Politecnico di Milano

Financial Engineering

Group 9
Assignment 8

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1 Certificate pricing

In order to compute the upfront we had to replicate the certificate's cash flow in the following way:

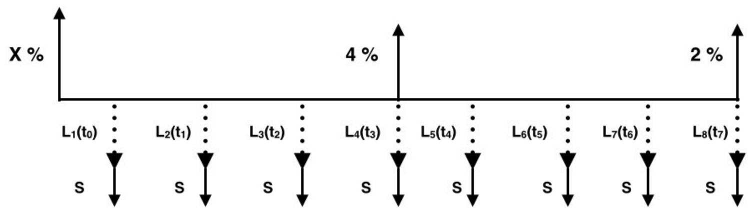

Figure 1: Cash flow

Two situations can occur: if the event $S_t > k$ occurs, the payment of the coupon at one year does not occur and there is the one at two years, otherwise the coupon is paid for the first year and not the second. The situations are shown in the following figures:

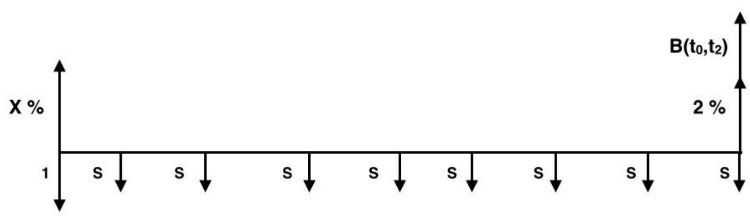

Figure 2: If: $(S_t > k)$

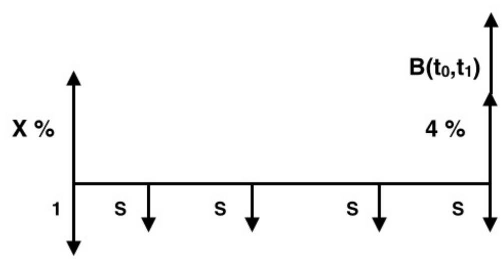

Figure 3: If: $(S_t < k)$

The value of the upfront is hence given by the following formula:

$$X = [S_{spol}BPV_2 + 1 - B(t_0, t_2) - B(t_0, t_2)\delta_{2\%}]E_0[\mathbb{I}_{S_{T1}>k}]+$$

$$[S_{spol}BPV_1 + 1 - B(t_0, t_1) - B(t_0, t_1)\delta_{4\%}](1 - E_0[\mathbb{I}_{S_{T1}>k}])$$

where BPV_2 is the basis point value until the second year and BPV_1 until the first.
And can be seen as a digital option of S_T where the notional of the digital is the second half of the flows.
In order to obtain the result we have to compute the $E_0[\mathbb{I}_{S_{T1}>k}]$

1.1 Upfront value using NIG model

To price the binary option we priced a narrow call spread around the strike of the digital, where the calls used are from the fft-NIG model, and hence we can price with arbitrary precision. We obtain an upfront of 6.13%. This could have been done by rewriting the Lewis formula for just one of the two expectations or by simulating the underlying.

1.2 Different model: Black adjusted Formula

We deemed also reasonable to compute the expected value with the Black adjusted formula for a digital option.
We can price a digital as:

$$CD_{Spread} = -\frac{dC_{Blk}(K)}{dK} = -\frac{\partial C_{Blk}}{\partial K} - \frac{\partial \sigma}{\partial K}\frac{\partial C_{Blk}}{\partial \sigma}$$

We obtain that the expected value can be computed as:

$$E_0[\mathbb{I}_{S_{T1}>k}] = \frac{CD_{Spread}}{B(t_0,t_1)}$$

In this case we obtain that the upfront is 6.09%.

1.3 Error using the Black model

Using the black model we observe that we obtain a much lower result than those obtained with the other two methods. This happens because in the Black model the result obtained for the expected value is simply given by:

$$E_0[I_{S_{T1}>k}] = \frac{CD_{Blk}}{B(t_0,t_1)}$$

where the CD_{Blk} is:

$$CD_{Blk} = N(d_2)B(t_0,t_1)$$

$$d_2 = \frac{log\left(\frac{F(t_0,t_1)}{K}\right)}{\sqrt{\sigma^2 \Delta T}} - \frac{1}{2}\sqrt{\sigma^2 \Delta T}$$

In this case we found that the upfront is 5.64%.

The absolute error using this method is: 50bps. This is not a negligible error and this is due to the fact that when we use the black model we are not considering the digital risk that comes from the vol surface.

1.4 3-Year Contract and Black adjusted Formula

The problem that occurs now in a three year contract is the concatenation of digital risk, since we have a strip of digitals. Indeed to do use the Black adjusted formula we would have to compute:

$$\begin{aligned}\mathbb{I}_{S_{T_1}>K}\mathbb{I}_{S_{T_2}>K} &= \lim_{\epsilon\to 0}\left(\frac{C_{T_1}(K+\epsilon)-C_{T_1}(K-\epsilon)}{2\epsilon}\frac{C_{T_2}(K+\epsilon)-C_{T_2}(K-\epsilon)}{2\epsilon}\right)\\ &\neq \frac{dC_{T_1}(K)}{dK}\frac{dC_{T_2}(K)}{dK}\end{aligned}$$

Hence expressing our product of indicator with volatility-smile-adjusted Black's formula can not be done.
The problem is indeed in the path dependency of the contract, and hence we would need the joint distribution of the the two times in order to price explicitly the contract.

1.5 3-Year contract and MonteCarlo simulation

If the structured bond has an expiry of three years the computations are similar to the previous one but in this case we have to compute two different expected values, were the event are clearly dependent.
Three situations can occur: if the event $S_{T_1} < k$ occurs, we have the payment of the coupon at one year and its value is 4% so after the first year Bank XX does not pay any floating leg.
If $S_{T_1} > k$ and $S_{T_2} < k$, the payment of the coupon at one year does not occur but we have a payment of a 4% coupon in the second year and after that Bank XX does not pay any floating leg.
Last situation which can occur is that $S_{T_1} > k$ and $S_{T_2} > k$, in this case the payment of the coupon in the first and second year does not occur but we have a payment of a 2% coupon in the third year.
We simulated using a NIG model where the parameters of the dynamics of the second year are supposed to be equal as the ones of the first year, assuming a vol surface constant in time.
The upfront is obtained as:

We obtain an upfront of 8.72%.

2 Swaption Price via Hull White

2.1 Jamshidian

The idea of Jamshidian (1989) is that a call on a coupon bond can be viewed as a linear combination of calls on zero coupon bonds, specifying some fictitious strikes for each payment coupon payment date, in formulas:

$$C^P_{\alpha\omega}(t_0, K) = \sum_{i=\alpha}^{\omega-1} c_i C^{ZC}_{\alpha i+1}(t_0, Ki)$$

We directly used an analogous formula for the put options, since we priced a receiver swaption which is equivalent to a put option on a coupon bond (actually since it is an ATM contract this would have been equivalent to a payer).
The quantity that we need in order to do the computation is K_i, i.e. the strikes of the zero coupon bonds.
Here we use the fact that, as suggested by Jamshidian, if we consider a sequence of monotonic functions f_i of one real variable (which map onto $[0, \infty)$), a random variable W, and a constant $K \geq 0$, since the function $\sum_i f_i$ is also increasing and maps onto $[0, \infty)$, there is a unique solution $w \in \mathbb{R}$ to the equation $\sum_i f_i(w) = K$.
Since the functions f_i are increasing:

$$\left(\sum_i f_i(W) - K\right)^+ = \left(\sum_i (f_i(W) - f_i(w))\right)^+ = \sum_i (f_i(W) - f_i(w)) 1_{\{W \geq w\}} = \sum_i (f_i(W) - f_i(w))^+.$$

In our case, each of the random variables $f_i(W)$ represents an asset value, the number K is the strike of the option on the portfolio of assets. We can therefore express the payoff of an option on

a portfolio of assets in terms of a portfolio of options on the individual assets $f_i(W)$ with corresponding strikes $f_i(w)$.
So we are interesting in finding $x^\star$, that is unique thanks to the discussion above, such that:

$$P(x^\star, T_\alpha; T_\alpha, T_\omega) = K$$

We know that P can be expressed as a function of $B_{\alpha i}(x_t, t)$, in particular:

$$P(x^\star, T_\alpha; T_\alpha, T_\omega) = \sum_{i=\alpha}^{\omega-1} c_i B_{\alpha i}(x_t, t)$$

Where $B_{\alpha i}(x_t, t)$ is a function of x_t which follows the Ornstein–Uhlenbeck dynamics, in particular:

$$B(t_i, x; t_i, t_i + \tau) = B(t_0; t_i, t_i + \tau)\text{exp}\left[-x\frac{\sigma(0,\tau)}{\sigma} - \frac{1}{2}\int_{t_0}^{t_i} \left[\sigma(u, t_i+\tau)^2 - \sigma(u, t_i)^2\right] du \right]$$

Once computed these quantities as a functions of x_t we can finally use the relation:

$$K_i = B_{\alpha i}(x^\star, T_\alpha)$$

With this method we obtain the following results:

Swaption	*Jamshidian*
2y8y	238
5y5y	219

Table 1: Values of the reciver swaption (bps)

2.2 Trinomial Tree

The trinomial tree is a way to price rate derivatives in a stochastic rate environment.
In particular the model is as above the Hull White single factor model. The tree implements the stochastic component of the rate, and we can avoid to calibrate the function $\phi(t)$, using the same formulas described above.

The transition probabilities $p_{i,j}$ of the tree are computed in order to simulate a mean reverting process, with zero mean, and of bounded width, hence x_t is bounded by l_min, l_max.

The most important step of the tree building procedure is to discount with stochastic discount each time step, i.e the discount to apply in each point x_i is given by the formula

$$D(t_i, t_i, t_{i+1}; x) = B(t_i, t_i, t_{i+1}; x)e^{f(dx_{i+1}, x_{i+1})}$$

where f is a function of the node in which we arrive starting from the node x, this again has to be used as the transition probabilities.
The tree procedure is the only way in which we could price path dependent rates options, as Bermudan, or Barrier. The results obtained are comparable to the results obtained with the closed formula given by the Jamshidian approach.
Looking at just the table seems to be that we were able to price the swaptions in the same precision.

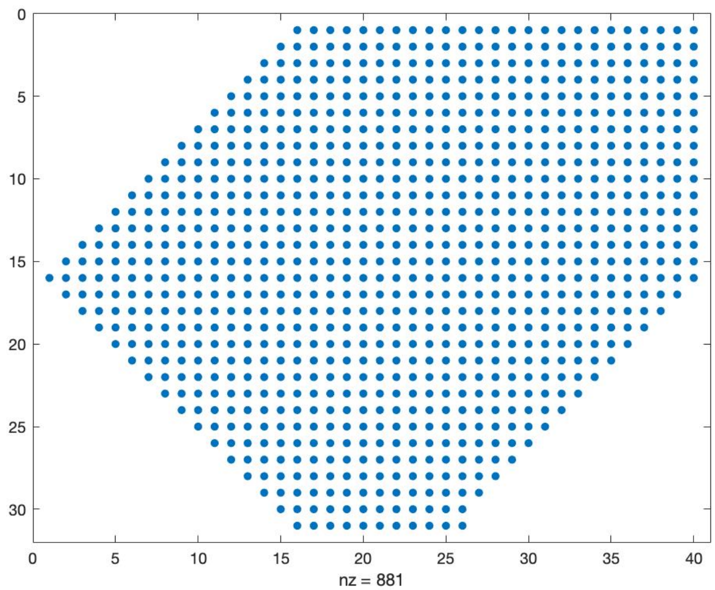

Figure 4: Tree structure, for swaption reciver payoffs.

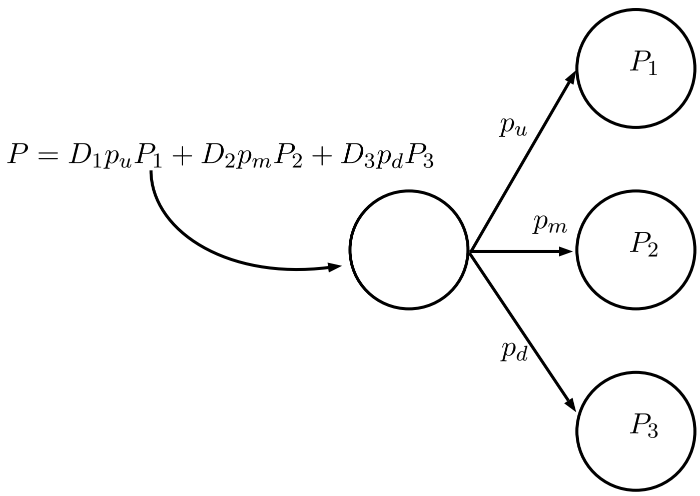

Figure 5: Tree Scheme for non boundary nodes.

$Swaption$	$Jamshidian$	$Tree$
2y8y	238	237
5y5y	220	220

Table 2: Values of the reciver swaption (bps)

Unfortunately this is not the case since we have a consistent, but small, bias in the price.
This can be seen by looking at the convergence of the price with respect to the number of time steps, again this is a small error of less that 1bp.
Probably this is due to some incorrect approximation for example rounding the times-temps in the

tree for the dates of the tree, or in some year-fraction used inconsistently between the two methods. However small the bias is not to look-over since we cannot know is this can be small only in this situation and not arbitrarily big for another contract.

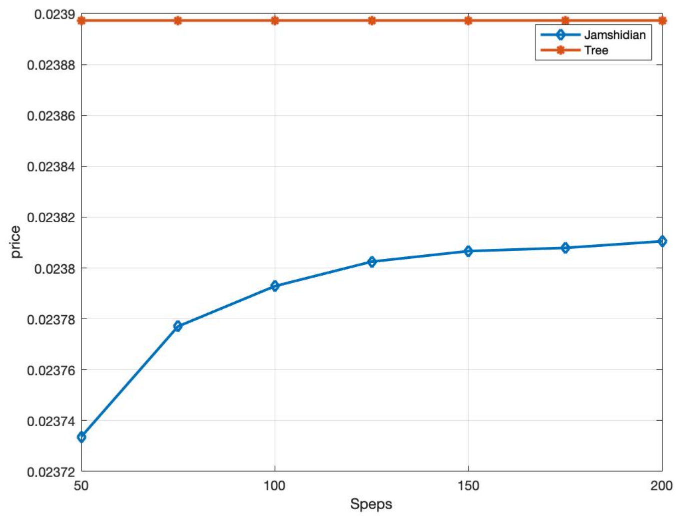

Figure 6: (Non) convergence of the tree price.