8.3 Markov Random Fields

Pattern Recognition And Machine Learning

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Outline

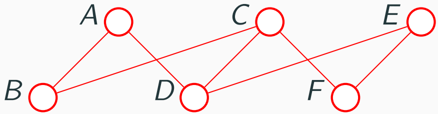
- Markov Random Field
- Conditional Independence Properties
- Factorization Properties
- Conditional Independence and Factorization

Markov Random Field

Markov Random Field

Markov Random Field (Markov Network, Undirected Graphical Model)

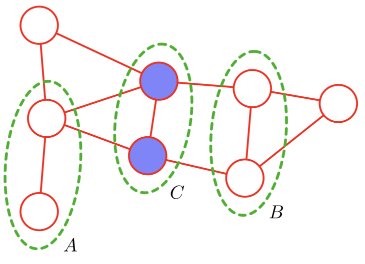
- Markov random field has a set of nodes each corresponding to a variable or group of variables as well as links between nodes.
- The links do not carry arrows. "No direction".



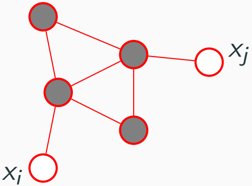
Conditional Independence Properties

Conditional Independence Properties

- A, B, C: A set of nodes.
- Consider $A \perp\!\!\!\perp B \mid C$.
    1. If all possible paths from A to B pass through one or more nodes in C, then all such paths are 'blocked'.
    2. Remove all nodes in C and related links. If two sets of nodes are disconnected, then it is conditional independence.



Example



- $x_i,\ x_j$: The variable.
- $\mathbf{x}_{\setminus\{i,\, j\}}$: The set of all variables with x_i and x_j removed.
- $x_i \perp\!\!\!\perp x_j \mid \mathbf{x}_{\setminus\{i,\, j\}}$.
  - ▷ There is no direct link between the two variables.
  - ▷ All other paths pass through variables that are observed.
    $\rightarrow$ Those paths are blocked.

$$\therefore p(x_i, x_j \mid \mathbf{x}_{\setminus\{i,\, j\}}) = p(x_i \mid \mathbf{x}_{\setminus\{i,\, j\}})p(x_j \mid \mathbf{x}_{\setminus\{i,\, j\}}).$$

Factorization Properties

Clique

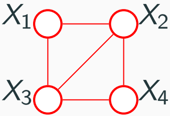
Clique

A subset of the nodes in a graph such that there exists a link between all pairs of nodes in the subset. Thus, the set of nodes in a clique is fully connected.

Maximal Clique

A maximal clique is a clique such that it is not possible to include any other nodes from the graph in the set without it ceasing to be a clique.

Example



- 1-node cliques: $\{X_1\}, \{X_2\}, \{X_3\}, \{X_4\}$.
- 2-node cliques: $\{X_1, X_2\}$, $\{X_1, X_3\}$, $\{X_2, X_3\}$,
  $\{X_2, X_4\}$, $\{X_3, X_4\}$.
- 3-node cliques (Maximal Clique): $\{X_1, X_2, X_3\}$, $\{X_2, X_3, X_4\}$.

Factorization Rule for Undirected Graph

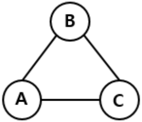
- The joint distribution is written as a product of *potential functions* $\psi_C(\mathbf{x}_C)$ over the maximal cliques of the graph

$$p(\mathbf{x}) = \frac{1}{Z} \prod_C \psi_C(\mathbf{x}_C).$$

  - ▷ C: A maximal clique.
  - ▷ $\mathbf{x}_C$: The set of variables in C.
  - ▷ $\psi_C(\mathbf{x}_C)$: A potential function, $\psi_C(\mathbf{x}_C) \geq 0$.
  - ▷ Z: A partition function, a normalization constant.

$$Z = \sum_{\mathbf{x}} \prod_C \psi_C(\mathbf{x}_C).$$

Example



$\psi_{A,B}(a, b) =$

A \ B	0	1
0	10	1
1	1	10

$\psi_{B,C}(b, c) =$

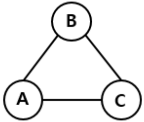
B \ C	0	1
0	10	1
1	1	10

$\psi_{A,C}(a, c) =$

A \ C	0	1
0	10	1
1	1	10

$$
\begin{aligned}
p(a, b, c) &= \frac{1}{Z} \prod_{C} \psi_C(\mathsf{x}_C). \\
&= \frac{1}{Z} \psi_1(a, b) \times \psi_2(b, c) \times \psi_2(a, c).
\end{aligned}
$$

Example



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$\psi_{A,C}(a, c)=$

A \ C	0	1
0	10	1
1	1	10

$$
\begin{aligned}
Z = & \sum_{a,\ b,\ c \in \{0,1\}^3} \psi_1(a,b) \times \psi_2(b,c) \times \psi_2(a,c). \\
& = \psi_1(0,0) \times \psi_2(0,0) \times \psi_2(0,0) \\
& \quad + \psi_1(1,0) \times \psi_2(0,0) \times \psi_2(1,0) \\
& \qquad \vdots \\
& \quad + \psi_1(1,1) \times \psi_2(1,1) \times \psi_2(1,1). \\
& = 2 \times 1000 + 6 \times 10 = 2060.
\end{aligned}
$$

Conditional Independence and Factorization

Conditional Independence and Factorization

UI

The set of such distributions that are consistent with the set of conditional independence statements that can be read from the graph using graph separation.

UF

The set of such distributions that can be expressed as a factorization of the form with respect to the maximal cliques.

- The Hammersley-Clifford theorem

$$\mathrm{UI} = \mathrm{UF}\ ,\ (\psi_C(\mathbf{x}_C) \geq 0).$$

Summary

- Markov Random Field: "No direction".
- Conditional Independence Properties.
- Factorization Properties.
  - ▷ Clique.
  - ▷ Factorization Rule.
- Conditional Independence and Factorization.


THANK YOU