5. p.310 p.431

IMPLIED VOLATILITY AND VOLATILITY SMILE.

HOW AND WHY IT IS USED. → Prices move very fast, driven by several quantities, while volatility is sticky (so, it is useful). MM compares derivatives w.r.t. their volatility.

8. VOLATILITY SMILE

r from the market
K, S_0 from the contract itself

The only parameter in the Black-Scholes-Merton pricing formula that can't be directly observed is the volatility of the stock price. (σ?)

In practice, traders usually work with what are known as IMPLIED VOLATILITIES. These are the volatilities implied by option prices observed in the market.

I have a Black model for the price, depending on one unknown parameter.

I calibrate the model: (implied volatility approach)

$$\boxed{\text{CALL PRICE}^{\text{Model}}(\sigma; \underbrace{K, TTM}_{\text{given}}) = \text{CALL PRICE}^{\text{MKT}}(K, TTM)}$$

⇒ I can get (σ) inverting the formula for the price that matches quoted prices present in the market

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price.

VOLATILITY SMILES

How close are the market prices of options to those predicted by the BSM model? Do traders really use this model when determining the price of an option?

Traders do use the BSM model, but not in exactly the way that it was intended. This is because they allow the volatility used to price an option to depend on its strike price and time to maturity.

A plot of the implied volatility of an option with a given TTM as a function of its strike price is known as VOLATILITY SMILE.

We describe the volatility smiles that traders use in equity and foreign currency markets.

The implied volatility of a European call option is always the same as the implied volatility of a European put option with the same strike price and maturity date.

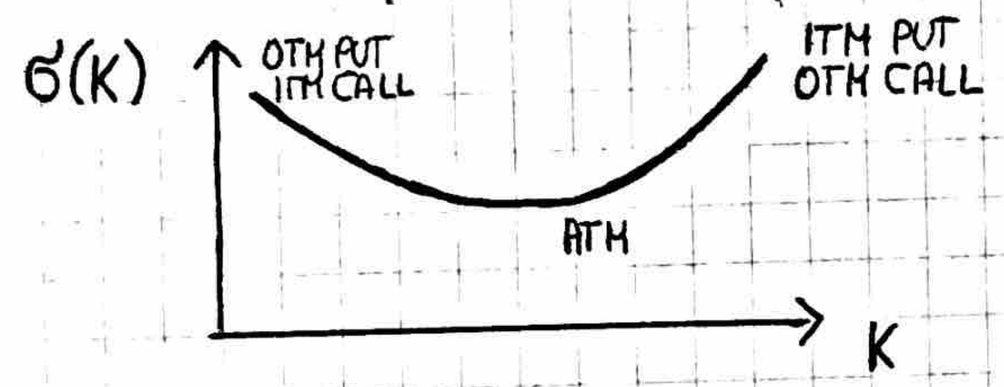
For a given strike price and maturity, the correct volatility to use in conjunction with the BSM model to price a European call should always be the same as that used to price a European put.

The VOLATILITY SMILE (i.e., the relationship between implied volatility and strike price for a particular maturity) is the same for European calls and European puts.

FOREIGN CURRENCY OPTIONS

Volatility smile not predicted by BS model that predicts that the implied volatility curve is flat.

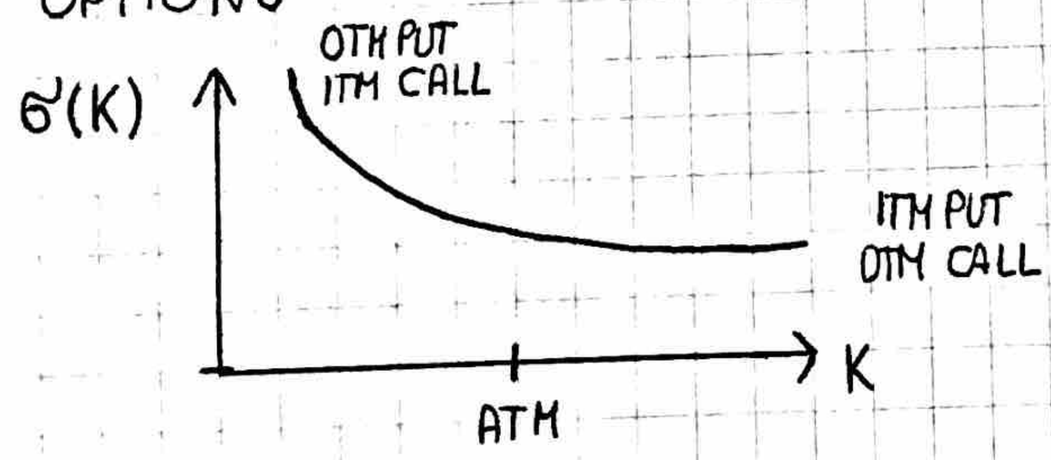
Fixed TTM:



General form of the volatility smile used by traders to price FX options.

Smile of volatility: the implied volatility is relatively low for ATM options; it becomes progressively higher as an option moves either into the money or out of the money.

EQUITY OPTIONS

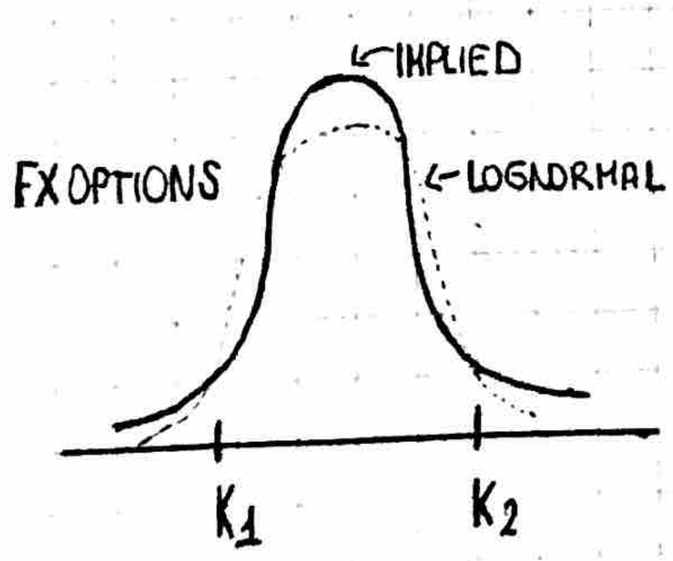


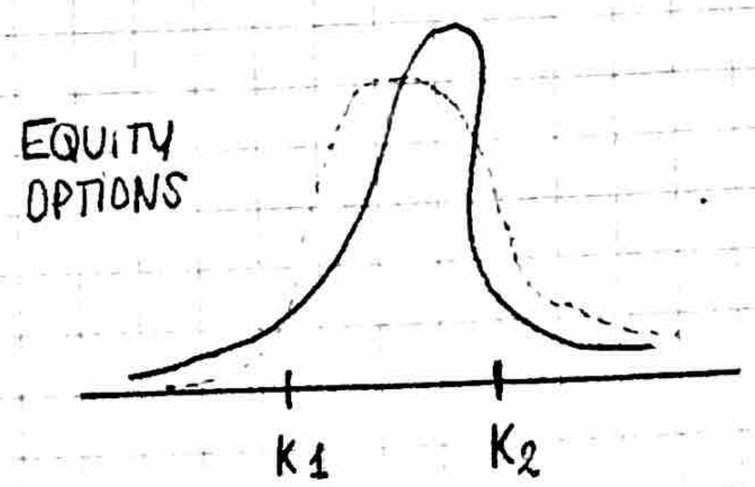
VOLATILITY SKEW / SMIRK

asymmetric behavior

The volatility used to price a low-strike-price option (deep OTM put or deep ITM call) is significantly higher than that used to price a high-strike-price option (deep ITM put or deep OTM call)

Implied and lognormal distribution:





We use the volatility smile in order to find, for the Black formula, the underlying's volatility's value correspondent to the strike of interest.

Price of a Call/Put based on many parameters: $C(t_0; t, K, S_0, \sigma, \ldots)$,
the greeks are the sensitivities w.r.t. parameters

6. GREEK LETTERS : DELTA AND VEGA

A financial institution that sells an option to a client in the OTC markets is faced with the problem of managing its risk.
If the option happens to be the same as one that is traded on an exchange, the financial institution can neutralize its exposure by buying on the exchange the same option as it has sold. But when the option has been tailored to the needs of a client and does not correspond to the standardized products traded by exchanges, hedging the exposure is far more difficult.

↓

GREEK LETTERS

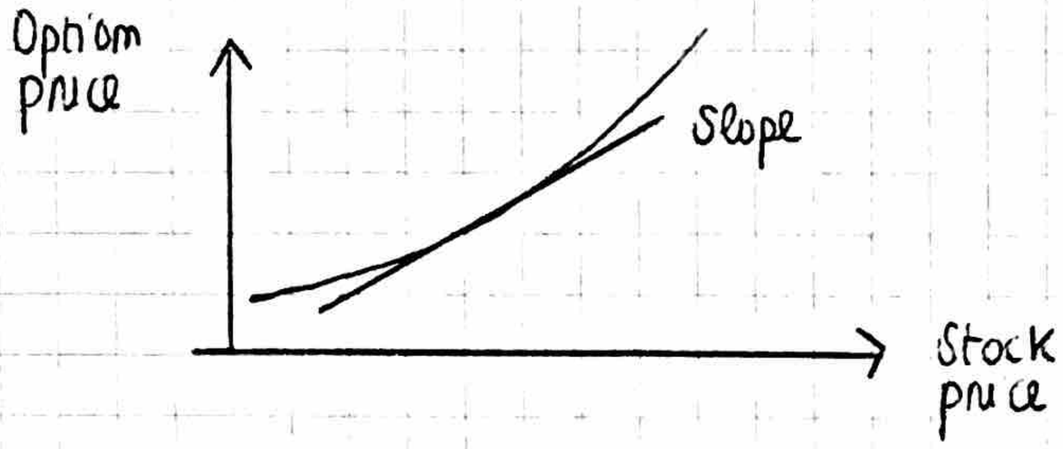
Each greek letter measures a different dimension to the risk in an option position and the aim of a trader is to manage the greeks so that all risks are acceptable.

FIRST ORDER GREEKS

- **DELTA** : rate of change of the option price w.r.t. the price of the underlying asset.

$$\Delta \equiv \frac{\partial C}{\partial S_0}$$

price SENSITIVITY to the movements in the underlying.



Slope of the curve that relates the option price to the underlying asset price.

Since the delta of an option does not remain constant, the trader's position remains delta hedged for only a relatively short period of time.

The hedge has to be adjusted periodically

→ DYNAMIC HEDGING vs. STATIC HEDGING, where a hedge is set up initially and never adjusted.

DELTA FOR A CALL OPTION

$$C(t_0,t) = B(t_0,t)\left\{F_0 N(d_1) - K N(d_2)\right\}$$

$$d_{1,2} = \frac{\ln F_0/K}{\sigma\sqrt{t-t_0}} \pm \frac{1}{2}\sigma\sqrt{t-t_0}$$

$$\Delta = \frac{\partial C}{\partial S_0} = \frac{\partial F_0}{\partial S_0}\cdot\frac{\partial C}{\partial F_0} =$$

$$= \frac{\partial F_0}{\partial S_0} B(t_0,t)\left\{N(d_1) + \overset{(1)}{\boxed{F_0\frac{\partial N(d_1)}{\partial F_0}}} - \overset{(2)}{\boxed{K\frac{\partial N(d_2)}{\partial F_0}}}\right\}$$

$$(1)\quad F_0\frac{\partial N(d_1)}{\partial F_0} = F_0\cdot\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\cdot\frac{\partial d_1}{\partial F_0}$$

$$\parallel$$

$$(2)\quad K\frac{\partial N(d_2)}{\partial F_0} = K\cdot\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\cdot\frac{\partial d_2}{\partial F_0}$$

since the only different part doesn't depend on F_0.

$$d_1^2 = (\ldots)^2 + \cancel{2}\,\frac{\ln F_0/K}{\sigma\sqrt{t-t_0}}\,\frac{1}{\cancel{2}}\sigma\sqrt{t-t_0} + (\ldots)^2$$

$$d_2^2 = (\ldots)^2 - \cancel{2}\,\frac{\ln F_0/K}{\sigma\sqrt{t-t_0}}\cdot\frac{1}{\cancel{2}}\sigma\sqrt{t-t_0} + (\ldots)^2$$

$$(1) = F_0\cdot e^{-\frac{1}{2}[\ln F_0/K]}\cdot(\ldots)$$

$$(2) = e^{+\frac{1}{2}[\ln F_0/K]}\cdot(\ldots)$$

equal in both cases.

$$(1) = F_0\cdot\left(\frac{F_0}{K}\right)^{-\frac{1}{2}}\cdot(\ldots)$$

$$(2) = \left(\frac{F_0}{K}\right)^{\frac{1}{2}}\cdot(\ldots)$$

$(1) = (2)$?

$$F_0\cdot\left(\frac{F_0}{K}\right)^{-\frac{1}{2}} = K\cdot\left(\frac{F_0}{K}\right)^{\frac{1}{2}}$$

$$\left(\frac{F_0^{\cancel{2}}}{\cancel{F_0}}\cdot K\right)^{1/2} = \left(\frac{K^{\cancel{2}}}{\cancel{K}}\cdot F_0\right)^{1/2} \quad \text{OK}$$

$$\Rightarrow \Delta = \frac{\partial F_0}{\partial S_0} B(t_0,t)\{N(d_1)\}$$

Long on the call:
$\Delta N \eta$ ΔS option

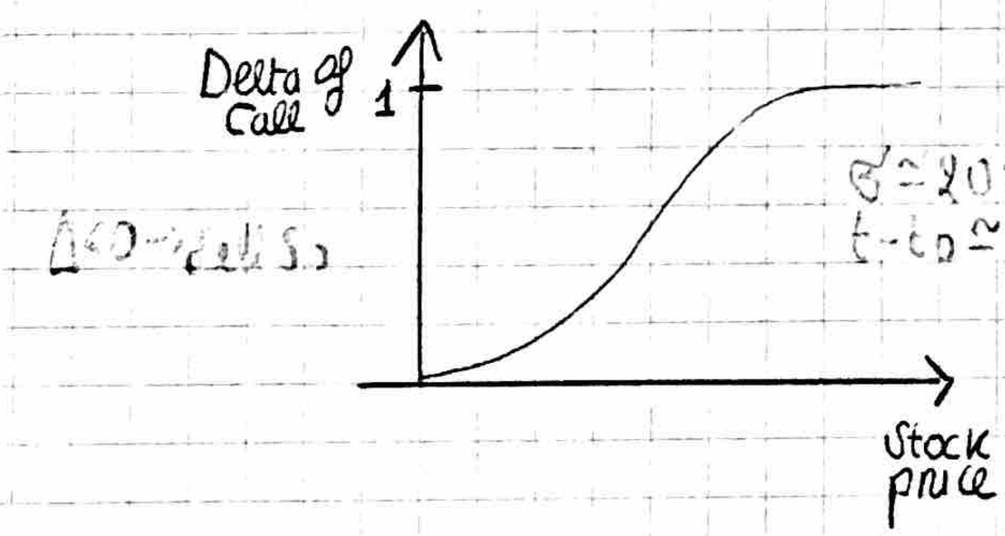
$-\Delta N \eta$ Realize what I sell on shares mkt

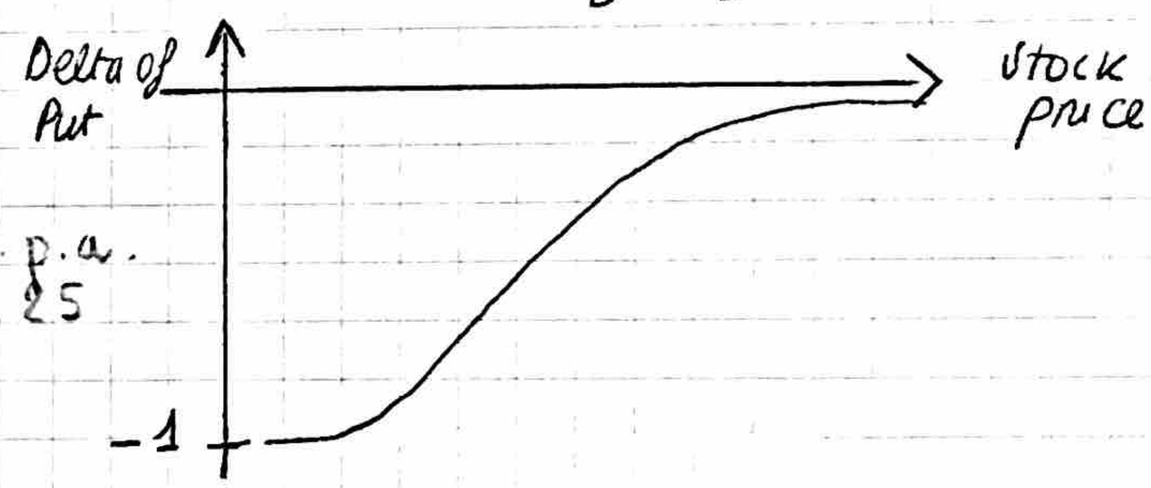
B&S : $F_0 = S_0 e^{r(t-t_0)}$ $\Rightarrow$ $\Delta_{call} = N(d_1)$

$\Delta_{put} = -N(-d_1)$

Using delta hedging for a short position in a European call option involves maintaining a long position of $N(d_1)$ for each option sold.
Similarly, using delta hedging for a long position in a European call option involves maintaining a short position of $N(d_1)$ shares for each option purchased.

For a European put option, delta is negative, which means that a long position in a put option should be hedged with a long position in the underlying stock and vice versa.





with dividends: $\Delta_{call} = e^{-d(t-t_0)} N(d_1)$

$\Delta_{put} = e^{-d(t-t_0)} [N(d_1) - 1]$

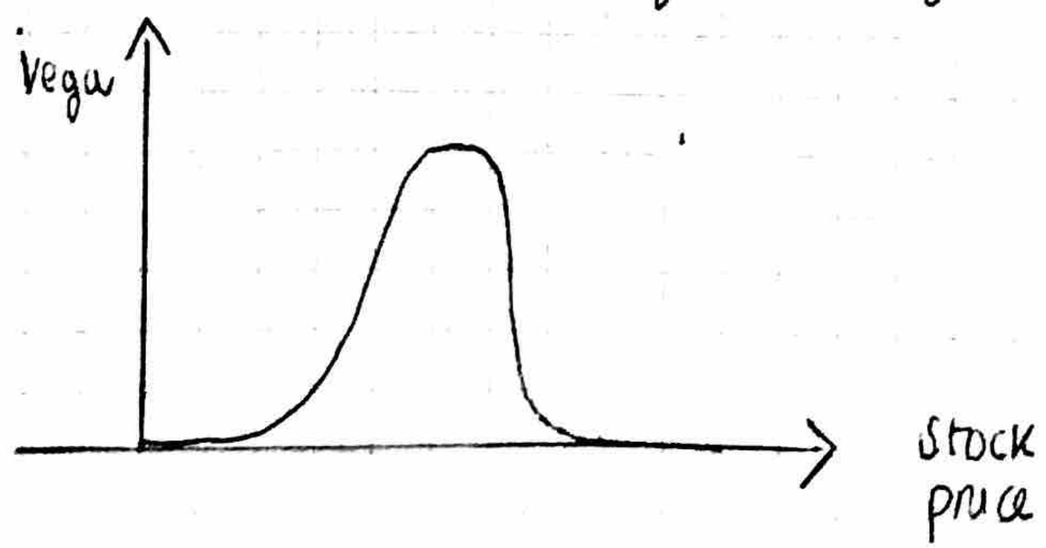
- VEGA : rate of change of the value of the price w.r.t. the volatility of the underlying asset.

hedge it trading other derivatives and options

$$V \equiv \frac{\partial C}{\partial \sigma} \Delta\sigma$$

price SENSITIVITY to the movements in volatility.

Volatility of the asset underlying a derivative is not constant, but changes over time: the value of a derivative is liable to change because of movements in volatility as well as because of changes in the asset price.



VEGA FOR A CALL OPTION

$$V = \frac{\partial C}{\partial \sigma} = B(t_0,t)\left\{ F_0 \frac{\partial N(d_1)}{\partial \sigma} - K \frac{\partial N(d_2)}{\partial \sigma} \right\}$$

$$F_0 \frac{\partial N(d_1)}{\partial \sigma} = F_0 \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \cdot \frac{\partial d_1}{\partial \sigma}$$

$$K \frac{\partial N(d_2)}{\partial \sigma} = K \frac{e^{-d_2^2/2}}{\sqrt{2\pi}} \cdot \frac{\partial d_2}{\partial \sigma}$$

$$\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \frac{1}{2}\sqrt{t-t_0} - \left(-\frac{1}{2}\sqrt{t-t_0}\right) \right)$$

$$\Rightarrow V = B(t_0,t)\, F_0 \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \cdot \sqrt{t-t_0}$$

Importance of vega: volatility doesn't move so fast as the underlying, it is more "sticky", so that I can handle it more easily.

Market Maker with:

<u>ATM-Spot</u>, $t-t_0 = 3m$

He buys the option (Call) ($N = 5$ Mln €)
If the underlying has a $\Delta S_0 = -2\%$, how much does the MM lose?

$$d_1 \approx 0 \rightarrow N(d_1) \sim \frac{1}{2} \rightarrow \text{loss} = \frac{1}{2}(-2\%)\cdot N = -50\text{ K €}$$

A MM does the HEDGING OF THE POSITION: he buys/sells other mkt instruments in order to cover the loss.

He can hedge using a replication strategy (static hedging). He models the underlying, but since agrees that he is not more in the mkt, the strategy is called static.

He is long on the call, when $\Delta S_0 = -2\%$, he enters into a short position on the underlying → he sells 2.5M € of underlying

7. GREEK LETTERS: GAMMA AND THETA. TIME VALUE.

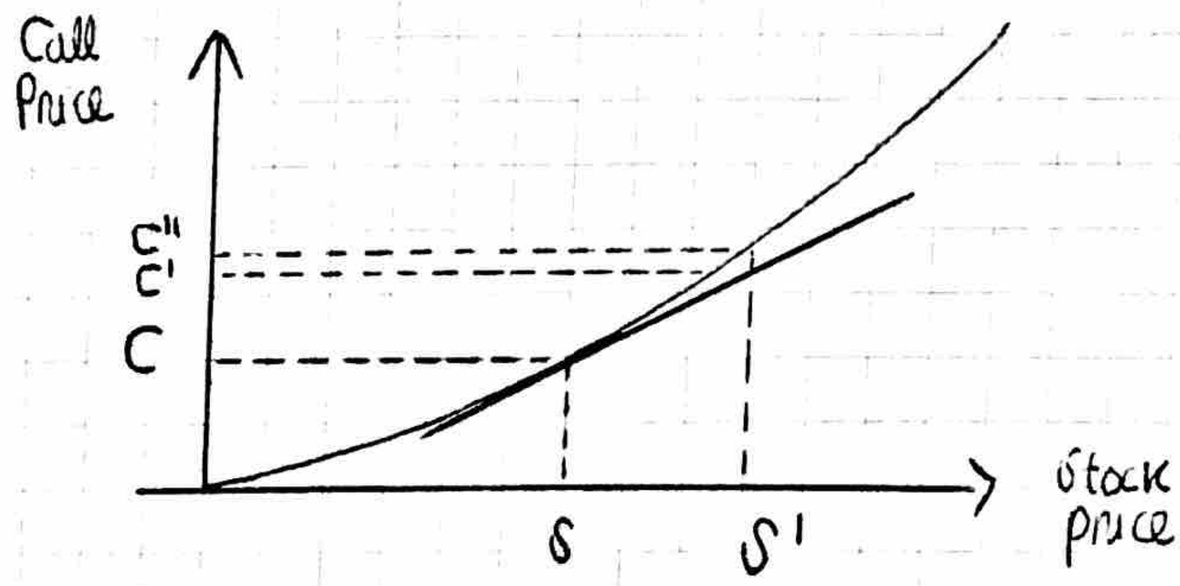
SECOND ORDER GREEKS

- GAMMA : Rate of change of the option price's delta w.r.t. the price of the underlying asset.

$$\Gamma \equiv \frac{\partial^2 C}{\partial S_0^2} = \frac{\partial \Delta}{\partial S_0}$$

If gamma is small, delta changes slowly and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.

However, if gamma is highly negative or highly positive, delta is very sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time.



When the stock price moves from S to S', delta hedging assumes that the option prices move from C to C', when in fact it moves from C to C''.

The difference between C' and C'' leads to a hedging error. The size of the error depends on the curvature of the relationship between the option price and the stock price.

If I'm long on a call and I see that the underlying price has gone upwards, as a market maker I should immediately manage the Δ.

SELL whenever S goes HIGH.
BUY whenever S goes DOWN.

LONG DERIVATIVE <-> make money managing position (hedging).

Neutralize ptf gamma: guarantee Δ-hedge effective under a bigger range of S measurements.

- THETA : Rate of change of the value of the portfolio w.r.t. the passage of time with all else remaining the same.

$$\Theta \equiv \frac{\partial C(S,t)}{\partial t} < 0$$

TIME DECAY of the portfolio

as time passes with all else remaining the same, the option is less valuable.

An option value can be decomposed into an intrinsic value + a time value. Going on with time we lose this time value, since we are paying for the previous right : we have to move fast in managing the Δ, otherwise we lose time value.

→ Γ strategy compensates the money I lose due to Θ.

Theta is not the same type of hedge parameter as delta. There is uncertainty about the future stock price, but there is no uncertainty about the passage of time. It makes sense to hedge against changes in the price of the underlying asset but it does not make any sense to hedge against the passage of time.

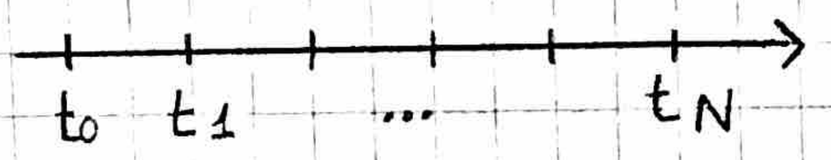
European Call and Put : most liquid derivatives, most traded in the market.

If a client asks for a customized derivative, the market maker can calibrate the model for a European option (closed formula) and then price something more complicated : he uses plain vanillas to evaluate more difficult stuff.

→ start from standardized derivatives to customized.

PRICING VIA SIMULATION : A GENERAL PROBLEM FORMULATION

Given a collection of reset dates :



we wish to price an exotic option :

$$\mathcal{O} = E\left[\sum_i D_{0i}\, h(t_i, \vec{F}(t_i)) \right]$$

D_{0i} stochastic discount in (t_0, t_i)

$h(t_i, \vec{F}(t_i))$ payoff

In general we use MC

Lezione di 9:

INTEREST RATES' CURVE

not more equity

- Interbank market is more complicated

FIXED OUTCOME PRODUCTS → LOANS
→ MORTGAGES
→ BONDS

← main exposure of banks is on FIXED INCOME PRODUCTS

In the equity world, dynamic of the underlying.
Now, dynamic of the forward.

For I.R., the <u>players</u> are the <u>banks</u> the assets are deposits of one bank to another bank.

Contracts:
- deposits
- FRA (Fwd rate agreements)
- swaps & fwd swaps

- bootstrap interbank mkt curve (single curve)

ingredients in the bootstrap:
- deposits
- sht futures
- swaps

- interest rate dynamics

plain vanilla options: Caps / floors & swaptions

- sensitivities measurements.

Remember that r is involved in $B(t_0,t)$, but banks are not interested in ZCB, we have to reconstruct r with the instruments of this mkt.

DEFINITIONS IN A SINGLE-CURVE FRAMEWORK

DISCOUNT FACTORS

$$D(t_0,t_1) \equiv e^{-\int_{t_0}^{t_1} r_s \, ds}$$

Stochastic discount, with r_s stochastic instantaneous IR at time s.

$$B(t_0,t_1) \equiv \mathbb{E}_0[D(t_0,t_1)]$$

discount factor

LIBOR

$$L(t_0; t_1, t_2)$$

t_0 → Value date FIXING

t_1 → IR starts to be applied CALCULATION STARTS

t_2 → IR stops to be applied CALCULATION ENDS

I establish in t_0 the rate for the period between t_1 and t_2.

FORWARD LIBOR established in t_0, calculation starts in t_1 and ends in t_2.

while with forwards we had only 2 dates.

YEAR FRACTION

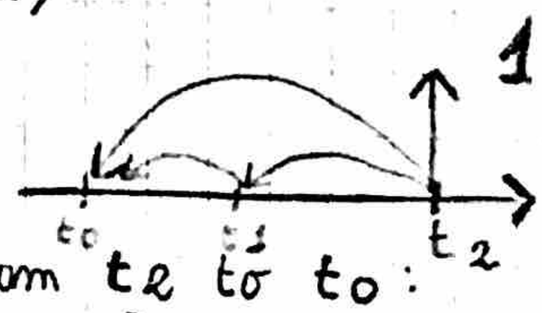
All rates are p.a., so I have to multiply by δ:

$\delta(t_0, t_1)$: year fraction between t_0 and t_1.

- ACT / 360 (depos, ...)
- ACT / 365 (TTM, zero rates, ...)
- 30 / 360 (fixed leg of a IRS, ...)

ACT = actual time interval

PROPERTY OF STOCHASTIC DISCOUNT



For a given stochastic pattern, if I want to discount from t_2 to t_0:

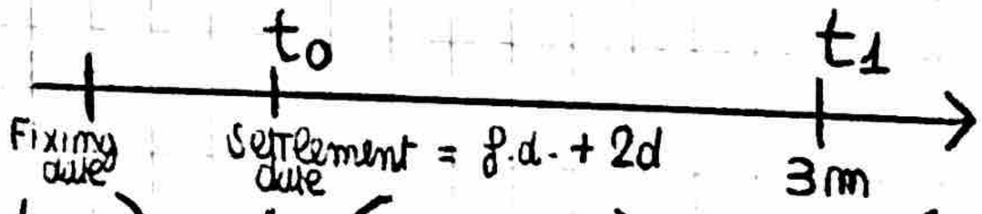
$$D(t_0, t_2) \equiv D(t_0, t_1)\, D(t_1, t_2)$$

This property doesn't hold for $B(t_0, t)$: only for stochastic paths!

Connection between QUOTED QUANTITIES (rates of depos, ...) and DF: ↓

First kind of assets in the interbank market:

INTERBANK DEPOSITS



→ $L(t_0; t_0, t_1) \equiv L(t_0, t_1) = L_1(t_0)$

deposit from one bank to another.

- LIBOR : London Inter Bank Offer Rate.
- EURIBOR : € " " " "

→ 24 greatest European banks: I.R. of the offers are collected by the renters; cutting off the extreme ones and taking the mean, we obtain the EURIBOR

→ Euribor rate fixing mechanism: each day at 11:00 C.E.T. of the Target Calendar.

There is one key rate every day: the fixing rate. For each day, for every Target Calendar, and in the past, there is an agency renter (nowadays, a market called ICE, Inter Continental Exchange) that calls every bank to ask about the rates, collects all the OFFER RATES and gets an AVERAGE ⇒ EURIBOR.

→ Interest rates are paid at the end of the calculation period. Depo day-count for year frac: ACT/360 (Europe & US)

E.G.: you do a mortgage, every 3m you have to pay to the bank

$\underbrace{E_{3m}}_{\text{established at the beginning}}$ + spread INTEREST RATE ON 3m

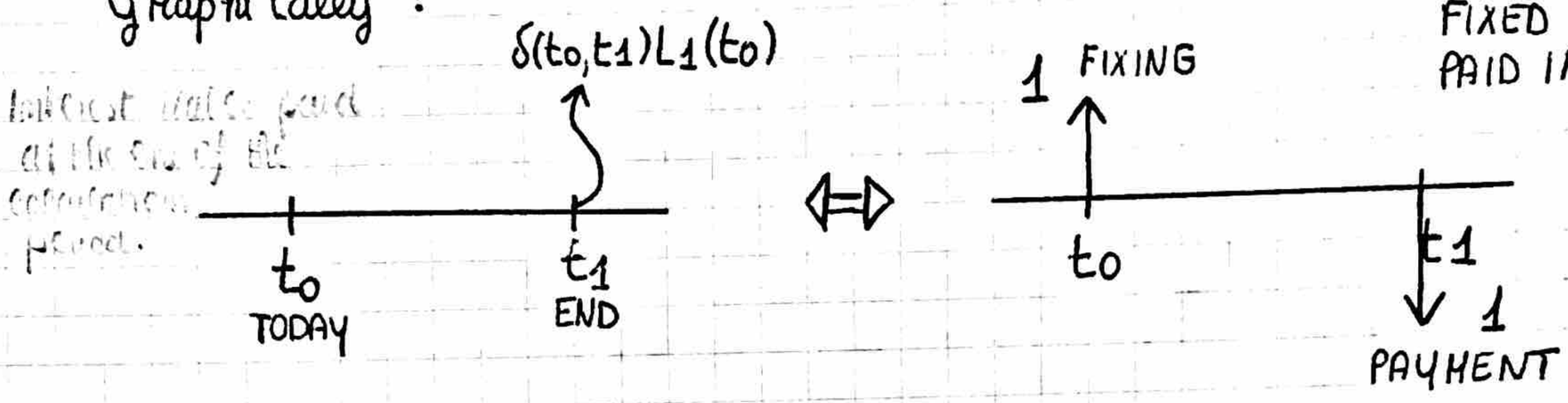
EURIBOR 1d = EONIA (European Over Night Index Average)

From the simple-compounding rule, we get:

DETERMINISTIC

$$B(t_0,t_1) = \frac{1}{1+\delta(t_0,t_1)\underbrace{L(t_0,t_1)}_{\text{known today, not stochastic.}}} \Leftrightarrow \delta(t_0,t_1)L(t_0,t_1)B(t_0,t_1) = \underset{\text{1 received today}}{1} - B(t_0,\underset{\text{1 paid in } t_1}{t_1})$$

Graphically:



FIXED IN ADVANCE, PAID IN ARREARS

$$\underbrace{\mathbb{E}[D(t_0,t_1)\,\delta(t_0,t_1)\,L(t_0,t_1)]}_{\text{exp of the flow today}} = \delta(t_0,t_1)L(t_0,t_1)B(t_0,t_1) = 1 - B(t_0,t_1)$$

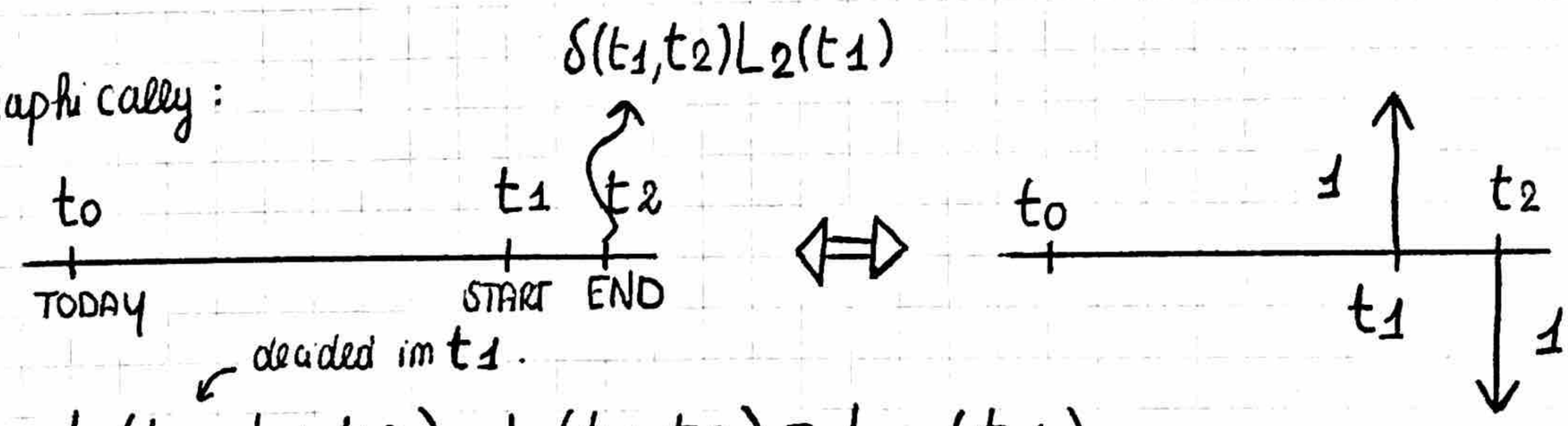
The value date is different from the date in which the transaction really happens: in practice, all transactions are thought to begin in 2 working days, at settlement date.

For a depo 3m, TTM = 3m − settlement date.

DEPOSITS AT A FUTURE DATE

← If I have a fixing date t_0, the real deposit will start 2 days after (because of physical transaction of money).

Graphically:



decided in t_1.

$$L(t_1;t_1,t_2) = L(t_1,t_2) = L_2(t_1)$$

The rate will change from t_0 to t_1 → I have to wait for t_1 to know the rate → STOCHASTIC.

Formally:

$$\boxed{\mathbb{E}_0[D(t_0,t_2)\underbrace{\delta(t_1,t_2)L(t_1;t_1,t_2)}_{\text{not known}}] = \underbrace{B(t_0,t_1) - B(t_0,t_2)}_{\text{known}}}$$

STOCHASTIC

CALIBRATION -> start from olura

PROOF. $\mathbb{E}_0[D(t_0,t_2)\,\delta(t_1,t_2)\,L(t_1;t_1,t_2)] =$

$$= \mathbb{E}_0[D(t_0,t_2)(-1+1+\delta(t_1,t_2)\,L(t_1,t_2))] =$$

$$= -B(t_0,t_2) + \mathbb{E}_0[\underbrace{D(t_0,t_1)}_{\text{Known}}\underbrace{D(t_1,t_2)}_{\text{stochastic}}[1+\delta(t_1,t_2)\underbrace{L(t_1,t_2)}_{\text{Known: libor fixes in } t_1.}]]$$

ALTIME RULE $= \left(\mathbb{E}[\cdot \mid \mathcal{F}_{t_0}] = \mathbb{E}[\mathbb{E}[\cdot \mid \mathcal{F}_{t_1}] \mid \mathcal{F}_{t_0}]\right) =$

$$= \mathbb{E}_0[D(t_0,t_1)\,\mathbb{E}[D(t_1,t_2)[1+\delta(t_1,t_2)L(t_1,t_2)]]] - B(t_0,t_2) =$$

$$= \mathbb{E}[D(t_0,t_1)\cancel{[1+\delta(t_1,t_2)L(t_1,t_2)]}\cdot \underbrace{\cancel{\mathbb{E}[D(t_1,t_2)\mid \mathcal{F}_{t_1}]}}_{B(t_1,t_2)} \mid \mathcal{F}_{t_0}] - B(t_0,t_2) =$$

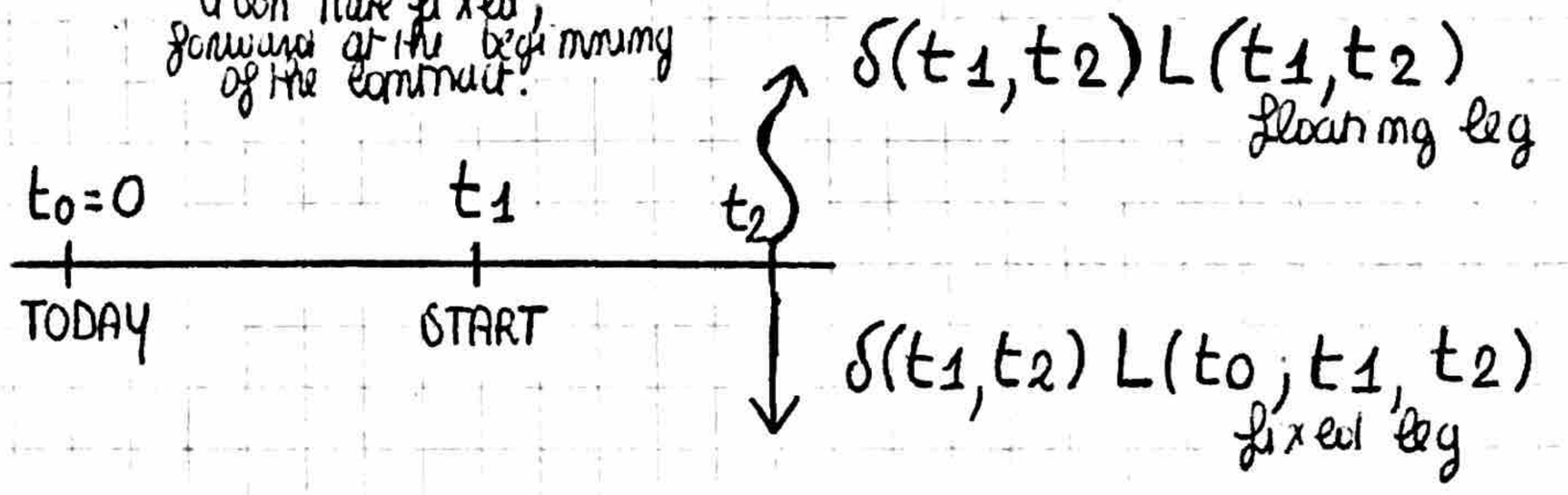
$$= B(t_0,t_1) - B(t_0,t_2) \qquad \blacksquare$$

Now we can start building the derivatives:

FORWARD RATE AGREEMENT (FRA)

Agreement between two banks that decide in t_0 to exchange $L(t_0;t_1,t_2)$ with $L(t_1;t_1,t_2)$ in t_2.

libor rate fixed, forward at the beginning of the contract.



It's the "same" of FORWARD but in this case there are rates.

The Forward rate satisfies, when FRA contract is written, the condition NPV = 0 (expected value of the flows, MEM). =>

$$NPV^{fixed\ leg} = \mathbb{E}_0\Big[D(t_0,t_2)\,\overbrace{\delta(t_1,t_2)\,L(t_0;t_1,t_2)}^{\text{deterministic, known in } t_0}\Big] =$$

$$= B(t_0,t_2)\,\delta(t_1,t_2)\,L(t_0;t_1,t_2)$$

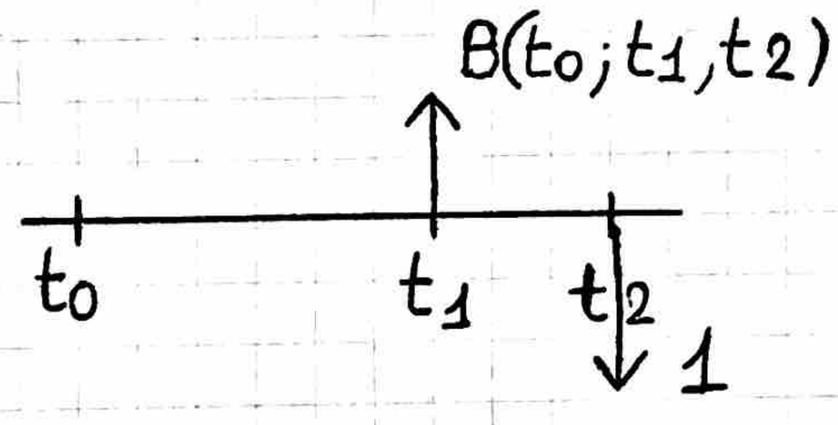
$$NPV^{floating\ leg} = \mathbb{E}_0\Big[D(t_0,t_2)\,\delta(t_1,t_2)\,\underbrace{L(t_1,t_2)}_{\text{stochastic quantity!}}\Big] =$$

like for a floating rate bond

$$= B(t_0,t_1) - B(t_0,t_2)$$

$$\Rightarrow \boxed{B(t_0,t_2)\,\delta(t_1,t_2)\,L(t_0;t_1,t_2) = B(t_0,t_1) - B(t_0,t_2)} \quad (*)$$

<u>FORWARD DISCOUNT FACTOR</u> : establish today the amount to pay in t_1 to receive 1 in t_2.



$$NPV = 0 \Rightarrow B(t_0,t_1)\,B(t_0;t_1,t_2) = B(t_0,t_2)$$

$$\Downarrow$$

$$B(t_0;t_1,t_2) = \frac{B(t_0,t_2)}{B(t_0,t_1)}$$

Hence, we can rewrite the libor (*)

$$\delta(t_1,t_2)\,L(t_0;t_1,t_2) = \left(\frac{1}{B(t_0;t_1,t_2)} - 1\right)$$

$$\boxed{L(t_0;t_1,t_2) = \frac{1}{\delta(t_1,t_2)}\left[\frac{1}{B(t_0;t_1,t_2)} - 1\right]}$$

$$\boxed{B(t_0;t_1,t_2) = \frac{1}{1+\delta(t_1,t_2)\,L(t_0;t_1,t_2)}}$$

CALIBRATION -> start from ois

STIR FUTURE

derivative exchange

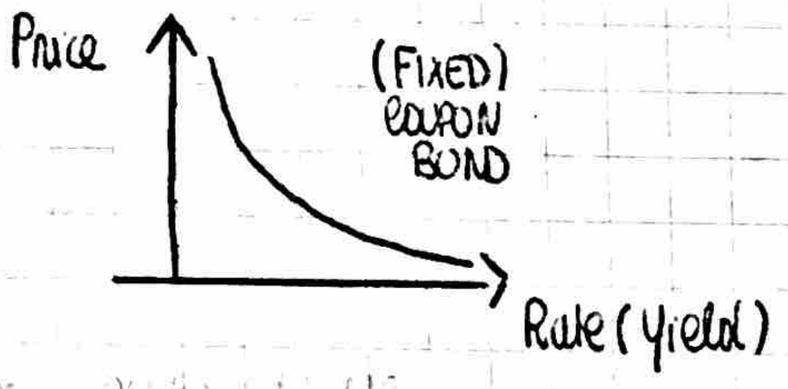
Short Term Interest Rates (STIR) Futures (or Eurodollar futures)

Standardization:

→ 3m Euribor rate UNDERLYING
→ only at IMM dates
→ price @ expiry = 100 − Libor rate $L(t_1; t_1, t_2)$ in % with day convention "ACT/360".
→ cash settlement

Price "=" 100 − Forward Libor rate $L(t_0; t_1, t_2)$ in %.

Delivery date: 3rd wednesday (rates established)
Fixing: 2 days before (3rd Monday)
End in 3m



Fixed CB goes down if the yield goes up.

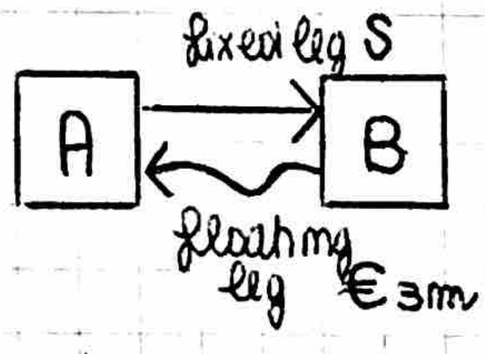
In order to get a future that decreases with the rates, we use 100 − LIBOR(%).
With very low rates, price ≈ 100.

IR are stochastic, so price is not exactly equal to 100 − LIBOR: there are convexity adjustments, but very little.

PLAIN VANILLA INTEREST RATE SWAP (IRS)

generalization with several flows as FRAs.

a swap can be either payer or receiver for a given counterparty!



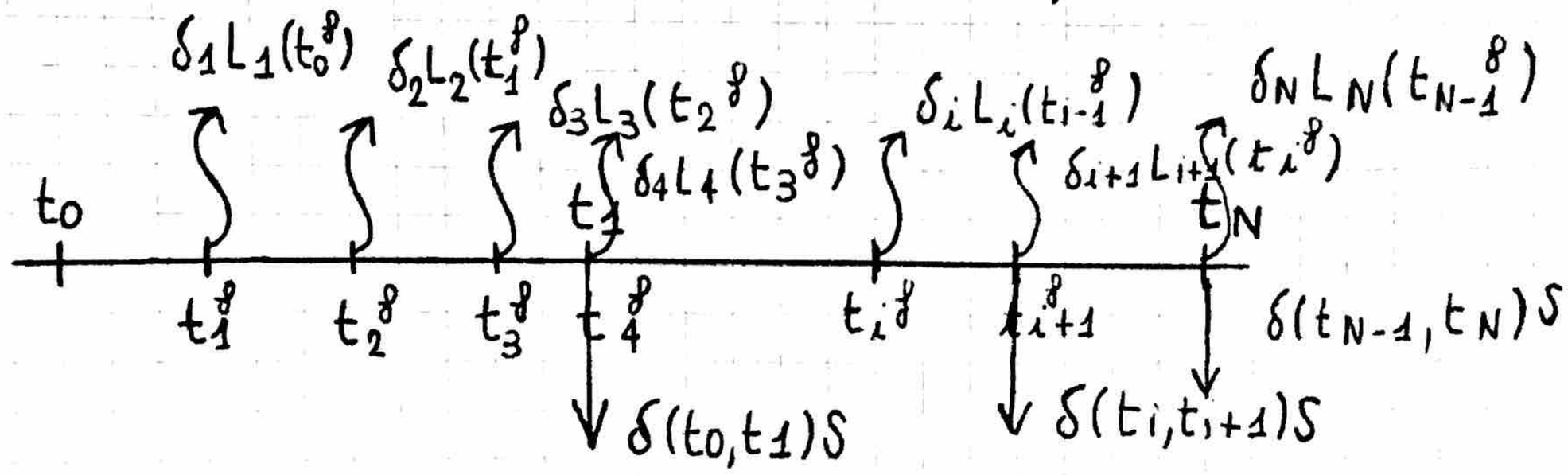
Most traded derivatives in the financial world.
Two counterparties (banks) decide to exchange A SET of flows.

A receives some floating rates (depo rates) (not known) and pays a fixed rate (known at time t_0):
A pays the swap (it's the PAYER).

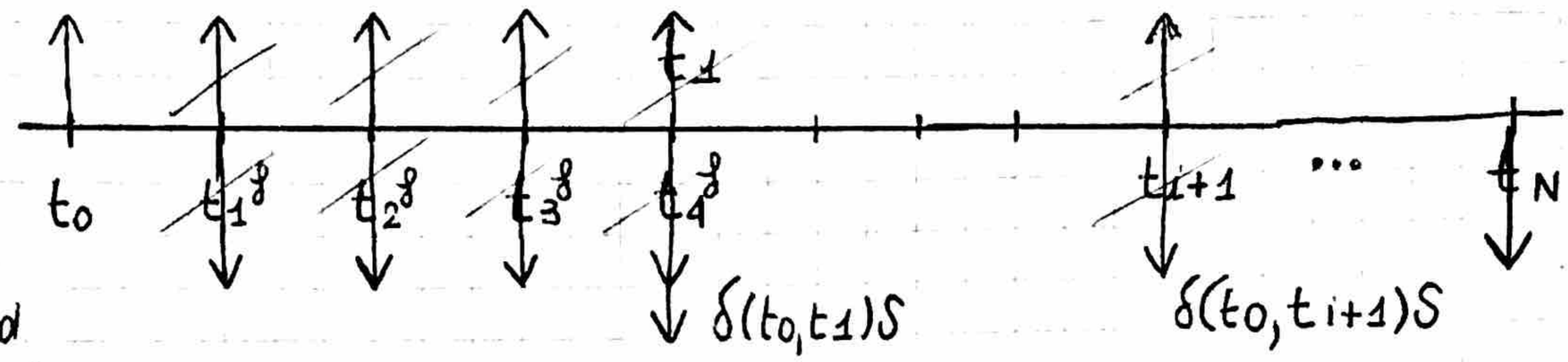
FLOATING LEG: payed quarterly
FIXED LEG: payed annually

the two legs in a swap could have different frequency and day-count.

some FIXED RATES are exchanged with FLOATING RATES, but in different moments not just in one moment as for FRAs



What is left are just the first and the last flows.



$$NPV^{fixed} = B(t_0,t_1)\,\delta(t_0,t_1)\,S + B(t_0,t_2)\,\delta(t_0,t_2)\,S + \ldots =$$

$$= S \sum_{i=1}^{N} B(t_0,t_i)\,\delta(t_{i-1},t_i)$$

$$NPV^{floating} = \underbrace{\mathbb{E}_0\left[D(t_0,t_1^{\delta})\,\delta(t_0,t_1^{\delta})\,L(t_0,t_1^{\delta})\right]}_{B(t_0,t_0)-B(t_0,t_1^{\delta}) = 1-\cancel{B(t_0,t_1^{\delta})}} +$$

dipo on a future date

$$+ \underbrace{\mathbb{E}_0\left[D(t_0,t_2^{\delta})\,\delta(t_1^{\delta},t_2^{\delta})\,L(t_1^{\delta},t_2^{\delta})\right]}_{\cancel{B(t_0,t_1^{\delta})}-B(t_0,t_2^{\delta})} + \ldots$$

telescopic sum

$$\ldots \overset{\swarrow}{=} 1 - B(t_0,t_N)$$

Find value of S imposing $NPV = 0$:

$$\boxed{S(t_0;t_0;t_N) = \frac{1 - B(t_0,t_N)}{\underbrace{\sum_{i=1}^{N} \delta(t_{i-1},t_i)\,B(t_0,t_i)}_{BPV_{0N}(t_0)}}}$$

Year fractions conventions $\neq$ for the two legs:

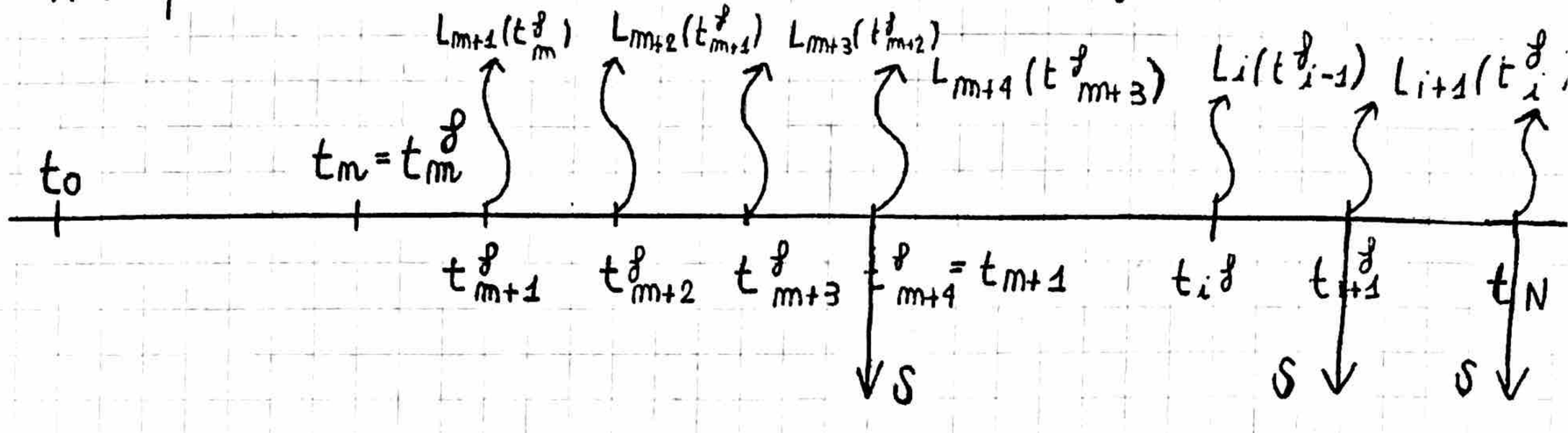
$\frac{ACT}{360}$ floating leg vs. $\frac{30}{360}$ for the fixed leg

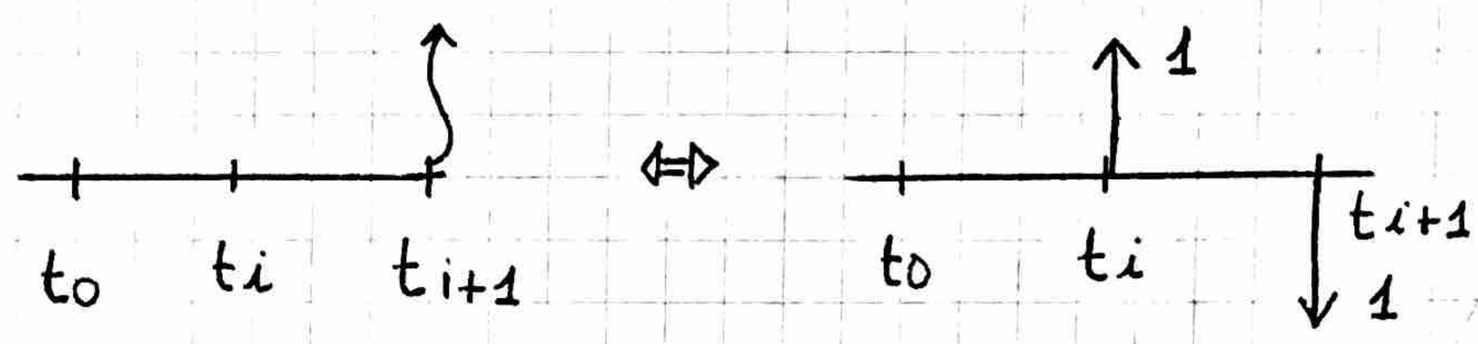
but in BPV we have $30/360$.

FORWARD (START) SWAP

The two counterparties decide in t_0 to exchange the fixed legs with the floating between t_m and t_N.

A swap established in t_0 that starts at a forward date





il pagamento è nel secondo anno, come prima che era in t_1. Data t_m e t_{m+1} sono floating.

$$NPV^{fixed} = S(t_0; t_m, t_N) \sum_{i=m+1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i)$$

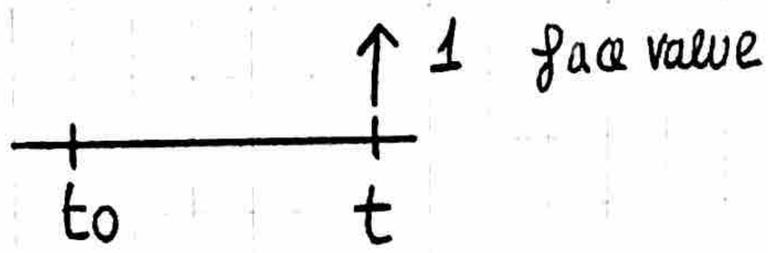
$$NPV^{floating} = E\Big[\underbrace{D(t_0, t^f_{m+1})\, \delta(t^f_m, t^f_{m+1})\, L(t^f_m, t^f_{m+1})}_{B(t_0, t^f_m) - B(t_0, t^f_{m+1})} \Big] + \dots \overset{\text{telescopic sum}}{=} B(t_0, t_m) - B(t_0, t_N)$$

$$\Rightarrow S(t_0; t_m, t_N) = \frac{B(t_0, t_m) - B(t_0, t_N)}{\sum_{i=m+1}^{N} \delta(t_{i-1}, t_i) B(t_0, t_i)} = \frac{1 - B(t_0; t_m, t_N)}{BPV_{mN}(t_0)}$$

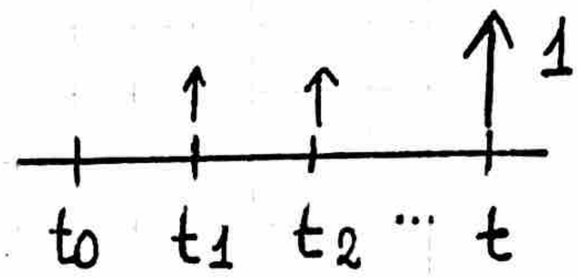
$$BPV_{mN}(t_0) = \sum_{i=m+1}^{N} \delta(t_{i-1}, t_i)\, B(t_0; t_m, t_i)$$

IDEALIZED BONDS

ZCB



Fixed CB

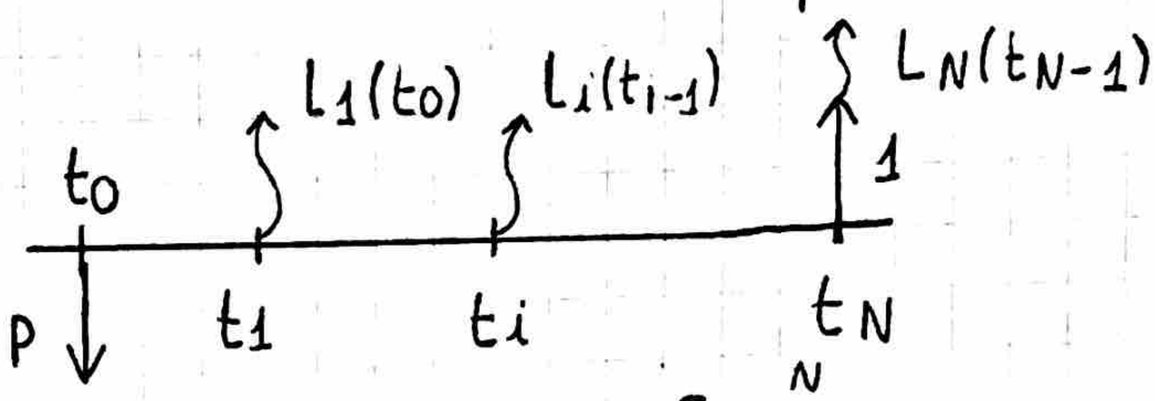


IDEAL WORLD: I'm supposing that the issuer of the bond will not default and will surely pay this bond. NO DEFAULT RISK.

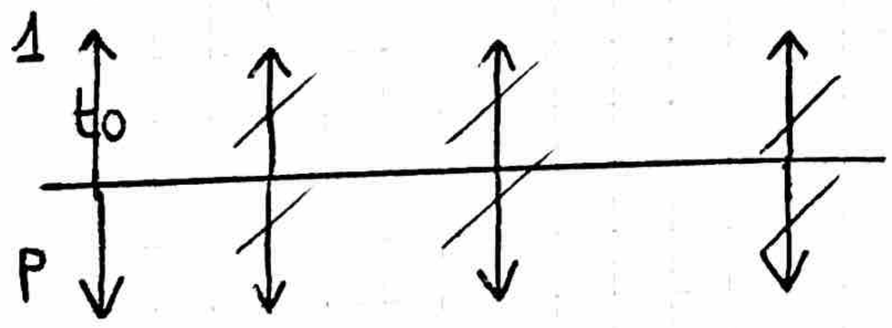
"INTER BANK" FLOATER

Not a derivative (not a real financial instrument, only a "fictitious" one)

Bond where each coupon has floating rates.



$$P = \mathbb{E}_0 \left\{ \sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, L(t_{i-1}, t_i)\, D(t_0, t_i) \right\} + B(t_0, t_N)$$

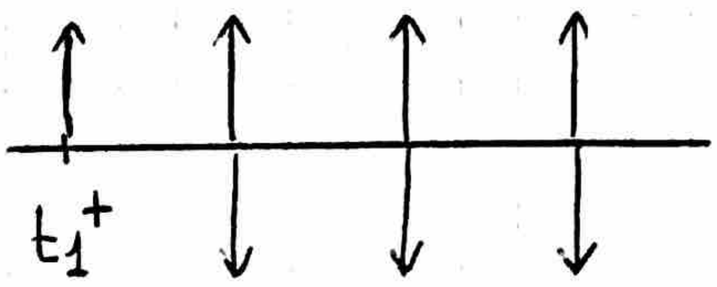


$$NPV^{\text{flows above}} = \underbrace{1 - B(t_0, t_N)}_{\text{floating rates}} + \underbrace{B(t_0, t_N)}_{\text{face value}}$$

$$\Rightarrow P = 1$$

What is the value of the I.B. floater immediately after having paid the first coupon (in t_1^+)?

I have already received the coupon, now it is in the past:



$$NPV = 1 - B(t_1, t_N) + B(t_1, t_N)$$

$$= 1 \quad \text{same value.}$$

The value will be exactly the same after each payment of the coupons, no risk.

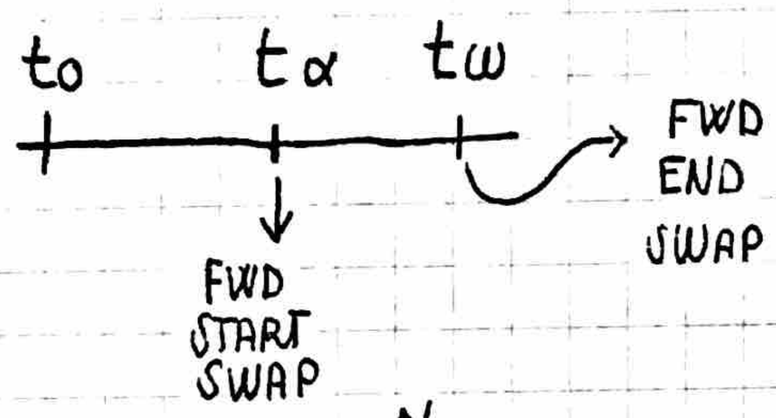
PLAIN VANILLA OPTION MARKETS

OPTION ON A FORWARD-SWAP

Swaption (Payer): right to enter in a "payer swap"

$$SP_{\alpha\omega} = B(t_0, t_\alpha)\, BPV_{\alpha\omega}(t_0) \left\{ S_{\alpha\omega}(t_0) \cdot N(d_1^S) - K N(d_2^S) \right\}$$

$$d_{1,2}^S = \frac{1}{\sigma_{\alpha\omega}\sqrt{t_\alpha - t_0}} \ln\left(\frac{S_{\alpha\omega}(t_0)}{K}\right) \pm \frac{1}{2}\sigma_{\alpha\omega}\sqrt{t_\alpha - t_0}$$



?

$$BPV_{\alpha\omega}(t_0) = \sum_{i=\alpha+1}^{N} \delta(t_{i-1}, t_i)\, B(t_i; t_\alpha, t_\omega)$$

OPTION ON A FORWARD-RATE

Caplet: Call on a Single Libor Rate
Floorlet: Put on a Single Libor Rate

$Cap = \sum Caplet_i$ — the first caplet is known thanks to euribor $\Rightarrow$ 2y cap = 7 caplets 3m

$$Caplet = B(t_0, t_{i+1})(t_{i+1} - t_i)\left\{ L_i(t_0) N(d_1^{cf}) - K N(d_2^{cf}) \right\}$$

$$d_{1,2}^{cf} = \frac{1}{\sigma_i \sqrt{t_i - t_0}} \ln\left(\frac{L_i(t_0)}{K}\right) \pm \frac{1}{2}\sigma_i \sqrt{t_i - t_0}$$

INTEREST RATE SENSITIVITIES: GREEKS.
THE SWAPTION EXAMPLE.

DELTA

$$\Delta \equiv \frac{\partial SP_{\alpha\omega}}{\partial S_{\alpha\omega}(t_0)} = B(t_0, t_\alpha)\, BPV_{\alpha\omega}(t_0)\, N(d_1^S)$$

If we move the underlying swap, we will get different discount factors: so, it doesn't make any sense to take all constant exept for $S_{\alpha\omega}$.

$$S_{\alpha\omega}(t_0) = \frac{1 - B(t_0; t_\alpha, t_\omega)}{BPV_{\alpha\omega}(t_0)}$$

The Delta for swaption is not used in practice.

What is really relevant is the VEGA:

$$V \equiv \frac{\partial SP_{\alpha\omega}}{\partial \sigma_{\alpha\omega}} \Delta\sigma_{\alpha\omega} = B(t_0, t_\alpha)\, BPV_{\alpha\omega}(t_0)\, S_{\alpha\omega}(t_0) \cdot \sqrt{t_\alpha - t_0}\, \frac{e^{-(d_1^S)^2/2}}{\sqrt{2\pi}} \cdot \Delta\sigma_{\alpha\omega}$$

Greeks not adequate to do RM

9. DV01, BPV AND MODIFIED DURATION.

WHAT ARE THEY AND HOW ARE THEY RELATED FOR AN IRS AND AN "IB" FIXED COUPON BOND (TRADED IN VALUE DATE)?

PARALLEL SHIFT & BUCKET DV01.

INTEREST RATE SENSITIVITIES

DV01 – parallel shift of mkt rates (Dollar value of 1 Basis Point) 0.01%

Variation in the NPV (Dollar Value) for a derivative's portfolio (e.g. a single swap) due to the increase of the rates used for the bootstrap of original curve for 1 bp in parallel shift. i.e. it is derivative's sensitivity for a (parallel shift) movement in the rate curve.

DEPOS, SWAP RATES +1bp
FUTURE RATES −1bp (Rates = 100 − price)

$$\boxed{DV01 := NPV\left(t_0, \text{with shifted IR curve upwards } 1bp\right) - NPV(t_0)}$$

sensitivity of my book after a parallel shift in mkt rates.

NPV obtained with the discount factors coming from the bootstrap after having shifted mkt rates of 1bp.

NPV obtained with the discount factors coming from the bootstrap with initial rates.

⇒ Variation in the NPV of a portfolio due to a shift of 1 bp of the rates of the bootstrap.

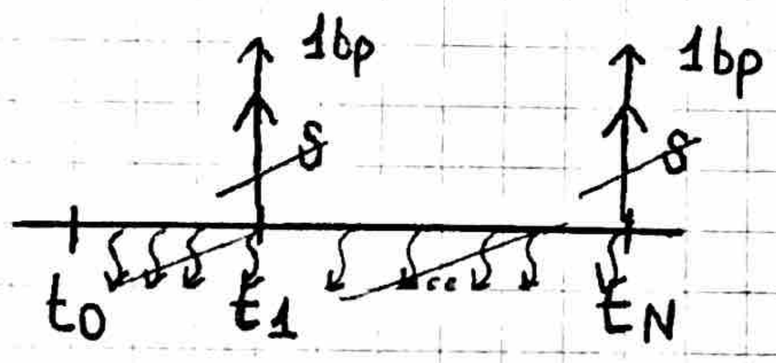
BPV – Basis Point Value

Net Present Value of 1 Basis Point (1 bp = 0.01%).

It is the difference between:

a. NPV of a receiver swap with a 1bp more in the fixed rate of the mid-mkt;

b. NPV of the original receiver swap with (mid-mkt) fixed rate S.



only the bps remain:

$$NPV = \sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, B(t_0, t_i) \cdot 1bp$$

BPV = NPV of 1bp at fixed leg dates.

$$\boxed{BPV := NPV(swap;\, S + 1bp) - NPV(swap;\, S)}$$

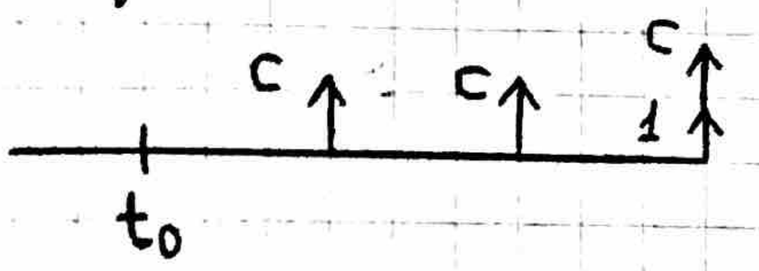
sensitivity of a swap approximation of DV01.

par swap = IBFCB

Remark: The two quantities are quite similar for a plain vanilla par swap (NPV = 0) at trade date.

DV01 is the financial quantity (true sensitivity), while BPV is a simple quantity to be computed.

Why is an "IB" fixed coupon bond connected to swaps?



$c = S_{ON}(t_0)$, fixed leg conv: 30/360

Connection between bond and swap.

$$P(t_0) = B(t_0, t_N) + \sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, c\, B(t_0, t_i)$$

$$P(t_0, c = S) = B(t_0, t_N) + S_{ON}(t_0)\, BPV_{ON}(t_0) =$$

$$= B(t_0, t_N) + 1 - B(t_0, t_N) = 1 \quad \text{"at par"}$$

BOND SENSITIVITIES (DURATION, connected to $DV01^{(z)}$)

$DV01^{(z)}$ – parallel shift of Zero Rates (Dollar Value of 1 bp on Zero Rates)

Variation in the NPV (Dollar Value) for a derivative's portfolio due to the increase of the zero rates for 1bp in parallel shift; i.e. it's derivative's sensitivity for a (parallel shift) movement in the zero rate curve.

The risk manager is not interested in the single book but in the whole market: instead of shifting mkt rates, he shifts zero rates (it's easier and faster than computing the previous DV01).

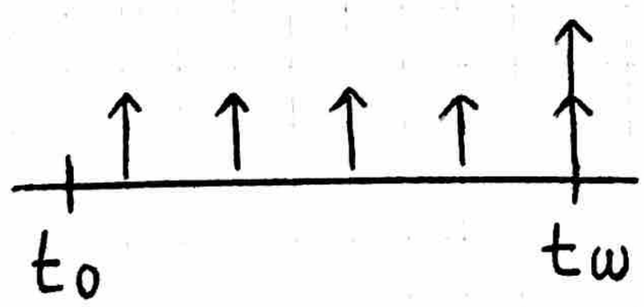
$DV01^{(z)}$ equivalent to DV01 but shifting zero rates: it is introduced for an easier computation, since it is no more necessary bootstrapping again. Infact, we get the discount factors for NPV shift (to) just inverting the formula of shifted zero rates.

MACAULAY DURATION

Variation in the value of an "I.B. coupon bond" for the parallel shift of the corresponding zero-rate curve of 1bp (i.e. the $DV01^{(z)}$ for this bond), normalized by bond price.

$$\boxed{\text{Mac } D(t_0) := -\frac{1}{P_{ow}(t_0)} \lim_{\Delta \to 0} \frac{\Delta P_{ow}(t_0)}{\Delta} = \frac{\sum_{i=1}^{m} c_i (T_i - t_0) B(t_0, T_i)}{\sum_{i=1}^{m} c_i B(t_0, T_i)}}$$

"IB" risk free fixed coupon bond (idealized object with no credit risk).



$$P_{ow}(t_0) = B(t_0, t_w) + \sum_{i=1}^{w} c_i B(t_0, t_i)$$

$$B(t_0, t) = e^{-R(t_0,t)(t-t_0)} \xrightarrow{\text{shift}} \tilde{B}(t_0, t) = e^{-[R(t_0,t)+\Delta](t-t_0)}$$

1st order expansion, $\Delta \to 0$

$$\tilde{B}(t_0, t) \cong e^{-R(t_0,t)(t-t_0)} [1 - \Delta(t-t_0)] =$$

$$= B(t_0, t) [1 - \underbrace{\Delta}_{1bp}(t-t_0)]$$

$$\Delta Pow(t_0) = Pow^{shifted}(t_0) - Pow(t_0) =$$

$$= \sum_{i=1}^{w} c_i e^{-[R(t_0,t_i)+\Delta](t_i-t_0)} - \sum_{i=1}^{w} c_i e^{-R(t_0,t_i)(t_i-t_0)} =$$

$$= \sum_{i=1}^{w} c_i e^{-R(t_0,t_i)(t-t_0)} (1-\Delta(t_i-t_0)) - \sum_{i=1}^{w} c_i e^{-R(t_0,t_i)(t_i-t_0)} =$$

$$= -\Delta \sum_{i=1}^{w} c_i e^{-R(t_0,t_i)(t_i-t_0)} (t_i-t_0)$$

Positive variation in the rates → negative variation in the value of the coupon.

Remark: for an "I.B. Coupon Bond" at par, (DEN = PRICE BOND ≃ 1)
the Modified duration is equal to the $DV01^{(z)}$ of the corresponding par swap (and then quite similar to the BPV of the corresponding par swap).

MacD is the $DV01^{(z)}$ for one fixed IBCB, exept for the denominator, since MacD is a relative measure, divided by the bond price) while $DV01^{(z)}$ is an absolute value).

$$DV01^{(z)} = MD \times \text{Face Value} \times 1\,bp$$

CURVE SENSITIVITIES

In practice, the IR curve moves as it wants (not necessarily parallel shift).

BUCKET DV01

Variation in the NPV (Dollar Value) for a derivative's portfolio due to the increase of (just) one rate among the ones used for the bootstrap of the original curve of 1 bp; i.e., it is portfolio's sensitivity for a movement of a single bucket in the rate curve.

Move only one bucket on curve rate, keeping all the remaining there.

Remark 1: The Bucket DV01, computed for every bucket, is the main IR sensitivity.
It is used as main indicator for delta hedging and in RM measurements.

Remark 2: The sum of Bucket DV01 is just the DV01 - parallel shift.

Several rates are very similar, generally they move together (aggregate information: often, sensitivities are relevant only 3rd set of macro-buckets).

↓

COARSE-GRAINED BUCKET DV01

Chose some macro-buckets generally 2y 5y 10y 15y 20y 30y 40y 50y

positive for a payer swap.

Sensitivities tell us how much money our positions will gain or lose after a 1bp movement in the mkt prices.