Politecnico di Milano

Financial Engineering

Group 9

Assignment 4

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1 MBS pricing

The Vasicek model assumes for an homogeneous portfolio a model for the value of each obbligor of

$$v_i = y\sqrt{\rho} + \epsilon_i\sqrt{1-\rho}, \quad y, \epsilon_i \overset{iid}{\sim} \mathcal{N}_{(0,1)}$$

where ρ is the imposed correlation among the reference portfolio constituents. Moreover the model assumes a single default probability p and recovery π among all constituents.
The model assumes a credit event whenever the value of the firm v_i drops below a value K equal among obbligors. The quantity y will be called "market" since models the market environment equal to all firms, while ϵ_i models the individual value to firms. The conditional probability, on the market y, of a single default under the Vasicek model is

$$p(y) = N\left(\frac{N^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$$

and so we can easily obtain the conditional probability of m defaults:

$$\mathbb{P}(m \mid y) = \binom{I}{m} p(y)^m (1-p(y))^{I-m} \implies \mathbb{P}(m) \int_{-\infty}^{\infty} \mathbb{P}(m \mid y)\phi(y)dy$$

This probability leads to the expected loss of the portfolio which is:

$$\epsilon_{HP} = \mathbb{E}\left[L_t\right] = \aleph_t \sum_{m=0}^{I} L\left(\frac{m}{I}\right)\mathbb{P}(m)$$

Where $L(z)$ is the loss of the tranche given $m = zI$ defaults. Due to the heavy computations involved in the calculations of the binomial we can take various approximations making feasible the computation of this quantity for sufficiently large I. By the CLT we can write that:

$$\epsilon_{LHP} = \aleph_t \int_0^1 L(z)\phi_L(dz)$$

where

$$\phi_L(z) = \frac{d[N(p^{-1}(z))]}{dz}$$

Another approximation is to approximate the binomial by the Stirling asymptotic expression of the factorial leading to the so called KL approach:

$$\epsilon_{KL} = \int_{-\infty}^{\infty}\int_0^1 L(z)C(z,y)e^{-IK(z;p(y))}dz\phi(dy)$$

where $K(z,p) = z\ln\left(\frac{z}{p}\right) + (1-z)\ln\left(\frac{1-z}{1-p}\right)$ and

$$C(z,y) = C^{(1)}(z) = \sqrt{\frac{2\pi}{(1-z)z}}$$

or

$$C(z,y) = C^{(1)}(z)/D(y), \quad D(y) = \int_0^1 C^{(1)}(z) e^{-IK(z;p(y))} dz$$

depending if we want to use the standard or exact normalization constant.
Intuitively the function $K(z,p)$ weights less when z is far from p and more when are close together. Then the price of the tranche would be $1-\epsilon$ since no coupons are paid and discount factors are deemed to be one.

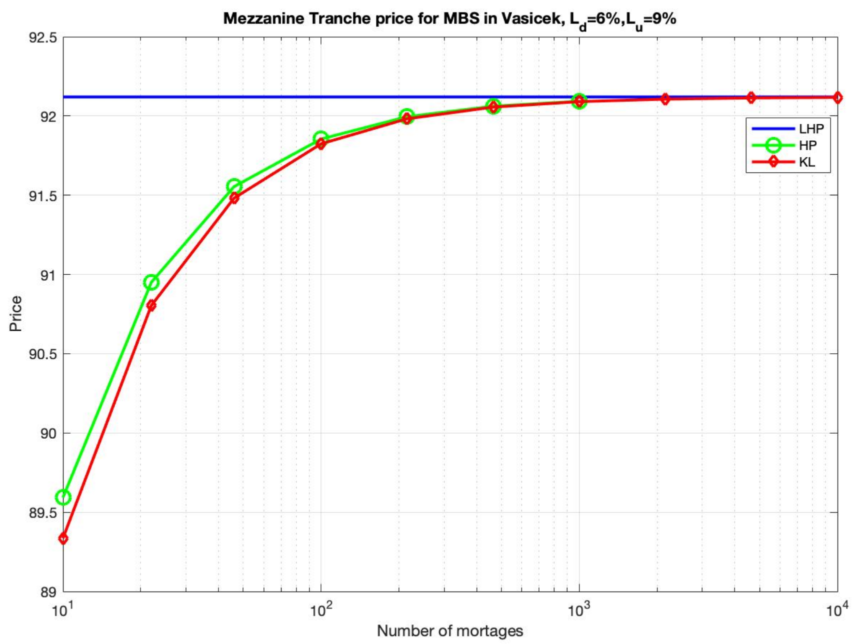

Figure 1: Mezzanine Tranche in Vasiceck model with $L_d = 6\%, L_u = 9\%, p = 6.5\%, \rho = 40\%$ and $\pi = 65\%$.

In the table there are reported the results obtained for I=60:

LHP	HP	KL
92.12	91.67	91.62

Table 1: KL is computed by correcting the normalization.

The error between KL and LHP is about 5bps with $I = 500$. This is somehow deemed a low level of error, but not completely negligible. While for the normalization error in the KL approach is clearly relevant for small number of mortgages as shown in figure 2.

1.1 Equity Tranche

The approach of approximating the binomial with the sterling formula is clearly inadequate as shown in figure 4

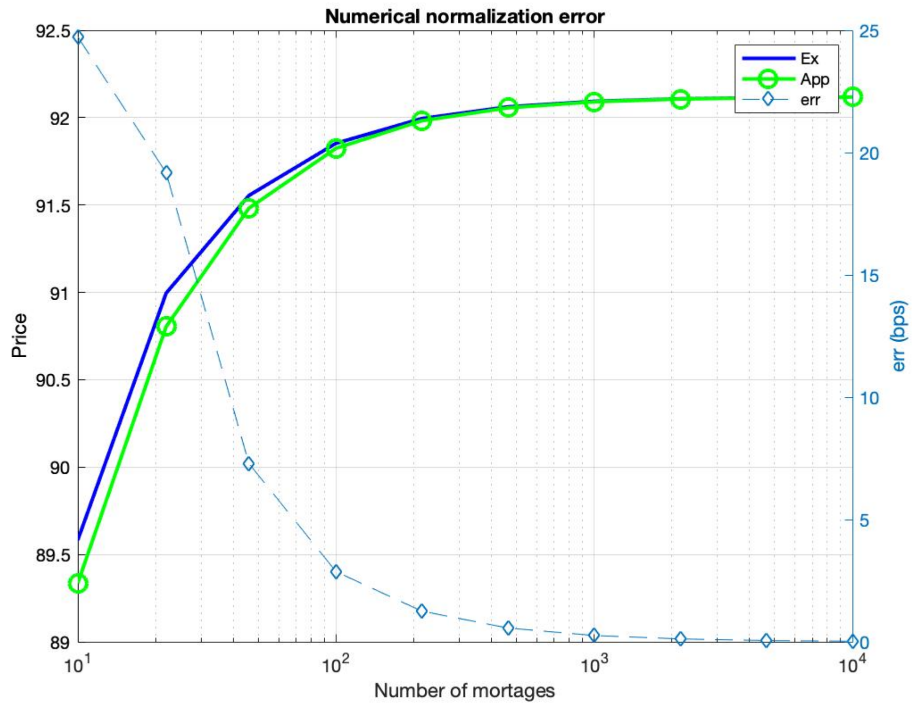

Figure 2: Error between approximate normalization and correct normalization.

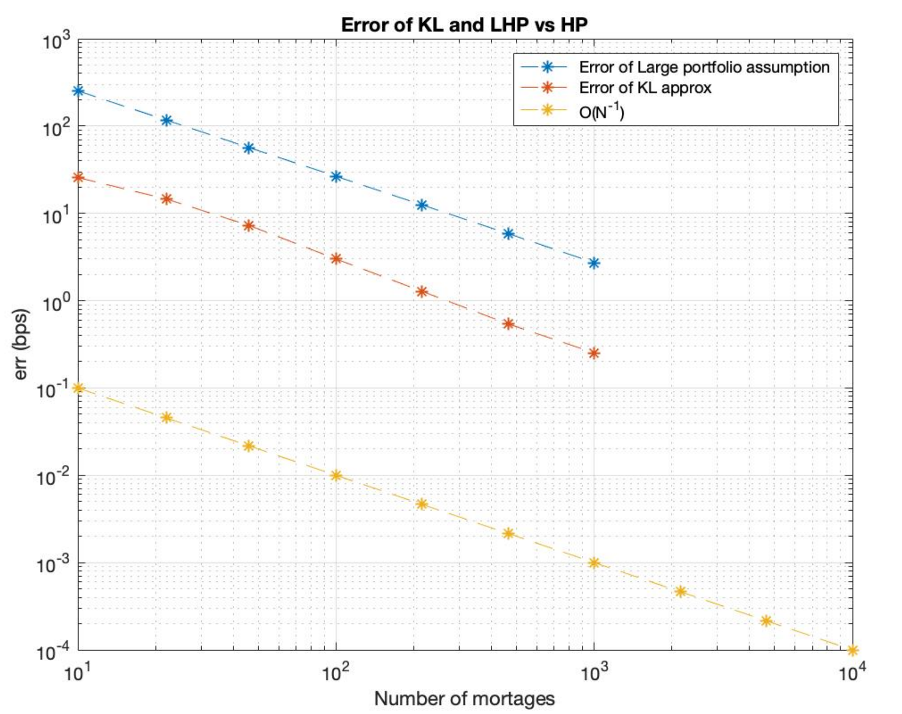

Figure 3: Convergence of error.

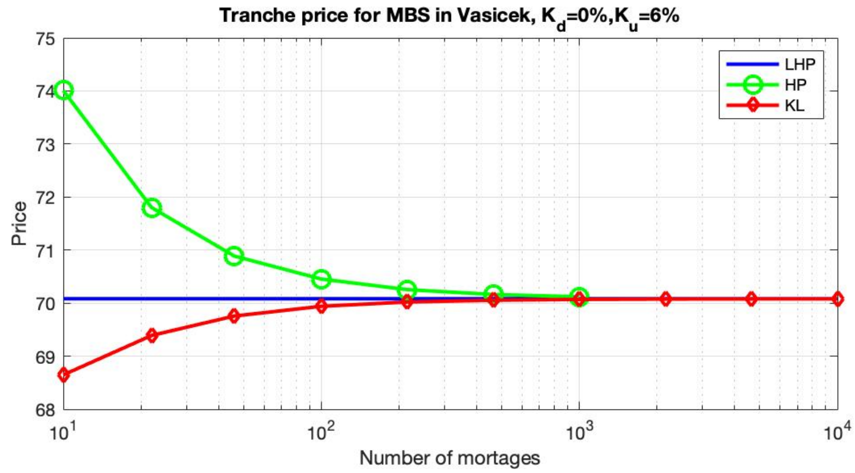

Figure 4: Equity tranche. No corrections.

In fact for the equity tranche, since it's the first to absorb losses, the probabilities of small number of payoffs are the most important, but with the KL approach we use the sterling approximation even for small m. It's well known that the binary entropy approximation of the binomial coefficient $\binom{n}{k}$ holds when k is linear in n and this happens for mid values of m as figure 5 shows.

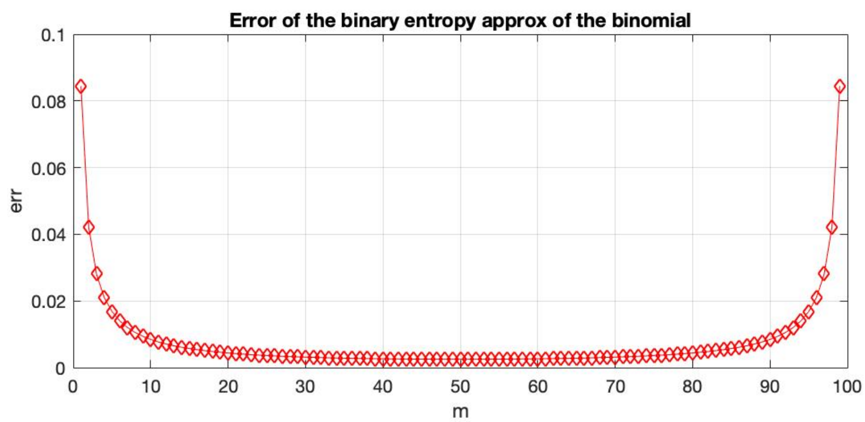

Figure 5: Binomial approx error. I=100.

For mezzanine tranches this is not a problem since for small m the loss of the tranche is zero while for large m this is also not relevant since in presence of large number of defaults the loss would already be 100% and error is the probability is this part of the space do not affect the price.

Now let's consider the payoff of the equity tranche which is $L_{0,6\%}(z)$ this can be written as

$$L_{0,6\%}(z) = C_1(L_{0,1}(z) - C_2 L_{6\%,1}(z))$$

for some suitable constant $C1, C2$ used to normalize the payoff of the tranche.
Now we can price $L_{0,1}$ using LHP or HP with just 1 mortgage and $L_{6\%,1}$ with KL. This is reasonable since $L_{0,1}$ is priced exactly since this is just the expected loss of the reference portfolio which is independent on the number of mortgages (the portfolio is homogeneous), and we can use KL on

$L_{6\%,1}$ for the aforementioned reasons.
This gives the following results:

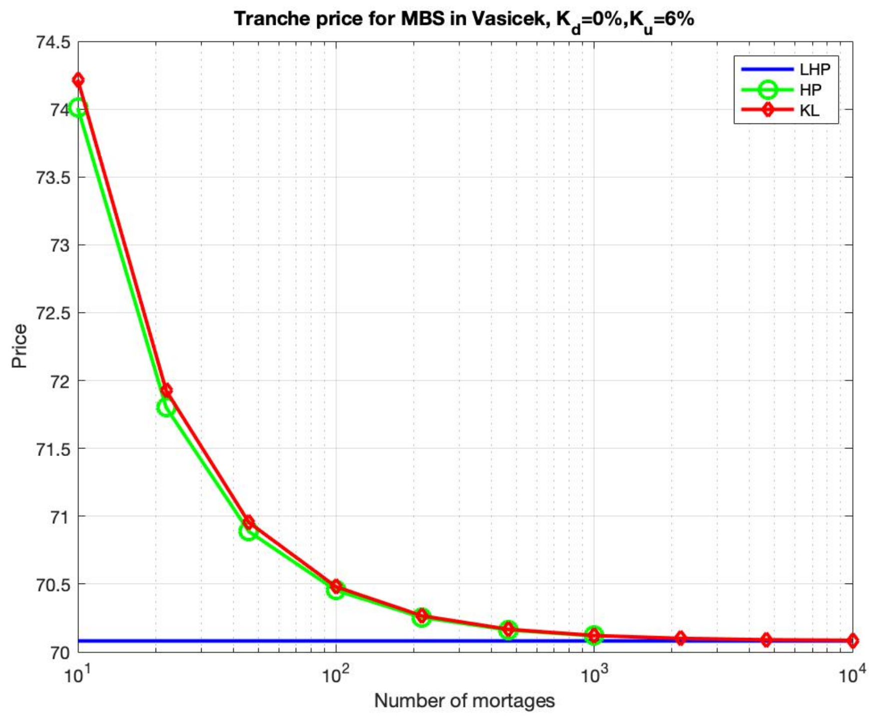

Figure 6: Equity tranche. KL corrected.

2 Pricing in presence of counterparty risk

The payoff at each payment date is :

$$\frac{[S(t_i) - S(t_{i-1})]^+}{S(t_{i-1})}$$

To price this Cliquet we simulated $5e5$ paths with antithetic variance reduction via Monte Carlo.

No def	Def
0.46	0.453

Table 2: Price of the Cliquet option with and without credit adjustment.

To find the credit adjustment we needed to obtain the time of default τ of each paths using survival probabilities from the intensity model, we furthermore assumed that if a credit event happens in t_{i-1}, t_i, at the next payment date t_i we will be payed the recovery times the actual payoff at t_i. As expected the price with credit adjustment is lower that the one without credit adjustment.

2.1 At what price ISP would try to sell?

Assuming the model to price the Clique is standard in industry or that the option is quoted (hence the fair price without the counterparty risk is known) then any difference in price would be explained by the default probabilities. This is why in our opinion ISP would try to sell at full price 0.46. The fair price, under non negligible credit event, can also be replicated by buying the option at full price and CDSs for every payment date with notional proportional to the expected payoff I'm loosing if a credit event happens, this. Then the price of the corrected option would be the full price minus the discounted fixed spreads of the CDSs.

3 Variance-covariance method for VaR ES

We are given a portfolio of 4 securities, that are all publicly traded common stocks, namely Santander, AXA, Generali and Bayer. We are required to give the 1 day VaR and ES, calibrating the distribution on a 3y horizon.
Back-filling the missing data points and rescaling the levels we obtain the following results: Since

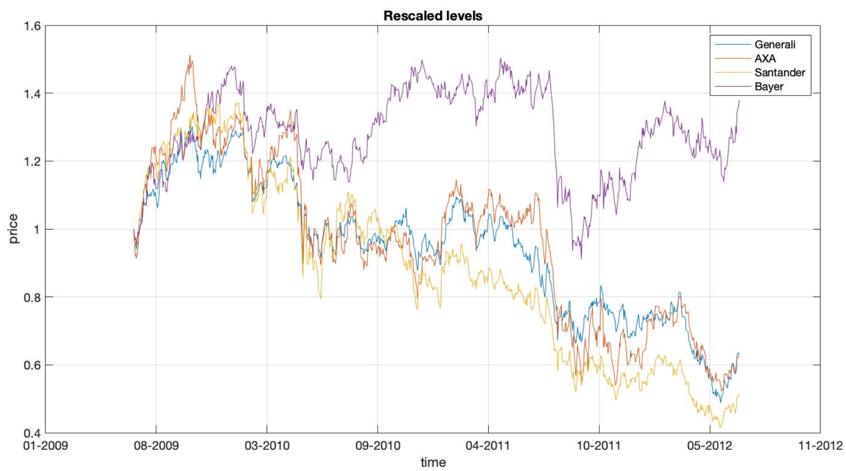

Figure 7: Normalized levels of the portfolio.

the time horizon is just one day we can linearize the portfolio returns as a first order expansion and assuming a multivariate normal distribution for daily returns of the levels.
So that we have a closed formula for the distribution Value at Risk and the Expected Shortfall of the constant equally weighted portfolio. Namely:

$$VaR_\alpha = \Delta t(\mu + \sigma N^{-1}(\alpha))$$

$$ES_\alpha = \Delta t \frac{\mu + \sigma\phi[N^{-1}(\alpha)]}{(1-\alpha)}$$

where $\mu = - < w, M >, \sigma = \sqrt{< w, \Sigma w >}$ with M, Σ as the annualized vector and matrix of mean and variance of the daily returns. Δt is the constant used to go back to a daily figure.
We obtain the following results:

Compared to the mean daily returns of the portfolio, which is about $-1.5bps$, we found a Value at Risk about 200 time larger, hence we are in fact looking at the tails of the distribution.

VaR_α	ES_α
4.57	5.24

Table 3: Daily VaR and ES in percent normalized by the today value of the portfolio.

The problem about this approach is twofold: first of all we are looking at tails event assuming a distribution which has exponentially decaying tails, this can be easily improved by assuming a power law for the tails, i.e. assuming a t-student distribution or other distribution. Better would be to assume a distribution calibrated only on extreme events, i.e Generalized Extreme Value Distribution [Sornette, Critical Phenomena in Natural Sciences].
The second problem, more difficult to overcome, is that the procedure is completely backward looking and hence sensitive to sampling error.
Clearly this problem are somewhat attenuated by the fact that this is a risk measure and only affects CR and not the only measure of risk that is considered internally.

4 Numerical Exercise

We want to compute the VaR & ES for a given portfolio of two stocks, in order to do this we use the following formulas:

$$Normal : VaR_\alpha = \mu + \sigma N^{-1}(\alpha), \qquad ES_\alpha = \mu + \sqrt{\sigma}\frac{\phi(N^{-1}(\alpha))}{1-\alpha}$$
$$t - Student : VaR_\alpha = \mu + \sigma t_\nu^{-1}(\alpha), \qquad {}_\alpha = \mu + \sqrt{\sigma}\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1}\frac{\phi(N^{-1}(\alpha))}{1-\alpha}$$

As we expected the ES is grater than the Var, moreover the VaR & ES related to the t-Student distribution is greater than the one with Normal distribution because of the fatter tails. In fact

	VaR_α	ES_α
Normal	2.46	2.82
t-Stud	4.82	7.44

Table 4: Daily VaR and ES in percent normalized by the today value of the portfolio in the framework.

the tails for a t-student with ν degrees of freedom are asymptotic to $x^{-(\nu+1)}$ while in a normal variable are asymptotic to e^{-x^2}. This exemplify the reason why extreme event assuming a t-student distribution are much more probable, or in our case, fixing the probability of the extreme event, we obtain more severe events.
This is shown by the shortfall-to-quantile ratio $\frac{ES_\alpha}{VaR_\alpha}$ that is 1.54 for the t-Student and 1.14 for the normal (according to the theory values of 1.5 and 1 for $\alpha \to 1$).