Politecnico di Milano

Financial Engineering

Group 9

Assignment 7

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1 Structured bond

The certificate we have to price can be also written as couterparty A being negative a strip of caplet on the 6m Euribor. As figure 1 and 2 show.

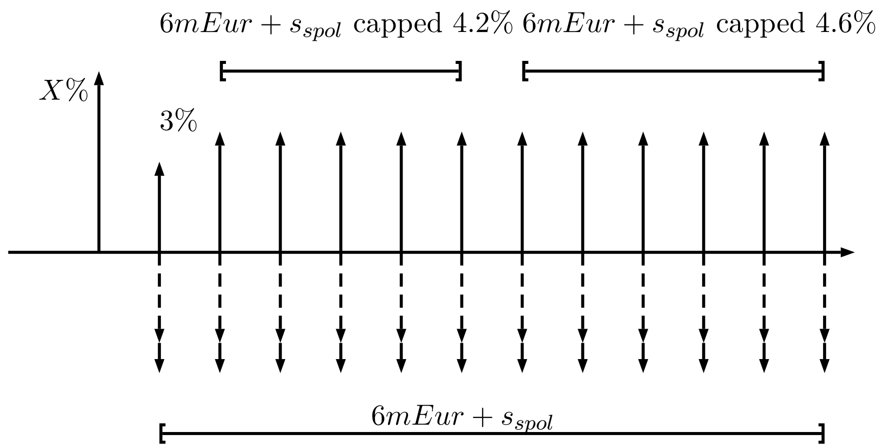

Figure 1: Certificate

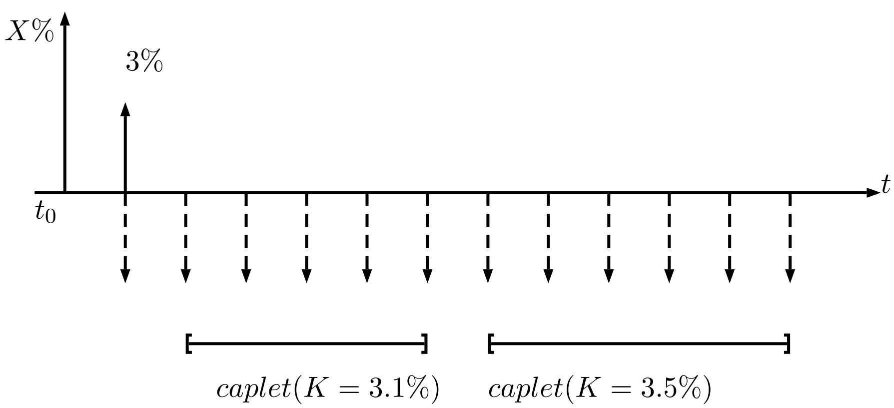

Figure 2: Equivalent certificate

We are given a volatility surface for the caps that is obtained by assuming the same volatilty in the caplets composing the cap. This is called flat volatility:

$$Cap_T = \sum_{i=6m}^{T-1} caplet\left(\sigma_T^{(F)}; t_0, t_i, t_{i+1}\right)$$

Since in our certificate we have to price caplets that fixes at different times then the caps, e.g. 1.5y, we want to have a way to price the caplets such that they remain consistent with the market prices for the caps. This is why we bootstrap the spot volatilities from the flat ones.
More in detail the bootstrapping procedure can be described as:

$$\Delta C = caplet1\left(\sigma_C^{(S)}, t_C\right) + caplet\left(\sigma_B^{(S)}, t_B\right)$$

with:

$$\sigma_B = \sigma_A + \frac{T_B - T_A}{T_C - T_A}(\sigma_C - \sigma_A)$$

hence we assume a linear relationship in the term structure of the volatility.
To solve the system we looked for the zero of the one dimensional function

$$F(\sigma_C) = \Delta C - caplet(\sigma_C, t_C) - caplet\left(\sigma_A + \frac{T_B - T_A}{T_C - T_A}(\sigma_C - \sigma_A), t_B\right)$$

with $T_A < T_B < T_C$ and we already bootstrap the curve up to T_A.
Since the strikes we are interested in are not part of the market strikes, we have to interpolate the caplet volatility surface, that we did with a spline interpolation.
The spot and flat vols are reported in the following heat maps:

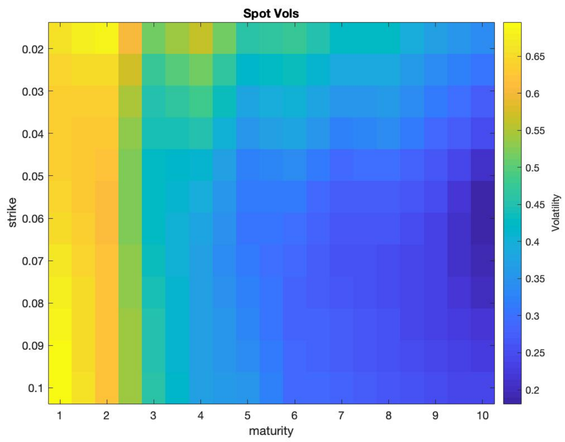

Figure 3: spot volatility

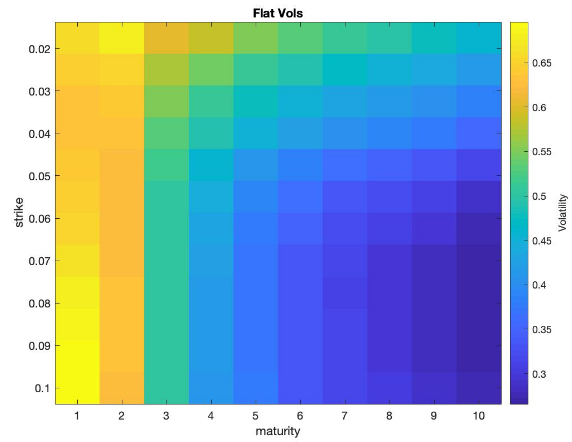

Figure 4: Flat volatility

The upfront value is therefore given by:

$$X\% = \delta_{t_0,t_{6m}}(1.736\% + 1.1\% - 3\%)B(t_0, t_{6m}) + Caplets_{t_0}$$

where as discussed the the strip of caplets is priced as

$$Caplets_{t_0} = \sum_{i=6m}^{6Y-1} caplet\left(K_i, \sigma_i^{(S)}; t_0, t_i, t_{i+1}\right)$$

The value that we obtained is 1.52%.

2 Pricing

2.1 NIG model

Our goal is to price a call according to the NIG model. The Laplace exponent and the characteristic function for the NIG model are:

$$ln\mathcal{L}[\omega] = \frac{\Delta t}{\kappa}(1 - \sqrt{1 + 2\kappa\omega\sigma^2})$$

$$\phi(x) = exp(-ix\ ln\mathcal{L}[\eta])\mathcal{L}\left[\frac{x^2 + i(1+2\eta)x}{2}\right]$$

For FFT and Quadrature we compute the call price using the Lewis formula.
For the Montecarlo simulation we need to simulate a sample of inverse Gaussian random variables, we can do it with the following algorithm:

- We generate independent u $\sim$ Uniform(0,1) and z$\sim$ χ_1^2
- Let $G^* = 1 - \frac{\kappa}{2\Delta t}(\sqrt{z^2 + \frac{4z\Delta t}{\kappa}} - 2)$.
- If $(1 + G^*)u > 1$ then let $G = 1/G^*$ else $G = G^*$.

or by using the provided probability distribution object provided by Matlab.
Then we can obtain the value of F_T with which we simulate the payoffs for the Montecarlo method as

$$f_t = ln(F_T/F_0) = \sqrt{t - t_0}\sigma\sqrt{G}g - \left(\frac{1}{2} + \eta\right)(t - t_0)\sigma^2 G - ln\mathcal{L}[\omega]$$

We then obtain the following results for the call:

Moneyness	FFT	Quad	MC
-0.2	125.59	125.59	125.81
-0.2:0.2	....	....	...
0.2	873.84	873.84	875.55

Table 1: Values of the call with different methods

2.2 VG model

Here we want to reproduce the previous computation in the VG framework, basically we just need to change our Laplace exponent formula in:

$$ln\mathcal{L}[\omega] = -\frac{\Delta t}{\kappa} ln[1 + \kappa\omega\sigma^2]$$

The results are shown in the table below:

Moneyness	FFT
-0.2	131.18
-0.2:0.2	....
0.2	896.50

Table 2: Values of the call with FFT in VG framework

The difference in the two models is negligible for small $kw\sigma^2$, indeed if we set $x = kw\sigma^2$ we can see that for small x the functions $1 - \sqrt{1 + 2x}$ and $-ln(1 + x)$ are asymptotic, so being the Laplace exponent the only difference in the two pricing models they give a very close results when $kw\sigma^2 \to 0$.

$$-ln(1 + x) \approx 1 - \sqrt{1 + 2x} = -x + \frac{x^2}{2} + o(x^2)$$

In the picture below we can see how the difference (in Euro) of the prices, obtained using the two different models, increases with the triplet $kw\sigma^2$. Moreover it is observed how the maximum difference appears when the moneyness is equal to zero, indeed in that case the integral used in the Lewis formula is maximized and so the difference:

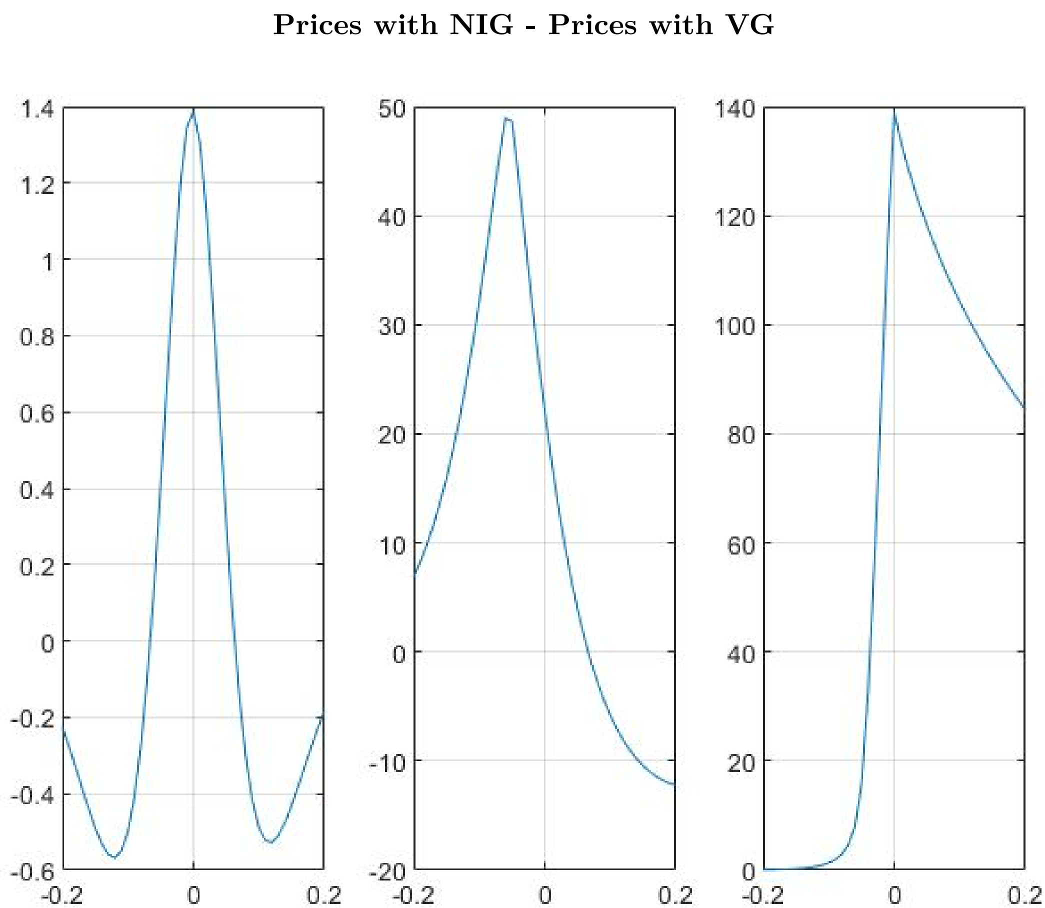

Figure 5: $kw\sigma^2 = 0.05$ vs $kw\sigma^2 = 1.25$ vs $kw\sigma^2 = 37500$

2.3 FFT and MC

Having a model with a characteristic function and a simulation algorithm is the best we can hope for. The reason is the following: the FFT pricing method is the best pricing method when calibrating on market prices, since it provides the whole surface at the same time, and since in the optimization procedure the pricing function gets called at each step, its very important to have a fast pricing method.
On the other hand we also need a simulable model to price exotic derivatives.
Both the VG and the NIG model have those characteristics.

3 Volatility surface calibration

Our goal is to calibrate a NIG model, with constant weights, starting from EUROSTOXX50 data. In order to do that we need to minimize the distance between market prices and model implied prices according with the following definition of distance:

$$d = \sum_{i=1}^{N} w_i |c(x_i, t_i; \mathbf{p}) - c_i^2|$$

C_i is obtained easily from the market implied volatility, then the model prices are a function of the parameters $\mathbf{p} = (\sigma, k, \eta)$. Following this idea we use the Fast Fourier Transform method to compute prices for all strikes/moneyness. Among the different optimization procedures available in Matlab, we choosed the function **fminsearch** together with a grid search to find the best starting point on a small $3x3x3$ grid.
The computation provides the following parameters:

σ	k	η	Distance d
0.1742	0.4553	10.1836	0.1223

Table 3: Parameter's Calibration

At the end, in order to verify the accuracy of the model, we show the market prices versus the values obtained with the NIG model we calibrated.

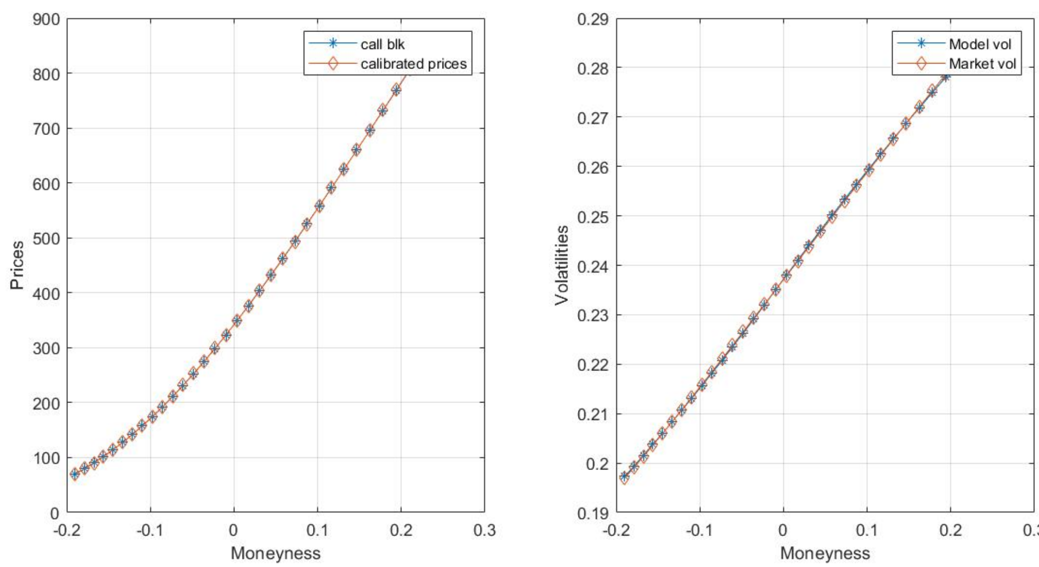

Figure 6: Moneyness vs (Prices, Volatilities)

As shown in the picture below (6) the maximum difference between market prices and calibrated prices is 0.8$, according to the FESE tick sizes guidelines (European guidelines for exchanges) this is comparable with the tick size for such a big contract, and since the bid ask would always be

greater then the minimum tick, we deemed this error reasonable.
Moreover the error at the ATM is less then 1bps, and in general we have no theoretical guarantees of finding the global minimum of the distance d.
Another technique would be to guarantee to price correctly the most liquid instrument by putting larger weights in correspondence of the most liquid instruments, or to calibrate the surface only in correspondence to fewer selected vanillas.

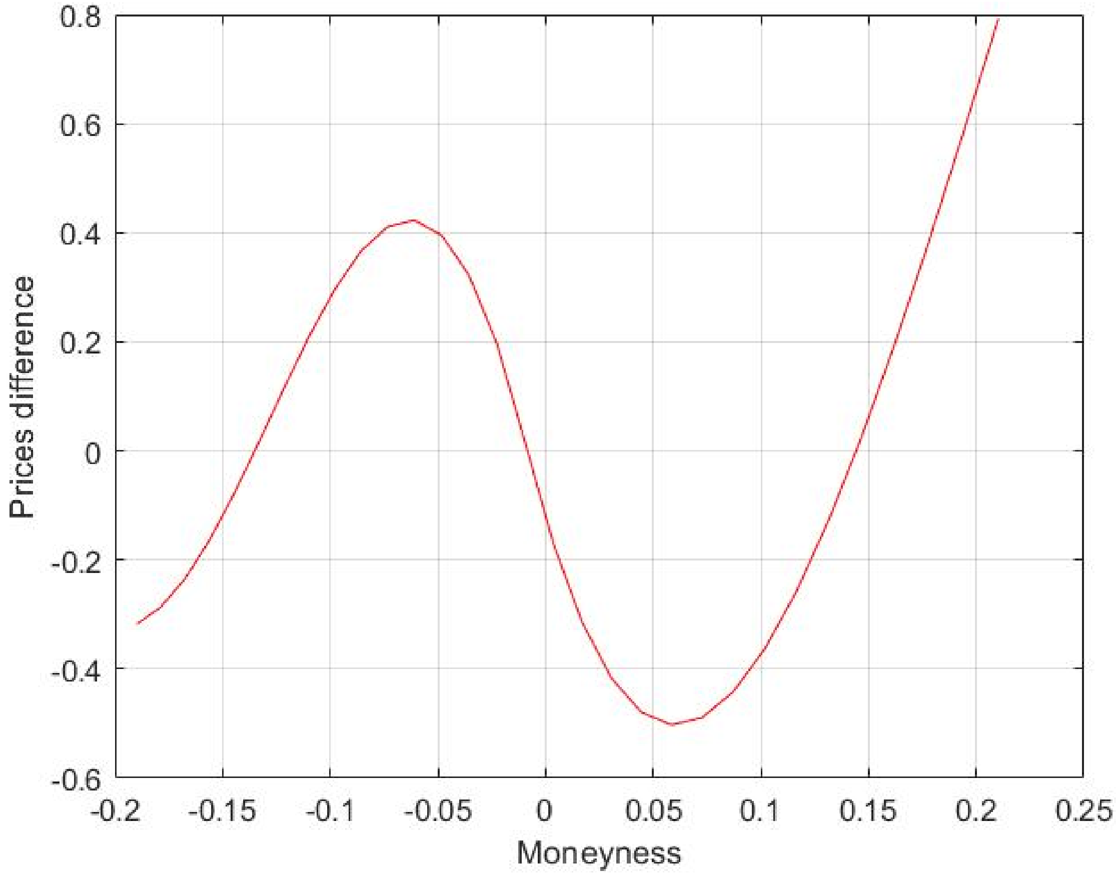

Figure 7: Moneyness vs Prices difference