Politecnico di Milano

Financial Engineering

Group 9
Assignment 6

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1 Case study: certificate pricing

We have to determine the upfront X of the Certificate issue neglecting counterparty risk since the two counterparties have signed an ISDA with CSA. The Libor floating payments can be converted into two fixed payments in the following way:

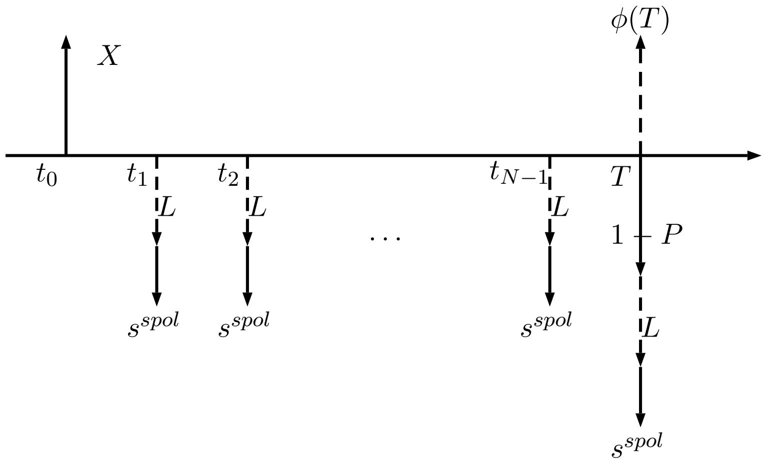

Figure 1: Exotic Swap

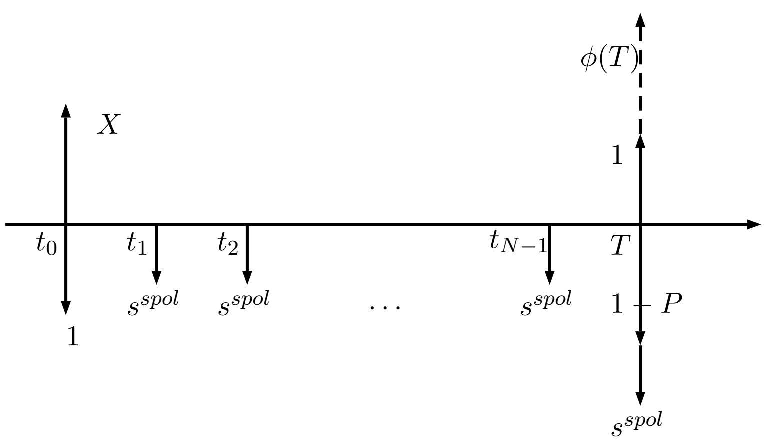

Figure 2: Equivalent Swap

In order to obtain the upfront X% we have to compute the NPV=0. So we have that X% is equal to:

$$X\% = 1 + S^{spol} BPV_{float} - B(\mathbb{E}[\phi(T)] + P)$$

where BPV is the basis point value for the floating leg, P is the protection and payoff is computed as:

$$\phi(T) = \alpha(S(T) - P)^{+}$$

and $S(t)$ is the equally weighted basket:

$$S(t) = 0.5 \left(\frac{S_t^1}{S_0^1} + \frac{S_t^2}{S_0^2} \right)$$

where S^i_t are GBM of dynamics:

$$S^i_t = S^i_0 e^{(r-d_i-\frac{1}{2}\sigma_i^2)t+\sigma_i Z^i\sqrt{t}}$$

Z_1 and Z_2 are standard normal random variable with correlation ρ, this can be done by using two uncorrelated normal standard random variable Z_1, Z_3 and defining Z_2 as $Z_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_3$.
To compute the fair value of the payoff we used a MC simulation.
At the end we obtain that the upfront is $X = 5.36\%$.

2 Digital option

We know that if we assume the Black dynamics for the underlying of the option then the price for a digital call is the following (assuming payment of one dollar if the option expires ITM):

$$CD_{Blk} = B(t_0,T)\mathbb{E}[\mathbb{I}_{F_T>K}] = B(t_0,T)\mathbb{P}[F_T>K] = B(t_0,T)N\left[\frac{\log\left(\frac{F_0}{K}\right)-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right]$$

For any vanilla product, given a vol surface, the corresponding price for vanilla is obtained by inverting the implied vols into prices. The pricing done with this method is exact for vanilla products, since the vols are inverted from prices using the same formula, in this sense we can use the Black formula in a presence of a non constant vol surface.
We can then use a replication argument to price digital option, buying a very narrow call spread around the strike of the digital.

$$\mathbb{I}_{F_T>K} = \lim_{\epsilon\to 0}\frac{(F_T>K+\epsilon)^+ - (F_T>K-\epsilon)^+}{2\epsilon}$$

We can then swap the limit and the expectation because the random variable is bounded by 1. Hence we can price a digital as:

$$CD_{Spread} = -\frac{dC_{Blk}(K)}{dK} = -\frac{\partial C_{Blk}}{\partial K}\frac{\partial\sigma}{\partial K}\frac{\partial C_{Blk}}{\partial\sigma}$$

We can then compute $\frac{\partial C_{Blk}}{\partial K}$ as CD_{Blk}, $\frac{\partial\sigma}{\partial K}$ as a numeric derivative of the vol surface and $\frac{\partial C_{Blk}}{\partial\sigma}$ as the vega of the option. As numerical derivative we used the centred scheme meaning $\frac{\partial\sigma}{\partial K} \approx \frac{\sigma(K_2)-\sigma(K_1)}{K_2-K_1}$ where K_2 is the lowest pillar of the vol surface greater then the K of the digital and K_1 is the greatest pillar smaller then the digital strike K, we will refer to this method as Numerical Call Spread.
To benchmark the obtained prices we also computed the prices obtained via a MC assuming a Black Dynamics with volatility $\sigma(K)$, and by buying the market call spread K_1, K_2 described above. The prices obtained are:

Blk	$BlkMC$	Num CSpread	Mkt CSpread
217K	217K	259K	260K

Table 1: Notional 10Mio

From this we can see that in a MC simulation for a fixed volatility we do not take into account the shape of the vol surface, to do a MC consistent with the vol surface we would need a local vol model. Moreover that the replicable call spread price accurately the digital option (since the implied probability of the underlying to expire inside the spread K_1, K_2 is negligible) and that the difference in pricing consistently with the vol surface and directly with the Black formula is not negligible at all.

3 Exercise: Complex integration

We have to compute the following integral:

$$\int_{-\infty}^{\infty} e^{-iz\xi}\frac{1}{\xi^2+\frac{1}{4}}$$

Which can be done in a number of ways:

3.1 Fast Fourier Transform

Our integral is of the form:

$$\hat{f}(z) = \int_{-\infty}^{\infty} e^{-iz\xi} f(\xi) d\xi$$

This integral can be approximated as

$$\hat{f}(z) \approx dx \sum_{j=1}^{N} f(x_j) e^{-ix_j z_k} = e^{-ix_1 z_k} FFT(k) dx$$

where FFT is the Fast Fourier Transformation defined as follows:

$$FFT(K) = \sum_{j=1}^{N} f_j w^{(j-1)(k-1)} \quad \text{where} \quad w = e^{-\frac{2\pi i}{N}}$$

The values in the table below (tab.2) are obtained choosing M=15 and $x_1 = -1000$, we have noticed that M has to be at least 12 in order to obtain a stable result, and at the same time x_1 has to be less than -60.
These results are shown in the figure above (fig.3).

3.2 Residual technique

The residual theorem is a strong tool in order to compute integrals in the complex plane because the residual are in general easier to compute, the theorem roughly states that:

$$\oint_{\gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{n} I_{z_k}(\gamma)\, \mathrm{Res}_{z_k}(f)$$

where γ is a closed circuit in $\mathbb{C}$, we can build the circuit γ such that it includes a pole. For example we can consider for $z = 1$ the upper semi circumference of radius R centred in the origin of the

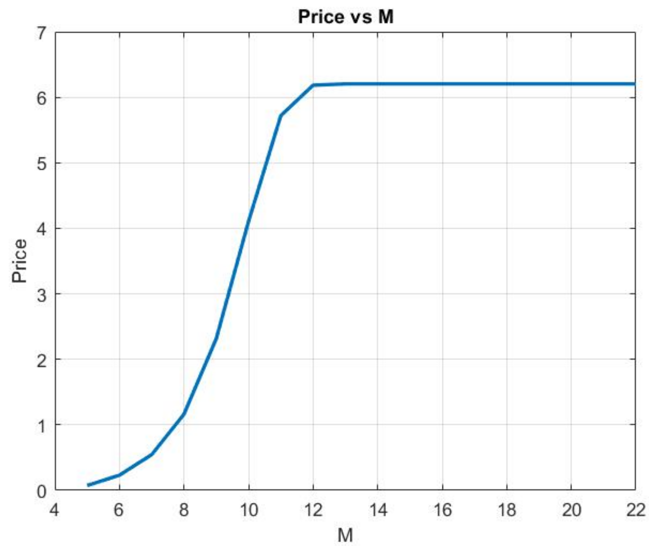

Figure 3: Price vs M

complex plan,e and using the Jordan lemma shows that the integral on the circumference tends to zero as R goes to infinity way so we can prove that the line integral over the semicircle is equal to the integral over the real axis and so the residual theorem lead us to:

$$\int_{-\infty}^{\infty} e^{-iz\xi} \frac{1}{\xi^2 + \frac{1}{4}} d\xi = \oint_\gamma f(z) = Res\left(f, \frac{i}{2}\right) = 2\pi e^{-|z|/2}$$

It's worthwhile noticing that the absolute value of z depends on the fact that we need two different versions of Jordan's Lemma, for positive and negative values of z, which results in using two different curves: one with an upper semicircle and the other with a lower semicircle.

	FFT	$Quad$	Res
$z = -2.5223$	6.2044	6.2044	6.2044
$z = 0$	6.2807	6.2812	6.2832
$z = 1$	6.2518	6.2518	6.2518

Table 2: Values of the integral with different z (in percentage)