8.3.4 Relation to Directed Graphs

Pattern Recognition And Machine Learning

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Outline

- Converting Directed Graph into Undirected Graph
  - Chain of Nodes
  - "Head-to-Head"
- Expression of Conditional Independence Properties

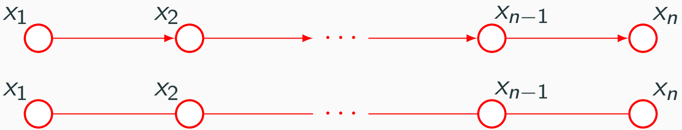
Converting Directed Graph into Undirected Graph

Converting Directed Graph into Undirected Graph

Convert any distribution specified by a factorization over a directed graph into one specified by a factorization over an undirected graph.

- The clique potentials of the undirected graph are given by the conditional distributions of the directed graph.
- The set of variables that appears in each of the conditional distributions is a member of at least one clique of the undirected graph.

Chain of Nodes



(a)

(b)

- In (b), the maximal cliques are the pairs of neighbouring nodes.

$$p(\mathbf{x}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)\cdots p(x_n \mid x_{n-1}) \tag{1}$$

$$p(\mathbf{x}) = \frac{1}{Z}\psi_{1,2}(x_1, x_2)\psi_{2,3}(x_2, x_3)\cdots\psi_{n-1,n}(x_{n-1}, x_n) \tag{2}$$

- If you want to convert from (1) to (2), the clique potentials of (b) are given by the conditional distributions of (a).

Chain of Nodes

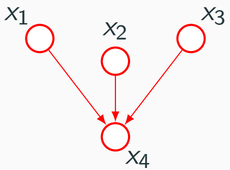
$$p(\mathbf{x}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)\cdots p(x_n \mid x_{n-1}) \tag{1}$$

$$p(\mathbf{x}) = \frac{1}{Z}\psi_{1,2}(x_1, x_2)\psi_{2,3}(x_2, x_3)\cdots\psi_{n-1,n}(x_{n-1}, x_n) \tag{2}$$

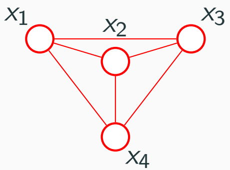
$$\Downarrow$$

$$\begin{aligned}
\psi_{1,2}(x_1, x_2) &= p(x_1)p(x_2 \mid x_1) \\
\psi_{2,3}(x_2, x_3) &= p(x_3 \mid x_2) \\
&\vdots \\
\psi_{n-1,n}(x_{n-1}, x_n) &= p(x_n \mid x_{n-1})
\end{aligned}$$

"Head-to-Head"



- $p(\mathbf{x}) = p(x_1)p(x_2)p(x_3)p(x_4 \mid x_1, x_2, x_3)$
- x_1, x_2, x_3 and x_4 must all belong to a single clique.

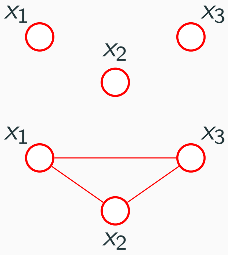


- Moralization: Marrying the parents.
- Moral graph: The resulting undirected graph.

Expression of Conditional Independence Properties

Definitions

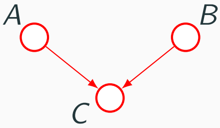
- $I(\cdot)$: A set of conditional independence properties.
- D-map (dependency map): $I(P) \subseteq I(G)$.
  e.g. A completely disconnected graph.
- I-map (independence map): $I(G) \subseteq I(P)$.
  e.g. A fully connected graph.
- Perfect map: $I(P) = I(G)$.



The two types of graph can express different conditional independence properties.

Example1: Directed Graph

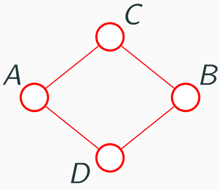
- $A \perp\!\!\!\perp B \mid \emptyset, \; A \not\perp\!\!\!\perp B \mid C$



  - ▷ This directed graph is a perfect map.
  - ▷ There is no corresponding undirected graph over three variables that is a perfect map.

Example2: Undirected Graph

- $A \perp\!\!\!\perp B \mid \emptyset,\ C \perp\!\!\!\perp D \mid \emptyset,\ A \not\!\perp\!\!\!\perp B \mid C \cup D,\ C \not\!\perp\!\!\!\perp D \mid A \cup B$



  - ▷ This undirected graph is a perfect map.
  - ▷ There is no directed graph over four variables that implies the same set of conditional independence properties.

Summary

- Converting Directed Graph into Undirected Graph:
  Moral graph.
  1. Add additional undirected links between all pairs of parents for each node in the graph.
  2. Drop the arrows on the original links.
- The two types of graph can express different conditional independence properties.

Thank You :D