CREDIT PRODUCTS

1

* Credit Risk is the risk for holding a risky bond. Counterparty risk is the risk that the counterparty will not be able to meet its contractual obligations if the credit event occurs. For ex, if you have a bond on AAPL and want to buy CDS to protect (buy from AIG, e.g.), then credit risk reflects the possibility AAPL will not fully repay the bond. Counterparty risk reflects the risk your counterparty will not be able to pay you the amount it owes you if AAPL goes bankrupt.

BASIC FI INSTRUMENTS IN PRESENCE OF CREDIT RISK

1. CREDIT RISK AND COUNTERPART RISK SIMILARITIES AND DIFFERENCES.

debitore non adempie ai suoi obblighi di pagamento.

CREDIT RISK is the risk that an obligor does not honour his payment obligations. Default risk is intrinsically linked to the payment obligation which the obligor ought to honour. An obligor who does not have any payment obligations, does not have any default risk either.

Both Credit Risk and Counterpart Risk refer to the same type of risk: the risk that the opposite side of a contract won't honour its obligations * to repay.

The Credit Risk is associated to traditional loans business (lending money to someone), one sided.

The Counterpart risk is used especially in OTC derivatives business: we have a two-sided risk since both of the counterparties assume the risk that the other may default. It is the risk of default of the derivative's counterparty.

COMPONENTS OF CREDIT RISK

→ single name:

- ARRIVAL RISK: term for the uncertainty whether a default will occur or not (specified w.r.t. a given time horizon) → mathematical description probability of default: distribution of indicator variable "default before the time horizon"
- TIMING RISK: uncertainty about the precise time of default → probability distribution function of the time to default.
- RECOVERY RISK: uncertainty about the severity of the losses if a default has happened. The uncertain quantity is the actual payoff that a creditor receives after a default. → probability distribution of recovery rate
- MARKET RISK → "correlation" with other risk factors.

→ multi name:

- DEFAULT DEPENDENCE → Joint default probabilities / default correlation
- TIMING RISK / CLUSTERING → clusters of events

CREDIT EVENT : default ⇒ a borrower doesn't pay back the money.

i.e. in case of:

→ bankruptcy

→ failure to pay (> threshold, after grace period)

→ obligation default

→ obligation acceleration (stop to pay all together)

The definition of a credit event has now been standardized by the

ISDA (International Swaps and Derivatives Association)

(In Italy, ABI).

ISDA Master Agreement : legal framework where all derivative contracts are written on.

LEGAL FRAMEWORK: BANKRUPTCY LAW (legge fallimentare)

Legislation modified in continental Europe similarly to the US Bankruptcy Law.

Anglosaxon legal framework

→ Default

→ Restructuring Plan

RESTRUCTURIVE PLANS

... and the Italian corresponding tool

→ Fallimento

→ Concordato preventivo

- Piano di risanamento (Art. 67) (between parties)
- Accordo di ristrutturazione (Art. 182 bis) (in front of the judge)
- Concordato preventivo (in the court)

CREDIT RATING → we will describe the [illegible] with [illegible] SPGL

(and [illegible]) [illegible] → independent [illegible]

AN (OVER) SIMPLIFIED BALANCE SHEET

ASSETS	LIABILITIES
firm value: material and immaterial assets	equity (shareholders should be payed) debt (loans / bonds / other debt vs. suppliers)

all the money that I have to give back to someone

$$A = L$$

$$V = S + B$$

A DEFAULT EVENT happens when firm value < total debt.

In case of default, debt is sewed in order of seniority → PRIORITY RULE.

In credit world, assets → Bonds vs. derivatives ("IB" coupon bond, "IB" floater)
→ Exotic

2. FUNDAMENTAL HYPOTESIS IN SINGLE NAME MODELS. COUPON BOND IN FUNCTION OF THE BUILDING BLOCKS.

2 main risks $\begin{cases} \text{recovery risk } \pi \\ \text{default risk (arrival / timing) } \tau \end{cases}$

only 1 information (SPOL)
→ we model only τ as r.v.
π parameter.

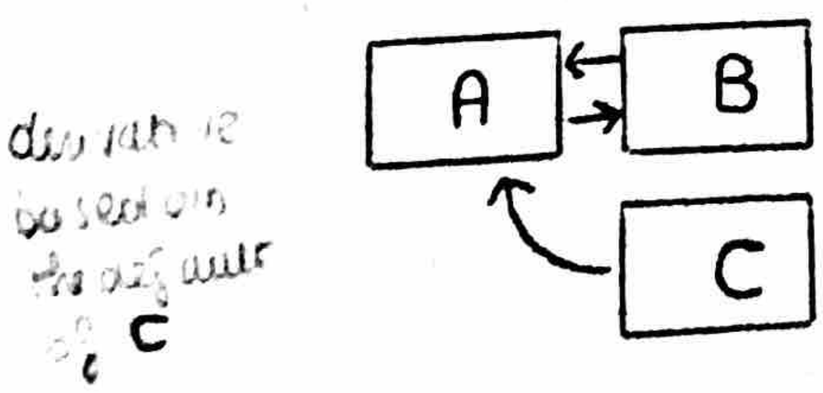
BASIC NOTATION

- π : recovery value → to model recovery risk
- τ : time to default
- $\mathbb{1}_{\tau>T}$: survival indicator function
- $P(t,T) = \mathbb{E}[\mathbb{1}_{\tau>T} \mid \mathcal{F}_t]$: survival probability between t and T
- $D(t,T)$: stochastic discount
- $B(t,T) \equiv \mathbb{E}[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t] = \mathbb{E}[D(t,T) \mid \mathcal{F}_t]$
  default free ZC bond (on "initial" discounts)
- $\overline{B}_0(t,T) \equiv \mathbb{E}[e^{-\int_t^T r_s ds} \mathbb{1}_{\tau>T} \mid \mathcal{F}_t]$: defaultable ZC with NO recovery
- $e(t;T,T+\Delta t) \equiv \mathbb{E}[e^{-\int_t^{T+\Delta t} r_s ds} (\mathbb{1}_{\tau>T} - \mathbb{1}_{\tau>T+\Delta t}) \mid \mathcal{F}_t]$
  value of 1 in $T+\Delta t$ if default happens in $(T, T+\Delta t]$.

BUILDING BLOCKS IN SINGLE-NAME CREDIT (NO RECOVERY MODEL)

FUNDAMENTAL HYPOTESIS IN SINGLE NAME MODELS:

- Constant recovery value π (so, I only have to model τ)
- NO default risk in derivative's counterpart (NO counterparty risk)



A insured counterparty (that receives a payoff if default happens on the party that is long the credit derivative)

B insurer (who has to pay or suffers a loss in default)

C will be the reference credit (obligor)

P/-B: counterparty risk

I'm interested in the default of C (credit risk)

→ I assume that B can't default. ?

It is not a strong assumption since I can ask for a guarantee (collateral), like in a future contract. This is regulated by CSA (credit support Annex).

- independence between rates and defaults (and so discounts)
- discretization: we consider a set of reset dates. If there is a default in $(t_m, t_{m+1}]$, we pay in t_{m+1}

Building blocks: $\tilde{\pi}$, $P(t,T)$, $B(t,T)$, $\bar{B}_0(t,T)$, $e(t;T,T+\Delta t)$

with these hypotesis:

- $e(t;T,T+\Delta t) = \mathbb{E}[D(t,T+\Delta t)(\mathbb{1}_{\tau>T} - \mathbb{1}_{\tau>T+\Delta t}) \mid \mathcal{F}_t] =$

$= B(t,T+\Delta t)\,\mathbb{E}[\{\mathbb{1}_{\tau>T} - \mathbb{1}_{\tau>T+\Delta t}\} \mid \mathcal{F}_t] =$

$= B(t,T+\Delta t)\big(P(t,T) - P(t,T+\Delta t)\big)$

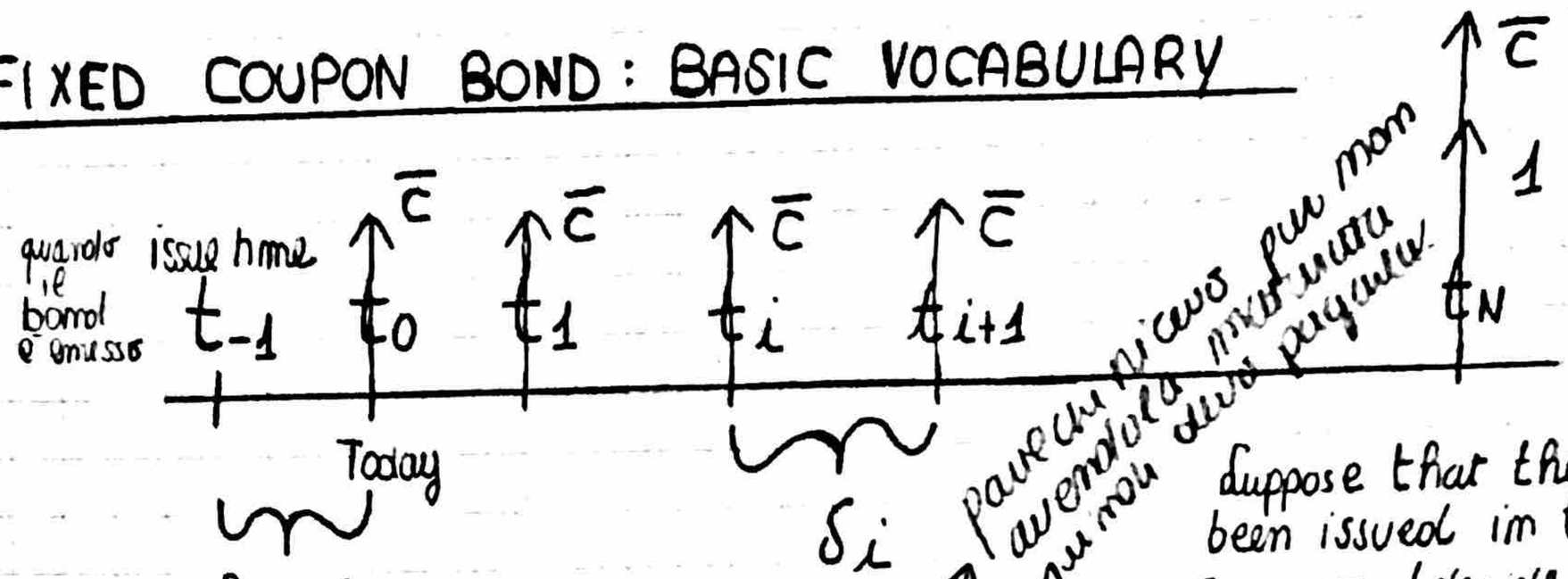
- $\bar{B}_0(t,T) = \mathbb{E}[D(t,T)\,\mathbb{1}_{\tau>T} \mid \mathcal{F}_t] =$

$= \mathbb{E}[D(t,T) \mid \mathcal{F}_t]\,\mathbb{E}[\mathbb{1}_{\tau>T} \mid \mathcal{F}_t] =$

$= B(t,T)\,P(t,T)$

⇒ 3 building blocks: $B(t,T)$, $P(t,T)$, $\tilde{\pi}$

FIXED COUPON BOND: BASIC VOCABULARY



parte di cui non può più avendola mkt una volta dopo pagato

Mkt for bonds in Italy: MOT, MTS (secondary mkts)

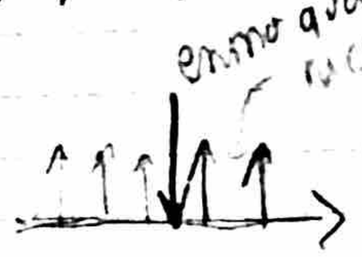
$$\boxed{\bar{C} = CP + \text{Accrual}}$$

Suppose that the bond has been issued in t_{-1} in the primary market. I decide to buy it in the secondary market in t_0. When I enter in t_0, I will pay not only CP, but also an accrual.

CP: Clean Price, Quoted Price

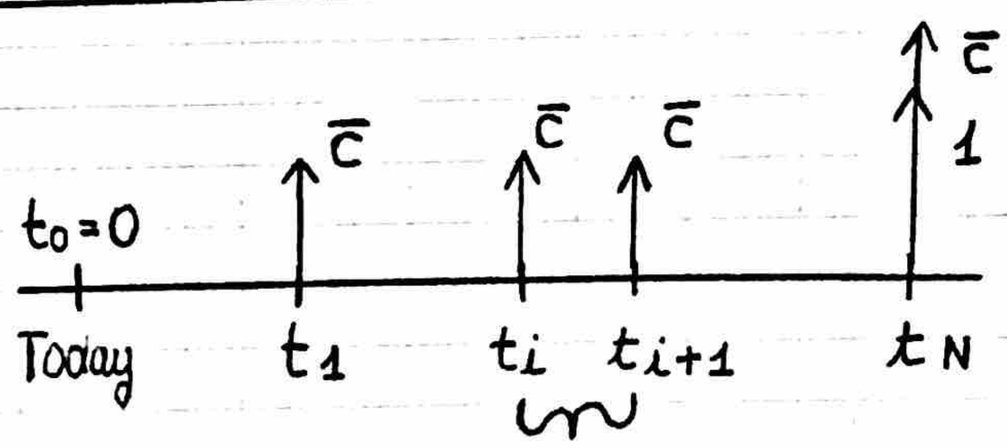
$\bar{C}$: Invoice (or Dirty) Price: the one relevant (also) for a modelling perspective.

Accrual: The part of the coupon that is recognised to the seller
(RATEO) $A = \delta(t_{-1}, t_0)\,\bar{C}$



Bond price is always dirty price.

FIXED COUPON BOND IN FUNCTION OF THE BUILDING BLOCKS



no default part

first $\bar{C}$ in t_1

no default!

valore ancora di D + 1 invece cum a perché default è + in là.

$$\bar{C}(0) = \bar{C}\,\delta(t_0,t_1)\,\underbrace{E_0\left[D(t_0,t_1)\,\mathbb{1}_{\tau>t_1}\right]}_{\bar{B}_0(t_0,t_1)} + \sum_{i=2}^{N} \bar{C}\,\delta(t_{i-1},t_i)\cdot \bar{B}_0(t_0,t_i) +$$

$$+ \bar{B}_0(0,t_N) +$$

$$+ \text{ default part}$$

default part:

$$\tau \text{ in } (t_0,t_1): \quad \pi\left[\frac{\delta_0}{2}\bar{C} + 1\right] e(t_0; t_0, t_1)$$

In case of default:

$e(t_0; t_i, t_{i+1})$ (default between t_i and t_{i+1})

$$\pi \sum_{i=1}^{N} E_0\left[D(t_0,t_{i+1})\left(\mathbb{1}_{\tau>t_i} - \mathbb{1}_{\tau>t_{i+1}}\right)\right] +$$

$$+ \pi\bar{C}\sum_{i=1}^{N} \frac{\delta_{i-1}}{2}\, e(t_0, t_{i-1}, t_i)$$ accrual term

non c'è alcun sarà il default! $E[D(1-1)]$

default è [illegible]

$$\Rightarrow \bar{C}(0) = \bar{c} \sum_{m=1}^{N} \delta(t_{m-1}, t_m)\, \bar{B}_0(t_0, t_m) +$$

$$+ \bar{B}_0(t_0, t_N) +$$

$$+ \tilde{\pi} \sum_{m=1}^{N} e(t_0; t_{m-1}, t_m) +$$

ACCRUAL: If $\tilde{\tau}$ t_i t_{i+1} $\rightarrow$

$$+ \tilde{\pi}\, \bar{c} \sum_{m=1}^{N} \frac{\delta(t_{m-1}, t_m)}{2}\, e(t_0; t_{m-1}, t_m)$$

I have to be payed for the coupon

two orders of magnitude lower.

3. SURVIVAL PROBABILITY AND ITS ROLE IN BASIC CREDIT PRODUCTS.

Survival probability is one of the building blocks in credit world:

$$P(t,T) = \mathbb{E}[\mathbb{1}_{\tau > T} \mid \mathcal{F}_t]$$

The implied survival probability from t to $T \geq t$ as seen from time t is the ratio of the defaultable to the default-free ZCB prices:

$$P(t,T) = \frac{\bar{B}_0(t,T)}{B(t,T)}$$

(single name models with hypotesis ...)

The implied default probability over $[t,T]$ is

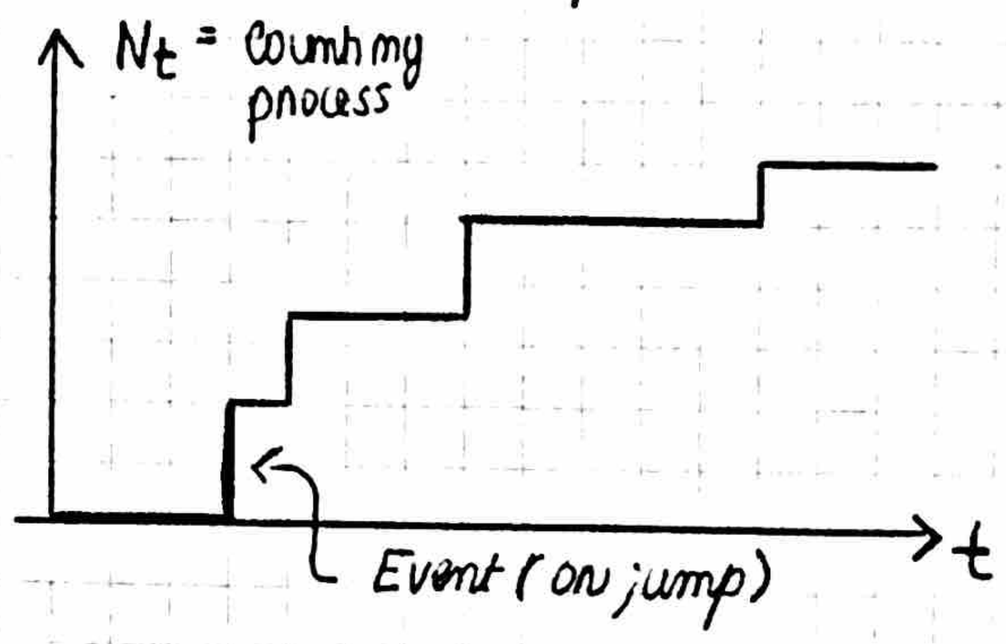
$$P^{def}(t,T) := 1 - P(t,T)$$

If we have zero-coupon bond prices for all maturities, we also have implied survival probabilities for all maturities. From this we can construct a term structure of survival probabilities.

The implied survival probability $P(t,T)$ is initially at one $P(t,t) = 1$, it is non-negative and decreasing in T. It is also reasonable to assume that eventually there is a default: $P(t,\infty) = 0$.

INTENSITY BASED MODELS : SURVIVAL PROBABILITY VS. INTENSITY

DETERMINISTIC INTENSITY BASED MODELS



Poisson Process in generale processo che si basa sul contare quante volte avviene una cosa. Se però il processo non è stazionario (si pensi agli orari di punta di un ristorante) allora non è un processo omogeneo

↓

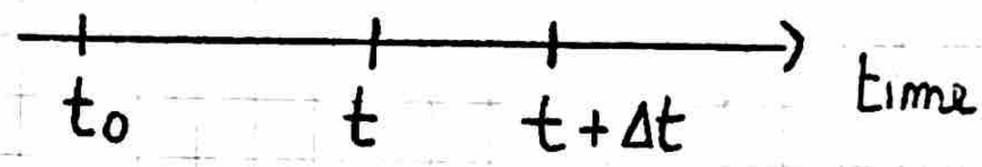
INHOMOGENEUS POISSON PROCESS con un tasso (ad ogni tempo) di $\lambda(t)\Delta t$

↑

prob. di jump

salto indip da storia precedente

Inhomogeneous Poisson process :



1) $P(\text{Event in } (t, t+\Delta t]) \propto \Delta t$

$$= \lambda(t)\Delta t$$

↑ probability to have one jump in $(t, t+\Delta t]$.

2) $P(\quad)$ ⊄ what has happened before t

independence of jumps

If $\lambda = \text{const}$ → Homogeneous Poisson process.

$N(t)$ counting process with intensity $\lambda(t)$

τ time of the first jump of $N(t)$

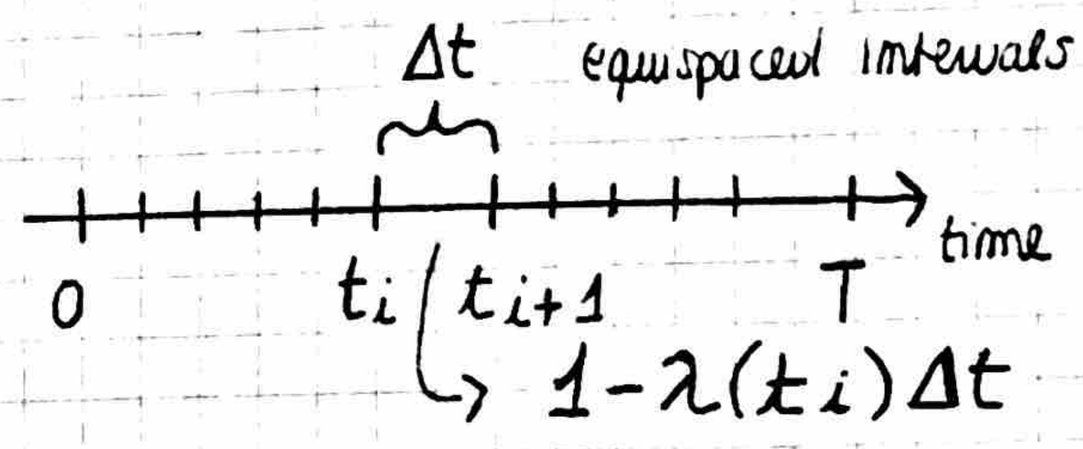
$P(0,T) = \mathbb{E}[N(T) = 0 \mid \mathcal{F}_0]$ survival probability

main property :

$N(t)$ purely discontinuous process ⟹ ZERO COVARIATION with continuous martingales (e.g. Wiener)

Building block : $P(t,T) = e^{-\int_t^T \lambda(s)\,ds}$

poisson with intensity



probability of NO event in $(t_i, t_{i+1}]$

$$P(0,T) = P\left(N(T)=0 \mid \mathcal{F}_0\right) = \prod_{i=1}^{N} \left(1-\lambda(t_i)\Delta t\right) =$$

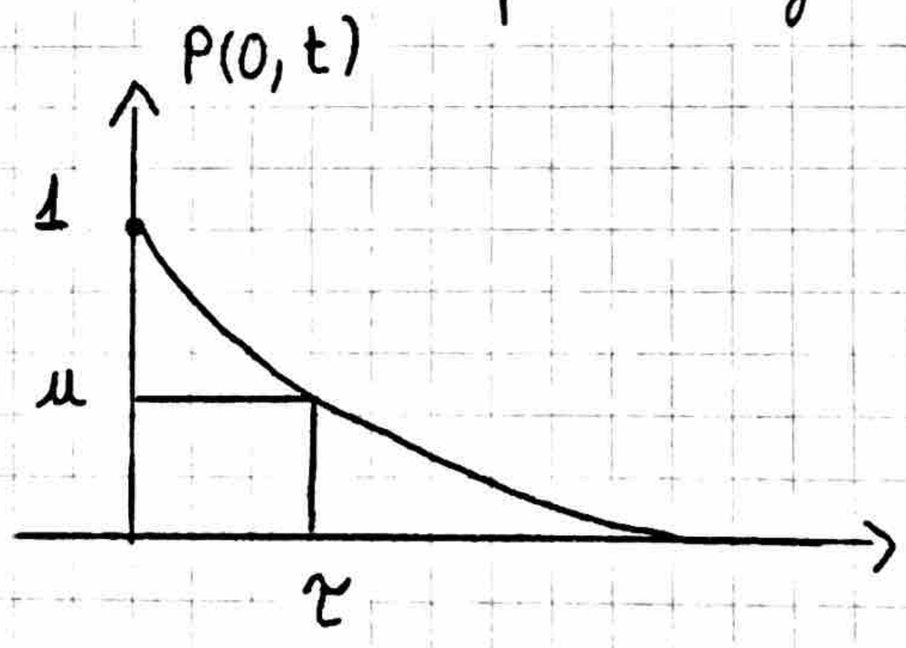
all events are independent

Δt very small → first order expansion

$$= \prod_{i=1}^{N} e^{-\Delta t\, \lambda(t_i)} = e^{-\sum_{i=1}^{N} \Delta t\, \lambda(t_i)} + \mathcal{O}(\Delta t^2)$$

$$\xrightarrow{\Delta t \to 0} \quad P(0,T) = e^{-\int_0^T \lambda(t)\,dt}$$

We can use survival probability to <u>simulate TIME TO DEFAULT</u> :



- draw $u \in [0,1]$ (» rand)
- τ : $e^{-\int_0^T \lambda(t)\,dt} = u$

we get the time to default τ of the simulation

→ In this way I can price whatever stuff, e.g. DEFAULTABLE BONDS (easy P, defaults $\perp\!\!\!\perp$ IR). Once I have τ, I can price with all flows up to τ, then I put π.

ASS. 5 : Given survival probabilities at time t up to T, $P(t,T) = e^{-\lambda(T-t)}$, $T-t = 30y$ and $\lambda = 4 \cdot 10^{-3}$, we simulate $\{\tau_i\}_{i=1:10^3}$ default times.

We generate a set of uniformly distributed random numbers $\{u_i\}_{i=1:10^3} \in [0,1]$, that represents our set of simulated survival probabilities $P_i(t,T)$;
from these quantities we can finally derive our simulated default times τ_i : $P_i(t,T) = e^{-\lambda(\tau_i - t)}$.

Once obtained this vector $\underline{\tau}$, it will be our input of the problem we want to solve: estimate the distribution of survival probabilities and so to provide a confidence interval for the parameter λ.

We group the simulated default times, dividing them year by year and then for each interval we compute the cumulate frequency, that will be our approximation of the distribution of default probabilities.

Interpolate with »fit to find an estimator $\hat{\lambda}$ and a confidence interval for the intensity.

Survival probability also used for the bootstrap (ASS. 3)

?

EUREX → Coupon $[S_{t_1} - K]^+$

To price: $\mathbb{E}_0[D(t_0,t_1)(S(t_1)-K)^+ \mathbb{1}_{\tau>t_1}] + \pi\, e(t_0,t_1,\dots)$

Simulate u to get τ

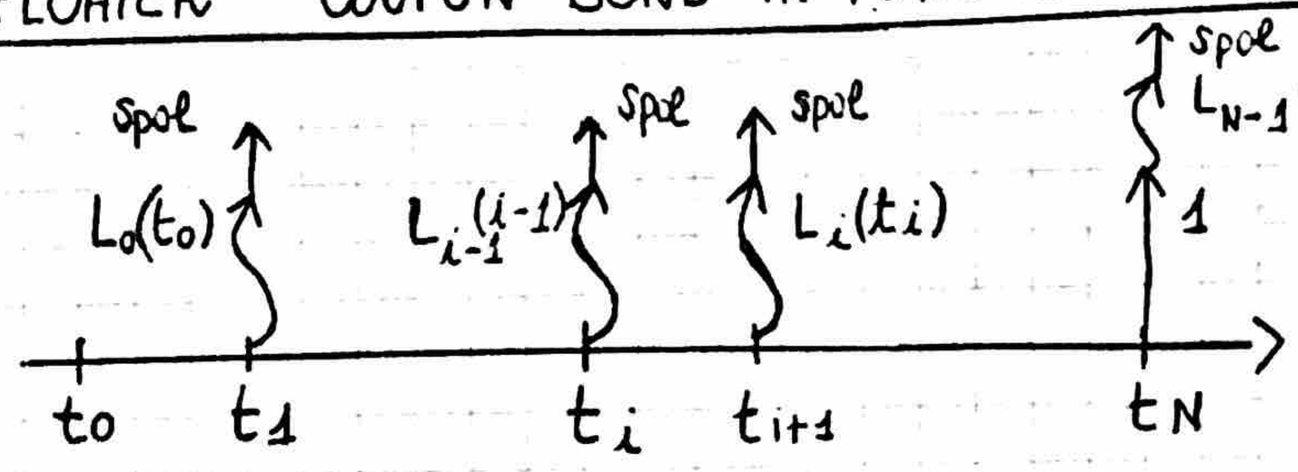
$$\begin{cases} \tau > t_1 \rightarrow \text{only discount back} \\ \tau < t_1 \rightarrow \text{introduce } \pi \end{cases}$$

Is this method correct? Consob uses it.

4. A FLOATER IN FUNCTION OF THE BUILDING BLOCKS.

Hypotesis in the single name models.

"FLOATER" COUPON BOND IN FUNCTION OF THE BUILDING BLOCKS



An issuer has to pay s spol because there is risk of default. It's constant, indeed it is a characteristic of the contract, decided at the beginning: I have to be sure of what I get, otherwise I won't buy it.

$$\overline{C}(0) = \underbrace{\text{part NO default}}_{\text{I}} + \underbrace{\text{part default}}_{\text{II}}$$

I NO DEFAULT

$$\mathbb{E}_0\Big[\big(L(\overbrace{t_{-1},t_1}^{\text{all life}})+s\big)\,\delta(t_{-1},t_1)\,D(t_0,\overset{\text{paid}}{t_1})\,\mathbb{1}_{\tau>t_1}\Big] +$$

$$+\,\mathbb{E}_0\Big[\big(L(t_{i-1},t_i)+s\big)\,\delta(t_{i-1},t_i)\,D(t_0,t_i)\,\mathbb{1}_{\tau>t_i}\Big]+\dots$$

$$=\big(L(t_{-1},t_1)+s\big)\,\delta(t_{-1},t_1)\,\overline{B}_0(t_0,t_1)\,+$$

$$+\,\delta(t_{i-1},t_i)\,\underline{\mathbb{E}_0}\Big[\big(L(t_{i-1},t_i)+\underline{s}\big)\,\underline{D(t_0,t_i)}\Big]\cdot$$

$$\cdot\underbrace{\mathbb{E}_0\big[\mathbb{1}_{\tau>t_i}\big]}_{P(t_0,t_i)}\quad+\dots=$$

$B(t_0,t_i)s$

libor rate that will be known in t_i

$$=\Big(\mathbb{E}_0\big[L(t_{i-1},t_i)\cdot D(t_0,t_i)\big]$$

$\|$ FRA

rate that appears in the FRA, known in t_0

$$B(t_0,t_i)\cdot L(t_0;t_{i-1},t_i)\Big) =$$

forward rates from forward rate agreement, known at value time.

$$=\sum_{i=1}^{n}\delta(t_{i-1},t_i)\Big[L(t_0;t_{i-1},t_i)+s\Big]\underbrace{B(t_0,t_i)\cdot P(t_0,t_i)}_{\overline{B}_0(t_0,t_i)}$$

fra gains is known $= L(t_{-1},t_1)$

$$\oplus\ \overline{B}_0(t_0,t_N)$$

II DEFAULT

if default happens in $(t_i, t_{i+1}]$

$$\pi \sum_{i=1}^{N} \mathbb{E}_0 \left[D(t_0, t_{i+1}) \left(\mathbb{1}_{\tau > t_i} - \mathbb{1}_{\tau > t_{i+1}} \right) \right] +$$

$$+ \pi \sum_{i=1}^{N} \frac{\delta(t_{i-1}, t_i)}{2} \mathbb{E}_0 \left[D(t_0, t_{i+1}) \left(\mathbb{1}_{\tau > t_i} - \mathbb{1}_{\tau > t_{i+1}} \right) \cdot \right.$$

accrual term, part to be payed before $\tau \in (t_i, t_{i+1}]$

$$\left. \cdot \left(L(t_{i-1}, t_i) + s \right) \right] =$$

$$= \pi \sum_{i=1}^{N} e(t_0; t_{i-1}, t_i) + \pi \sum_{i=1}^{N} e(t_0; t_i, t_{i+1}) \left(L(t_0; t_i, t_{i+1}) + s \right) \cdot \frac{\delta(t_{i-1}, t_i)}{2}$$

$$\bar{C}(0) = \sum_{m=1}^{N} \delta(t_{m-1}, t_m) \left(L(t_0; t_{m-1}, t_m) + s \right) \bar{B}_0(t_0, t_m) + \bar{B}_0(t_0, t_N) +$$

$$+ \pi \sum_{m=1}^{N} e(t_0; t_{m-1}, t_m) + \pi \sum_{m=1}^{N} \cancel{\frac{\delta(t_{m-1}, t_m)}{2} \left(L(t_0; t_{m-1}, t_m) + s \right) \cdot e(t_0; t_{m-1}, t_m)}$$

... generally traded at par at issue date.

recovery value for a company is fixed by the mkt: it only can change if business plans of the bank changes.

CREDIT PRODUCTS	
→ single name	– BONDS : floater & fixed coupon – CREDIT DERIVATIVES : ASW, CDS

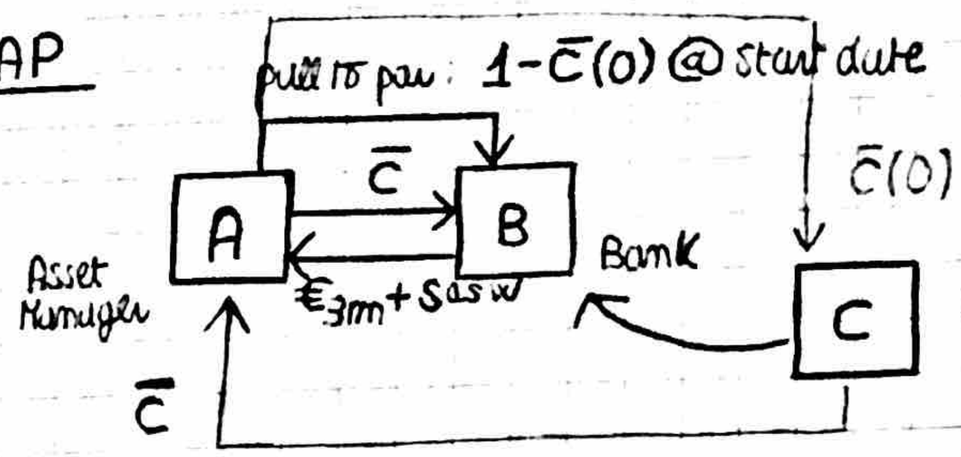
floater doesn't exist → ASW

No s on floater coupon!

5. ASW : DESCRIPTION
ASW SPREAD IN FUNCTION OF THE BUILDING BLOCKS.

ASSET SWAP

Derivative contract : transform fixed in floater



Whatever happens to the obligor C, nothing happens to the contract. C no more involved.

The Asset Manager, that works for a fund and manages money, wants to invest, for example, in ENI:

he wants a ENI floater 5Y (that doesn't exist)
→ he has to resort to an ASW package.

In the market, ∃ a fixed Coupon Bond: he can enter in an ASW Package.

pays $\bar{C}(0)$ (<1) for a fixed CB and with the amount left he enters a swap.

ASW Package { FCB $\bar{C}(0)$ given price (C)
ASW (A,B) (SWAP derivative)
↳ replicates exactly the floater. }

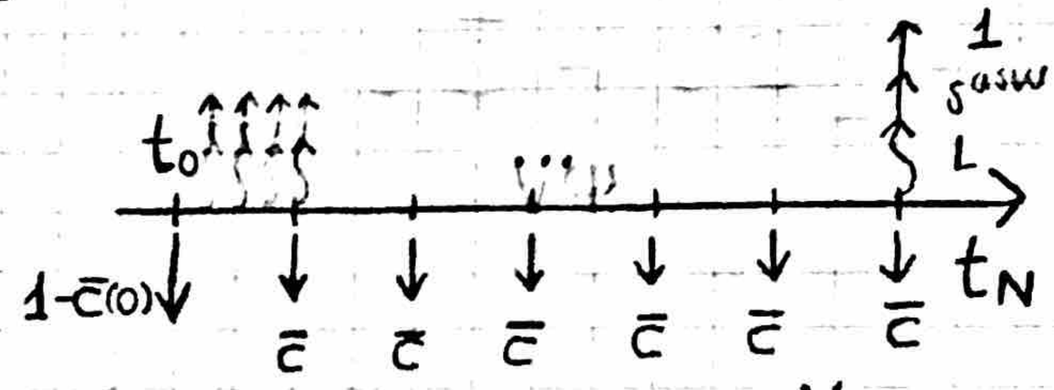
@ Start date : $-(\underbrace{1-\bar{C}(0)}_{\text{banca}}) - \underbrace{\bar{C}(0)}_{\text{obligor}} = 1$

In to, A pays 1

@t_i : A receives $€3m + S^{asw}$ and pays $\bar{C}$ — It replicates a floater if no default happens.

@t_N : A receives $€3m + S^{asw} + 1$ del FCB

(A) SIDE



Price the ASW NPV = 0.

$$NPV^{fixed} = 1 - \bar{C}(0) + \bar{C}\sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i)$$

(!) We use B, not $\bar{B}$, because there is no counterpart risk!
([illegible] not to be bisogni di fare $E[D\mathbb{1}_{\tau>}]$)

NO counterparty risk
↓
NO $\bar{B}$ (defaultable ZC bond and a recovery)

$$NPV^{float} = \sum_{i=1}^{N^{\delta}} \delta(t_{i-1}^{\delta}, t_i^{\delta})\, \mathbb{E}_0\Big[\big(L(t_{i-1}^{\delta}, t_i^{\delta}) + S^{asw}\big) \cdot D(t_0, t_i^{\delta})\Big] =$$

no default but I use $\mathbb{E}[\,\cdot\,]$, not known L!

stochastic, so I need $\mathbb{E}[L \cdot D]$ but not defaulted!

FRA

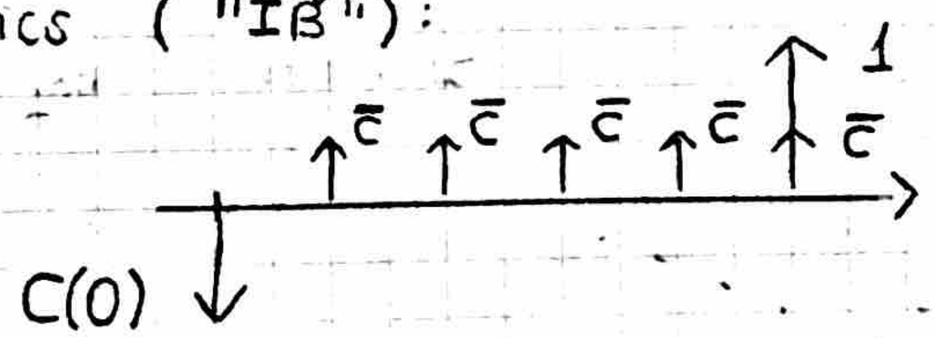
$$= \sum_{i=1}^{N^{\delta}} \delta(t_{i-1}^{\delta}, t_i^{\delta})\big(L(t_0, t_{i-1}^{\delta}, t_i^{\delta}) + S^{asw}\big)\, B(t_0, t_i^{\delta}) =$$

$$= 1 - B(t_0, t_N) + S^{asw} \cdot \underbrace{\sum_{i=1}^{N^{\delta}} \delta(t_{i-1}^{\delta}, t_i^{\delta})\, B(t_0, t_i^{\delta})}_{BPV^{\delta\ell}(t_0)}$$

$$\Downarrow$$

$$S^{asw} = \frac{\cancel{1} - \bar{C}(0) + \bar{c} \cdot \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i) \cancel{-1} + B(t_0, t_N)}{BPV^{\delta\ell}(t_0)}$$

If we consider a not-defaultable bond with the same characteristics ("IB"):



risk free bond - idealized bond.

$$C(0) = \bar{c} \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i) + B(t_0, t_N)$$

è questo defaultable, NO CE sul C!

$$\Rightarrow \boxed{S^{asw} = \frac{C(0) - \bar{C}(0)}{BPV^{\delta\ell}(t_0)}}$$

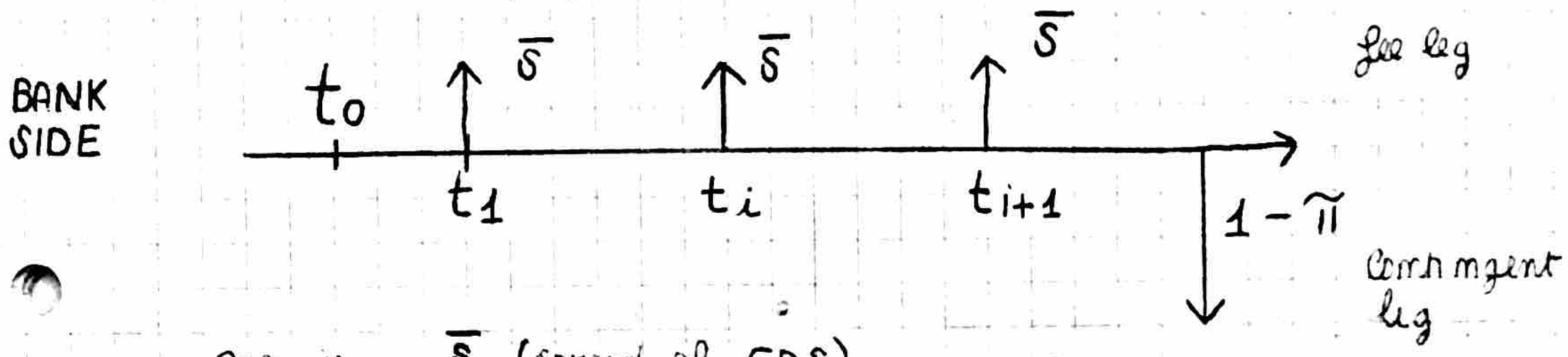
6. CDS : DESCRIPTION
CDS SPREAD IN FUNCTION OF THE BUILDING BLOCKS.

CREDIT DEFAULT SWAP : Credit derivative in single name models.

Again two counterparties : A, B

B will receive some flows $\bar{S}$ and, in case of default of C, will pay $1-\tilde{\pi}$: insurance contract.

Insurance contract between A and B to cover a possible default of C.



Premium $\bar{S}$ (spread of CDS)
Loss Given Default (LGD) $= 1-\tilde{\pi}$.

no default, default occurs after t_m.

$$\underbrace{NPV^{\text{fee leg}}}_{\substack{\text{NO DEFAULT}\\+\\\text{DEFAULT}}} = \sum_{m=1}^{N} \delta(t_{m-1},t_m)\,\mathbb{E}\Big[\bar{S}\cdot \underbrace{D(t_0,t_m)}_{B(t_0,t_m)}\cdot \underbrace{\mathbb{1}_{\tau>t_m}}_{P(t_0,t_m)}\Big] +$$

$\bar{B}_0(t_0,t_m)$

$$+\ \bar{S}\sum_{m=1}^{N} \cancel{\frac{\delta(t_{m-1},t_m)}{2}}\ \underbrace{e(t_0;t_{m-1},t_m)}_{\mathbb{E}[D(t_0,t_m)(\mathbb{1}_{\tau>t_{m-1}}-\mathbb{1}_{\tau>t_m})]}$$

negligible compared to $1-\tilde{\pi}$

default in $(t_{m-1},t_m]$: I pay the accrual.

$$NPV^{\text{contingent}} = (1-\tilde{\pi})\sum_{m=1}^{N} \mathbb{E}\big[D(t_0,t_{m+1})[\mathbb{1}_{\tau>t_m}-\mathbb{1}_{\tau>t_{m+1}}]\big] =$$

$$= (1-\tilde{\pi})\sum_{m=1}^{N} e(t_0;t_{m-1},t_m)$$

$$\Rightarrow \boxed{\bar{S}\sum_{m=1}^{N}\delta(t_{m-1},t_m)\,\bar{B}_0(t_0,t_m) = (1-\tilde{\pi})\sum_{m=1}^{N} e(t_0;t_{m-1},t_m)}$$

7. RELATION BETWEEN SPOL, CDS SPREAD, ASW SPREAD.

MAIN RELATIONS BETWEEN THE THREE SPREADS

In the market, we have the SPOL (spread over libor), spread of a floater.
but there exist three spreads, that are correlated:

- $\bar{S} = S^{spol}$
- $\bar{S} \sim S^{asw}$ if $\bar{C}(0) \sim 1$

$\boxed{\bar{S} = S^{spol}}$

SINGLE NAME → only 1 obligor, 1 firm

- **PORTFOLIO I**: buy N_Y FLOATER OF YY + CDS PROTECTION ON YY
- **PORTFOLIO II**: buy "IB" FLOATER → NO Spol

NO DEFAULT	NO DEFAULT
t_0: $\overset{\text{pay the floater}}{-1} \; \overset{\text{CDS}}{-0}$	t_0: -1
t_i: $\overset{\text{receive flows of floater}}{\left(L(t_{i-1},t_i)+S^{spol}\right)\delta} \; \overset{\text{pay CDS's}}{-\bar{S}\delta}$	t_i: $\delta\left(L(t_{i-1},t_i)\right)$
t_N: $1+\delta\{L(t_{N-1},t_N)+S^{spol}-\bar{S}\}$	t_N: $1+\delta L(t_{N-1},t_N)$
DEFAULT in $\tau = t_i^+$	DEFAULT in $\tau = t_i^+$
t_i^+: $\overset{\text{floater}}{\pi} + \overset{\text{CDS}}{(1-\pi)} = 1$	t_i^+: 1 (sell it, "IB" floater is always valued 1) if YY defaults, I sell the floater.

The PORTFOLIO I (risk-free) is equivalent to a risk-free floater and this happens when $\bar{S} = S^{spol}$.

$\boxed{\bar{S} \sim S^{asw}}$

- PORTFOLIO I : ASP OF YY $\rightarrow$ { FIXED CB, ASW }
  +
  CDS PROTECTION ON YY

- PORTFOLIO II : "IB" FLOATER

NO DEFAULT

t_0: $-\overset{bond}{\bar{C}(0)} - (\overset{ASW}{1-\bar{C}(0)}) - \overset{CDS}{0} = -1$

t_i: $\delta(\overset{ASW}{L(t_{i-1}, t_i)} + S^{asw})$
$- \underset{CDS}{\delta\bar{S}} + \underset{BOND}{\delta[\bar{C}} - \underset{ASW}{\bar{C}]}$

t_N: $\delta(L(t_{-1}, t) + S^{asw})$
$- \delta\bar{S} + \underbrace{1}$
from the fixed coupon bond, we receive the face value.

DEFAULT in $\tau = t_i^+$

t_i^+: $\underset{BOND}{\pi} + \underset{CDS}{(1-\pi)} + \underbrace{NPV(\tau)}$
NPV of my ASW, flows would continue after the default: I sell at fair value.

NO DEFAULT

t_0: -1

t_i: $\delta(L(t_{i-1}, t_i))$

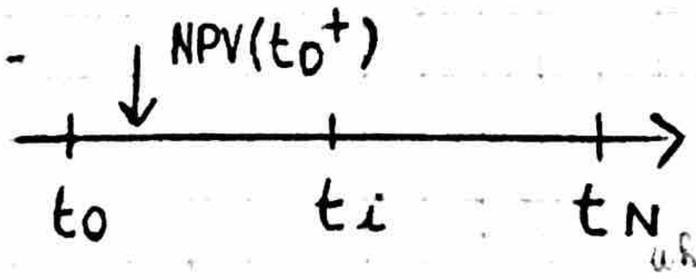
t_N: $1 + \delta L(t_{N-1}, t_N)$

DEFAULT in $\tau = t_i^+$

t_i^+: 1

This situation looks very similar to the one before if with now $\bar{S} = S^{asw}$

NPV ~ 0 :

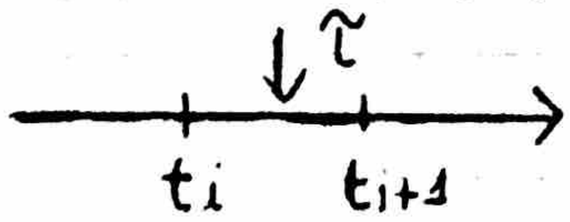


$t_0^+ \rightarrow$ already paid $1 - \bar{C}(0)$ (in t_0)

NPV can be neglected when compared to the other flows.

what we have just paid in t_0 to invest to buy ASW

$NPV(t_0^+) = 1 - \bar{C}(0) \sim 0$ if $\bar{C}(0) \sim 1$



$\mathbb{E}[NPV(\tau) : \tau \in (t_i, t_{i+1}]]$ = independence between rates and defaults

$= \mathbb{E}_0[NPV(\tau)] \, (P(t_0, t_i) - P(t_0, t_{i+1}))$

8. FALLING ANGEL. RELATION BETWEEN CDS SPREAD AND ASW SPREAD IN THIS CASE.

- $\bar{S} > S^{asw}$ if $\bar{C}(0) \ll 1$

We have already seen that, if $\bar{C}(0) \sim 1$, $\bar{S} \sim S^{asw}$.
Instead, if $\bar{C}(0) \ll 1 \Rightarrow$ <u>FALLING ANGEL.</u> (bond that was once considered a good investment is now not worth.)

EXAMPLE: France Telecom was a company considered with no risk, sure (angel) (own by France state)
At the end of the 90's, it went through an IPO (Initial Public Offer) and it became a quoted company.
With that money, it bought ORANGE, the first operator of UK mobile, second in Germany.
The valuation of this company went high.

In March 2001, BUBBLE: re-evaluation of DOT-COM company, quality of credit deteriorated.
DOT-COM companies collapse.

Coupon bond sold at 14.15 ≪ 100!

→ Close to default.

Rates ↓ → pay fix, receive float NPV < 0
Rates ↑ → NPV > 0

↓

NPV huge, can't be neglected

France Telecom theoretically (→ it's defaulting!) wants to issue a new bond, with same maturity and same payment dates.

NEW ISSUE AT PAR $\boxed{\bar{C}'(0) \sim 1}$ $\overset{\text{I should have much larger coupons}}{\Longrightarrow}$ $\bar{C}' \gg \bar{C}$

$\Downarrow$

$\boxed{S^{asw'} \sim \bar{S}}$

$$S^{asw'} - S^{asw} = \frac{C'(0) - \bar{C}'(0)}{BPV^{\delta}(t_0)} - \frac{C(0) - \bar{C}(0)}{BPV^{\delta}(t_0)} =$$

$$= \frac{\overset{(1)}{C'(0) - C(0)} - \overset{(2)}{[\bar{C}'(0) - \bar{C}(0)]}}{BPV^{\delta}(t_0)}$$

Risk-free bond: No default $\rightarrow x \sum_{i=1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i)$

could default, same FCB with same maturity. $x \sum_{i=1}^{N} \delta(t_{i-1}, t_i) \bar{B}(t_0, t_i)$

difference between coupons $\chi := \bar{C}' - \bar{C} \gg 0$ if $\bar{C}' \gg \bar{C}$

high probability of default

$$(1) \gg (2) \quad \left(B(t_0, t_i) \gg \bar{B}_0(t_0, t_i) \right)$$

$$\Downarrow$$

$$S^{asw'} - S^{asw} \gg 0$$

$$\bar{S} \sim S^{asw'} \gg S^{asw} \Longrightarrow \bar{S} > S^{asw}$$

SINGLE NAME

9. FIRM VALUE MODELS. WHY PREFERRING FVM vs. INTENSITY MODELS? MERTON MODEL.

(INTENSITY BASED MODELS → DETERMINISTIC (D.13) → / → STOCHASTIC)

equity → dynamic for call/put
IB → dynamic for swaptions
for DERIVATIVES

STOCHASTIC INTENSITY BASED MODELS

The deterministic models are not enough for OPTIONS ON CDS: we need the dynamic of λ.

⇒ CDS swaptions (not traded anymore after the financial crisis).

We will MODEL THE INTENSITY, considering a dynamic for instantaneous intensity $\lambda(t)$.

not necessary to be too precise

⇒ COX PROCESS (ONE DIMENSIONAL: as ~~less~~ as we can since LARGE BID-ASK → only 1 risk factor)
we can be more rough than with IR

$$d\lambda(t) = \mu_\lambda(t)\,dt + \sigma_\lambda(t)\,dW_\lambda(t) \qquad \text{BM}$$

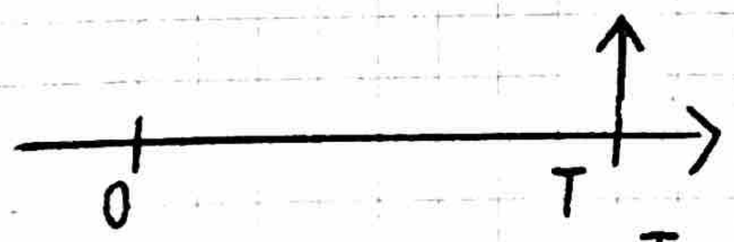
s.t. $\lambda(t) > 0$

and, conditional on $\lambda(t)$, $N(t)$ is a inhomogeneous Poisson process with intensity $\lambda(t)$.

λ stoch; indipendence between rates and defaults is not valid in general: it is valid only given a particular path of $\lambda(t)$.

NO indipendence between intensity and rates (both are based on a b.m. and they may be correlated).

BASIC PROPERTIES



$$B(0,T) \equiv \mathbb{E}\left[e^{-\int_0^T r_s\,ds} \,\middle|\, \mathcal{F}_0 \right]$$

I can't immediately use indipendence

$$\bar{B}_0(0,T) \equiv \mathbb{E}\left[D(0,T)\, \mathbb{1}_{\tau > T} \,\middle|\, \mathcal{F}_0 \right] =$$

$$= \mathbb{E}_0 \left[\mathbb{E} \left[e^{-\int_0^T \kappa_s ds} \mathbb{1}_{\tau > T} \mid \lambda(t) \right] \right] =$$

$$= \mathbb{E}_0 \left[\mathbb{E} \left[e^{-\int_0^T \kappa_s ds} \mid \lambda(t) \right] \mathbb{E} \left[e^{-\int_0^T \lambda_s ds} \mid \lambda(t) \right] \right] =$$

$$= \mathbb{E}_0 \left[\mathbb{E} \left[e^{-\int_0^T (\kappa_s + \lambda_s) ds} \mid \underbrace{\{\lambda(t)\}_{t \in (0,T)}}_{\mathcal{F}_0^N} \right] \right]$$

$$\Rightarrow \bar{B}_0(0,T) = \mathbb{E} \left[e^{-\int_0^T (\kappa_s + \lambda_s) ds} \mid \mathcal{F}_0^N \right]$$

→ Advantages of the <u>INTENSITY BASED MODELS</u> :

- direct <u>calibration</u> to <u>credit mkt spreads</u> (via bootstrap, JT, ...)
- <u>simple simulation</u> (to price very complicated products)
- it is possible to see impact of mkt risks (e.g. effects of IR correlations)

→ Disadvantages of the INTENSITY BASED MODELS :

We have simplified our financial problem (firm could default)

$$\begin{cases} 0 & \text{Business As Usual} \\ 1 & \text{Default} \end{cases}$$

only two situations

Problem to connect EQUITY and DEBT INSTRUMENTS
↳ prob. default 1 equity / debt
(dopo $V_t > D$!)

FIRM VALUE MODELS

Motivations :
- link equity ↔ debt instruments.
- pricing $\underbrace{\text{convertible bonds}}$

  bonds where the issuer gives an additional right : convert the bond into SHARES of the firm in a certain period. ⇒ OPTION ON THE CREDIT.

  convert amount in equity.
- corporate finance questions (strategic defaults, capital structure optimization)

  ↓ value of the firm gets DOWN, lower than the debt → firm may ask a RESTRUCTURING PLAN with strategic defaults question (put some money again in the firm)

  FIRM VALUE $V(t)$ { EQUITY, DEBT

  analising corporate finance issues.

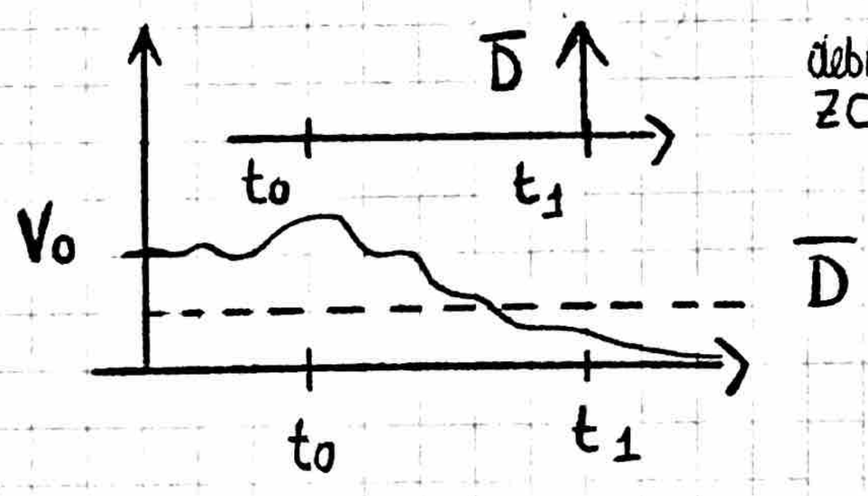
FIRM VALUE DYNAMIC

$$\begin{cases} dV_t = V_t \mu \, dt + V_t \sigma_V \, dW_t \\ V_0 > 0 \end{cases}$$

one dimensional GBM.

LEASING ?

MODEL THE FIRM VALUE & THE BOND



debt modelled as ZC bond at final date.

Controllo a scadenza perché ho modellato il debito come un bond alla data finale.

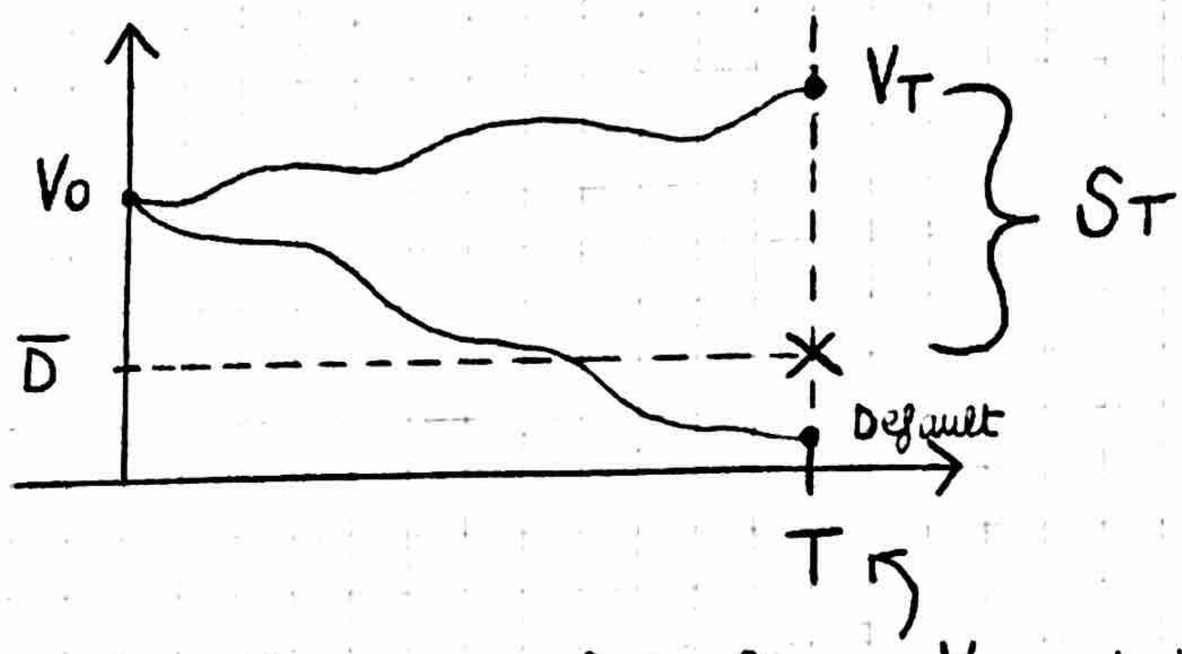
LEASING : I need a tool → I can buy it, I can lend it and, at the end, I can buy it back.
(es. Uni rent Leasing)

Financially, it is a LOAN, indeed the leg is €3m + Spread.

The difference is legal : if you don't pay with leasing, the tool goes back to the owner IMMEDIATELY (in loan, there could be legal consequences).

default trigger :
- value @ maturity less than debt $\bar{D}$: $V(T) < \bar{D}$ (Merton model)
- ∃ safety covenants : $V(t) < K(t)$ (Black-Cox model)

MERTON MODEL (1973)



We check the value of the firm V_T at the END T.

In the model, we can intervene only at the maturity T of the debt.

If the value of the firm is less than the debt, there is a default:

$V_T \leq \bar{D} \Rightarrow$ DEFAULT $\rightarrow$ equity valued 0.

$\Rightarrow$ <u>condition at maturity</u> is:

$$S(T) = \max\{V(T) - \bar{D}, 0\} \quad \text{EQUITY}$$

Considering deterministic IR:

$$S_0 = \mathbb{E}[B(0,T)\{V_T - \bar{D}\}^+]$$

V_t has g.b.m. dynamic

<u>EQUITY IS A CALL ON THE VALUE OF THE FIRM</u> (!)

<u>BASIC DYNAMIC</u>:

$$\begin{cases} dV_t = V_t(\mu dt + \sigma_V dW_t) \\ V_{t=0} = V_0 \\ \mu_t = \mu \end{cases}$$

$$d\ln V_t = \frac{1}{V_t} dV_t - \frac{1}{2}\cdot\frac{1}{V_t^2}\langle dV_t^2\rangle =$$

$\langle dV_t^2\rangle \rightarrow V_t^2\sigma_V^2 dt$

$$= (\mu dt + \sigma_V dW_t) - \frac{1}{2}\sigma_V^2 dt =$$

$$= \left(\mu - \frac{\sigma_V^2}{2}\right)dt + \sigma_V dW_t$$

$$\int_{t_0}^{T} \Rightarrow \ln\frac{V_T}{V_{t_0}} = \left(\mu - \frac{\sigma_V^2}{2}\right)(T - t_0) + \sigma_V W_T$$

$$\Rightarrow \quad V_T = V_0 \exp\left\{ \left(\mu - \frac{\sigma_V^2}{2}\right)(T-t_0) + \sigma_V \, dW_T \right\}$$

$$\Rightarrow \quad S(t_0) = B(t_0,T)\, \mathbb{E}_0 \left[(V_T - \bar{D})^+ \right]$$

the underlying is the value of the firm.

<u>SOLUTION</u>

$$\begin{cases} S(V_t, t) = C^{BS}(V_t, t; T, \bar{D}, \sigma_V) \\ \bar{B}(V_t, t) = V_t - S(V_t, t) \end{cases}$$

$V > D \rightarrow D$
$V < D \rightarrow V$

at each time, value firm = equity + debt

↳ value of the debt today, taking into account that the firm could default.

C^{BS} : Black & Scholes Call solution

So:
- we model the equity / share as a Call.
- we model the debt as a bond

Merton model done in 1973 but, since it didn't have a calibration, it wasn't used.

In the 90's a model for calibration : KMV.

Introduce anche Merton Model (!).

10. FIRM VALUE MODELS. KMV CALIBRATION OF MERTON MODEL.

MERTON MODEL : CALIBRATION (KMV APPROACH)

Vasicek

It's almost impossible to infer firm value V from balance sheet. It's even more difficult to detect the volatility of the firm value.

σ_V ?

V_0 ?

Ingredients of Merton model :

- → σ_V
- → T (estimated through duration)
- → V_0
- → $\bar{D}$ (from TOTAL DEBT of the firm, from balance sheet)

K_t deterministic

produced once a year (31/12), published in April

→ delay problem
→ MATERIAL ASSETS (warehouse, magazzino) : estimation can be difficult!

The company is quoted in the derivative market.

From the stock exchange, I can get :

- value of the stock S_0 share value
- implied volatility of the stock σ_S for some options of the firm in its mind

I need 2 equations, to connect S_0, σ_S with V_0, σ_V:

$$\begin{cases} S = C^{BS}(V, T, \bar{D}, \sigma_V) \\ dS_t = \dots dt + \underbrace{S_t \sigma_S}\, dW_t \end{cases}$$

already known

we have to obtain this second equation

I consider $S_t = f(V_t, t, \dots)$.

Knowing the dynamic of V_t : $dV_t = V_t(r dt + \sigma_V dW_t)$,

we study the dynamic of S_t :

$$dS_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial V_t}(V_t r dt + V_t \sigma_V dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial V_t^2} V_t^2 \sigma_V^2 dt =$$

$$= \{\dots\} dt + \frac{\partial f}{\partial V_t} V_t \sigma_V dW_t$$

$$\Rightarrow dS_t = \{\dots\} dt + \frac{\partial C^{BS}}{\partial V_t} V_t \sigma_V dW_t$$

$$\Rightarrow \quad dS_t = \{\dots\}dt + S_t \sigma_S dW_t \qquad \Rightarrow \quad S_t \sigma_S = V_t \sigma_V \frac{\partial C^{BS}}{\partial V_t}$$

$$dS_t = \{\dots\}dt + \frac{\partial C^{BS}}{\partial V_t} V_t \sigma_V dW_t$$

Set of equations for V, σ_V:

osservazione alli debiti

$$\begin{cases} S = C^{BS}(V; T, \bar{D}, \sigma_V) \\ S\sigma_S = V\sigma_V \dfrac{\partial C^{BS}}{\partial V_t} \end{cases}$$

11. FIRM VALUE MODELS. OTHER CALIBRATION OF MERTON MODEL

introduco ancora Merton model, spiego brevemente l'altra calibrazione e perché si passa a questa.

In case we don't have the information above (not present in the market, firm not quoted), how can I calibrate the Merton model?

$$V_T = V_0 \exp\left\{\left(r - \frac{\sigma_V^2}{2}\right)(T-t_0) + \sigma_V W_T\right\} \qquad W_T \sim N(0, T-t_0)$$

$$\overset{\text{"law"}}{\sim} V_0 \exp\left\{\left(r - \frac{\sigma_V^2}{2}\right)(T-t_0) + \sigma_V \sqrt{T-t_0}\, g\right\} \qquad g \sim N(0,1)$$

Merton condition for default $\rightarrow$ $V_T \leq \bar{D}$

$$\Rightarrow \quad e^{\left(r - \frac{\sigma_V^2}{2}\right)(T-t_0) + \sigma_V g \sqrt{T-t_0}} \leq \frac{\bar{D}}{V_0}$$

$$\Rightarrow \quad g \leq \underbrace{\left\{\ln \frac{\bar{D}}{V_0} - \left(r - \frac{\sigma_V^2}{2}\right)(T-t_0)\right\} \cdot \frac{1}{\sigma_V \sqrt{T-t_0}}}_{K^{eff}}$$

cumulate distribution function

$$\mathbb{E}_0[\mathbb{1}_{V_T \leq \bar{D}}] = N(K^{eff}) \quad \Rightarrow \quad K^{eff} = N^{-1}(p)$$

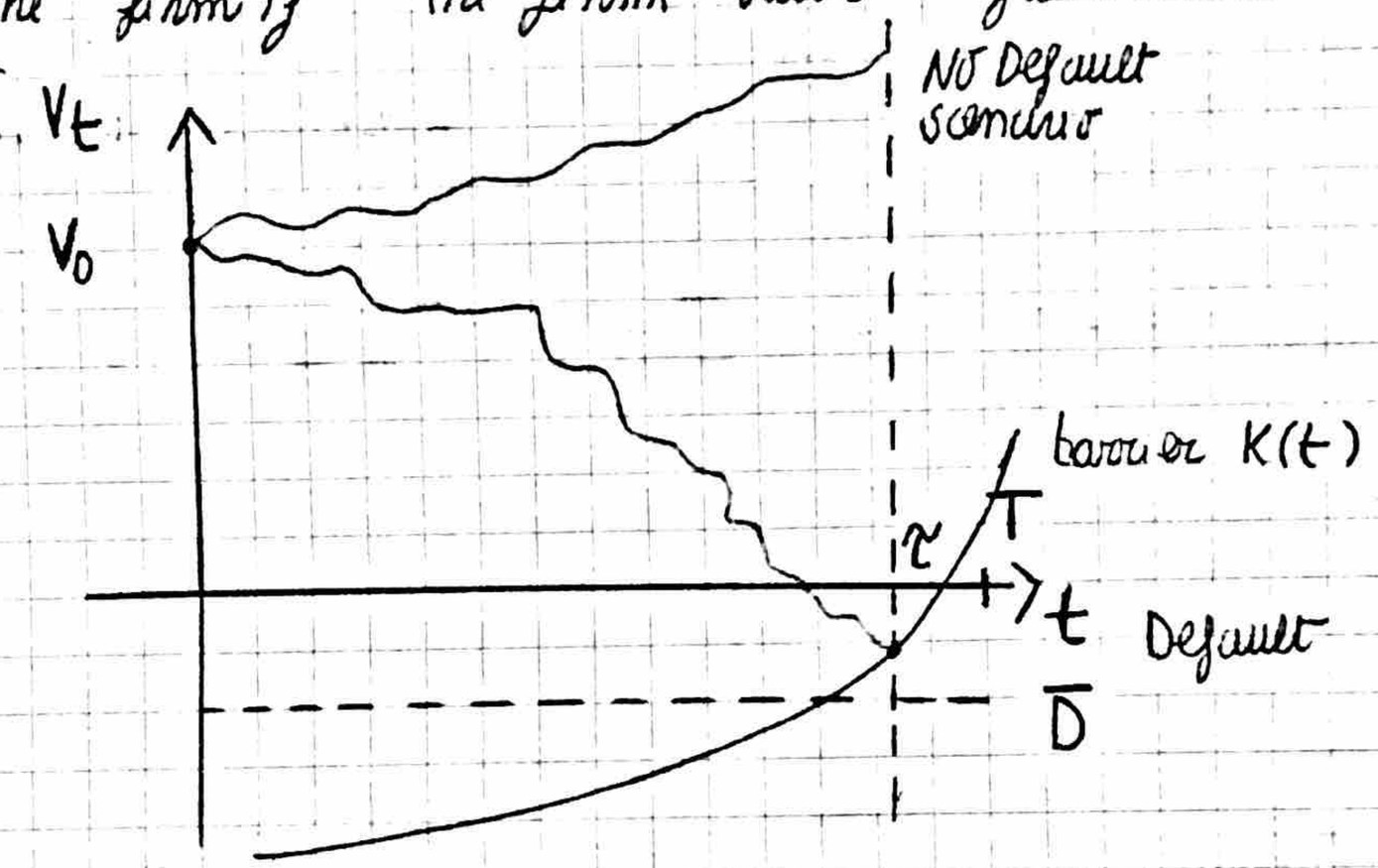
is the only parameter I need to calibrate the model

In general, average probability of default p of the obligor is an input (credit scoring)

12. FIRMVALUE MODELS.
BLACK-COX MODEL [deduction with a constant barrier].
EMPIRICAL MARKET EVIDENCE ON THE SHAPE OF CREDIT SPREADS AND LIMITS OF THE APPROACH.

BLACK-COX MODEL

Generally, in loan agreements, there will be some COVENANTS, i.e. constraints, indications of how things are managed: measured as ability to make liquidity
the bank verifies the value of the firm V_t continuously, not just at the expiry of the loan, so that creditors can liquidate the firm if the firm value falls below a THRESHOLD.



Barrier built by the bank to control the value of the firm: default happens when we reach the barrier → the bank asks for the liquidation of the company, and it will get the recovery of the liquidation

Default occurs as soon as:

$$\tau = \inf\{t \geq 0 \mid V(t) \leq K(t)\}$$

The bond is equal to:

$$\bar{B}(t,T) = \begin{cases} \bar{D}\,B(t,T) & \text{NO DEFAULT} \\ \bar{D}\,B(t,T)\,\tilde{\pi} & \text{DEFAULT} \end{cases}$$

value of the debt

quello che posso recuperare in caso di default

Firm value dynamic is Merton's one.

How to chose the barrier ?

The simplest case is a constant threshold :

$$K(t) = \overline{K}$$

But there are other cases in which one can find closed formulas :

- $K(t) = B(t,T)\,\overline{D}$
- Stoch. IR (Gaussian HJM) → impact of correlations Rates <–> defaults.

CONSTANT BARRIER : $\boxed{K(t) = \overline{K}}$ and CONSTANT IR r

$$\overline{B}(t,T) = \overline{D}\,B(t,T)\,P(t,T) + \pi\,\overline{D}\,B(t,T)\,[1 - P(t,T)] =$$

$$= \overline{D}\,B(t,T)\,\{(1-\pi)\,P(t,T) + \pi\}$$

where $P(t,T) = \mathbb{E}_t[\mathbb{1}_{\tau > T}] =$

$$= N\left[\frac{\Delta(t)}{\sigma\sqrt{T-t}} + \frac{\mu}{2}\sigma\sqrt{T-t}\right] - e^{-\mu\Delta(t)} \cdot N\left[\frac{-\Delta(t)}{\sigma\sqrt{T-t}} + \frac{\mu}{2}\sigma\sqrt{T-t}\right]$$

$\Lambda(t) = -\dfrac{\ln P(0,T)}{T}$ spread in the zero recovery case

$\mu = \dfrac{2r}{\sigma^2} - 1$; $\Delta(t) = \ln \dfrac{V(t)}{K}$

PROOF OF BLACK-COX APPROACH FOR FIRM VALUE

Deduction of the Black-Cox survival probability in the case of a constant barrier using the REFLECTION PRINCIPLE.

Black-Cox is the natural extension of Merton firm-value model when one wants to describe the effects of safety covenants.

The underlying firm value V_t, with $t \in [0,T]$, is modelled as in Merton model with a Geometric-Brownian-Motion and the covenants as a continuous comparison with a barrier.

One of the cases that can be solved analytically is when the barrier assumes a constant value $\bar{K}$.

Firm value dynamic: $\begin{cases} dV_t = V_t\,[\mu\,dt - \sigma\,dW_t] \\ V_0 \end{cases}$ μ, σ constant in $\mathbb{R}^+$

$W_{t_0} = W_0 = 0$.

$$V_t = V_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t - \sigma W_t\right\}$$

Survival condition: $V_t > \bar{K} \quad \forall t \in [0,T]$

$$\Leftrightarrow \left(\mu - \frac{\sigma^2}{2}\right)t - \sigma W_t > \ln\frac{\bar{K}}{V_0}$$

$$\Leftrightarrow W_t - \underbrace{\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)}_{:=\eta}t < \underbrace{\frac{\ln V_0/\bar{K}}{\sigma}}_{:=a}$$

$$-\mu\frac{\sigma}{2}t = -\frac{\sigma}{2}\underbrace{\left(\frac{2\mu}{\sigma^2} - 1\right)}_{:=\mu}t$$

$$\boxed{\tilde{W}_t := W_t + \eta t < a \quad \forall t \in [0,T] \quad \text{SURVIVAL CONDITION}}$$

$a \to \Delta/\sigma$

drifted brownian motion

DEF. Max & First-Passage-Time

$$X_T := \max_{t\in[0,T]} W_t \quad \& \quad \tau_a := \inf\{t \in \mathbb{R}^+ : W_t = a\}$$

$$W_t < a \ \forall t \leftrightarrow X_T < a \leftrightarrow \tau_\alpha > T \text{ (no default)}$$

$$X_T > a \Leftrightarrow \tau_\alpha < T$$

process without drift

$W_t < a$

It is of particular interest the joint probability $\boxed{\mathbb{P}(X_T > a,\ W_T \leq b)}$ and the associated density $f_{X_T, W_T}(a,b)$

Clearly interesting when $b \in (-\infty, a]$ because $X_T \geq W_T$.

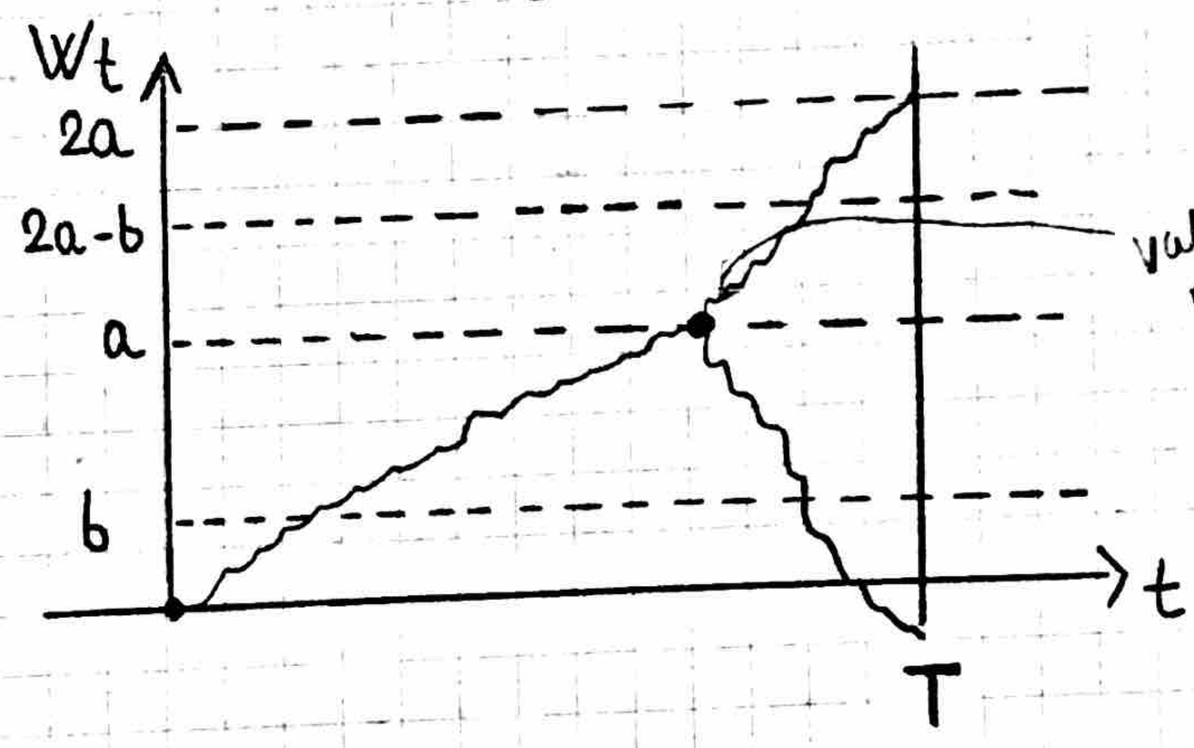
REFLECTION PRINCIPLE

$$\mathbb{P}(W_T > \overset{a}{2a} - b,\ \overbrace{\tau_\alpha < T}^{X_T > a}) = \mathbb{P}(W_T < b,\ \overset{X_T > a}{\tau_\alpha < T})$$

$a \in \mathbb{R}^+$
$b \in (-\infty, a]$

$2a - b \geq a$ for $b \in (-\infty, a]$

$b < a$ since $W_t \leq X_t$



valore ho raggiunto a mio suo percorso, quindi raggiungere 2a-b o b è uguale (distanza = a-b)

LEMMA The joint probability density is:

$$f_{X_T, W_T}(a,b) = \sqrt{\frac{2}{\pi T}}\ \frac{2a-b}{T}\ e^{-\frac{(2a-b)^2}{2T}}\ \mathbb{1}_{a \geq b \vee 0}$$

for $a \in \mathbb{R}^+$, $b \in (-\infty, a]$ it's enough $W_T > 2a - b$ since $X_T > 2a - b$ when $W_T > 2a - b$ (X_T is the max W_T)

PROOF.

$$\mathbb{P}(W_T < b,\ X_T > a) = \mathbb{P}(W_T < b,\ \overset{X_T > a}{\tau_\alpha < T}) =$$

$$= \mathbb{P}(W_T > 2a - b,\ \overset{X_T > 0}{\tau_\alpha < T}) = \mathbb{P}(\underbrace{W_T}_{g\sqrt{T}} > 2a - b) =$$

non serve $X_T > \alpha$

$$= \int_{(2a-b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}\ dx =$$

$$= \int_{(2a-b)}^{+\infty} \frac{1}{\sqrt{2\pi T}}\ e^{-x^2/2T}\ dx$$

$$\frac{\partial}{\partial a}\left(\int_{(2a-b)}^{+\infty} \frac{dx}{\sqrt{2\pi T}}\, e^{-x^2/2T}\right) = \frac{\partial}{\partial a}\left(\dots - F(2a-b)\right) =$$

$$= \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{(2a-b)^2}{2T}} \cdot (-2) \qquad = -\frac{\partial}{\partial a}\left(F(2a-b)\right) = -F'(2a-b)\cdot 2$$

$$-\frac{\partial}{\partial b}\left(-2\cdot\frac{1}{\sqrt{2\pi T}}\, e^{-\frac{(2a-b)^2}{2T}}\right) =$$

$$= -\left(-\sqrt{\frac{2}{\pi T}}\, e^{-\frac{(2a-b)^2}{2T}} \cdot -\frac{\cancel{2}}{\cancel{2}T}(b-2a)\right) =$$

$$= \sqrt{\frac{2}{\pi T}}\, \frac{2a-b}{T}\, e^{-\frac{(2a-b)^2}{2T}}\, \mathbb{1}_{a>b \vee 0}$$

$$f(X_T = a, W_T = b) = -\frac{\partial}{\partial b}\frac{\partial}{\partial a}\, \mathbb{P}\left(W_T > 2a-b,\ X_T > a\right) \blacksquare$$

$(\leq \leq) \to (+)$

ma ho $\mathbb{P}(W_T < b, X_T > a)$

Now, let's consider the drifted Brownian Motion and the associated maximum

$$\tilde{X}_T := \max_{t\in[0,T]} \tilde{W}_t$$

drifted process

In order to deduce the survival probability we use the Girsanov Theorem

We call with $\tilde{P}$ the measure where $\tilde{W}_t$ is a Brownian motion

The Radon Nikodym derivative between the two measures $\tilde{P}$ and P is:

$$\frac{d\tilde{P}}{dP} = e^{-\eta W_T - \eta^2 T/2} = e^{-\eta \tilde{W}_T + \eta^2 T/2}$$

$$\vdots$$

$$= e^{-\frac{1}{2}\eta^2 T - \eta(\tilde{W}_T - \eta T)}$$

PROP. The cumulative probability distribution is:

$$\mathbb{P}(\tilde{X}_T < a) = N\left(\frac{a-\eta T}{\sqrt{T}}\right) - e^{2\eta a} N\left(\frac{-a-\eta T}{\sqrt{T}}\right)$$

PROOF. The joint probability density for $b \leq a$ is:

$$\mathbb{P}(\tilde{X}_T < a, \tilde{W}_T < b) = \mathbb{E}[\mathbb{1}\{\tilde{X}_T < a, \tilde{W}_T < b\}] = \mathbb{E}\left[\cancel{dP} \cdot \frac{d\tilde{P}}{\cancel{dP}} \cdot \frac{dP}{d\tilde{P}}\right]$$

Girsanov $\rightarrow$ $= \tilde{\mathbb{E}}[e^{+\eta \tilde{W}_T - \eta^2 T/2} \mathbb{1}\{\tilde{X}_T < a, \tilde{W}_T < b\}] =$

Lemma $\rightarrow$ $= \sqrt{\frac{2}{\pi T}} \int_0^a dx \int_{-\infty}^b dy \; e^{-\frac{\eta^2}{2}T + \eta y} \; \frac{2x-y}{T} \cdot e^{-\frac{(2x-y)^2}{2T}} \mathbb{1}_{x > y \vee 0}$.

In the particular case $b = a$, we get:

$$\mathbb{P}(\tilde{X}_T < a) = \mathbb{P}(\tilde{X}_T < a, \tilde{W}_T < a) = \int_{-\infty}^a \overbrace{dy \, e^{-\frac{\eta^2}{2}T + \eta y}}^{W} \sqrt{\frac{2}{\pi T}} \int_0^a \overbrace{dx \, \frac{2x-y}{T} e^{-\frac{(2x-y)^2}{T}}}^{X} \cdot \cancel{\mathbb{1}_{x > y \vee 0}} =$$

$$= \int_{-\infty}^a dy \, e^{-\frac{\eta^2}{2}T + \eta y} \sqrt{\frac{2}{\pi T}} \underbrace{\int_{y \vee 0}^a dx \, \frac{2x-y}{T} e^{-\frac{(2x-y)^2}{2T}}}_{(*)} =$$

$$= \sqrt{\frac{2}{\pi T}} \int_{-\infty}^a dy \int_{y \vee 0}^a dx \, \frac{2x-y}{T} e^{-\frac{(2x-y)^2}{2T} + \eta y - \frac{\eta^2 T}{2}}$$

Because

$$(*) = \int_{y \vee 0}^a dx \, \frac{2x-y}{T} e^{-\frac{(2x-y)^2}{2T}} = \left(\begin{array}{l} \tilde{x} = \frac{2x-y}{\sqrt{T}} \\ d\tilde{x} = \frac{2}{\sqrt{T}} dx \end{array} \right) =$$

$$= \frac{1}{2} \int_{|y|/\sqrt{T}}^{(2a-y)/\sqrt{T}} d\tilde{x} \, \tilde{x} \, e^{-\tilde{x}^2/2} = \frac{1}{2}\left[e^{-\frac{y^2}{2T}} - e^{-\frac{(2a-y)^2}{2T}}\right]$$

one gets

$$\mathbb{P}(\tilde{X}_T < a) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a} dy \; e^{-\frac{1}{2T}[y^2 - 2T\eta y + \eta^2 T^2]} +$$

$$- \int_{-\infty}^{a} dy \; e^{-\frac{1}{2T}[(2a-y)^2 - 2T\eta y + \eta^2 T^2]} =$$

$$= \frac{1}{\sqrt{2\pi T}} \left\{ \int_{-\infty}^{a} dy \; e^{-\frac{1}{2T}[y - \eta T]^2} - \int_{-\infty}^{a} dy \; e^{-\frac{1}{2T}\underbrace{[(2a-y)^2 - 2T\eta y + \eta^2 T^2]}} \right\}$$

$$= -\frac{1}{2}[(2a-y)^2 - 2\eta T(y-2a) + \eta^2 T^2] + 2a\eta T$$

$$\left(\begin{aligned} y' &= \frac{y - \eta T}{\sqrt{T}} \\ dy' &= \frac{1}{\sqrt{T}} dy \end{aligned} \right) \qquad \left(\begin{aligned} y' &= \frac{y - 2a - \eta T}{\sqrt{T}} \\ dy' &= \frac{1}{\sqrt{T}} dy \end{aligned} \right)$$

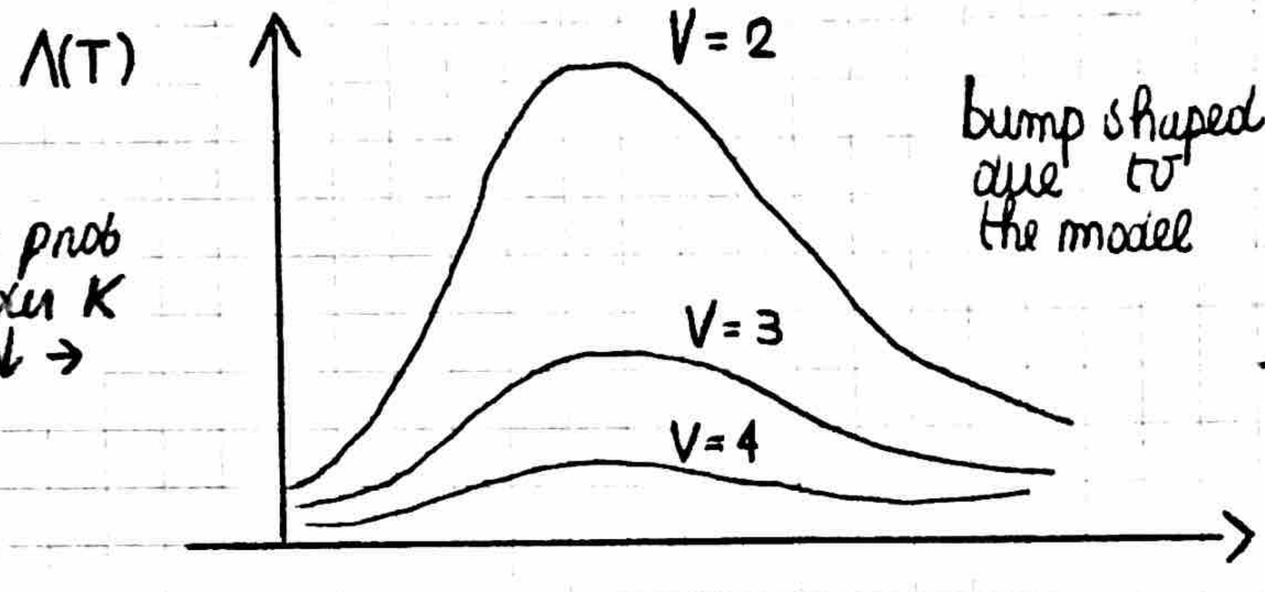
$$= \int_{-\infty}^{\frac{a-\eta T}{\sqrt{T}}} \frac{dy'}{\sqrt{2\pi}} e^{-y'^2/2} - e^{-\frac{1}{2T}[-2\eta T 2a]} \int_{-\infty}^{\frac{a-2a-\eta T}{\sqrt{T}}} \frac{dy'}{\sqrt{2\pi}} e^{-y'^2/2}$$

$$\Longrightarrow \mathbb{P}(\tilde{X}_T < a) = N\left(\frac{a - \eta T}{\sqrt{T}}\right) - e^{2\eta a} N\left(\frac{-a - \eta T}{\sqrt{T}}\right) \blacksquare$$

SHAPE OF CREDIT SPREADS

$\Lambda_T = \frac{\bar{S}}{1-\pi} \Rightarrow \Lambda_T = \bar{S}$

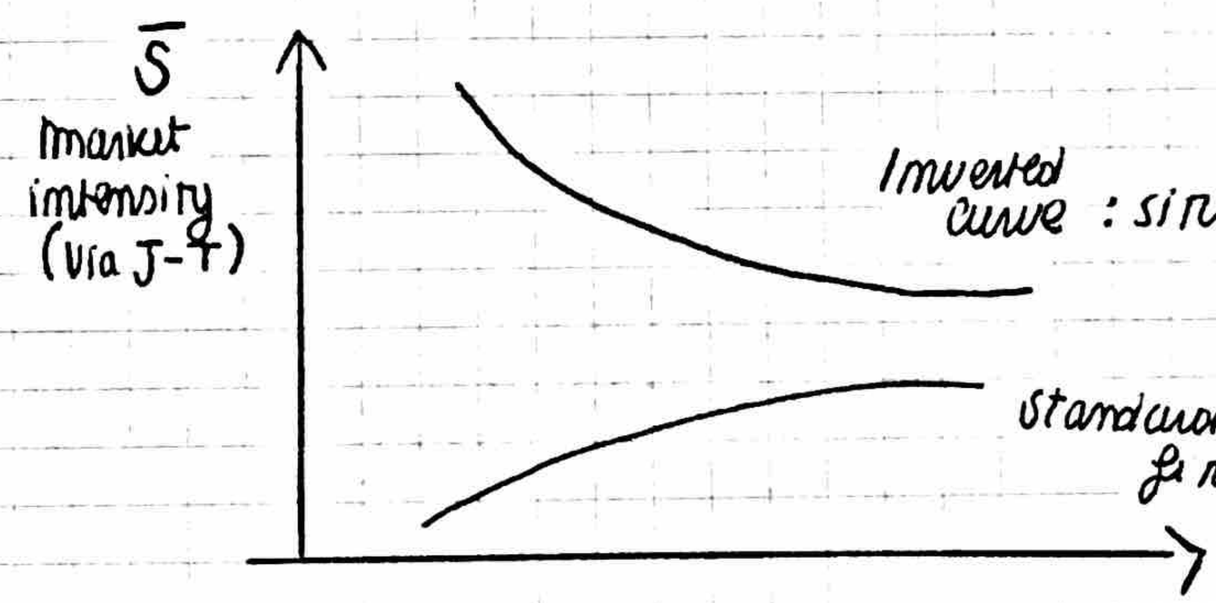
$\Lambda(T) = -\frac{1}{T} \ln P(0,T)$ in the zero recovery case $(\mu = 2, \bar{K} = 1)$



bump shaped due to the model

$V\uparrow \rightarrow$ less prob to go under K ($V\uparrow \rightarrow \Lambda\downarrow \rightarrow P\uparrow$)

→ Se iniziamo lontani da K, P molto alta → limite del modello.



In the market spreads, there is not a bump shape!

Due to empirical market evidence on the shape of credit spreads → limits of the approach.

↓

disadvantage of FVM.

13. (DETERMINISTIC) INTENSITY BASED MODEL: INHOMOGENEOUS POISSON PROCESS.

MODEL FOR DEFAULT ARRIVAL RISK

The following assumption describes the way in which <u>default arrival risk</u> is modelled in all <u>intensity-based default risk models</u>:

INTENSITY MODEL DEFAULT ARRIVALS

Let $N(t)$ be a COUNTING PROCESS (non-decreasing, integer-valued process $N(t)$ with $N(0)=0$) with (possibly stochastic) intensity $\lambda(t)$.

The time of default τ is the time of the first jump of N, i.e.

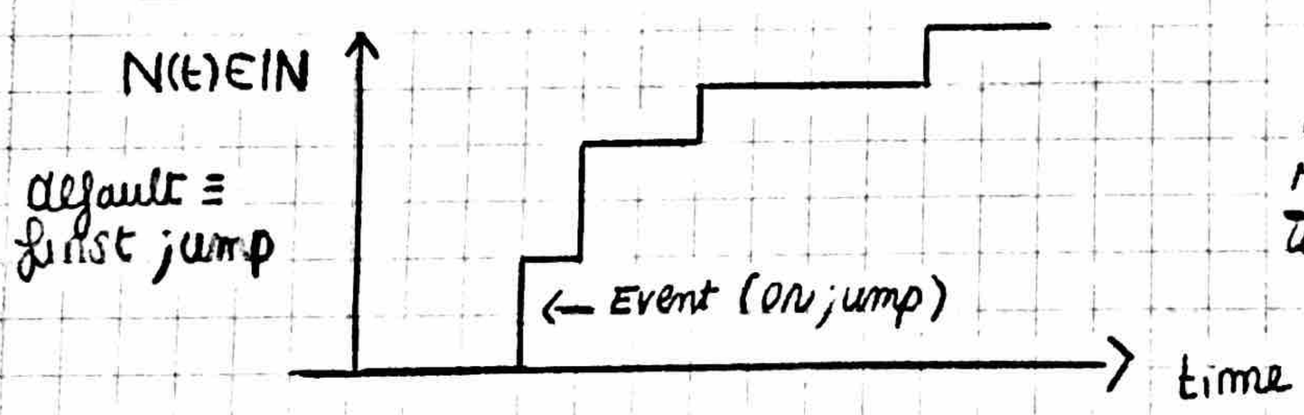
$$\tau = \inf\{t \in \mathbb{R}_+ \mid N(t) > 0\}$$

TIME OF THE FIRST JUMP OF A POISSON PROCESS.

The survival probabilities in this setup are given by:

$$P(0,T) = \mathbb{P}[N(T) = 0 \mid \mathcal{F}_0]$$

CONSTRUCTION OF POISSON PROCESS



* Credit spread curves are not flat in reality, although a constant-intensity Poisson process implies this. In order to reach a more realistic shape of the spread curve, we must allow the intensity to change over time.

1) $P[N(t+\Delta t) - N(t) = 1] = \lambda(t)\Delta t$

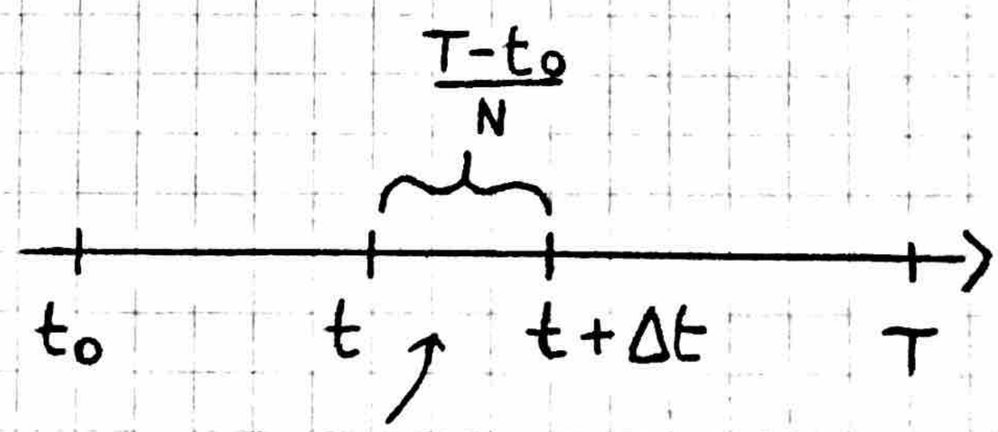
↖ event in $(t, t+\Delta t]$

2) $P(\quad) \leftarrow$ ✱ What has happened before t

(jumps in disjoint time intervals happen independently of each other)

POISSON PROCESS has no memory

* To construct a <u>inhomogeneous Poisson process</u> →



In each of these subintervals the process N has a jump with probability $\Delta t\,\lambda(t)$ deterministic function of time.

Poisson process: defaults are rare (hopefully) and they are discrete.

main property:

$N(t)$ purely discontinuous process $\rightarrow$ zero covariation with continuous martingales (e.g. rates)

PROBABILITY OF NO JUMP in $(t_i, t_{i+1}] = 1 - \lambda(t_i)\Delta t$

PROBABILITY OF NO JUMP in $(0, T]$:

$$\text{Prob}(N(T)=0) = \prod_{i=1}^{N} (1 - \lambda(t_i)\Delta t) =$$

Δt very small: first order expansion $\hookrightarrow e^{-\lambda(t_i)\Delta t}$

$$= \prod_{i=1}^{N} e^{-\lambda(t_i)\Delta t + o(\Delta t^2)} = e^{-\sum_{i=1}^{N} \lambda(t_i)\Delta t} \xrightarrow{\Delta t \to 0} \exp\left\{ -\int_0^T \lambda(t)\,dt \right\}$$

... OTHER BUILDING BLOCKS.

SIMULATION OF $\tilde{\tau}$ (<-D3)

introduce DET. IBM-IPP, $P(t,T) = e^{-\int_0^T \lambda(s)\,ds}$

14. JARROW-TURNBULL RELATION BETWEEN CDS SPREAD, RECOVERY AND INTENSITY.

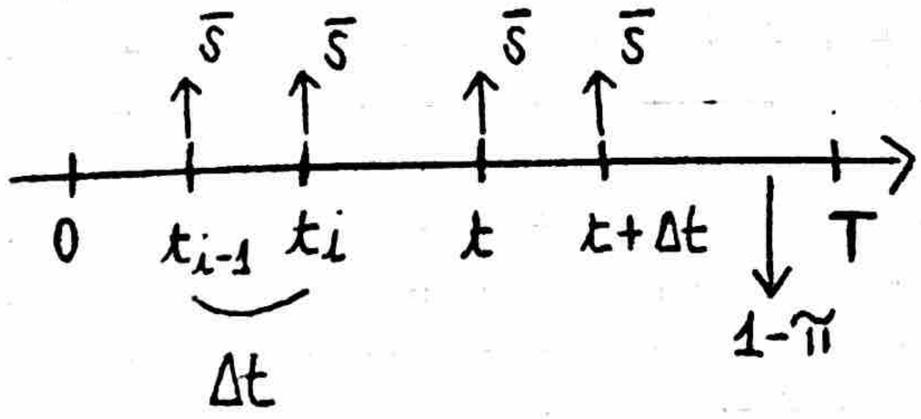
CALIBRATION OF CREDIT SPREAD CURVE in a simplified example (flat λ, $\bar{s}$ paid continuously)

(1995): JARROW & TUMBULL in the case of a very simple CDS:

- fee leg received continuously (very small yf)
- λ constant

NO ACCRUAL TERM, because the yf is very small but we have exactly the fee and the contingent legs as before.

mathematical and technical assumption in order to totally avoid accrual



$$NPV^{fee}(t_0) = \bar{s} \sum_{i=1}^{N} \Delta t \, \mathbb{E}_0[D(t_0, t_i) \mathbb{1}_{\tau > t_i}] =$$

$$= \bar{s} \sum_{i=1}^{N} \Delta t \, \bar{B}_0(t_0, t_i)$$

$$NPV^{contingent}(t_0) = (1-\pi) \sum_{i=1}^{N} \mathbb{E}_0[D(t_0, t_i)(\mathbb{1}_{\tau > t_{i-1}} - \mathbb{1}_{\tau > t_i})] =$$

$$= (1-\pi) \sum_{i=1}^{N} B(t_0, t_i)\left(P(t_0, t_i - \Delta t) - P(t_0, t_i)\right) =$$

$$= (1-\pi) \sum_{i=1}^{N} B(t_0, t_i)\left(e^{-\lambda(t_i - \Delta t)} - e^{-\lambda t_i}\right) =$$

$$= (1-\pi) \sum_{i=1}^{N} B(t_0, t_i) e^{-\lambda t_i}\left(e^{\lambda \Delta t} - 1\right) =$$

$$= (1-\pi) \sum_{i=1}^{N} B(t_0, t_i) P(t_0, t_i) \underbrace{\left(e^{\lambda \Delta t} - 1\right)}_{\lambda \Delta t \quad \Delta t \to 0}$$

$$\Rightarrow NPV^{contingent} = (1-\pi)\lambda \sum_{i=1}^{N} \Delta t \, \underbrace{B(t_0, t_i) P(t_0, t_i)}_{\bar{B}_0(t_0, t_i)}$$

$$NPV^{fee} = NPV^{contingent} \Rightarrow \bar{s} = \lambda(1-\pi)$$

from mkt we get $\bar{s}$, $1-\pi$

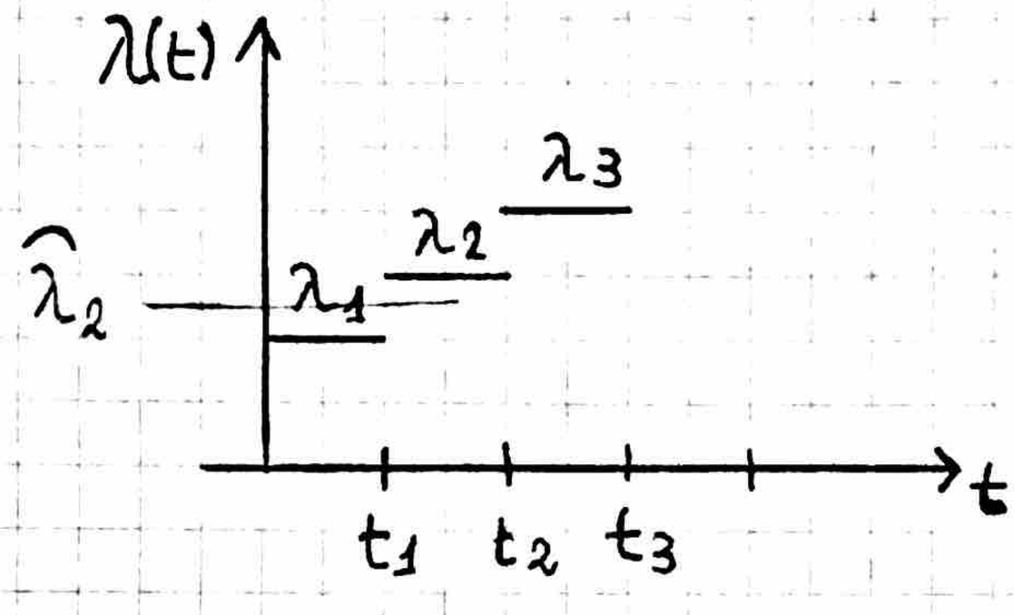
$$\Rightarrow \boxed{\lambda = \frac{\bar{s}}{1-\pi}}$$

calibration for an intensity based (deterministic) model

15. CALIBRATING CREDIT SPREAD CURVE: BOOTSTRAP VS. JT APPROACH.

In reality, CDS are not paid continuously and λ is not constant.

In practice:



PIECEWISE CONSTANT
↓
market info come each year ⇝ the value changes every year.

In order to find $\bar{S}_{1Y}$, $\bar{S}_{2Y}$, ... I need bootstrap.

For a CDS, we computed:

$$\bar{S} = \frac{(1-\pi)\sum_{i=1}^{N} e(t_0; t_{i-1}, t_i)}{\sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, \bar{B}_0(t_0, t_i)}$$

negleting the accrual

that, rewritten in function of the building blocks, becomes:

$$\bar{S} = \frac{(1-\pi)\sum_{i=1}^{N} B(t_0, t_i)\left[P(t_0, t_{i-1}) - P(t_0, t_i)\right]}{\sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, B(t_0, t_i)\, P(t_0, t_i)}$$

→ For $\bar{S}_{1Y}$, the only unknown variable is $P(t_0, t_1)$, that is a linear function of other things:

$$P(t_0, t_1) = e^{-\lambda_1 \underbrace{(t_1 - t_0)}_{\text{ACT/365}}}$$

$\bar{S}$ → Quoted Mid Market Value

⇒ λ_1

→ For $\bar{S}2y$: $P(t_0, t_2)$ unknown.

no dynamics

$$P(t_0, t_2) = P(t_0; t_0, t_2) = P(t_0; t_0, t_1) \cdot \underline{P(t_0; t_1, t_2)} =$$

$$= \underline{P(t_1, t_2)} \cdot P(t_0, t_1) =$$

$$= e^{-\lambda_1(t_1 - t_0) - \lambda_2(t_2 - t_1)} \qquad P(t_0, t_m) = e^{-\sum_{i=1}^{m} \lambda_i \delta(t_{i-1}, t_i)}$$

$\Rightarrow \lambda_2$

...

CDS BOOTSTRAP

Considering a CDS, given the discount factors $B(t_0, t_i)$, we want to evaluate values of survival probabilities $P(t_0, t_i)$ and intensities $\lambda(t_i)$.

Neglecting the accrual factor

for each year $i \in [1, N]$:

SIMPLIFIED $\bar{S}$

$$* \quad \bar{S}_i \sum_{j=1}^{i} \overset{30/360}{\delta(t_{j-1}, t_j)} B(t_0, t_j) P(t_0, t_j) =$$

$$= (1 - \pi) \sum_{j=1}^{i} B(t_0, t_j) [P(t_0, t_{j-1}) - P(t_0, t_j)]$$

Once computed the survival probability of interest $P(t_0, t_i)$, we can get the value of the corresponding intensity:

$$P(t_0, t_i) = \exp\left\{ -\sum_{j=1}^{i} \lambda(t_j) \delta(t_{j-1}, t_j) \right\}$$

$$\Rightarrow \lambda(t_i) = -\frac{1}{\underbrace{\delta(t_{i-1}, t_i)}_{ACT/365}} \ln\left(\frac{P(t_0, t_i)}{P(t_0, t_{i-1})} \right)$$

Considering the accrual factor

We get $P(t_0, t_i)$ from the previous formula, only adding to the left of * :

EXACT $\bar{S}$

$$\bar{S}_i \sum_{j=1}^{i} \frac{\delta(t_{j-1}, t_j)}{2} B(t_0, t_j) [P(t_0, t_{j-1}) - P(t_0, t_j)]$$

and we compute $\lambda(t_i)$ as did before.

ASSIGNMENT 3 → accrual factor really negligible since the difference between P is 10^{-5} (gr. mkt quantities)

<u>Jarrow - Turnbull approximation</u> (λ const. + continuously paid CDS spread) → not true in mkt yearly!

$$\lambda(t_i)^{JT} = \frac{\bar{S}_i}{1-\pi}$$

good approximation with few computations

MOST USED RULE OF THUMB FOR INTENSITIES.

$\lambda_1 \sim \lambda_1^{JT}$

... year by year, the values obtained with J-T are less precise w.r.t. the ones found without J-T. J-T intensities at year i turn out to be - as we expected - approximately the average of the ones from t_1 to t_i previously obtained without this approximation.

We should have bootstrapped from bond, but the problem is that the dates of the bond are not regular: not always annual, depends on the mkt → I need to interpolate.

MULTI NAME

16. STRUCTURING OF AN ABS.

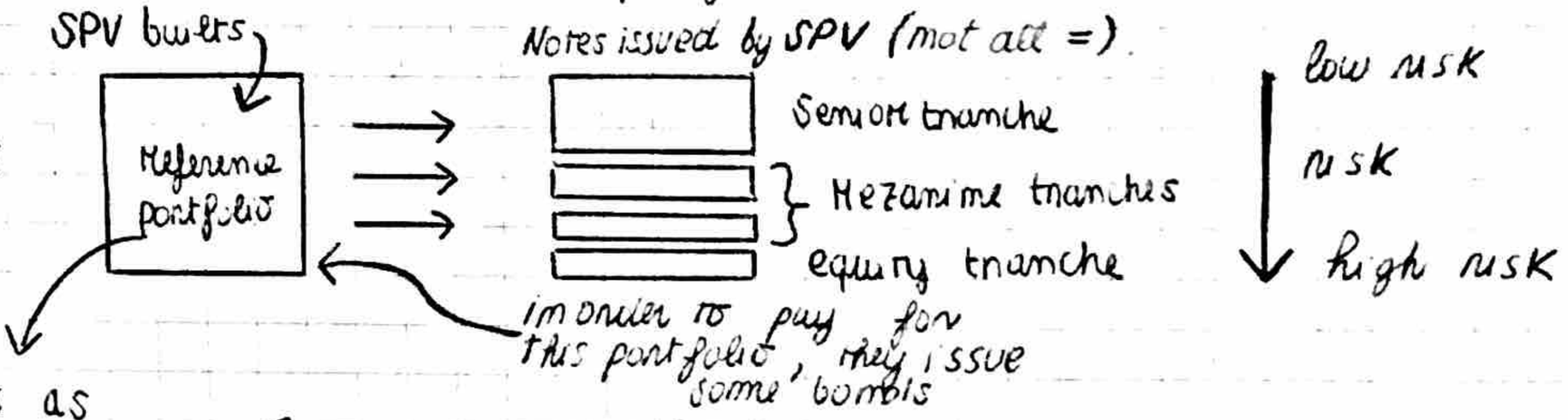
CARTOLARIZZAZIONE: un'azienda con un portfolio crediti li cede alla SPV che emetterà obbligazioni per ripagare i crediti comprati.
ABS: strumento finanziario che paga cedole a scadenze prefissate → una società sbilancia dal bilancio dei crediti, li impacchetta, li cede grazie a SPV

Let's introduce a new player:

SPV = Special Purpose Vehicle

This is an "Automatic" Company.

WATERFALL STRUCTURE



- Assets as mortgages: ABS (Asset Backed Security) MBS
- Bonds: CDO (Collateralized Debt Obligation) (in portfolio, bonds or loans)

SPV notes have different names depending on the r.p.: ABS (mortgage loans), CDO (bonds)

SPV buys a r.p. of CREDIT ASSETS a 1, in order to pay it, issues some NOTES (bonds related to a r.p.)

BASIC STRUCTURING:

→ Reference portfolio (r.p.) is transferred to a SPV (ASSET SIDE)

→ SPV issues notes dividend in different tranches (LIABILITY SIDE)

→ r.p. income goes toward paying tranches' coupons according to their seniority.

→ at maturity, r.p. is liquidated and proceeds distributing to the tranches according to their seniority

→ if a default occurs, recovery payments are reinvested in default-free securities.

TRANCHE BASIC CHARACTERISTICS

→ subordination (K_d): amount of losses a portfolio can suffer before tranche's notional is eroded.

→ tranche cumulative loss $L_{tranche}(t) = \min\{\max[L_{rp}(t) - K_d, 0], K_u - K_d\}$

SPV buys a portfolio, with differents notes, divided into different tranches.

The income goes toward paying tranches' coupons according to their seniority.

If someone cannot pay, default occurs. The SPV tries to get back the money. If it doesn't happen, the SPV takes the guarantee (ex. the house for a mortgage) and puts it on the mkt.

ABS: strumenti finanziari, emessi a seguito di operazioni di cartolarizzazione, del tutto simili alle normali obbligazioni; come queste, pagano al detentore una serie di cedole a scadenze prefissate per un ammontare determinato sulla base di tassi di interesse fissi o variabili.

CARTOLARIZZAZIONE: azienda che ha in ptf crediti, li cede ad una **SPV** che emette obbligazioni per ripagare acquisto dei crediti. Crediti sono garanzia del pagamento delle obbligazioni emesse. (ammontare crediti > valore nominale obbligazioni emesse, a beneficio rischio).

"SECURITISATION" = creazione di un **ABS**

↓

Società scorpora dal suo bilancio una serie di crediti, li "impacchetta" adeguatamente e li cede sul mercato, assieme ai flussi finanziari che essi generano, per le mani ne della SPV con l'obiettivo di generare liquidità

SMOBILIZZARE CREDITI CON SECURITISATION

ES. BANCA
- Che dispone di MUTUI
- Che necessita di LIQUIDITÀ

→ CREA SPV a cui conferisce crediti e le garanzie date dai clienti sulle ipoteche relative ai mutui

↓ il quale

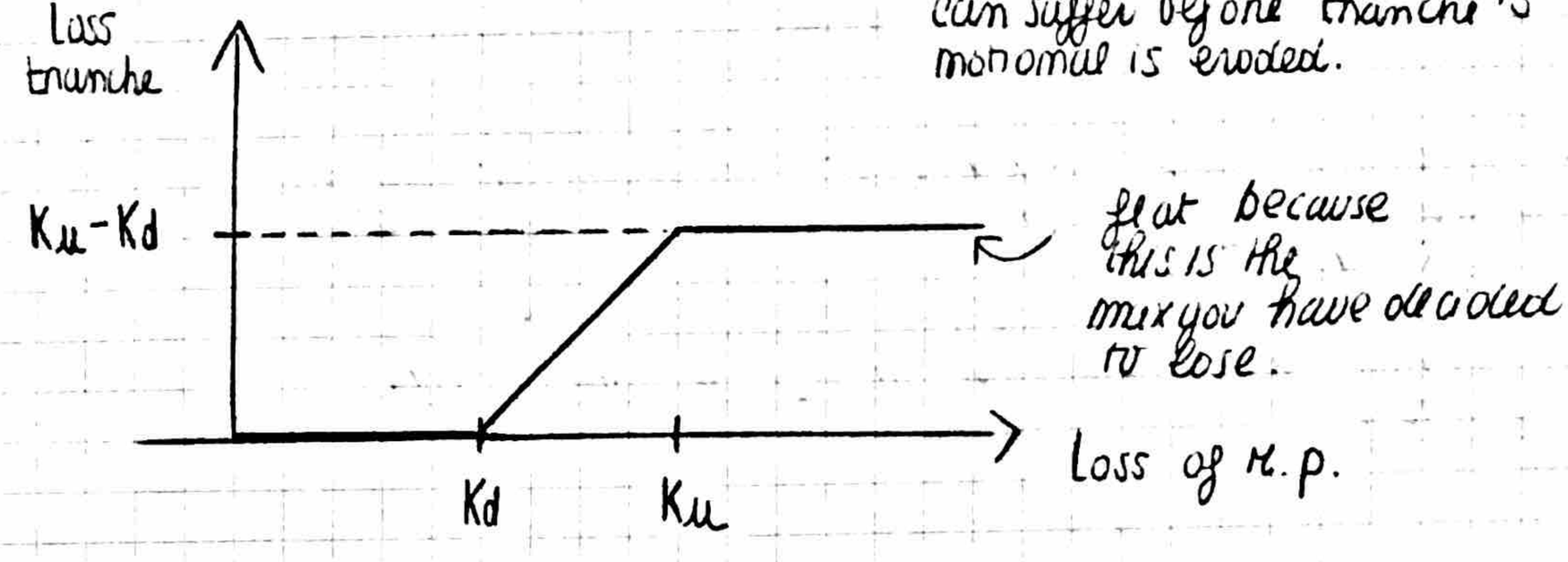
- emette obbligazioni con V.N. < crediti ← liquidità
- paga cedole con rate dei mutui che riceve

Originate (mortgages) & Distribute (notes)

MBS

17. ABS TRANCHE, FUNDAMENTALS INGREDIENTS TO PRICE IT? HOMOGENEOUS PORTFOLIO (HP) AND LARGE HOMOGENEOUS PORTFOLIO (LHP) APPROACHES.

The notes have a SUBORDINATION
UPPER → K_u and LOWER → K_d ← amount of losses a portfolio can suffer before tranche's nominal is eroded.



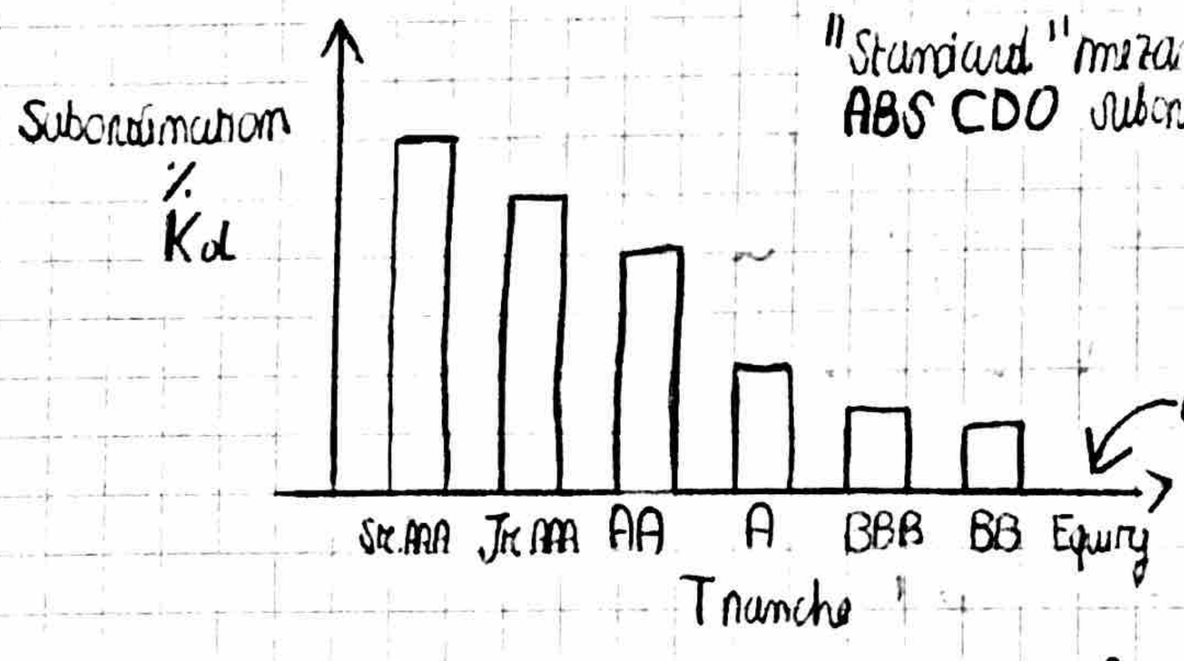
TRANCHE CUMULATIVE LOSS:

$$L_{tranche}(t) = \min(\max[L_{HP}(t) - K_d, 0], K_u - K_d)$$

$$= \begin{cases} 0 & L_{HP} < K_d \\ L_{HP} - K_d & L_{HP} > K_d \end{cases}$$

$$\begin{cases} L - K_d & \text{eventual Loss} \\ L > K_u \rightarrow & \text{perdo tutto} \\ L < K_d \rightarrow & \text{guadagno il possibile} \\ \text{Più } K_u \downarrow \rightarrow & \text{più guadagno (più rischio)} \end{cases}$$

ABS & CDO : SUBORDINATION EXAMPLE



The subordination is given by a rating.

K_d Equity = 0,
K_u Equity = 5%
(K_d of the following = K_u of the previous)

The equity tranche has a very high coupon, since it's very risky. The senior tranches are easy to be sold since they are similar to risk-free. But the mezzanine tranches are difficult to be sold.

However, you have to sell everything, so you pack CDO → CDO^2.

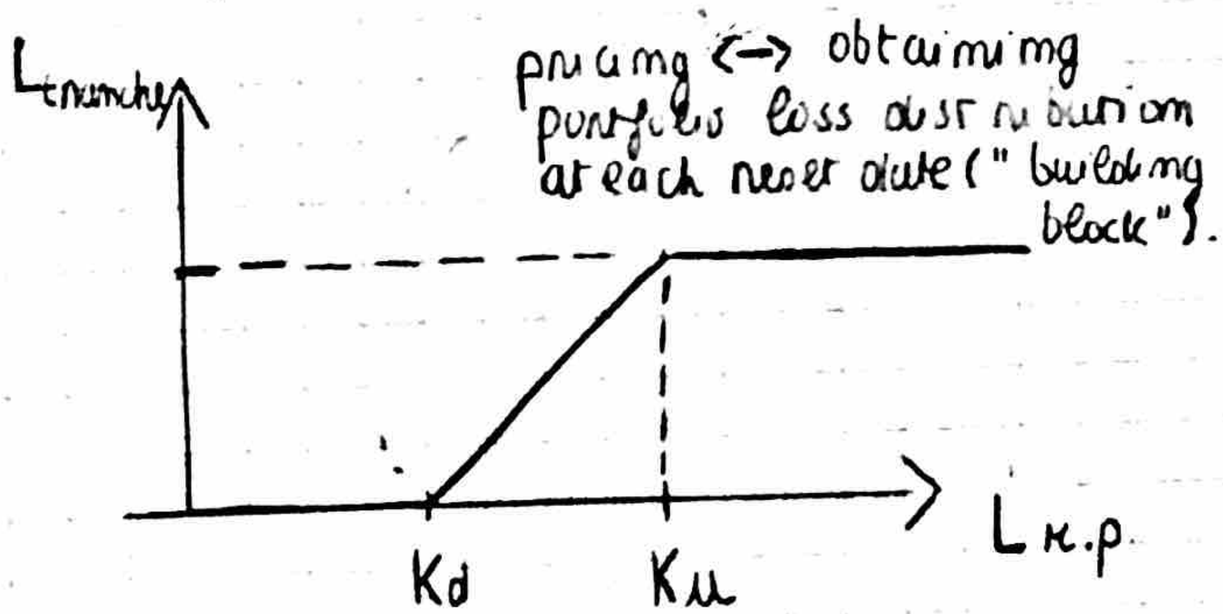
iTraxx subordination (for all tranches except for the first one, not traded, bid-ask prices).
↓ as EuroStoxx in equity
index in the Credit world! created by JP Morgan, made of:

- I.G. (Investment Grade) 75 — firms with a very high rating
- CrossOver (not IG) 125

The index is decided as an equally weighted r.p.

We are making these examples in order to understand which are the building blocks used in order to PRICE ABS and CDO.

For the pricing, we need the distribution of the Loss of the r.p.



If I have the distribution, this is the payoff, EQUIVALENT TO A BULL SPREAD (CALL SPREAD) (buy a call with K_d, sell a call with K_u).

ingredients { Default distribution; K_d ok; K_u ok }

$N_{tranche} = N(K_u - K_d)$

only missing

We have a Call Spread on the $L_{r.p.}$

Main problem in multi-name default: price models of the r.p. (to do it, we need loss distribution of the r.p.)

⇒ We need $L_{r.p.}$ in order to price.

Unique building block in the multi-name case: CDF of L_{rp}!

We will use different models, always considering two cases:

- HOMOGENEOUS PORTFOLIO (HP)
- LARGE HOMOGENEOUS PORTFOLIO (LHP)

approaches.

HP assumptions:
i) $N_i \leftarrow i \Rightarrow N$
ii) $\pi_i = \pi$
iii) $p_i = p \Rightarrow p_i = N(K_i)$
so we have just one $K = N^{-1}(p)$.

posso usare CDF(m)

LHP with large number of obligons (to use CLT) $\varepsilon_{HP} = N_t \sum_{m=0}^{I} \ell\left(\frac{m}{I}\right) f_{HP}(m)$
grande ed omogeneo di obligazioni!

... LHP e HP descritti in D.19! →

$L_{rp} = (1-\pi)m$

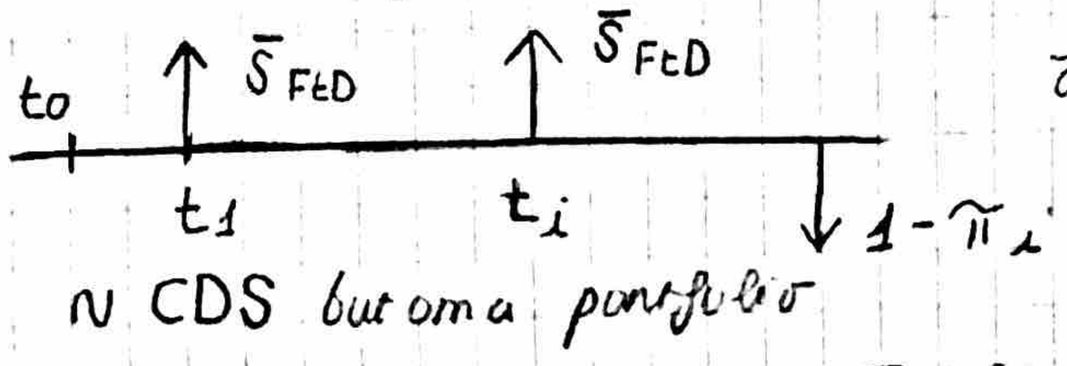
modellare questo non CDF(LHP)!
quindi ricavo ε expected loss di un investitore

18. WHAT IS A FIRST TO DEFAULT, WHICH ARE THE FUNDAMENTAL INGREDIENTS TO PRICE IT?

PORTFOLIO CREDIT DERIVATIVES ?
in MULTI - NAME

FIRST TO DEFAULT

on a portfolio of ($i = 1, .., I$) reference credits (generally with the same weight)



fee leg

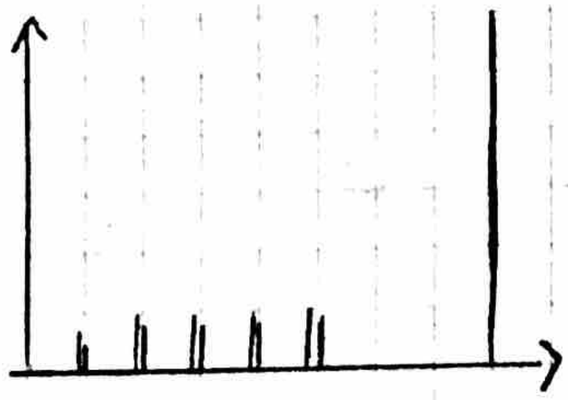
contingent leg: if the i-th credit defaults

~ CDS but on a portfolio

On a m.p. receive spread $\bar{S}^{FtD}$ until the i-th credit of BASKET DERIVATIVES.

(LOSS LAYER (equivalence of tranches in equity.)
derivative on a m.p. of reference credits $\{C_i\}$ with monomials $\{K_i\}$
mp loss → cumulative layer loss L_{layer}.) ~ ABS

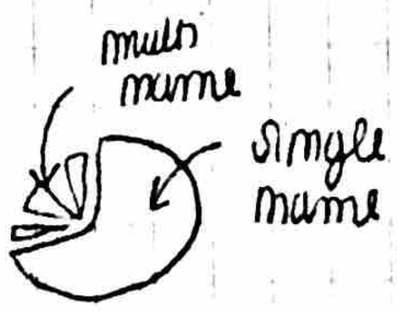
Credit Market:



Cash bonds
Credit derivatives

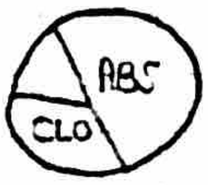
Volume of credit derivatives superior than the volume of bonds

derivatives by product:



2/3 of derivatives product are CDS (single name).

multinames:



CLO → loans
CDO connected to loans, most risky ones are LBO.

FUNDAMENTAL INGREDIENTS TO PRICE A FtD → Serve come i tempi to default, la loro distribuzione, ma non sono in grado → serve la copula (L. Model → price FtD).

19. ONE TIME STEP MODEL.

AN ABS TRANCHE IN THE VASICECK MODEL (HP, LHP)

ONE TIME STEP MODEL $\quad \tau \in (0,T]$

Each obligor ($i = 1, \dots, I$) defaults in the lag T (time window of interest) with a default probability p_i.

HP Assumptions:
- $p_i = p \qquad K_{eff} = \forall i,\ p_i = N(K_{eff_i})$
- $\pi_i = \pi$ — loss distribution $\leftrightarrow$ default distribution
- $N_i = N$

$$\Rightarrow L_{HP} = \underbrace{(1-\pi)}_{LGD}\, \underbrace{m}_{\text{n° of obligons defaulted in } (0,T]}$$

LHP Assumptions:
- $I \gg 1$
- loss fraction $x = \dfrac{\#\text{ defaults}}{I}$

... factor models allow simple computation

FACTOR MODEL: VASICECK

multi name
Limn Valve model
a questo cond. a eravamo arrivati con la calibrazione alternativa a quella KMV

Each firm modeled as in Merton model:

$V_i \leq K_{eff}$ — default of the i-th obligor
std. n. — $i = 1, \dots, I$ n° of obligons in the r.p.

I consider V_i as the sum of two different standard normal independent r.v.:

FACTOR MODEL $\Rightarrow \quad V_i = \sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i$

ε_i → idiosincratic component, characteristic of the i-th obligor

y → market component: average credit market.

$y, \{\varepsilon_i\}$ i.i.d. std n rvs.

$\mathbb{E}[V_i] = 0$

$Var[V_i] = \mathbb{E}[V_i^2] = \rho + 1 - \rho = 1$

$\rightarrow V_i$ std. n

Why this split? Consider $i \neq j$, $V_i \sim N(0,1)$, $V_j \sim N(0,1)$; $y \sim N(0,1)$, $\varepsilon_i \perp y$; $\varepsilon_i \sim N(0,1)$, $\varepsilon_i \perp \varepsilon_j$

Given y, $\{V_i\}_{i=1,\dots,I}$ are independent $\rightarrow$

↳ we can use conditional independence !!

$$\mathrm{Corr}(V_i, V_j) = \mathrm{Cov}(V_i, V_j) = \mathbb{E}[V_i V_j] =$$
$$= \mathbb{E}[(\sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i)(\sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_j)] =$$
$$= \rho \cdot \mathrm{Var}(y) + \emptyset = \rho$$

$$\Rightarrow \mathrm{Corr}(V_i, V_j) = \rho \quad \forall\, i \neq j$$
$$M^{I \times I} \ni \Sigma^2 = \begin{bmatrix} 1 & & \rho \\ & \ddots & \\ \rho & & 1 \end{bmatrix}$$

single factor model.

given y, V_i and V_j are independent.

We have said that, in order to price an ABS tranche, we have to model the $L_{k.p.}$.

$$L_{k.p.} = (1 - \tilde{\pi})\, m$$

$(1-\tilde{\pi})$: loss in each obligor

m: number of obligors defaulted in the k.p.

Under our assumptions (HP, LHP), since all notes in the k.p. have the same loss

modeling $L_{k.p.} \equiv$ modeling m $\quad$ CDF(m) instead of CDF of $L_{k.p.}$

We want to find the CDF of m:

$\mathbb{P}(m < m)$ number of default up to m

<u>HP</u>

Given y, default happens if

$$V_i < K \iff \sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i < K$$
$$\iff \boxed{\varepsilon_i < \frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}}$$

So, the <u>probability to observe exactly one default</u> given y is: $p(y) = N\left(\frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)$

Which will be the probability to observe exactly m defaults, given y ?

$$\mathbb{P}(X=m \mid y) \underset{\text{Binomial}}{=} \binom{I}{m} \underbrace{p(y)^m}_{\text{independent given } y} [1-p(y)]^{I-m}$$

$$\mathbb{P}(X \leq m) = \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{m} \mathbb{P}(X=m \mid y) =$$

$$= \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{m} \binom{I}{m} \underbrace{p(y)^m}_{\text{normalized probability}} [1-p(y)]^{I-m}$$

CDF for m defaults

$$p(y) = N\left(\frac{N^{-1}(p) - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)$$

TCL $\{X_i\}$ iid, $\mathbb{P}(\lim_{m\to\infty} \bar{X}_m = \mu) = 1$

$$\bar{X}_m = \frac{X_1 + X_2 + \ldots + X_m}{m}$$

LHP

Given y,

we consider the fraction of defaulted obligons $Z := \frac{m}{I}$,

$x := \frac{m}{I}$.

We want that $Z < x$

$Z = p(y)$ we are conditioning w.r.t. y → given y

$= N\left(\frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)$ monotonic decreasing function of y

TCL $Z \xrightarrow{I\to\infty} p(y)$ frequency will tend to the probability!

$\exists!\, y^* : x = p(y^*)$

$$\Rightarrow \quad p(y) \leq p(y^*) \quad \Rightarrow \quad \boxed{y \geq y^*}$$

where y^* is the value obtained as:

$$x = p(y^*) \rightarrow \frac{K - \sqrt{\rho}\, y^*}{\sqrt{1-\rho}} = N^{-1}(x)$$

$$\Rightarrow y^* = -\frac{N^{-1}(x)\sqrt{1-\rho} - K}{\sqrt{\rho}}$$

$$\Rightarrow \mathbb{P}(Z < x) = \int_{-\infty}^{+\infty} dy\, \phi(y)\, \mathbb{1}_{y \geq y^*} = \mathbb{E}_y[\mathbb{1}_{y \geq y^*}]$$

$$= \int_{y^*}^{+\infty} dy\, \underbrace{\phi(y)}_{\text{simmetric}} = \int_{-\infty}^{-y^*} dy\, \phi(y) =$$

$$= N(-y^*) = N\left(\frac{N^{-1}(x)\sqrt{1-\rho} - K}{\sqrt{\rho}}\right)$$

where $K = N^{-1}(p)$

So, we have the loss cumulative distribution function :

$$\mathbb{P}(X \leq x) = N\left\{\frac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\, N^{-1}(x) - N^{-1}(p)\right]\right\}$$

prob to observe a fraction of default up to x

$\underbrace{p, \pi, \rho}_{\text{Key parameters}}$

Vasicek model in credit world $\equiv$ B&S in equity world

20. ONE TIME STEP MODEL. KULLBACK-LEIBLER APPROXIMATION FOR AN ABS TRANCHE IN THE HP VASICECK MODEL.

Consider now a Homogeneous reference Portfolio in an ABS.

I want to estimate the <u>expected loss of a tranche</u>

$\mathbb{E}[L_{tranche}]$.

$$L_{HP} = (1-\pi)\frac{m}{I}N$$

number of obligors defaulted in the HP.

$$H(1-\pi)m = \underbrace{HI}_{N}(1-\pi)\frac{m}{I}$$

$$\Rightarrow L_{tranche} := \underbrace{\min\left\{\max\left(\frac{m}{I} - d, 0\right), u-d\right)\right\}}_{\mathcal{L}_t\left(\frac{m}{I}\right)} \cdot (1-\pi)N$$

$$d := \frac{K_d}{1-\pi}, \quad u := \frac{K_u}{1-\pi}$$

$$\Rightarrow \mathbb{E}[L_t] = (1-\pi)\cdot N \sum_{m=1}^{I} \mathcal{L}_t\left(\frac{m}{I}\right)\cdot \underbrace{\mathbb{P}(m)}$$

expected loss for the tranche

probability density of having m defaults in the HP

$$\rightarrow \int_{-\infty}^{+\infty} dy\, \phi(y) \binom{I}{m} p(y)^m [1-p(y)]^{I-m}$$

probability density of variable y

This is a closed formula, but what is the problem?

It's a computational problem: $\binom{I}{m}$ too difficult to be done by the machine.

So, on one hand, HP pricing can be used up to I quite small, on the other we don't know the quality of LHP pricing formula and what is the error we commit for a finite I.

HP $\Rightarrow$ LARGE DEVIATION APPROACH

We consider the CDF of observing a number of defaulted obligons up to m in the HP case:

$$\mathbb{P}(m \leq m) = \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{m} \underbrace{\binom{I}{m}}_{\text{STIRLING FORMULA to approximate it.}} (p(y))^{m} [1-p(y)]^{I-m}$$

$$\binom{I}{m} = \frac{I!}{m!(I-m)!}$$

expand the Newton coefficient with Stirling formula:

$$m! \cong \sqrt{2\pi m}\, m^{m} e^{-m}$$

$$\binom{I}{m} = \frac{I!}{m!(I-m)!} = \frac{\sqrt{2\pi I}\; I^{+I} \cancel{e^{-I}}}{\sqrt{2\pi m}\; m^{m} \cancel{e^{-m}} \cdot \sqrt{2\pi(I-m)}\,(I-m)^{(I-m)} \cancel{e^{-(I-m)}}} =$$

$$= \left(z = \frac{m}{I}\right) = \frac{1}{\sqrt{2\pi \frac{m}{I}\left(1-\frac{m}{I}\right)}\sqrt{I}\left(\frac{m}{I}\right)^{m}\left(\frac{I-m}{I}\right)^{I-m}} =$$

$$\frac{I-m}{I^{I}}^{I-m} = \left(\frac{I-m}{I}\right)^{I-m} \frac{1}{I^{m}}$$

$$= \frac{1}{\sqrt{I}} \frac{1}{\sqrt{2\pi z(1-z)}} \frac{1}{z^{m}(1-z)^{I-m}} =$$

$$= \frac{1}{\sqrt{I}} \frac{1}{\sqrt{2\pi z(1-z)}}\, z^{\underbrace{-zI}_{-\frac{m}{I}\cdot I}} \cdot (1-z)^{-I(1-z)}$$

and then, since $p(y)^{m} = p(y)^{zI} = e^{Iz\ln(p(y))}$

$$\binom{I}{m} = \frac{1}{I} C^{1}(z) \exp\{-I[z\ln z + (1-z)\ln(1-z)]\}$$

$$C^{1}(z) = \sqrt{\frac{I}{2\pi(1-z)z}}$$

We obtain:

$$\mathbb{P}(m \leq n) \simeq \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{n} \frac{1}{I} C^{(1)}\left(\frac{m}{I}\right) \left(\frac{m}{I}\right)^{-m} \left(\frac{I-m}{I}\right)^{-(I-m)} \cdot p(y)^{m} \left(1-p(y)\right)^{I-m} =$$

$$= \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{n} \frac{1}{I} C^{(1)}\left(\frac{m}{I}\right) \underbrace{\left(\frac{m}{I\,p(y)}\right)^{-m} \left(\frac{I-m}{I(1-p(y))}\right)^{-(I-m)}}_{\left(\frac{z}{p(y)}\right)^{-zI} \left(\frac{(1-z)}{(1-p(y))}\right)^{-I(1-z)}} =$$

$$= \exp\Big\{ \underbrace{-I\left[z \ln \frac{z}{p(y)} + (1-z) \ln\left(\frac{1-z}{1-p(y)}\right)\right]}_{K(z,\, p(y))} \Big\}$$

KULLBACK-LEIBLER ENTROPY

$$= \int_{-\infty}^{+\infty} dy\, \phi(y) \sum_{m=0}^{n} \frac{1}{I} C^{(1)}\left(\frac{m}{I}\right) \exp\left\{-I\, K\left(\frac{m}{I}; p(y)\right)\right\}$$

$$\left(\begin{array}{l} \mathbb{P}(m) = \displaystyle\int_{-\infty}^{+\infty} dy\, \phi(y)\, \frac{C^{(1)}(z, I)}{I} \exp\{-I[\, K(z, p(y))\,]\} \\ \Rightarrow \mathbb{E}[L_t] = N(1-\pi) \displaystyle\sum_{m=0}^{I} \mathcal{L}\left(\frac{m}{I}\right) \mathbb{P}(m) \end{array} \right)$$

For a value of I sufficiently large, the sum $\sum_{m=0}^{n}$ can be approximated with the corresponding integral, i.e.

defining $dz = \frac{\Delta m}{I}$ (così che z varia da 0 a x)

we get:

$$\mathbb{P}(z \leq x) = \int_{-\infty}^{+\infty} dy\, \phi(y) \int_{0}^{x} dz\, C^{(1)}(z) \exp\{-I\, K(z; p(y))\}$$

We know that, given y,

$$\int_0^1 dz\, C^{(1)}(z, y)\, e^{-IK(z,p(y))} \text{ should be one}$$

$$\left(\mathbb{P}(m|y) = \binom{I}{m} p(y)^m (1-p(y))^{I-m} \right.$$

$$\left. \sum_{m=0}^{\infty} \mathbb{P}(m|y) = 1 \right)$$

With Stirling formula, it is not one.

For every given value of y, we can consider a more precise normalization function

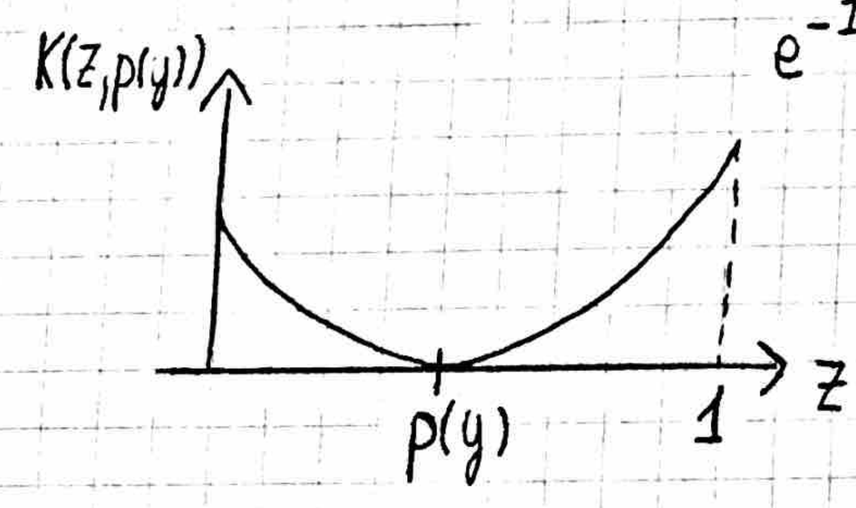
$$C(z,y) = C^{(1)}(z)/D(y) \quad \int_0^1 \frac{C^{(1)}(z)}{D(y)} e^{-IK(z;p(y))} dz = 1$$

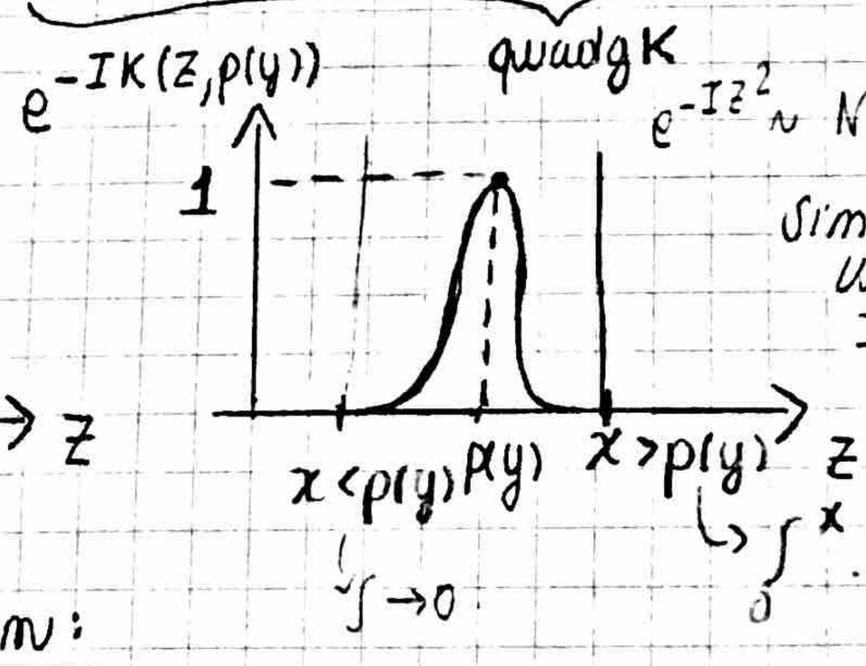
$$D(y) = \int_0^1 dz\, C^{(1)}(z) \exp\{-IK[z; p(y)]\}$$

$$\left(1 = \sum_{m=0}^{\infty} \mathbb{P}(m|y) = \sum_{m=0}^{\infty} \frac{C^{(1)}(z, I)}{I} e^{-IK(z,p(y))} \cdot C^*(y) \right)$$

In conclusion,

$$\mathbb{P}(x \leq m) = \int_{-\infty}^{+\infty} dy\, \phi(y) \underbrace{\int_0^x dz\, C(z, I, y) \exp\{-IK[z, p(y)]\}}_{\text{quadgK}}$$





$e^{-Iz^2} \sim N(\ \ , \mathcal{O}(1/\sqrt{I}))$

Since we will usually consider I large, we will consider the peak.

<u>Saddle point approximation:</u>

$$\int_\Omega dz\, C(z) e^{-IK(z)} = e^{-IK(z_0)}$$

z_0 minimum of the function $K(z, p(y))$ on the domain $0 \leq z \leq 1$.

$$\begin{cases} x > p(y) \rightsquigarrow z_0 = p(y) \rightarrow K(z_0) = 0 \Rightarrow \text{quadgK approx} = 1 \\ x < p(y) \rightsquigarrow z_0 = x \rightarrow K(z_0) = K(x, p(y)) \rightarrow e^{-IK(x,p(y))} \end{cases}$$

$$I \to \infty \begin{cases} x > p(y) \to 1 \\ x < p(y) \to 0 \end{cases} \qquad x > p(y) \rightsquigarrow \underset{\text{LHP}}{p(y^*)} > p(y)$$

$I \to \infty$, KL $\to$ LHP

21. ABS PRICING WITH O'KANE & SCHLOEGL MODEL (HP, LHP).

DEFAULT DEPENDENCY AND NEW QUESTIONS

Let's describe what happened in the last financial crises.

There was a serious problem in the ABS market (mortgages), in particular in the SUB-PRIME mortgages.

The prime mortgage mkt was invented in 1929 after the great depression by Roosevelt as part of the New Deal. They needed to start again the economy

They needed a way to classify obligons in order to allow them to have enough money to buy an house.

Agency: Fanny Mae.

1) Credit scoring in order to detect their "probability to default" based on some questions to people (income, family)... Above a given threshold, you are considered a bad borrower.

2) Classify the loan (< Money Threshold)

3) $LtV = \frac{\text{Loan}}{\text{Value House}} < 80\%$

1+2+3 → Prime

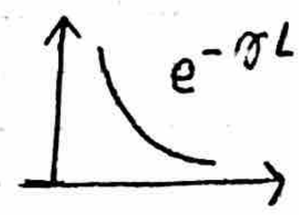
Fannie Mae was created in order to financiate banks in order to allow prime mortgages.
(vietnam war)
But in the 70's this company was sold and another one was created, Freddy Mac.

SUB-PRIME Mortgages - for people excluded from prime mkt.
Banks started to lend them when Insurance was born (Basel 2).
The commercial target was originate & distribute from 2004
→ very aggressive mkt.
to NINJA (No Income No Job)
The assumption was that the house value would have been increasing forever: so, if you can't repay the mortgage, you can sell the house. But when the house prices went down, there was a problem.

All houses were taken as a collateral → "Foreclosure"
→ even prime mortgages started to default, and Alt-A, between prime and sub-prime, car loans...

The situation becomes one that the tail of a normal distribution wasn't able to describe.
→ CONTAGION EFFECT.

In Vasicek, exponential decreasing tail:



thin tails (Loss function is an Inverse Exponential)

The Contagion effect can't be described → need to generalize Vasicek.

During the crises, FAT TAILS (as in t-Student)
Indeed, we saw that, at first, the problem was on the subprime mkt, a subproduct. But then, with a cascade effect, it produced problems in the whole mkt.

MODELING OBJECTIVES

→ reproduce default dependency of realistic magnitude

→ reproduce timing of default and clustering

e.g. Saving & Loan US Banks 1980-82: 962 bankruptcies on 4000 existing in 1980

... and the model should present:
→ straight calibration
→ simple implementation
→ parsimony

GENERALIZATION: t-Student (O'KANE - SCHLOEGL, 2005)

The IDEA is the same "like" Normal Mixture in market (!) risk.

To have a t-Student: $t_\nu = \dfrac{\text{Gaussian}}{\sqrt{\chi^2_\nu / \nu}}$

(same if I multiply by $W \sim IG(\nu)$).

So, we add now a positive r.v. $W \sim \chi^2_\nu$ (ν dof)

$$\Rightarrow \quad V_i = \frac{\sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i}{\sqrt{W/\nu}}, \quad y, \varepsilon_i \text{ std m r.v.}$$

same thing used to model fat tails in mkt risk.

Vasicek: obligor i defaults iff $V_i < K$

$$\Rightarrow \quad p_i = t_V(K) \qquad p = \int_{-\infty}^{K} \phi_V(v)\,dv$$

First, we used the trick to condition on y (so that $\{V_i\}_{i=1:I}$ were independent) to find $\mathbb{P}(m \leq m \mid y)$.

Now, we have two random variables y and W: given y and W, $\{V_i\}_{i=1:I}$ are independent.

<u>HP</u>

Given y and W, default:

$$V_i \leq K \iff \frac{\sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i}{\sqrt{W/\nu}} \leq K$$

$$\iff \varepsilon_i \leq \frac{K\sqrt{W/\nu} - \sqrt{\rho}\, y}{\sqrt{1-\rho}} \rightarrow \eta := K\sqrt{\frac{W}{\nu}} - \sqrt{\rho}\, y$$

weighted linear combination of V_i + stdm

Given η, probability of default:

$$p(\eta) = \overset{\varepsilon_i \text{ stdm}}{N}\left(\frac{\eta}{\sqrt{1-\rho}}\right) \quad \text{increasing in } \eta$$

So we condition on η instead of y:

$$\mathbb{P}(m \leq m) = \int_{-\infty}^{+\infty} d\eta\, f(\eta) \sum_{m=0}^{m} \underbrace{\binom{I}{m} p(\eta)^m (1-p(\eta))^{I-m}}_{\mathbb{P}(m \mid \eta)}$$

We have to rebuild numerically the distribution function of η (convolution)

$\mathbb{P}(\eta \leq t) \rightarrow f(\eta)$

LHP

In Vasicek, we considered the fraction of defaulted obligons z given y : $z = p(y)$.

Now, we consider the fraction of defaulted obligons given η : $p(\eta) = z$ (TCL).

We have $z \leq x$.

$\exists!\ \eta^*$ s.t. $x = p(\eta^*) = N\left(\frac{\eta^*}{\sqrt{1-\rho}}\right)$

$\Rightarrow \eta^* = \sqrt{1-\rho}\, N^{-1}(x)$.

We have: $p(\eta) \leq p(\eta^*) \overset{\eta\uparrow,\ p(\eta)\uparrow}{\longleftrightarrow} \eta \leq \eta^*$

So, $\mathbb{P}(z \leq x) = \int_{-\infty}^{+\infty} d\eta\, g(\eta)\, \mathbb{1}_{\eta \leq \eta^*} \overset{\mathbb{E}[\mathbb{1}_{\eta \leq \eta^*}]}{=} \int_{-\infty}^{\eta^*} d\eta\, g(\eta) =$

$= F(\eta^*) = F\left(\sqrt{1-\rho}\, N^{-1}(x)\right)$

We have found the loss distribution

22. ABS PRICING WITH DOUBLE t-Student (HP, LHP).

GENERALIZATION 2: DOUBLE t-Student (Hull and White, 2004)

We have generalized the Vasiceck model with the t-Student, but the problem is compute $f(m)$ numerically.

There is another generalization: <u>double t-Student.</u>

$$V_i = \sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i \quad , \quad y, \{\varepsilon_i\}_{i=1:I} \text{ iid t-student}$$

We want to find the loss distribution in HP and LHP.

HP

Given y,

obligor i defaults iff $V_i \leq K \iff \sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i \leq K$

$$\iff \varepsilon_i \leq \frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}$$

And so the probability of default will be:

$$p(y) = \overset{\varepsilon_i \text{ t-St}}{t_\nu}\left(\frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right) \quad \text{decreasing in } y$$

So the probability for m defaults given y will be the binomial distribution with probability $p(y)$:

$$\mathbb{P}(m \mid y) = \binom{I}{m} p(y)^m (1 - p(y))^{I-m}$$

$$\Rightarrow \mathbb{P}(m \leq n) = \int_{-\infty}^{+\infty} dy\, \underbrace{\varphi_\nu(\overset{y \text{ t-St}}{y})}_{\text{density of a t-student.}} \sum_{m=0}^{n} \binom{I}{m} p(y)^m (1-p(y))^{I-m}$$

LHP

For $I \to \infty$, the frequency of default will tend to the probability of default.

So, we consider the fraction of defaulted obligons

$$z = \frac{m}{I} = p(y).$$

We have that $z \leq x$.

$$\exists! y^* \text{ s.t. } x = p(y^*) = tr\left(\frac{K - \sqrt{\rho}\, y^*}{\sqrt{1-\rho}}\right)$$

$$p(y) \leq p(y^*) \iff y \geq y^*$$

So, $\mathbb{P}(z < x) = \mathbb{E}[\mathbb{1}_{y \geq y^*}] = \int_{-\infty}^{+\infty} dy\, \phi_r(y)\, \mathbb{1}_{y \geq y^*} \overset{y' = -y}{=}$

$$= \int_{-\infty}^{+\infty} dy'\, \phi_r(y')\, \mathbb{1}_{y' \leq -y^* = \tilde{y}}$$

$$y^* = -\frac{\sqrt{1-\rho}\; tr^{-1}(x) - K}{\sqrt{\rho}}$$

$$\Rightarrow \mathbb{P}(z \leq x) = \int_{-\infty}^{\tilde{y}} dy'\, \phi_r(y') = tr(\tilde{y}) =$$

$$= tr\left\{\frac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\; tr^{-1}(x) - K\right]\right\}$$

CALIBRATION OF K: $p : N(K) \to K = N^{-1}(p)$

$\mathbb{P}(V_i \leq K) = p$ (from the mkt)

$\Rightarrow K$

Given the ingredients seen in mkt risk we can do the same computations also in credit risk.

23. SKLAR THEOREM. GAUSSIAN COPULA & ARCHIMEDEAN COPULA.

COPULA new mathematical concept in credit world, extremely powerful instrument.

DEF. A copula is a function $C(u_1, u_2, \dots, u_I)$,

$C : \overset{\text{HYPERCUBE}}{[0,1]^I} \to [0,1]$ s.t.

- C is a distribution function of r.v. $\{u_i\}_{i=1:I}$
- $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$

MAIN PROPERTY : SKLAR'S THEOREM (CTS case weak form)

$\forall\ F(x_1, \dots, x_I)$ multi-variate continuous distribution,
$\exists!\ C(u_1, \dots, u_I)$ s.t.

Correlates marginal and total.

$$F(x_1, \dots, x_I) = C\left(F_1(x_1), \dots, F_I(x_I)\right)$$

aspect related to the single name, $F_i(x_i)$ single name CDF.

with $\{F_i(x_i)\}_{i=1:I}$ set of univariate marginal distribution function

So, with the copula, we are splitting the multi-name model into I single name models.
The problem in multi name credit is when all the firms start defaulting at the same time. This information is hold by a unique $C(u_1, \dots, u_I)$.

Split → each obligor default dynamic $F_i(x_i)$ single obligor default dynamic

→ relations among obligors (correlation) dependency structure

CREDIT INFO OF EACH OBLIGOR (SINGLE NAME)

VS.

COPULA DEPENDENCY (MULTI-NAME)

2-D , FRECHET BOUNDS

u, v independent r.v., the distribution function must be the product of the marginals.

$$C_I(u, v) = u * v$$

If we consider the copula minimum $W(u,v) = \max[u-(1-v), 0]$ and the copula maximum $M(u,v) = \min(u,v)$:

$$W(u,v) \leq C(u,v) \leq M(u,v)$$

GAUSSIAN COPULA

$\{u_i\}$ iid, $u_i = N(x_i)$

$$C(\underline{u}) = N_I\left(N^{-1}[u_1], \dots, N^{-1}[u_I]\right)$$

with :

$$N[u] \equiv \int_{-\infty}^{u} \phi(y)\,dy$$

$$\phi(y) \equiv \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(y-\mu_i)^2/2\sigma_i^2}$$

$N_I(\cdot)$ I-d cumulated normal

How can I get a vector $\underline{u} \in [0,1)^I$ distributed according to this Gaussian copula?

We can simulate $\underline{x} = \underline{\mu} + \Sigma^{1/2}\underline{y}$,

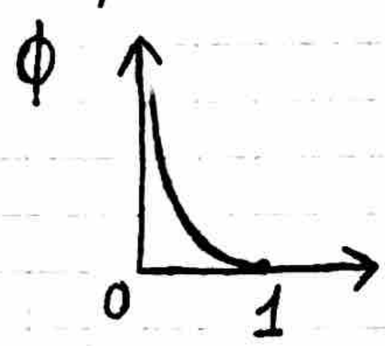
with $\{y_i\}$ iid $\sim N(0,1)$

Then, knowing that $u_i = N(x_i)$, we get the corresponding $\underline{u}$.

ARCHIMEDEAN COPULA

$$C(\underline{u}) = \phi^{-1}\left(\sum_{i=1}^{I} \phi(u_i)\right)$$

where $\phi : [0,1] \to \mathbb{R}^+$ a decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$, called COPULA GENERATOR.



This copula is OK, since

$$C(1, \dots 1, u_i, 1, \dots, 1) = u_i$$
$$= \phi^{-1}(\cancel{\phi(1)} + \cancel{\phi(1)} + \dots + \phi(u_i) + \cancel{\phi(1)} + \dots + \cancel{\phi(1)}) =$$
$$= \phi^{-1}(\phi(u_i)) = u_i$$

24. NARSHALL OLKIN THEOREM. ARCHIMEDEAN COPULA EXAMPLE.

We want to obtain a vector $\underline{u}$ distributed according to this copula.

TEO. drawing ε_i iid flat in $[0,1]$

+ $y \in \mathbb{R}^+ \sim f(y)$ s.t. $\mathcal{L}_f(s) = \int_0^{+\infty} f(y) e^{-ys} dy = \phi^{-1}(s)$
independent from the other r.v.

$$\Rightarrow \boxed{u_i = \phi^{-1}\left(-\frac{1}{y} \ln \varepsilon_i\right)}$$

follows an archimedean copula function with generator ϕ.

$$\phi^{-1}(s) = \mathcal{L}_f(s) = \int_0^{+\infty} dy \underbrace{f(y)}_{\text{density}} \underbrace{e^{-sy}}_{\in [0,1]} \in [0,1]$$

$\forall s$ $\mathcal{L}_f(s) \leq 1$, $\mathcal{L}_f(0) = 1$.

given y, $\{u_i\}$ iid

PROOF. $\mathbb{P}(\{u_i \leq q_i\}_{i=1,\dots,I}) = \mathbb{E}_y[\mathbb{P}(\{u_i \leq q_i\} \mid y)]$

$q_i \in [0,1]$

Given y:

$$u_i = \phi^{-1}\left(-\frac{1}{y} \ln \varepsilon_i\right)$$

follows the archimedean copula

$$u_i = \phi^{-1}\Big(\overbrace{-\frac{1}{y}\underbrace{\ln \overbrace{\varepsilon_i}^{\in[0,1]}}_{<0}}^{>0}\Big) \leq q_i$$

$$\Longleftrightarrow \quad -\frac{1}{y}\ln \varepsilon_i \geq \phi(q_i)$$

$$\Longleftrightarrow \quad \underbrace{\frac{1}{y}}_{>0}\ln \varepsilon_i \leq -\phi(q_i)$$

$$\Longleftrightarrow \quad \underbrace{\varepsilon_i}_{\in[0,1]} \leq e^{-y\phi(q_i)} \quad \rightarrow \mathbb{P}(u_i \leq q_i \mid y) = e^{-y\phi(q_i)}$$

$$\mathbb{P}(\{u_i \leq q_i\}) = \mathbb{E}_y\left[e^{-y\sum_{i=1}^{I}\phi(q_i)}\right] = \quad \varepsilon_i \text{ iid}$$

$$= \int_0^{+\infty} dy\, f(y)\, e^{-y\underbrace{\sum_{i=1}^{I}\phi(q_i)}_{s}} =$$

$$= \mathcal{L}\left[\sum_{i=1}^{I}\phi(q_i)\right] = \phi^{-1}\left[\sum_{i=1}^{I}\phi(q_i)\right] \quad \text{CDF of } q_i \qquad \blacksquare$$

EXAMPLES OF ARCHIMEDEAN COPULA

We need to specify the generator for an Archimedean Copula:

CLAYTON

$$\phi(t) = t^{-\vartheta} - 1$$

$$\phi^{-1}(s) = (1+s)^{-1/\vartheta} = \mathcal{L}_f$$

$$\vartheta \geq 0$$

$$f(y) = \frac{1}{\Gamma(1/\vartheta)}\, e^{-y}\, y^{(1-\vartheta)/\vartheta}$$

gamma distribution

$$F(y) = \int_0^y f(u)\,du$$

GUMBEL

$$\phi(t) = (-\ln t)^{\vartheta}$$

$$\phi^{-1}(s) = e^{-s^{1/\vartheta}}$$

$$\vartheta \geq 1$$

$f(y) =$ positive α-stable

$\alpha = 1/\vartheta$

NB. Generalized CTL if Variance not finite. The limit will be a positive α-stable distribution.

25. GENERAL THRESHOLD MODEL (GTM). VASICECK AND GORDY MODELS AS EXAMPLES.

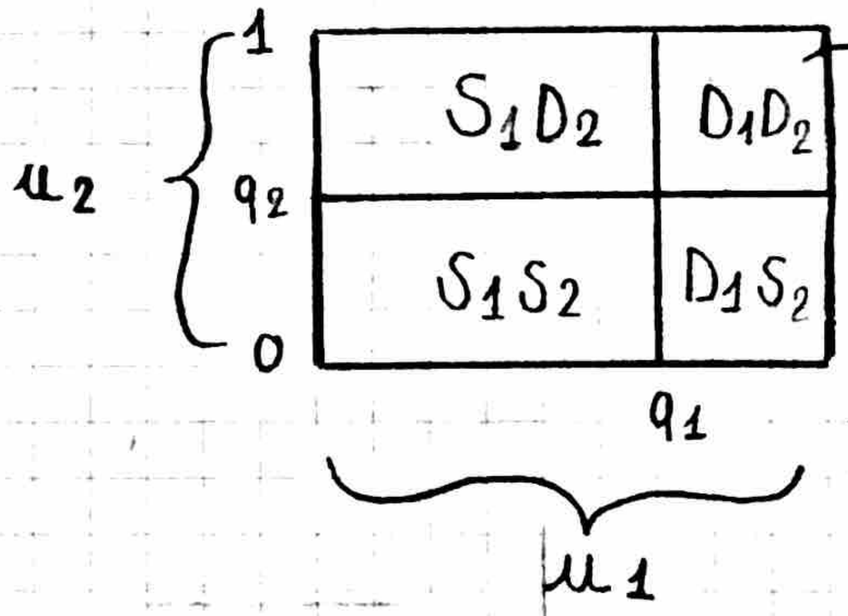
COPULA: THE MOST GENERAL ONE-TIME STEP DEFAULT MODEL (GENERAL THRESHOLD MODEL)

1. Given $\begin{cases} \{q_i\}_{i=1:I} & \text{: individual (marginal) survival probabilities (1 time step)} \\ C(\underline{u}) & \text{copula that describes default dependency} \end{cases}$

2. Draw $\{u_i\}_{i=1,\dots,I}$ according to copula $C(\underline{u})$ (set of numbers $\in [0,1]$)
3. Obligor i survives iff $u_i \leq q_i$
4. Select the default obligors

In this way, we can price ∀ bond, tranche portfolio or derivative product (ex. FtD).

2 dim case:



most interesting case, when we have a CLUSTERING of defaults: obligons begin to default at the same time.

Remarks:
- $\mathbb{P}[u_i \leq q_i] = q_i$
- prob. of no default: $\mathbb{P}[u_i \leq q_i, \forall i \leq I] = C(q_1, \dots, q_I)$
- prob. of survival of the first K obligons:
  $\mathbb{P}[u_i \leq q_i, \forall i \leq K] = C(q_1, \dots, q_K, 1, \dots, 1)$

EXAMPLES OF GTM:

- VASICECK

$\underline{V}$ multivariate gaussian | $\underline{\mu} = 0$, $\Sigma = \underline{\underline{\rho}} = \begin{bmatrix} 1 & & \rho \\ & \ddots & \\ \rho & & \end{bmatrix} \in \mathbb{R}^{I \times I}$

default condition: $V_i < K$

$p = N(K)$ def. probability $\forall i$

$V_i < K \Rightarrow N(V_i) < N(K)$ default (increasing function)

def. $\underset{\in[0,1]}{u_i'} := N(V_i)$, $\underset{\in[0,1]}{u_i} := 1 - u_i'$

V_i gaussian $\rightarrow$ u_i gaussian copula $\rightarrow$ cum $N(V_i) < p$

Defaut : $u_i = 1 - u_i' = 1 - N(V_i) > 1 - p = q$

i^{th} obligor survives if $u_i \leq q$ cvd

- GORDY

$$\underline{\underline{\rho}} = \begin{pmatrix} 1 & \rho_i\rho_j \\ \rho_i\rho_j & \ddots \; 1 \end{pmatrix}, \quad V_i \leq K_i, \; N(K_i) = PD_i$$

26. ARCHIMEDEAN COPULA IN GTM (HP, LHP)

GTM

Obligor i survives iff $\boxed{u_i \leq q}$

with $u_i = \phi^{-1}\left(-\frac{1}{y} \ln \varepsilon_i\right)$ (M&O)

with ε_i iid flat in $[0,1]$
$y \in \mathbb{R}^+ \sim f(y)$ s.t. $\mathcal{L}_F[\cdot] = \phi^{-1}[\cdot]$
ϕ decreasing function.

<u>Remark</u> : given y, $\{u_i\}_{i=1:I}$ iid (we condition on y)

<u>HP</u>

$\mathbb{P}(m \leq n)$?

Given y,

probability of having one default $p(y)$:

Let's start from the survival condition for i^{th}-obligor :

$u_i \leq q \iff \phi^{-1}\left(-\frac{1}{y} \ln \varepsilon_i\right) \leq q \iff$ ϕ decreasing function

$\iff -\frac{1}{y} \ln \varepsilon_i \geq \phi(q) \iff \frac{1}{y} \ln \varepsilon_i \leq -\phi(q) \iff$

$\iff \varepsilon_i \leq e^{-y\phi(q)} \Rightarrow Q(y) = e^{-y\phi(q)}$ HP surv. prob. given y

$p(y) = 1 - e^{-y\phi(q)}$ HP default prob. given y

$$\Rightarrow \mathbb{P}(m \leq n) = \mathbb{E}_Y[\mathbb{P}(m \leq n \mid y)] = \int_0^{+\infty} y f(y) \sum_{m=0}^{n} \binom{I}{m} p(y)^m [1 - p(y)]^{I-m}$$

LHP

Consider the fraction of defaulted obligons

$$Z := \frac{m_v}{I} \leq x := \frac{n}{I}$$

TCL: $Z = \underbrace{p(y)}_{1-e^{-y\phi(q)}}$ increasing in y

$\exists!\ y^*$ s.t. $x = p(y^*)$

$\Rightarrow p(y) \leq p(y^*) \iff y \leq y^*$

$\Rightarrow \mathbb{P}(Z \leq x) = \mathbb{E}_Y[\mathbb{1}_{y \leq y^*}] = F(y^*)$

$x = p(y^*) = 1 - e^{-y^*\phi(q)} \iff \ln(1-x) = -y^*\phi(q)$

$\iff y^* = -\dfrac{\ln(1-x)}{\phi(q)}$

$$\Rightarrow \mathbb{P}(Z \leq x) = \int_0^{+\infty} dy\, f(y)\, \mathbb{1}_{y \leq y^*} = \underbrace{F}_{\text{CDF of } Y}\left\{-\frac{\ln(1-x)}{\phi(q)}\right\}$$

Why copula? gives dependence of obligons + Li Model.

27. Li MODEL FOR CONTINUOUS TIME CREDIT MODELING. HOW TO PRICE A CREDIT PRODUCT, e.g. A FtD OR A TRANCHE IN A (REAL) CDO.

MULTI-NAME DEFAULT TIME MODEL (Li MODEL) MULTI NAME CONTINUOUS TIME

Before, we have presented 1 time step models (GTM), now CONTINUOUS TIME for credit modeling.

whatever credit product in continuous time can be priced thanks to Li approach.

We have:

$\{P_i(0,T)\}_{i=1,\dots,I}$: individual (marginal) survival probabilities. (with an intensity model) ingredients.

$C(\underline{u})$: copula which describes default dependency. for names in my n.p.

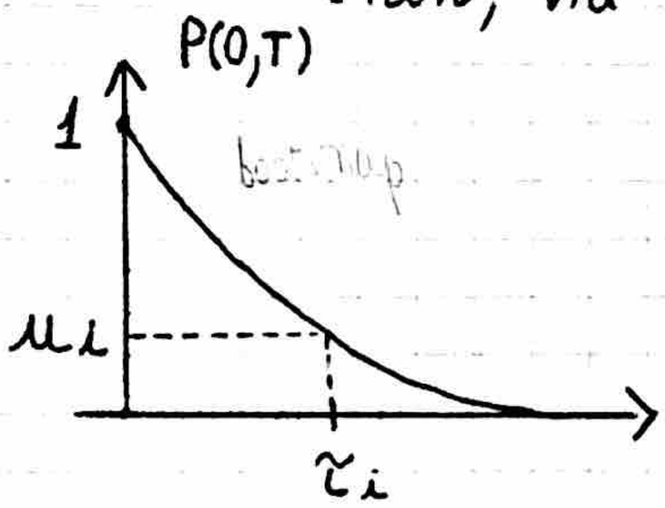
Remark : for each fixed T, the model is a (one-step) copula default model with surv. prob. $\{P_i(0,T)\}_{i=1,\dots,I}$.

Default correlation between two credit risks as the correlation coefficient between their survival times.

Default correlation <u>not</u> well defined : based on discrete events.
<u>But</u> default correlation is a time-dependent quantity.
Arbitrarily focusing on a one year period wastes important information.
In the majority of credit derivative valuations, what we need is not the default correlation of two entities over the next year, we may need to have a joint distribution survival times for the next 10 years.

PRICING:

(1) I want a vector $\underline{\tau}$ of default times, I draw $\{u_i\}$ (NB: $\underline{u}$ generated accordingly to the copula $C(\underline{u})$): then, via surv. prob., I obtain $\tau_i \ \forall i$.

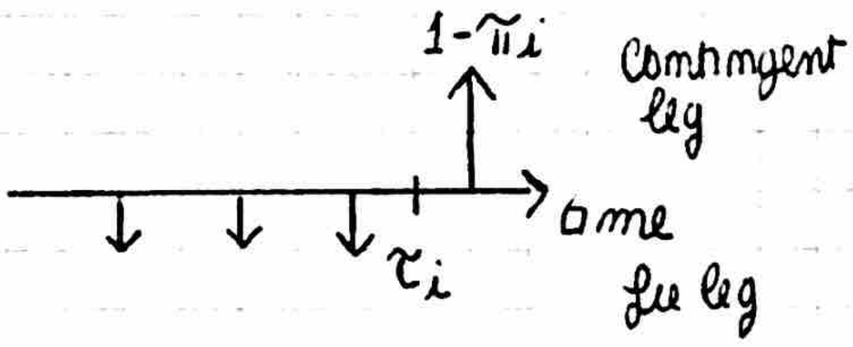


$\tau_i = P_i^{-1}(u_i)$
↳ use them to price whatever product.

(2) Evaluate the payoff given the scenario (I can price whatever credit product)

Ex :

FIRST TO DEFAULT : derivative contract where, if I buy the protection, I have to pay the fee leg.



2 mod NO DEFAULT.
2 mod all τ are independent

MC

$$\tau^* = \min_{i \in I} \{\tau_i\}$$

if $\tau^* > T \Rightarrow$ only the fee leg
if $\tau^* \leq T \Rightarrow$ fee leg + contingent

TRANCHE IN A REAL CDO :

CALIBRATION
→ calibrate the marginal survival probability of each obligor.

28. CALIBRATION OF MULTINAME MODELS.

<u>IMPLIED CORRELATION</u> (as implied volatility in equity, we have only 1 parameter ρ)

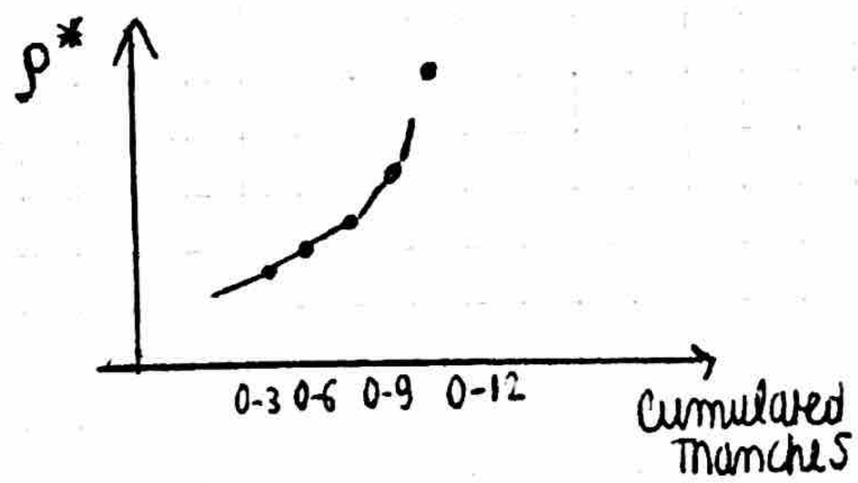
In multi-name credit, it is often considered an implied correlation, considering the "Vasicek" model (or its cont. time Li generalization), i.e. the correlation value ρ^* for "liquid" products (e.g. iTraxx tranches) s.t.:

$$\text{Mkt price} = \text{Model Price}(\rho^*)$$

Supposing that the model is correct, ρ is flat.

We consider the CUMULATED TRANCHES ($K_u = 0$ always): ← tranches quoted on the mkt, we use CUMULATED TRANCHES.

In credit world, we don't consider



Why ρ^* changes?

I'm considering rare events → need bigger tails.

Quoted tranches
↓
Cumulated tranches price
↓
Compute implied correlation of the cumulated tranches.

(0-3%, Equity)

1. For the first tranche,

Model Price = MKt Price $\Rightarrow \rho^*$ (in general, the parameter of my model, e.g. θ for Clayton)

gaussian
Clayton
t-student
double t-student

2. Fixed parameter, price other cumulated tranches: (0-6, 0-9, 0-12)

model price cumulated tranches = model price cumulated tranches (Vasicek)

↑ Li model

↑ with the corresponding correlation matrix.

→ I obtain the parameter for every tranche (correlation smile)

The ρ^* for the first tranche will be the same for all models (Vasicek, double t, t-student and market).

double t-student : ν, ρ^{ts} two free parameters

Fix ν,

I compute ρ with ν fixed (ρ for that particular ν s.t. MKt price = Model price)

For every ν, I compute the MSE:

$$d = \sum_{i=2 \text{ (1st equal)!}}^{5} \left(\rho_i^{Mkt} - \rho_i^{Model}\right)^2$$

← select ν s.t. the distance w.r.t. the mkt is minimum

$$\nu = \underset{\nu \in [1, +\infty]}{\operatorname{argmin}} \; d(\nu)$$

Calibrate 2 parameters with 5 points!