Chapter 6: Bayesian Inference for Gaussian Distribution

강의 목표

- 정규분포를 중심으로 베이지안 추론의 이해
- Parameter Estimation (모수 추정)
  - Point Estimation (점추정)
  - Confidence Interval (구간추정)
- Prediction (예측)

Why Normal Models?

- Approximately normally distributed quantities appear often in nature.
- CLT tells us any variable that is basically a sum of independent components should be approximately normal.
- Note $\bar{X}$ and S^2 are independent when sampling from a normal population - so if beliefs about the mean are independent of beliefs about the variance, a normal model may be appropriate.

Why Normal Models?

- The normal model is analytically convenient.
- Inference about the population mean based on a normal model will be correct as $n \to \infty$ even if the data are truly non-normal.

Conjugate analysis with Normal Data (variance known)

- Simple situation: Assume data $X_1, ..., X_n$ are iid $N(\mu, \sigma^2)$, with μ unknown and σ^2 known.
- We will make inference about μ.
- The likelihood is

$$L(\mu \mid X) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(x_i - \mu)^2}$$

- A conjugate prior for μ is $\mu \sim N(\delta, \tau^2)$:

$$p(\mu) = \frac{1}{\sqrt{(2\pi\tau^2)}} e^{-\frac{1}{2\tau^2}(\mu - \delta)^2}.$$

Conjugate analysis with Normal Data (variance known)

- The posterior is:

$$
\begin{aligned}
\pi(\mu \mid x) &\propto L(\mu \mid x)p(\mu) \\
&\propto \prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} e^{-\frac{1}{2\tau^2}(\mu-\delta)^2} \\
&= \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{1}{\tau^2}(\mu-\delta)^2\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i^2 - 2x_i\mu + \mu^2)\right.\right. \\
&\qquad \left.\left. + \frac{1}{\tau^2}(\mu^2 - 2\mu\delta + \delta^2)\right)\right\}
\end{aligned}
$$

Conjugate analysis with Normal Data (variance known)

- Continued:

$$
\begin{aligned}
\pi(\mu \mid x) \quad &\propto \quad \exp\left\{-\frac{1}{2\sigma^2\tau^2}\left(\tau^2\sum x_i^2 - 2\tau^2\mu n\bar{x} + n\mu^2\tau^2\right.\right. \\
&\qquad\qquad \left.\left. +(\sigma^2\mu^2 - 2\sigma^2\mu\delta + \sigma^2\delta^2)\right)\right\} \\
&= \quad \exp\left\{-\frac{1}{2\sigma^2\tau^2}\left(\mu^2(\sigma^2 + n\tau^2) - 2\mu(\delta\sigma^2 + \tau^2 n\bar{x})\right.\right. \\
&\qquad\qquad \left.\left. +(\sigma^2\delta^2\tau^2\sum x_i^2)\right)\right\} \\
&\propto \quad \exp\left\{-\frac{1}{2}\mu^2\left(\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right) - 2\mu\left(\frac{\delta}{\tau^2} + \frac{n\bar{x}}{\sigma^2} + k\right)\right\}
\end{aligned}
$$

where k is a constant.

Conjugate analysis with Normal Data (variance known)

- Therefore, the posterior is

$$\begin{aligned}
\pi(\mu \mid x) &\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)\left(\mu^2 - 2\mu\left(\frac{\frac{\delta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\right)+k\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)\left(\mu - \frac{\frac{\delta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\right)^2\right\}
\end{aligned}$$

Conjugate analysis with Normal Data (variance known)

- The posterior expectation:

$$\frac{\frac{\delta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} = \frac{\sigma^2}{\sigma^2 + n\tau^2}\delta + \frac{n\tau^2}{\sigma^2 + n\tau^2}\bar{x}.$$

- The posterior variance:

$$\left(\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right)^{-1}$$

- The precision is the reciprocal of the variance.
- Prior precision: $\frac{1}{\tau^2}$.
- Data precision: $\frac{n}{\sigma^2}$.
- Posterior precision: $\frac{1}{\tau^2} + \frac{n}{\sigma^2}$.

Conjugate analysis with Normal Data (variance known)

- The posterior mean $E(\mu \mid x)$:

$$\frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\delta + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\bar{x}$$

- The mean is a weighted average of the **prior mean** and the **sample mean**.
- If the prior is highly precise, the weight is large on δ.
- If the data are highly precise (e.g., when n is large), the weight is large on $\bar{x}$.
- Clearly as $n \to \infty$, $E(\mu \mid x) \asymp \bar{x}$ and **var**$(\mu \mid x) \asymp \frac{\sigma^2}{n}$ if we choose a large prior variance τ^2.

Conjugate analysis with Normal Data (variance known)

- The posterior mean $E(\mu \mid x)$:

$$\frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\delta + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\bar{x}$$

- This implies that for τ^2 large and n large, Bayesian and Frequentist inference about μ will be nearly identical.

Example for Normal Data (variance known)

과거의 자료로부터 $\theta \sim N(370, 21^2)$ 사전분포를 가정하고, $\bar{X}\bar{\theta} \sim N(\theta, 8^2)$ 분포를 따르는 $\bar{X}$의 관측치는 $x = 421$이었다. 이 경우 θ의 사후분포는 평균과 분산으로 각각

$$
\begin{aligned}
\mu_\pi &= \frac{1/8^2 \times 421 + 1/21^2 \times 370}{1/8^2 + 1/21^2} = 413 \\
\sigma_\pi^2 &= \frac{1}{1/8^2 + 1/21^2} = 7^2.
\end{aligned}
$$

을 갖는 정규분포이다. 정규분포는 symmetric이므로 95% HPD interval은 $413 \pm 1.96 \times 7$이 된다.

Example for Normal Data (variance known)

과거의 자료로부터 $\theta \sim N(370, 50^2)$ 사전분포를 가정하고, $\bar{X}\bar{\theta} \sim N(\theta, 8^2)$ 분포를 따르는 $\bar{X}$의 관측치는 $x = 421$이었다. 이 경우 θ의 사후분포는 평균과 분산으로 각각

$$
\begin{aligned}
\mu_\pi &= \frac{1/8^2 \times 421 + 1/50^2 \times 370}{1/8^2 + 1/50^2} = 419 \\
\sigma_\pi^2 &= \frac{1}{1/8^2 + 1/50^2} = 8^2.
\end{aligned}
$$

을 갖는 정규분포이다. 이 경우 prior variance가 sample variance에 비해 매우 크므로, posterior mean이 sample mean에 더욱 가까워진다.

Prediction Distribution (variance known)

- Prediction distribution

$$
\begin{aligned}
& f(x_{n+1} \mid x_1, ..., x_n) \\
= & \int f(x_{n+1} \mid \mu, x_1, x_2, ..., x_n)\pi(\mu \mid x_1, x_2, ..., x_n)d\mu \\
= & \int f(x_{n+1} \mid \mu)\pi(\mu \mid x_1, x_2, ..., x_n)d\mu
\end{aligned}
$$

- Since we know $f(x_{n+1} \mid \theta) \sim$ Normal distribution, and its posterior is also normal, the prediction distribution is normal.

Prediction Distribution (variance known)

- Using the independent property,

$$
\begin{aligned}
X_{n+1} \mid x_1, ..., x_n &=_d X_{n+1} - \mu + \mu \mid x_1, ..., x_n \\
X_{n+1} - \mu \mid x_1, ..., x_n &\sim N(0, \sigma^2) \\
X_{n+1} \mid x_1, ..., x_n &\sim N(\mu_\pi, \sigma^2 + \sigma_\pi^2)
\end{aligned}
$$

Example for Normal Data (variance known)

우리나라 전체 가구의 월수입을 로그변환하면 자료의 분포가 대략적으로 대칭적인 정규분포 모양을 가진다. 따라서 가구 당 로그 월수입을 변수 X라 한다면 X의 분포를 정규분포로 가정해도 큰 무리가 없다. 우리의 관심은 어떤 특정 도시 A의 가구당 월평균 수입을 알아보는 것이다. 정규성 가정이 편리하므로 로그 월수입을 X라 하고, X의 평균 μ를 추정한 뒤 이를 지수 변환하여 원하는 가구당 월 평균 수입을 얻도록 하자.

Example for Normal Data (variance known)

θ에 대한 사전정보를 얻기 위하여 알아보니 해당 연도 전체 한국 국민의 가구당 월수입은 평균이 294만 원, 표준편차가 200만 원이었다. 이를 로그변환하면 로그 월수입은 평균이 $\log 294 = 5.68$ 만 원, 표준편차는 $\log 200 = 5.30$이다. 이로부터 $N(5.68, 5.30^2)$를 θ의 사전 분포로 사용한다.

Example for Normal Data (variance known)

실제로 도시 A에서 50개의 랜덤 표본을 뽑아 조사하니 로그 월수입의 평균은 5.5, 표준편차는 $s = 4.8$이었다. X의 실제 표준편차는 모르지만 표본의 수가 작지 않으므로 표 본 표준편차 $s = 4.8$을 실제 모집단의 표준편차 $\sigma = 4.8$로 사용하도록 하자. 이에 따라 $X_1, ..., X_{50}$의 평균 $\bar{X}$는 $N(\mu, 4.8^2/50)$ 분포를 따르며 관측치는 $\bar{x} = 5.5$가 된다. 이 들을 앞의 식에 대입하면, μ의 사후분포는 $N(\mu_\pi, \sigma_\pi^2)$으로

$$
\begin{aligned}
\mu_\pi &= \frac{50/4.8^2 \times 5.5 + 1/5.3^2 \times 5.68}{50/4.8^2 + 1/5.3^2} = 5.503 \\
\sigma_\pi^2 &= \frac{1}{50/4.8^2 + 1/5.3^2} = 0.4533.
\end{aligned}
$$

Example for Normal Data (variance known)

μ는 로그 월수입의 평균이므로 이를 지수변환하면 A도시의 가구당 월평균 수입을 추정할 수 있는데, 추정치는 $e^{5.503} = 245.42$만 원이고, 이 추정치의 표준오차는 $e^{\sqrt{5.503}} = 1.96$이다.

$$\begin{aligned} \mu_\pi &= \frac{50/4.8^2 \times 5.5 + 1/5.3^2 \times 5.68}{50/4.8^2 + 1/5.3^2} = 5.503 \\ \sigma_\pi^2 &= \frac{1}{50/4.8^2 + 1/5.3^2} = 0.4533. \end{aligned}$$

이 예에서는 사전분포 μ의 분산이 표본평균 $\bar{x}$의 분산에 비해 상당히 큰 값으로 사전분포의 영향이 미미한 경우이다

Example for Normal Data (variance known)

새로운 관측치 X_{new}의 예측 분포도 앞의 식을 통해 찾을 수 있다.

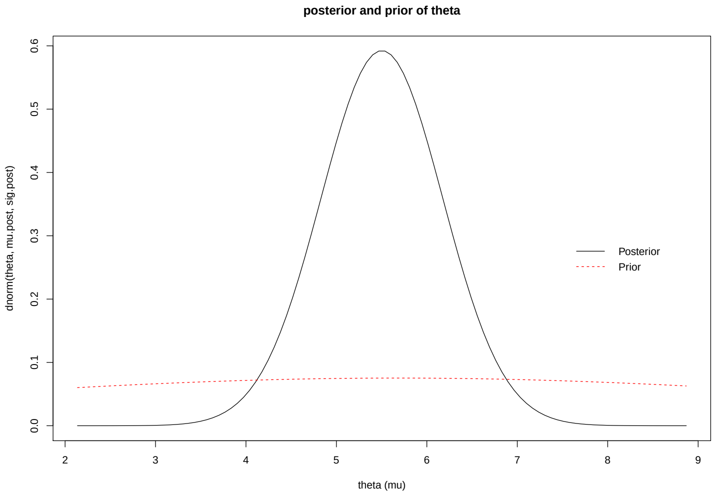
앞의 식에서 예측 분포는 $N(\mu_\pi, \sigma^2 + \sigma_\pi^2)$이므로

$$X_{new} \mid x_1, ..., x_n \sim N(5.503, 4.8^2 + 0.4533).$$

Posterior vs. Prior

```
mu0 = 5.68; sig0 = 5.30; n = 50; xbar = 5.5; s = 4.8
#posterior
c= n/s^2 ; c0=1/sig0^2; w = c/(c+c0)
mu.post = w * xbar + (1-w)* mu0; sig.post = sqrt( 1/(c+c0) )
theta=seq(mu.post-5*sig.post, mu.post+5*sig.post, length=100)
plot(theta, dnorm(theta,mu.post, sig.post), type= "l",
main= "posterior and prior of theta", xlab = "theta (mu)" )
lines(theta,dnorm(theta,mu0,sig0),lty=2, col = 2)
legend(7.5,0.3,legend= c( "Posterior", "Prior"),lty= c(1,2),
col= c(1,2), bty= "n")
```

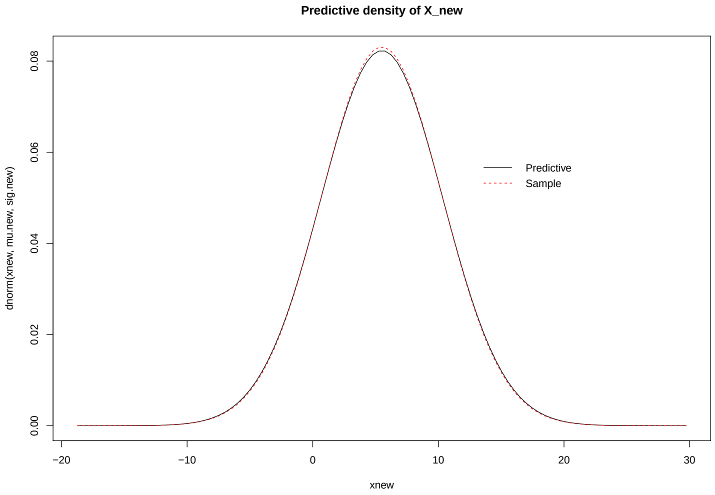
Posterior v.s. Prior



Predictive vs. Data

```
#predictive
mu.new = mu.post; sig.new = sqrt( s^2 + sig.post^2 )
xnew=seq(mu.new - 5* sig.new , mu.new + 5* sig.new, length=100)
plot(xnew, dnorm(xnew, mu.new, sig.new),type= "l",
main= "Predictive density of X_new")
lines(xnew,dnorm(xnew,xbar,s),lty=2, col = 2)
legend(12.5,0.06,legend= c( "Predictive", "Sample"),lty= c(1,2),
col= c(1,2), bty= "n")
```

Predictive vs. Data



Conjugate analysis with Normal Data (mean known)

- $X_1, ..., X_n \sim N(\mu, \sigma^2)$ with μ known and σ^2 unknown.
- We will make inference about σ^2.
- The likelihood of σ^2:

$$L(\sigma^2 \mid x) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{\frac{n}{2\sigma^2}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2\right]\right\}$$

- Let W denote the sufficient statistic: $\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$.
- The conjugate prior for σ^2 is the inverse gamma distribution.

Conjugate analysis with Normal Data (mean known)

- The conjugate prior for σ^2 is the inverse gamma distribution.
- If a r.v. $Y \sim$ gamma, then $Y^{-1} \sim$ inverse gamma (IG).
- The prior for σ^2:

$$p(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-(\alpha+1)} e^{-(\frac{\beta}{\sigma^2})} \quad \text{for} \quad \sigma^2 > 0$$

where $\alpha, \beta > 0$.

Conjugate analysis with Normal Data (mean known)

- The prior mean

$$\mathbb{E}(\sigma^2) = \frac{\beta}{\alpha - 1} \quad \text{provided that} \quad \alpha > 1.$$

- The prior variance

$$\textbf{var}(\sigma^2) = \frac{\beta^2}{(\alpha - 1)^2(\alpha - 2)} \quad \text{provided that} \quad \alpha > 2.$$

- The posterior for σ^2:

$$\begin{aligned} \pi(\sigma^2 \mid x) &\propto L(\sigma^2 \mid x) p(\sigma^2) \\ &\propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n}{2\sigma^2} w} (\sigma^2)^{-(\alpha+1)} e^{-(\beta/\sigma^2)} \\ &= (\sigma^2)^{-(\alpha + \frac{n}{2} + 1)} e^{-\frac{\beta + \frac{n}{2} w}{\sigma^2}}. \end{aligned}$$

where $\alpha, \beta > 0$.

Conjugate analysis with Normal Data (mean known)

- Conjugate: The posterior is an $IG(\alpha + \frac{n}{2}, \beta + \frac{n}{2} w)$ distribution, where $w = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$.

A Conjugate analysis with Normal Data (mean known)

- How to choose the prior parameters α and β?
- Note that common estimators are

$$\alpha = \frac{\mathbb{E}(\sigma^2)^2}{var(\sigma^2)} + 2$$

and

$$\beta = \mathbb{E}(\sigma^2)\left\{\frac{\mathbb{E}(\sigma^2)^2}{var(\sigma^2)} + 1\right\}.$$

- So we could make guesses about $\mathbb{E}(\sigma^2)$ and $var(\sigma^2)$ and use these to determine α and β.

Model for Normal Data (both unknown)

- $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ with both μ, σ^2 **unknown**.
- The conjugate prior for the mean explicitly depends on the variance:

$$
\begin{aligned}
p(\sigma^2) &\propto (\sigma^2)^{-(\alpha+1)} e^{\beta/\sigma^2} \\
p(\mu \mid \sigma^2) &\propto (\sigma^2)^{-\frac{1}{2}} e^{\frac{1}{2\sigma^2/s_0}(\mu-\delta)^2}
\end{aligned}
$$

- The prior parameter s_0 measures the analyst's confidence in the prior specification.
- When s_0 is large, we strongly believe in our prior.

A Model for Normal Data (both unknown)

- The joint posterior for (μ, σ^2) is:

$$
\begin{aligned}
& \pi(\mu, \sigma^2 \mid x) \propto L(\mu, \sigma^2 \mid x) p(\sigma^2) p(\mu \mid \sigma^2) \\
\propto \quad & (\sigma^2)^{-\alpha - \frac{n}{2} - \frac{3}{2}} \exp\left\{ -\frac{\beta}{\sigma^2} - \frac{1}{\sigma^2} - \sum_{i=1}^{n} (x_i - \mu)^2 - \frac{1}{2\sigma^2/s_0} (\mu - \delta)^2 \right\} \\
= \quad & (\sigma^2)^{-\alpha - \frac{n}{2} - \frac{3}{2}} \exp\left\{ -\frac{\beta}{\sigma^2} - \frac{1}{\sigma^2} - \sum_{i=1}^{n} (x_i^2 - 2x_i\mu + \mu^2) - \frac{1}{2\sigma^2/s_0} (\mu^2 - 2\mu\delta + \delta^2) \right\} \\
= \quad & \left\{ (\sigma^2)^{-\alpha - \frac{n}{2} - 1} \exp\left\{ -\frac{\beta}{\sigma^2} - \frac{1}{2\sigma^2} \left(\sum_{i=1}^{n} (x_i^2 - n\bar{x}) \right) \right\} \right\} \\
& \quad \times \left\{ (\sigma^{-1} \exp\left\{ -\frac{1}{2\sigma^2} ((n + s_0)\mu^2 - 2(n\bar{x} + \delta s_0)\mu + (n\bar{x} + s_0\delta^2)) \right\} \right\}.
\end{aligned}
$$

Note the second part is simply **a normal kernel** for μ.

A Model for Normal Data (both unknown)

- To get the posterior for σ^2, we integrate out μ:

$$\begin{aligned}\pi(\sigma^2 \mid x) &= \int_{-\infty}^{\infty} \pi(\mu, \sigma^2 \mid x) d\mu \\ &\propto (\sigma^2)^{-\alpha-\frac{n}{2}-1} \exp\left\{-\frac{1}{\sigma^2}(\beta + \frac{1}{2}(\sum x_i^2 - n\bar{x}) + \frac{1}{2}\frac{s_0 n}{s_0+n}(\bar{x}-\delta)^2\right\}\end{aligned}$$

since the second part (which depends on μ) just integrates to a normalizing constant.

- Hence, we see the posterior for σ^2 is inverse gamma:

$$\sigma^2 \mid x \sim IG\left(\alpha + \frac{n}{2}, \beta + \frac{1}{2}\left(\sum (x_i - \bar{x})^2 + \frac{s_0 n}{s_0 + n}(\bar{x}-\delta)^2\right)\right).$$

A Model for Normal Data (both unknown)

- Find the the posterior for μ:

$$\pi(\mu \mid \sigma^2, x) = \frac{\pi(\mu, \sigma^2 \mid x)}{\pi(\sigma^2 \mid x)}.$$

- After some cancellations,

$$\begin{aligned} \pi(\mu \mid \sigma^2, x) &\propto \sigma^{-2} \exp\left\{ -\frac{1}{2\sigma^2}((n+s_0)\mu^2 - 2(n\bar{x} + \delta s_0)\mu + (n\bar{x} + s_0\delta^2)) \right\} \\ &= \sigma^{-2} \exp\left\{ -\frac{1}{2\sigma^2/(n+s_0)} \left(\mu^2 - 2\frac{n\bar{x} + \delta s_0}{n+s_0}\mu + \frac{n\bar{x}^2 + s_0\delta^2}{n+s_0} \right) \right\}. \end{aligned}$$

- Hence, $\pi(\mu \mid \sigma^2, x)$ is **Normal**.

$$\mu \mid \sigma^2, x \sim N\left(\frac{n\bar{x} + \delta s_0}{n+s_0}, \frac{\sigma^2}{n+s_0} \right).$$

A Model for Normal Data (both unknown)

- Note as $s_0 \to 0$, (the level of belief of prior information is very weak)

$$\mu \mid \sigma^2,\ x \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right).$$

- Note also the conditional posterior mean is

$$\frac{n}{n+s_0}\bar{x} + \frac{s_0}{n+s_0}\delta.$$

- The relative sizes of n and s_0 determine the weighting of the sample mean $\bar{x}$ and the prior mean δ.

A Model for Normal Data (both unknown)

- The marginal posterior for μ is:

$$\begin{aligned}\pi(\mu|x) &= \int_0^\infty \pi(\mu, \sigma^2 \mid x) d\sigma^2 \\ &= \int_0^\infty (\sigma^2)^{-\alpha-\frac{n}{2}-\frac{3}{2}} \exp\left\{\frac{2\beta + (s_0+n)(\mu-\delta)^2}{2\sigma^2}\right\} d\sigma^2\end{aligned}$$

- Letting $A = 2\beta + (s_0+n)(\mu-\delta)^2$, $z = \frac{A}{2\sigma^2}$.
- Then $\sigma^2 = \frac{A}{2z}$ and $d\sigma^2 = -\frac{A}{2z^2} dz$

A Model for Normal Data (both unknown)

- The marginal posterior for μ is:

$$
\begin{aligned}
\pi(\mu \mid x) &\propto \int_0^\infty \left(\frac{A}{2z}\right)^{-\alpha-\frac{n}{2}-\frac{3}{2}} \frac{A}{2z^2} e^{-z} dz \\
&= \int_0^\infty \left(\frac{A}{2z}\right)^{-\alpha-\frac{n}{2}-\frac{1}{2}} \frac{1}{z} e^{-z} dz \\
&\propto A^{-\alpha-\frac{n}{2}-\frac{1}{2}} \int_0^\infty z^{-\alpha-\frac{n}{2}-\frac{1}{2}-1} e^{-z} dz
\end{aligned}
$$

- This integrand is the kernel of a **gamma density** and thus the integral is a constant.

A Model for Normal Data (both unknown)

- The marginal posterior for μ is:

$$
\begin{aligned}
\pi(\mu \mid x) &\propto A^{-\alpha-\frac{n}{2}-\frac{1}{2}} \\
&= \left(2\beta + (s_0+n)(\mu-\delta)^2\right)^{\frac{-2\alpha+n+1}{2}} \\
&\propto \left(1 + \frac{(s_0+n)(\mu-\delta)^2}{2\beta}\right)^{\frac{-2\alpha+n+1}{2}}
\end{aligned}
$$

- which is a (scaled) noncentral $t-$kernel having noncentrality parameter δ and degrees of freedom $n + 2\alpha$.

Example

서울시 거주 학생들의 2009년 1인당 월평균 사교육비 지출액수를 추정하고자 한다. 2009년 서울시 거주 학생들의 1인당 월별 사교육비 지출액이 정규분포를 따르고 평균이 μ, 표준편차가 σ 라고하자. 2008년 100명을 대상으로 한 조사에 따르면 학생 1인당 사교육비 지출은 월평균 22.5만원이고, 표준편차는 20.4 만원이었다. 2008년의 자료 는 100명에 대한 조사지만 과거의 자료를 현재 자료와 동등하게 취급하는 것은 문제가 있으므로, 과거의 자료를 약 5개의 현재자료와 동등한 정보를 준다고 간주하자.

Example

μ와 σ^2의 사전분포를 $\sigma^2 \sim IG(\alpha, \beta)$, $\mu \mid \sigma^2 \sim N(\delta, \sigma^2/s_0)$ 로 가정하고, 과거 자료를이용하여 사전분포의 모수를결정한다. μ에 대한 사전 분포에서 $\delta = 22.5$, $s_0 = 5$로 놓는 것이 적당할 것으로 보인다. α와 β의 경우 일반화된 estimator는 없으나 posteior 분포를 통해 $\frac{\text{관측치수}}{2} = \frac{s_0}{2} = \frac{5}{2}$라고 할 수 있다. β의 경우에도 $\frac{3}{2} \times 20.4^2$라고 하자

Example

종합하면

$$\sigma^2 \sim IG(\frac{5}{2}, \frac{3}{2}20.4^2),$$

$$\mu \mid \sigma^2 \sim N(22.5, \sigma^2/5)$$

로 가정할 수 있다.

Example

2009년 서울시 거주 학부모 20명을 대상으로 자녀의 사교육비 현황을 조사한 결과 학생 1 인당 월평균 27.4만 원을 지출하였다고 답하였고 표준편차 $s = 22.5$만 원이었다. 이로 부터 사후분포를 구하면,

$$
\begin{aligned}
\mu_\pi &= \frac{s_0\delta + n\bar{x}}{s_0 + n} = \frac{5 \times 22.5 + 20 \times 27.4}{5 + 20} = 26.42. \\
\alpha_\pi &= \frac{n}{2} + \alpha = \frac{20}{2} + \frac{5}{2} \\
\beta_\pi &= \frac{1}{2}\{9618.75 + \frac{5 \times 20}{5 + 20}(27.4 - 22.5)^2 + \frac{3}{2} \times 20.4^2 = 74.03^2
\end{aligned}
$$

Example

종합하면 다음과 같은 사후 분포를 얻을 수 있다.

$$\sigma^2 \mid x_1, ..., x_n \sim IG(\frac{5}{2}, \frac{3}{2} \times 20.4^2),$$

$$\mu \mid \sigma^2, \mid x_1, ..., x_n \sim N(22.5, \sigma^2/5)$$

즉,

$$\mu, \sigma^2 \mid x_1, ..., x_n \sim N(22.5, \sigma^2/5) \times IG(\frac{5}{2}, \frac{3}{2} \times 20.4^2),$$

R-code for Multivariate Normal Distribution

```
## joint posterior distribution

install.packages("MCMCpack")

library(MCMCpack)

#prior

mu0=22.5; s0=5; sig0=20.4

a=s0/2; b=(a-1)*sig0^2

#data

xbar=27.4; n=20; s=22.5

#parameters

mu.theta.post=(s0* mu0+n*xbar)/(s0+n)

a.post=n/2+a

b.post=1/2*((n-1)*s^2+s0*n/(s0+n)*(xbar-mu0)^2)+b
```

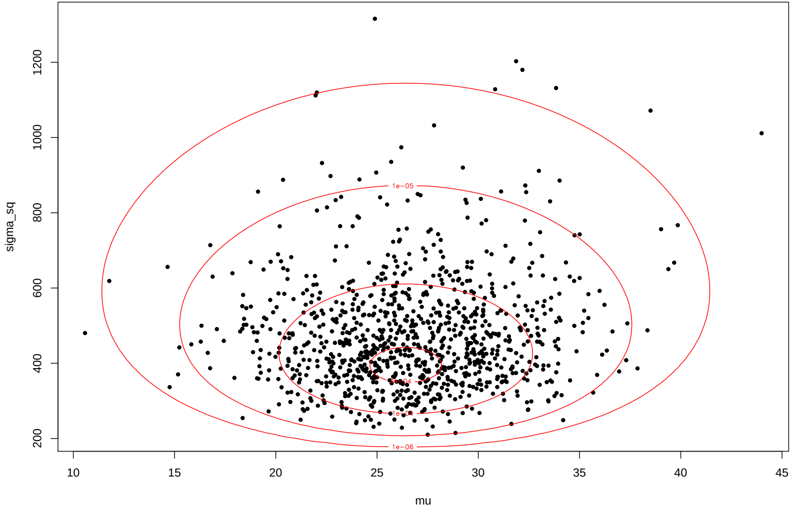
R-code for Multivariate Normal Distribution

```
f=function(theta,sigsq,a.post,b.post,mu.theta.post,n,s0){
    f= dinvgamma(sigsq,a.post,scale=b.post)*
    dnorm(theta,mu.theta.post, sqrt ( sigsq/ ( n+s0)) )
}


post.joint=
outer(theta.grid,sigsq.grid,f,a.post,b.post,mu.theta.post,n,s0)
#Monte Carlo using 1000 samples
Nsim=1000
sigsq.sim=1/rgamma(Nsim,a.post,b.post)
theta.sim=rnorm(Nsim,mu.theta.post,sqrt(sigsq.sim/(s0+n)))
```

R-code for Multivariate Normal Distribution

```
#Contour Plot

par(mfrow=c(1,1))

plot(theta.sim, sigsq.sim, xlab="mu",ylab="sigma_sq", pch =20)

contour(theta.grid,sigsq.grid,post.joint,

level= c(1.e-6,1.e-5,1.e-4,3.e-4),add= T, col = "red")
```

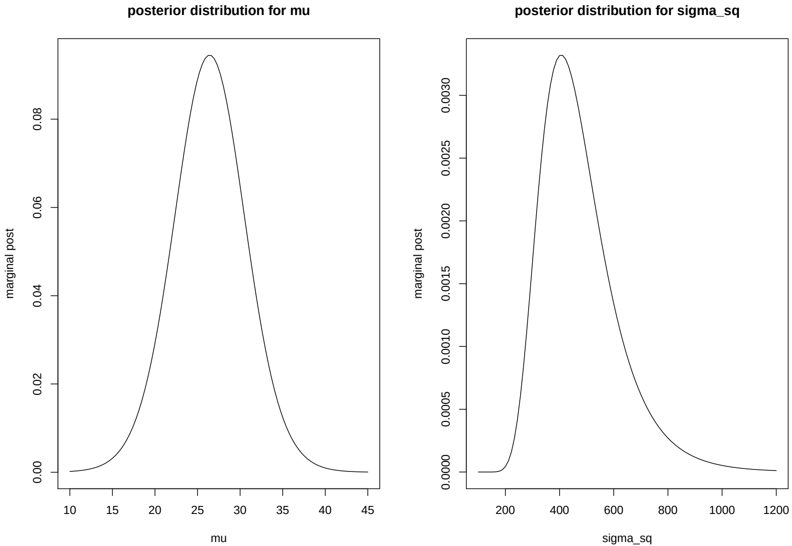
R-code for Multivariate Normal Distribution



R-code for Multivariate Normal Distribution

```
#marginal posterior.
post.sigsq= dinvgamma(sigsq.grid,a.post, scale=b.post)
post.theta=theta.grid*0
for(j in 1:length(theta.grid) ){
post.theta[j]= mean( dnorm(theta.grid[j],mu.theta.post,
sqrt(sigsq.sim/(n+s0)) ))
}

par(mfrow=c(1,2))
plot(theta.grid, post.theta, type="l",xlab="mu",ylab="marginal post")
title("posterior distribution for mu")
plot(sigsq.grid, post.sigsq, type="l",xlab="sigma_sq",ylab="marginal post")
title("posterior distribution for sigma_sq")
```

R-code for Multivariate Normal Distribution



Noninformative (Vague) Priors with Normal Data

- The conjugate priors we have discussed include a certain amount of **subjective** prior information.
- Another approach is to use a noninformative or vague prior.
- Consider $(X_1, ..., X_n) \sim_{iid} N(\mu, \sigma^2)$, with μ, σ^2 unknown.
- We can use the vague priors for μ and σ^2

$$
\begin{aligned}
p(\mu) &= 1, \text{ if } -1 < \mu < 1 \text{ independent priors here} \\
p(\sigma) &= 1/\sigma, \quad 0 < \sigma < 1
\end{aligned}
$$

- Clearly these priors are improper since they integrate to ∞ and thus are **not** true densities.
- This is OK, as long as the resulting posteriors are proper densities.

Vague Priors with Normal Data

- The joint posterior for μ and σ^2 is:

$$\pi(\mu, \sigma^2 \mid x) \propto L(\mu, \sigma^2 \mid x)p(\mu)p(\sigma^2).$$

- Note that

$$\begin{aligned} L(\mu, \sigma^2 \mid x) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\} \\ = \quad & (\sqrt{2\pi}\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum[(x_i-\bar{x})-(\mu-\bar{x})]^2\} \\ = \quad & (\sqrt{2\pi}\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\{\sum(x_i-\bar{x})^2 - 2\sum(x_i\mu - x_i\bar{x} - \bar{x}\mu + \bar{x}^2) + n(\bar{x}-\mu)^2\} \\ \propto \quad & (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{x}-\mu)^2]\right\} \end{aligned}$$

- Hence the posterior is

$$\pi(\mu, \sigma^2 \mid x) \propto L(\mu, \sigma^2 \mid x)(1)\frac{1}{\sigma^2} \propto (\sigma^2)^{-\frac{n}{2}-1} \exp\left\{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\mu-\bar{x})^2]\right\}$$

Vague Priors with Normal Data

- To get the marginal posterior for μ, integrate out σ^2 using the formula:
$$\int_0^\infty x^{-(b+1)}\exp\left\{-\frac{a}{x^2}\right\}dx = \frac{1}{2}a^{-\frac{b}{2}}\Gamma\left(\frac{b}{2}\right).$$
- For ease of calcuation, let us use the
$$\pi(\mu,\sigma \mid x) \propto L(\mu,\sigma \mid x)(1)\frac{1}{\sigma} \propto \sigma^{-n-1}\exp\left\{-\frac{1}{2\sigma^2}[(n-1)s^2+n(\mu-\bar{x})^2]\right\}$$
- Then
$$\begin{aligned}
\pi(\mu \mid x) &= \int_0^\infty \pi(\mu,\sigma \mid x)d\sigma \\
&\propto \frac{1}{2}\left\{\frac{1}{2}\left[(n-1)s^2+n(\mu-\bar{x})^2\right]\right\}^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) \\
&= \frac{1}{2}[(n-1)s^2]^{-\frac{n}{2}}\left[1+\frac{n(\mu-\bar{x})^2}{(n-1)s^2}\right]^{-\frac{n}{2}}\left(\frac{1}{2}\right)^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) \\
&\propto \left\{\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{n/s^2}{(n-1)\pi}\right]^{-\frac{1}{2}}\right\}\left[1+\frac{1}{n-1}\left(\frac{\mu-\bar{x}}{s/\sqrt{n}}\right)^2\right]^{-\frac{n}{2}}
\end{aligned}$$

Vague Priors with Normal Data

- Making the transformation $t = \frac{\mu - \bar{x}}{s/\sqrt{n}}$ with Jacobian $J = \frac{s}{\sqrt{n}}$:

$$\pi(t \mid x) = \frac{\Gamma\left(\frac{n-1+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{[(n-1)\pi]^{1/2}\left[1 + \frac{t^2}{n-1}\right]^{\frac{n-1+1}{2}}}$$

- This is clearly a t-distribution with $n - 1$ degrees of freedom.
- Since the posterior distribution for μ is t-distribution, bayes inference and classical inference are the same.

Vague Priors with Normal Data

- To get the marginal distribution of σ^2,

$$\begin{aligned}
\pi(\sigma^2 \mid x) &= \int_{-\infty}^{\infty} \pi(\mu, \sigma^2 \mid x) d\mu \\
&\propto (\sigma^2)^{\frac{-(n+1)}{2}} \exp\{-\frac{1}{2\sigma^2}(n-1)s^2\} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}\sigma^2 n(\mu - \bar{x})^2\} d\mu \\
&= (\sigma^2)^{\frac{-(n+1)}{2}} \exp\{-\frac{1}{2\sigma^2}(n-1)s^2\} \left[\sqrt{2\pi \frac{\sigma^2}{n}}\right].
\end{aligned}$$

- Hence

$$\sigma^2 \mid x \sim \mathrm{IG}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right).$$

Vague Priors with Normal Data

- Both of the posteriors (for μ and for σ^2) are proper.
- Compared to the posteriors in the conjugate analyses, they are more diffuse (spread).
- This is because we had vague prior information.
- For a large sample size, there is little difference between the conjugate analysis and the "noninformative" analysis.

Bayesian Model for Multivariate Data

- Suppose each individual has q variables observed on it, so that $(X1, ..., X_n)$ are q-dimensional random vectors.
- Assume the random vectors are iid multivariate normal, with mean vector μ and variance-covariance matrix Σ.
- Then a set of conjugate priors for μ and Σ are:

$$\mu \mid \Sigma \sim N_q\left(\delta, \frac{1}{n_0}\Sigma\right), \quad \Sigma^{-1} \sim \mathrm{Wishart}$$

- The Wishart distribution is a multivariate generalization of the gamma.
- n_0 is a tuning parameter that reflects confidence in the prior.

Bayesian Model for Multivariate Data

- If $\frac{n_0}{n}$ is a larger, the analyst has more confidence in the prior.
- The posterior distributions are:

$$\mu \mid \Sigma, x \sim N_q\left(\frac{n0\delta + n\bar{x}}{n0 + n}, \frac{1}{n0 + n}\Sigma\right), \quad \Sigma^{-1} \mid x \sim \text{anotherWishart}$$