Politecnico di Milano

Financial Engineering

Group 9
Assignement 1

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1 Option pricing

The model considered is the Black model that is a monoparametic model which assigned to the Forward the martingale dynamics of

$$dF(t_0,t) = \sigma F(t_0,t)dW_t$$

1. Closed formula:
   A closed formula exists for prices of plain vanilla on Forward, for the call the price is:
   $$C(t_0,t) = B(t_0,t)[F_0 N(d_1) - KN(d_2)]$$
   with
   $$d_{1,2} = \frac{ln(F_0/K)}{\sigma\sqrt{t-t_0}} \pm \frac{1}{2}\sigma\sqrt{t-t_0}$$
2. CRR tree approach:
   In this way we modeled the underlying dynamic with a binomial tree described by an up movement of $u = e^{\Delta x}$ and a down return of $d = \frac{1}{u}$ (where $\Delta x = \sigma\sqrt{\Delta t}$ so that the model is consistent with the Black model). Then we compute the payoffs in the maturity and then moving backward we are able to price the option today, updating values at nodes as the expected price in the future nodes using an up probability of $q = \frac{1-d}{u-d}$ so that the Forward is a matringale.
3. Monte Carlo approach:
   This method simulate a large number of possible paths of the underlying and then takes the expectation of the all possible discounted payoffs, and does this computing numerically the following quantity:
   $$C(t_0,t) = B(t_0,t)\mathbb{E}\left[(F(t_0,t) - K)^+\right] \tag{1}$$

1.1 Option Pricing results

Here are reported the prices for Call options with notional of 1 unit.

Closed Formula	0.0941
CRR approach	0.0940
MC approach	0.0942

2 Errors

With the M assigned ($M = 100$ for CRR and $M = 10^6$ for MC) and using the errors suggested, the error is about $10\mathrm{e}{-4} = 1bps$.
This is considered adequate because about the smallest amount you can quote prices and hence is commonly deemed a reasonable error.

2.1 Error Plot

Numerically we verified the theoretical convergence rate of the Monte Carlo pricing method. Let $\theta_n = \frac{1}{n}\sum_{i=1}^{N} X_i$ be the Monte Carlo estimator for E(X). Letting $\sigma^2 = Var(X)$, by the CLT, $\sqrt{n}(\theta_n - \mathbb{E}(X)) \xrightarrow{d} \mathcal{N}(0, Var(X))$ and this implies that Monte Carlo estimator converges with rate $\sqrt{n}$.
Instead for CRR we can show that the convergence rate is O(1/n) apart for some noise in the monotonicity of the error. The non monotonicity of the error is due to two factors regarding CRR. One is that the binomial approximation of a normal variable has alternating behaviour w.r.t. the number of points used to approximate it, this is called the odd-even effect. Since we are using number of point of the form 2^k this problem is not present in our case.
The other effect present in the convergence of the binomial tree is the fact that depending on the number of points used, interesting quantity, such as the Strike in plain valilla, can be present or not in the discretization, this produces periodic non linearity in the error. In plain vanilla this is not such a big problem but can be relevant when pricing exotic products. Both behaviours are shown in the following picture:

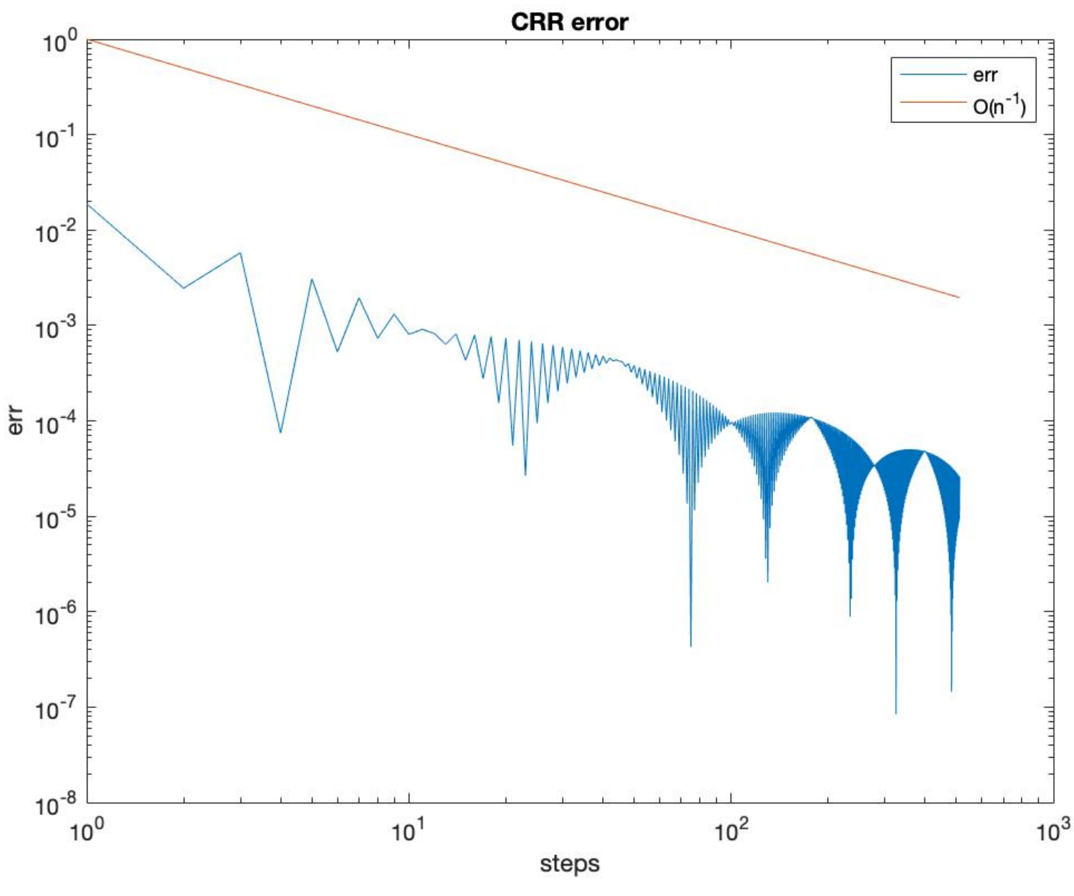

Figure 1: Fine CRR error

3 Delta and Vega

Delta is the price sensitivity to the movement in the underlying, in this case it's measured percentage. Vega is the price sensitivity to the movement in volatility, corresponding to a movement in the vol of 100bps, this is measured in dollar terms. Vega is the same for Calls and Puts, this can be seen from the Put Call parity since the difference in price of a call and a put does not depend on sigma.

$$\Delta \equiv \frac{\partial C}{\partial S_0} = B(t_0,t)N(d_1)\frac{\partial F_0}{\partial S_0} \qquad \nu \equiv \frac{\partial C}{\partial \sigma} = B(t_0,t)F_0\sqrt{t-t_0}\frac{e^{-d_1^2}}{\sqrt{2\pi}}\Delta\sigma \tag{2}$$

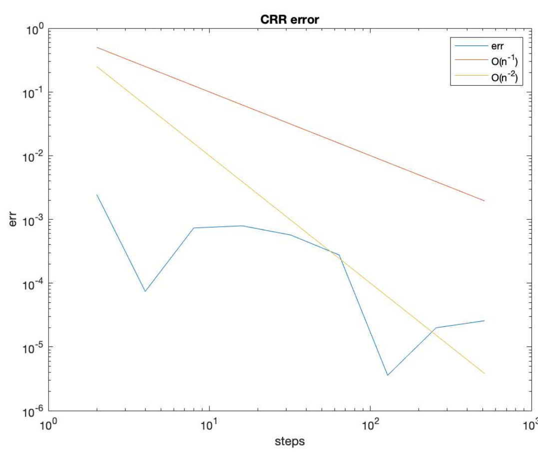

Figure 2: CRR error

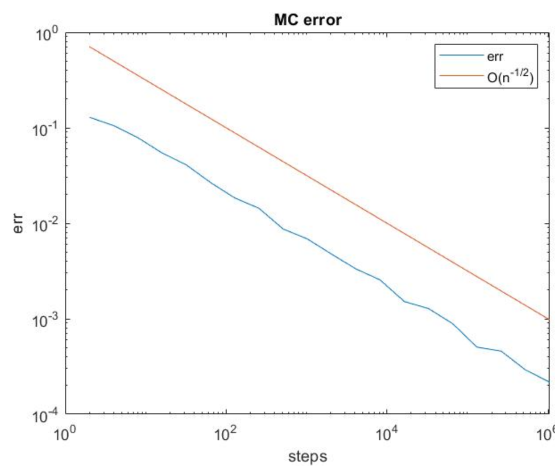

Figure 3: MC error

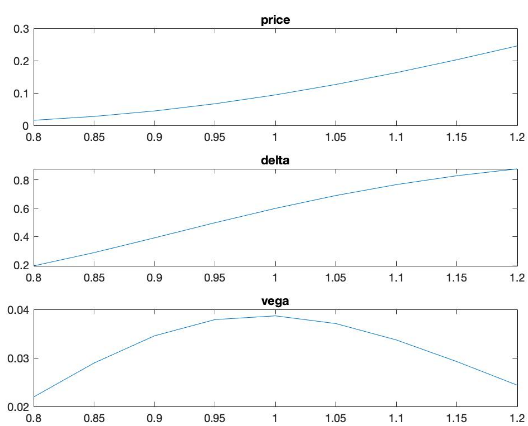

Figure 4: Delta and Vega Call

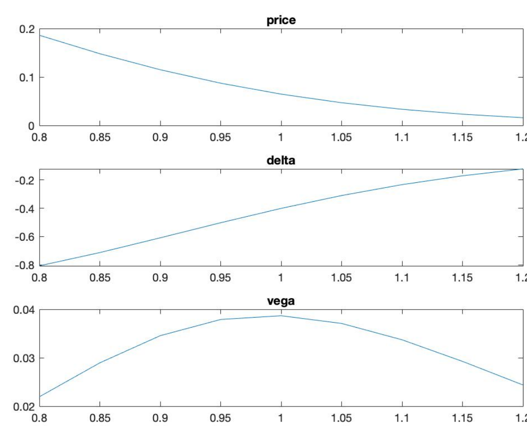

Figure 5: Delta and Vega Put