9. PRINCIPAL COMPONENT ANALYSIS: VaR FOR A REDUCED FORM PORTFOLIO

The aim is to reduce the dimensionality of the problem.

Indeed, if, for example, $\underline{x} \in \mathbb{R}^5$, we have to estimate 10^{10} parameters (10^5 for μ, $\frac{10^{10}}{2}$ for Σ).

$\frac{(n+1)n}{2}$ parameters

We can diagonalize Σ:

$$\Sigma = \Gamma \Lambda \Gamma'$$

Ordered eigenvalues: $\Lambda = diag(\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$

Ortonormal eigenvectors: $\Gamma = (\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_d)$ s.t. $\Gamma'\Gamma = \Gamma\Gamma' = I_d$.

Cholesky decomposition: $\Sigma = AA^T \Rightarrow A = \Gamma\Lambda^{1/2}$

$$\Rightarrow \underline{x} = \underline{\mu} + A\underline{y} = \underline{\mu} + \Gamma\Lambda^{1/2}\underline{y} =$$

$$= \underline{\mu} + \sum_{i=1}^{d} \sqrt{\lambda_i}\, \gamma_i\, y_i$$

d is the number of dimension.

We can see that not all λ_i, γ_i have the same importance.

$\rightarrow$ I can sum up only the RELEVANT K ones:

$$\underline{x} = \underline{\mu} + \sum_{i=1}^{K<d} \sqrt{\lambda_i}\, \gamma_i\, y_i + \underline{\varepsilon}$$

idiosincratic error term

PCA : REDUCED FORM PORTFOLIO

Defining $\underline{\hat{\mu}} := \Gamma' \underline{\mu}$

the projected portfolio on principals is : $\underline{\hat{w}} := \Gamma' \underline{w}$

Original portfolio $\qquad L \sim N(\underline{w} \cdot \underline{\mu}, \underline{w} \cdot \Sigma \underline{w})$

$$\begin{cases} \sigma^{2\,ptf} = \underline{w} \cdot \Sigma \underline{w} \overset{\underline{\hat{w}} = \Gamma' \underline{w}}{=} (\Gamma \underline{\hat{w}}) \cdot \Sigma \Gamma \underline{\hat{w}} = (\Gamma \underline{\hat{w}}) \cdot (\Gamma \Lambda \cancel{\Gamma'}) \cancel{\Gamma} \underline{\hat{w}} = \\ = \underline{\hat{w}} \cancel{\Gamma' \Gamma} \Lambda \underline{\hat{w}} = \underline{\hat{w}} \cdot \Lambda \underline{\hat{w}} = \sum_{i=1}^{d} \hat{w}_i^{\,2} \lambda_i^{\,2} \\ \mu^{ptf} = \underline{\mu} \cdot \underline{w} \overset{\underline{\hat{w}} = \Gamma' \underline{w},\ \underline{\hat{\mu}} = \Gamma' \underline{\mu}}{=} \Gamma \underline{\hat{\mu}} \cdot \Gamma \underline{\hat{w}} = \\ = \underline{\hat{\mu}}\, \Gamma' \Gamma\, \underline{\hat{w}} = \underline{\hat{\mu}} \cdot \underline{\hat{w}} = \sum_{i=1}^{d} \hat{\mu}_i \hat{w}_i \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \sigma^{2\,red} = \sum_{i=1}^{K} \hat{w}_i^{\,2} \lambda_i \\ \mu^{red} = \sum_{i=1}^{K} \hat{w}_i \hat{\mu}_i \end{cases} \qquad \text{reduced form portfolio}$$

Remark: if asset volatilities differ significantly, PCA is applied to correlation matrixes.

$$\Rightarrow VaR_\alpha^\pi = \Delta \mu^{red} + \sqrt{\Delta}\, \sigma^{red}\, VaR_\alpha^{std}$$

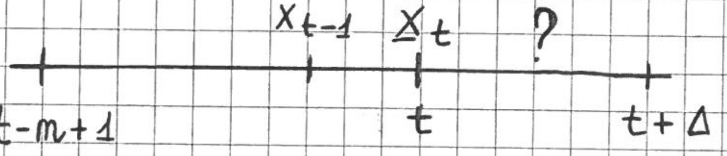
→ no assumption on the distribution of risk factors.

$\boxed{10.}$ NON PARAMETRIC APPROACHES: HS, WHS, BOOTSTRAP

• HISTORICAL SIMULATION

Returns are not gaussian ⟹ let's use REALIZED RETURNS

Consider d-time series for a time window m:



$Loss_t$ = pesi · ritorni$_t$
∀ asset — ritorni al tempo t ∀ asset

I can value my portfolio considering that the future return of my portfolio will be one of the ones I have observed in the past.

Under the assumption of FROZEN PORTFOLIO:

simulation of possible future losses

$$L_{t+\Delta} = -V_t \left(\underline{W}_t \cdot \underline{X}_{t+\Delta} + C_t \right)$$

ptf data NOW — mkt data

$(\underline{X}_t) = -V(t)\left(C_t + \underline{W}_t \cdot \underline{X}_t\right)$

same weights → we are assuming that we don't trade → same weights!

vettore di pesi di dim = # asset

$$W_{i,t} = \frac{n_{i,t}\, S_i(t)}{V(t)}$$

$$V(t) = \sum_{i=1}^{d} n_{i,t} \cdot S_i(t)$$

↑ # shares in the i-th asset.

In order to simulate possible losses in the future, you use past data (knowing the weights now).

① Value losses under the empirical dist of realized values for factors:

$$L_s = \left\{ L(\underline{X}_s)_{s=t-m+1,\dots,t} \right\}$$

② Consider the (decreasing) ordered sequence $L^{(i,m)}$: $L^{(m,m)} \leq \dots \leq \underbrace{L^{(1,m)}}_{\text{largest loss}}$ (order losses)

③ Risk measurements:

$$VaR_\alpha = L^{([m(1-\alpha)],m)}$$

$$ES_\alpha = \text{mean}\left\{ L^{(i,m)},\ i = [m(1-\alpha)], \dots, 1 \right\}$$

$[m(1-\alpha)]$ largest integer not exceeding $m(1-\alpha)$.

Remark: 1. HS preserves dependence among factors

2. The technique can be applied to an interval using the scaling rule.

EXAMPLE

daily losses

$m = 500$ (2y) and $\alpha = 99\%$ — 5^{th} worst case!

→ VaR corresponds to the 5^{th} highest value;

→ ES corresponds to the average of the 5 largest values.

Main advantages:

→ No statistical estimation of the parameters;

→ No assumptions about the dependence structure of risk factors.

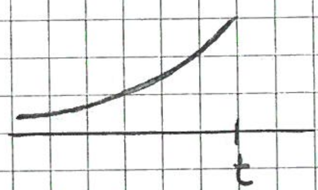
WEIGHTED HISTORICAL SIMULATION

stressed VaR required by the regulator — weight more a certain time window.

Instead of considering uniform weights, one can decide to associate to the loss sequence (w.r.t. time) a set of new weights that are decreasing in the past.

In this way, recent losses have a bigger impact.

So, we consider the following set of weights:



$$\{w_s\}_{s=t-m+1,\dots,t}$$

where $w_s = C\lambda^{t-s}$

with $0 < \lambda < 1$ (typically within 0.95, 0.99) and C a normalization factor s.t. $C = \dfrac{1-\lambda}{1-\lambda^m}$.

We can compute the loss sequence and order it in descend way and, not to loose correspondance with losses, we create a new vector of weights, the first (last) of which will be related to the biggest (lowest) loss.

Using these weights, we look for i^*, the largest value s.t.

$$\sum_{i=1}^{i^*} w_i \le 1-\alpha$$

W_ord = W(K)

[Loss, K] = sort(Loss, descend)

and then we can compute risk measurements:

$$VaR_\alpha = L^{(i^*, m)}$$

$$ES_\alpha = \frac{\sum_{i=1}^{i^*} w_i L^{(i,m)}}{\sum_{i=1}^{i^*} w_i}$$

WHS is more sensitive to recent conditions but generates more volatility in VaR.

- STATISTICAL BOOTSTRAP : random sampling.

For the Statistical Bootstrap, the idea is the same of the HS, but, instead of considering all the realized returns in the past years, we choose only a specified number of elements randomly (very powerful when you have a too large dataset).

⇒ Random sampling with replacement from the original data set (composed of m returns).

1. Sample an integer number between 1 and m, randi (1, m) M times;
2. Select the corresponding set of returns;
3. Value the loss distribution on the set and risk measures as in the HS case;
4. Adjust with the scaling factor.

As in HS, we preserve dependence among risk factors, since it is applied to all factors at the same time.

These are three non-parametric methods (most used in a bank)

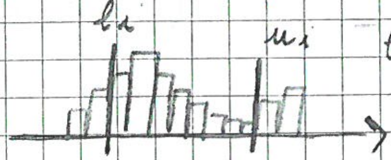
As an engineer, I want to have the order of magnitude as a check.

↓

11. PLAUSIBILITY CHECK FOR VaR.

Rule of thumb.

Sometimes, one just needs an estimation of the order of magnitude of portfolio VaR.



left and right quantiles of distr. of returns.

In this case, set of risk factors $\underline{X}$: plot $\forall i$

1. Value risk-factor correlation C and lower / upper percentiles :

$$l_i \equiv VaR_{X_i}(1-\alpha), \quad u_i \equiv VaR_{X_i}(\alpha)$$

2. Compute signed-VaR for each risk factor :

$$sVaR_i = sens_i \frac{|l_i| + |u_i|}{2}$$

sensitivities for asset i, weight for stock → linear ptf ($sens_i = w_i$)

3. Portfolio VaR can be estimated by :

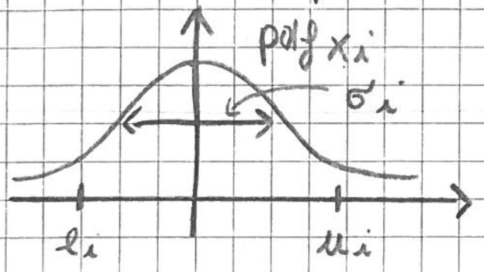
$$VaR^{ptf} = \sqrt{sVaR \cdot C \, sVaR}$$

Is this check good?

We know that, if returns are driftless and Gaussian in a linear portfolio, the described approach is equivalent to the Analytical valuation via a Variance-Covariance method.

Let's test if it's true.

driftless $\rightarrow \mu = 0$ (expected loss). We want to model UL.



$$l_i = -u_i \ , \ u_i > 0$$

$$u_i = \sigma_i \underbrace{N^{-1}(\alpha)}_{\text{equal } \forall i = A}$$

linear ptf $\Rightarrow$ dens $i = w_i$

$$\Rightarrow \text{sVaR} \cdot \text{CsVaR} = A^2 \sum_{i,j=1}^{d} w_i \underbrace{\sigma_i C_{ij} \sigma_j}_{\Sigma_{ij}} w_j$$

$$\Rightarrow \text{VaR}^{ptf} = A \sqrt{\underline{w} \cdot \Sigma \underline{w}}$$

In the analytical case with Variance / Covariance method:

$$\text{VaR} = \cancel{\mu} + \underbrace{\sqrt{\underline{w} \cdot \Sigma \underline{w}}}_{\text{Var}(L)} \underbrace{\text{VaR}_\alpha^{std}}_{N^{-1}(\alpha)}$$

$\boxed{12.}$ FULL VALUATION VS. DELTA-NORMAL (DELTA-GAMMA) METHOD.

NON LINEAR PORTFOLIO

- **FULL-VALUATION MONTE-CARLO METHOD**

In precence of derivatives portfolio loss, portfolio is valued with a Monte-Carlo approach with starting value in t:

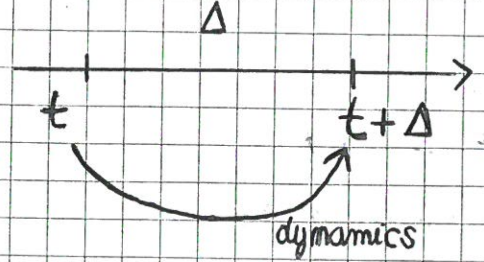
$$L^{der}(\underline{X}_t ; \Delta) = - \sum_i [C_i(t+\Delta) - C_i(t)]$$

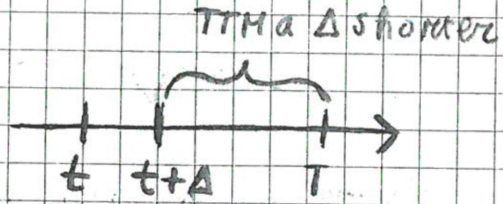
$C_i(t)$: value in t of the i-th derivative.

Then loss distribution can be obtained via a MC approach considering a simulated set of values for the risk factors.

EXAMPLE :

Let's consider the case where the derivatives' portfolio is just composed by one Call.





In a simple model, underlying value in $t + \Delta$:

$$S_{t+\Delta} = S_t \underbrace{e^{X_t}}_{\text{exponential return between } t \text{ and } t+\Delta}$$

$X_t \rightarrow$ usually normally distributed

Option value should be computed just considering the new starting value and updating the TTM (that is a Δ shorter).

One obtains the Loss distribution simulating several values the returns up to $t + \Delta$.

ASS 5 : portfolio composed by stocks + same number of put options.

daily VaR via a Full Monte Carlo and a Delta normal approaches.

$\rightarrow$ 2y HS for the underlying

With Full MC, we obtain the loss distribution considering a simulated set of values for the risk factor.

$\rightarrow$ In this case, HS to find the price of the underlying on the next day.

$$\underbrace{S_{t+\Delta}}_{\text{simulated underlying value in } t+\Delta} = \underbrace{S_t}_{\text{underlying value today}} e^{X_{t+\Delta}}$$

$X_{t+\Delta}$ → mkt risk factor in $t+\Delta$ (unknown)
↳ we use all risk factors observed in the last 2y to simulate it.

↓

We use daily risk factors of the last 2y to create an array that represents a set of simulated underlying prices in $t+\Delta$.

We price the Put today $P(t)$.
With the vector $S_{t+\Delta}$, we obtain a vector with put option's prices in $t+\Delta$ (TTM is Δ-shorter!). $P(t+\Delta)$

Vector of losses for the derivative part of the portfolio

↳ $L_p(X_t;\Delta) = -n_p\,(P(t+\Delta) - P(t))$ (Σ)

For the linear part of the portfolio, another vector of losses:

$$L_S(X_t;\Delta) = -n_S\,(S_{t+\Delta} - S_t)$$ ($S_i \sim \Sigma$, numero di shares)

$L_\pi = L_p + L_S$

This approach is time consuming! → I have to deal with derivatives separately.

↓

1^{st} order Taylor expansion to linearize the ptf.

- **DELTA - NORMAL METHOD**

For a derivatives portfolio, an expansion up to the first order loss derivatives:

$$L(\underline{X}_t) = -\sum_{i=1}^{d} \text{sens}_i(t)\, X_{t,i}$$

1^{st} order greeks : delta / vega

where $\text{sens}_i(t) = \delta_i \underbrace{S_i(t)}_{\omega_i V_t}$ in case of a single derivative and in general it is ptf sensitivity (in $ terms) w.r.t. i-th risk factor.

Then Loss distribution can be obtained either via an analytical approach or via a simulation (HS, WHS, Bootstrap) approach. (parametric or non parametric approach)

DELTA - GAMMA (MONTE CARLO) METHOD

Sometimes the regulator may ask for a second-order expansion.

For a derivatives' portfolio, derivatives loss can be obtained by an expansion in its Greeks:

$$L(\underline{X}_t) = -\left\{ \sum_{i=1}^{d} \delta_i S_i(t) X_{t;i} + \right.$$
$$\left. + \frac{1}{2} \sum_{i,j=1}^{d} \gamma_{ij} S_i(t) S_j(t) X_{t;i} X_{t;j} \right\}$$

Then loss distribution can be obtained via a Monte Carlo (HS, WHS, Bootstrap) approach.

In case of NO CROSS GAMMA ($\gamma_{ij} = 0 \ \forall \ i \neq j$), CORNISH - FISHER approximation (situation rarely observed in the mkt).
↳ analytical formula for VaR

Ass 5:

- DELTA NORMAL APPROACH

Compute the $\delta_{PUT} = e^{-d(T-t_0)} N(-d1)$

$\rightarrow L(X_{t}, i) = -(n_p \delta_p + n_s) S_t X_{t,i}$

loss for the i-th risk factor (Vuoi simularlo con HS)

- DELTA - GAMMA

γ_{PUT}

$\rightarrow L(X_{t,i}) = -[(n_p \delta_p + n_s) S_t X_{t,i} + n_p \gamma_p S_t^2 X_{t,i}^2]$

$\boxed{13.}$ NORMAL MIXTURE : MAIN PROPERTIES (MOMENTS, CONDITIONAL MEAN & CF) AND EXAMPLES

OTHER MULTINOMIAL MODELS: NORMAL (VARIANCE) MIXTURE

The parametric approach may not work, because we have to introduce the volatilities' movements.

NVM

MODEL : $\underline{x} = \underline{\mu} + \sqrt{W}\, A\, \underline{y}$

with
$$\begin{cases} A = \Sigma^{1/2} \text{ s.t. } AA' = \Sigma : \text{dispersion matrix} \\ \underline{y} \sim N_d(0, I_d) \\ W \geq 0 \text{ mixing variable indipendent from } y \text{ with } \mathbb{E}[W] < +\infty \end{cases}$$

Remarks:

- $\underline{x} \mid W = \omega \sim N_d(\underline{\mu}, W\Sigma)$
- $\mathbb{E}[\underline{x}] = \underline{\mu}$
- $$\begin{aligned} Cov(\underline{x}) &= \mathbb{E}[(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu}^T)] = \\ &= \mathbb{E}[(\underbrace{\sqrt{W}}_{\text{scalar}} A\underline{y})(\sqrt{W} A\underline{y})^T] = \\ &= \mathbb{E}[W A\underline{y}\underline{y}^T A^T] = \\ &\overset{(W \perp \underline{y})}{=} \mathbb{E}[W]\, A\, \underbrace{\mathbb{E}[\underline{y}\underline{y}^T]}_{I} A^T = \mathbb{E}[W] \cdot \Sigma \end{aligned}$$
- $$\begin{aligned} \phi^c(\underline{t}) &= \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = e^{i\underline{t}\cdot\mu}\,\mathbb{E}[e^{i\sqrt{W}\underline{t}\cdot A\underline{y}}] = \\ &= e^{i\underline{t}\cdot\underline{\mu}}\, \underbrace{\mathbb{E}_W[\mathbb{E}[e^{i\sqrt{W}\underline{t}\cdot A\underline{y}} \mid W]]}_{\bigstar} = \end{aligned}$$

if I condition w.r.t. W I can do the same computations as for the characteristic function of Gaussian multinomial $e^{-\frac{1}{2}\underline{t}\cdot\underline{t}}$

$\hookrightarrow$

$$\mathbb{E}[e^{i\sqrt{W}\,\underline{t}\cdot A\underline{y}}|W] = (\underline{t}\cdot A\underline{y} = \underbrace{A^T\underline{t}}_{\underline{\tilde{t}}}\cdot\underline{y}) =$$

$$= \mathbb{E}[e^{i\sqrt{W}\,\underline{\tilde{t}}\cdot\underline{y}}|W] = \int_{-\infty}^{+\infty}\prod_{i=1}^{d}\frac{dy_i}{\sqrt{2\pi}}\,e^{-y_i^2/2}\,e^{i\sqrt{W}\tilde{t}_i y_i} = \ast$$

$$e^{\underbrace{-\frac{y_i^2}{2}+i\sqrt{W}\tilde{t}_i y_i}} = e^{-\frac{1}{2}(y_i - i\sqrt{W}\tilde{t}_i)^2 - \frac{\tilde{t}_i^2}{2}W}$$

$$-\underbrace{\left(\frac{y_i^2}{2} - i\sqrt{W}\tilde{t}_i y_i - \frac{W\tilde{t}_i^2}{2}\right)}_{-\frac{1}{2}(\underbrace{y_i - i\sqrt{W}\tilde{t}_i}_{y_i'})^2} - \frac{\tilde{t}_i^2}{2}W$$

$$\int_{-\infty}^{+\infty}\frac{dy_i}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(y_i - i\sqrt{W}\tilde{t}_i)^2} =$$

$$= \int_{-\infty}^{+\infty}\frac{dy_i'}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(y_i')^2} = 1$$

$$\ast = e^{-\frac{1}{2}W\underline{\tilde{t}}\cdot\underline{\tilde{t}}}$$

$$= \mathbb{E}_W[e^{-\frac{1}{2}W\underline{\tilde{t}}\cdot\underline{\tilde{t}}}] = (\underline{\tilde{t}}\cdot\underline{\tilde{t}} = (A^T\underline{t})\cdot(A^T\underline{t}) = \underline{t}\,\underbrace{AA^T}_{\Sigma}\underline{t}) =$$

$$= \mathbb{E}_W[e^{-W\frac{1}{2}\underline{t}\cdot\Sigma\underline{t}}] =$$

$$= \int_{\substack{0\\ W\geq 0}}^{+\infty} dW\, f(W)\, e^{-W\underbrace{(\frac{1}{2}\underline{t}\cdot\Sigma\underline{t})}_{s}} = \mathcal{L}[\tfrac{1}{2}\underline{t}\cdot\Sigma\underline{t}]$$

$$\Rightarrow \phi^c(\underline{t}) = e^{i\underline{t}\cdot\mu}\,\mathcal{L}[\tfrac{1}{2}\underline{t}\cdot\Sigma\underline{t}]$$

We have introduced W in order to take into account the movements of volatilities.

LEMMA

x_1, x_2 uncorrelated normal r.v.
W not constant
$\Rightarrow$ x_1, x_2 not indipendent

PROOF. (driftless case) $\rightarrow$ $\underline{x} = \sqrt{W} A \underline{y}$

- $\mathbb{E}[x_1 x_2] = \mathbb{E}[W \sigma_1 y_1 \sigma_2 y_2] \overset{W \perp y}{=} \mathbb{E}[W]\, \mathbb{E}[\sigma_1 y_1 \sigma_2 y_2] \overset{\text{uncorr}}{=}$

- $\mathbb{E}[|x_1||x_2|] = \mathbb{E}[W \sigma_1 \sigma_2 |y_1||y_2|] \overset{W \perp y}{=}$

$= \sigma_1 \sigma_2\, \mathbb{E}[W]\, \mathbb{E}[|y_1||y_2|]$

Jensen inequality: $\sqrt{\mathbb{E}[W]} \geq \mathbb{E}[\sqrt{W}]$ (strictly if W not constant, i.e. a r.v.)

$\sigma\sigma \geq \sigma_1 \sigma_2\, \mathbb{E}[\sqrt{W}]^2\, \mathbb{E}[|y_1|]\, \mathbb{E}[|y_2|] =$

$= \mathbb{E}[\sqrt{W} \sigma_1 |y_1|]\, \mathbb{E}[\sqrt{W} \sigma_2 |y_2|] =$??

$= \mathbb{E}[|x_1|]\, \mathbb{E}[|x_2|]$

$\Rightarrow \mathbb{E}[|x_1||x_2|] \geq \mathbb{E}[|x_1|]\, \mathbb{E}[|x_2|]$

$\Rightarrow \mathrm{Corr}(|x_1|, |x_2|) \geq 0$ $\blacksquare$

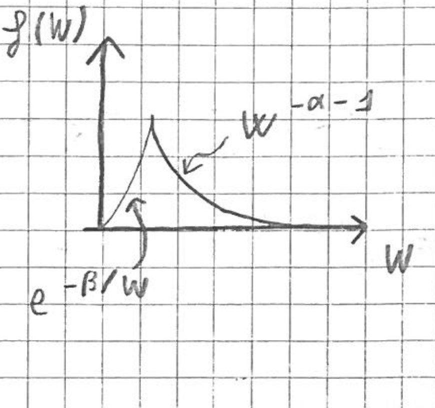
If W is not constant, $\mathrm{Corr}(|x_1|, |x_2|) > 0$
$\Rightarrow x_1, x_2$ not indipendent.

EXAMPLES : t - Student

$$x = \underbrace{\sqrt{\frac{1}{G}}}_{W} A y \qquad G \sim \chi^2_r \qquad y \sim N(0,1) \qquad z \perp y$$

$W \sim$ Inv. Gamma $\overset{1/\chi^2}{,}$ $\beta = \frac{1}{2}$, $\alpha = \frac{r}{2}$

$$f(W; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} W^{-\alpha-1} \exp\left(-\frac{\beta}{W}\right) \qquad W > 0$$



$\underline{x} = \underline{\mu} + \sqrt{W} A \underline{y} \sim t_r$

$\boxed{14.}$ NORMAL MEAN-VARIANCE MIXTURE : MAIN PROPERTIES AND EXAMPLES.

NMVM

$$\underline{\text{MODEL}} : \quad \underline{x} = \underline{\mu} + W\underline{\gamma} + \sqrt{W} A \underline{y}$$

where, in addition to NM parameters, we have introduced the constant vector $\underline{\gamma}$ and $W \geq 0$ r.v. called mixing variable independent from $\underline{y}$ with $Var[W] < +\infty$.

Remarks:

- $\underline{x} \mid W = \omega \sim N(\underline{\mu} + \overset{W}{\omega}\underline{\gamma}, \overset{W}{\omega}\Sigma)$
- $\mathbb{E}[\underline{x}] = \underline{\mu} + \underline{\gamma}\,\mathbb{E}[W]$
- $COV(\underline{x}) = \mathbb{E}[(\underline{x} - \mathbb{E}[\underline{x}])(\underline{x} - \mathbb{E}[\underline{x}])^T] =$

$$= \mathbb{E}[(\underline{x} - \underline{\mu} - \mathbb{E}[W]\underline{\gamma})(\underline{x} - \underline{\mu} - \mathbb{E}[W]\underline{\gamma})^T] =$$

$$= \mathbb{E}[((W - \mathbb{E}[W])\underline{\gamma} + \sqrt{W}A\underline{y})((W - \mathbb{E}[W])\underline{\gamma} + \sqrt{W}A\underline{y})^T]$$

$$= \mathbb{E}[(W - \mathbb{E}[W])^2]\underline{\gamma}\,\underline{\gamma}^T + \mathbb{E}[WA\underline{y}\,\underline{y}^T A^T] =$$

$$= Var(W)\,\underline{\gamma}\,\underline{\gamma}^T + \mathbb{E}[W]\Sigma$$

- $\phi^c(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] \overset{|W}{=} e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}_W[e^{i\underline{t}\cdot W\underline{\gamma}}\,\mathbb{E}[e^{i\underline{t}\cdot\sqrt{W}A\underline{y}}|W$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}_W[e^{iW\underline{t}\cdot\underline{\gamma}}\,\mathbb{E}[e^{i\sqrt{W}\underline{t}\cdot A\underline{y}}|W]] =$$

$$\underline{t}\cdot A\underline{y} = \underbrace{A^T\underline{t}}_{\underline{\tilde{t}}}\cdot\underline{y}$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}_W[e^{iW\underline{t}\cdot\underline{\gamma}}\,e^{-\frac{1}{2}W\underline{\tilde{t}}\cdot\underline{\tilde{t}}}] =$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}_W[e^{iW\underline{t}\cdot\underline{\gamma}}\,e^{-\frac{1}{2}W\underline{t}\cdot\Sigma\underline{t}}] =$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}_W[e^{-W(-i\underline{t}\cdot\underline{\gamma} + \frac{1}{2}\underline{t}\cdot\Sigma\underline{t}}] =$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathcal{L}[-i\underline{t}\cdot\underline{\gamma} + \frac{1}{2}\underline{t}\cdot\Sigma\underline{t}]$$

EXAMPLES

- $W \sim$ Generalized Inverse Gaussian (GIG)

$$f(W; a, b, p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} W^{(p-1)} e^{-(aW + b/W)/2}$$

$W > 0$

$a, b \in \mathbb{R}^+$, $p \in \mathbb{R}$ and $K_p(\lambda)$ is a Bessel function of the second kind, i.e.:

$$K_p(\lambda) = \frac{1}{2}\int_0^{+\infty} W^{p-1} e^{-\lambda(W + 1/W)/2} \, dW$$

Special cases

W	condition	normal MVM
IG	$p = -\frac{1}{2}$	NIG (Normal Inverse Gaussian)
Gamma	$b \to 0$	VG (Variance Gamma)
Ig (inverse gamma)	$a \to 0$	skewed t

$\boxed{15.}$ SPHERICAL: DEFINITION AND DISTRIBUTION PROPERTIES

SPHERICAL DISTRIBUTIONS

DEF. $\underline{x}$ (spherical distribution) s.t. $\underline{x} \overset{law}{=} U\underline{x} \quad \forall U \in \mathbb{R}^{d\times d}$
s.t. $U^TU = UU^T = Id$, $\rightarrow$ (U is a rotation)
i.e. spherical random vectors are distributionally invariant under rotation.

THEO. Spherical distribution's properties:

1. x is spherical
2. $\phi_{\underline{x}}(\underline{t}) \equiv \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = \psi(\|t\|^2)$ with ψ scalar function
3. $a'\underline{x} \overset{law}{=} \|a\| x_1 \quad \forall a \in \mathbb{R}^d$

PROOF. $(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1)$

$\boxed{1 \Rightarrow 2}$

$$\phi_{\underline{x}}(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = \mathbb{E}[e^{i\underline{t}\cdot U\underline{x}}] =$$
$$= \mathbb{E}[e^{iU'\underline{t}\cdot\underline{x}}] = \phi_{\underline{x}}(U'\underline{t})$$

since $\phi_{\underline{x}}(\underline{t}) = \phi_{\underline{x}}(U'\underline{t}) \quad \forall U$ rotation,
the characteristic function must be a function of the norm (invariant under rotation).
$\hookrightarrow \|U'\underline{t}\|^2 = \|\underline{t}\|^2$

$$\Rightarrow \phi_{\underline{x}}(\underline{t}) = \psi(\|\underline{t}\|^2)$$

$\boxed{2 \Rightarrow 3}$

$$\phi_{x_1}(b) \overset{b>0}{=} \mathbb{E}[e^{ibx_1}] = \mathbb{E}[e^{i b \underline{e}_1 \cdot \underline{x}}] = \psi(b^2) \quad * \qquad (x_1 = \underline{e}_1\cdot\underline{x})$$

$$\phi_{\underline{a}'\underline{x}}(t) = \mathbb{E}[e^{it\cdot a'\underline{x}}] = \mathbb{E}[e^{ita\,\underline{x}}] = \psi(t^2\|a\|^2) =$$
$$\overset{*}{=} \phi_{x_1}(t\|a\|) = \mathbb{E}[e^{it\|a\|x_1}] = \phi_{\|a\|x_1}(t)$$

$$\Rightarrow \underline{a}'\cdot\underline{x} \overset{law}{=} \|a\| x_1$$

$\Big(\phi_{\underline{a}'\underline{x}}(1) = \mathbb{E}[e^{i\underline{a}\cdot\underline{x}}] = \psi(\|a\|^2) \overset{*}{=} \phi_{x_1}(\|a\|) = \mathbb{E}[e^{ix_1\|a\|}] \Rightarrow a'\underline{x}$

$\boxed{3 \Rightarrow 1}$ $\qquad (3): \underline{t} \cdot \underline{x} = \|\underline{t}\| x_1$

$$\phi^c_{\underline{x}}(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = \mathbb{E}[e^{i\|t\| x_1}] \overset{\|U't\| = \|t\|}{=}$$

$$= \mathbb{E}[e^{i\|U't\| x_1}] \overset{(3)}{=} \mathbb{E}[e^{iU't\cdot\underline{x}}] =$$

$$= \mathbb{E}[e^{i\underline{t}\cdot U\underline{x}}] \quad \forall U$$

$$\Rightarrow \quad U\underline{x} \overset{law}{=} \underline{x} \quad \forall U$$

Example: Normal mixture with $\mu = 0$, $\Sigma = I_d$

$$\phi^c_{\underline{x}}(\underline{t}) = \mathcal{L}[\tfrac{1}{2}\|\underline{t}\|^2] = \psi(\|t\|^2)$$

$$= e^{i\underline{t}\cdot\mu}\mathcal{L}[\tfrac{1}{2}\underline{t}\cdot\Sigma\underline{t}]$$

spherical using $(2) \Rightarrow (1)$

$\boxed{16.}$ ELLIPTICAL DISTRIBUTION & CF

ELLIPTICAL DISTRIBUTIONS : multivariate affine transform of spherical distributions.

DEF. $\underline{x}$ is ELLIPTICAL if $\underline{x} = \mu + A\underline{y}$, $\underline{y} \in \mathbb{R}^d$ spherical distribution with $\begin{cases} \mu \in \mathbb{R}^d \\ AA' = \Sigma \in \mathbb{R}^d \text{ positive definite} \end{cases}$

The characteristic function is:

$$\phi_{\underline{x}}(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = e^{i\underline{t}\cdot\mu}\mathbb{E}[e^{i\underline{t}\cdot A\underline{y}}] =$$

$$= e^{i\underline{t}\cdot\mu}\mathbb{E}[e^{i\underbrace{A'\underline{t}}_{\tilde{\underline{t}}}\cdot\underbrace{\underline{y}}_{\text{spherical}}}] \overset{(2)}{=} e^{i\underline{t}\cdot\mu}\psi(\|\tilde{\underline{t}}\|^2) =$$

$$= e^{i\underline{t}\cdot\mu}\psi(\underline{t}\cdot\Sigma\underline{t})$$

$$\|\tilde{\underline{t}}\|^2 = \tilde{\underline{t}}\cdot\tilde{\underline{t}} = (A'\underline{t})\cdot(A'\underline{t}) = \underline{t}\,\Sigma\,\underline{t}$$

Example: Normal mixture

$$\phi_{\underline{x}}(\underline{t}) = e^{i\underline{t}\cdot\mu}\mathcal{L}(\underline{t}\cdot\Sigma\underline{t}/2)$$

17. COHERENT RISK MEASURE : AXIOMS

COHERENT MEASURE OF RISK

axiomatic approach to risk measures

DEF. Convex cone $\mathcal{M}$ s.t. rvs $L_1, L_2 \in \mathcal{M}$

$$\Rightarrow \begin{cases} L_1 + L_2 \in \mathcal{M} \\ \lambda L_1 \in \mathcal{M} \quad \forall \lambda \in \mathbb{R}^+ \end{cases}$$

$\mathbb{R}^+$ set of non-negative real numbers.

AXIOMS FOR A COHERENT RISK MEASURE :

Risk measure $\rho : \mathcal{M} \to \mathbb{R}$:

1. TRANSLATION INVARIANCE : $\rho(L + \ell) = \rho(L) + \ell$, $\forall L \in \mathcal{M}, \forall \ell \in \mathbb{R}$
2. SUBADDITIVITY : $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$, $\forall L_1, L_2 \in \mathcal{M}$
3. POSITIVE HOMOGENEITY : $\rho(\lambda L) = \lambda \rho(L) \quad \forall \lambda \in \mathbb{R}^+$
4. MONOTONICITY : $\rho(L_1) \leq \rho(L_2) \quad \forall L_1 \underset{a.s.}{\leq} L_2 \in \mathcal{M}$

Financial point of view :

1. Impact of an amount of cash on my ptf :
   if I have a given ptf that generates a loss and a cash amount, my R.m. will be the R.m. of my loss + cash amount.
2. Diversification principle for risk measures.
3. λ times ptf $\to$ λ times risk (bigger ptf $\to$ bigger risk).
4. If one ptf has a loss lower than the other, also the R.m. will have the same relation.

In the positive homogeneity assumption we consider an INFINITE LIQUIDITY : you can't actually multiply for any number.
In order to handle liquidity problems, new theory of convex risk measures, where 3 holds with

$$\rho(\lambda L) \geq \rho(L)\lambda$$

Remarks :

1. Given 2 & 3, 4 can be rewritten as:

$$\rho(L) \leq 0 \quad \forall L \leq 0 \text{ a.s.} \quad (4^*)$$

PROOF. $4 \overset{2+3}{\longleftrightarrow} 4^*$

($\Rightarrow$) $\rho(0) = \rho(0 \cdot L) \overset{3}{=} 0 \cdot \rho(L) = 0$

Consider $L_2 = 0$, $L_1 \leq 0 \Rightarrow L_1 \leq L_2$

$\xrightarrow{2} \rho(L_1) \leq \rho(L_2) = 0$

($\Leftarrow$) $L_1 \leq L_2$ a.s. $\longleftrightarrow \underbrace{L_1 - L_2}_{L} \leq 0$ a.s.

$\overset{Rp}{\Rightarrow} \rho(L) \leq 0$

$(2) \rightarrow \rho(L_1) \leq \underbrace{\rho(L_1 - L_2)}_{\rho(L)} + \rho(L_2)$

$\rightarrow \rho(L_1) - \rho(L_2) \leq \rho(L) \leq 0$

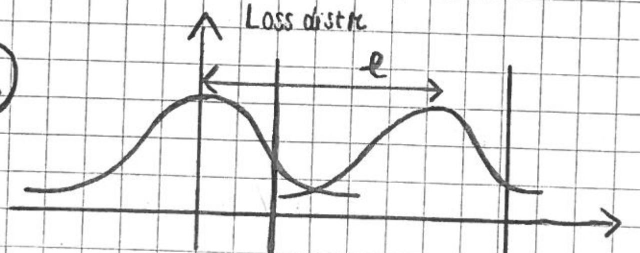
$\rightarrow \rho(L_1) \leq \rho(L_2)$

?

2. Convexity axioms holds (relaxes axioms 2 & 3)
↓
convex measure of risk.

$\boxed{18.}$ VaR: COHERENT RISK MEASURE AXIOMS THAT ARE SATISFIED

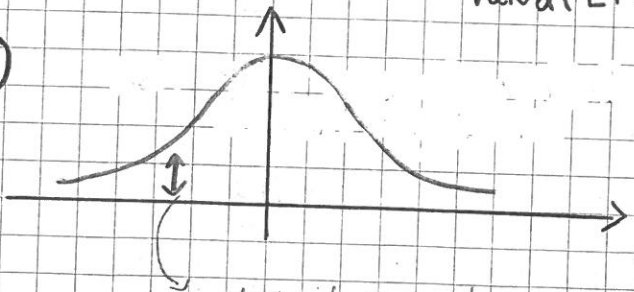
1.



$VaR_\alpha(L + \ell) = VaR_\alpha(L) + \ell$

OK

3.



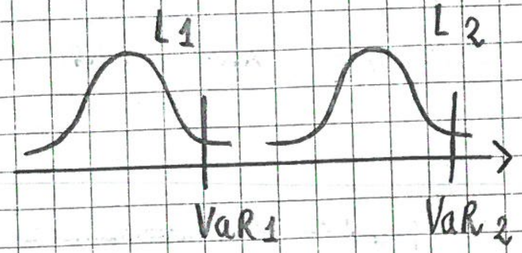
$VaR(\lambda L) = \lambda \, VaR(L)$

OK

distribution becomes fatter and VaR becomes fatter as well.

$$\int_{-\infty}^{\alpha} L\lambda = L \int_{-\infty}^{\alpha} L$$

(4) OK



Translation invariance (1)
Positive Homogeneity (3)
Monotonicity (4)

} guaranteed by VaR Quantile representation.

(2) Subadditivity ? $L = -V(\underline{c} + \underline{w}\cdot\underline{X})$, $\underline{X} = \underline{\mu} + A\underline{y}$

Only for a LINEAR PORTFOLIO and ELLIPTICAL DISTRIBUTION.

$$= \underbrace{-V(\underline{c} + \underline{w}\cdot\underline{\mu})}_{EL} - \underbrace{V\underline{w}\cdot A\underline{y}}_{UL}$$

19. VaR SUB-ADDITIVE ELLIPTICAL CASE.

THEO. VaR sub-additivity for elliptical risk factors and linear portfolio

$L_1, L_2 \in \mathcal{M}$, $0.5 \leq \alpha \leq 1$

$$\Downarrow$$

$$VaR_\alpha(L_1+L_2) \leq VaR_\alpha(L_1) + VaR_\alpha(L_2)$$

$$\mathcal{M} = \left\{ L : L = \lambda_0 + \sum_{i=1}^{d} \lambda_i x_i,\ \lambda_i \in \mathbb{R} \right\}$$

$\underline{x} = \underline{\mu} + A\underline{y}$, $\underline{y} \in \mathbb{R}^d$ spherical distribution.

PROOF. Loss for a linear portfolio:

$$L = -V(C + \underline{w}\cdot\underline{x}) = \underbrace{-V\underline{c}}_{\lambda_0} - \underbrace{V\underline{w}}_{\underline{\lambda}}\cdot\underline{x}$$

$$\Rightarrow L = \lambda_0 + \sum_{i=1}^{d} \lambda_i x_i,\ \lambda_i \in \mathbb{R}$$

Elliptical risk factors: $\underline{x} = \underline{\mu} + A\underline{y}$

$$\Rightarrow L = \lambda_0 + \underline{\lambda}\cdot(\underline{\mu} + A\underline{y}) = \underbrace{\lambda_0 + \underline{\lambda}\cdot\underline{\mu}}_{\text{EXPECTED LOSS}} + \underbrace{\underline{\lambda}\cdot A\underline{y}}_{\text{UNEXPECTED LOSS}}$$

$$L = EL + \underline{\lambda} \cdot A\underline{y} = EL + \overbrace{A'\underline{\lambda}}^{a} \cdot \underline{y} =$$

prop. (3) spherical distributions

$$= EL + \|\underline{a}\| y_1$$

translation invariance + positive homogeneity

$$\Rightarrow \mathrm{VaR}_\alpha(L) = EL + \|a\| \, \mathrm{VaR}_\alpha(y_1)$$

Now take L_1, L_2:

$$L_1 = EL^{(1)} + \underline{\lambda}_1 \cdot A\underline{y} \quad , \quad L_2 = EL^{(2)} + \underline{\lambda}_2 \cdot A\underline{y}$$

Let's try subadditivity:

$$\mathrm{VaR}_\alpha(L_1 + L_2) = EL^{(1)} + EL^{(2)} + \mathrm{VaR}_\alpha\left((\underline{\lambda}_1 + \underline{\lambda}_2) \cdot A\underline{y}\right) =$$

$$= EL^{(1)} + EL^{(2)} + \|A'\underline{\lambda}_1 + A'\underline{\lambda}_2\| \underbrace{\mathrm{VaR}_\alpha(y_1)}_{*} \leq$$

$*$ $\mathrm{VaR}_\alpha(y_1) \geq 0$ for $0.5 \leq \alpha \leq 1$,
since y_1 is simmetrical ($\underline{y}$ spherical)
$\Rightarrow$ mean $= 0$
$\mathrm{VaR}_{0.5}(y_1) = 0$

$$\leq \left(\text{triangolar inequality} : \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|\right) \leq$$

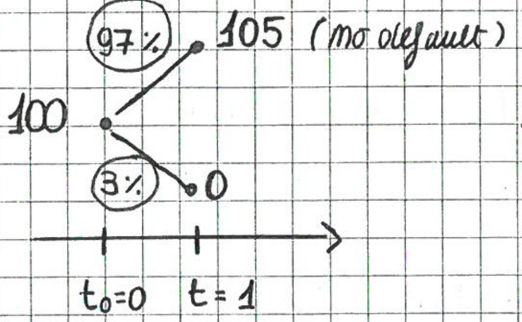
$$\leq EL^{(1)} + EL^{(2)} + \left(\|A'\underline{\lambda}_1\| + \|A'\underline{\lambda}_2\|\right) \mathrm{VaR}_\alpha(y_1) =$$

$$= \underbrace{EL^{(1)} + \|A'\underline{\lambda}_1\| \, \mathrm{VaR}_\alpha(y_1)}_{\mathrm{VaR}_\alpha(L_1)} + \underbrace{EL^{(2)} + \|A'\underline{\lambda}_2\| \, \mathrm{VaR}_\alpha(y_2)}_{\mathrm{VaR}_\alpha(L_2)}$$

20. VaR SUB-ADDITIVE : COUNTER EXAMPLE

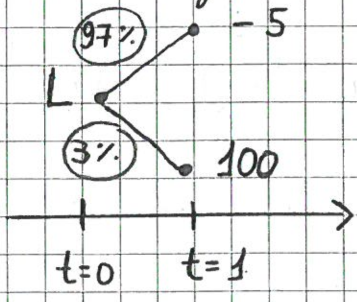
Economy with <u>1 time step</u> and <u>2 indipendent defaultable bonds</u> with zero recovery, discount factors = 1.

Survival probability = 97%.



$\longleftrightarrow$

In terms of loss:



Consider now two possible portfolios:

1. undiversified: 200 € in the first bond.
2. diversified: 100 € invested in each bond.

Losses:

1. $\begin{cases} 200 & p = 3\% \\ -10 & p = 97\% \end{cases}$ default / no default

2. $\begin{cases} 200 & p = 0.09\% \\ 95 & p = 2\cdot(97\%)\cdot(3\%) \simeq 6\% \\ -10 & p = (97\%)^2 \simeq 94.1\% \end{cases}$

both bonds default; one default, the other not; any of them default

$$\text{VaR} = \inf_{\ell \in \mathbb{R}} \underbrace{\{P(L > \ell) \leq 1-\alpha\}}_{F(\ell) \geq \alpha}$$

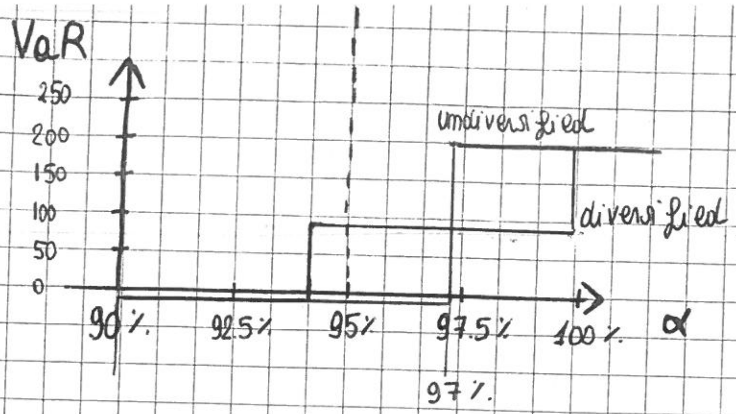
$\alpha = 95\%$

: VaR lowest loss s.t. probability $\leq 1-\alpha = 5\%$.

$F(\ell)$: $P(L \leq \ell)$

1. $P(L > \ell) = \begin{cases} 100\% & \ell \leq -10 \\ 3\% & -10 \leq \ell \leq 200 \\ 0\% & \ell > 200 \end{cases}$

2. $P(L > \ell) = \begin{cases} 100\% & \ell \leq -10 \\ 6\% & -10 \leq \ell \leq 95 \\ 0.09\% & 95 \leq \ell \leq 200 \\ 0\% & \ell > 200 \end{cases}$



For example, for $\alpha = 95\%$, $VaR_{diversified} > VaR_{undiversified}$

$\Longrightarrow$ subadditivity doesn't hold.

But if we compute the ES subadditivity holds:

$$ES_1(95\%) = \frac{1}{5\%}\int_{95\%}^{1} VaR_u \, du =$$

somma losses dalla + piccola in poi fino ad arrivare alla prob $1-\alpha$!

$$= \frac{1}{5\%}\left(3\% \cdot (200) + 2\% \cdot (-10)\right) = 116.0$$

$$ES_2(95\%) = \frac{0.09\% \cdot 200 + 4.91\% \cdot 95}{5\%} = 96.9$$

21. ES: COHERENT RISK MEASURE.

ES was subadditive in the previous counterexample: is it, in general, a COHERENT MEASURE OF RISK?

THEO. ES is a coherent risk measure.

PROOF.

- 1, 3, 4 come from 1, 3, 4 of VaR, thanks to ES representation $\int$.

  In fact, for example:

  1: $$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u \, du$$

  $$L \to (L+\ell), \quad \ell \in \mathbb{R}$$

  $$VaR(L+\ell) = \ell + VaR(L)$$

  $$ES_\alpha(\ell+L) = \frac{1}{1-\alpha}\int_\alpha^1 [\ell + VaR_u(L)] \, du =$$

  $$= \ell + ES_\alpha(L)$$

  ...

- 2: SUBADDITIVITY?

  Take m simulations of losses of the financial firm: $L_1, \dots, L_m$

  $\{L_i\}_{i=1,\dots,m}$

  We order them: $L^{(i,m)}$, $L^{(1,m)} > \dots > L^{(m,m)}$.

  Then, i can sum the largest $1 \leq m \leq n$:

  $$\sum_{i=1}^{m} L^{(i,m)}$$

  $$\shortparallel$$

  $$\sup_{\{i_j\}} \{ \underbrace{L_{i_1} + L_{i_2} + \dots + L_{i_m}}_{\text{This is any sum}} \} \qquad 1 \leq i_1, \dots, i_m \leq m$$

  possible sampling

I can write the same for another $\tilde{L}$:

$$\sum_{i=1}^{m} \left(L + \tilde{L}\right)^{(i,m)} =$$

First sum
Then order

$$= \sup_{i_j} \left\{ (L_{i1} + \tilde{L}_{i1}) + \dots + (L_{im} + \tilde{L}_{im}) \right\}$$

$$\leq \sup_{i_j} \left\{ L_{i1} + \dots + L_{im} \right\} + \sup_{i_j} \left\{ \tilde{L}_{i1} + \dots + \tilde{L}_{im} \right\} =$$

$$= \sum_{i=1}^{m} L^{(i;m)} + \sum_{i=1}^{m} \tilde{L}^{(i,m)}$$

We can normalize by $m = [m(1-\alpha)]$ (HIS)

$$\frac{1}{[m(1-\alpha)]} \sum_{i=1}^{[m(1-\alpha)]} \left(L + \tilde{L}\right)^{(i,m)} \leq$$

$$\leq \frac{1}{[m(1-\alpha)]} \sum_{i=1}^{m} L^{(i,m)} + \frac{1}{[m(1-\alpha)]} \sum_{i=1}^{m} \tilde{L}^{(1,m)}$$

$$\Rightarrow ES_\alpha(L + \tilde{L}) \leq ES_\alpha(L + \tilde{L}) \quad \blacksquare$$

For practical aims, the VaR is coherent, because we usually use linear (or linearized) portfolios and elliptical distributions.

The problem of ES is connected to a precise measure of the tails, that have to be accurately modelled

22. CAPITAL ALLOCATION & EULER PRINCIPLE. ELLIPTICAL CASE & CONTRIBUTION TO RISK MEASURE.

Used to decompose a risk measure in the Capital Associated (AC) to each i-th investment:

$$\rho(L) = \underline{\lambda}' \, \tilde{\pi}^{\rho}(\underline{\lambda}) = $$
$$= \sum_{i=1}^{d} \lambda_i \, \tilde{\pi}_i^{\rho}(\lambda)$$

possible for positive homogeneous risk measures.

d: number of risk factors in the ptf.

PRINCIPLE: Euler Capital Allocation Principle

if $\rho(L)$ positive homogeneous & cont. differentiable

$$\Rightarrow \rho(L) = \sum_{i=1}^{d} \lambda_i \frac{\partial}{\partial \lambda_i} \rho(L)$$

$$\Leftrightarrow \boxed{\tilde{\pi}^{\rho}(\underline{\lambda}) = \nabla_{\lambda} \rho(L)}$$

PROOF. $\rho(L)$ is function of $\underline{\lambda}$

$$L = EL + \underbrace{UL}_{\underline{\lambda} \cdot \underline{x}}$$

$$\frac{\partial}{\partial \alpha} \rho(\alpha \underline{\lambda}) = \sum_{i=1}^{d} \lambda_i \frac{\partial \rho(\alpha \underline{\lambda})}{\partial \alpha \lambda_i}$$

$$\frac{\partial}{\partial \alpha} \rho(\alpha \underline{\lambda}) = \frac{\partial}{\partial \alpha} \left(\alpha \rho(\underline{\lambda})\right) = \rho(\underline{\lambda})$$

$$\xRightarrow{\alpha = 1} \rho(\underline{\lambda}) = \sum_{i=1}^{d} \lambda_i \frac{\partial \rho(\underline{\lambda})}{\partial \lambda_i}$$

Homogeneous function $f: \mathbb{R}^d \to \mathbb{R}$

$f(\underline{x}) = \alpha^K f(\underline{x})$ homogeneous of degree K

PROP. $f(\underline{x}) = \underline{x} \cdot \nabla f(\underline{x})$ homogeneous degree one, differentiable.

CORROLLARY : For risk factors described by <u>elliptic</u> distributions (& linear) ptf :

$$\tilde{\pi}^{\rho}(\underline{\lambda}) = \frac{\Sigma\underline{\lambda}}{\underline{\lambda}'\Sigma\underline{\lambda}}\,\rho(UL)$$

$$\parallel$$

$$\nabla_{\lambda}\rho(L)$$

PROOF. Loss in elliptical distributions :

$$L = \underbrace{\lambda_0 + \underline{\lambda}\cdot\underline{\mu}}_{EL} + \underbrace{\underline{\lambda}\cdot A\underline{y}}_{UL} \quad , \underline{y} \text{ spherical distribution}$$

prop. (3) spherical distributions

$$UL = \underline{\lambda}\cdot A\underline{y} = A'\underline{\lambda}\cdot\underline{y} = \|A'\underline{\lambda}\|\, y_1$$

$$\rho(UL) = \|A'\lambda\|\,\rho(y_1)$$

L^2 norm

$$\|A'\lambda\| = \sqrt{(A'\underline{\lambda})(A'\underline{\lambda})} = \sqrt{\underline{\lambda}\,AA'\underline{\lambda}} = \sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}$$

$$\Rightarrow \rho(L) = \sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}\;\rho(y_1) \rightarrow \rho(y_1) = \rho(L)/\sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}$$

$$\frac{\partial}{\partial\lambda_i}\sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}} = \frac{\cancel{2}\,\Sigma\underline{\lambda}}{\cancel{2}\sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}} \Rightarrow \nabla\rho(\underline{\lambda}) = \frac{\Sigma\underline{\lambda}}{\sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}}\,\rho(y_1)$$

$$\rho(y_1) = \frac{\rho(UL)}{\sqrt{\underline{\lambda}\cdot\Sigma\underline{\lambda}}}$$

$$\Downarrow$$

$$\nabla_{\lambda}\rho(\underline{\lambda}) = \tilde{\pi}^{\rho}(\underline{\lambda}) = \frac{\Sigma\underline{\lambda}}{\underline{\lambda}\cdot\Sigma\underline{\lambda}}\,\rho(UL) \qquad \blacksquare$$

Contribution to Risk Measure → We can associate to each risk a contribution to VaR & ES, so I can select the business to fine (the ones that absorbing too much capital), instead of the whole bank!

23. BACKTESTING VaR : BASEL APPROACH & UNCONDITIONAL COVERAGE TEST.

BACKTESTING VaR

Basel II, Pillar II → supervisors' review of bank's capital adequacy.

How to compute $K \in [3,4]$ present in RC in Basel II ?

→ BACKTESTING : evaluation EX-POST of the quality of a VaR measurement.

Why is it relevant ?

Basel Committee requires that VaR is regularly tested :
- predictable ability via a comparison of daily estimates and "actual" losses;
- in order to determine mkt risk capital requirements.

Main Idea : Check consistency between number of exceptions and confidence level, e.g.

with VaR (99%) we expect that losses will be higher in 2.5 days a year.
$1y \simeq 250d$

↓

Monitor not only the VaR daily but also the exceptions.

Which P&L should be considered ?

- P&L coming from actual positions and trades, P&L gest detected from Business Control Unit.
- P&L obtained revaluing end of previous day positions under new mkt conditions (static P&L). (FROZEN PTF)

$VaR_{99\%}$, $99\% = 1-\alpha$, $\alpha = 1\%$

PROBABILITY EXCEPTIONS

Measure the Number of Exceptions x (# times that the Loss is larger than the VaR).

Prob. to observe x exceptions is binomial:

$$\mathbb{P}(x \mid \alpha, \underset{\substack{\uparrow \\ \text{\# points 1y}}}{N}) = \binom{N}{x} \alpha^{x} (1-\alpha)^{N-x}$$

For $x \geq 5$, $\mathbb{P} < 5\%$.

BASEL APPROACH

In Basel 2:

$$\text{if you have} \begin{cases} 0 \leq x \leq 4 & \text{OK} \\ 5 \leq x \leq 9 & \text{not good, the supervisor will implement } K \\ x \geq 10 & \text{very bad} \rightarrow \text{re-think the model for VaR} \end{cases}$$

Build a quantitative method to check the accuracy of our risk measure:

LIKELIHOOD RATIO

$$LR \equiv -2 \ln \frac{L(x \mid \text{null hypotesis})}{L(x \mid \text{model to be tested})}$$

our VaR is correct

We apply Wilks' theorem:

$$N \rightarrow \infty, \quad LR \sim \chi^2_{g\,\text{model} - g\,\text{null}} \quad (\text{if } H_0 \text{ is correct})$$

As in standard LR test,

$$p < \beta \rightarrow H_0 \text{ rejected}$$

$$p = 1 - \phi_{\chi^2}(LR)$$

p-value associated to the observed LR

$$p \uparrow \quad , 1 - \phi_{\chi^2}(LR) \uparrow$$

$$\phi_{\chi^2} \downarrow \rightarrow LR \uparrow \rightarrow \frac{L()}{L()} \simeq \rightarrow L \uparrow$$

PROPORTION OF FAILURES TEST OR UNCONDITIONAL COVERAGE TEST (Kupiec, 1995)

$$H_0 : L(x / \overset{1\%}{\alpha}) = \binom{N}{x} \alpha^{x} (1-\alpha)^{(N-x)}$$

Model to be tested: $L(x/\hat{\alpha}) = \binom{N}{x} \hat{\alpha}^{x} (1-\hat{\alpha})^{(N-x)}$

$\hat{\alpha} \equiv \dfrac{x}{N}$ frequency of empirical exceptions in the back test.

$$LR_{UC} = -2 \ln \frac{L(x/\alpha)}{L(x/\hat{\alpha})} \xrightarrow{N \to \infty} \chi^2_1 \quad 1 \text{ dof } (1-0)$$

[illegible] verificare che $\alpha = \hat{\alpha}$

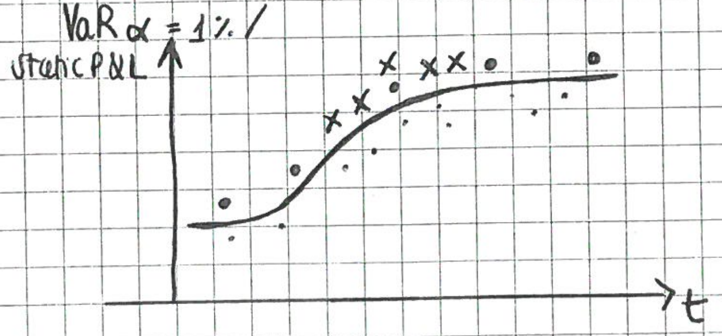
Example $N = 500$

x	$p(N=500)$
5	16.2 %
6	5.9 %
7	1.9 %
⋮	

In this case also 5 and 6 are acceptable!

$\boxed{24.}$ BACKTESTING VaR: CONDITIONAL (CHRISTOFFERSEN) TEST.

CONDITIONAL TEST (CHRISTOFFERSEN, 1998)



Reaction of VaR to volatility in the mkt

- • 1^{st} bank
- x 2^{nd} bank

In both cases, I have 5 exceptions.

Are both banks risky the same? The 2^{nd} is more risky.

(you are forgetting some risk very relevant in the VaR measure, i.e. risk coming from a SHADOW BANKING)

↳ the VaR is not really informing of the response to volatility of the mkt.

We have to do a conditional test to detect situations like the second one:

Consider an <u>elementary Markov chain</u> for exceptions:

$t-1$	t
1	1
0	0

$N_{01} = \#(0 \to 1)$
$N_{00} = \#(0 \to 0)$
$N_{10} = \#(1 \to 0)$
$N_{11} = \#(1 \to 1)$

We want to see if there's a connection between exceptions:
if I have $1 \to 1$, probably I have some trades that I am hiding (shadow banking).

FRACTIONS (of exeptions):

$$\hat{\alpha}_{01} = \frac{N_{01}}{N_{01}+N_{00}}, \quad \hat{\alpha}_{11} = \frac{N_{11}}{N_{11}+N_{10}} \qquad 2\,dof$$

$$\hat{\alpha}_{00} = 1 - \hat{\alpha}_{01}, \quad \hat{\alpha}_{10} = 1 - \hat{\alpha}_{11}$$

Again, we do a likelihood test
In the null hypotesis, $\alpha_{01} = \alpha_{11} = \alpha$

$$LR_{cc} = -2\ln \frac{L(x|\alpha)}{L(x|\hat{\alpha}_{00}, \hat{\alpha}_{01}, \hat{\alpha}_{10}, \hat{\alpha}_{11})} \qquad (2\ d.o.f.)$$

$$L(x|\{\hat{\alpha}_{ij}\}) = \frac{N!}{N_{00}!N_{01}!N_{10}!N_{11}!}(1-\hat{\alpha}_{01})^{N_{00}}\hat{\alpha}_{01}^{N_{01}} \cdot$$

$$\cdot (1-\hat{\alpha}_{11})^{N_{10}}\hat{\alpha}_{11}^{N_{11}}$$

...?