preceding definitions imply that the columns (or rows) of an $N \times N$ unitary matrix are orthogonal and form a complete set of basis vectors in an N-dimensional vector space.

Kronecker Products

If A and B are $M_1 \times M_2$ and $N_1 \times N_2$ matrices, respectively, then their Kronecker product is defined as

$$A \otimes B \triangleq \{a(m,n)B\} = \begin{bmatrix} a(1,1)B & \cdots & a(1,M_2)B \\ \vdots & & \vdots \\ a(M_1,1)B & \cdots & a(M_1,M_2)B \end{bmatrix}$$

This is an $M_1 \times M_2$ block matrix of basic dimension $N_1 \times N_2$. $A \otimes B \neq B \otimes A$

Eg :→

$A_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $A_2 = \begin{bmatrix} \sqrt{2} & j \\ -j & \sqrt{2} \end{bmatrix}$ $A_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$

A_1 is orthogonal and unitary
A_2 is not unitary
A_3 is unitary with orthogonal rows.

Eg :→
Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 3 & 4 & -3 & -4 \end{bmatrix}$$

$$B \otimes A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 2 & -2 \\ 3 & 3 & 4 & 4 \\ 3 & -3 & 4 & -4 \end{bmatrix}$$

IMAGE ENHANCEMENT :→

Image enhancement refers to accentuation, or sharpening, of image features such as edges, boundaries, or contrast to make a graphic display more useful for display and analysis. The enhancement process does not increase the inherent information content in the data. But it does increase the dynamic range of the chosen features so that they can be detected easily. Image enhancement includes gray level and

Haar Transform when N=2

$\therefore k = 0, 1$

k	0	1
p	0	0
q	0	1

⇒ The first row of 2×2 Haar matrix is computed by using $h_0(x)$ with $x = 0/2, 1/2$

⇒ $h_0(x)$ is equal to $1/\sqrt{2}$, independent of x

⇒ So the first row of the matrix has two identical elements $1/\sqrt{2}$

⇒ The second row is obtained by computing $h_1(x)$ for $x = 0/2, 1/2$

⇒ When $k=1$, $p=0$ & $q=1$

So,

$$h_1(x) = h_{01}(x) = \frac{1}{\sqrt{2}} \begin{cases} 2^0 = 1 & 0 \le x < 1/2 \\ -2^0 = -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise for } x \in [0,1] \end{cases}$$

$\therefore h_1(0/2) = h_1(0) = 1/\sqrt{2}$

$h_1(1/2) = -1/\sqrt{2}$

$$\therefore A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Following similar procedure yield the matrix for N=4

$$A_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

The ith row of an NXN Haar transformation matrix contains the elements of $h_i(x)$ for $x = 0/N, 1/N, 2/N, \ldots, (N-1)/N$.

If $N=2$, for eg:→ Here $x = 0, 1/2$

$$h_0(x) = h_{00}(x) = \frac{1}{\sqrt{2}}, \quad x \in [0,1]$$

and

$$h_1(x) = h_{01}(x) = \frac{1}{\sqrt{2}} \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise}, x \in [0,1] \end{cases}$$

$$H_2(x) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

i.e. $h_0(0) = 1/\sqrt{2}$

$h_0(1/2) = 1/\sqrt{2}$

$h_1(0) = \frac{1}{\sqrt{2}}$

$h_1(1/2) = -\frac{1}{\sqrt{2}}$

Again, if $N=4$ then

$x = 0, 1/4, 2/4, 3/4$

$$h_0(x) = h_{00}(x) = \frac{1}{\sqrt{4}}, \quad x \in [0,1]$$

i.e. $h_0(0) = 1/\sqrt{4}$

$h_0(1/4) = 1/\sqrt{4}$

$h_0(2/4) = 1/\sqrt{4}$

$$h_1(x) = h_{01}(x) = \frac{1}{\sqrt{4}} \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise}, x \in [0,1] \end{cases}$$

$$\Rightarrow h_1(0) = 1 \;,\; h_1(1/4) = 1 \;,\; h_1(2/4) = -1$$

$$h_3(3/4) = -1$$

$$h_2(x) = h_{11}(x) = \frac{1}{\sqrt{4}} \begin{cases} \sqrt{2} & 0 \le x < 1/4 \\ -\sqrt{2} & 1/4 \le x < 1/2 \\ 0 & \text{otherwise}, x \in [0,1] \end{cases}$$

$$\Rightarrow h_2(0) = \sqrt{2} \;,\; h_2(1/4) = -\sqrt{2} \;,\; h_2(2/4) = 0$$

$$h_2(3/4) = 0$$

$$h_3(x) = h_{12}(x) = \frac{1}{\sqrt{4}} \begin{cases} \sqrt{2} & 1/2 \le x < 3/4 \\ -\sqrt{2} & 3/4 \le x < 1 \\ 0 & \text{otherwise}, x \in [0,1] \end{cases}$$

$h_3(0) = 0 \;,\; h_3(1/4) = 0,\; h_3(2/4) = \sqrt{2}$ and

$$h_3(3/4) = -\sqrt{2}$$

Hence

$$H_4(x) = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

The Cosine Transform :→

The $N \times N$ cosine transform matrix $C = \{c(k,n)\}$ also called the discrete cosine transform (DCT) is defined as

$$c(k,n) = \begin{cases} \dfrac{1}{\sqrt{N}} & k=0,\ 0 \le n \le N-1 \\ \sqrt{\dfrac{2}{N}} \cos \dfrac{\pi(2n+1)k}{2N} & 1 \le k \le N-1,\ 0 \le n \le N-1 \end{cases}$$

The one-dimensional DCT of a sequence $\{u(n), 0 \le n \le N-1\}$ is defined as

$$v(k) = \alpha(k) \sum_{n=0}^{N-1} u(n) \cos\left[\frac{\pi(2n+1)k}{2N}\right], \quad 0 \le k \le N-1$$

where

$$\alpha(0) = \sqrt{\frac{1}{N}} \qquad \alpha(k) = \sqrt{\frac{2}{N}} \text{ for } 1 \le k \le N-1$$

The inverse transformation is given by

$$u(n) = \sum_{k=0}^{N-1} \alpha(k) v(k) \cos\left[\frac{\pi(2n+1)k}{2N}\right], \quad 0 \le n \le N-1$$

Properties :→

1. The cosine transform is real and orthogonal, i.e., $C = C^* \Rightarrow C^{-1} = C^T$

2. The cosine transform has excellent energy compaction for highly correlated data