Chapters 3: Inference

Inference

- Model: $Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$
- Estimates: $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$
- Draw conclusions about $\beta_0, \beta_1, \ldots, \beta_p$

Inference

- Two main inference tools:

  (1) Hypothesis tests

  (2) Confidence intervals

- A distribution assumption is now required.

Savings Example

- 50 different countries
- Data from 1960 – 1970
- Response: aggregate personal savings divided by disposable income (*sr*)
- Predictors: per capital disposable income (*dpi*), percentage rate of change in per capita disposable income (*ddpi*), percentage of population under 15 (*pop15*), percentage of population over 75 (*pop75*)

```
> data(savings)
> savings
                   sr pop15 pop75     dpi  ddpi
Australia       11.43 29.35  2.87 2329.68  2.87
Austria         12.07 23.32  4.41 1507.99  3.93
... ...
> result <- lm(sr ~ pop15 + pop75 + dpi + ddpi,
    savings)
```

```
> summary(result)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 28.5660865  7.3545161   3.884 0.000334
pop15       -0.4611931  0.1446422  -3.189 0.002603
pop75       -1.6914977  1.0835989  -1.561 0.125530
dpi         -0.0003369  0.0009311  -0.362 0.719173
ddpi         0.4096949  0.1961971   2.088 0.042471

Residual standard error: 3.803 on 45 degrees of freedom
Multiple R-Squared: 0.3385, Adjusted R-squared: 0.2797
F-statistic: 5.756 on 4 and 45 DF,  p-value: 0.0007904
```

Savings Example Ctd

- Is *dpi* significant in the full model?
- Estimated from data:

$$\begin{aligned}\widehat{sr} &= 28.6 - 0.46 \times pop15 - 1.69 \times pop75 \\ &\quad - 0.0003 \times dpi + 0.41 \times ddpi\end{aligned}$$

- Is "0.0003" random fluctuation due to chance, or does it indicate that the true coefficient β_{dpi} is different from 0?

Savings Example Ctd

- Is *pop75* significant in the full model?
- Estimated from data:

$$\begin{aligned} \widehat{sr} &= 28.6 - 0.46 \times pop15 - 1.69 \times pop75 \\ &\quad - 0.0003 \times dpi + 0.41 \times ddpi \end{aligned}$$

- Is "-1.69" random fluctuation due to chance, or does it indicate that the true coefficient β_{pop75} is different from 0?

Hypothesis Tests

- Testing: use probability to decide whether data is consistent with hypothesis
- Null hypothesis H_0 (e.g., $\beta_j = 0$)
- Alternative hypothesis H_A (e.g., $\beta_j \neq 0$)
- Decide whether data is consistent with H_0:
  - If not, reject H_0 (there is an evidence that we reject H_0)
  - Otherwise, fail to reject H_0

Errors in Hypothesis Testing

		True State	
		H_0 true	H_0 false
Our Decision	Not reject H_0	-	Type II error
	Reject H_0	Type I error	-

Procedure

- Set $\alpha = P(\text{type I error})$. Typically $\alpha = 0.05$ or 0.01. α is called the significance level.
- Compute p-value: the probability of observed or more extreme departure from H_0 (in favor of H_A) when H_0 is true.
- If p-value $< \alpha$, reject H_0.
- How to calculate p-value?

Savings Example

Full model:

$$sr = \beta_0 + \beta_{pop15} \times pop15 + \beta_{pop75} \times pop75 + \beta_{dpi} \times dpi + \beta_{ddpi} \times ddpi$$

- Null hypothesis: $\beta_{pop75} = 0$
- Alternative hypothesis: $\beta_{pop75} \neq 0$

We observe that

$$\widehat{sr} = 28.6 - 0.46 \times pop15 - 1.69 \times pop75 - 0.0003 \times dpi + 0.41 \times ddpi$$

Compute the p-value:

$$P(|\hat{\beta}_{pop75}| \geq 1.69 \mid \beta_{pop75} = 0)$$

Distribution

Examples

- Bernoulli distribution: binary outcome (e.g. T/F, S/F, 0/1)
- Binomial distribution: generalized Bernoulli distribution (multiple trials)
- Normal distribution: continuous variable

Further Assumption on Errors

We have only assumed $E(\epsilon) = 0$ for LSE.

To compute the *p*-value, we also need to assume a distribution for the errors ϵ. The usual assumption is

$$\epsilon \sim Normal_n(0, \sigma^2 \mathbb{I})$$

where σ^2 is a constant and $\mathbb{I}$ is a n by n identity matrix.

Summary of Assumptions

- Linearity

$\epsilon = (\epsilon_1, ..., \epsilon_n)$ should satisfy the following assumptions.

- Normal distribution
- Constant variance
- Independence

Distribution of $\hat{\beta}$

If $\epsilon \sim N_n(0, \sigma^2 \mathbb{I})$, then

$$\begin{aligned}\hat{\beta} &\sim N_{p+1}(\beta, (X^T X)^{-1}\sigma^2) \\ \hat{\beta}_j &\sim N(\beta_j, (X^T X)^{-1}_{jj}\sigma^2)\end{aligned}$$

Property of Normal Distn

Suppose that $Z \sim N_{p+1}(\mu, \Sigma)$. Then,

- For any $a \in \mathbb{R}^{p+1}$,

$$a^T Z \sim N(a^T \mu, a^T \Sigma a).$$

Distribution of $\hat{\beta}$

- The standard error is

$$se(\hat{\beta}_j) = \sqrt{(X^T X)^{-1}_{jj} \sigma^2}$$

- In practice, we use the **approximation**

$$\widehat{se}(\hat{\beta}_j) = \sqrt{(X^T X)^{-1}_{jj} \hat{\sigma}^2}$$

- Recall that $\hat{\sigma}^2 = \frac{\sum_i (y_i - \hat{y}_i)^2}{n-(p+1)}$

Distribution of $\hat{\beta}$ Ctd

Under the normal assumption on the errors,

$$\begin{aligned} \frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} &\sim N(0,1) \\ \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} &\sim t_{n-(p+1)} \end{aligned}$$

The t-distribution

$$T_\nu = \frac{Z}{\sqrt{V/\nu}}$$

where

- Z is a standard normal with expected value 0 and variance 1;
- V has a chi-squared distribution with ν degrees of freedom;
- Z and V are independent.

$$\frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\frac{\sum_i (y_i - \hat{y}_i)^2}{n-(p+1)}}} = \frac{(\hat{\beta}_j - \beta_j)/\sqrt{(X^T X)^{-1}_{jj}\sigma^2}}{\sqrt{\frac{\sum_i (y_i - \hat{y}_i)^2}{(X^T X)^{-1}_{jj}\sigma^2}/n - (p+1)}}.$$

The t-distribution

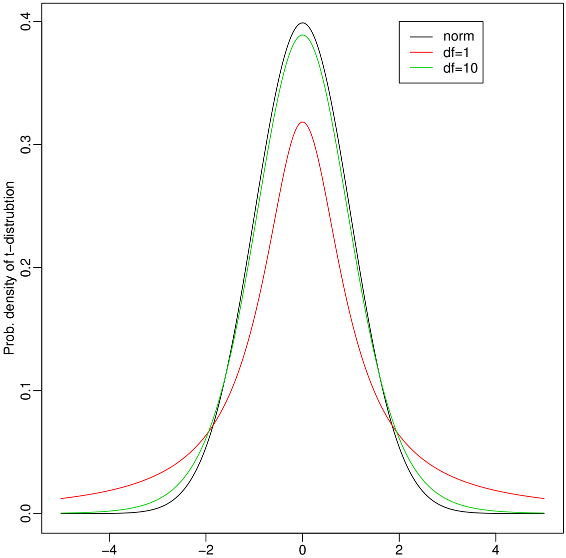
Probability density function (pdf):

$$N(0,1) \sim \frac{1}{\sqrt{2\pi}} e^{-1/2z^2}$$

$$t_n \sim \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \cdot \Gamma\left(\frac{n}{2}\right)} (1 + t^2/n)^{-(n+1)/2}$$

- Has a single parameter n called **degrees of freedom**
- Symmetric around 0, "bell-shaped", but heavier tails than normal
- As $n \to \infty$, $t_n \to N(0,1)$

The t density



Like normal distribution with wider tails

The t-statistic (Savings Example)

If the null is true, i.e., $\beta_{pop75} = 0$, then

$$\frac{\hat{\beta}_{pop75} - \beta_{pop75}}{\widehat{se}(\hat{\beta}_{pop75})} \sim t_{50-(4+1)}$$

From the R output, we have (t-statistic)

$$\frac{\hat{\beta}_{pop75} - 0}{\widehat{se}(\hat{\beta}_{pop75})} = \frac{-1.69 - 0}{1.08} = -1.56$$

```
Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) 28.5660865  7.3545161   3.884 0.000334
pop15       -0.4611931  0.1446422  -3.189 0.002603
pop75       -1.6914977  1.0835989  -1.561 0.125530
dpi         -0.0003369  0.0009311  -0.362 0.719173
ddpi         0.4096949  0.1961971   2.088 0.042471

Residual standard error: 3.803 on 45 degrees of freedom
Multiple R-Squared: 0.3385, Adjusted R-squared: 0.2797
F-statistic: 5.756 on 4 and 45 DF,  p-value: 0.0007904
```

t-statistic (Savings Example) Ctd

Is this value extreme for the t_{45} distribution? i.e., need to compute the the probability

$P(\text{observe "-1.56" or more extreme}|\beta_{pop75} = 0) = P(|t_{45}| \geq 1.56) = ?$

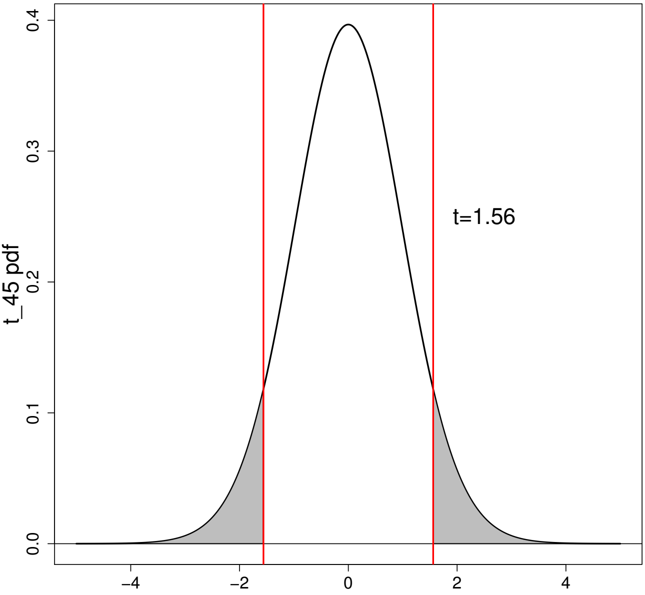
What if the test is one-sided?
e.g., $H_A : \beta_{pop75} > 0$ or $H_A : \beta_{pop75} < 0$

What if we are interested in whether $\beta_{pop75} = -1$ or not?
e.g., $H_0 : \beta_{pop75} = -1$ or $H_A : \beta_{pop75} \neq -1$

t-test

- Two-sided test: $P(|t_{45}| \geq 1.56) = 0.13 > \alpha = 0.05$, therefore we fail to reject H_0.
- Thus *pop75* is not significant in the full model at level $\alpha = 0.05$.
-
```
## CDF of t-distribution
> 2*(pt(1.56, df=45, lower.tail = F))
[1] 0.1257658
```

t-statistic (Savings Example) Ctd



t-statistic (Savings Example) Ctd

What if the test is one-sided?

e.g., $H_A : \beta_{pop75} > 0$ or $H_A : \beta_{pop75} < 0$

What if we are interested in whether $\beta_{pop75} = -1$ or not?

e.g., $H_0 : \beta_{pop75} = -1$ or $H_A : \beta_{pop75} \neq -1$

t-statistic (Savings Example) Ctd

$H_0 : \beta_{pop75} = 0 \quad v.s. \quad H_A : \beta_{pop75} > 0.$

If the null is true, i.e., $\beta_{pop75} = 0$, then

$$\frac{\hat{\beta}_{pop75} - 0}{\widehat{se}(\hat{\beta}_{pop75})} \sim t_{50-(4+1)}$$

Is this value extreme for the t_{45} distribution? (follow the H_A)

$P(\text{observe "-1.56" or more extreme}|\beta_{pop75} = 0) = P(t_{45} > 1.56) =?$

t-statistic (Savings Example) Ctd

$H_0 : \beta_{pop75} = -1 \;\; v.s. \;\; H_A : \beta_{pop75} \neq -1.$

If the null is true, i.e., $\beta_{pop75} = 0$, then

$$\frac{\hat{\beta}_{pop75} - 1}{\widehat{se}(\hat{\beta}_{pop75})} = \frac{-1.69 - (-1)}{1.08} = -0.64$$

Is this value extreme for the t_{45} distribution? (follow the H_A)

$$P(\text{observe "-0.64" or more extreme}|\beta_{pop75} = 0) = P(|t_{45}| > 0.64)$$

Why in regression?

Why in regression?

In the following full model,

$$Weight = \beta_0 + \beta_1 Height + \beta_2 Gender + \epsilon$$

$$H_0 : \beta_1 = 0 \;\; v.s. \;\; H_A : \beta_1 \neq 0$$

.

In the following full model,

$$Weight = \beta_0 + \beta_1 Height + \epsilon$$

$$H_0 : \beta_1 = 0 \;\; v.s. \;\; H_A : \beta_1 \neq 0$$

.

Why in regression? Example

'Nuisance' variables.

- Age
- Gender
- Ethnicity

This nuisance variables effects cannot be simply ignored.

Predictor ordering

In the ordering of predictor influence p-value?

Test for multiple parameters

In the following full model,

$$Weight = \beta_0 + \beta_1 Height + \beta_2 Gender + \epsilon$$

$$H_0 : \beta_1 = \beta_2 = 0 \ \ v.s. \ \ H_A : H_0 \text{ is not true.}$$

Test for multiple parameters

For example,

$$Weight = \beta_0 + \beta_1 Height + \beta_2 Gender + \epsilon$$

Test 1:

$$H_0 : \beta_1 = 0 \quad v.s. \quad H_A : \beta_1 \neq 0$$

Test 2:

$$H_0 : \beta_2 = 0 \quad v.s. \quad H_A : \beta_2 \neq 0$$

Test for multiple parameters: Type I Error

$$Weight = \beta_0 + \beta_1 Height + \beta_2 Gender + \epsilon$$

Test 1:

$$H_0 : \beta_1 = 0 \quad v.s. \quad H_A : \beta_1 \neq 0$$

Test 2:

$$H_0 : \beta_2 = 0 \quad v.s. \quad H_A : \beta_2 \neq 0$$

The maximum type I error is

$$\# \text{ of tests} \times \alpha.$$

Another (General) Approach

- Recall *RSS*: residual sum of squares $\sum_i \hat{\epsilon}_i^2$
- Fit a model under H_0, compute RSS_{H_0} (e.g. with β_{pop75} set equal to 0)
- Fit another model under $H_0 \cup H_A$, compute $RSS_{H_0 \cup H_A}$ (e.g. no restriction on β_{pop75})
- Compute

$$F = \frac{(RSS_{H_0} - RSS_{H_0 \cup H_A})/(df_{H_0} - df_{H_0 \cup H_A})}{RSS_{H_0 \cup H_A}/df_{H_0 \cup H_A}}$$

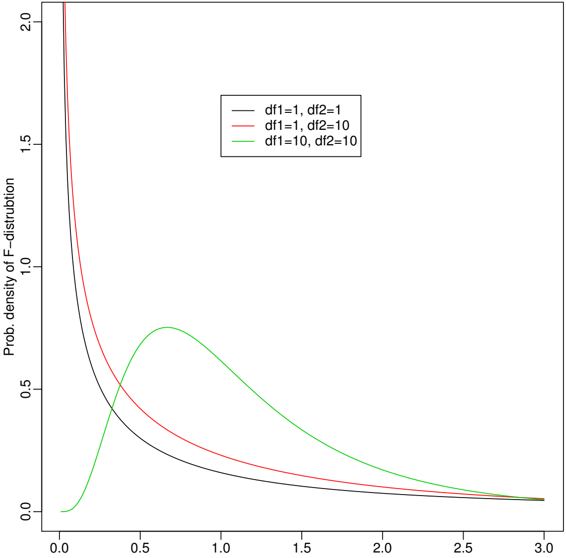
General Approach Ctd

- If H_0 is true,

$$F \sim F_{df_1, df_2}; \quad df_1 = df_{H_0} - df_{H_0 \cup H_A}, df_2 = df_{H_0 \cup H_A}$$

- Compute p-value $= P(F_{df_1, df_2} > F)$

F-distribution



Important facts: (1) $F_{df_1,df_2} > 0$ (2) $t_{df}^2 \sim F_{1,df}$

F-distribution

Important fact (1) $F_{df_1,df_2} > 0$

$$F = \frac{(RSS_{H_0} - RSS_{H_0 \cup H_A})/(df_{H_0} - df_{H_0 \cup H_A})}{RSS_{H_0 \cup H_A}/df_{H_0 \cup H_A}}$$

- $RSS_{H_0 \cup H_A} > 0$
- $RSS_{H_0} - RSS_{H_0 \cup H_A} > 0$.

Theory

F-distribution

- $Z_1, \ldots, Z_n$ i.i.d. Normal(0,1). Then

$$U = Z_1^2 + \cdots + Z_n^2$$

  has χ^2 (chi-square) distribution with n degrees of freedom.
- χ_n^2 is the same as $Gamma(n/2, 2)$.
- Suppose $U \sim \chi_n^2$, $W \sim \chi_m^2$ are independent. Then

$$\frac{U/n}{W/m} \sim F_{n,m}$$

  F-distribution with n and m degrees of freedom.

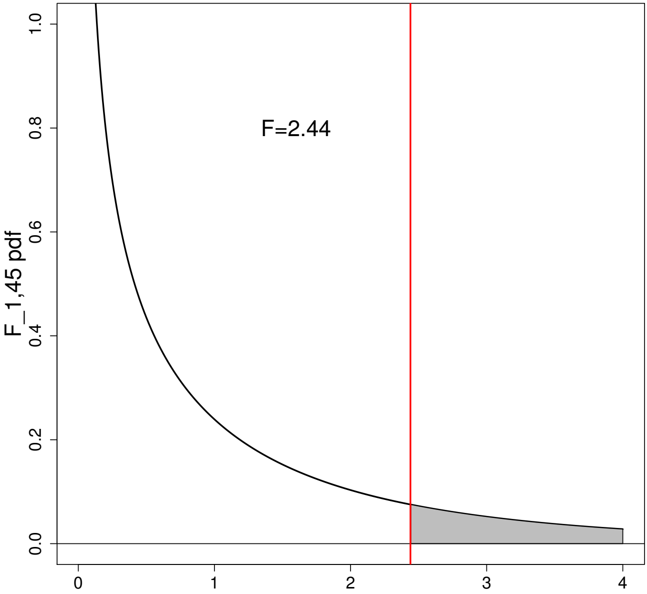
F-test: Savings Example

```
## Model under H0
> h0 <- lm(sr ~ pop15 + dpi + ddpi, savings)
> summary(h0)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 19.2771687  4.3888974   4.392 6.53e-05
pop15       -0.2883861  0.0945354  -3.051  0.00378
dpi         -0.0008704  0.0008795  -0.990  0.32755
ddpi         0.3929355  0.1989390   1.975  0.05427
```

F-test: Savings Example

```
## Model under (H0 U HA)
> h0a <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
> anova(h0, h0a)
Analysis of Variance Table

Model 1: sr ~ pop15 + dpi + ddpi
Model 2: sr ~ pop15 + pop75 + dpi + ddpi
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     46 685.95
2     45 650.71  1     35.24 2.4367 0.1255
```

F-test Ctd



F-test and t-test

- $P(F_{1,45} > 2.44) = 0.13 > \alpha = 0.05$, therefore we fail to reject H_0.
- Notice $t^2 = 1.56^2 = 2.44 = F$
- F-test and two-sided t-test are equivalent for testing a single predictor.

Test a Pair

- Whether both *pop75* and *dpi* can be excluded from the model.
- H_0: $\beta_{pop75} = \beta_{dpi} = 0$; H_A: H_0 is not true.

```
> h0 <- lm(sr ~ pop15 + ddpi, savings)
> h0a <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
> summary(h0)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 15.59958    2.33439   6.682 2.48e-08
pop15       -0.21638    0.06033  -3.586 0.000796
ddpi         0.44283    0.19240   2.302 0.025837
```

```
> anova(h0, h0a)
Analysis of Variance Table

Model 1: sr ~ pop15 + ddpi
Model 2: sr ~ pop15 + pop75 + dpi + ddpi
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     47 700.55
2     45 650.71  2     49.84 1.7233 0.1900
```

- What if we want to test whether any of the predictors are useful in predicting the response?
- H_0: $\beta_{pop15} = \beta_{pop75} = \beta_{dpi} = \beta_{ddpi} = 0$

```
> h0 <- lm(sr ~1, savings)
> h0a <- lm(sr ~ pop15 + + ddpi + pop75 + dpi , savings)
>anova(h0, h0a)

Analysis of Variance Table

Model 1: sr ~ 1
Model 2: sr ~ pop15 + +ddpi + pop75 + dpi
Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1      49 983.63
2      45 650.71  4    332.92 5.7557 0.0007904 ***
```

- What if we want to test whether any of the predictors are useful in predicting the response?
- H_0: $\beta_{pop15} = \beta_{pop75} = \beta_{dpi} = \beta_{ddpi} = 0$

```
> h0 <- lm(sr ~1, savings)
> h0a <- lm(sr ~ pop15 + + ddpi + pop75 + dpi , savings)
>summary(h0a)

Residual standard error: 3.803 on 45 degrees of freedom
Multiple R-squared:  0.3385,Adjusted R-squared:  0.2797
F-statistic: 5.756 on 4 and 45 DF,  p-value: 0.0007904
```

Test a Subspace

- Whether the effect of young people and the effect of old people on the savings rate are the same.
- H_0: $\beta_{pop15} = \beta_{pop75}$; H_A: $\beta_{pop15} \neq \beta_{pop75}$

```
> h0 <- lm(sr ~ I(pop15 + pop75) + dpi + ddpi, savings)
> h0a <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
> summary(h0)
Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)      21.6093051  4.8833633   4.425 5.87e-05
I(pop15 + pop75) -0.3336331  0.1038679  -3.212  0.00241
dpi              -0.0008451  0.0008444  -1.001  0.32212
ddpi              0.3909649  0.1968714   1.986  0.05302

Residual standard error: 3.827 on 46 degrees of freedom
Multiple R-Squared: 0.3152, Adjusted R-squared: 0.2705
```

```
> anova(h0, h0a)
Analysis of Variance Table

Model 1: sr ~ I(pop15 + pop75) + dpi + ddpi
Model 2: sr ~ pop15 + pop75 + dpi + ddpi
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     46 673.63
2     45 650.71  1     22.91 1.5847 0.2146
```

Test another Subspace

- Test whether β_{ddpi} is equal to 0.5
- H_0: $\beta_{ddpi} = 0.5$; H_A: $\beta_{ddpi} \neq 0.5$

```
> h0 <- lm(sr ~ pop15 + pop75 + dpi + offset(0.5*ddpi),
    savings)
> summary(h0)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 27.9287866  7.1608589   3.900 0.000311
pop15       -0.4543714  0.1426430  -3.185 0.002596
pop75       -1.7187908  1.0726662  -1.602 0.115923
dpi         -0.0002274  0.0008925  -0.255 0.800004
```

```
> h0a <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
> anova(h0, h0a)
Analysis of Variance Table

Model 1: sr ~ pop15 + pop75 + dpi + offset(0.5 * ddpi)
Model 2: sr ~ pop15 + pop75 + dpi + ddpi
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     46 653.78
2     45 650.71  1      3.06 0.2119 0.6475
```

- What about using t-test?

Confidence Intervals

Why do we care about CI?

- Hypothesis test: yes/no only
- Statistical significance vs. practical significance (size of the effect)

Confidence Intervals

What is the confidence interval (C.I)?

- "Were this procedure to be repeated on multiple samples, the calculated confidence interval (which would differ for each sample) would encompass the true population parameter 90% of the time."

Confidence Intervals

True or False.

- 95% C.I for β_j contains the true parameter β_j with probability 95%.

Confidence Intervals

True or False.

- 95% C.I for β_j contains the true parameter β_j with probability either 0% or 100%.

Confidence Intervals

How to interpret 95% C.I for β_j?

- We are 95% confident that the C.I for β_j contains the true parameter β_j.

Confidence Intervals for β_j

Consider each parameter individually.

$$\text{Recall} \quad \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} \sim t_{n-(p+1)}$$

Hence

$$P\left(-t_{n-(p+1)}^{(\alpha/2)} \leq \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} \leq t_{n-(p+1)}^{(\alpha/2)}\right) = 1 - \alpha$$

Or with probability $1 - \alpha$, i.e. confidence $100(1 - \alpha)\%$

$$\hat{\beta}_j - t_{n-(p+1)}^{(\alpha/2)} \cdot \widehat{se}(\hat{\beta}_j) \leq \beta_j \leq \hat{\beta}_j + t_{n-(p+1)}^{(\alpha/2)} \cdot \widehat{se}(\hat{\beta}_j)$$

$t^{(\alpha)}$ is the tail probability: $P(t > t^{(\alpha)}) = \alpha$.

Confidence Intervals for β_j Ctd

- General form:

$$\text{estimate} \pm \text{critical value} \times \text{s.e. of estimate}$$

- Two-sided t-test and CI

Savings Example

```
> result <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
> summary(result)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 28.5660865  7.3545161   3.884 0.000334
pop15       -0.4611931  0.1446422  -3.189 0.002603
pop75       -1.6914977  1.0835989  -1.561 0.125530
dpi         -0.0003369  0.0009311  -0.362 0.719173
ddpi         0.4096949  0.1961971   2.088 0.042471
```

Savings Example

```
## Convenient way to compute CIs
> conf <- confint(result)
> conf
                  2.5 %       97.5 %
(Intercept) 13.753330728 43.378842354
pop15       -0.752517542 -0.169868752
pop75       -3.873977955  0.490982602
dpi         -0.002212248  0.001538444
ddpi         0.014533628  0.804856227
## Quantile of t-distribution
> qt(0.975, 45)
[1] 2.014
> c(-0.461 - 2.01*0.145, -0.461 + 2.01*0.145)
[1] -0.753 -0.169
```

Savings Example

What is 90% C.I for each parameter?

```
## Quantile of t-distribution
> qt(0.950, 45)
[1] 1.680
> c(-0.461 - 1.680*0.145, -0.461 + 1.680*0.145)
[1] -0.7046 -0.2174
```

Simultaneous Confidence Regions

Similarly,

$$\frac{(\hat{\beta}-\beta)^T X^T X(\hat{\beta}-\beta)}{(p+1)\hat{\sigma^2}} \sim F_{p+1,n-(p+1)}$$

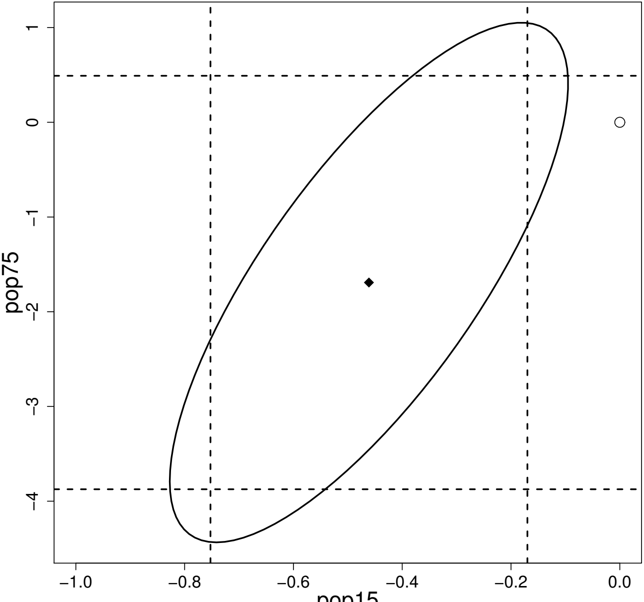
With probability $1-\alpha$, i.e. confidence $100(1-\alpha)\%$

$$(\hat{\beta}-\beta)^T X^T X(\hat{\beta}-\beta) \leq (p+1)\hat{\sigma^2} F^{(\alpha)}_{p+1,n-(p+1)}$$

Savings Example

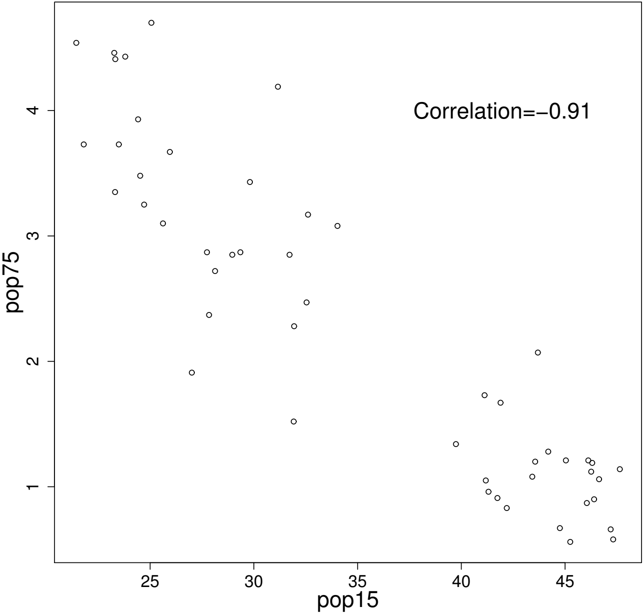
```
## Need to install the "ellipse" package
> library(ellipse)
## Plot the confidence region
> plot(ellipse(result, c('pop15', 'pop75')),
    type=="l", xlim=c(-1,0))
## Add the estimates to the plot
> points(result$coef['pop15'], result$coef['pop75'],pch=18)
## Add the origin to the plot
> points(0, 0, pch=1)
## Add the confidence interval for pop15
> abline(v=conf['pop15',], lty=2)
## Add the confidence interval for pop75
> abline(h=conf['pop75',], lty=2)
```

Savings Example: Confidence region



```
## Correlation between pop15 and pop75
> plot(x=savings$pop15, y=savings$pop75)
> cor(savings$pop15, savings$pop75)
[1] -0.9084787
```

Correlation between predictors



Simultaneous Confidence Regions

What if they are independent?

Prediction

Confidence Intervals for Predictions

- Given new predictors, x_0, what is the predicted response?

$$\hat{y}_0 = x_0^T \hat{\beta}$$

- Two types of predictions:
  - Prediction of a future observation
  - Prediction of the future mean response
- Prediction intervals vs. confidence intervals

Confidence Intervals for Predictions Ctd

For a future observation: $y_{new} = x_{new}^T \beta + \epsilon$

$$\hat{y}_0 \pm t_{n-(p+1)}^{(\alpha/2)} \hat{\sigma} \sqrt{1 + x_0^T (X^T X)^{-1} x_0}$$

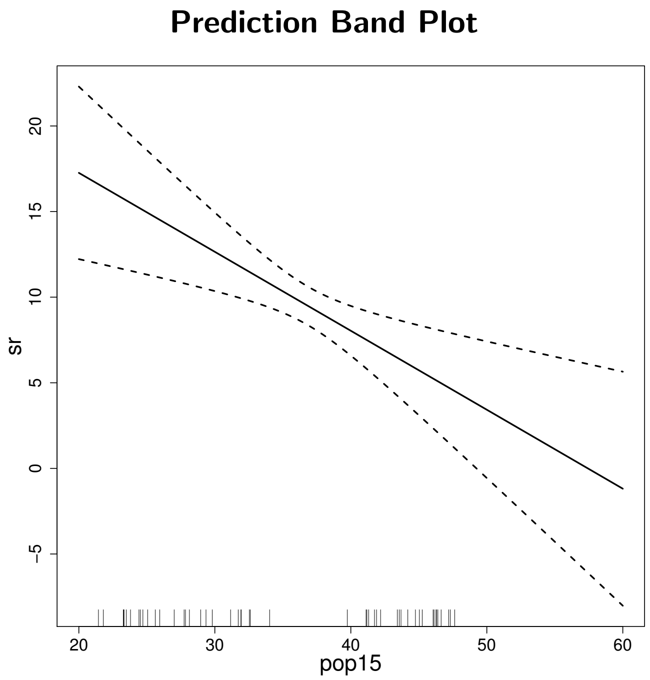
For the future mean response: $E(y_{new}) = E(x_{new}^T \beta) + E(\epsilon)$

$$\hat{y}_0 \pm t_{n-(p+1)}^{(\alpha/2)} \hat{\sigma} \sqrt{x_0^T (X^T X)^{-1} x_0}$$

Savings Example

```
> result <- lm(sr ~ pop15 + pop75 + dpi + ddpi, savings)
## Convenient way to compute CI and PI
> x0 <- data.frame(pop15=35, pop75=2, dpi=1000, ddpi=4)
> predict(result, x0, interval="confidence")
fit      lwr      upr
[1,] 10.34321 9.093452 11.59297
> predict(result, x0, interval="prediction")
fit      lwr      upr
[1,] 10.34321 2.582946 18.10347
```

```
## Generate a sequence of points
> grid <- seq(20, 60, 1)
> pred <- predict(result, data.frame(pop15=grid, pop75=2,
dpi=1000, ddpi=4), interval="confidence")
## Plot a matrix
> matplot(grid, pred, lty=c(1,2,2), col=1, type="l",
xlab="pop15", ylab="sr")
> rug(savings$pop15)
```



Prediction Band Plot
sr
pop15
20
30
40
50
60
−5
0
5
10
15
20

Prediction Band Plot

Why is prediction band getting wider?

$$\hat{y}_0 \pm t_{n-(p+1)}^{(\alpha/2)} \hat{\sigma} \sqrt{1 + x_0^T (X^T X)^{-1} x_0}$$

$$\hat{y}_0 \pm t_{n-(p+1)}^{(\alpha/2)} \hat{\sigma} \sqrt{1 + a + 2bx_0 + cx_0^2}$$

where

$$(X^T X)^{-1} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

```
## Data
age = c(rep(20, 10), rep(50, 10))
ht = c( rep(170, 5), rep(180, 5), rep(170, 5), rep(180, 5) )
wt = 0.5 * age + 0.8 * ht + round( rnorm(20, 0, 15), 0 )
data = data.frame(age, ht, wt)
xtabs(wt~age+ht,data)/5
```

age | ht	170	180
20	140.8	165.4
50	171.6	151.2

- Can you use this table to predict the weight for a age of 35 and a height of 170 cm?

```
## Data
age = c(rep(20, 10), rep(50, 10))
ht = c( rep(170, 5), rep(180, 5), rep(170, 5), rep(180, 5) )
wt = 0.5 * age + 0.8 * ht + round( rnorm(20, 0, 15), 0 )
data = data.frame(age, ht, wt)
xtabs(wt~age+ht,data)/5
```

age | ht	170	180
20	150.6	164.2
50	174.0	186.4

- Can you use this table to predict the weight for a age of 35 and a height of 170 cm?

```
## Data
age = c(rep(20, 10), rep(50, 10))
ht = c( rep(170, 5), rep(180, 5), rep(170, 5), rep(180, 5) )
wt = 0.5 * age + 0.8 * ht + round( rnorm(20, 0, 15), 0 )
data = data.frame(age, ht, wt)
xtabs(wt~age+ht,data)/5
```

age | ht	170	180
20	150.6	164.2
50	174.0	186.4

- Can you use this table to predict the weight for a age of 35 and a height of 175 cm?

age | ht	170	180
20	150.6	164.2
50	174.0	186.4

- Can you use this table to predict the weight for a age of 35 and a height of 175 cm?

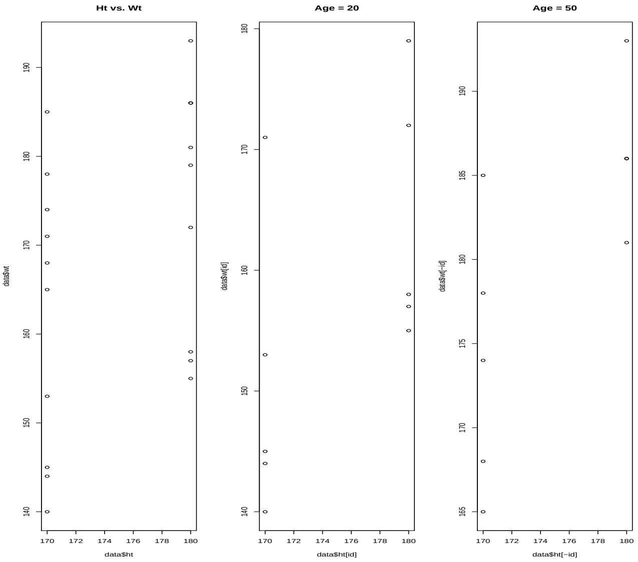
$$\frac{150.6 + 164.2 + 174.0 + 186.4}{4} = 168.8.$$

```
lm0 = lm(wt~age+ht,data)
summary(lm0)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -85.3000    71.0303  -1.201  0.24626
age           0.7600     0.1349   5.632 2.99e-05 ***
ht            1.3000     0.4048   3.211  0.00512 **
```

- Can you use this linear model to predict the weight for a age of 35 and a height of 175 cm?

$$\hat{Wt} = -85.30 + 0.76 \times 35 + 1.3 \times 175 = 168.8$$

```
id = which(data$age ==20)
par(mfrow = c(1,3))
plot(data$ht, data$wt)
plot(data$ht[id], data$wt[id] )
plot(data$ht[-id], data$wt[-id] )
```


Ht vs. Wt
data$wt
data$ht
Age = 20
data$wt[id]
data$ht[id]
Age = 50
data$wt[−id]
data$ht[−id]

```
lm0 = lm(wt~age+ht,data)
summary(lm0)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -85.3000    71.0303  -1.201  0.24626
age           0.7600     0.1349   5.632 2.99e-05 ***
ht            1.3000     0.4048   3.211  0.00512 **
```

- Can you use this linear model to predict the weight for a age of 35 and a height of 175 cm?

Interpreting Predictions

- True model and parameter values may never be known
- Concentrating on predicting future responses removes the need for interpretation of β
- Conceptually simpler, but need to worry about extrapolation

Extrapolation Example: Ht v.s. Wt

- Data: weights(lb) and heights (cm)
- Estimated from data:

$$\widehat{wt} = 34.61 + 0.82 \times ht$$

ht	185	195	184	175	184	169	182	181	175	178	161	157
wt	184	198	178	171	188	172	189	191	174	186	172	164

- New Obs: $ht = 220$ or $ht = 145$.

Example

We performed an experiment concerned with assessing the toxic effect of dioxin. Every separate fish tanks were maintained with different dioxin concentrations (x). A single fish was placed in each tank, and the length of time until the fish died was recorded in days (y). Of interest is a single linear regression model of the form $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$. We assume $\epsilon_i \sim_{i.i.d.} N(0, \sigma^2)$.

```
Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) 506.4864    50.1074  10.108   <2e-16 ***
x            -0.9281     0.5140  -1.806   0.07406
```

(a) The following italicized statement is either true or false. *When hypotheses are $H_0 : \beta_1 = 0$ v.s. $H_A : \beta_1 \neq 0$, we can reject the H_0* Circle whether the statement is True or False and explain your choice.

```
Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) 506.4864    50.1074  10.108   <2e-16 ***
x            -0.9281     0.5140  -1.806   0.07406
```

(b) The following italicized statement is either true or false. *When $H_0 : \beta_1 = 0$ v.s. $H_A : \beta_1 < 0$, we reject the H_0 with $\alpha = 0.05$* Circle whether the statement is True or False and explain your choice.

```
Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) 506.4864    50.1074  10.108   <2e-16 ***
x            -0.9281     0.5140  -1.806   0.07406
```

(c) The following italicized statement is either true or false. *there is no relationship between the dioxin and response variable in the simple linear model.* Circle whether the statement is True or False and explain your choice.