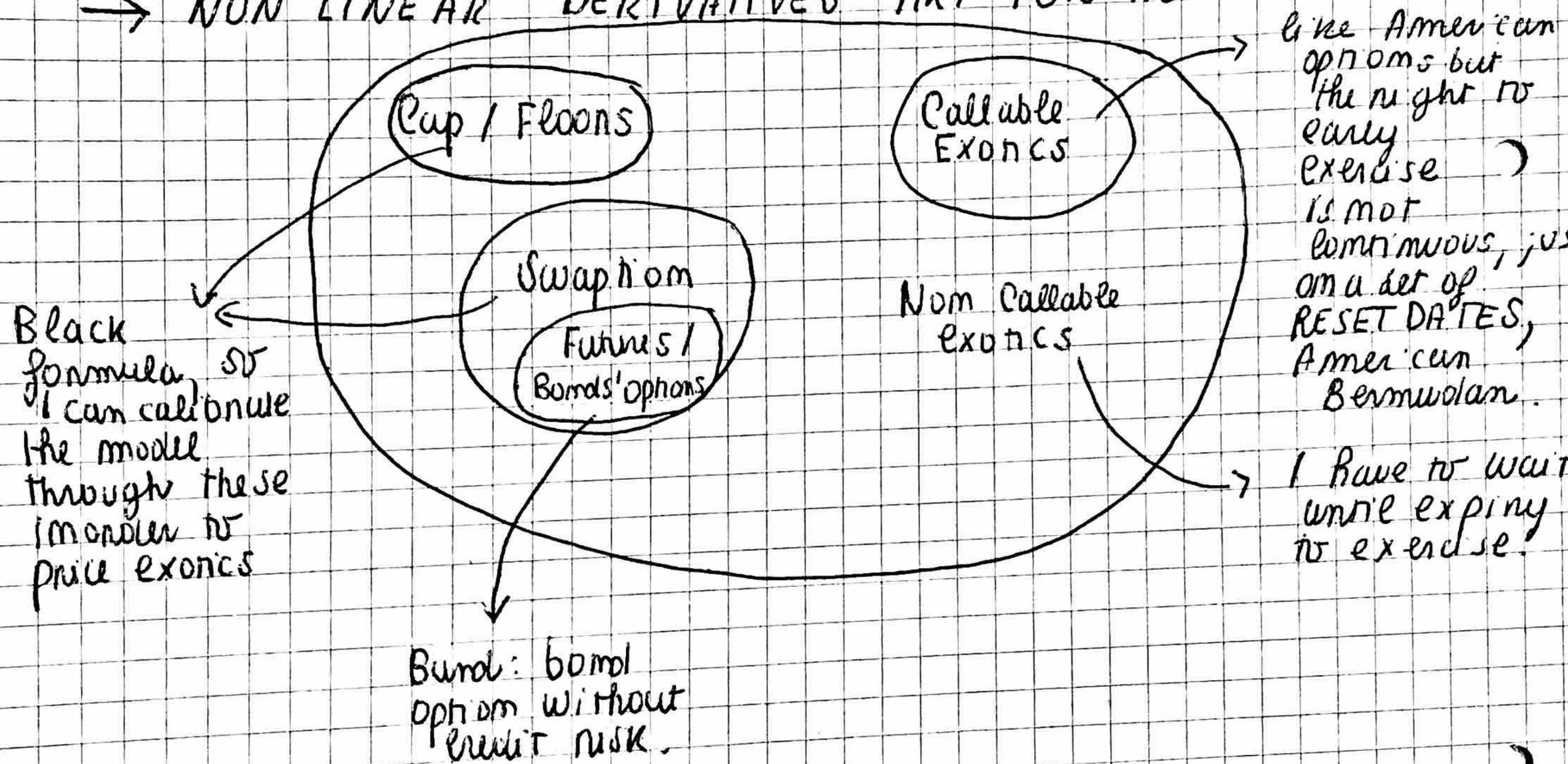
STRUCTURED INTEREST RATES

Linear products on IR:

- IRS
- FRA
- Depos
- Futures

} linear since, in order to hedge the position, I can consider only the 1st order greeks.

→ NON LINEAR DERIVATIVES' MKT FOR IR



IR MKT : BASIC RELATIONS

$$B_i(T_0) = B(T_0, t_i, t_{i+1})$$

(Forward) ZC Bond between T_α and T_ω : $B_{\alpha\omega}(T_0) = \prod_{i=\alpha}^{\omega-1} B_i(T_0)$

(Forward) Coupon Bond between T_α and T_ω : $P(c;T_0) = c \sum_{i=\alpha-1}^{\omega} \vartheta_{i-1} B_{\alpha i}(T_0) + B_{\alpha\omega}(T_0)$

(Forward) swap : $S_{\alpha\omega}(T_0) = \dfrac{1 - B_{\alpha\omega}(T_0)}{BPV_{\alpha\omega}(T_0)}$

with $BPV_{\alpha\omega}(T_0) = \sum_{i=\alpha-1}^{\omega} \vartheta_i B_{\alpha i}(T_0)$

PV MKT: 3 main formulas

Black formulas for $\begin{cases} \text{Swaption} \\ \text{Cap/Floor} \\ \text{Bond Options} \end{cases}$

- **SWAPTION PAYER**

$$SP_{\alpha\omega} = B_{0\alpha}(T_0)\, BPV_{\alpha\omega}(T_0)\left\{ S_{\alpha\omega}(T_0)\cdot N(d_1^S) - K\cdot N(d_2^S)\right\}$$

with: $$d_1^S = \frac{1}{\sigma_{\alpha\omega}\sqrt{T_\alpha}} \ln\left(\frac{S_{\alpha\omega}(T_0)}{K}\right) + \frac{1}{2}\sigma_{\alpha\omega}\sqrt{T_\alpha}$$

Option on a Swap: I have the right to enter a swap in which I pay a fixed rate K.

If $S_{\alpha\omega}(t_\alpha)$ (in the mkt, I see the quoted price) $> K$, I will exercise; otherwise, I will enter into a swap in the mkt.

Payoff: $$(S_{\alpha\omega}(t_\alpha) - K)^+ \cdot \sum_{i=\alpha}^{\omega-1} \delta_i B(t_0, t_i)$$

always comparing the two swaps (with same floating legs), so we are comparing the fixed legs.

that's why we have $BPV_{\alpha\omega}(T_0)$.

- **CAP / FLOOR**

1st unknown

$11 \neq$ implied vol.
assumption of FLAT VOLATILITIES

$$Cap = \sum_{i=\alpha}^{\omega-1} Caplet$$

$$Caplet_i = B_{i+1}(T_0)(T_{i-1} - T_i)\left\{ L_i(T_0) N(d_1^{cf}) - K N(d_2^{cf})\right\}$$

$$d_1^{cf} = \frac{1}{\sigma_i\sqrt{T_i}} \ln\left(\frac{L_i(T_0)}{K}\right) + \frac{1}{2}\sigma_i\sqrt{T_i}$$

PAYOFF CAPLET: $[L(T_i, T_{i+1}) - K]^+$

PAYOFF FLOORLET: $[K - L(T_i, T_{i+1})]^+$

BOND OPTIONS

$$BO_{\alpha\omega} = B_{0\alpha}(T_0)\left\{ P_{\alpha\omega}(c;T_0)\, N(d_1^B) - K\, N(d_2^B)\right\}$$

$$d_1^B = \frac{1}{V_{\alpha\omega}\sqrt{T_\alpha}} \ln\left(\frac{P_{\alpha\omega}(c;T_0)}{K}\right) + \frac{1}{2} V_{\alpha\omega}\sqrt{T_\alpha}$$

options on bonds not involved in credit mkt (IB bond), not "real", there is no a mkt.

There are mkts approximately considered with zero credit risk

ex. German Bonds — BUND (10y or larger), BOBL (5y), SHATZ (~2y)

The closed formulas above are <u>not coherent</u>.

Black formula for Cap/Floors:

$$L \to L(t_i, t_i, t_{i+1}) \quad \text{GBM}$$

... but as function of bonds:

$$L_i(t) = \frac{1}{\delta_i}\left[\frac{1}{B_i(t)} - 1\right] \quad \text{NO GBM}$$

If (2) is correct, (3) is not and viceversa.

So, if I have to price exotic options, <u>I want a coherent model</u>.

Whenever we consider IR, we can always look at 2 faces of the problem:

$\Longrightarrow$ <u>Options on Rates vs. Options on Bonds</u>

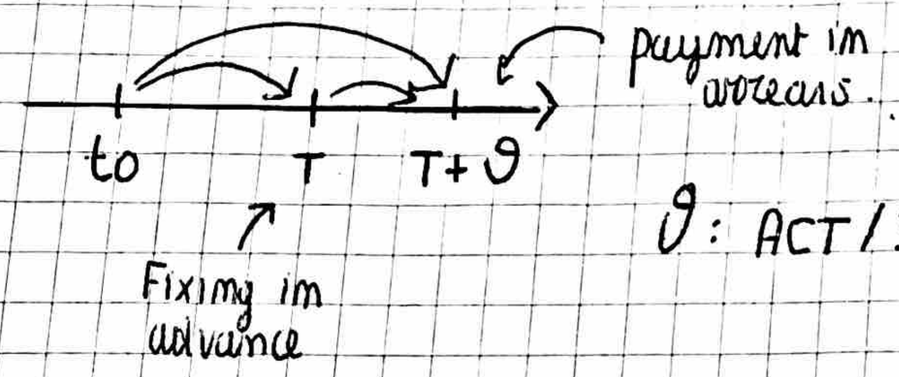
20. EQUIVALENCE : CAPLET VS. PUT ON ZC ; PAYER SWAPTION VS. PUT ON COUPON BOND.

Option on Rates vs. Option on Bonds

BASIC EQUIVALENCES

1. CAPLET VS. ZC PUT OPTION

Caplet



ϑ : ACT/360

Payoff Caplet : $\vartheta[L(T,T+\vartheta)-K]^+$

$$\Rightarrow \text{Caplet} = \mathbb{E}_0\Big[D(t_0,T+\vartheta)\big[\vartheta \underbrace{L(T,T+\vartheta)}_{L(T;T,T+\vartheta)} - \vartheta K\big]^+\Big] =$$

payoff exchanged at last date

$$\left(\begin{array}{l} D(t_0,T+\vartheta) = D(t_0,T)\,D(T,T+\vartheta) \\ \mathbb{E}_0[\cdot] = \mathbb{E}_0[\mathbb{E}_T[\cdot]] \\ \qquad\qquad \text{only not constant element in } T \\ \mathbb{E}_T[D(T,T+\vartheta)] = B(T,T+\vartheta) \end{array}\right)$$

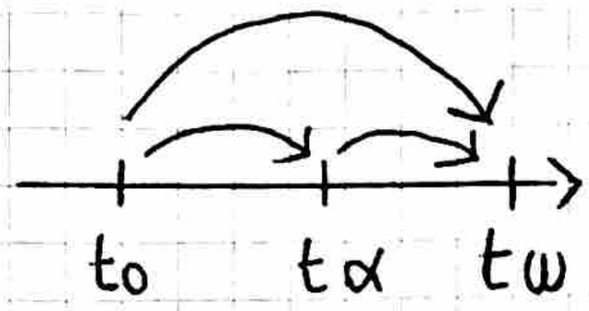
$$= \mathbb{E}_0\Big[D(t_0,T)\,B(T,T+\vartheta)\big(\underbrace{1+\vartheta L(T,T+\vartheta)}_{1/B(T,T+\vartheta)} - 1 - \vartheta K\big)^+\Big] =$$

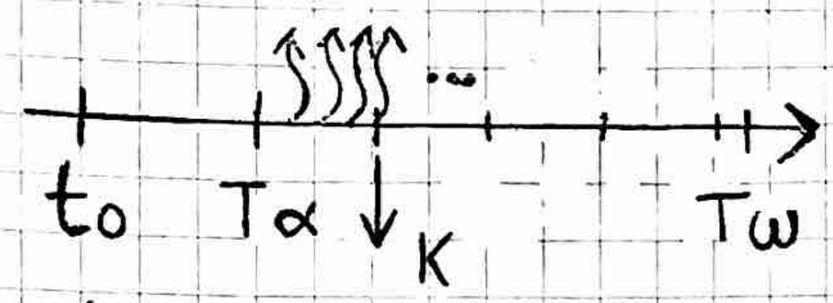
$$= (1+\vartheta K)\,\underbrace{\mathbb{E}_0\Big[D(t_0,T)\Big(\frac{1}{1+\vartheta K} - \underbrace{B(T,T+\vartheta)}_{B(T;T,T+\vartheta)}\Big)^+\Big]}_{\text{this is a PUT OPTION on a ZCB with } K' = 1/\vartheta K+1}$$

(no credit risk considered)

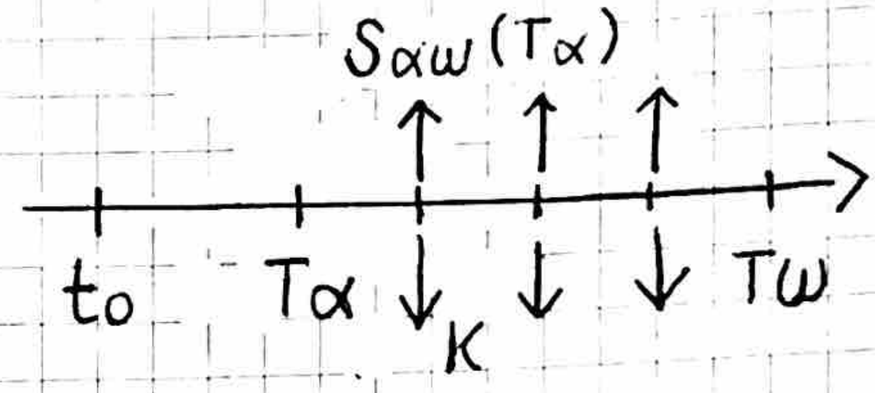
2. (RECEIVER) PAYER SWAPTION VS. (CALL) PUT ON COUPON BOND

Swaption payer





$\Longleftrightarrow$



(Swaption: option on a swap. I have the right to enter into an IRS to receive some floating legs (payer). K is the exercise price to exercise the right. Do I exercise?)

(In T_α, the floating legs 3m are equal to the yearly swap rate fixed at T_α (NPV=0)
↳ I substitute $S_{\alpha\omega}(T_\alpha)$ and I compare with the strike price.)

If $S_{\alpha\omega}(T_\alpha) > K$, I

$$P_{\alpha\omega}(t_0) = E_0\left[D(t_0, t_\alpha)\, BPV_{\alpha\omega}(t_\alpha)\, [S_{\alpha\omega}(t_\alpha) - K]^+ \right] =$$

$$= E_0\left[D(t_0, t_\alpha) \left[BPV_{\alpha\omega}(t_\alpha) S_{\alpha\omega}(t_\alpha) + B(t_\alpha, t_\omega) + \right.\right.$$

$$\left.\left. - \left(BPV_{\alpha\omega}(t_\alpha) K + B(t_\alpha, t_\omega)\right)\right]^+ \right] =$$

$$= \left(S_{\alpha\omega}(t_\alpha) = \frac{1 - B(t_\alpha, t_\omega)}{BPV_{\alpha\omega}(t_\alpha)} \right) =$$

$$= E_0\left[D(t_0, t_\alpha) \left[1 - \underbrace{\left(BPV_{\alpha\omega}(t_\alpha) K + B(t_\alpha, t_\omega)\right)}_{\substack{\text{Coupon bond with} \\ \text{coupon } = K \\ P_{\alpha\omega}(K, t_\alpha)}} \right]^+ \right] =$$

$$= E_0\left[D(t_0, t_\alpha) \underbrace{\left[1 - P_{\alpha\omega}(K, t_\alpha) \right]^+}_{\substack{\text{put option on a CB} \\ \text{with } K = 1}} \right]$$

21. HJM: RN FORMULATION WITH CONTINUOUS FORWARD VS WITH ZC BOND AND ZERO RATES.

CURVE DYNAMIC: HJM FRAMEWORK

Introduzione X. 2022 un modello basato per L'otic. of Trans! (Closed Black formulas not current).

Standard way to model discount factors:

$$\begin{cases} dB(t,T) = B(t,T)\left[r_t\, dt - \underline{\sigma}(t,T)\cdot d\underline{W}_t \right] \\ B(t_0,T) \quad \text{initial condition} \end{cases}$$

where $\underline{\sigma}(t,T) \in \mathbb{R}^d$ not just a function adapted process of $W(t)$ (in general),

$W(t)$ BM in $\mathbb{R}^d$;

$dW_i(t)\, dW_j(t) = \rho_{ij}\, dt$

$\rightsquigarrow B(t,T)$

driver of the dynamic — Parameter

When $\underline{\sigma}(t,T)$ is a deterministic function of time in $\mathbb{R}^d$, the model is called GAUSSIAN HJM.

In particular, within this frame, the simplest 1-d example is the Extended Vasicek model:

$$\sigma(s,t) = \frac{\sigma}{a}\left(1 - e^{-a(t-s)}\right) \in \mathbb{R}$$

DEF. Continuous fwd rate:

$$f(t,T) \equiv -\frac{\partial \ln B(t,\overset{\text{parameter}}{T})}{\partial T} \iff B(t,T) = e^{-\int_t^T f(t,u)\,du}$$

DEF. Instantaneous rate:

$$r_t := \lim_{T\to t} f(t,T)$$

NOTATION $\quad \underline{z}^2 := \underline{z}\cdot \rho\, \underline{z} \quad \forall\, \underline{z} \in \mathbb{R}^d$

matrix of correlation

HJM FRAMEWORK : DYNAMIC AND MAIN PROPOSITIONS

(All models will come from this one, making different assumptions on volatility.)

The (R.N.) dynamic for the curve of continuous forward rate is:

$$df(t,T) = \frac{1}{2}\frac{\partial}{\partial T}\left[\underline{\sigma}(t,T)\cdot\rho\,\underline{\sigma}(t,T)\right]dt + \frac{\partial}{\partial T}\underline{\sigma}(t,T)\cdot d\underline{W}(t)$$

with $\begin{cases} \underline{\sigma}(t,T) \text{ an adapted process } \in \mathbb{R}^d \text{ s.t. } \underline{\sigma}(t,t)=0 \ \forall t \\ dW_i(t)\,dW_j(t) = \rho_{ij}\,dt \end{cases}$

and the initial condition $f(t_0,T) = -\frac{\partial}{\partial T}\ln B(t_0,T)$

(HJM, 1992)

dynamic stops in T!

Requirement of HJM : $\underline{\sigma}(t,t)=0 \ \forall t \Rightarrow \sigma(T,T)=0$

driver of the dynamic, which is σ, must be zero at the end.

dynamic of f is zero from $t \geq T$.
→ "frozen" dynamic.

PROP. Above dynamic is equivalent for the ZC bond:

$$\begin{cases} dB(t,T) = B(t,T)\left[r_t\,dt - \underline{\sigma}(t,T)\cdot d\underline{W}_t\right] \\ r_t = f(t_0,t) + \frac{1}{2}\int_{t_0}^{t}\frac{\partial}{\partial t}\left[\underline{\sigma}(t',t)\cdot\rho\,\underline{\sigma}(t',t)\right]dt' + \int_{t_0}^{t}\frac{\partial}{\partial t}\underline{\sigma}(t',t)\cdot dW(t') \end{cases}$$

For the proof,

ITÔ CALCULUS IN $\mathbb{R}^d$:

$$g(t, \underline{X}_t) \qquad \underline{X} \in \mathbb{R}^N$$

$$d\underline{X}_t = \underline{\mu}(t, \dots)\, dt + \underline{\sigma}(t, \dots) \cdot dW_t$$

$$dg(t, \underline{X}_t) = \frac{\partial g}{\partial t} dt + \sum_{i=1}^{N} \frac{\partial g}{\partial X_i} \cdot dX_i +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 g}{\partial X_i \partial X_j} \langle dX_i(t), dX_j(t) \rangle$$

$$\langle dX_i(t), dX_j(t) \rangle = [\sigma_i(x,t) \cdot d\underline{W}_t][\underline{\sigma}_j(x,t) \cdot d\underline{W}_t]$$

PROOF.

$$\boxed{(B \Rightarrow f)}$$

$$d \ln B(t,T) = \frac{dB(t,T)}{B(t,T)} - \frac{1}{2} \frac{\langle dB(t,T) \rangle^2}{B(t,T)^2}$$

$$\langle dB(t,T) \rangle^2 = B^2(t,T)\, \mathbb{E}[\underline{\sigma}(t,T) \cdot d\underline{W}_t\ \underline{\sigma}(t,T) \cdot d\underline{W}_t] =$$

$$= B^2(t,T) \sum_{i,j=1}^{d} \sigma_i(t,T)\, \underbrace{dW_i(t)\, dW_j(t)}_{\rho_{ij} dt}\, \sigma_j(t,T) =$$

$$= B^2(t,T)\, \underbrace{\underline{\sigma}(t,T) \cdot \rho\, \underline{\sigma}(t,T)}_{\underline{\sigma}^2(t,T)}\, dt$$

$$\Rightarrow d \ln B(t,T) = (\mu_t\, dt - \underline{\sigma}(t,T) \cdot d\underline{W}_t) - \frac{1}{2} \underline{\sigma}^2(t,T)\, dt =$$

$$= \left(\mu_t - \frac{1}{2} \underline{\sigma}^2(t,T)\right) dt - \underline{\sigma}(t,T) \cdot d\underline{W}_t$$

Then let's do $-\frac{\partial}{\partial T}$:

$$d f(t,T) = -\frac{\partial}{\partial T} d \ln B(t,T) = \frac{1}{2} \frac{\partial}{\partial T} \underline{\sigma}^2(t,T)\, dt + \frac{\partial}{\partial T} \underline{\sigma}(t,T) \cdot d\underline{W}_t$$

$\boxed{(f \Rightarrow B)}$ $\quad E(t,T) = e^{-\int^T f(t,u)\,du}$ → du f ragio n

$$df(t,T) = +\frac{1}{2}\frac{\partial}{\partial T}\underline{\sigma}(t,T)^2\,dt + \frac{\partial}{\partial T}\underline{\sigma}(t,T)\cdot d\underline{W}_t$$

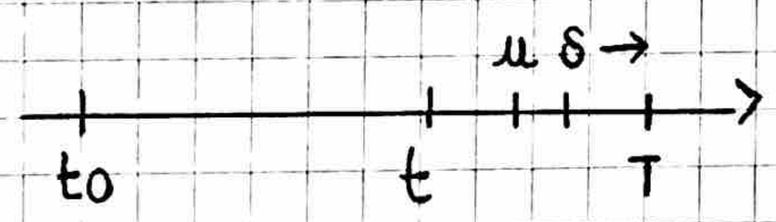
↳ dynamic driven by t, T is a parameter

$$\rightarrow \int_{t_0}^{s} df(t,T) = f(s,T) - f(t_0,T) = \int_{t_0}^{s} \text{per curve } f(s,T),\ f(t_0,T)$$

$$= \frac{1}{2}\int_{t_0}^{s}\frac{\partial}{\partial T}\underline{\sigma}(t,T)^2\,dt + \int_{t_0}^{s}\frac{\partial}{\partial T}\underline{\sigma}(t,T)\cdot d\underline{W}_t$$

$$\rightarrow r_s = \lim_{T\to s} f(s,T) = f(t_0,s) + \frac{1}{2}\int_{t_0}^{s}\frac{\partial}{\partial s}\underline{\sigma}(t,s)^2\,dt + \int_{t_0}^{s}\frac{\partial}{\partial s}\underline{\sigma}(t,s)\cdot d\underline{W}_t$$

→ Let's make the integral between t and T:



$$I = \int_t^T r_s\,ds = \int_t^T f(t_0,s)\,ds + \frac{1}{2}\int_t^T ds\int_{t_0}^{s}\frac{\partial}{\partial s}\underline{\sigma}(u,s)^2\,du + \int_t^T ds\int_{t_0}^{s}\frac{\partial}{\partial s}\underline{\sigma}(u,s)\cdot d\underline{W}_u \quad *$$

We can split the integral from t_0 to s in two parts:

$$t_0 \xrightarrow{a} t \xrightarrow{b} s$$

$$* \quad \frac{1}{2}\int_t^T ds \left\{ \int_{t_0}^{t} \dots du + \int_t^s \dots du \right\} +$$

$$+ \int_t^T ds \left\{ \int_{t_0}^{t} \dots dW_u + \int_t^s \dots d\underline{W}_u \right\} + \int_t^T f(t_0, s)\, ds$$

Let's compute (da inserire dentro un unico $\int_t^T \dots$)

$$\int_{t_0}^{t} du \frac{\partial}{\partial s} \underline{\sigma}(u,s)^2 + \int_{t_0}^{t} d\underline{W}_u \cdot \frac{\partial}{\partial s} \underline{\sigma}(u,s) + f(t_0, s) =$$

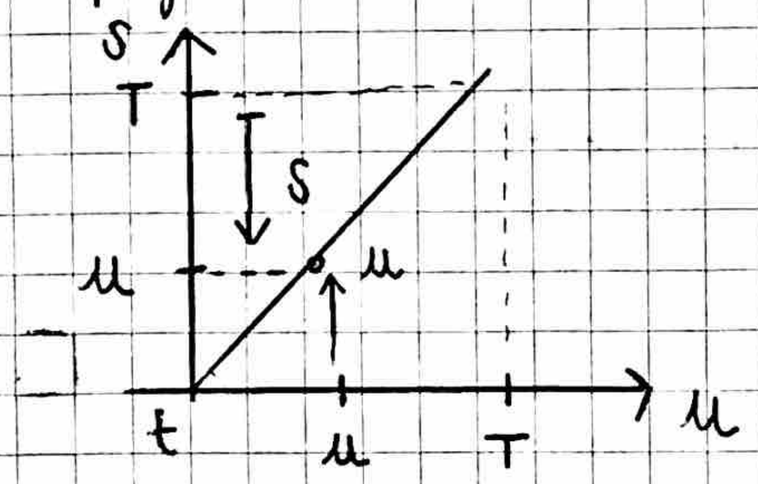
$$= f(t,s)$$

And so the integral becomes:

$$I = \int_t^T f(t,s)\, ds + \frac{1}{2}\int_t^T ds \int_t^s \frac{\partial}{\partial s} \underline{\sigma}(u,s)^2 du +$$

$$+ \int_t^T ds \int_t^s \frac{\partial}{\partial s} \underline{\sigma}(u,s) \cdot d\underline{W}_u$$

We would like to switch the integrals, but we have to pay attention to the domain of interpretation:



fixed u, s is between u and T

$$I \overset{\text{Fubini}}{=} \int_t^T f(t,s)\,ds + \frac{1}{2}\int_t^T du \underbrace{\int_u^T ds \frac{\partial}{\partial s} \underline{\sigma}(u,s)^2}_{\sigma(u,T)^2 - \cancel{\sigma(u,u)^2}} +$$

$$+ \int_t^T d\underline{W}_u \underbrace{\int_u^T \frac{\partial}{\partial s} \underline{\sigma}(u,s)\,ds}_{\sigma(u,T) - \cancel{\sigma(u,u)}} =$$

$$= \underbrace{\int_t^T f(t,s)\,ds}_{-\ln B(t,T) = \int_t^T d(\ln B(u,T)) \quad \forall t,T} + \frac{1}{2}\int_t^T du\, \underline{\sigma}(u,T)^2 + \int_t^T d\underline{W}_u \cdot \sigma(u,T)$$

$$\Rightarrow \int_t^T ds\, r_s = \int_t^T d \ln B(s,T) + \frac{1}{2}\int_t^T du\, \underline{\sigma}(u,T)^2 +$$

$$+ \int_t^T d\underline{W}_u \cdot \underline{\sigma}(u,T) \quad \forall t,T \text{ s.t. } t_0 \le t < T$$

And so the equation will hold also for the integrand:

$$d \ln B(s,T) = r_s\,ds - \frac{1}{2}\underline{\sigma}(s,T)^2 ds - \underline{\sigma}(s,T)\,d\underline{W}_s$$

$$\Downarrow \text{ Itô in reverse}$$

$$dB(s,T) = B(s,T)\{ r_s\,ds - \underline{\sigma}(s,t)\cdot d\underline{W}_s \} \quad \blacksquare$$

Why B instead of f?
We want to express the dynamic for B
for ZCB.

$\boxed{22.}$ HJM : LEMMA 1 + $\boxed{23.}$ HJM : LEMMA 2

For an option, there is a payoff at a reset date $h(\cdot)$ and a discount factor D : $\mathbb{E}_0[D(0,T^*)h(T^*)]$.

For example,

* $\delta_i[L_i(t)-K]^+ \leftarrow B_i(t)$
* $BPV_{\alpha\omega}(T_\alpha)[S_{\alpha\omega}(T_\alpha)-K]^+ \leftarrow B_\alpha(t)$

... both are functions of ZC bond.

Trick : write the payoff as function of ZC bond

$\rightarrow B(t,T,T+\vartheta)$

- <u>LEMMA 1</u> : Dynamic for a forward ZC bond.

$$dB(t;T,T+\vartheta) = -B(t;T,T+\vartheta)[\underline{\sigma}(t,T+\vartheta)-\underline{\sigma}(t,T)]\cdot[dW_t+\rho\,\underline{\sigma}(t,T)dt]$$

- <u>COROLLARY</u> :

$$\frac{B(t;T,T+\vartheta)}{B(t_0;T,T+\vartheta)} = \exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left(\underline{\sigma}(u,T+\vartheta)^2-\underline{\sigma}(u,T)^2\right)du + \right.$$
$$\left. -\left(\underline{\sigma}(u,T+\vartheta)-\underline{\sigma}(u,T)\right)\cdot dW(u)\right\}$$

- <u>LEMMA 2</u> : Discount

$$D(t,T) = B(t,T)\exp\left\{-\frac{1}{2}\int_t^T \underline{\sigma}(u,T)^2\,du + \right.$$
$$\left. -\int_t^T \underline{\sigma}(u,T)\cdot dW(u)\right\}$$

Are these dynamics coherent with HJM framework ?

PROOF.

LEMMA 2

$$D(t,T) = \exp\left\{ -\int_t^T r_s \, ds \right\}$$

Since $\int_t^T r_s \, ds \overset{\text{(from prima)}}{=} -\ln B(t,T) + \frac{1}{2}\int_t^T \underline{\sigma}(s,T)^2 ds + \int_t^T \underline{\sigma}(s,T) \cdot d\underline{W}_s$

$$\Rightarrow D(t,T) = B(t,T) \exp\left\{ -\frac{1}{2}\int_t^T \underline{\sigma}(u,T)^2 du - \int_t^T \sigma(u,T) \cdot d\underline{W}_u \right\}$$

LEMMA 1

Before proving it, let's do an exercise of Itô Calculus for the function $g(\underline{X}, t) = \dfrac{X_1(t)}{X_2(t)}$

with the dynamics for X_1, X_2:

$$\begin{cases} dX_1(t) = X_1(t)\left(\mu_1(t)\,dt + \underline{\sigma}_1(t) \cdot d\underline{W}_t\right) \\ dX_2(t) = X_2(t)\left(\mu_2(t)\,dt + \underline{\sigma}_2(t) \cdot d\underline{W}_t\right) \end{cases}$$

$$dg(\underline{X}, t) = d\frac{X_1}{X_2}(t) = \frac{1}{X_2} dX_1(t) - X_1 \cdot \frac{1}{X_2^2} dX_2(t) +$$

$$+ \frac{1}{2}\left\{ +\cancel{2} X_1 \frac{1}{X_2^3} \langle dX_2^2 \rangle - \frac{1}{X_2^2} \langle dX_1(t), dX_2(t) \rangle \right.$$

$$\left. - \cancel{\frac{1}{X_2^2} \langle dX_1(t), dX_2(t) \rangle} \right\} =$$

$$= \frac{X_1}{X_2}\left\{ \mu_1(t)\,dt + \underline{\sigma}_1(t) \cdot d\underline{W}_t - \left(\mu_2(t)\,dt + \underline{\sigma}_2(t) \cdot d\underline{W}_t\right) + \right.$$

$$\left. + \left(\underline{\sigma}_2(t) \cdot \rho\, \underline{\sigma}_2(t)\right) dt - \underline{\sigma}_1(t) \cdot \rho\, \underline{\sigma}_2(t)\, dt \right\}$$

$$= \frac{X_1(t)}{X_2(t)} \left\{ (\mu_1(t) - \mu_2(t))\,dt - (\underline{\sigma}_1(t) - \underline{\sigma}_2(t)) \cdot \rho \underline{\sigma}_2(t) + \right.$$
$$\left. + (\underline{\sigma}_1(t) - \underline{\sigma}_2(t)) \cdot dW_t \right\} =$$

$$= \frac{X_1(t)}{X_2(t)} \left\{ (\mu_1(t) - \mu_2(t))\,dt + (\underline{\sigma}_1(t) - \underline{\sigma}_2(t)) \cdot (d\underline{W}_t - \rho \sigma_2(t)) \right.$$

Let's apply this formula to the forward ZC bond:

$$B(t; T, T+\theta) = \frac{B(t, T+\theta)}{B(t,T)} = \frac{B1(t)}{B_2(t)}$$

where $\quad dB(t,T) = B(t,T)\{ r_t\,dt - \underline{\sigma}(t,T) \cdot dW_t \}$

$$dB(t; T, T+\theta) = B(t; T, T+\theta) \{ (-\underline{\sigma}(t, T+\theta) +$$
$$* \qquad + \sigma(t,T)) \cdot (dW_t + \rho \underline{\sigma}(t,T)\,dt) \}$$

■

COROLLARY : solve * line in the equity case with Itô and log function.

$$d \ln B(t; T, T+\theta) \leftarrow \text{Itô}$$

$$\int_{t_0}^{t} d \ln B(s; T, T+\theta)\,ds = \ln B(t; T, T+\theta) - \ln B(t_0; T, T+\theta)$$

...

24. CHANGE OF MEASURE & GIRSANOV THEOREM. PRICES REWRITTEN FOR PAYOFF AT PAYMENT DATE DISCOUNTED. EQUITY EUROPEAN DERIVATIVE WITH STOCHASTIC IR.

CHANGE OF MEASURE

Consider a b.m. with a drift

$$\underbrace{d\underline{W}_t^{(\eta)}}_{\text{New measure}} = \underbrace{d\underline{W}_t}_{\text{b.m. w.r.t. a measure}} + \eta(\dots)\,dt$$

In the current measure $d\underline{W}_t^{\eta}$ is not a b.m. (no zero mean), but we can always find a new measure w.r.t. which $d\underline{W}_t^{\eta}$ is a b.m., thanks to

GIRSANOV THEOREM: $\eta(t) = \rho\,\underline{\sigma}(t,T)$

So, we have an original measure $\tilde{\Pi}$, $d\underline{W}_t$.

We can pass to a new measure $\tilde{\Pi}^{(t_i)}$:

t_i- fwd measure:

$$\left[dW_t^{(i)} = dW_t + \rho\,\underline{\sigma}(t,t_i)\,dt\right]$$

Radom - Nikodym derivative (Girsanov theorem):

$$\frac{d\tilde{\Pi}^{(i)}}{d\tilde{\Pi}} = \exp\left\{-\frac{1}{2}\int_{t_0}^{t_i} \underline{\sigma}(u,t_i)^2\,du - \int_{t_0}^{t_i} \underline{\sigma}(u,t_i)\cdot dW_u\right\}$$

$B = \frac{1}{1+\delta L}$ $\quad (\frac{1}{B} + \delta L) = \frac{1}{B} \rightarrow \frac{1}{\delta B}$

$\forall$ payoff we have, we can write it as a function of ZCB, related to corresponding fwd ZCB.

<u>EX.</u> * Caplet payoff

$$\delta(T, T+\vartheta)\left(L(T; T, T+\vartheta) - K\right)^+$$

$$= \frac{1}{\delta}\left(\frac{1}{B(T, T, T+\vartheta)} - 1\right)$$

* Swaption payoff with:

$$S(T_\alpha; T_\alpha, T_\omega) = \frac{1 - B_{\alpha\omega}(T_\alpha)}{BPV_{\alpha\omega}(T_\alpha)}$$

$\Rightarrow$ payoff will be a function of my ZCB at expiry.

$$\text{Payoff} \quad h\left(T, \{B(t_0, T)\}\right)$$

<u>PRICES REWRITTEN FOR PAYOFF AT PAYMENT DATE DISCOUNTED</u>

$$\mathcal{O}(t_0) = \mathbb{E}_0\left[D(t_0, T)\, h(T)\right] \overset{\text{Lemma 2}}{=}$$

$$= B(t_0, T)\, \mathbb{E}_0\left[\exp\left\{-\frac{1}{2}\int_{t_0}^{T} \underline{\sigma}(u, T)^2\, du + \right.\right.$$

$$\left.\left. - \int_{t_0}^{T} \underline{\sigma}(u, T) \cdot d\underline{W}_u\right\} \cdot h(T)\right] \overset{\text{Girsanov theorem}}{=}$$

$$= B(t_0, T)\, \mathbb{E}_0\left[\frac{d\tilde{\pi}^{(T)}}{d\tilde{\pi}}\, h(T)\right] \overset{\text{Change of Measure}}{=}$$

$$= B(t_0, T)\, \mathbb{E}^{(T)}\left[h(T)\right]$$

$\uparrow$ payoff at time T

the final expression under T-measure !!

EXAMPLES :

* EQUITY EUROPEAN DERIVATIVE WITH STOCHASTIC IR

B&S formula with stochastic IR

dynamic for the fwd :
$$\begin{cases} dF(t,T) = F(t,T)\,\underline{\sigma}\cdot d\underline{W}_t \\ \text{with } \underline{\sigma} = \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} \end{cases}$$
σ → for equity; $d+1$ components used in order to model dynamic for IR.

F_0 : initial condition

$$+\ dB(t,T) = B(t,T)\left[r_t\,dt - \underline{\sigma}(t,T)\cdot d\underline{W}_t\right]$$

(if we consider the dynamic for the underlying :
$$dS_t = S(t)\left[r_t\,dt - \underline{\sigma}\cdot d\underline{W}_t\right]$$
)

Call Payoff in T : $[F(T,T) - K]^+$

We want to consider the dynamic of the forward, selecting it in such a way that we will consider the T-forward measure :

So we have $F(t,T) = S_t\, E_t\left[e^{\int_t^T r\,dt}\right]$ → $\frac{1}{B(t,T)}$

$$dF(t,T) = d\left(\frac{S(t)}{B(t,T)}\right)$$

So we can consider the dynamics for the underlying and the discount :

$$\begin{cases} dS_t = S_t\left[r_t\,dt - \underline{\sigma}\cdot d\underline{W}_t\right] \\ dB(t,T) = B(t,T)\left[r_t\,dt - \underline{\sigma}(t,T)\cdot d\underline{W}_t\right] \end{cases}$$

$$d\left(\frac{X_1(t)}{X_2(t)}\right) = \frac{X_1(t)}{X_2(t)}\left\{\left(\mu_1(t) - \mu_2(t)\right)dt + \left(\underline{\sigma}_1(t) - \underline{\sigma}_2(t)\right)\left(d\underline{W}_t - \rho\,\underline{\sigma}_2(t)\right)\right\}$$

$$* \quad d\frac{S(t)}{B(t,T)} = \frac{S(t)}{B(t,T)}\Big\{-\underbrace{[\underline{\sigma} - \underline{\sigma}(t,T)]}_{\underline{\sigma}_F(t)}\underbrace{[d\underline{W}_t + \rho\,\underline{\sigma}(t,T)]}_{d\underline{W}_t^{(T)}}\Big\} \star$$

$$= F(t,T)\Big\{-\underline{\sigma}_F(t)\cdot d\underline{W}_t^{(T)}\Big\}$$

For the B&S model, call payoff will be:

$$C = \mathbb{E}_0\left[D(t_0,T)[F(T,T)-K]^+\right] = \quad \text{Lemma 2 + Girsanov theorem}$$

$$= B(t_0,T)\,\mathbb{E}^{(T)}\left[(F(T,T)-K)^+\right]$$

* is a geometric b.m. in the T-measure → mg ok for the forward. (if $\underline{\sigma}_F$ deterministic)

$$d\ln F(t,T) = \frac{1}{\cancel{F(t,T)}}\cancel{F(t,T)}\Big\{-\underline{\sigma}_F(t)\cdot d\underline{W}_t^{(T)}\Big\} -$$

$$-\frac{1}{2}\frac{1}{\cancel{F(t,T)^2}}\Big\{\cancel{F(t,T)^2}\,\underline{\sigma}_F^2\,dt\Big\}$$

$$= -\underline{\sigma}_F(t)\cdot d\underline{W}_t^{(T)} - \frac{1}{2}\underline{\sigma}_F^2\,dt$$

$$F(T,T) \overset{\text{"law"}}{=} F_0\exp\Big\{-\frac{1}{2}\int_{t_0}^{T}\underline{\sigma}_F^2\,dt - \Sigma\sqrt{T-t_0}\,g\Big\}$$

$$\Sigma^2 = \frac{1}{T-t_0}\int_{t_0}^{T}\underline{\sigma}_F^2\,dt$$

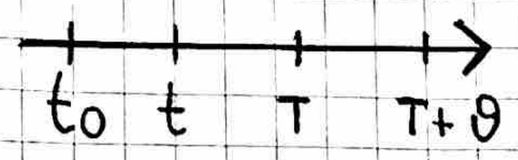
$$\Rightarrow C(t_0) = B(t_0,T)\Big\{F_0N(d_1) - KN(d_2)\Big\}$$

$$d_{1,2} = \frac{\ln F_0/K}{\Sigma\sqrt{T-t_0}} \pm \frac{1}{2}\Sigma\sqrt{T-t_0}$$

Another example of prices rewritten, with another measure, in function of the payoff at payment date:

25. GAUSSIAN HJM: ZC OPTION SOLUTION

CALL OPTION ON ZC FOR GAUSSIAN HJM



$$C = \mathbb{E}_0\left[D(t_0,T)\left(B(T,T+\vartheta)-K\right)^+\right] \overset{\text{Lemma 2 + Girsanov}}{=}$$

$$= B(t_0,T)\,\mathbb{E}^{(T)}\left[\left(B(T,T+\vartheta)-K\right)^+\right]$$

Lemma 1 $dB(t;T,T+\vartheta) = -B(t;T,T+\vartheta)\left[\underline{\sigma}(t;T+\vartheta) - \underline{\sigma}(t,T)\right]\cdot\left[d\underline{W}_t + \rho\,\underline{\sigma}(t,T)\,dt\right] =$

$$= -B(t;T,T+\vartheta)\,\underbrace{\underline{V}(t)}_{\substack{\text{deterministic}\\ \text{vector as}\\ \text{function of time}}}\cdot d\underline{W}_t^{(T)}$$

$$\Rightarrow B(t,T,T+\vartheta) = B(t_0,T,T+\vartheta)\, e^{-\int_{t_0}^{t}\underline{V}(u)\,d\underline{W}_u^{(T)} - \frac{1}{2}\int_{t_0}^{t}\underline{V}(u)^2\,du} =$$

$$= B(t_0,T,T+\vartheta)\, e^{-\Sigma_t\sqrt{T-t_0}\,g - \frac{1}{2}\Sigma_t^2(t-t_0)}$$

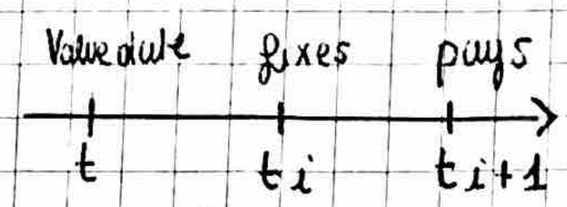
$$C = B(t_0,T)\left\{B(t_0,T,T+\vartheta)\,N(d_1) - K\,N(d_2)\right\}$$

$$\Sigma_t^2 = \frac{1}{t-t_0}\int_{t_0}^{t}\underline{V}(u)^2\,du$$

26. LMM & BMM. AN ITO APPLICATION: CAPLET SOLUTION

MARKET MODELS: BOND MARKET MODEL

BMM



$$B_i(t) := B(t; t_i, t_{i+1})$$

mean maybe given in (t) fwd measure but also in $\mathbb{Q}$ fwd measure.

LEMMA 1 $$dB_i(t) = -B_i(t)\underbrace{\left(\underline{\sigma}(t,t_{i+1}) - \underline{\sigma}(t,t_i)\right)}_{:=\underline{V}_i(t)} \cdot \underbrace{\left(d\underline{W}_t + \underline{\underline{\rho}}\,\underline{\sigma}(t,t_i)dt\right)}_{d\underline{W}_t^{(t_i)}}$$

BMM DYNAMIC: SPOT MEASURE

$$dB_i(t) = B_i(t)\,\underline{V}_i(t) \cdot \left\{ -\rho \sum_{j=K+1}^{i-1} \underline{V}_j(t)dt - dW_t \right\}$$

$$t_K < t \leq t_{K+1}$$

with $dW_i(t)\,dW_j(t) = \rho_{ij}\,dt$

$\boxed{\text{Fixing}} \Rightarrow \underline{V}_i(t) = 0 \quad \forall t \geq t_i$

Since $\sigma(t_i, t_i) = 0$ by assumption in any HJM model, this means that:

$$V_i(t_i) = \sigma(t_i, t_{i+1}) - \cancel{\sigma(t_i, t_i)} = 0$$

Why? Because the dymamic for the forward will end at t_i, since at this time the libor rate will be fixed → no dymamic any more (financial reasons).

dymamic for $B(t; t_i, t_{i+1})$ up to t_i

... HJM: $$\underline{\sigma}(t; t_i) = \underline{V}_{i-1}(t) + \underline{\sigma}(t, t_{i-1}) = \sum_{j=K+1}^{i-1} \underline{V}_j(t)$$

since $\underline{V}_K(t) = 0$

$$\underline{\sigma}(t, t_i) = \underline{V}_{i-1}(t) + \underline{\sigma}(t, t_{i-1}) =$$

$$= \underline{V}_{i-1}(t) + \underline{V}_{i-2}(t) + \dots + \underline{V}_{K+1}(t) =$$

$t_K < t \leq t_{K+1}$

BMM DYNAMIC : t_i FORWARD MEASURE

lemma 1 +
- $\sigma(t, T_i)$ det f of time
- $V_i(t) = 0 \quad \forall t > t_i$

$$dB_i(t) = -B_i(t) \cdot \underline{V}_i(t)\, d\underline{W}_t^{(i)}$$

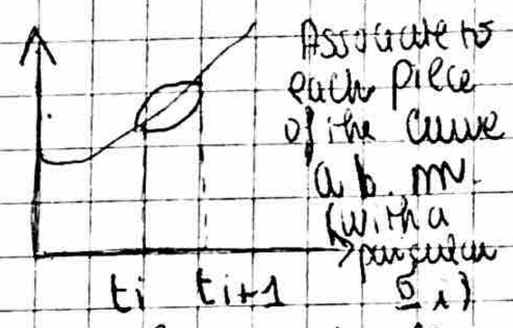
Gaussian HJM perché è g.b.m. sia nella spot che nella fwd measure.

Discounts:

$$D(t_0, t_i) = ?$$

$$D(t_0, t_i) = \prod_{j=0}^{i-1} D(t_j, t_{j+1})$$

$$D(t_i, t_{i+1}) = B(t_i, t_{i+1}) \exp\left\{ -\frac{1}{2} \int_{t_i}^{t_{i+1}} \underline{\sigma}(u, t_{i+1})^2\, du - \int_{t_i}^{t_{i+1}} \underline{\sigma}(u, t_{i+1}) \cdot d\underline{W}_u \right\}$$



Associate to each piece of the curve a b.m. (with a particular $\underline{\sigma}_i$)

Fixing mechanism: $\forall$ interval $[t_i, t_{i+1}]$,

$\underline{\sigma}(u, t_{i+1}) = 0 \Rightarrow$ the exponential $= 1$

$$\Rightarrow D(t_0, t_i) = \prod_{j=0}^{i-1} B(t_j, t_{j+1})$$

problem originally ∞ dim, but thanks to this mechanism reduced to a d-dim.

Markov on reset dates.

Fixing reduces to a finite dimensional problem (there was the dynamic of a curve that has now become the dynamic of a finite det)

We can price a put on ZC bond with HJM (Gaussian):

$$P(t_0) = \mathbb{E}_0\left[D(t_0, t_i) \left(K - B(t_i, t_{i+1}) \right)^+ \right] \overset{\text{Lemma 2+ Girsanov}}{=}$$

$$= B(t_0, t_i)\, \mathbb{E}^{(t_i)}\left[\left(K - B(t_i, t_{i+1}) \right)^+ \right] \overset{\text{dynamic of discount thanks to Lemma 1}}{=}$$

$$= B(t_0, t_i)\left[K N(-d_2) - B(t_0, t_i, t_{i+1}) N(-d_1) \right]$$

$\underline{V}(t)$ deterministic function of time.

$$d_{1,2} = \frac{\ln \frac{B(t_0; t_i, t_{i+1})}{K}}{v\sqrt{t_{i+1} - t_i}} \pm \frac{1}{2} v \sqrt{t_{i+1} - t_i}$$

$$v^2 = \frac{1}{t - t_0} \int_{t_0}^{t} \underline{V}(u)^2\, du$$

Since we can price a put on a ZC, we can also price a caplet! $\left(L_i(t) := L(t; t_i, t_{i+1})\right)$

$$\delta_i \mathbb{E}_0 \left[D(t_0, t_{i+1}) \left[L_i(t_i) - K \right]^+ \right] =$$

$$= (1+\delta_i K) \mathbb{E}_0 \left[D(t_0, t_i) \left[\frac{1}{1+\delta_i K} - B_i(t_i) \right]^+ \right]$$

Equivalence Caplet vs. ZC Put Option

CAPLET SOLUTION (BMM)

$$\Rightarrow C_i(t_0) = (1+\delta_i K) B(t_0, t_i) \left[\frac{1}{1+\delta_i K} N(-d_2) - B_i(t_0) N(-d_1) \right]$$

$$= \left(B_i(t_0) = \frac{B(t_0, t_{i+1})}{B(t_0, t_i)} \right) =$$

$$= B(t_0, t_{i+1}) \left[\frac{1}{B_i(t_0)} N(-d_2) - (1+\delta_i K) N(-d_1) \right]$$

$$= B(t_0, t_{i+1}) \left[(1+\delta_i L_i(t_0)) N(\overset{d_1^B}{\overset{\shortparallel}{-d_2}}) - (1+\delta_i K) N(\overset{d_2^B}{\overset{\shortparallel}{-d_1}}) \right]$$

$$d_{1,2} = \frac{\ln \frac{1+\delta_i K}{1+\delta_i L_i(t_0)}}{v \sqrt{t_{i+1} - t_i}} \pm \frac{1}{2} v \sqrt{t_{i+1} - t_i}$$

$$d_{2,1}^{(B)} = -d_{1,2} = \frac{\ln \frac{1+\delta_i L_i(t_0)}{1+\delta_i K}}{v \sqrt{t_{i+1} - t_i}} \mp v \sqrt{t_{i+1} - t_i}$$

$$d_{1,2}^{(B)} = \frac{\ln \frac{1+\delta_i L_i(t_0)}{1+\delta_i K}}{v \sqrt{t_{i+1} - t_i}} \pm v \sqrt{t_{i+1} - t_i}$$

V (M) alternativo

MARKET MODELS: LIBOR MARKET MODEL

LMM

$$L_i(t) := L(t, t_i, t_{i+1})$$

$$L_i(t) = \frac{1}{\delta_i}\left(\frac{1}{B_i(t)} - 1\right)$$

$$d\left(\delta_i L_i(t)\right) = -\frac{1}{B_i(t)^2} dB_i(t) + \frac{1}{\cancel{2}} \cdot \cancel{2} \frac{1}{B_i(t)^3} \overbrace{\langle dB_i(t)^2 \rangle}^{B_i(t)^2 (\underline{v}_i(t)\, d\underline{W}_t)(\underline{v}_i(t)\, d\underline{W}_t)} =$$

$$= \frac{1}{B_i(t)} \left\{ \underline{v}_i(t) \cdot [d\underline{W}_t + \rho\, \underline{\sigma}(t, t_i)\, dt] \right\} +$$

$$+ \frac{1}{B_i(t)} \left\{ \underline{v}_i(t)^2 dt \right\} =$$

$$= \frac{\underline{v}_i(t)}{B_i(t)} \cdot \left\{ d\underline{W}_t + \rho\, \underline{\sigma}(t, t_i)\, dt + \rho\, \underline{v}_i(t)\, dt \right\} =$$

$$= \frac{\underline{v}_i(t)}{B_i(t)} \cdot \left\{ d\underline{W}_t + \rho\, \underline{\sigma}(t, t_{i+1})\, dt \right\}$$

$$\Rightarrow dL_i(t) = \underbrace{\frac{\underline{v}_i(t)}{\delta_i B_i(t)}}_{:= L_i(t)\, \underline{\sigma}_i^{LMM}(t)} \left\{ d\underline{W}_t + \rho\, \underline{\sigma}(t, t_{i+1})\, dt \right\} =$$

$$= L_i(t)\, \underline{\sigma}_i^{LMM}(t) \cdot d\underline{W}_t^{(t_{i+1})}$$

martingale in the t_{i+1} - fwd measure

$$\boxed{L_i(t)\, \underline{\sigma}_i^{LMM}(t) = \frac{\underline{v}_i(t)}{\delta_i B_i(t)}}$$

$\underline{\sigma}_i^{LMM}(t)$ is a deterministic function of time.
$= 0 \ \forall t \geq t_i$ (forward dynamic stops at t_i)

LMM DYNAMIC : SPOT MEASURE

NO GBM

$$dL_i(t) = L_i(t)\, \underline{\sigma}_i^{LMM}(t) \cdot \left\{ \rho \sum_{j=k+1}^{i} \overbrace{\underline{\sigma}_j^{LMM}(t) \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}}^{V_j(t)} dt + d\underline{W}_t \right\}$$

$$t_k < t \leq t_{k+1}$$

$$dW_i(t)\, dW_j(t) = \rho_{ij}\, dt$$

Fixing $\Rightarrow$ $\underline{\sigma}_i(t) = 0 \quad \forall t \geq t_i$

$$\underline{\sigma}(t, t_{i+1}) = \sum_{j=k+1}^{i} \underline{V}_j(t)$$

$$\underline{V}_i(t) = 0 \Rightarrow \underline{\sigma}_i^{LMM}(t) = 0 \quad \forall t \geq t_i$$

$$\underline{V}_j(t) = \frac{\underline{\sigma}_j^{LMM}(t)\, L_j(t)\, \delta_j}{1+\delta_j L_j(t)}$$

In the R.N. measure, $L_i(t)$ NOT b.m. (we couldn't write the solution)

GBM

LMM DYNAMIC : t_{i+1} FORWARD MEASURE

$$\begin{cases} dL_i(t) = L_i(t)\, \underline{\sigma}_i^{LMM}(t) \cdot d\underline{W}_t^{(t_{i+1})} \\ L_i(t_0) \end{cases}$$

The libor forward rate is a martingale in the t_{i+1} - fwd measure.

CAPLET SOLUTION (LMM)

$$C_i(t_0) = \mathbb{E}_0\left[D(t_0, t_{i+1}) \left(\delta_i L_i(t_i) - \delta_i K \right)^+ \right] =$$

$$= \delta_i B(t_0, t_{i+1})\, \mathbb{E}_0^{(i+1)}\left[(L_i(t_i) - K)^+ \right] =$$

$$= \delta_i B(t_0, t_{i+1}) \left[L_i(t_0) N(d_1^L) - K N(d_2^L) \right]$$

Black solution ($L_i(t_i)$ in t_{i+1} measure is g.b.m.)

$$d_{1,2}^L = \frac{\ln L_i(t_0)/K}{\Sigma \sqrt{t_i - t_0}} \pm \frac{1}{2} \Sigma \sqrt{t_i - t_0}$$

$$\Sigma^2 = \frac{1}{t_i - t_0} \int_{t_0}^{t_i} \underline{\sigma}_i^{LMM}(t)^2\, dt$$

With BMM and LMM I can price also swaptions:

MM SWAPTION SOLUTIONS

- LM
  - → Exact : Not Available
  - → Approximated : Not so good...
- BM
  - → Exact : Available
  - → Approximated : Excellent precision

  With LMM geometric b.m. → no negative rates.

27. NORMAL AND SHIFTED LMM : CAPLET SOLUTION

N-LMM We have written the dynamic of the libor in the t_{i+1} - fwd measure

$$dL_i^{(i+1)}(t) = \underbrace{\frac{\underline{v}_i(t)}{\delta_i B_i(t)}}_{\text{defined } := \underline{\sigma}_i^N(t)} \cdot d\underline{W}_t^{(t_{i+1})}$$

Request: $\underline{\sigma}_i^N(t)$ deterministic function of time.

$$\Rightarrow \begin{cases} dL_i(t)^{(i+1)} = \underline{\sigma}_i^N(t) \cdot d\underline{W}_t^{(t_{i+1})} \\ L_i(t_0) \end{cases}$$

$$\Rightarrow L_i(t) = L_i(t_0) + \int_{t_0}^{t} \underline{\sigma}_i^N(t) \, d\underline{W}_t^{(t_{i+1})} =$$

$$\overset{\text{"law"}}{=} L_i(t_0) \cdot \tilde{\Sigma}^N \sqrt{t_i - t_0} \; g$$

$$\tilde{\Sigma}^N = \frac{1}{t_i - t_0} \int_{t_0}^{t_i} \underline{\sigma}_i^{(N)}(t) \cdot \rho \, \underline{\sigma}_i^{(N)}(t) \, dt$$

$$\text{Caplet}_i(t_0) = \delta_i B(t_0, t_{i+1}) \mathbb{E}_0^{(i+1)}[(L_i(t_i)-K)^+]$$

$$\mathbb{E}_0^{(i+1)}[L_i(t_i)-K]^+ = \mathbb{E}_0[(L_i(t_0)-K-\sqrt{t_i-t_0}\,\tilde{\Sigma}^{(N)} g)^+]$$

$$\sqrt{t_i-t_0}\,\tilde{\Sigma}^N g < L_i(t_0)-K \iff g < \frac{L_i(t_0)-K}{\sqrt{t_i-t_0}\,\tilde{\Sigma}^N} := \alpha^{(N)}$$

$$\mathbb{E}_0[(L_i(t_0)-K-\sqrt{t_i-t_0}\,\tilde{\Sigma}^{(N)} g)\mathbb{1}_{g<\alpha^{(N)}}] =$$

$$= (L_i(t_0)-K)N(\alpha^{(N)}) - \sqrt{t_i-t_0}\,\tilde{\Sigma}^{(N)}\mathbb{E}_0[g\,\mathbb{1}_{g<\alpha^{(N)}}]$$

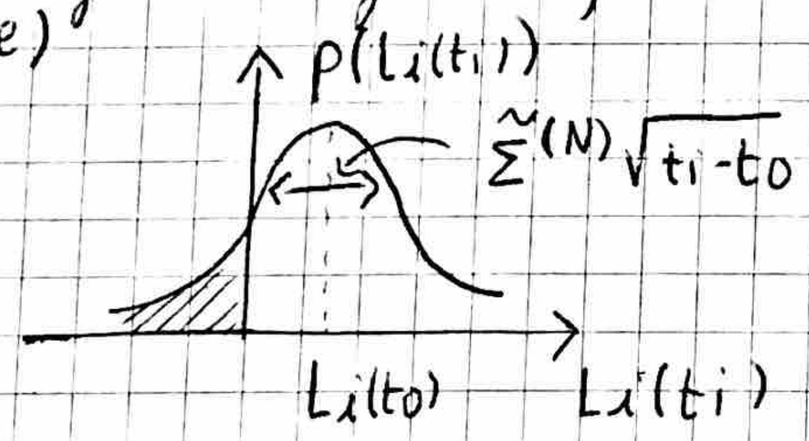
$$\mathbb{E}[g\,\mathbb{1}_{g<\alpha^{(N)}}] = \int_{-\infty}^{\alpha^{(N)}} g\,\frac{e^{-g^2/2}}{\sqrt{2\pi}} = -\frac{e^{-g^2/2}}{\sqrt{2\pi}}\Bigg|_{-\infty}^{\alpha^{(N)}} =$$

$$= \frac{-e^{-\alpha^{(N)2}/2}}{\sqrt{2\pi}}$$

$$\Longrightarrow \text{Caplet}_i(t_0) = \delta_i B(t_0, t_{i+1})\Big\{(L_i(t_0)-K)N(\alpha^{(N)}) + \tilde{\Sigma}^{(N)}\sqrt{t_i-t_0}\,\phi(\alpha^{(N)})\Big\}$$

Why NLMM?

In this way, negative libor rates are allowed (not only the dynamic, but also L_i can be negative) before, L_i GBM



SHIFTED - LMM

Another way to allow negative rates:

$$X_i(t) = L_i(t) + \alpha_i, \quad \alpha_i \in \mathbb{R}^+ \quad (\alpha_i = 0 : \text{usual LMM})$$

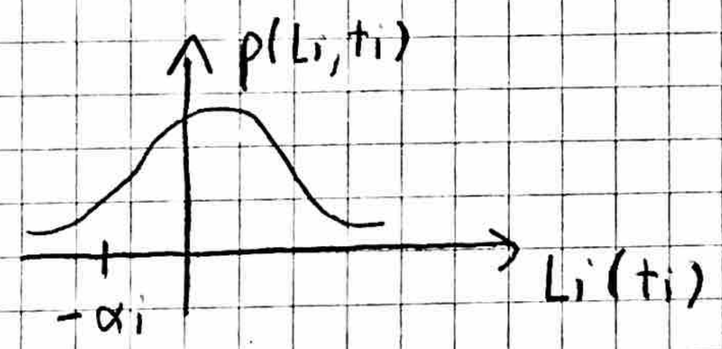
$$\Rightarrow dX_i(t) = dL_i(t) = \underbrace{\frac{\underline{V}_i(t)}{\delta_i B_i(t)}}_{=: X_i(t)\,\underline{\sigma}_i^{SL}(t)} \cdot d\underline{W}_t^{(t_{i+1})}$$

<u>Request</u>: $\underline{\sigma}_i^{SL}(t)$ deterministic function of time

$$\text{Caplet}_i(t_0) = B(t_0, t_{i+1})\,\delta_i\, \mathbb{E}_0^{(i+1)}\left[(X_i(t_i) - \alpha_i - K)^+ \right] =$$

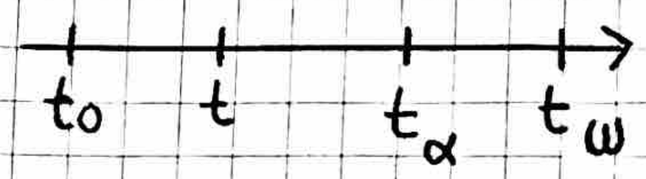
$$= B(t_0, t_{i+1})\,\delta_i \left[X_i(t_0)\, N(d_1^{SL}) - (K + \alpha_i)\, N(d_2^{SL}) \right]$$

$$d_{1,2}^{SL} = \frac{\ln \frac{L_i(t) + \alpha_i}{K + \alpha_i}}{\Sigma^{SL}\sqrt{t_i - t_0}} \pm \frac{1}{2}\Sigma^{SL}\sqrt{t_i - t_0}$$



28. SMM: SWAPTION FORMULA

SWAP MARKET MODEL



$$B_{\alpha i}(t) = B(t_0; t_\alpha, t_i)$$

Let's write the formula for the FORWARD SWAP as a function of the forward ZC bond in order to use Lemma 1:

$$\Longrightarrow \quad S_{\alpha\omega}(t) = \frac{1 - B_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)}$$

Remember Lemma 1:

$$dB_{\alpha\omega}(t) = -B_{\alpha\omega}(t)\underbrace{[\underline{\sigma}(t,t_\omega) - \underline{\sigma}(t,t_\alpha)]}_{\underline{V}_{\alpha\omega}(t)} \cdot \underbrace{[d\underline{W}_t + \rho\underline{\sigma}(t,t_\alpha)dt]}_{d\underline{W}_t^{(\alpha)}}$$

Payer swaption:

$$\Longrightarrow \quad PS(t_0) = \mathbb{E}_0[D(t_0,t_\alpha)BPV_{\alpha\omega}(t_\alpha)[S_{\alpha\omega}(t_\alpha) - K]^+] =$$

$$\overset{\text{Lemma 2 + Girsanov}}{=} B(t_0, t_\alpha)\mathbb{E}_0^{(\alpha)}[BPV_{\alpha\omega}(t_\alpha)[S_{\alpha\omega}(t_\alpha) - K]^+]$$

fwd BPV: $BPV_{\alpha\omega}(t_\alpha) = \sum_{i=\alpha}^{\omega-1} \delta(t_i, t_{i+1}) B(t; t_\alpha, t_{i+1})$

dynamic for BPV $\Longrightarrow$

$$\begin{cases} dBPV_{\alpha\omega}(t) = +\sum_{i=\alpha}^{\omega-1} \delta(t_i, t_{i+1})\, dB_{\alpha,i+1}(t) = \\ \qquad = -\sum_{i=\alpha}^{\omega-1} \delta_i B_{\alpha,i+1}(t)\, \underline{V}_{\alpha,i+1}(t) \cdot d\underline{W}_t^{(\alpha)} = \\ \qquad = -BPV_{\alpha\omega}(t)\, \underline{\Omega}_{\alpha\omega}(t) \cdot d\underline{W}_t^{(\alpha)} \\ BPV_{\alpha\omega}(t_0) \end{cases}$$

But, we are not imposing any condition on $\underline{\Omega}_{\alpha\omega}(t)$ (not a deterministic function for sure, can be an adapted process).

$$dBPV_{\alpha\omega}(t) = -\underline{\Omega}_{\alpha\omega}(t)\, BPV_{\alpha\omega}(t) \cdot d\underline{W}_t^{(\alpha)}$$

$$\Rightarrow BPV_{\alpha\omega}(t_\alpha) = BPV_{\alpha\omega}(t_0) \exp\left\{ -\int_{t_0}^{t_\alpha} \underline{\Omega}_{\alpha\omega}(t) \cdot d\underline{W}_t^{\alpha} + \right.$$

$$\left. -\frac{1}{2}\int_{t_0}^{t_\alpha} \underline{\Omega}_{\alpha\omega}^2(t)\, dt \right\}$$

We want the dynamic for $S_{\alpha\omega}(t) = \dfrac{1 - B_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)}$

$$d(1 - B_{\alpha\omega}(t)) = B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t) \cdot d\underline{W}_t^{(\alpha)} = -dB_{\alpha\omega}(t)$$

$$dS_{\alpha\omega}(t) = \frac{-dB_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)} - \frac{1 - B_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)^2}\, dBPV_{\alpha\omega}(t) +$$

$$+ \frac{1}{\cancel{2}}\left\{ +\cancel{2}\frac{\langle dB_{\alpha\omega}(t), dBPV_{\alpha\omega}(t)\rangle}{BPV_{\alpha\omega}(t)^2} + \cancel{2} \cdot \frac{(1 - B_{\alpha\omega}(t))}{BPV_{\alpha\omega}(t)^3} \langle dBPV_{\alpha\omega}(t)\rangle^2 \right\}$$

$$\langle dB_{\alpha\omega}(t), dBPV_{\alpha\omega}(t)\rangle = \left(-B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t) \cdot d\underline{W}_t^{\alpha}\right) \left(-\underline{\Omega}_{\alpha\omega}(t)\, BPV_{\alpha\omega}(t) \cdot d\underline{W}_t^{\alpha}\right) =$$

$$= B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t)\, \rho\, \underline{\Omega}_{\alpha\omega}(t)\, BPV_{\alpha\omega}(t)\, dt$$

$$\langle dBPV_{\alpha\omega}(t)\rangle = \underline{\Omega}_{\alpha\omega}(t)^2\, BPV_{\alpha\omega}(t)^2\, \rho\, dt$$

$$\Rightarrow dS_{\alpha\omega}(t) = \frac{1}{BPV_{\alpha\omega}(t)} \left\{ \left[B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t) + (1 - B_{\alpha\omega}(t))\, \underline{\Omega}_{\alpha\omega}(t) \right] \cdot d\underline{W}_t^{\alpha} \right.$$

$$\left. + \left[B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t) \cdot \rho\, \underline{\Omega}_{\alpha\omega}(t) + (1 - B_{\alpha\omega}(t))\, \rho\, \underline{\Omega}_{\alpha\omega}^2(t) \right] dt \right\}$$

$$= \frac{1}{BPV_{\alpha\omega}(t)} \left\{ \left[B_{\alpha\omega}(t)\, \underline{V}_{\alpha\omega}(t) + (1 - B_{\alpha\omega}(t))\, \underline{\Omega}_{\alpha\omega}(t) \right] \cdot \left[d\underline{W}_t^{\alpha} + \rho\, \underline{\Omega}_{\alpha\omega}(t)\, dt \right] \right.$$

$$S_{\alpha w}(t) = \frac{1 - B_{\alpha w}(t)}{BPV_{\alpha w}(t)}$$

$$d(1 - B_{\alpha w}(t)) = B_{\alpha w}(t)\, \underline{v}_{\alpha w}(t) \cdot d\underline{W}_t^{(\alpha)} = -dB_{\alpha w}(t)$$

$$dBPV_{\alpha w}(t) = -BPV_{\alpha w}(t)\, \underline{\Omega}_{\alpha w}(t) \cdot d\underline{W}_t^{(\alpha)}$$

$$dS_{\alpha w}(t) = d\left(\frac{1}{BPV_{\alpha w}(t)}\right) - d\left(\frac{B_{\alpha w}(t)}{BPV_{\alpha w}(t)}\right) =$$

$$= \frac{1}{BPV_{\alpha w}(t)} \left\{ + \underline{\Omega}_{\alpha w}(t) \right\} \left\{ d\underline{W}_t^{(\alpha)} + \rho\, \underline{\Omega}_{\alpha w}(t)\, dt \right\}$$

$$- \frac{B_{\alpha w}(t)}{BPV_{\alpha w}(t)} \left\{ -\underline{v}_{\alpha w}(t) + \underline{\Omega}_{\alpha w}(t) \right\} \left\{ d\underline{W}_t^{(\alpha)} + \rho\, \underline{\Omega}_{\alpha w}(t)\, dt \right\} =$$

$$= \underbrace{\frac{\left(B_{\alpha w}(t)\, \underline{v}_{\alpha w}(t) + (1 - B_{\alpha w}(t))\, \underline{\Omega}_{\alpha w}(t)\right)}{BPV_{\alpha w}(t)}}_{S_{\alpha w}(t)\ \Sigma_{\alpha w}(t)} \underbrace{\left\{ d\underline{W}_t^{\alpha} + \rho\, \underline{\Omega}_{\alpha w}(t)\, dt \right\}}_{d\underline{W}_t^{\alpha w}}$$

$$= \underbrace{\frac{\left[B_{\alpha w}(t)\,\underline{v}_{\alpha w}(t) + (1-B_{\alpha w}(t))\,\underline{\Omega}_{\alpha w}(t)\right]}{BPV_{\alpha w}(t)}}_{:= S_{\alpha w}(t)\,\Sigma_{\alpha w}(t)} \cdot \underbrace{\left[dW_t^{\alpha} + \rho\,\underline{\Omega}_{\alpha w}(t)dt\right]}_{:= d\underline{W}_t^{\alpha w}}$$

$$d\underline{W}_t^{\alpha w} = d\underline{W}_t^{\alpha} + \rho\,\underline{\Omega}_{\alpha w}(t)\,dt$$ fwd αw-swap measure

$\downarrow$

It's exactly the change of measure that generates the RADON-NYKODIN derivative found in $BPV_{\alpha w}(t_\alpha)$.

$$\frac{d\pi^{\alpha w}}{d\pi^{\alpha}} = e^{-\int_{t_0}^{t_\alpha} \underline{\Omega}_{\alpha w}(t)\cdot d\underline{W}_t^{\alpha} - \frac{1}{2}\int_{t_0}^{t_\alpha} \underline{\Omega}_{\alpha w}^2(t)\,dt}$$

$$\Rightarrow P_{\alpha w}(t_0) = B(t_0, t_\alpha)\,BPV_{\alpha w}(t_0)\,\mathbb{E}_0^{\alpha w}\left[(S_{\alpha w}(t)-K)^+\right]$$

with $dS_{\alpha w}(t) = S_{\alpha w}(t)\,\underline{\Sigma}_{\alpha w}(t)\cdot d\underline{W}_t^{\alpha w}$

$\triangleright$

<u>SWAP MARKET MODEL</u>: $\underline{\Sigma}_{\alpha w}(t)$ deterministic function of time, $\underline{\Sigma}_{\alpha w}(t) = 0$ $\forall t \geq t_\alpha$

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$$\Rightarrow P_{\alpha w}(t_0) = B(t_0, t_\alpha)\,BPV_{\alpha w}(t_0)\left\{S_{\alpha w}(t_0)\,N(d_1^S) - K\,N(d_2^S)\right\}$$

$$d_{1,2}^S = \frac{\ln \frac{S_{\alpha w}(t_0)}{K}}{\sqrt{t_\alpha - t_0}\;\hat{\Sigma}_{\alpha w}} \pm \frac{1}{2}\sqrt{t_\alpha - t_0}\;\hat{\Sigma}_{\alpha w}$$

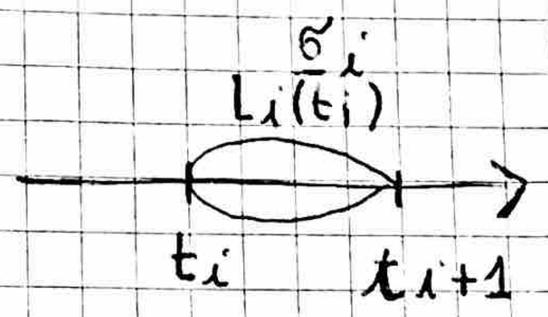
$$\left(\hat{\Sigma}_{\alpha w} = \frac{1}{t_\alpha - t_0}\int_{t_0}^{t_\alpha} \underline{\Sigma}_{\alpha w}(t)^2\,dt\right)$$

29. FLAT VOL vs SPOT VOL

Price a caplet fixed in t_i and payed in t_{i+1} → BMM / LMM
(... we need to calibrate the spot volatilities).

The problem in calibrating MM is that, when pricing according to the market, we consider FLAT VOLS, given by the tables.

But in those mkt models we consider a function of time $\underline{\sigma}_i$; the tables only give us some values.



$\underline{\sigma}_i(t) = 0 \ \forall t \geq t_i$

In general, $\underline{\sigma}_i(t)$ is a vector:

$$\underline{\sigma}_i(t) = \sigma_i(t)\, \underline{e}_i$$

⇒ I select the b.m. between t_i and t_{i+1} using the scalar $\sigma_i(t)$

So, each component of the b.m. vector controls a certain bucket of the dynamic.

So, the volatility:

$$\underline{\sigma}_i(t)^2 = \underline{\sigma}_i(t) \cdot \rho \underline{\sigma}_i(t) = \sigma_i(t)^2\, \underline{e}_i \cdot \rho \underline{e}_i = \sigma_i(t)^2$$

scalar

We want to calibrate SPOT VOLATILITIES:

$$\Sigma_i^2 = \frac{1}{t_i - t_0} \int_{t_0}^{t_i} \sigma_i(t)^2 \, dt$$

In the mkt, FLAT VOL:

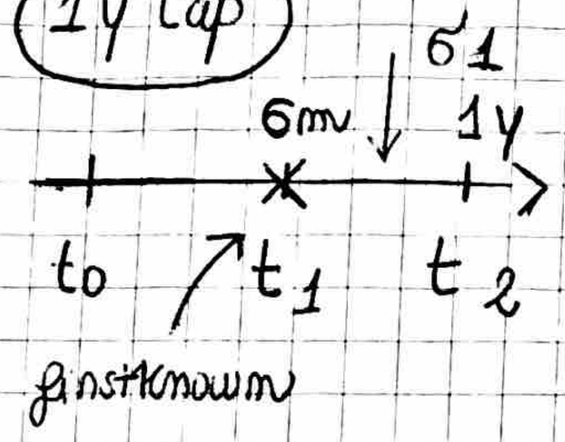
$$Cap(T) = \sum_{i;\, T_i \leq T} Caplet(i, \sigma^F)$$

We want to obtain SPOT VOL:

$$Cap(T) = \sum_{i,\, T_i \leq T} Caplet(i, \sigma_i)$$

Euribor 6m

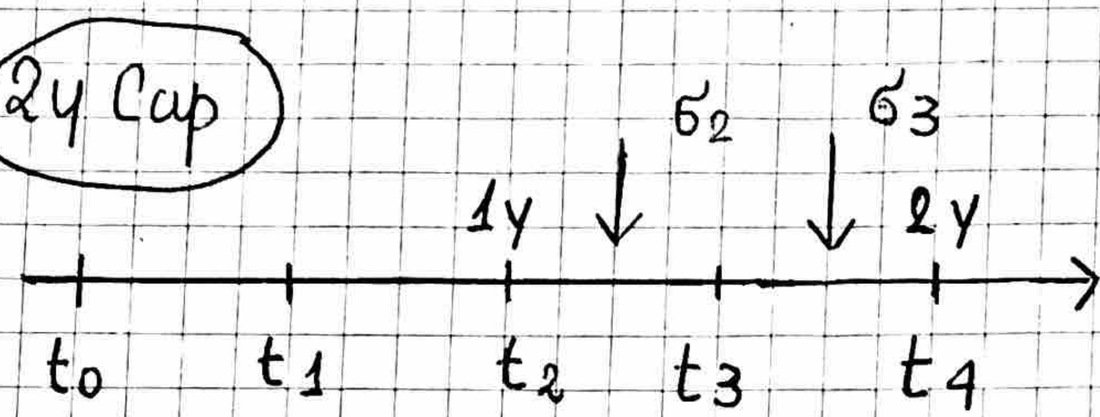
1y Cap



$$\sigma_1^{spot} = \sigma_{1y}^{FLAT}$$

bootstrap the flat volatilities provided by the mkt to get the spot volatilities.

2y Cap

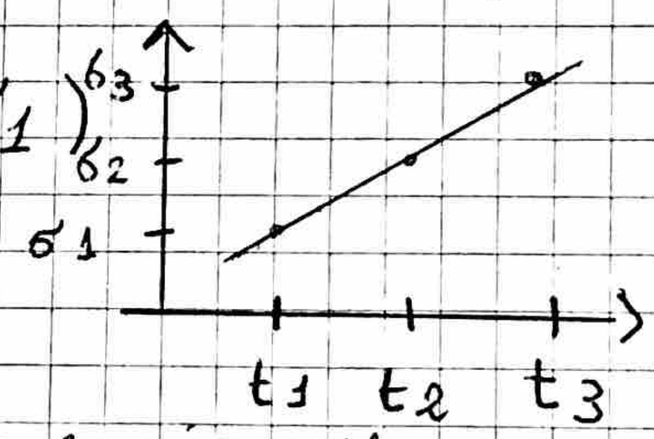


$$\Delta C_{1y,2y} = C_{2y}(\sigma_{2y}^{FLAT}) - C_{1y}(\sigma_{1y}^{FLAT}) =$$

$$= \sum_{i=2}^{3} \text{Caplet}(\sigma_i)$$

For the second equation in the system, we know that $\sigma_1, \sigma_2, \sigma_3$ are related linearly:

$$\sigma_2 = \sigma_1 + \frac{T_2 - T_1}{T_3 - T_1}(\sigma_3 - \sigma_1)$$



Now, we don't need to consider the correlation, since we are considering the same bucket.

For correlation matrix $\underline{\underline{\rho}}$, we can only use swaptions.

- σ_{FLAT} computed with LMM Black formula, so if we want to calibrate the BMM it's a bit more tricky since we have 2 different models to compare.
  - * σ_1^{FLAT} from table $\Rightarrow$ price of C_{1y} with LMM. Now we know the price $\rightarrow$ we can find V_1^{spot} (an average, not all the function $V_1(t)$) inverting BMM formula.
  - * $\sigma_2^{FLAT}, \sigma_3^{FLAT}$ ... in the same way.

30.

HULL-WHITE MODEL (EXTENDED VASICECK):

DYNAMIC, FWD BOND, SATISFY LEMMA 1 & 2.

GAUSSIAN HJM : $\sigma(t,T)$ deterministic function of time.

HULL-WHITE MODEL

dynamic: $$r_t = \underbrace{\varphi_t}_{\text{deterministic function, from zero curve}} + \underbrace{X_t}_{\text{OU process}}$$

ORNSTEIN UHLENBACH
$$\begin{cases} dX_t = -a X_t \, dt + \sigma \, dW_t \\ X_{t_0} = 0 \end{cases}$$

Normalized version: $$r_t = \varphi_t + \sigma \xi_t$$

$$\begin{cases} d\xi_t = -a \xi_t \, dt + dW_t \\ \xi_{t_0} = 0 \end{cases}$$

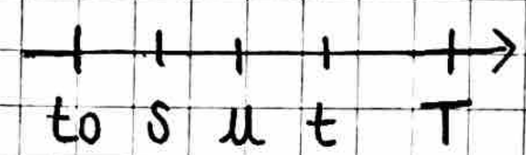
This process is interesting since it's a MEAN REVERTING process: the farest I get from the mean the strongest the drift will revert the process to its mean.

First, let's solve the SDE:

$$\begin{cases} dX_t = -X_t a\,dt + \sigma dW_t \\ X_{t_0} = 0 \end{cases}$$

Let's define $\boxed{\tilde{X}_t = e^{at} X_t}$

$$d\tilde{X}_t = \cancel{a e^{at} X_t} + \cancel{e^{at}[-X_t a\,dt} + \sigma dW_t] =$$
$$= e^{at}\sigma dW_t$$



Let's integrate between s and t:

$$\tilde{X}_t - \tilde{X}_s = \sigma \int_s^t e^{au} dW_u = e^{at} X_t - e^{as} X_s$$

$$\leftrightarrow \boxed{X_t = X_s e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u}$$

We want to show that HW model satisfies Lemma 1 and 2 of HJM models.

▷ <u>PROOF.</u>

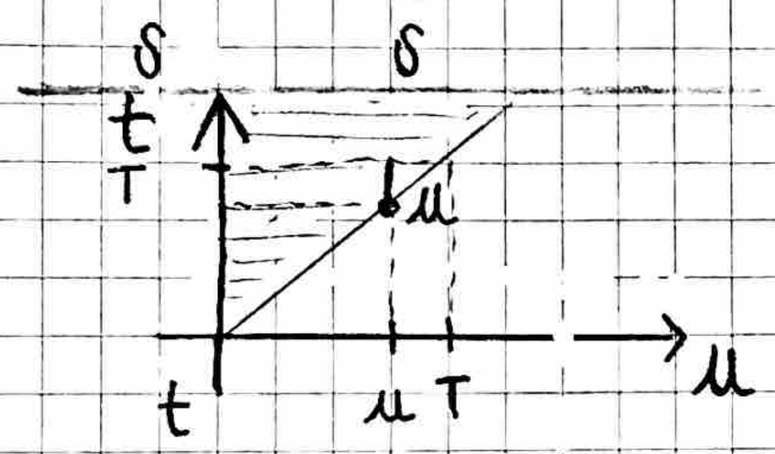
$$\int_s^T H_t\,dt = \int_s^T \varphi_t\,dt + \overbrace{X_s \int_s^T e^{-a(t-s)}\,dt}^{1} +$$
$$+ \underbrace{\sigma \int_t^T dt \int_s^t e^{-a(t-u)} dW_u}_{2}$$

$$① = \int_s^T e^{-a(t-s)}\,dt = e^{as}\left.\frac{e^{-at}}{-a}\right]_{t=s}^{t=T} =$$

$$= e^{as}\,\frac{e^{-aT} - e^{-as}}{-a} =$$

$$= \frac{1 - e^{-a(T-s)}}{a}$$

$$② = \sigma \int_s^T dt \int_s^t e^{-a(t-u)}\,dW_u =$$



We can use Fubini, since until T everything is bounded.
fixed u, t is between u and T

$$= \sigma \int_s^T dW_u \underbrace{\int_u^T e^{-a(t-u)}\,dt}_{= \frac{1 - e^{-a(T-u)}}{a}} =$$

$$= \int_s^T dW_u \underbrace{\frac{\sigma}{a}\left(1 - e^{-a(T-u)}\right)}_{:= \sigma(u,T)}$$

associated Gaussian HJM volatility

$$\int_s^T r_t \, dt = \int_s^T \varphi_t \, dt + x_s \frac{\sigma(s,T)}{\sigma} + \int_s^T dW_u \, \sigma(u,T)$$

$$D(s,T) = e^{-\int_s^T r_t \, dt} =$$

$$= \exp\left\{ -\int_s^T \varphi_t \, dt - x_s \frac{\sigma(s,T)}{\sigma} - \int_s^T dW_u \, \sigma(u,T) \right\}$$

$$B(s,T) = \mathbb{E}_s [D(s,T)] =$$

$$= e^{-\int_s^T \varphi_t \, dt - x_s \frac{\sigma(s,T)}{\sigma}} \mathbb{E}_s [e^{-\int_s^T dW_u \, \sigma(u,T)}] =$$

$$\int_s^T dW_u \, \sigma(u,T) \sim \mathcal{N}(0, \Sigma^2)$$

$$\Sigma^2 = \int_s^T \sigma(u,T)^2 \, du$$

$$\mathbb{E}[e^{-\gamma g}] = e^{\gamma^2/2}$$

$$\mathbb{E}_s [e^{-\Sigma^2 g}] = e^{\Sigma^2/2}$$

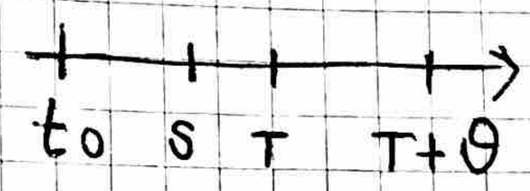
$$= e^{-\int_s^T \varphi_t \, dt - x_s \frac{\sigma(s,T)}{\sigma}} e^{\frac{1}{2} \int_s^T \sigma(u,T)^2 \, du}$$

$$\Rightarrow \frac{D(s,T)}{B(s,T)} = \exp\left\{ -\int_s^T dW_u \, \sigma(u,T) - \frac{1}{2} \int_s^T \sigma(u,T)^2 \, du \right\}$$

<u>LEMMA 2</u> OK

$$B(s,T) = \exp\left\{ -\int_s^T \varphi_t \, dt - x_s \frac{\sigma(s,T)}{\sigma} + \frac{1}{2}\int_s^T \sigma^2(t,T)\, dt \right\}$$

Let's compute the forward:



$$B(s;T,T+\theta) = \frac{B(s,T+\theta)}{B(s,T)} =$$

$$= \exp\left\{ -\int_T^{T+\theta} \varphi_t \, dt - \frac{x_s}{\sigma}\left(\sigma(s,T+\theta) - \sigma(s,T)\right) + \right.$$

$$\left. + \frac{1}{2}\int_s^{T+\theta} \sigma^2(t,T+\theta)\, dt - \frac{1}{2}\int_s^{T} \sigma^2(t,T)\, dt \right\}$$

$$B(t_0;T,T+\theta) = \exp\left\{ -\int_T^{T+\theta} \varphi_t \, dt + \frac{1}{2}\int_{t_0}^{T+\theta} \sigma^2(t,T+\theta)\, dt + \right.$$

det. function of time: there to calibrate initial term structure.

$$\left. -\frac{1}{2}\int_{t_0}^{T} \sigma^2(t,T)\, dt \right\}$$

$$B(s;T,T+\theta) = B(t_0;T,T+\theta)\exp\left\{ \overbrace{-\frac{x_s}{\sigma}\left(\sigma(s,T+\theta) - \sigma(s,T)\right)}^{*} + \right.$$

$$\left. -\frac{1}{2}\int_{t_0}^{s} \left(-\sigma^2(t,T+\theta) + \sigma^2(t,T)\right) dt \right.$$

$$x_s = \cancel{x_{t_0} e^{-a(s-t_0)}} + \sigma \int_s^t e^{-a(s-u)}\, dW_u$$

$$* = \sigma \int_{t_0}^{s} e^{-a(s-u)} dW_u \frac{1}{a}\left(-e^{-a(T+\vartheta-s)} + e^{-a(T-s)}\right) =$$

$$= \frac{\sigma}{a} \int_{t_0}^{s} \left(1 - e^{-a(T+\vartheta-u)}\right)\left(1 - e^{-a(T-u)}\right) dW_u =$$

$$= \int_{t_0}^{s} [\sigma(u, T+\vartheta) - \sigma(u, T)] dW_u$$

$$\Rightarrow B(s; T, T+\vartheta) = B(t_0; T, T+\vartheta) \exp\left\{ - \int_{t_0}^{s} [\sigma(u, T+\vartheta) - \sigma(u, T)] dW_u \right.$$

$$\left. + \frac{1}{2} \int_{t_0}^{s} [-\sigma^2(u, T+\vartheta) + \sigma^2(u, T)] du \right.$$

LEMMA 1 OK ∎

So, we have seen that his is a Gaussian HJM: we can price a caplet.

31. HW: CAPLET SOLUTION

We have shown that HW-model is a Gaussian HJM: we can price a caplet, pricing a put on a ZC bond. (infact, if $\sigma(t,T)$ is deterministic, the discount follows a GBM and I can use Black formula → BMM).

Price of a ZC Put with Gaussian HJM:

$$P(t_0) = B(t_0, t_i) \mathbb{E}_0^{(t_i)}[(K - B(t_i, t_{i+1}))^+] =$$

$$= B(t_0, t_i)[K N(-d_2) - B(t_i, t_{i+1}) N(-d_1)]$$

Caplet vs. ZC Put

$$\delta_i \mathbb{E}_0[D(t_0, t_i)(L_i(t_i) - K)^+] =$$

$$= (1 + \delta_i K) \mathbb{E}_0\left[D(t_0, t_i)\left(\frac{1}{1+\delta_i K} - B_i(t_i)\right)^+\right]$$

BMM formula

$$\Rightarrow C_i(t_0) = (1+\delta_i K)(B(t_0,t_i))\left[\frac{1}{1+\delta_i K} N(-d_2) - B_i(t_0) N(-d_1)\right]$$

$$= \left(B_i(t_0) = \frac{B(t_0, t_{i+1})}{B(t_0, t_i)} \right) =$$

$$= (1+\delta_i K)\, B(t_0, t_{i+1}) \left[\frac{1}{1+\delta_i K} \frac{N(-d_2)}{B_i(t_0)} - N(-d_1) \right] =$$

$$= B(t_0, t_{i+1}) \left[\frac{N(\overset{d_1^{(B)}}{\overset{\shortparallel}{-d_2}})}{B_i(t_0)} - (1+\delta_i K)\, N(\overset{d_2^{(B)}}{\overset{\shortparallel}{-d_1}}) \right] =$$

$$= B(t_0, t_{i+1}) \left[(1+\delta_i L_i(t_0)) N(d_1^{(B)}) - (1+\delta_i K) N(d_2^{(B)}) \right]$$

$$d_{1,2}^{(B)} = \frac{1}{v_i \sqrt{t_i - t_0}} \ln \frac{1+\delta_i L_i(t_0)}{1+\theta_i K} \pm \frac{1}{2} v_i \sqrt{t_i - t_0}$$

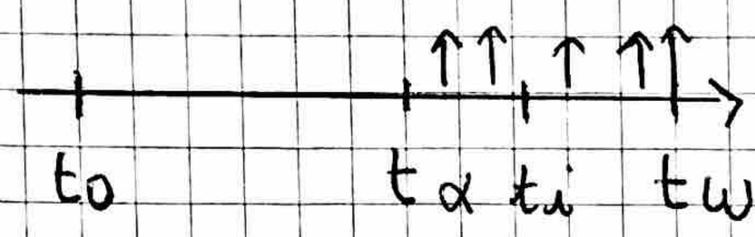
$$v_i = \frac{1}{t_i - t_0} \int_{t_0}^{t_i} \underline{v}_i(t)^2\, dt \,, \quad \underline{v}_i(t) = \sigma(t, t_{i+1}) - \sigma(t, t_i)$$

32. HW : JAMSHIDIAN APPROACH FOR SWAPTION SOLUTION.

Price of a swaption = price of a coupon "bond" option

Let's focus on the price of a coupon bond option

$$C(T_0) = \mathbb{E}_0 [D(t_0, t_\alpha) (P(t_\alpha; \underline{c}, \underline{t}) - K)^+]$$



$$P(t; \underline{c}, t_\alpha, t_\omega) = \sum_{i=\alpha+1}^{\omega} c_i B(t; t_\alpha, t_i)$$

$$c_i = \begin{cases} c\delta_i, & i < \omega \\ 1 + c\delta_i, & i = \omega \end{cases}$$

$$B(t; t_\alpha, t_i) = B(t_0; t_\alpha, t_i)\, e^{-\frac{x_{t_\alpha}}{\sigma}(\sigma(t, t_i) - \sigma(t, t_\alpha))}$$

$e^{-\alpha}$

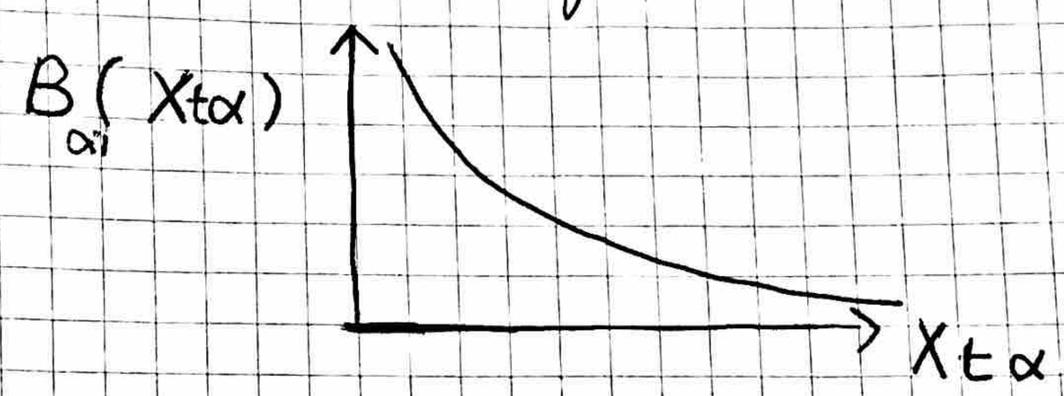
stochastic driven only by X_{t_α}

Using the starting fwd, we can compute the value of $B(t_\alpha, t_i)$ without using φ_t.

Why is not a good idea to write things from the dynamic of r_t?

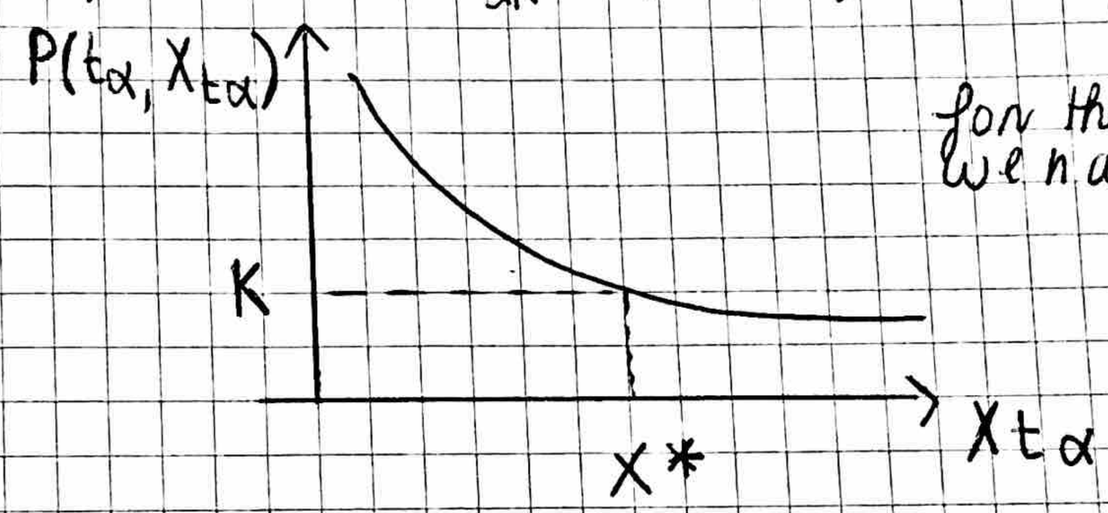
Since we have $r_t = \varphi_t + X_t$, but we should do $\frac{\partial B(t,T)}{\partial T}$ → but this is a derivative, while we have constructed $B(t,T)$ with linear interpolation.

Now, I'm claiming that :



is a negative exponential

$P(t_\alpha, X_{t\alpha})$ is a linear combination of convex functions $B_{\alpha i}(t_\alpha, X_{t\alpha})$.



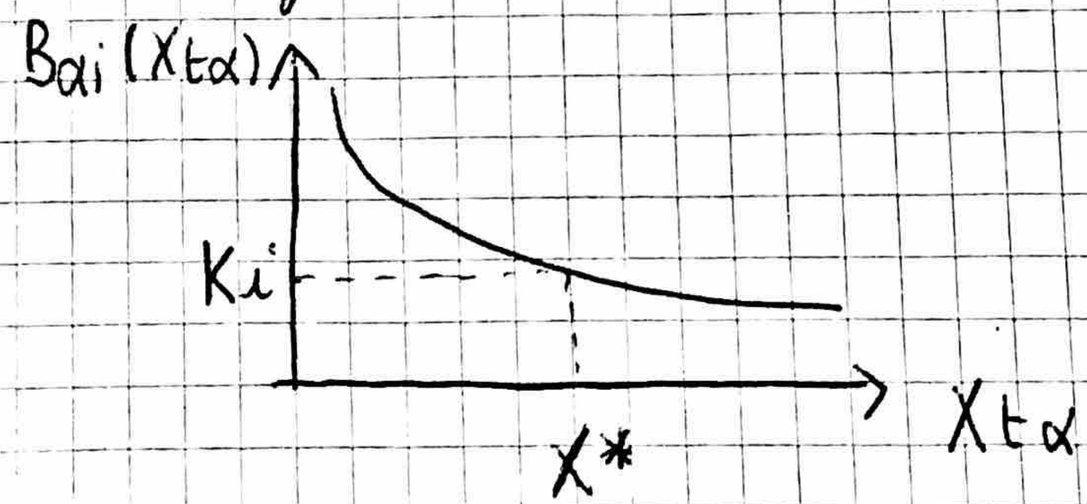
for the strike K, we have a particular X^*.

$$\exists!\ X^* \text{ s.t. } P(t_\alpha; \dots) = K$$

$$\Rightarrow C(t_0) = \mathbb{E}_0\left[D(t_0, t_\alpha)\left(P(t_\alpha; \underline{c}, \underline{t}) - K\right)^+ \right] =$$

$$= \mathbb{E}_0\left[D(t_0, t_\alpha)\left[\sum_{i=\alpha+1}^{\omega} c_i B_{\alpha i}(X_\alpha) - K\right] \mathbb{1}_{X_{t\alpha} < X^*} \right]$$

Considering X^* over the $B_{\alpha i}(X_{t\alpha})$ plot :



$$K_i = B_{\alpha i}(t_\alpha; X_{t\alpha} = X^*)$$

$$K = P(t_\alpha; X_{t\alpha} = X^*) = \sum_{i=\alpha+1}^{\omega} c_i K_i$$

$$= \mathbb{E}_0 \Big[D(t_0, t_\alpha) \Big[\sum_{i=\alpha+1}^{\omega} c_i \big(B_{\alpha i}(t_\alpha, X_{t_\alpha}) - K_i \big) \mathbb{1}_{X_{t_\alpha} < X^*} \Big]$$

$$= \sum_{\substack{i=\alpha+1 \\ i:\, t < t_i < t_\omega}}^{\omega} c_i \, \mathbb{E}_0 \Big[D(t_0, t_\alpha) \Big[B_{\alpha i}(t_\alpha; X_{t_\alpha}) - K_i \Big]^+ \Big]$$

Call option on a ZCB
(we know the solution)

(Newton-Rapson method)

I compute X^* numerically, then I put it into the expression of $B(t_\alpha, t_i)$ to get K_i and finally we compute the price.

34.

VASICECK MODEL: ZERO RATE AS A FUNCTION OF THE EXPIRY AND LIMITS OF THE APPROACH. POSSIBLE INITIAL CONDITION.

(short) Rate Dynamic:

$$\begin{cases} dr_t = -a(r_t - b)\,dt + \sigma dW_t \\ r_s \end{cases}$$

$$r_t = b + x_t \rightarrow x_t = r_t - b$$

b doesn't depend on time!

$$x_t = (r_t - b) = (r_0 - b)\,e^{-a(t-t_0)} + \sigma \int_{t_0}^{t} e^{-a(t-u)}\,dW_u$$

derived from the expression for X_t in the extended case.

$$B(s,T) = \mathbb{E}_s[D(s,T)] =$$

$$= \mathbb{E}_s\left[e^{-\int_s^T r_t\,dt}\right] =$$

$$= e^{-b(T-s)} - (r_s - b)\frac{\sigma(s,T)}{\sigma} + \frac{1}{2}\int_s^T \sigma^2(t,T)\,dt$$

$$B(t_0,T) = e^{-b(T-t_0)} - (r_0 - b)\frac{\sigma(t_0,T)}{\sigma} + \frac{1}{2}\int_{t_0}^T \sigma^2(t,T)\,dt$$

Vasicek model can't be considered an HJM model, where we had a dynamic for $B(t,T)$ + initial condition in t_0, that could be whatever function, curve.

Now, <u>initial condition</u> has a <u>particular shape in terms of the parameters</u>: we can't start from $\forall$ condition and match that $B(t_0,T)$.

→ we can't match initial condition!

Problems not from the curve but from initial condition: missing of some important shapes.

Let's try to find the possible initial shapes:

Let's start from the zero rate as function of expiry:

$$R(T) = -\frac{1}{T} \ln B(0,T) =$$

$$= b + (r_0 - b)\frac{1-e^{-a(T)}}{aT}$$

$$- \frac{1}{2T}\int_0^T \sigma^2(t,T)\,dt = *$$

$$\int_0^T \sigma^2(t,T)\,dt = \left(\frac{\sigma}{a}\right)^2 \int_0^T \left[1 - e^{-a(T-t)}\right]^2 dt =$$

$$= \frac{\sigma^2}{a^2}\left(1 - 2\,\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a}\right)$$

$$\int_0^T e^{-a(T-t)}\,dt = \frac{e^{-aT}(e^{aT}-1)}{a}$$

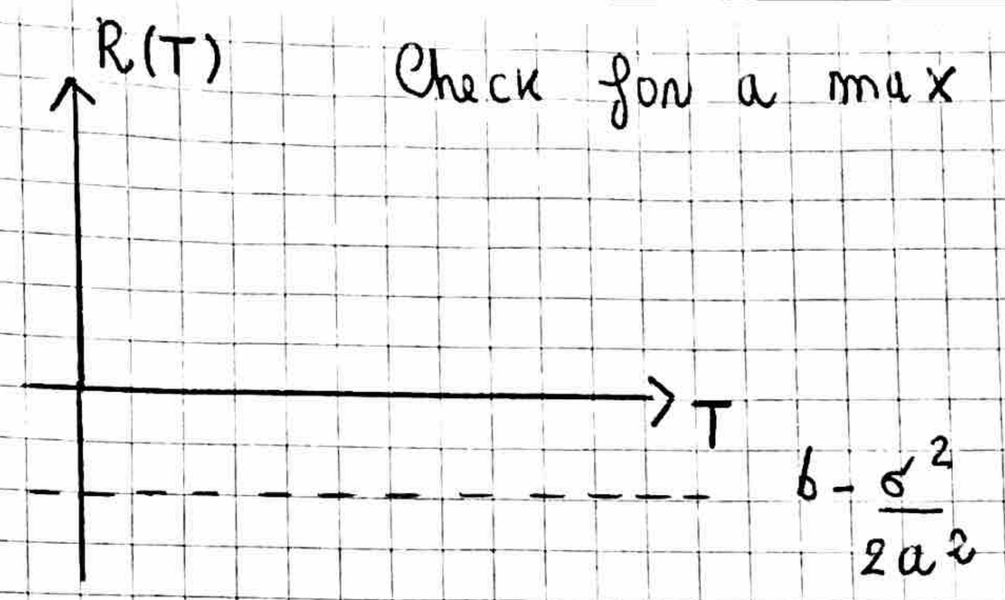
$$* = b + (r_0 - b)\frac{1-e^{-aT}}{aT} - \frac{\sigma^2}{2a^2}\left(1 - 2\,\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a}\right)$$

$$\frac{1-e^{-aT}}{aT} = \frac{aT}{aT} + \mathcal{O}(T)$$

$$\begin{cases} R(0) = \left(\cancel{b} - \cancel{\frac{\sigma^2}{2a^2}}\right) + (r_0 - \cancel{b}) - \cancel{\frac{\sigma^2}{2a^2}}(-2+1) = r_0 \\ R(\infty) = \left(b - \frac{\sigma^2}{2a^2}\right) \end{cases}$$

Today we know the rate (reasonable 1st. fwd rate in 0 = zero rate)

Check for a max



$$\underset{\tau = aT}{R(T)} \rightarrow R(\tau) = \left(b - \frac{\sigma^2}{2a^2}\right) + \left(r_0 - b + \frac{\sigma^2}{a^2}\right)\frac{1-e^{-\tau}}{\tau} +$$

$$- \frac{\sigma^2}{4a^2}\,\frac{1-e^{-2\tau}}{\tau} =$$

$$= \dots + \frac{\sigma^2}{4a^2}\,\frac{1}{\tau}\left\{\underbrace{\frac{4a^2}{\sigma^2}\left(r_0 - b + \frac{\sigma^2}{a^2}\right)}_{c} - (1+e^{-\tau})\right\}\cdot$$

$$\cdot(1-e^{-\tau})$$

$$F(\tau) = (c - 1 - e^{-\tau})\left(\frac{1-e^{-\tau}}{\tau}\right)$$

$$F'(\tau) = e^{-\tau}\left(\frac{1-e^{-\tau}}{\tau}\right) + (c-1-e^{-\tau})\underbrace{\left[\frac{e^{-\tau}\tau - (1-e^{-\tau})}{\tau^2}\right]}_{\frac{(\tau+1)e^{-\tau}-1}{\tau^2}}$$

$$(\tau+1)e^{-\tau} - 1 < 0 \leftrightarrow (\tau+1) < e^{\tau}$$

$$F'(\tau) = [\ldots]\left\{ \underbrace{e^{-\tau}\frac{(1-e^{-\tau})}{\cancel{\tau}}\frac{\tau^{\cancel{2}}}{(\tau+1)e^{-\tau}-1} + (c-1-e^{-\tau})}_{\text{if there is a zero, it will be here.}} \right\}$$

$$= e^{-\tau}\left[\tilde{G}(\tau) + (c-1)e^{\tau} - 1\right]$$

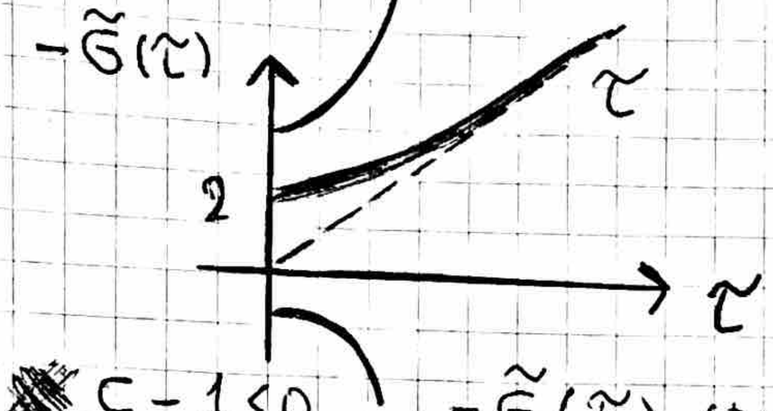
$$\tilde{G}(\tau) = \tau\,\frac{e^{-\tau}(1-e^{-\tau})}{\tau e^{-\tau} - (1-e^{-\tau})} =$$

$$= \tau\,\frac{e^{-\tau}}{\frac{\tau e^{-\tau}}{1-e^{-\tau}} - 1} = \tau\,\frac{1}{\frac{\tau}{1-e^{-\tau}} - e^{\tau}} \longrightarrow$$

$$\xrightarrow{\tau\to 0} \tau\,\frac{1}{\frac{\cancel{\tau}}{\cancel{\tau}-\frac{1}{2}\tau^{\cancel{2}}} - (1+\tau)} =$$

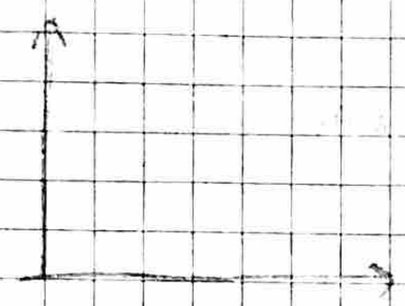
$$= \tau\,\frac{1}{1+\frac{1}{2}\tau - 1 - \tau} = \frac{\tau}{-\frac{\tau}{2}} = -2$$

Zero : $(C-1)\,e^{\tau} - 1 = -\tilde{G}(\tau)$ (asymptotic to a straight line)



$C-1<0$ $\quad -\tilde{G}(\tau)$ starts from 2 and it is almost like a straight line
$\rightarrow$ NO INTERSECTION

$C-1>0$



$(C-1)-1>2 \quad \longleftrightarrow \quad C>4$

No solution: left side starts after $-\tilde{G}(\tau)$

$1<C<4$ $\quad$ 1 solution

$$C = \frac{4a^2}{\sigma^2}\left(r_0 - b + \frac{\sigma^2}{a^2}\right)$$

$$\Updownarrow$$

$$\frac{\sigma^2}{4a^2} < r_0 - b + \frac{\sigma^2}{a^2} < \frac{4\sigma^2}{4a^2}$$

$$\Updownarrow$$

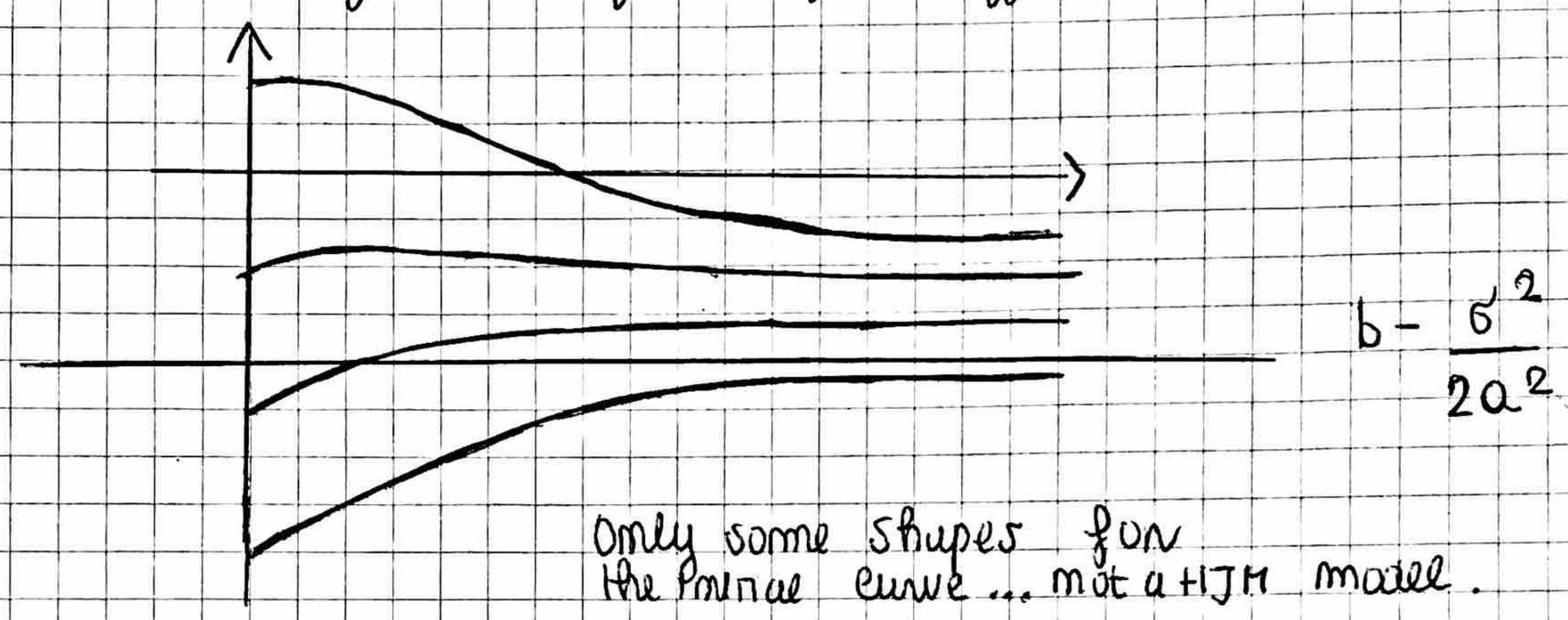
$$-\frac{3}{4}\frac{\sigma^2}{a^2} < r_0 - b < 0$$

...but this never happens.

Only some shapes are allowed for the curve between r_0 and $R(\infty)$:

$$\begin{cases} R(\infty) = b - \dfrac{\sigma^2}{2a^2}, & \sigma(0,t) \xrightarrow{t\to\infty} \sigma/a \\ R(0) = r_0, & \sigma(0,t) \xrightarrow{t\to 0} \sigma t \end{cases}$$

$R(T)$ as a function of time for different values of r_0:

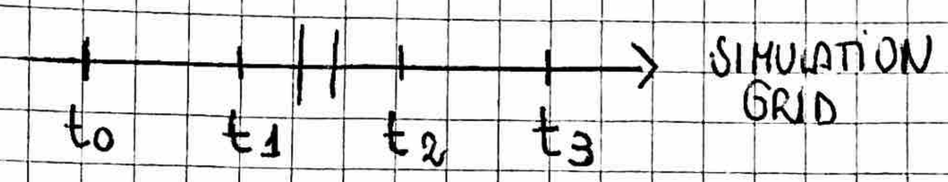


Only some shapes for the Primal curve ... not a HJM model.

33. HW: MC AND TREE

In order to simulate exotics, we have to get $B(t, T, T+\vartheta)$ and so we have to simulate X_t.

Set of reset dates:



Since we don't have a Markov model, we can't consider only reset dates → we will have a much smaller grid.

$dX = \ldots$

$$\Rightarrow X_{(t_{i+1})} = X_{(t_i)} e^{-a(t_{i+1}-t_i)} + \underbrace{\sigma \int_{t_i}^{t_{i+1}} e^{-a(t_{i+1}-u)} dW_u}_{\downarrow \; N(0, \hat{\sigma}^2)}$$

$$\hat{\sigma}^2 = \sigma^2 \int_{t_i}^{t_{i+1}} e^{-2a(t_{i+1}-u)} du$$

Simulate discretized grid:

$$\Delta x(i) := x(t_{i+1}) - x(t_i) =$$
$$= -x(t_i)\left(1 - e^{-a(t_{i+1}-t_i)}\right) + \hat{\sigma}\, g(i)$$

$$\Rightarrow \boxed{\Delta x(i) = -\hat{\mu}\, x(i) + \hat{\sigma}\, g(i)}$$

From x_1, we can simulate the whole $B(t_1, T, T+\vartheta)$ curve ⇒ then we simulate x_2 from x_1 → $B(t_2, T, T+\vartheta)$ and so on.

Thanks to $B(t, T, T+\theta)$, we can handle the payoff, but, in order to price, we need to handle the stochastic discount $\rightsquigarrow$ Lemma 2.

$$D(t_i, t_{i+1}) = B(t_i, t_{i+1}) \exp\left\{ -\frac{1}{2} \underbrace{\int_{t_i}^{t_{i+1}} \sigma^2(u, t_{i+1})\, du}_{\tilde{\sigma}^{*2}} + \underbrace{-\int_{t_i}^{t_{i+1}} \sigma(u, t_{i+1})\, dW_u}_{\text{gaussian variable similar to the previous } g(i)} \right\}$$

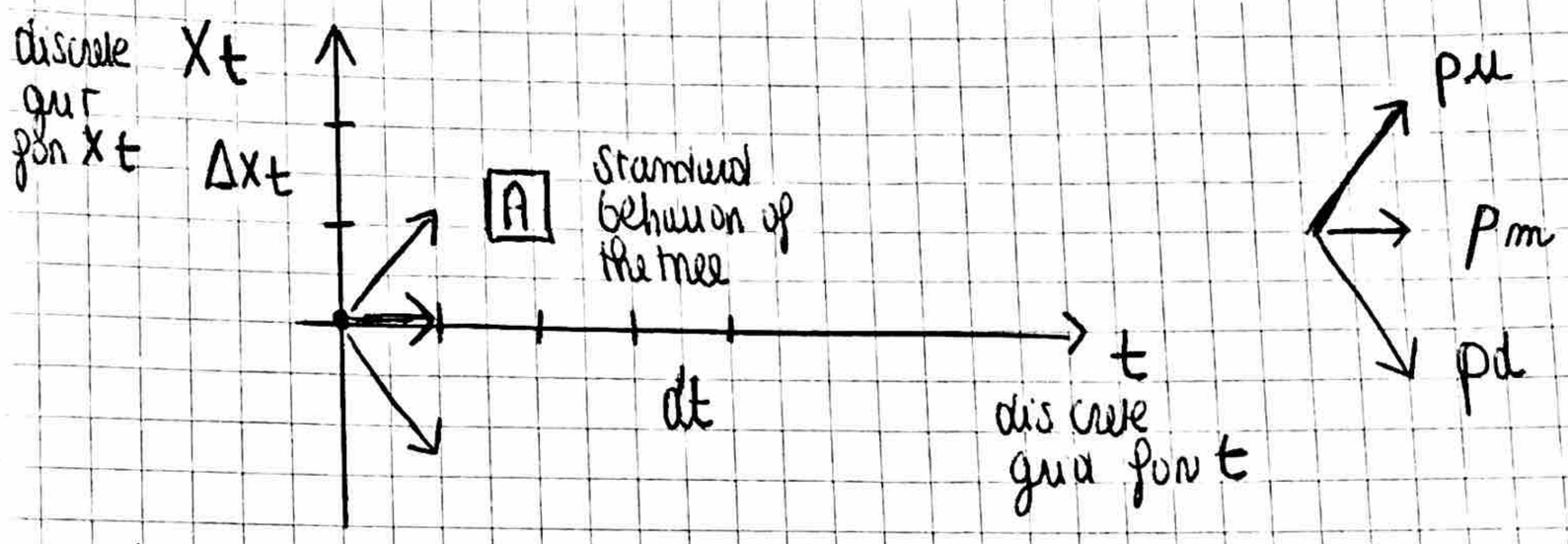
(related to the g of the ZCB, they are different (exp, 1-exp), but we can make an approximation)

$$\frac{\tilde{\sigma}^*}{\tilde{\sigma}} [\Delta x(i) + \hat{\mu}\, x(i)]$$

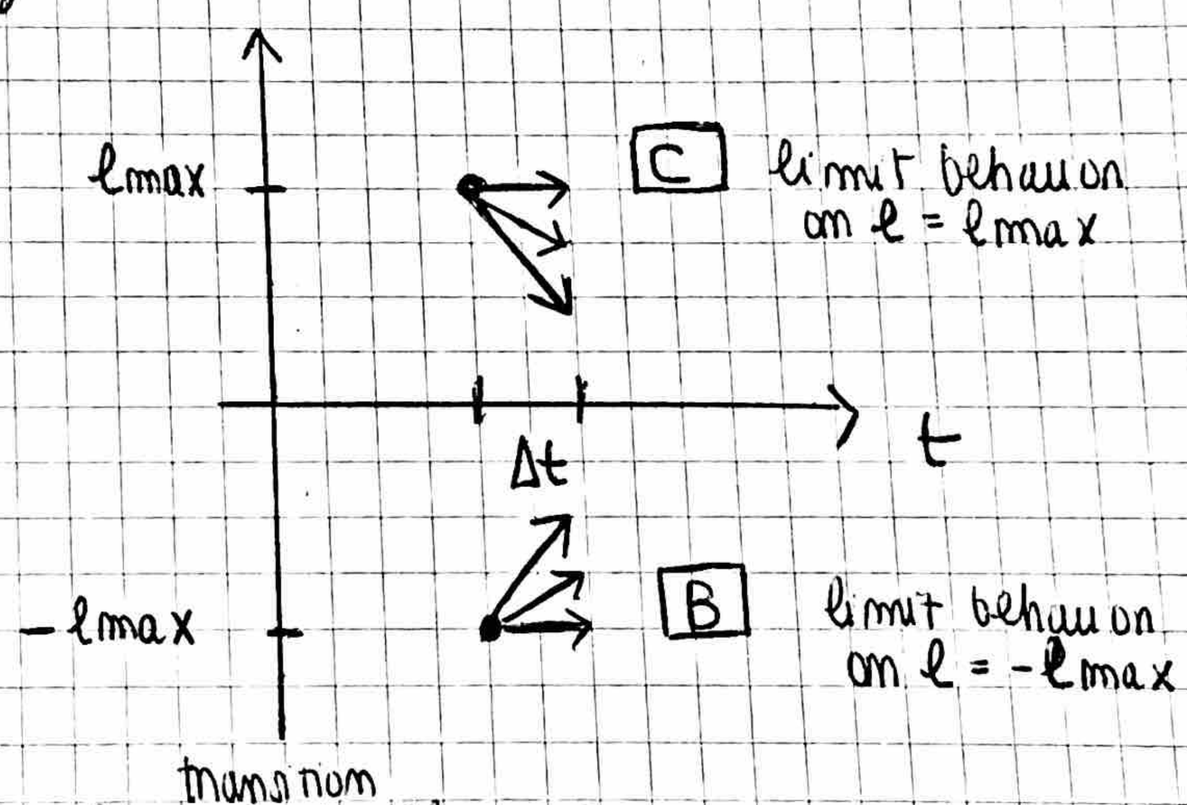
We can handle it as a function of the previous process X_t.

TREE APPROACH

HULL-WHITE : TRINOMIAL TREE



We will have this kind of behavior until a MAXIMUM, and MINIMUM, level where we can't go up or down any more.



We want to get the transition probabilities for alternative branching processes.

We have a grid for X_t, we want to relate Δx_t and Δt.

discretization
$$\begin{cases} \Delta x = \sqrt{3}\,\hat{\sigma} \\ x = l\,\Delta x \end{cases} \qquad \begin{cases} \hat{\sigma} \equiv \sigma\sqrt{\dfrac{1-e^{-2a\,dt}}{2a}} \\ l = -l_{max}, \dots, -1, 0, 1, \dots, +l_{max} \end{cases}$$

$$\Delta x(i) = -\hat{\mu}\,x(i) + \hat{\sigma}\,g(i)$$

$\boxed{A}$:

1. $\mathbb{E}[\Delta x] = p_u \Delta x + \cancel{p_m 0} + p_d(-\Delta x) \overset{\text{impose}}{=}$

$$= -\hat{\mu}\, \ell\, \Delta x$$

2. $\mathbb{E}[\Delta x^2] = p_u \Delta x^2 + p_d \Delta x^2 =$

$$\frac{\Delta x^2}{3} = \tilde{\sigma}^2 + \hat{\mu}^2 \ell^2 \Delta x^2$$

$$\left(Var(\Delta x) = \mathbb{E}[\Delta x^2] - \mathbb{E}[\Delta x]^2\right)$$

3. $p_d + p_u + p_m = 1$

$$\Downarrow$$

$$p_u - p_d = -\hat{\mu}\ell$$
$$p_u + p_d = \frac{1}{3} + \hat{\mu}^2 \ell^2$$
$$p_d + p_u + p_m = 1$$

$$\begin{cases} p_u = \frac{1}{2}\left(\frac{1}{3} - \hat{\mu}\ell + \hat{\mu}^2\ell^2\right) \\ p_m = \frac{2}{3} - \hat{\mu}^2\ell^2 \\ p_d = \frac{1}{2}\left(\frac{1}{3} + \hat{\mu}\ell + \hat{\mu}^2\ell^2\right) \end{cases}$$

Till which point ℓ those probabilities will be consistent (i.e., non negative) ?

$p_d, p_u > 0 \quad \forall \ell \qquad \tilde{\ell} = \hat{\mu}\ell$

$$p_m = \frac{2}{3} - \hat{\mu}^2\ell^2 > 0 \Rightarrow \left(\frac{2}{3} - \tilde{\ell}^2\right) > 0$$

$$\Rightarrow \tilde{\ell}_{1,2} = \pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow -\sqrt{\frac{2}{3}} < \underset{\hat{\mu}\ell}{\underset{\parallel}{\tilde{\ell}}} < \sqrt{\frac{2}{3}}$$

$\boxed{C}$ 1. $\mathbb{E}[\Delta x] = -p_m \Delta x - p_d 2\Delta x =$

$= -\hat{\mu} \ell \Delta x$

2. $\mathbb{E}[\Delta x^2] = p_m \Delta x^2 + 4 p_d \Delta x^2 =$

$\frac{\Delta x^2}{3} = \hat{\sigma}^2 + \hat{\mu}^2 \ell^2 \Delta x^2$

3. $p_u + p_m + p_d = 1$

$\Downarrow$

$$\begin{cases} p_m + p_d = \hat{\mu} \ell \Delta x \\ p_m + 4 p_d = \frac{1}{3} + \hat{\mu}^2 \ell^2 \Delta x^2 \\ p_u + p_m + p_d = 1 \end{cases}$$

$p_u, p_d > 0 \ \forall \ell$

$p_m > 0 \Rightarrow \quad 1 - \sqrt{\frac{2}{3}} < \ell \hat{\mu} < 1 + \sqrt{\frac{2}{3}}$

$\downarrow$

$1 - \sqrt{\frac{2}{3}} < \ell_{max} \hat{\mu} < \sqrt{\frac{2}{3}}$