Politecnico di Milano

Financial Engineering

Group 9
Assignement 2

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1 Bootstrap for Euribor 3M Interbank curve

Bootstrapping the zero rate curve means to construct it in an forward iterative way, and in such terms it's an iterative algorithm that uses already computed nodes to infer new ones.
This procedures may vary in the kind of instruments used to construct it, thereafter we describe in detail the particular bootstrapping used to infer the discount curve.
According to Ron's paper we follow the the thereafter general concept: the short term is constructed with interbank deposit, the middle area of the curve is derived from IR future. The long term is cover by IR swaps.
Throughout the procedure we consider as relevant quantities the mid price, when we are given bid and asks prices. All the instruments have as underlying the 3M Euribor rate.

1. **Inter-bank deposit rates**
   We computed the discount factor from depos is this way:
   $$B(t_0, t_i) = \frac{1}{1 + \delta(t_0, t_i)L(t_0, t_i)}$$
   This procedures is done up to the first future settlement date, we prefer to use futures on the 3m Euribor when the two overlap since they are most liquid than deposits.

2. **Futures**
   Futures are quoted s.t.
   $$L(t_0, t_i, t_{i+1}) = (100 - future.price_{t_i,t_{i+1}})/100$$
   In order to obtain the discount factor $B(t_0, t_{i+1})$ from those quotes we should interpolate values on the already computed curve as the following relationship between forward discounts and discounts indicates.
   $$B(t_0, t_{i+1}) = B(t_0, t_i)B(t_0; t_i, t_{i+1})$$
   So $B(t_0, t_i)$ is obtained by interpolation and we use the following relationship to obtain the forward discount
   $$B(t_0; t_i, t_{i+1}) = \frac{1}{1 + \delta(t_i, t_{i+1})L(t_0; t_i, t_{i+1})}$$
   This highlights even further the iterative feature of the techinque.
   Our procedure is defined to use the first 7 futures, because they are the most liquid ones and the last inferred date should, in this case, be later that the 1y swap, that we will use for the long term part of the curve.

3. **Swaps**
   Swaps are in some sense independent instrument, meaning that they provide us directly the discount $B(t_0, t_i)$ where t_i is a swap payment date. This come from the iterative relationship between swaps rates of different years. In theory for the first one we could use:
   $$B(t_0, t_1) = \frac{1}{1 + \delta(t_0, t_1)S(t_0, t_1)}$$
   and the the following ones using:
   $$B(t_0, t_n) = \frac{1 - S(t_0, t_n)\sum_{i=1}^{N-1} \delta(t_{i-1}, t_i)B(t_0, t_i)}{1 + \delta(t_{i-1}, t_i)S(t_0, t_n)}$$

In this sense the swaps could be "independent" of the already built curve. In order to make the all curve consistently we instead infer $B(t_0, t_1)$ from the already built curve with interpolation and then use the iterative relationship described for the swaps with maturity grater that 1 year.

4. **Interpolation and extrapolation**
When we need to query a point in the already built curve we usually use interpolation, the interpolation used can affect the all curve but we can consider this a second order effect, in industry a lot of thought is given on the interpolation used as Hagan, West 2006 on Willmot 2006 shows.
As we shall see, bootstrap fixes exactly prices of chosen instruments, but interpolation can affect the error committed in replicating the ones not used in the procedure.
We relay heavily on interpolation during the curve construction, for example when we convert the forward discounting factor inferred from futures into actual discounts.
The chosen interpolation scheme chosen should guarantee the arbitrage free - down sloping property of the curve, otherwise one could sell the bond at longer maturity (and higher discount) and buy the one at smaller maturity. This self financing strategy would generate a free dollar in between the two maturities. As a sanity check our curve is in fact decreasing.
We implemented two interpolations, mainly: linear interpolation of discount and log-linear interpolation of the zero rate.
This picture shows the absolute error of the discount curve having changed the interpolation scheme:

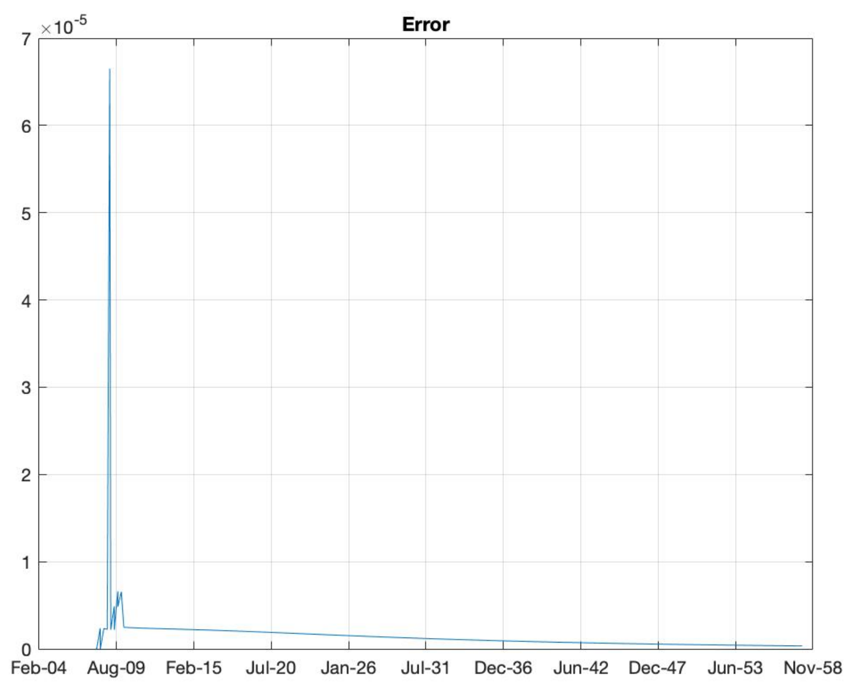


Figure 1: Error between log linear interpolation and linear interpolation.

The biggest error is at date '19-Feb-2009' which is the first date for the swaps, the bigger error is due to the fact that the interpolation in this case in quite large, 1 month approximately, while in other points it's much smaller. This error is below 1e-4 and at other nodes the error is less than 1e-6.

It can also happen that the expiry of a future can be 1 or two days prior to the settlement of the following one. In this case we deemed immaterial to use linear extrapolation of the curve in order to sustain the procedure, this by the software is only allowed for just two days.

Here there is the discount curve and the zero rate curve of the bootstrapped curve.

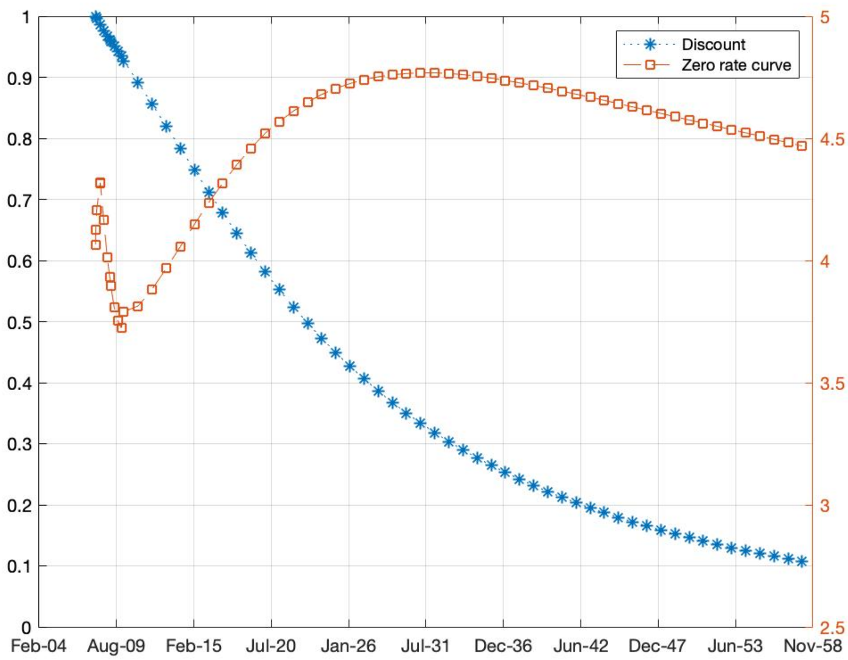


Figure 2: Discounts vs Zero rate curve

1.1 Question

Following what we read in Ametrano, Bianchetti 2009, we understood that bootstrapping is very relevant in industry since it can replicate exactly prices of chosen instruments. We can see this effect if we reprice forward rates for the futures and swap rates, we can observe that they replicate perfectly the mids of the instruments used. This property is independent on the interpolation scheme used.
The error is zero compared to the machine-ϵ for a fixed finite set of instrument. For example the the first swap is not replicated exactly since we used interpolation to obtain the corresponding discount and we did not used the inverse formula to find the corresponding discount. This is also true for the depos. Since we used only the ones needed to sustain the procedure, we can see a zero err on the ones used, but a non negligible error in the later ones.

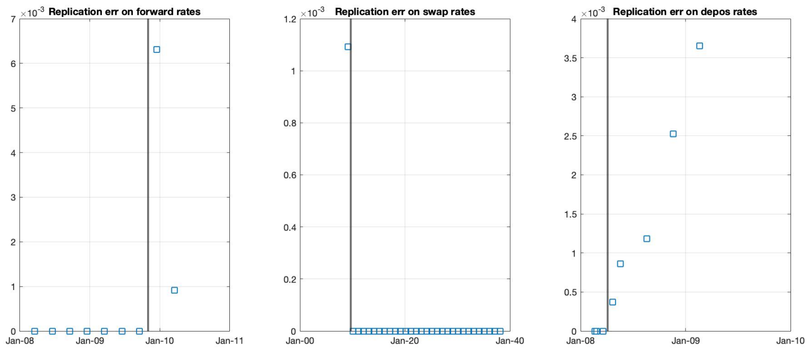


Figure 3: Error on quoted instruments. Vertical lines separate "used" and "not used" instruments.

2 DV01 for an IRS, Modified duration for a coupon bond

The reason for making a DV01 calculation is to quantify the sensitivity of a interest rate derivative with respect to a market movement of the rate.
In order to obtain DV01 we should shift rates up by 1bp and then recompute another bootstrap on these new data, obtaining a new shifted discount curve. Then DV01 is the difference of the NPVs of the instrument after and befor this shift in rates.

$$DV01 = NPV_{shifted} - NPV_{original}$$

We assumed the Swap was a payer swap paying the fixed leg since it's paying the bid rate. Another shift which we compute is the $DV01_z$, less precise than the DV01, but more convenient to calculate because it does not need bootstrapping again the curve after for shift, and it's just a parallel shift of the zero rates.
The BPV is defined as

$$BPV = \sum_i \delta(t_{i-1}, t_i) B(t_0, t_i)$$

and as we shall see it's a related quantity to the DV01.
These are the results obtained with these calculations.

DV01	4.4535e-04
DV01$_z$	4.6281e-04
BPV	4.4560

Note that in the case of a swap paying the par fixed leg we have that the following relationship holds:

$$DV01 = 1bp\ BPV_{shift} \approx 1bp\ BPV$$

In our case the NPV of the swap is initially not zero since we are not paying the mid and the relationship should not hold, however we tested the procedure assuming the swap was paying the mid rate and we got the exact relationship $DV01 = 1bp\ BPV_{shift}$.
Bond sensitivities are calculated in a different way. A measure for the sensitivity is given by:

$$-\frac{1}{P(t_0, t_n)} \lim_{\Delta \to 0} \frac{\Delta P(t_0, t_N)}{\Delta}$$

which is the derivative of the price w.r.t. the rate normalized by the price of the bond and the negative sign is given to correct the inverse relationship that exists between rates and bond prices

$\Delta P(t_0, t_N)$ its calculated by a $DV01_z$ shift. This quantity reduces after calculations to the Maculay Duration that has the following form:

$$MacD(t_0) = \frac{\sum_{i=1}^{n} c_i \delta(t_0, t_i) B(t_0, t_i)}{\sum_{i=1}^{n} c_i B(t_0, t_i)}$$

The result that we obtain is reasonable because we expected a value between 4 and 5, this value represent the weighted average time until cash flows are received (measured in years).
The result obtained is the following:

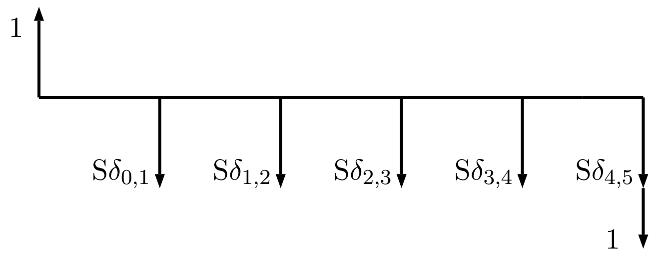
Mac	4.6246

To conclude we give some useful remark:

1. DV01 is the change in price in dollars, and hence it's a preferred measure for zero upfront instruments since this is a dollar figure.
2. Duration it's used to measure sensitivity after being multiplied by the change in rate. In our case a variation in rate of +0.2% would result in a percentage change in the contract of -0.927% in value.
3. We can observe that the numerator of the MacDuration is the $DV01_z$ of the contract and hance we have the following relationship: $DV01_z = 1bp\ MacD\ BondPrice$

3 Theoretical Exercise

Using the telescopic sum for the floating leg part of a IRS we know that the cash flows of a payer-swap (5y in this case) are equivalent to the following ones:



In formulas we can write:

$$NPV_{flt} = NPV_{fix}$$

$$1 - B(t_0, t_5) = S(t_0, t_5) \sum_{i=1}^{5} \delta(t_i, t_{i-1}) B(t_0, t_i)$$

$$\underbrace{\sum_{i=1}^{5} \delta(t_i, t_{i-1}) B(t_0, t_i) + B(t_0, t_5) S(t_0, t_5)}_{\text{Bond Cash flows}} = 1$$

So the price of a fixed coupon bond with fixed rate equal to the swap rate is 1. This is confirmed by the pricing done by the code, this is exact since the pricing basically inverts the steps done to infer the discounts.