STRUCTURED & DERIVATIVE PRODUCTS ③

STRUCTURED EQUITY

STRUCTURED PRODUCTS & DIGITAL RISK IN EQUITY

STRUCTURED PRODUCTS :

STRUCTURED PRODUCTS

→ CREDIT LINKED NOTES: issues guaranteed by m.p. (ABS, CLO, ...)

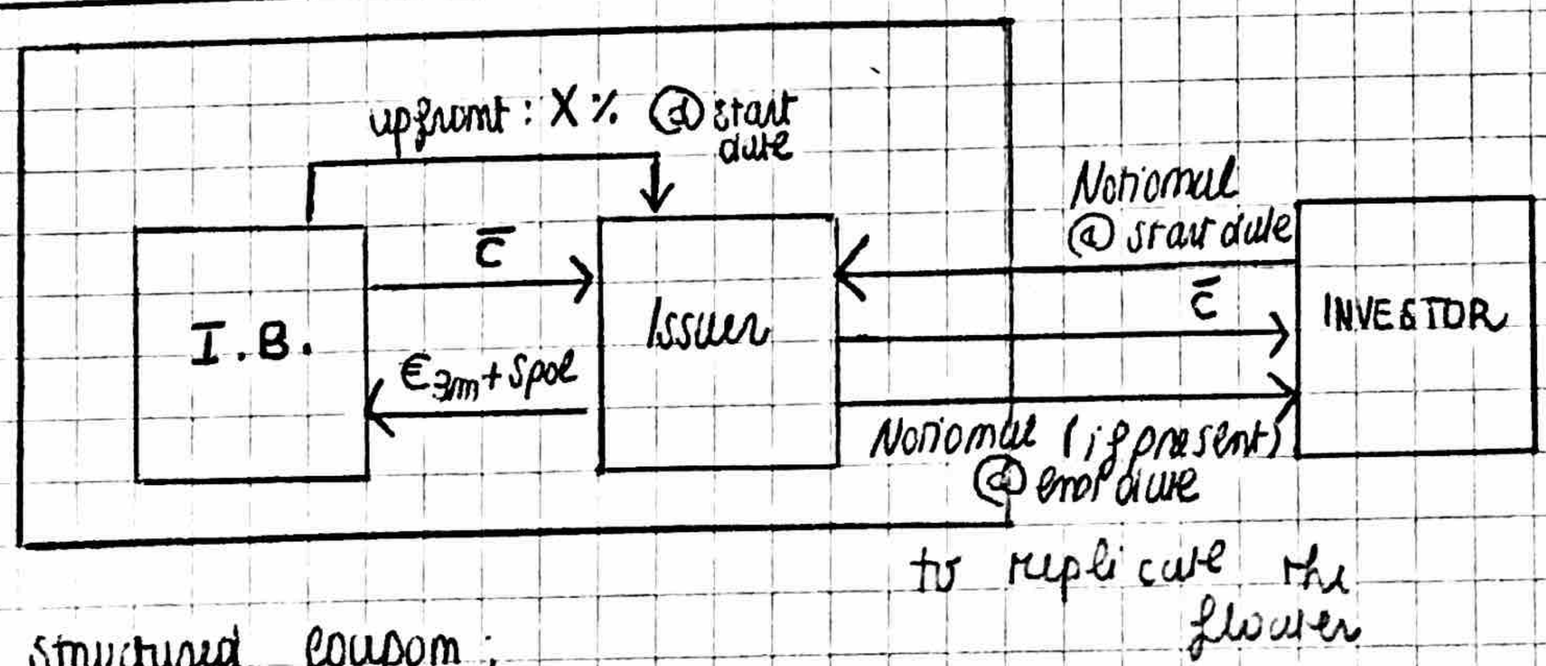
→ STRUCTURED BONDS: bonds issued by a real issuer, capital guaranteed at expiry if issuer doesn't default. Coupon not fixed in advance, based on the payoff of an option of another underlying, but pays the face value for sure (if no default).

↳ CERTIFICATES: as structured bonds, but face value not guaranteed.

Bank:

Liabilities	Assets
Floater ($€_{3m}$ + S^{spre})	Loans, Leasing ($€_{3m}$ + $S^{obligor}$) Mortgages ($€_{3m}$ + spread)

GENERAL SCHEME OF SWAPS IN STRUCTURED PRODUCTS



→ $\bar{C}$ structured coupon;

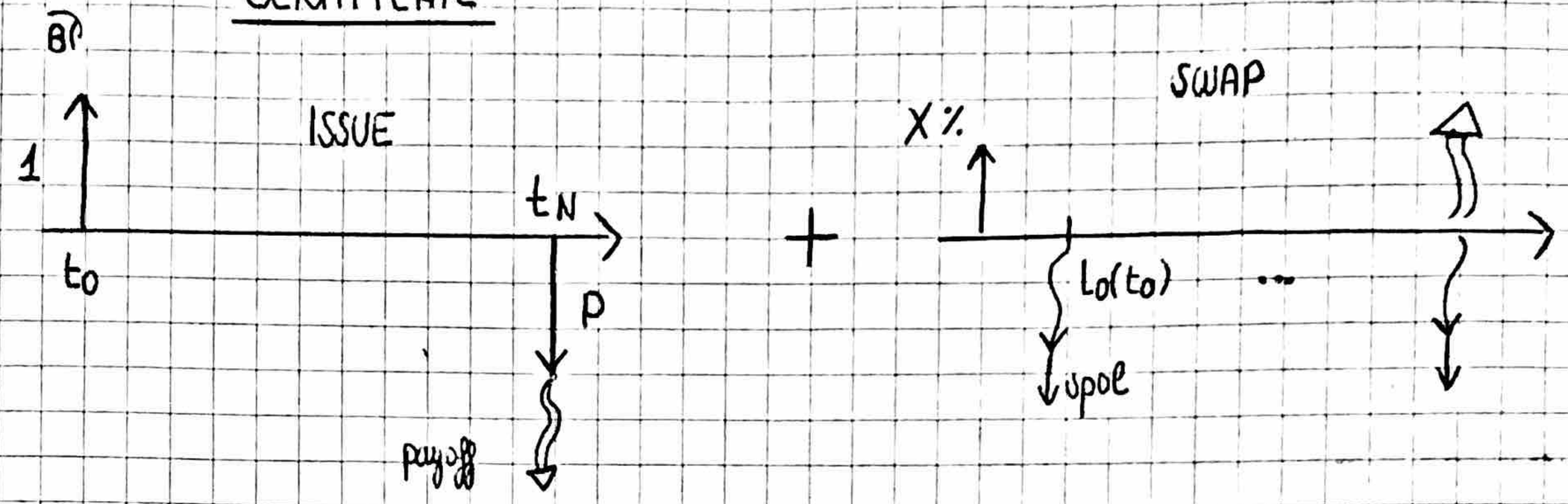
→ the structured product does not (in general) guarantee the invested capital.

INCOME STATEMENT (Conto Economico)

↳ Interest Margin

↳ Collected Fees ← X% (after the eliminated costs).

STRUCTURING SCHEME FOR AN "EQUITY PROTECTION" CERTIFICATE



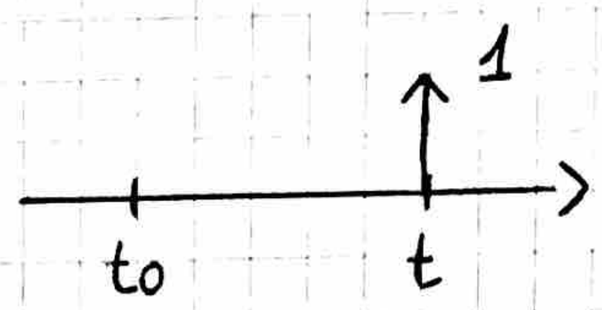
⇒ floater + X%

established in the contract, it appears in "conto economico" (used in order to repay the costs faced to create the structured bonds)

10. DIGITAL OPTION IN PRESENCE OF SMILE. DIGITAL RISK AND CORRECTED BLACK FORMULA.

DIGITAL OPTION

$$\text{Payoff} \begin{cases} 1, & \text{if } S_t \geq K \\ 0, & \text{otherwise} \end{cases}$$



Before, we had $[S_t - K]^+ = S_t \mathbb{1}_{S_t \geq K} - K \mathbb{1}_{S_t \geq K}$.

Now, the price for a digital option becomes:

$$dc_B = B(t_0,t)\, \mathbb{E}[\mathbb{1}_{S_t \geq K}] = B(t_0,t)\, N(d_2)$$

BLACK DIGITAL OPTION

It can be proven that:

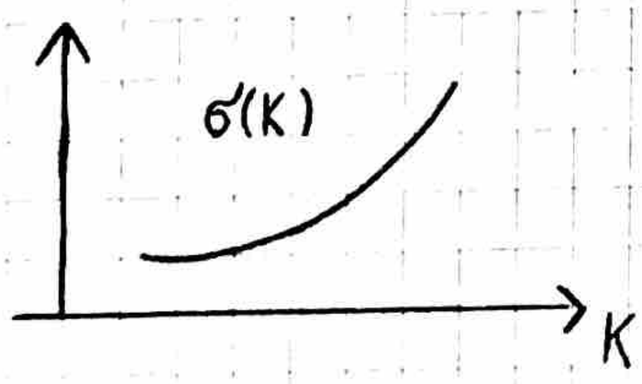
$$dc_B(K) = -\frac{\partial c_B(K)}{\partial K}$$

IMPLIED VOLATILITY APPROACH

A volatility $\sigma(K)$ s.t.

$$C(K, \sigma(K)) = \text{Quoted Price}$$

Once calibrated implied volatility, I can use this model only for plain vanillas.



"SMILE": I don't have the same volatility with ≠ strikes, so the B&S model isn't correct.

Implied volatility is the main tool to price PLAIN VANILLAS.
We can't use the implied volatility directly for a digital option.

How to price a digital option with an implied volatility approach?

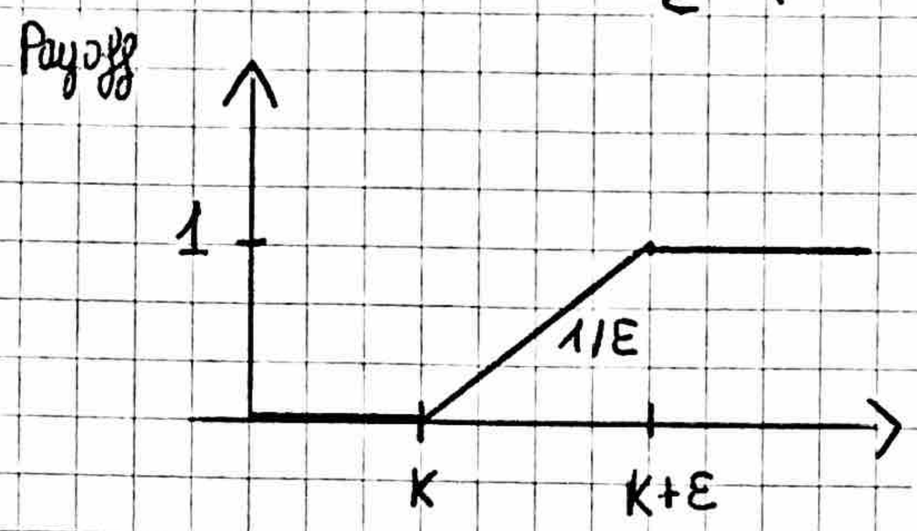
A digital option is well approximated by a CALL SPREAD / BULL SPREAD.

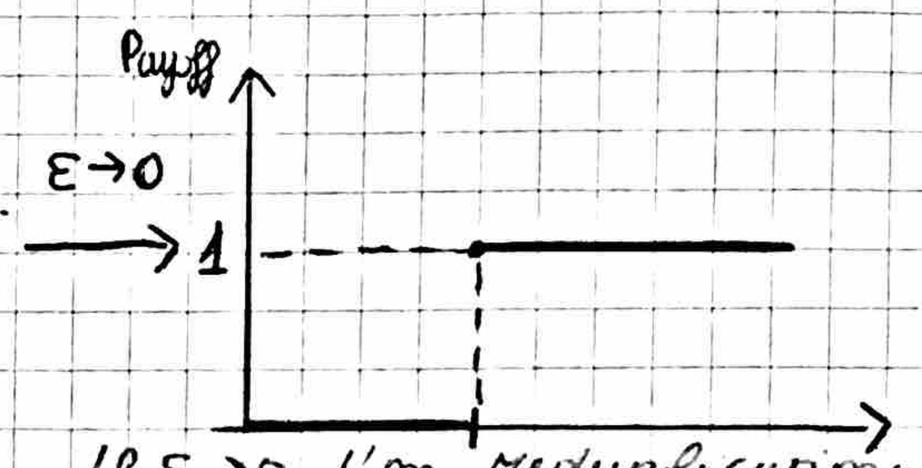
DIGITAL OPTION IN AN IMPLIED VOLATILITY APPROACH

8) Let's consider a Call Spread with a Notional of $1/\varepsilon$:

$$\text{Call Spread} = \frac{1}{\varepsilon}\left(\text{Call}(K) - \text{Call}(K+\varepsilon)\right)$$

same underlying with two ≠ strikes.

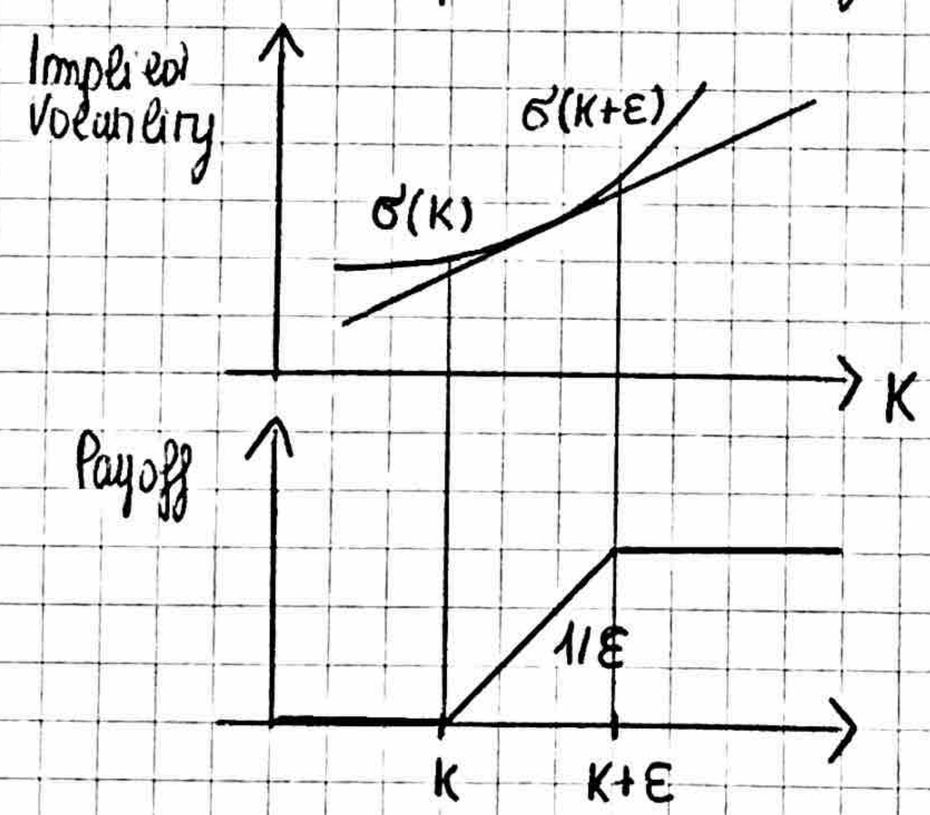




If $\varepsilon \to 0$, I'm reduplicating a digital option.

⟹ digital option as a combination of two plain vanillas.

Now, I can use implied volatility approach:



CORRECTED BLACK FORMULA:

$$dC = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[C(K,\sigma(K)) - C(K+\varepsilon,\sigma(K+\varepsilon))\right] =$$

$$= -\frac{d}{dK} C[K,\sigma(K)] =$$

$$= \underbrace{-\frac{\partial}{\partial K} C[K,\sigma(K)]}_{\text{Black}} \frac{\partial \sigma(K)}{\partial K} \underbrace{\frac{\partial C[K,\sigma(K)]}{\partial \sigma}}_{\text{Vega PV}}$$

slope impact: digital "directs" the difference Black vs Implied.

If I consider the Black model, I can't see the digital risk.

ASS.6 : $dc = dc_B - m\nu$

price of a digital option taking into account the smile in the curve of the implied vol.

$m = \frac{\partial \sigma(K)}{\partial K}$: slope of the tangent line at the curve of the implied volatility passing exactly from the point (K, σ).

computed as the slope of the straight line passing for the two closest point at K that we have in the dataset

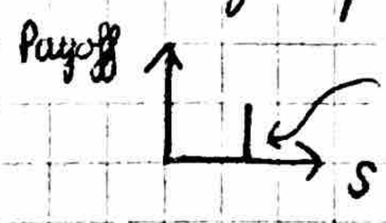
$$m = \frac{\sigma_2 - \sigma_1}{K_2 - K_1}$$

We have introduced the Black model and the implied volatility in order to price a digital option.

Why don't we use a better model? Because this one is precise enough (we don't introduce further models if it is not necessary, since this will affect the whole bank, not only RM section)

When do we start thinking about new models?

If I have a product with, for example, a discontinuous payoff, I can't price with a replication through plain vanillas:



we can replicate straight line this, not more complicated.

I must be able to → calibrate → use few parameters → simulate easily

In order to price OPTIONS, we use the characteristic function.

We used to price options as $f(K)$, but usually what is relevant is the RATIO:

$x := \ln \frac{F_0}{K}$ LOG - MONEYNESS

11. LEWIS FORMULA.

OPTION PRICE

Given the return's characteristic function

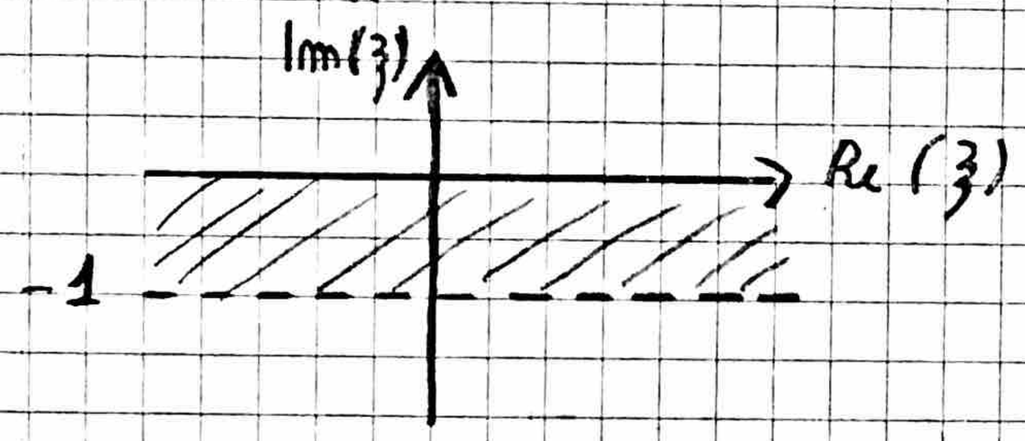
$$\phi(\xi),\ \xi \in \mathbb{C}$$

and an option with log-moneyness $x := \ln \frac{F_0}{K}$

PROP. European Call Price is:

$$\frac{C(x)}{B(t_0,t)F_0} = 1 - e^{-x/2} \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \phi\left(-\xi - \frac{i}{2}\right) \cdot \frac{1}{\xi^2 + \frac{1}{4}}$$

with $\phi(\xi)$ analytic in the horizontal strip $\{\xi \in \mathbb{C} : -1 \leq \mathrm{Im}(\xi) \leq 0\}$ (Carr Madan, 1998, Lewis, 2001)

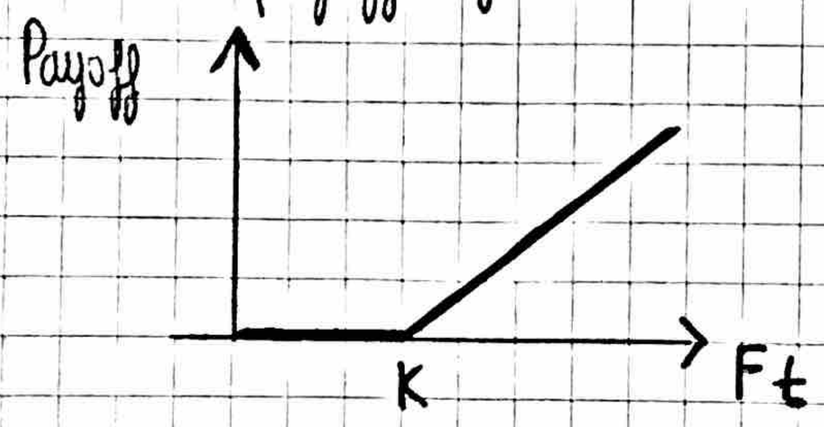


Because the integral is within the analicity strip → 1-d Fourier Transform.

PROOF.

(1) Remark:

the payoff for a Call is $[F_t - K]^+$

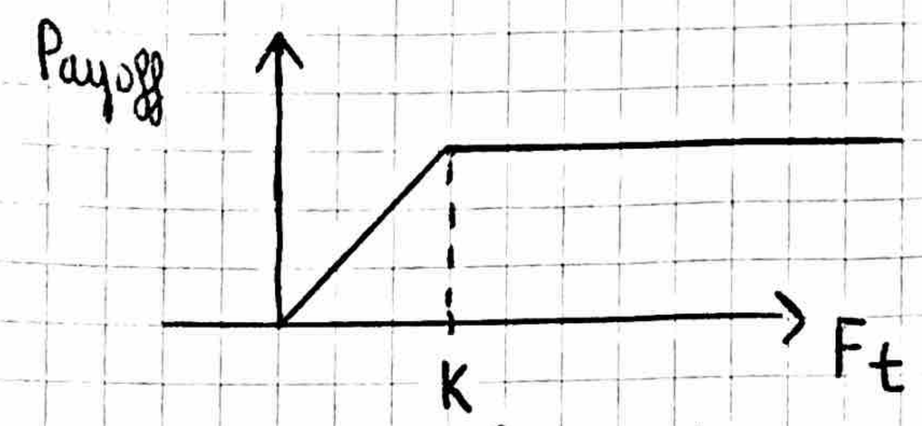


Problem: unbounded payoff

For this reason, it's better to see the Payoff for a Call as:

$$\text{Payoff Call} = F(t,t) - \underbrace{\min\left(F(t,t), K\right)}_{\text{Payoff Covered Call}}$$

Covered Call:



Why do we use a covered call? it's bounded, so we will not have problems with Fubini's theorem and Parseval.

BOUNDED PAYOFF.

$$\min(F_t, K) = F_t - [F_t - K]^+$$

(2) Compute price in function of the moneyness:

$x := \ln F_0 / K \Rightarrow K = F_0 e^{-x}$

$$C(x) = B(t_0, t)\,\mathbb{E}[(F(t,t) - K)^+]$$

Let's introduce the risk factor $\psi = \ln(F(t,t)/F_0)$ Margin of the fund

$\Rightarrow F(t,t) = F_0 e^{\psi}$ r.v.

$$\Rightarrow C(x) = B(t_0,t)\,\mathbb{E}[(F_0 e^{\psi} - F_0 e^{-x})^+]$$

Considering the covered call:

$$\mathbb{E}[(F_t - K)^+] = \mathbb{E}[F_0 e^{\psi} - \min(F_0 e^{\psi}, F_0 e^{-x})] =$$

$\left(F(t,t) \text{ is a martingale} \Rightarrow \mathbb{E}_0[F(t,t)] = F_0\,\mathbb{E}[e^{\psi}] \leftrightarrow \mathbb{E}[e^{\psi}] = 1\right)$

$$= F_0\left(1 - \mathbb{E}[\underbrace{\min(e^{\psi}, e^{-x})}]\right)$$

$\omega_{cc}(\psi)$: payoff of a covered call (normalized, divided by F_0) function of x

$$\Rightarrow \frac{C(x)}{B(t_0,t)F_0} = 1 - \underbrace{\mathbb{E}[\omega_{cc}(\psi)]}$$

We want to prove it is =

$$= e^{-x/2}\int_{-\infty}^{+\infty} \frac{d\xi}{2\pi}\, e^{-i\xi x}\, \phi\left(-\xi - \frac{i}{2}\right) \frac{1}{\xi^2 + \frac{1}{4}}$$

(3) Aim: compute the expectation

$$\mathbb{E}[\omega_{cc}(\psi)] = \int_{-\infty}^{+\infty} d\psi\, \omega_{cc}(\psi)\, \underbrace{p(\psi)}_{\text{density of } \psi}$$

Remark:

$$\begin{cases} \hat{\omega}(z) = \int_{-\infty}^{+\infty} e^{-iz\psi}\, \omega(\psi)\, d\psi & \text{Fourier transform} \\ \omega(\psi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\psi z}\, \hat{\omega}(z)\, dz & \text{Inverse Fourier transform} \end{cases}$$

$$\omega \underset{\text{IFT}}{\overset{\text{FT}}{\rightleftarrows}} \hat{\omega}$$

$$\mathbb{E}[\omega_{cc}(\psi)] = \int_{-\infty}^{+\infty} d\psi\, \underline{\omega_{cc}(\psi)}\, p(\psi) \overset{\text{IFT}}{=}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\psi\, p(\psi) \int_{-\infty}^{+\infty} e^{i\psi z}\, \hat{\omega}_{cc}(z)\, dz \overset{\text{Fubini theorem (since } \omega_{cc}(\psi) \text{ is bounded)}}{=}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz\, \hat{\omega}_{cc}(z) \underbrace{\int_{-\infty}^{+\infty} d\psi\, p(\psi)\, e^{i\psi z}}_{\phi^c(z)\ :\ \text{characteristic function}}$$

$$\Rightarrow \mathbb{E}[\omega_{cc}(\psi)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz\, \hat{\omega}_{cc}(z)\, \phi^c(z) \qquad \text{PARSEVAL RELATION}$$

(4) Now, we have to compute the Fourier Transform:

$$\widehat{\omega}_{CC}(z) = \int_{-\infty}^{+\infty} e^{-iz\psi} \underbrace{\omega_{CC}(\psi)}_{\min[e^{\psi}, e^{-x}]} d\psi =$$

$$= \underbrace{\int_{-\infty}^{+\infty} e^{-iz\psi} e^{\psi} \mathbb{1}_{\psi < -x} d\psi}_{(2)} + \underbrace{\int_{-\infty}^{+\infty} e^{-iz\psi} e^{-x} \mathbb{1}_{\psi \geq -x} d\psi}_{(1)}$$

$$(1) = \int_{-x}^{+\infty} d\psi\, e^{-i\psi z} e^{-x} = e^{-x} \left[\frac{e^{-i\psi z}}{-iz} \right]_{-x}^{+\infty} = (*)$$

$$= e^{-x} \frac{e^{izx}}{iz}$$

per far sì che
$e^{b\psi}$ decresca in $+\infty$
$b < 0$

(*) $\phi^{c}(\cdot)$ must be analytical.

In order to have a well-defined integral:

$z = a + ib$, $(a, b \in \mathbb{R})$

$e^{-iz\psi} = e^{-ia\psi + b\psi}$ $(-iz = -ia + b)$

I'm integrating at $+\infty$, so, in order not to have $+\infty$ as result of the integral, I need

$b < 0 \Rightarrow \boxed{\mathrm{Im}(z) < 0}$

$$② = \int_{-\infty}^{-x} e^{-iz\psi} e^{\psi} \, d\psi =$$

$$= \int_{-\infty}^{-x} e^{\psi(1-iz)} \, d\psi = \left[\frac{e^{\psi(1-iz)}}{1-iz} \right]_{-\infty}^{-x} = (*)$$

$$= \frac{e^{izx-x}}{1-iz}$$

$(*)$ $\quad z = a + ib \rightarrow (1-iz) = 1 - ia + b$

$$e^{\psi(1-iz)} = e^{-ia\psi + (b+1)\psi}$$

$$b + 1 > 0 \Rightarrow \boxed{Im(z) > -1}$$

$$\Rightarrow \hat{w}_{cc}(z) = e^{-x} \frac{e^{izx}}{iz} + \frac{e^{izx-x}}{1-iz} =$$

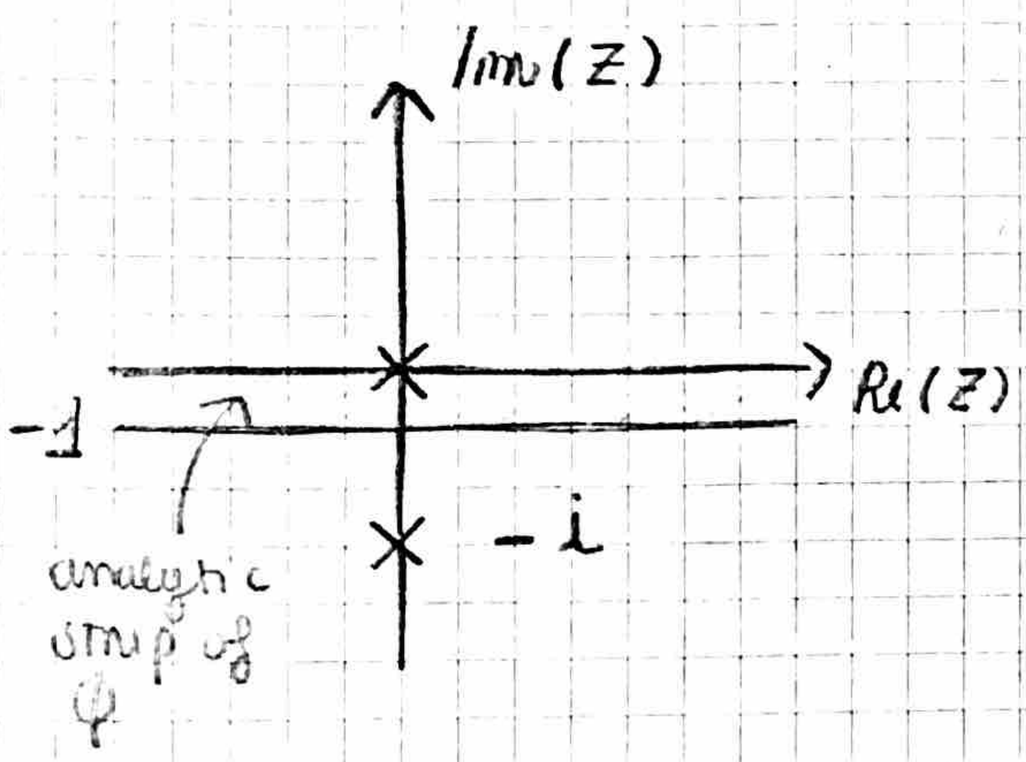
$$= e^{(iz-1)x} \left[\frac{1}{iz} + \frac{1}{1-iz} \right] =$$

$$= \frac{e^{(iz-1)x}}{iz(1-iz)} = \frac{e^{(iz-1)x}}{z(z+i)}$$

$$-1 < Im(z) < 0$$

(5) $$\frac{C(x)}{B(t_0,t)F_0} = 1 - \frac{e^{-x}}{2\pi}\int_{-\infty}^{+\infty} dz\, \underbrace{\frac{e^{izx}}{z(z+i)}}\, \phi^c(z)$$

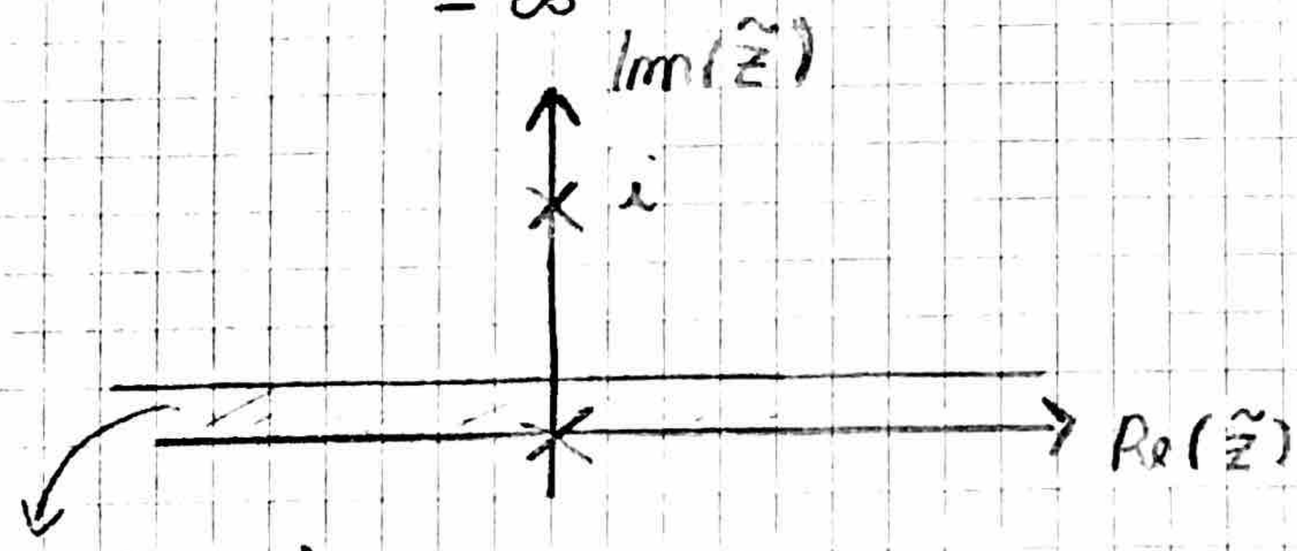
2 poles : $\begin{cases} z = 0 \\ z = -i \end{cases}$



Change of variable to have "nice" axis :

$\tilde{z} = -z$

$$\frac{C(x)}{B(t_0,t)F_0} = 1 - \frac{e^{-x}}{2\pi}\int_{-\infty}^{+\infty} d\tilde{z}\, \frac{e^{-i\tilde{z}x}}{\tilde{z}(\tilde{z}-i)}\, \phi^c(-\tilde{z})$$

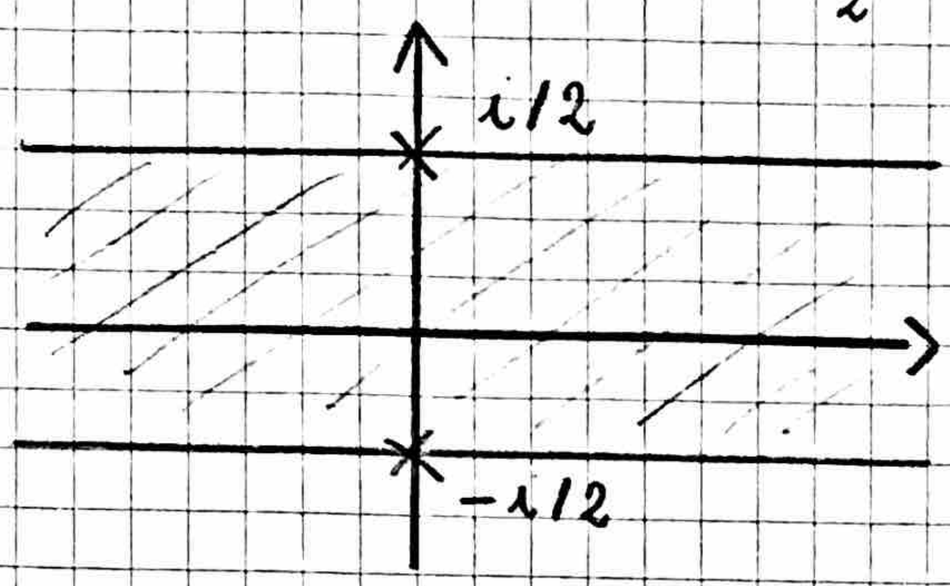


$0 < \mathrm{Im}(\tilde{z}) < 1$

→ PROBLEM: $\int$ very near to zero pole

Another change of variables: $\xi = \tilde{z} - \frac{i}{2} \Rightarrow \tilde{z} = \xi + \frac{i}{2}$

$$\frac{C(x)}{B(t_0,t)F_0} = 1 - \frac{e^{-x}}{2\pi}\int_{-\infty - i/2}^{+\infty - i/2} d\xi \, \frac{e^{-ix(\xi + i/2)}}{(\xi + \frac{i}{2})(\xi - \frac{i}{2})} \phi_C\left(-\xi - \frac{i}{2}\right)$$



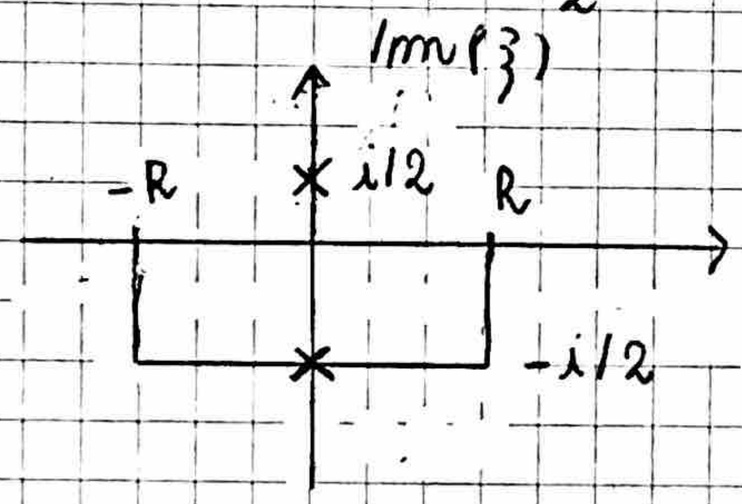
NO ZERO POLE

We need that $\phi^C(z)$ is analytic in the strip

$-1 \leq Im(z) \leq 0$

original variable

$\left(\phi^C\left(-\xi - \frac{i}{2}\right)\right.$ analytic in $\left.-\frac{1}{2} \leq Im(\tilde{z}) \leq \frac{1}{2}\right)$



$\int_{\text{in } \mathbb{R}}$

$$\frac{C(x)}{B(t_0,t)F_0} = 1 - \frac{e^{-x/2}}{2\pi}\int_{-\infty}^{+\infty} d\xi \, \frac{e^{-ix\xi}}{\xi^2 + \frac{1}{4}} \phi_C\left(-\xi - \frac{i}{2}\right)$$

The only thing we need to compute $C(x)$ is the characteristic function

Lewis formula is only for Call and Put (European), because those are the most liquid contracts (so for calibration), then you can price exotic...

12. FFT

FAST FOURIER TRANSFORM

FOURIER TRANSFORM $\quad \hat{f}(z) = \int_{-\infty}^{+\infty} f(x)\, e^{-izx}\, dx$

To handle the integral in the Lewis formula, we need a discrete Fourier Transform.

We map a vector into another vector:

$$\{f_j\}_{j=1:N} \xrightarrow{DFT} \{\hat{f}_k\}_{k=1:N} \qquad N = 2^M \ (\text{OR } 3^M, 4^M, \dots)$$

$$\text{"DFT}(z_k)\text{"} = \sum_{j=1}^{N} e^{-i z_k x_j} f(x_j)$$

$$\text{DFT}(z_k) = \sum_{j=1}^{N} f_j\, \omega^{(j-1)(k-1)}, \quad \omega = e^{-i\frac{2\pi}{N}}$$

linear transformation

Complexity $\Rightarrow \mathcal{O}(N^2)$

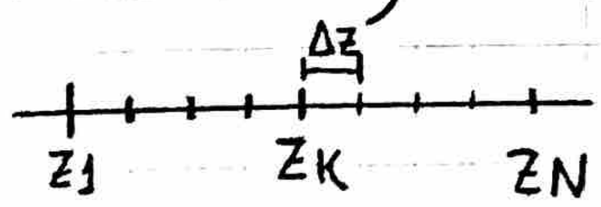
FFT is like DFT but uses a better algorithm, with complexity $\mathcal{O}(N \ln N)$

(The algorithm is designed in order to compute prices for all strikes at the same time).

FOURIER TRANSFORM VS. FFT

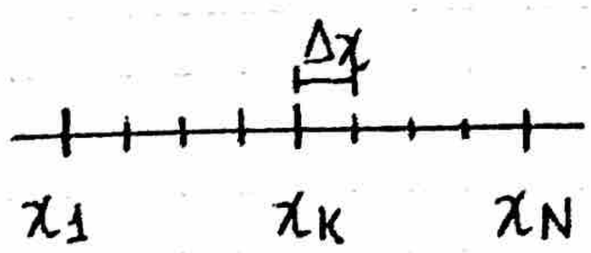
Using the discretization for z and x s.t.

$$\begin{cases} z = z_k \equiv z_1 + (k-1)dz \\ x = x_j \equiv x_1 + (j-1)dx \end{cases} \quad \text{s.t.} \quad \begin{matrix} k = 1, \dots, N \\ j = 1, \dots, N \end{matrix}$$



$$dz = \Delta z = \frac{z_N - z_1}{N}$$

s.t. $\boxed{dx \cdot dz = \frac{2\pi}{N}}$



$$dx = \Delta x = \frac{x_N - x_1}{N}$$

$$\hat{f}(z) = \int_{-\infty}^{+\infty} f(x)\, e^{-ixz}\, dx \overset{\text{approximate extremes}}{\simeq} \int_{x_1}^{x_N} f(x)\, e^{-ixz}\, dx$$

Then pass to Riemann sums:

$$\hat{f}(z_k) = \sum_{j=1}^{N} dx \cdot f(x_j)\, e^{-i x_j z_k} =$$

$$= \left(x_j z_k = [x_1 + (j-1)dx][z_1 + (k-1)dz] \right) =$$

$$= \underbrace{dx\, e^{-i x_1 z_k}}_{\text{prefactor}} \sum_{j=1}^{N} \underbrace{f(x_j)\, e^{-i z_1 dx (j-1)}}_{f_j} \cdot \underbrace{e^{-i \frac{2\pi}{N}(j-1)(k-1)}}_{\omega^{(j-1)(k-1)}} =$$

$$= dx\, e^{-i x_1 z_k} \underbrace{FFT(k)}$$

on $f_j = f(x_j)\, e^{-i z_1 dx (j-1)}$

So, we have seven parameters and five relations between them:

$$\begin{cases} N = 2^M \\ x_1 = -x_N \\ z_1 = -z_N \\ dx = \dfrac{x_N - x_1}{N} \\ dz = \dfrac{z_N - z_1}{N} \\ dx \cdot dz = \dfrac{2\pi}{N} \end{cases}$$

In order not to have imaginary part:

x_N ... e^{-ixz} $e^{i\theta} = \cos(\theta) + i \underbrace{\sin(\theta)}$

In a symmetric interval, the imaginary part goes to $\emptyset$.

x_1

$z_1 = -z_N \longrightarrow$ by momentum for A.M-F equal to $\emptyset \rightarrow$ symmetric interval

2 free parameters: M + one to choose

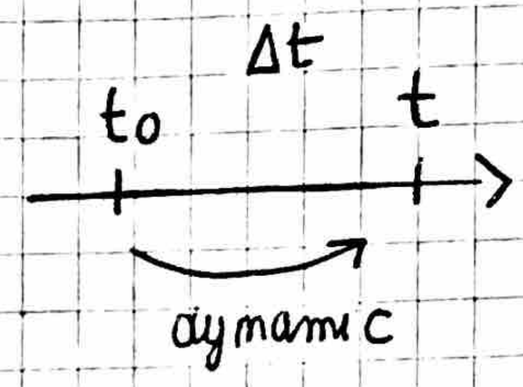
Ass. 6: (M, dz), (M, x_1)

13. EXP-LEVY MODEL: INTRODUCTION FROM A BLACK MODEL; CHARACTERISTIC FUNCTION

EXPONENTIAL LÈVY VIA NORMAL-MEAN VARIANCE MIXTURE

We have to model the risk factors, always in a 1-time step model:

LÈVY MODEL FOR THE RETURN : $\boxed{\xi_t \equiv \ln \frac{F_t}{F_0}}$



$\rightarrow F_t = F_0 \, e^{\xi_t}$

$$F_0 = \mathbb{E}[F(t,t)] = F_0 \, \mathbb{E}[e^{\xi}] \Leftrightarrow \mathbb{E}[e^{\xi}] = 1$$

Black model: $F_t = F_0 \, e^{-\frac{\sigma^2}{2}(t-t_0) + \sigma\sqrt{t-t_0}\, g}$

and so $\xi_t = -\frac{\sigma^2}{2}(t-t_0) + \sigma\sqrt{t-t_0}\, g$

By the Normal Mean Variance Mixture (NMVM)

$\xi = \underline{\mu} + \underline{\gamma} W + \sqrt{W} A \underline{Z}$

W mixing variable, $\perp \underline{Z}$, $Var[W] < \infty$, $\underline{Z} \sim N_d(0, I_d)$

$$\Rightarrow \xi_t = -\frac{\sigma^2}{2} G (t-t_0) + \sigma\sqrt{t-t_0}\sqrt{G}\, g$$

But this model can reproduce only simmetric smiles, so we have to introduce also a drift term:

$$\Rightarrow \xi_t = \mu - \left(\frac{1}{2} + \eta\right)\sigma^2 G (t-t_0) + \sigma\sqrt{t-t_0}\sqrt{G}\, g$$

r.v.:
- $g \rightarrow$ std. normal
- $G \rightarrow G > 0$, positive r.v. with unitary mean and $\frac{K}{\Delta t}$ variance

g, G i.d. r.v.

parameters (most parsimony extension of B&S):
- $\sigma \in \mathbb{R}^+ \rightarrow$ Mean volatility (not B&S vol)
- $\eta \in \mathbb{R} \rightarrow$ asymmetry
- $K \in \mathbb{R}^+ \rightarrow$ Volatility of the volatility

$\rightarrow$ SUBORDINATED BROWNIAN MOTION

In order to use Black formula, we have to check the analytical strip.

Let's consider the Laplace transform of G:

$$\hat{\mathcal{L}}_G(\omega) = \int_0^{+\infty} dG\, \underbrace{p(G)}_{\text{pdf of } G} e^{-\omega G}$$

Then we can introduce a constant $C = (t-t_0)\sigma^2 > 0$:

$$* \Big\| \; \hat{\mathcal{L}}_G(\omega) = \int_0^{+\infty} dG\, p(G)\, e^{-\omega (t-t_0)\sigma^2 G}$$

G "Laplace" Transform

Remember the Normal Mean-Variance Mixture

$$\underline{x} = \underline{\mu} + W \underline{\gamma} + \sqrt{W}\, A\, \underline{y} \quad \text{(multidimensional)}$$

In 1-dim: $x = \mu + \underbrace{W\gamma}_{-(\frac{1}{2}+\eta)\sigma^2 G(t-t_0)} + \underbrace{\sqrt{W}\sqrt{\Sigma}\, y}_{\sigma\sqrt{t-t_0}\sqrt{G}\, g}$

We introduced its characteristic function

$$\left(\phi^c(\underline{t}) = e^{i\underline{t}\cdot\underline{\mu}}\, \mathcal{L}\left[-i\underline{t}\cdot\underline{\gamma} + \frac{1}{2}\underline{t}\cdot\Sigma\underline{t}\right] \right)$$

So now we have:

$$\phi^c_{(g)}(\xi) = e^{i\xi\mu}\, \mathcal{L}\left[-i\xi\gamma + \frac{1}{2}\Sigma\xi^2\right] =$$

$$= e^{i\xi\mu} \int_0^{+\infty} dW\, p(W)\, e^{-W\left[-i\xi\gamma + \frac{1}{2}\Sigma\xi^2\right]}$$

So, rewriting in our parameters:

$$\phi^c_{(g)}(\xi) = \mathbb{E}\left[e^{ig\xi}\right] =$$

$$= e^{i\xi\mu} \int_0^{+\infty} dG\, p(G)\, e^{-G\left[i\xi(\frac{1}{2}+\eta) + \frac{1}{2}\xi^2\right]\sigma^2(t-t_0)} =$$

$$= e^{i\xi\mu}\, \mathcal{L}\left[\frac{\xi^2 + i(2\eta+1)\xi}{2}\right]$$

accordingly to * "Laplace" transform

We'll have to ask:

1. Well posed financial problem:

Impose Martingale condition on the forward.

$$\Rightarrow \mathbb{E}[e^{f}] = 1$$

$$\parallel$$

$$\phi_f^c(-i) = e^{+\mu}\,\mathcal{L}\left[\frac{\cancel{-1}+\cancel{1}+2\eta}{2}\right] =$$

$$= e^{\mu}\,\mathcal{L}[\eta] = 1 \rightarrow \mu = -\ln(\mathcal{L}[\eta])$$ related to η!

So, the dynamic up to time t can be written as:

$$\boxed{f_t = \sqrt{t-t_0}\,\sigma\sqrt{G}\,g - \left(\frac{1}{2}+\eta\right)(t-t_0)\,\sigma^2 G - \ln(\mathcal{L}[\eta])}$$

We can choose the distribution of G:

- Inverse Gaussian
- Gamma

Random Variable	Laplace exponent $\ln \mathcal{L}[\omega]$	f
IG	$\frac{\Delta t}{K}\left[1-\sqrt{1+2K\omega\sigma^2}\right]$	NIG
G	$-\frac{\Delta t}{K}\ln\left[1+K\omega\sigma^2\right]$	VG

2. When is the Laplace transform well defined? $\omega \in \mathbb{R}^+$.

But, sometimes, we will also have $-\omega$:

$$\mathcal{L}_G(\omega) = \int_0^{+\infty} p(G)\,e^{-D\omega G}\,dG$$

In order to have convergence of the integrale, we can also use a negative ω if the density has an exponential tail.

IG : $1 + 2K\omega\sigma^2 > 0$

$$\rightarrow \omega > -\frac{1}{2K\sigma^2} =: -\underline{\omega}$$

BRANCH CUT PROBLEMS

G : $1 + K\omega\sigma^2 > 0$

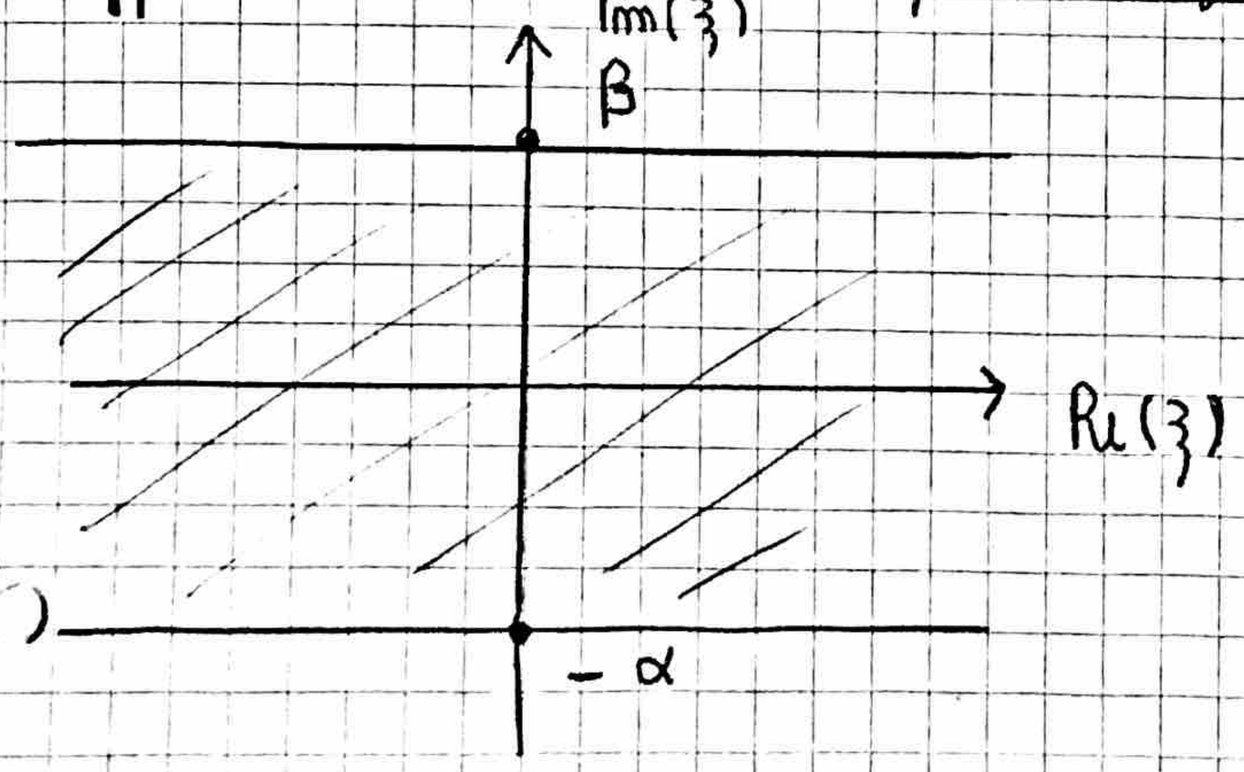
$$\rightarrow \omega > -\frac{1}{K\sigma^2} =: -\underline{\omega}$$

In order to have a well-defined quantity, we must have $\boxed{\eta \geq -\underline{\omega}}$ (to have a well defined quantity in $\Phi^c_{(g)}(\xi)$)

14 EXP-LEVY MODEL: ANALYCITY STRIP.

LUCKAS THEOREM

If a characteristic function is regular in a neighborhood of the origin, it is regular in a horizontal strip and can be represented in this strip by a Fourier Integral. Also, the purely imaginary points on the boundary of the strip of regularity (if this strip is not the whole plane) are singular points of $f(z)$.



The strip of analycity is bounded by two imaginary points.

$$-\alpha < \mathrm{Im}(z) < \beta \quad (\alpha, \beta > 0)$$

Strip of regularity of an analytic c. function, whose distribution function is $F(x)$.

In our Exp-Lèvy, we have a c.f. on a r.v. well defined in the origin: $\phi_f^c(\xi=0)=1$.

→ analyticity of $\phi_f^c(\xi)$ in the strip $-1 \leq Im(\xi) \leq 0$.

We have problems on the imaginary line of the boundary (since we have poles in $\xi=0$, $\xi=-i$).

I should not check for all the ξ, but only for $\xi = ia$, $a \in \mathbb{R}$ (since Luckas theorem says we have two bounds on the imaginary axis).

In order to have that $\phi_f^c(\xi)$ is well defined in its $\mathcal{L}$-transform on a real argument, we should have $\eta \geq -\underline{\omega}$.

$$\mathcal{L}\left[\frac{\xi^2 + i(2\eta+1)\xi}{2}\right] \overset{\xi = ia}{=}$$

$$= \mathcal{L}\left[\underbrace{\frac{-a^2 - a(1+2\eta)}{2}}_{\eta}\right]$$

argument of the Laplace transform in $\phi_f^c(\xi)$ (real).

$$\eta = \frac{-a^2 - a(1+2\eta)}{2} > -\underline{\omega}$$

2nd order inequality ⟹

$$a^2 + a(1+2\eta) - 2\underline{\omega} < 0$$

$$a_{1,2} = \frac{-(1+2\eta) \pm \sqrt{(1+2\eta)^2 - 4(-2\underline{\omega})}}{2}$$

$a_1 > 0$ (since $\underline{\omega} > 0$)

$a_2 < 0$

we have a_1, a_2 that limit our boundary.

$$\Rightarrow \quad a_2 < a < a_1$$

$$\Rightarrow \quad a_2 < \operatorname{Im}(\xi) < a_1$$

ANALYTICAL STRIP

We should check that our first constraint (Lewis: $-1 \leq \operatorname{Im}(\xi) \leq 0$) is into the analytical strip:

$a_1 > 0$, $a_2 < -1$

- $a_1 \geq 0$ OK
- $a_2 \leq -1$?

$$a_2 = -\frac{(1+2\eta)}{2} - \sqrt{\left(\frac{1+2\eta}{2}\right)^2 + 2\underline{\omega}} \leq -1$$

$$\Leftrightarrow \quad \frac{1-2\eta}{2} \leq \sqrt{\left(\frac{1+2\eta}{2}\right)^2 + 2\underline{\omega}}$$

Cases:

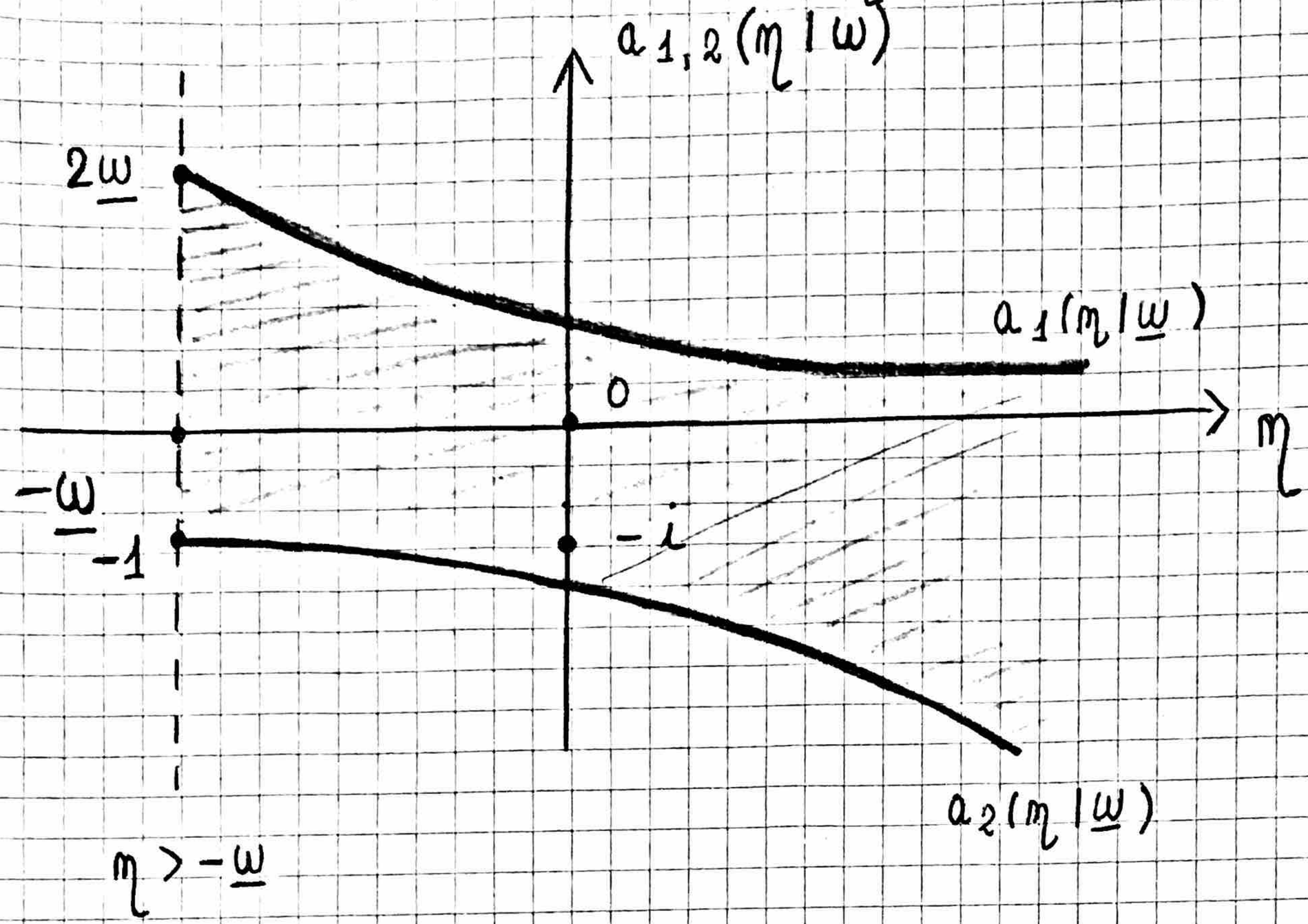
* $\frac{1-2\eta}{2} < 0 \rightarrow$ condition satisfied
* $\frac{1-2\eta}{2} > 0 \xrightarrow{\text{square}} \left(\frac{1-2\eta}{2}\right)^2 \leq \left(\frac{1+2\eta}{2}\right)^2 + 2\underline{\omega}$

$$\Leftrightarrow \quad -2\underline{\omega} \leq 2\eta \quad \Leftrightarrow \quad \eta > -\underline{\omega}$$

OK condition satisfied.

We aren't adding any new conditions!

Let's draw a_1, a_2 as $a_{1,2}(\eta \mid \underline{w})$



$\eta > -\underline{w}$

$$a_{1,2}(\eta = -\underline{w} \mid \underline{w}) = -\frac{(1-2\underline{w})}{2} \pm \sqrt{\frac{(1-2\underline{w})^2}{4} + 2\underline{w}} =$$

$$= -\frac{(1-2\underline{w})}{2} \pm \sqrt{\frac{(1+2\underline{w})^2}{4}} =$$

$$= -\frac{(1-2\underline{w})}{2} \pm \frac{1+2\underline{w}}{2}$$

$\cancel{-1} + 2\underline{w} + \cancel{1} + 2\underline{w}$

$-\cancel{1} + 2\underline{w} - \cancel{1} - 2\underline{w}$

$$= \begin{cases} 2\underline{w} \\ -1 \end{cases}$$

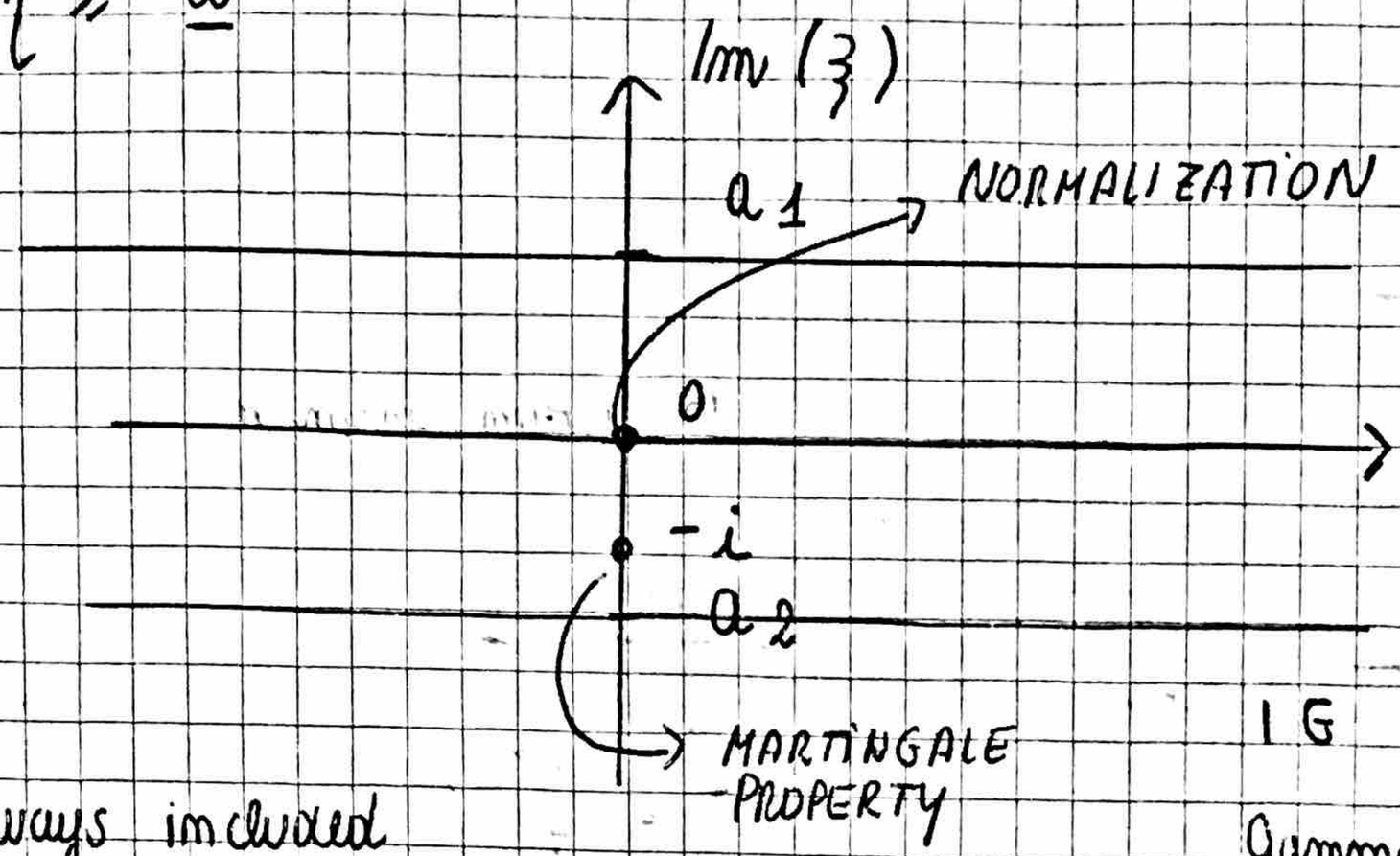
$\eta \to \infty$? $\quad \sqrt{1+x} \sim 1 + \frac{x}{2} + \dots$

$$a_{1,2} = \frac{1+2\eta}{2}\left[-1 \mp \sqrt{1 + \frac{8\underline{w}}{(1+2\eta)^2}}\right]$$

$$\sim \frac{1+2\eta}{2}\left[-1 \mp \left(1 + \frac{4\underline{w}}{(1+2\eta^2)}\right)\right] \begin{cases} 0^+ & a_1 \\ -\infty & a_2 \end{cases}$$

ANALYCITY STRIP: SUBORDINATED BROWNIAN MOTION

$\eta \geq -\underline{\omega}$



always included

	$\underline{\omega}$
IG	$1/2\kappa\sigma^2$
Gamma	$1/\kappa\sigma^2$

EXPONENTIAL TAIL OF MIXING VARIABLE'S DENSITY

$\underline{\omega} \geq 0$ is related to exponential tail of G's pdf

$$\underline{\omega} = -\frac{1}{\Delta t\,\sigma^2} \lim_{G\to\infty} \frac{1}{G}\, p(G)$$

"Laplace" transform : $\mathcal{L}_G(\omega) := \int_0^{+\infty} dG\, p(G)\, e^{-\sigma^2 \Delta t\, G\omega}$

pdf ≠ for IG, Gamma...

p(G)'s tails behavior is important for the Laplace transform.

15. EXP-LEVY MODEL : PARSIMONY, FINANCIAL MEANING OF PARAMETERS AND SMILE SIMMETRY FOR ETA = 0.

PARSIMONY

Black : only 1 parameter σ (then, from mkt, smile) — not too precise

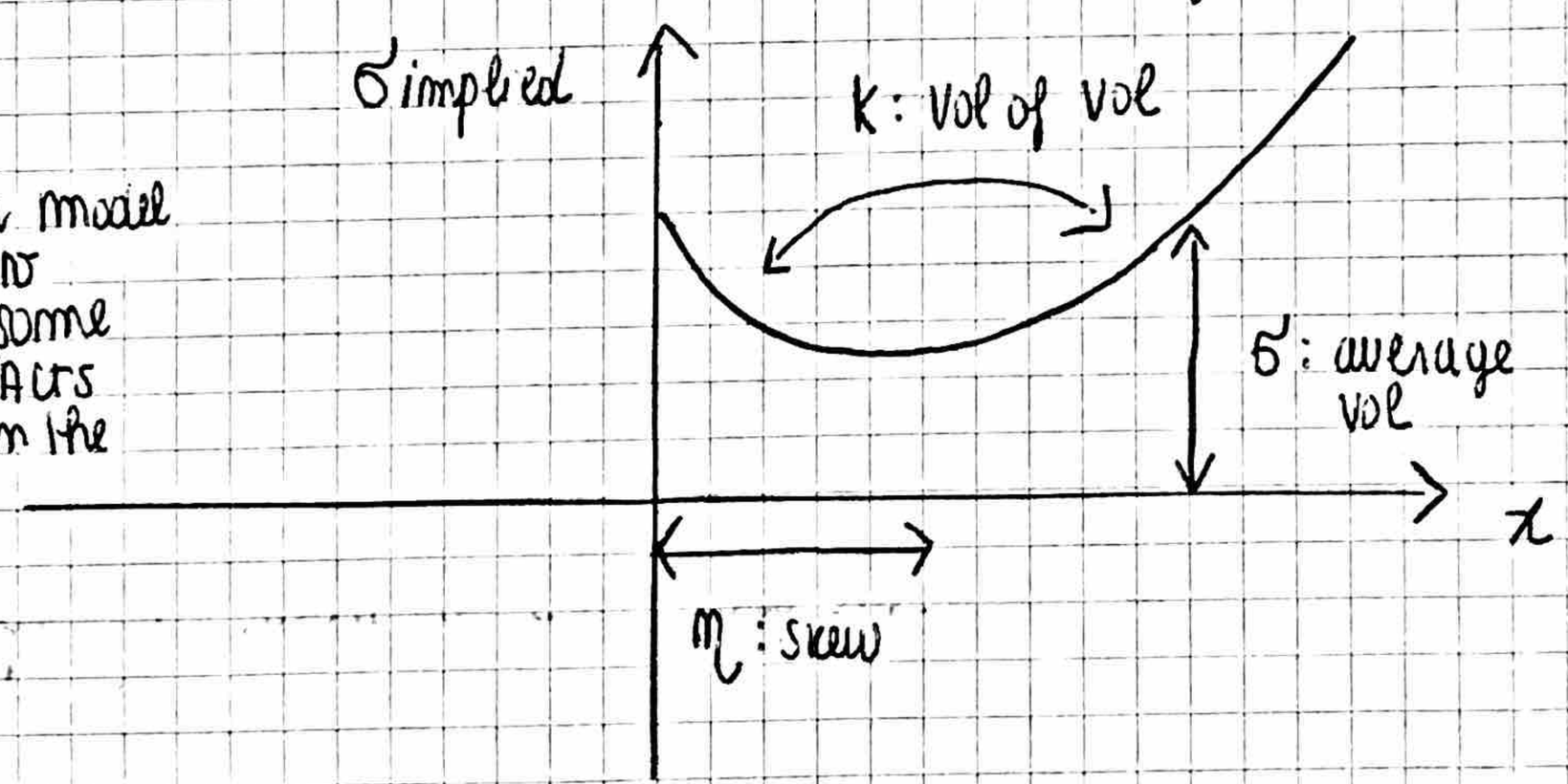
Exp-Levy : 3 parameters control the implied curve

most parsimony extension of B&S.

FINANCIAL INTERPRETATION OF PARAMETERS

- σ : average of implied volatility
- η : asimmetry
- K : volatility of volatility convexity of our implied volatility

Extend our model in order to immorale some STYLIZED FACTS observed in the mkt.



If σ is given, $K \ll 1$ and the smink becomes a straight line (again Black Model)

Let's now show that η really controls asymmetry. We are going to compute the implied volatility of a model, i.e., given a model, I obtain prices from the model and then we compare them with Mkt prices (Black) to obtain the implied volatility.

SMILE ASIMMETRY

$$\eta = 0 \implies \sigma_B(x) = \sigma_B(-x)$$

We will prove that the presence of η implies asimmetry; otherwise, if $\eta = 0$, the implied volatility, as function of the moneyness is simmetric.

$$\phi\left(-\xi - \frac{i}{2}\right) \overset{NMVM}{=} \mathcal{L}\left[\frac{(\xi + i/2)^2 - i(\xi + i/2)}{2}\right] =$$

$$= \mathcal{L}\left[\frac{\xi^2 + \cancel{i\xi} - 1/4 - \cancel{i\xi} + 1/2}{2}\right] =$$

$$= \mathcal{L}\left[\frac{1}{2}\left(\xi^2 + \frac{1}{4}\right)\right] \quad \text{in lewis formula}$$

$$\frac{C(x)}{B(t_0,t)\,F_0} = 1 - e^{-x/2} \underbrace{\int_{-\infty}^{+\infty} d\xi \, \frac{e^{-i\xi x}}{2\pi} \, \frac{\mathcal{L}\left[\frac{1}{2}\left(\xi^2 + \frac{1}{4}\right)\right]}{\xi^2 + 1/4}}_{I(x)}$$

To find the implied volatility of the model, we have to impose: $C(x) = C_B(x; \sigma_B(x))$

For Black price, we consider as density for G the delta function around 1 (since we consider σ const. in Black model)

$$\mathcal{L}_B[\xi] = \int_0^{+\infty} dG \, \underbrace{p(G)}_{\delta(G)} \, e^{-\xi \sigma^2 \Delta t G} = e^{-\xi \sigma^2 \Delta t}$$

G r.v. che esprime volatilità, quando si vuole σ costante o prendere $G = 1$ tornare a Black

$G = 1$ (come ξt ...)

$$\Rightarrow \frac{C_B(x)}{B(t_0,t)\,F_0} = 1 - e^{-x/2} \underbrace{\int_{-\infty}^{+\infty} d\xi \, \frac{e^{-i\xi x}}{2\pi} \cdot \frac{e^{-\sigma_B^2 \Delta t \left(\frac{1}{2}\left(\xi^2+\frac{1}{4}\right)\right)}}{\xi^2+\frac{1}{4}}}_{I_B(x)}$$

$$\Rightarrow I(x) = I_B(x, \sigma_B(x))$$

$$I(x) = \int_{-\infty}^{+\infty} d\xi \, e^{-ix\xi} \, \frac{\mathcal{L}\left[\frac{\xi^2+1/4}{2}\right]}{\xi^2+\frac{1}{4}}$$

$$I_B(x) = \int_{-\infty}^{+\infty} d\xi \, e^{-ix\xi} \, \frac{e^{-\sigma_B^2 \Delta t \left(\frac{1}{2}\left(\xi^2+\frac{1}{4}\right)\right)}}{\xi^2+\frac{1}{4}}$$

$$I(-x) = \int_{-\infty}^{+\infty} d\xi \, e^{+ix\xi} \, \frac{\mathcal{L}\left[\frac{1}{2}\left(\xi^2+\frac{1}{4}\right)\right]}{\xi^2+\frac{1}{4}} =$$

$$\overset{\xi' = -\xi}{=} \int_{-\infty}^{+\infty} d\xi' \, e^{-ix\xi'} \, \frac{\mathcal{L}\left[\frac{1}{2}\left(\xi'^2+\frac{1}{4}\right)\right]}{\xi'^2+\frac{1}{4}} =$$

$$= I(x) \Rightarrow I(x) = I(-x)$$

For Black formula, it's the same but we must be careful :

$$I_B(-x) = \int_{-\infty}^{+\infty} d\xi' \, e^{-ix\xi'} \, \frac{e^{-\sigma_B^2(-x)\Delta t \left(\xi^2+\frac{1}{4}\right)\frac{1}{2}}}{\xi'^2+\frac{1}{4}}$$

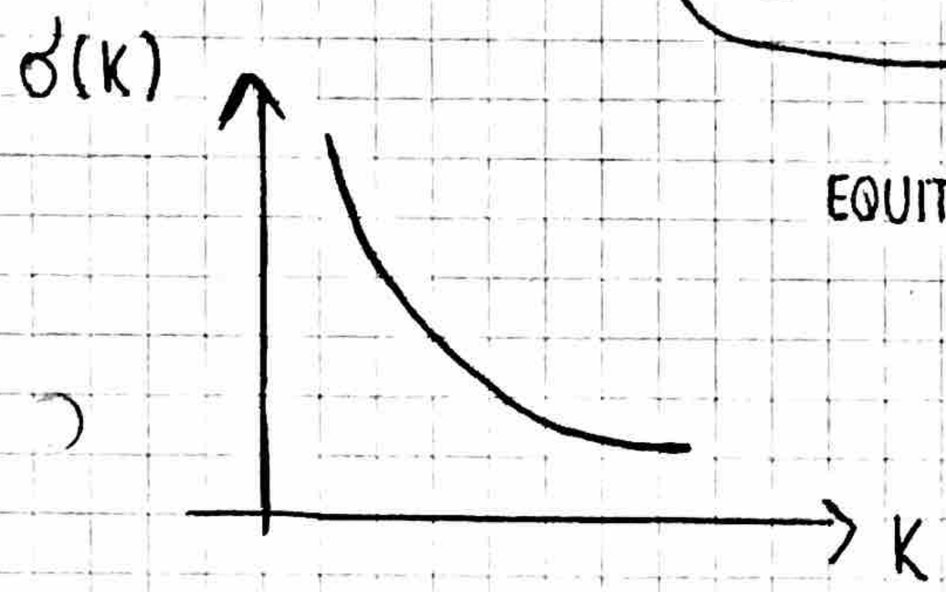
Since we have:

$$I(x) = I_B(x, \sigma_B(x))$$
$$\parallel$$
$$I(-x) = I_B(-x, \sigma_B(-x))$$
$$\Downarrow$$
$$I_B(x, \sigma_B(x)) = I_B(-x, \sigma_B(-x))$$
$$\Downarrow$$
$$\boxed{\sigma_B^2(x) = \sigma_B^2(-x)}$$

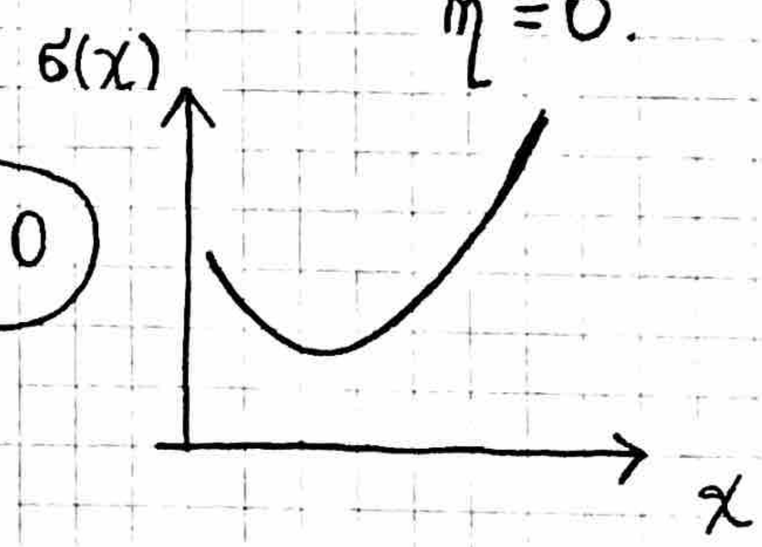
it's equivalence of σ^2 !!!

Since we consider the implied volatility always positive, we can write

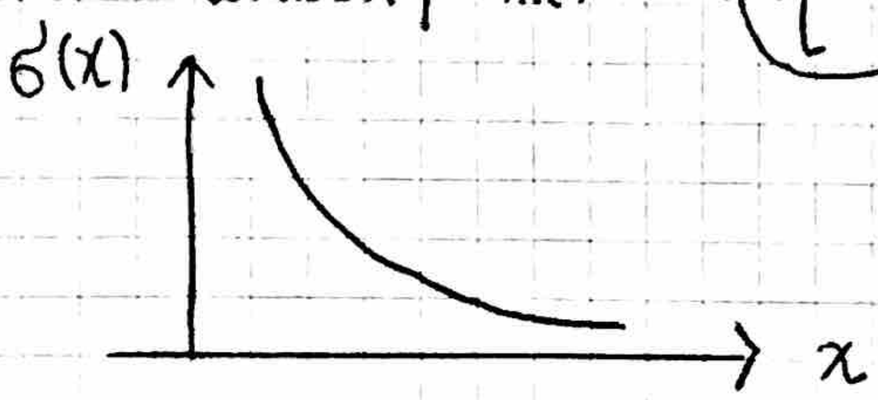
$$\sigma_B(x) = \sigma_B(-x)$$: symmetric smile with $\eta = 0$.



EQUITY MKT $\eta > 0$



COMMODITY MKT → $\eta < 0$



Why do we stress the η?

In our model, we have the constraint:

$\eta > -\omega$ → so, if we are in the equity world, we won't have any problem in calibration.

16. EXP-LEVY MODEL : HULL-WHITE MODEL.

HULL-WHITE APPROACH : OPTIONS' CLOSED FORMULAS

BLACK		EXP-LÈVY
$\dfrac{C_B(x)}{B(t_0,t)\cdot K}$	$= e^x N(d_1(x;\sigma^2)) - N(d_2(x;\sigma^2))$	$\rightarrow e^x I_1(x) - I_2(x)$
$\dfrac{dc_B}{B(t_0,t)}$	$= N(d_2(x;\sigma^2))$	$\rightarrow I_2(x)$

$$d_{1,2}(x;\sigma^2) = \frac{x}{\sqrt{(t-t_0)\sigma^2}} \pm \frac{1}{2}\sqrt{(t-t_0)\sigma^2}$$

where

$$\begin{cases} e^x I_1(x) \equiv \mathbb{E}_G[e^{x_G} N(d_1(x_G;\sigma^2 G))] \\ I_2(x) \equiv \mathbb{E}_G[N(d_2(x_G;\sigma^2 G))] \end{cases}$$

$$x_G = x - \eta \Delta t\, \sigma^2 G - \ln \mathcal{L}[\eta]$$

I_1, I_2 simple 1-d integrals: CDF corrected for the presence of fat tails.

(Hull-White, 1988)

$$\frac{C_B(x)}{B(t_0,t)} = \mathbb{E}[F_T - K]^+ = \mathbb{E}\left[\frac{F_T}{K} - 1\right]^+ =$$

$$= \mathbb{E}\left[\frac{F_0 e^{\delta}}{K} - 1\right]^+ \rightarrow \frac{C_B(x)}{B(t_0,t)F_0} = \mathbb{E}[e^{x+\delta} - 1]^+$$

...

17. STICKY STRIKE AND STICKY DELTA

Implied volatility is so useful in the mkt since it doesn't move so fast as the prices do : it's sticky

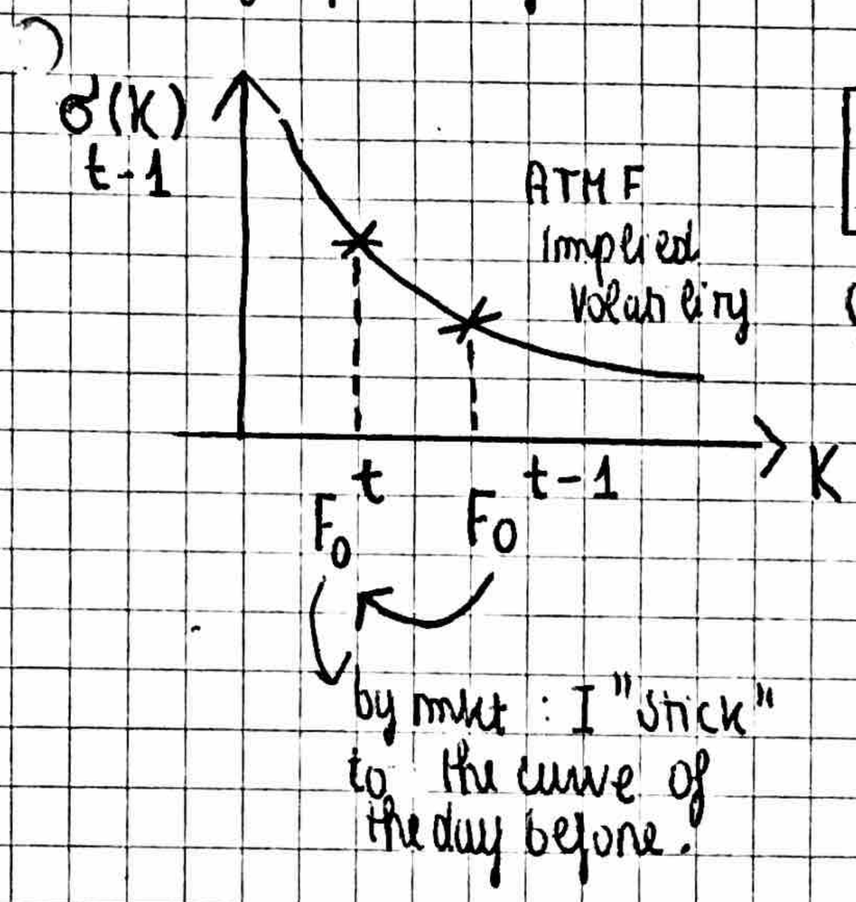
We trade on the volatility and then we derive the price.

There are two typical behaviors of the volatility surface :

1. STICKY STRIKE : $\sigma_B^{(t+1)}(K) = \sigma_B^{(t)}(K)$ (equal on the strike)

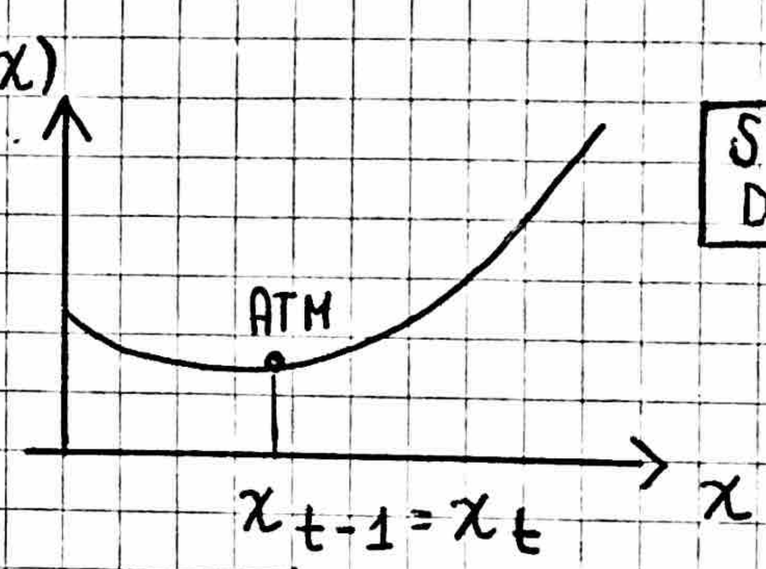
2. STICKY DELTA : $\sigma_B^{(t+1)}(x) = \sigma_B^{(t)}(x)$ (equal on the moneyness)

Graphically :



STICKY STRIKE

Commodities

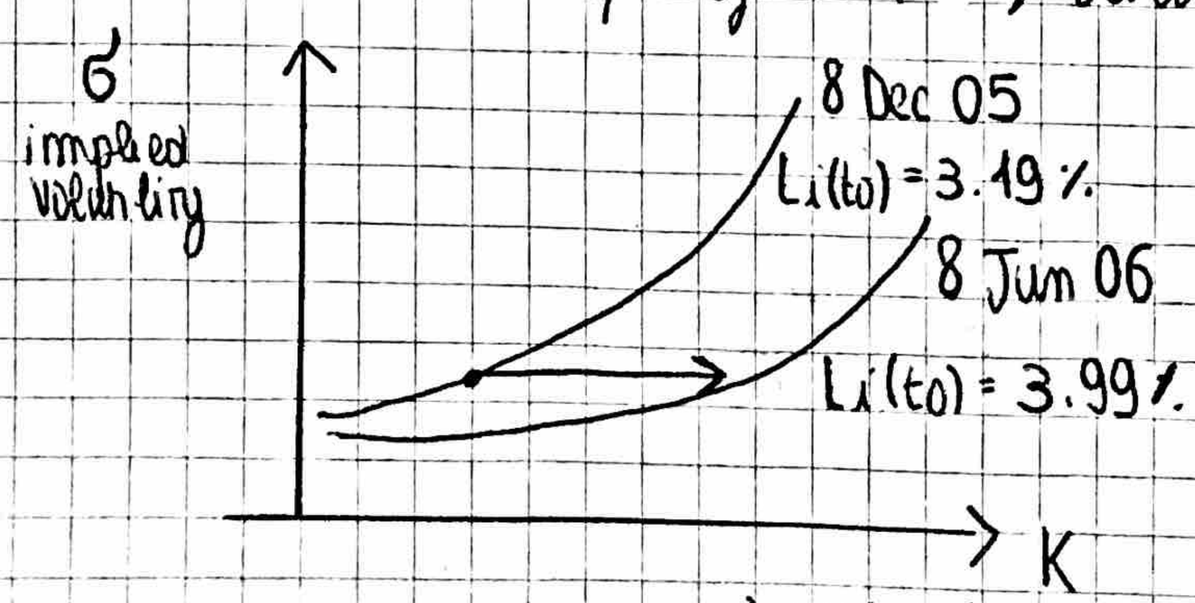


STICKY DELTA

The implied volatility as a function of the moneyness is constant. (main mkts are sticky delta) → equity, IR derivatives' mkt.

In case of sticky delta, even in case of movement of F_0, the volatility as a function of x will be the same (it doesn't move after F_0's movements).

Let's observe the slope of $\sigma(\cdot)$ between 8 Dec 05 and 8 Jun 06.



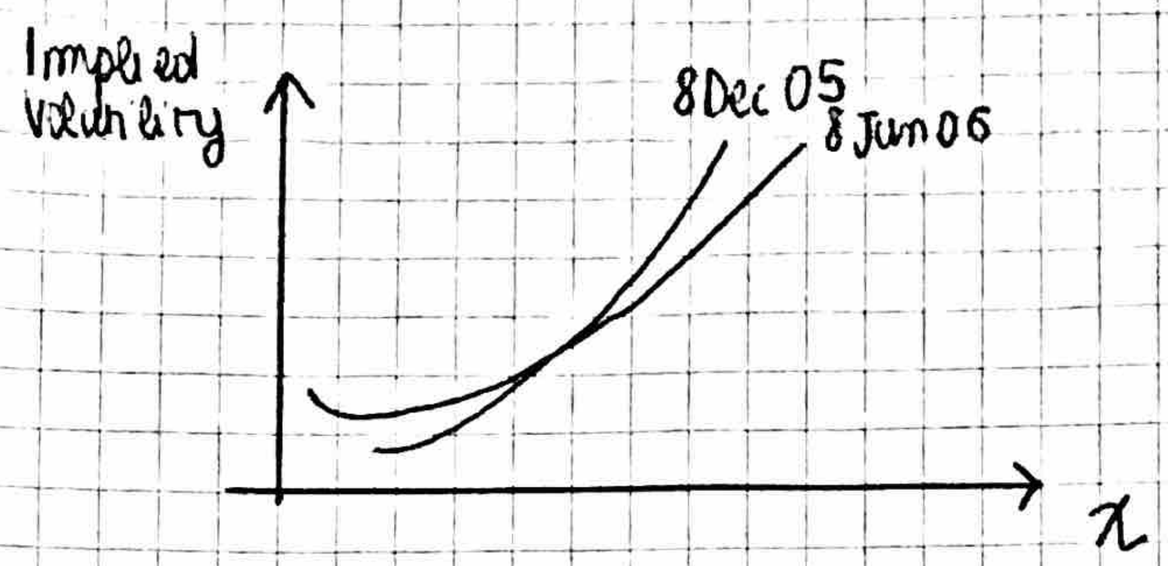
Computed on a 2x3 yearly cap (1 caplet)

σ and underlying move in the same direction of the rates

→ NO sticky strike!

correlated underlying dynamics

instantaneous volatility



Qualitatively, they collapse on each other.

⊥ instantaneous volatility, underlying dynamics

We are considering an implied volatility curve after one year:

8 Jun 05: 4 y after the dot-com crises.

CB cut rates in order to let companies pay less IR for loans, mortgages... motivation to increase amount of debt.

8 Dec 05: 6 months after ECB started increasing rates, after a long period of low rates.

8 Jun 06: Middle of this increasing period.

We are describing 3 situations that are completely different and what happens to σ is that, in case of sticky delta, volatilities almost collapse one on the other as a function of x →

Very sticky delta even after this dramatic change in IR!

The cap are also the part of the curve that moves faster → so, it's actual good STICKY DELTA!

$\sigma(x)$ → I can model underlying and volatility independently.

G → volatility as r.v.
g → underlying risk

} We have modelled as ⊥ r.v. → this can model properly sticky delta moves.

We have built a model (Exp-Lévy) that satisfies what it's seen in the mkt (implied volatility).

Is it a sticky delta one? Yes, it is automatically a sticky delta → the price, and then σ obtained as implied vol. won't depend from the underlying.

18. EXP. LEVY: GLOBAL CALIBRATION & MC FOR IG

Summing up, we have:

- introduced mean-variance mixture (model)
- priced plain vanillas (even digitals)
- we have to CALIBRATE

(so, then, we can price exotic options and manage risk).

GLOBAL CALIBRATION

* $(t_i, K_i) \rightarrow C_i$, price in the mkt
* $C(x_i, t_i, \underline{p})$, price in the model where for NIG and VG the set of parameters is: $\underline{p} = (\sigma, \kappa, \eta)$, $\eta > -\underline{\omega}$

LEAST SQUARE CALIBRATION:

$$d = \sum_{i=1}^{N} w_i \left[C(x_i, t_i; \underline{p}) - C_i \right]^2$$

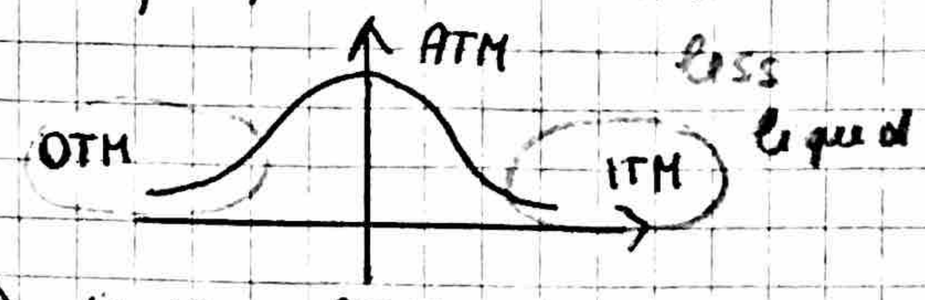
Usually, weights are chosen all equal:

$$w_i = 1$$

An alternative:

$$w_i = \frac{1}{\text{Vega}_B(x_i, t_i)^2}$$

NOT SUGGESTED since we have to calibrate on the most liquid, but we have:



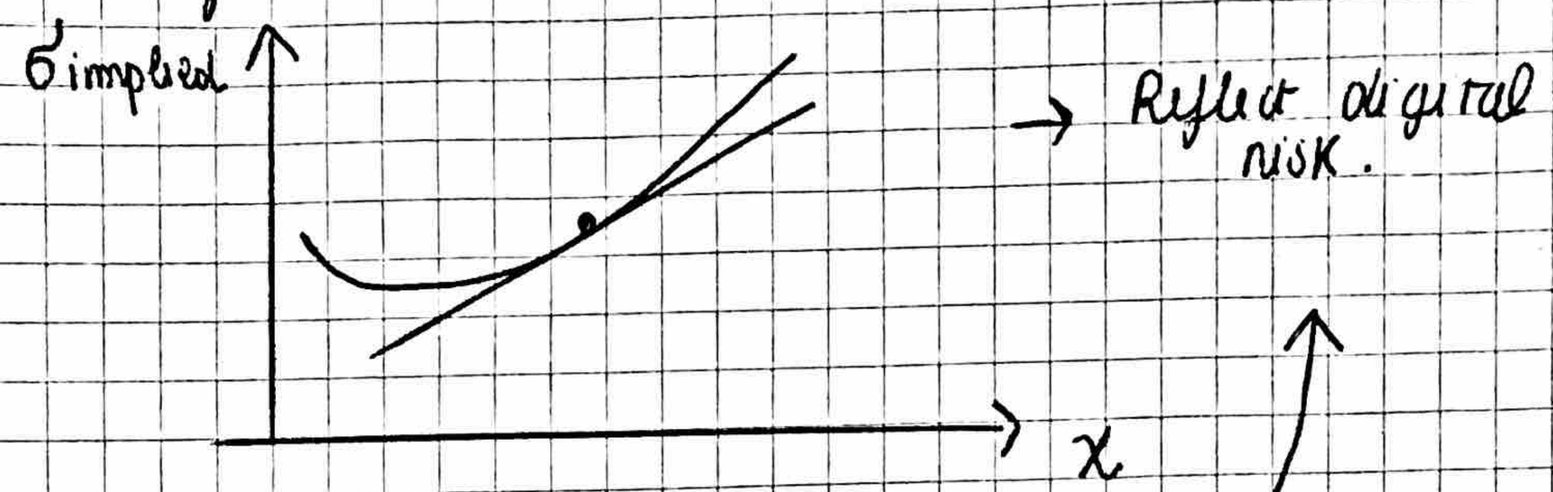
In IR, ATM options are the most liquid, but they have a high vega.

We put the lowest weight to the most liquid derivatives and the highest to the less liquid ones.

LOCAL CALIBRATION

Another possibility can be used if I know a book where the trader has a huge digital risk on a particular strike and at a particular TTM.

If I calibrate volatility in that particular situation, my book will respect the mkt in its changes.



Imposing that mkt and calibrated implied volatility have the same slope on the relevant strike.

SIMULATION (MC)

→ For NIG Lévy models, MC is simple between reset dates, since it is immediate to generate the variables in the dynamics (Gaussian and IG)

$$f \leftarrow g, \boxed{G}$$ two iid r.v.

std. n.

for a NIG, IG r.v. $\perp\!\!\!\perp$ g

The IG(1, k) case: (Inverse Gaussian)

1. Generate independent $u \sim$ Uniform (0,1), $z \sim \chi^2_1$
2. Let $G^* \equiv 1 - \frac{k}{2}\left(\sqrt{z^2 + \frac{4z}{k}} - z\right)$
3. If $(1+G^*)u > 1$, then $G = \frac{1}{G^*}$, else $G = G^*$

VARIANCED GAMMA

→ Also VG Lèvy Models allow a simple MC, since it is immediate to generate Gamma r.v.s.

Exp. Lèvy not needed for plain vanillas.
For structured bonds, we shall use MC.

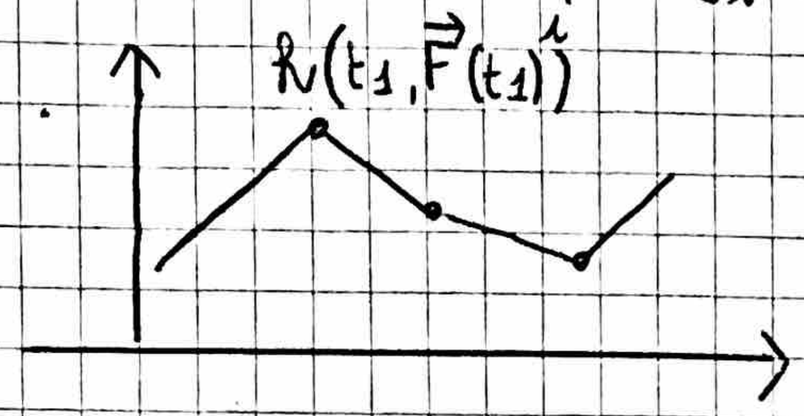
19. MC FOR A STRUCTURED PRODUCT.

We can simulate our MVM and we can price whatever exotic.

$$\Rightarrow O = \mathbb{E}\left[\sum_i D_{0i}\, h(t_i, \vec{F}(t_i))\right]$$

payoff

given a collection of reset dates, we wish to price and manage risk in an exotic.

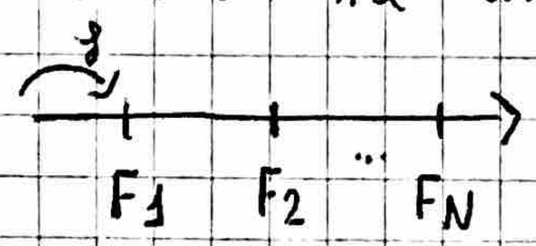


... if this option presents digital risk!

If we have several reset dates it's better to use the dynamic for underlying (not Fwd).
↳ because it has maturity dates too.

We will start pricing derivatives without credit risk; in case of credit risk, Li Model.

We simulate the underlying at each reset date.



simulate g, σ at each date.

NON CALLABLE, CALLABLE & AUTO CALLABLE

We prefer analytical formulas (if present), since we have to compute greeks and since MC can have some noise.