STRUCTURED & DERIVATIVE PRODUCTS ③

BASIC EQUITY / IR

A FORWARD contract is a relatively simple derivative: it is an AGREEMENT to buy or sell an asset at a certain future time for a certain price (vs. SPOT, to buy or sell today).

1. FORWARD PRICE FOR A NOT-DIVIDEND-PAYING STOCK. [Hull Ch. 5.1-5.4] [deduction].

PUT - CALL PARITY [Hull Ch. 9.4].

Two time-step modelling: VALUE DATE t_0, Δt, t (dynamics)

Assumption: deterministic discount factor $B(t_0,t)$ → value of 1 (at time t) in t_0.

A FORWARD is a contract between two counterparties (that decide today the price of a derivative) to buy/sell an asset at a specified price on a future specified date.

$$F(s,t) \text{ s.t. } \begin{cases} F(t,t) = S_t \\ \text{money exchange vs. delivery in } t \end{cases}$$

t_0 — t: $F(t,t)$, S_t

(proof in the next pages.)

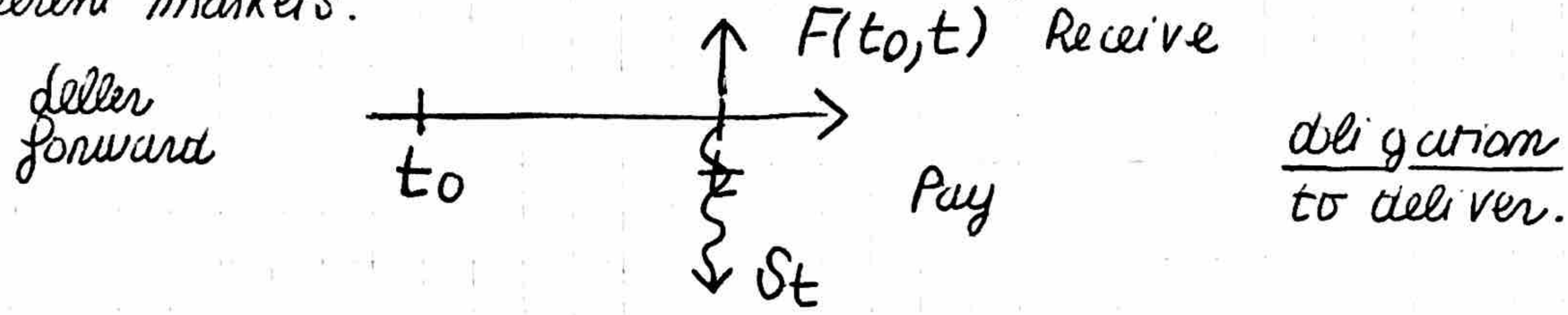
Cash Settlement (pay in amount)
vs.
Physical Delivery (share, commodity, currency)

$$\text{NO ARBITRAGE} \leftrightarrow F(s,t) = \mathbb{E}_s[\underbrace{F(t,t)}_{S_t}] = \mathbb{E}[F(t,t) \mid \mathcal{F}_s]$$

i.e. $F(s,t)$ is a Martingale Process in the R.N. measure.

FORWARD	vs.	FUTURE
• Between two counterparties (OTC);		• Exchange; → { CBOT, Eurex
• payoff customized: specific contract; (private contracts)		• payoff standardized: the delivery will be on a GIVEN DAY of a GIVEN TYPE; → contract terms standardized by the exchange.
• counterparty risk.		• NO counterparty risk. organize trading so that contract defaults are avoided.

Forward and future contracts are the principal LONG-TERM DERIVATIVE CONTRACTS: there is a contract between two counterparties to the delivery of a determined quantity of a given underlying at a price and at a time (maturity date). They are traded in different markets.



FWD : OTC, customized, counterparty risk.

FUTURE : Exchange, standardized, (*) NO counterparty risk.

(*) thanks to a CLEARING HOUSE (subsidiary of Exchange), intermediary between buyers and sellers, responsible to settle accounts, clear trades, maintain margins.
At the end of each day, the exchange publishes a SETTLEMENT PRICE (price adapted at each day) + initial margin, at the start of the exchange, to pay to avoid any other risk during the participation.

HULL : A fwd contract is settled at the end of its life, while a future contract is settled daily (at the end of each day, the investor's gain (loss) is added to (subtracted from) the margin account.

DERIVATIVE : financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value is dependent on the price of a stock. However, derivatives can be dependent on almost any variables (from prices of stk to amount of snow...)

OTC MKT : A key advantage of the OTC market is that the terms of a contract do not have to be those specified by an exchange. Mkt participants are free to negotiate any mutually attractive deal. A disadvantage is that there is usually some credit risk in an OTC-trade (there is a small risk that the contract will not be honored).

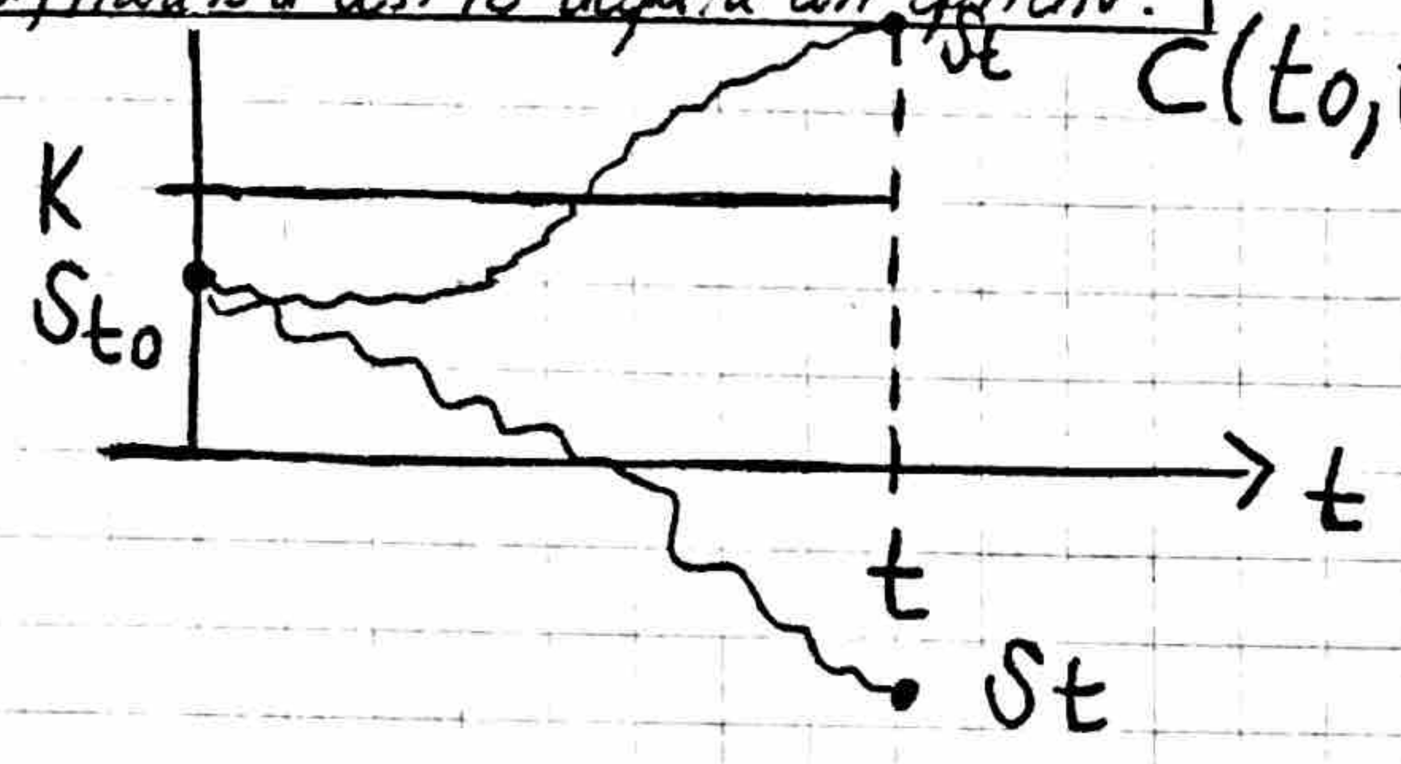
FWD CONTRACT : Agreement between two counterparties. One of the parties assumes a LONG POSITION and agrees to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a SHORT POSITION and agrees to sell the asset on the same date for the same price.

FUTURES CONTRACT : To make trading possible, the exchange specifies certain standardized features of the contract, and provides a mechanism that gives the two parties a guarantee that the contract will be honored.
Closed up and rewritten at a new price each day.

Other basic derivaties' contracts: → options, traded both on exchanges and in the OTC-mkt.

- EUROPEAN CALL OPTION : contract that gives you the RIGHT (not obligation) to buy at delivery time (maturity / expiry) at the strike price K.
  ↳ price at which I have the right to buy the underlying S_t.

An option gives the holder the RIGHT to do something but the holder does not have to exercise this right. This is what distinguishes options from forwards and futures, where the holder is obligated to buy or sell the underlying. Whereas it costs nothing to enter into a fwd or futures contract, there is a cost to acquire an option.



$$C(t_0,t) \text{ s.t. } C(t,t) = \begin{cases} [S_t - K]^+ & \text{(payoff)} \\ \text{Premium paid in } t_0 \end{cases}$$

↑ premium paid in advance

$$\mathbb{E}_0 = \mathbb{E}[\cdot \mid \mathcal{F}_{t_0}]$$

No arbitrage $\Rightarrow$ $C(t_0,t) = \mathbb{E}_0[B(t_0,t)C(t,t)] =$

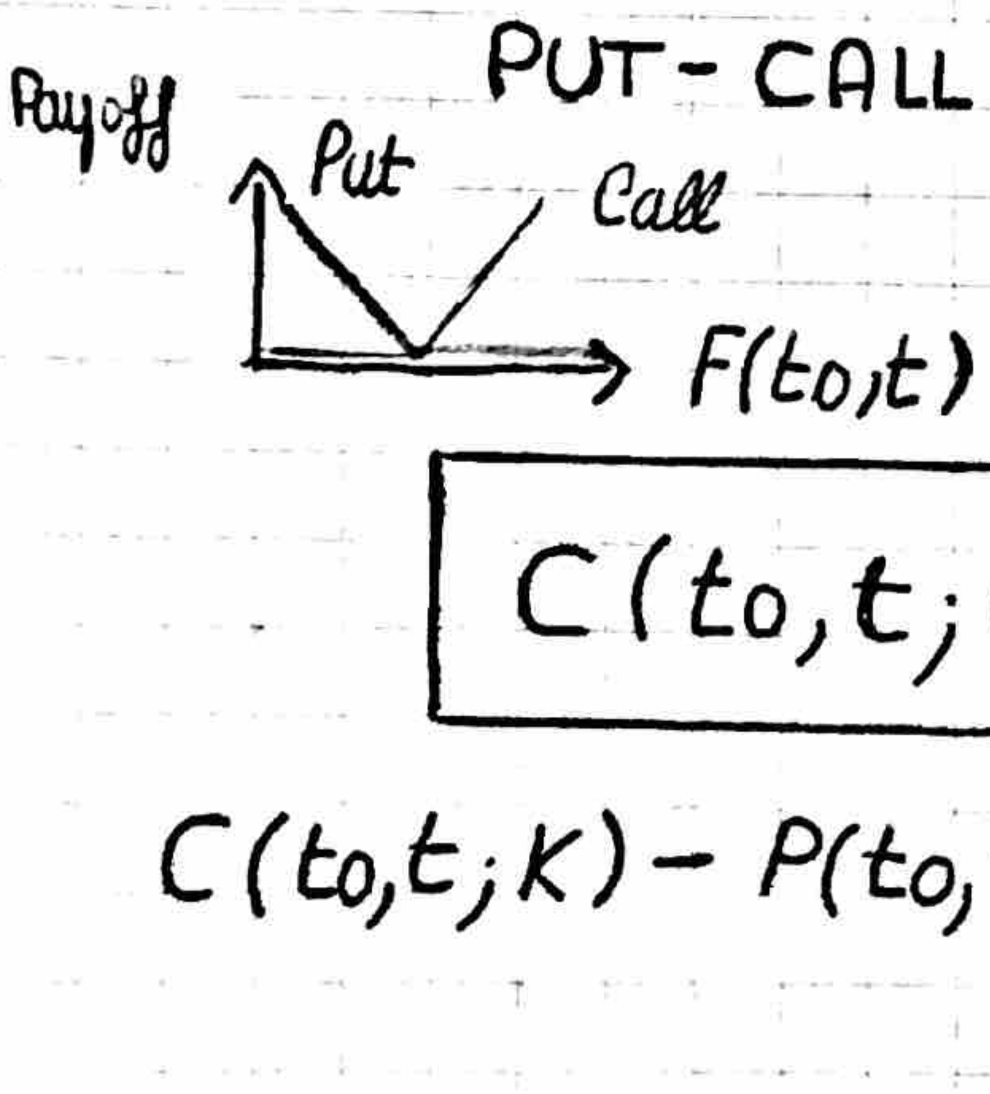
$$= B(t_0,t)\,\mathbb{E}_0\Big[\underbrace{\big(F(t,t)-K\big)^+}_{\text{payoff of European Call option.}}\Big]$$

- EUROPEAN PUT OPTION : contract that gives you the RIGHT (not obligation) to sell the underlying.

$$P(t_0,t) = \mathbb{E}_0\big[B(t_0,t)\,[K - F(t,t)]^+\big] \quad \text{(APT)}$$

[10.4]

PUT-CALL PARITY: equality that involves call, put with same expiry, strike price, value date.



⇓

$$\boxed{C(t_0,t;K) - P(t_0,t;K) = B(t_0,t)\,[F(t_0,t) - K]}$$

$$C(t_0,t;K) - P(t_0,t;K) = B(t_0,t)\,\mathbb{E}_0\Big[\underbrace{[F(t,t)-K]^+}_{(F(t,t)-K)\mathbb{1}_{F(t,t)\geq K}} - \underbrace{[K-F(t,t)]^+}_{(K-F(t,t))\mathbb{1}_{F(t,t)<K}}\Big] =$$

$$= B(t_0,t)\,\mathbb{E}_0[F(t,t) - K] = B(t_0,t)\,[F(t_0,t) - K]$$

HULL:

FACTORS AFFECTING OPTION PRICES: $S_0, K, TTM, \sigma, r, d$

	$C_€$	$P_€$	C	P
$S_0 \uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
$K \uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$
$TTM \uparrow$	?	?	$\uparrow$	$\uparrow$
$\sigma \uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$r \uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
$d \uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$

→ The long-life option must be worth at least as the short-life option.

PUT-CALL PARITY

: important relationship between the prices of European put and call options with the same K and TTM.

Consider the following two portfolios:

1. one European call option + a ZCB that provides a payoff of K at time T
2. one European put option + one share of the stock

Stock pays no dividends.

Values of portfolio 1 and portfolio 2 at time T

		$S_T > K$	$S_T < K$
portfolio 1	Call Option	$S_T - K$	0
	ZCB	K	K
	TOTAL	S_T	K
portfolio 2	Put Option	0	$K - S_T$
	Share	S_T	S_T
	TOTAL	S_T	K

⇒ both portfolios are worth $\max(S_T, K)$ at time T.

Because they are European, the options cannot be exercised prior to time T. Since the portfolios have identical values at time T, they must have identical values today. If this were not the case, an arbitrageur could buy the less expensive portfolio and sell the more expensive one. Because the portfolios are guaranteed to cancel each other out at time T, this trading strategy would lock in an arbitrage profit equal to the difference in the values of the two portfolios.

⇓

$$C + Ke^{-rT} = P + S_0$$

For American Options: $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$

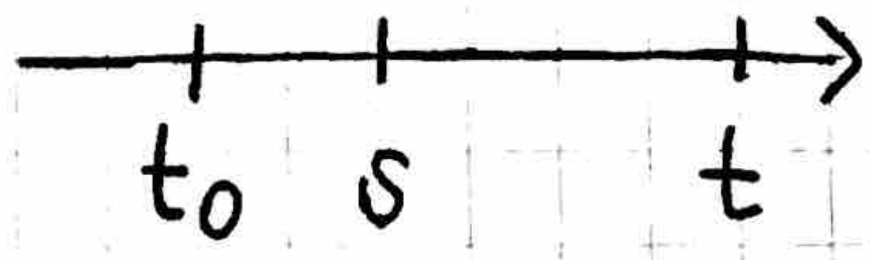
2. BLACK-SCHOLES & BLACK MODELS: FORWARD OPTION PRICE. CASE WITH DETERMINISTIC FUNCTION OF TIME FOR IR, DIVIDEND YIELD AND VOLATILITY.

W_t BM is a mg $\mathbb{E}[W_t] = W_{t_0} = 0$, but it has both positive and negative values (no good)

BLACK SOLUTION ('76) (BLACK-SCHOLES-MERTON)

Forward has to be a positive amount ⇒ Forward: (martingale) Geometric Brownian Motion

$t_0 \leq s \leq t$



$$\begin{cases} dF(s,t) = F(s,t)\,\sigma\, dW_s \\ F(t_0,t) = F_0 \end{cases}$$

with σ the volatility of a forward (of a stock), i.e. the standard deviation of the fwd-price RETURN in a short period of time.

To solve the SDE, we use the Itô differential rule:

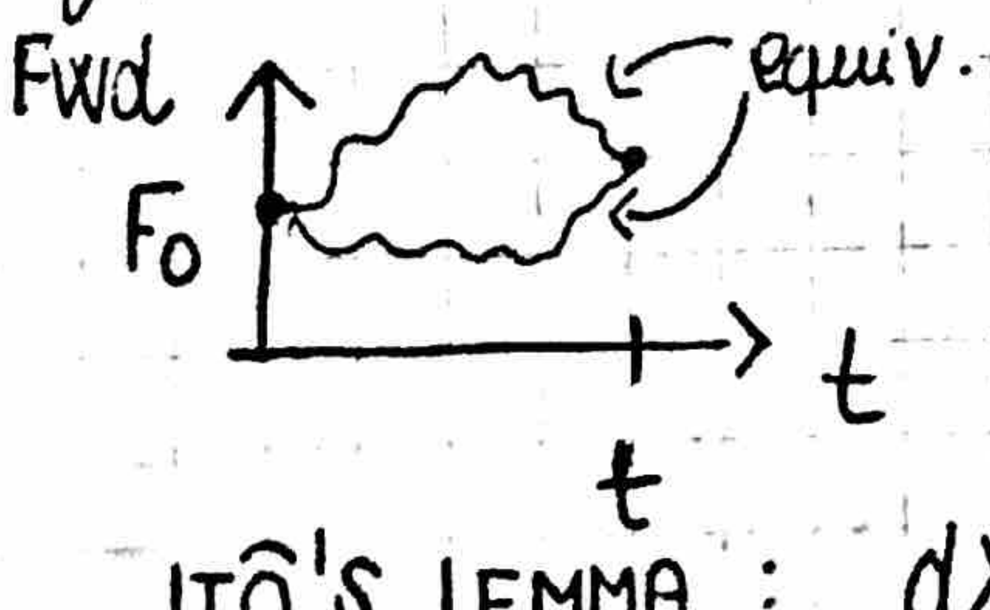
FORMULA ITO-CALCULUS 1-DIMENSIONAL

$$\begin{cases} f(s, X_s) \\ dX_s \end{cases} \Rightarrow \boxed{df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial x_s} dx_s + \frac{1}{2} \frac{\partial^2 f}{\partial x_s^2} \langle dX_s^2 \rangle}$$

differential of a stochastic quantity : $dF_s = F_s\,\sigma\, dW_s$

↑ F_s stochastic process, Wiener process

We are not interested in the path, but only in the final value.



ITÔ'S LEMMA : $dX_t = \mu(X_t,t)\,dt + \Sigma(X_t,t)\,dW_t$, $h(X_s,t)$

$$\Rightarrow dh(X_t,t) = \frac{\partial h(X,t)}{\partial t} dt + \frac{\partial h(X,t)}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 h}{\partial X_t^2} \langle dX_t^2 \rangle$$

$$\langle dX_t^2 \rangle = \Sigma^2(X_t,t)\,dt$$

$$\begin{cases} dF_s = F_s \sigma dW_s \\ F_0 \end{cases}$$

$$F(\overset{\text{time}}{s}, \underset{\text{parameter}}{t})$$

Let's consider $h = \ln(F_s)$:

$$d\ln(F_s) = 0 + \frac{1}{F_s} dF_s + \frac{1}{2}(-)\frac{1}{F_s^2}\langle dF_s^2\rangle =$$

$$= \left(\langle dF_s^2\rangle = F_s^2\sigma^2 ds\right) =$$

$$= \frac{1}{F_s} dF_s - \frac{1}{2}\frac{1}{\cancel{F_s^2}}\cancel{F_s^2}\sigma^2 ds =$$

$$= -\frac{1}{2}\sigma^2 ds + \sigma dW_s$$

$$\downarrow \int_{t_0}^{t}$$

Brownian motion starts from zero.

$$\ln F_t - \ln F_{t_0} = -\frac{1}{2}\sigma^2(t-t_0) + \sigma(W_t - \cancel{W_{t_0}})$$

$$\Rightarrow \boxed{\underset{F(t,t)}{F_t} = \underset{F(t_0,t)}{F_0} \exp\left\{-\frac{1}{2}\sigma^2(t-t_0) + \sigma W_t\right\}}$$

$$C(t_0,t) = B(t_0,t)\mathbb{E}_0[(F(t,t)-K)^+] = ?$$

$$F_t = F_0 \exp\left\{-\frac{1}{2}\sigma^2(t-t_0) + \sigma W_t\right\} = \quad W_t \sim N(0, t-t_0)$$

$$\overset{\text{"law"}}{=} F_0 \exp\left\{-\frac{1}{2}\sigma^2(t-t_0) \pm \sigma\sqrt{t-t_0}\, g\right\}, \quad g \sim N(0,1)$$

So it's positive. $F(t,t) > 0$!

Check if it's a martingale:

$$\mathbb{E}_0[F(t,t)] = \mathbb{E}_0\left[F(t_0,t)\, e^{-\frac{1}{2}\sigma^2(t-t_0) - \sigma\sqrt{t-t_0}\, g}\right] =$$

$$= F(t_0,t)\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}[g+\sigma\sqrt{t-t_0}]^2}}{\sqrt{2\pi}} dg = \left(g' = g + \sigma\sqrt{t-t_0}\right) =$$

$$= F(t_0,t)\underbrace{\int_{-\infty}^{+\infty} \frac{e^{-g'^2/2}}{\sqrt{2\pi}} dg'}_{1}$$

$$\Rightarrow \mathbb{E}_0[F(t,t)] = F_0 \quad \text{still martingale!}$$

$$\boxed{C(t_0,t) = B(t_0,t)\,\mathbb{E}_0\left[F(t,t)\,\mathbb{1}_{F(t,t)\geq K} - K\,\mathbb{1}_{F(t,t)\geq K}\right]}$$ price of a Call Option

- $\mathbb{E}_0[\mathbb{1}_{F(t,t)\geq K}]$?

$$F_0 e^{-\frac{1}{2}\sigma^2(t-t_0) - \sigma\sqrt{t-t_0}\,g} \geq K \leftrightarrow$$

$$\leftrightarrow -\frac{1}{2}\sigma^2(t-t_0) - \sigma\sqrt{t-t_0}\,g \geq \ln\frac{K}{F_0} \leftrightarrow$$

$$\leftrightarrow g \leq \left\{ \ln\frac{F_0}{K} - \frac{1}{2}\sigma^2(t-t_0)\right\}\cdot\frac{1}{\sigma\sqrt{t-t_0}} := d_2$$

$$\mathbb{E}_0[\mathbb{1}_{F(t,t)\geq K}] = \int_{-\infty}^{+\infty} dg\,\frac{e^{-g^2/2}}{\sqrt{2\pi}}\,\mathbb{1}_{g\leq d_2} = N(d_2)$$ CDF of std gaussian

- $\mathbb{E}_0[F_t\,\mathbb{1}_{F(t,t)\geq K}]$?

$$= F_0\,\mathbb{E}_0\left[e^{-\frac{1}{2}\sigma^2(t-t_0) - \sigma\sqrt{t-t_0}\,g}\,\mathbb{1}_{g\leq d_2}\right] =$$

$$= F_0\int_{-\infty}^{d_2} dg\,\frac{e^{-(g^2/2 + \frac{1}{2}\sigma^2(t-t_0) + \sigma\sqrt{t-t_0}\,g)}}{\sqrt{2\pi}} =$$

$$= F_0\int_{-\infty}^{d_2} dg\,\frac{e^{-\frac{1}{2}(g+\sigma\sqrt{t-t_0})^2}}{\sqrt{2\pi}} = \left(g' = g + \sigma\sqrt{t-t_0}\right) =$$

$$= F_0\int_{-\infty}^{d_2+\sigma\sqrt{t-t_0}} dg'\,\frac{e^{-\frac{g'^2}{2}}}{\sqrt{2\pi}} = F_0 N(d_1)$$

$$\Rightarrow \boxed{C(t_0,t) = B(t_0,t)\left\{F_0 N(d_1) - K N(d_2)\right\}}$$

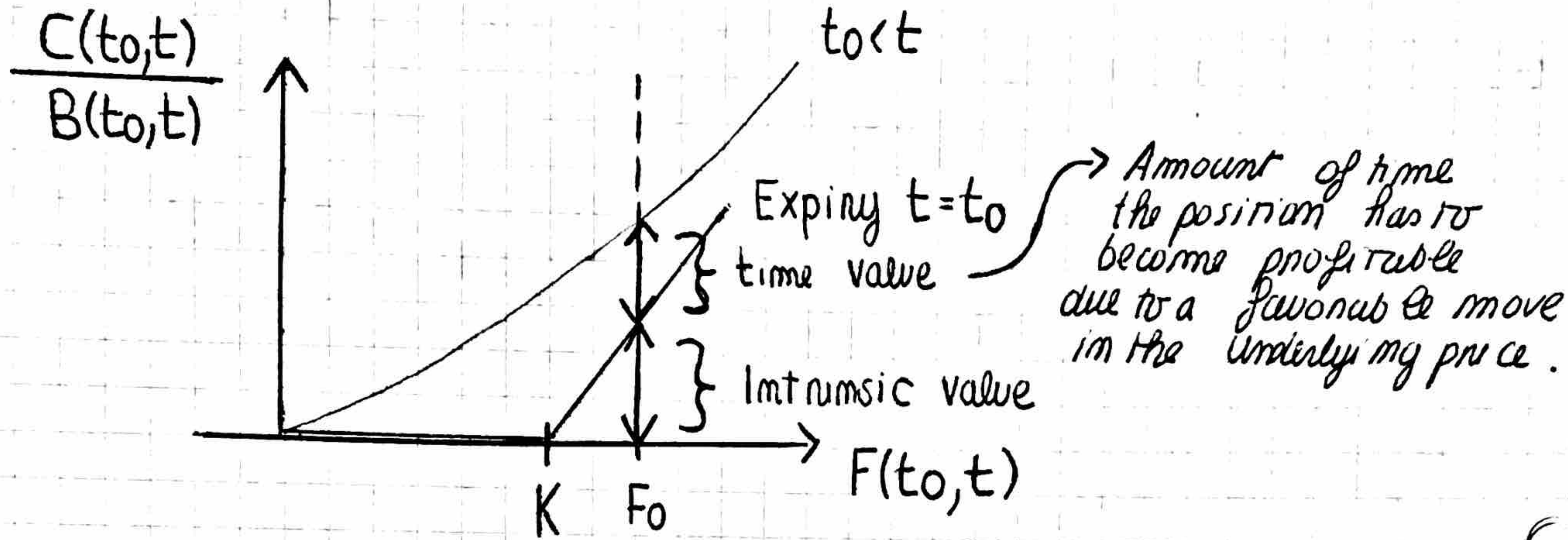
$$d_1 = \frac{1}{\sigma\sqrt{t-t_0}}\ln\frac{F(t_0,t)}{K} + \frac{1}{2}\sigma\sqrt{t-t_0}$$

$$d_2 = \frac{1}{\sigma\sqrt{t-t_0}}\ln\frac{F(t_0,t)}{K} - \frac{1}{2}\sigma\sqrt{t-t_0}$$

For the Put, use PUT-CALL PARITY:

$$P(t_0,t) = B(t_0,t)\{K\,N(-d_2) - F_0\,N(-d_1)\}$$

DECOMPOSITION IN INTRINSIC VALUE AND TIME VALUE:



INTRINSIC VALUE: $S_t - K$ for a Call (only in the money options $\Rightarrow S_t > K$)
↳ only for ITM option.

TIME VALUE: Forward not deterministic ⇒ possibility of the option to have higher value.
Greater $t-t_0$, greater time value.

ANSWER (1): → hypotesis: • No transaction costs • participants s.t. the same tax rate and can borrow money at the same r.f. rate as they can lend, take adv. of arbitrage as soon as possible (so, no arbitrage opportunities).

RELATION BETWEEN THE FORWARD AND THE UNDERLYING
(depending on the chosen model).

(ex. GOOGLE, ENI: 2 for year ...)

- Equity stocks with NO DIVIDENDS and constant instantaneous interest rate r (continuous compounding)

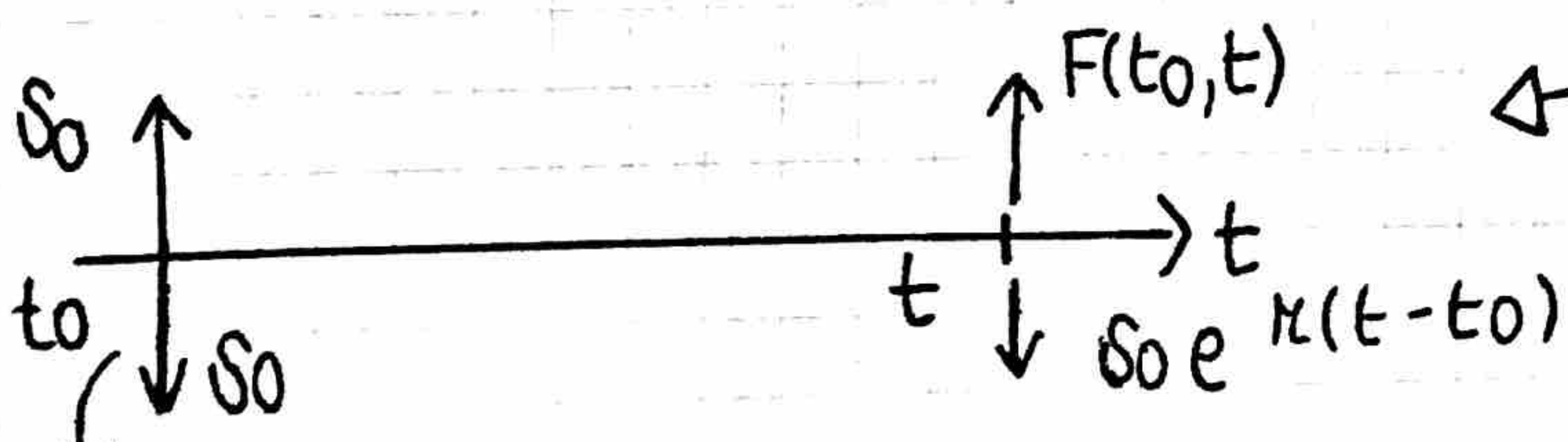
⇓

BLACK & SCHOLES

$$B(t_0,t) = e^{-r(t-t_0)}$$

$$\Rightarrow \boxed{F(t_0,t) = S_0\,e^{r(t-t_0)}}$$

Related to the No-arbitrage theory.



I spend S_0 at time t_0 and I buy the stock: ↓ S_0, borrowing from a lender the needed quantity: ↑ S_0.

I will receive $F(t_0,t)$ in t and I'll have to pay $S_0 e^{r(t-t_0)}$ for having borrowed S_0.

If we suppose $F(t_0,t) > S_0 e^{r(t-t_0)}$
or $F(t_0,t) < S_0 e^{r(t-t_0)}$ → ARBITRAGE.

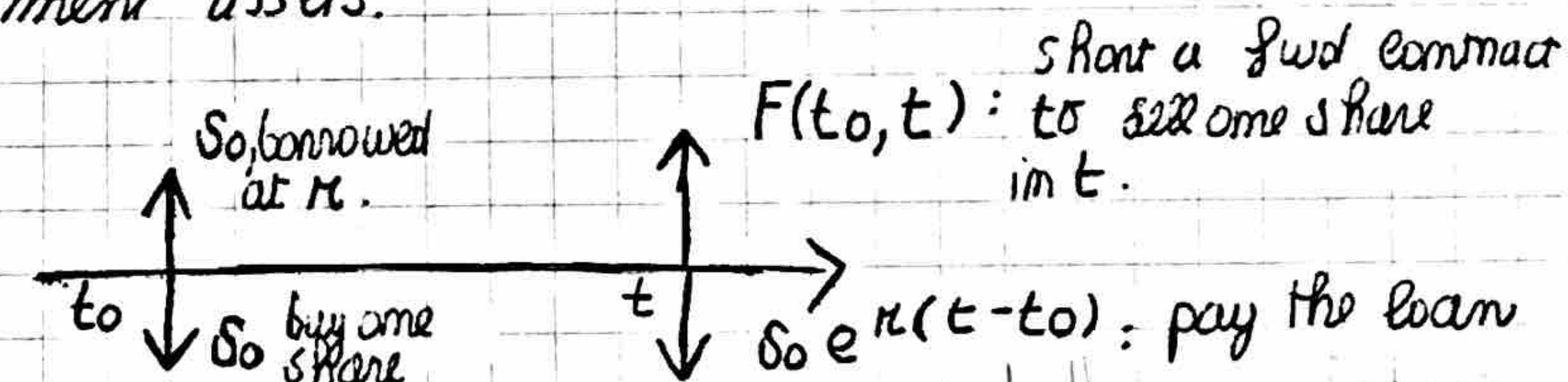
Due to arbitrage, the only possible strategy is the REPLICATION STRATEGY (STATICAL REPLICATION, OR HEDGE):
$F(t_0,t) = S_0 e^{r(t-t_0)}$ not to have arbitrage.

HULL: FORWARD PRICE FOR AN INVESTMENT ASSET: IMPLICATION OF ABSENCE OF ARBITRAGE IN A SIMPLE MODEL [5.4]
↳ Fwd contract on individual STOCK.

The easiest forward contract to value is one written on an investment asset that provides the holder with no income. Non-dividend-paying stocks and zero-coupon bonds are examples of such investment assets.

$t-t_0$, S_0, r, $F(t_0,t)$

$\boxed{F(t_0,t) > S_0}$

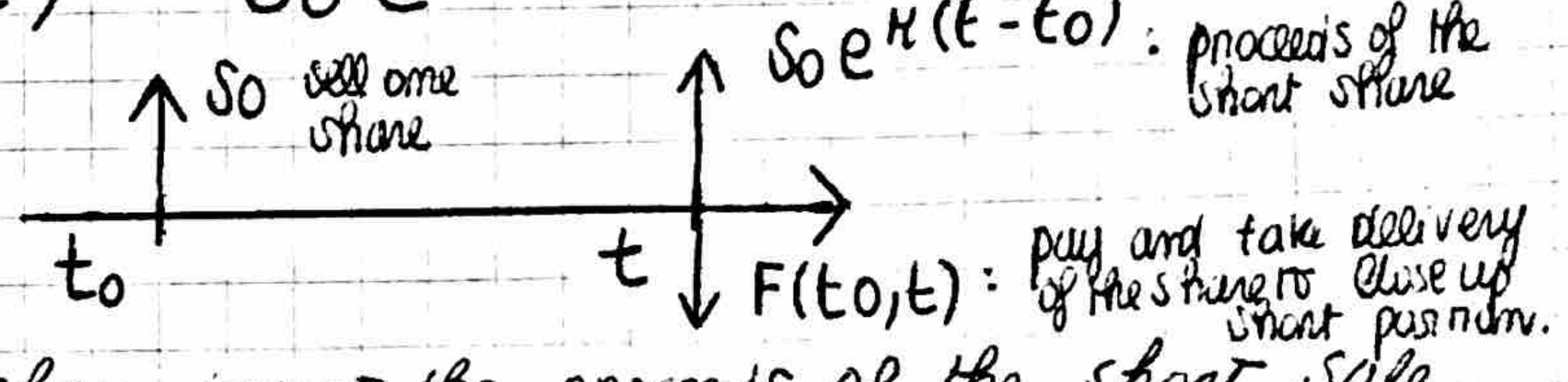


An arbitrageur can borrow S_0 at the risk free interest rate r, buy one share and short a forward contract to sell one share in $t-t_0$.
At the end of the TTM, the arbitrageur delivers the share and receives $F(t_0,t)$.

The sum of the money required to pay off the loan is:
$$S_0 e^{r(t-t_0)}.$$

By following this strategy, the arbitrageur locks in a profit of:
$$F(t_0,t) - S_0 e^{r(t-t_0)}$$
at the end of the TTM.

$\boxed{F(t_0,t) < S_0}$



An arbitrageur can short one share, invest the proceeds of the short sale at r for $t-t_0$, and take a long position in a $t-t_0$ fwd contract.
The proceeds of the short share grow to $S_0 e^{r(t-t_0)}$ in $t-t_0$.
At the end of the TTM, the arbitrageur pays $F(t_0,t)$, takes delivery of the share under the terms of the fwd contract and uses it to close up the short position.
At the end of the TTM, there is a net gain of:
$$S_0 e^{r(t-t_0)} - F(t_0,t).$$

Arbitrage opportunities when fwd price is out of line with spot price for asset providing no income.

$\boxed{F(t_0,t) > S_0}$

IN t_0:

→ Borrow S_0 at r for $(t-t_0)$.
→ Buy one unit of asset
→ Enter into a fwd contract to sell asset in $t-t_0$ for $F(t_0,t)$.

IN t:
→ Sell asset for $F(t_0,t)$.
→ Use $S_0 e^{r(t-t_0)}$ to repay loan with interest

Profit realized: $F(t_0,t) - S_0 e^{r(t-t_0)}$

$\boxed{F(t_0,t) < S_0}$

IN t_0:

→ Short one unit of asset to realize S_0.
→ Invest S_0 at r for $t-t_0$
→ Enter into a fwd contract to buy asset in $t-t_0$ for $F(t_0,t)$

IN t:
→ Buy asset for $F(t_0,t)$.
→ Close short position.
→ Receive $S_0 e^{r(t-t_0)}$ from investment

Profit realized: $S_0 e^{r(t-t_0)} - F(t_0,t)$

$$\Downarrow$$

$$F(t_0,t) = S_0 e^{r(t-t_0)}$$

- KNOWN YIELD: dividend yield model d (or a foreign interest rate) and a costant instantaneous interest rate r.

  Situation where the asset underlying a forward contract provides a known yield (the income is known when expressed as a percentage of the asset's price at the time the income is paid).

$$\Rightarrow \boxed{F(t_0,t) = S_0 e^{(r-d)(t-t_0)}}$$

The most liquid mkt is the mkt of the SHARES, not the forwards' one.
A share gives a return to the owner, usually once a year, this return is called DIVIDEND.
Not all companies pay dividends (ex. Google doesn't), ENI pays twice a year.
If dividends are paid continuously, they are called DIVIDEND YIELD.

GARMAN - KOHLHAGEN FORMULA

Garman - Kohlhagen formula holds for a European Call option with an underlying with interest rates, continuous dividends and volatility deterministic function of time. In particular, the formula is exactly the same considering their average value over the time-to-maturity instead of the constant values in the "standard formula".

The dynamic of the underlying is a Geometric Brownian Motion:

$$\begin{cases} dS_t = S_t\left[(r(t)-d(t))\,dt + \sigma(t)\,dW_t\right] \\ S_0 \end{cases}$$

$$d\ln(S_t) = \frac{1}{S_t}\,dS_t - \frac{1}{2}\frac{1}{S_t^2}\langle dS_t^2\rangle =$$

$$= \left[(r(t)-d(t))dt + \sigma(t)dW_t\right] - \frac{1}{2}\sigma(t)^2 dt =$$

$$= \left((r(t)-d(t)) - \frac{1}{2}\sigma(t)^2\right)dt + \sigma(t)\,dW_t$$

$$\Big\downarrow \int_{t_0}^{t}$$

$$\int_{t_0}^{t} d\ln(S_u) = \int_{t_0}^{t}\left[(r(u)-d(u)) - \frac{1}{2}\sigma(u)^2\right]du + \int_{t_0}^{t}\sigma(u)\,dW_u$$

Defining:

$$D(t) = \frac{1}{t-t_0}\int_{t}^{t_0} d(u)\,du, \quad R(t) = \frac{1}{t-t_0}\int_{t}^{t_0} r(u)\,du,$$

$$\Sigma^2(t) = \frac{1}{t-t_0}\int_{t}^{t_0}\sigma(u)^2\,du$$

we obtain:

$$\ln(S_t) - \ln(S_{t_0}) = \left(R(t) - D(t) - \frac{1}{2}\Sigma^2(t)\right)(t-t_0) + \underbrace{\int_{t_0}^{t}\sigma(u)\,dW_u}_{*}$$

$*$: stochastic integral with a deterministic integrand, so it's a centred Gaussian process.

We can use Itō's isometry:

$$Var\left(\int_{t_0}^{t}\sigma(u)dW_u\right) = \mathbb{E}\left[\left(\int_{t_0}^{t}\sigma(u)dW_u\right)^2\right] = \mathbb{E}\left[\left(\int_{t_0}^{t}\sigma(u)^2du\right)\right] = (t-t_0)\Sigma(t)^2$$

$$\int_{t_0}^{t} \sigma(u)\,dW_u \sim N(0,(t-t_0)\Sigma^2(t))$$

$$\Rightarrow S_t = S_0 \exp\left[(t-t_0)\left(R(t)-D(t)-\frac{\Sigma^2(t)}{2}\right) + \int_{t_0}^{t} \sigma(u)\,dW_u\right]$$

$$\Rightarrow S_t \overset{\text{"law"}}{\sim} S_0 \exp\left[(t-t_0)\left(R(t)-D(t)-\frac{\Sigma^2(t)}{2}\right) \pm \sqrt{t-t_0}\,\Sigma(t)\,g\right]$$

$$g \sim N(0,1)$$

$$C(t_0,t) = B(t_0,t)\,\mathbb{E}_0\left[(S_t-K)^+\right] =$$

$$= B(t_0,t)\left(\mathbb{E}\left[S_t \mathbb{1}_{S_t \geq K}\right] - K\,\mathbb{E}\left[\mathbb{1}_{S_t \geq K}\right]\right)$$

- $\mathbb{E}\left[\mathbb{1}_{S_t \geq K}\right] = ?$

$$S_t \geq K \leftrightarrow S_0 \exp\left[(t-t_0)\left(R(t)-D(t)-\frac{\Sigma^2(t)}{2}\right) - \sqrt{t-t_0}\,\Sigma(t)\,g\right] \geq K$$

$$\leftrightarrow \left[(t-t_0)\left(R(t)-D(t)-\frac{\Sigma^2(t)}{2}\right) - \sqrt{t-t_0}\,\Sigma(t)\,g\right] \geq \ln\left(\frac{K}{S_0}\right)$$

$$\leftrightarrow g \leq \left[\ln\left(\frac{S_0}{K}\right) - \frac{1}{2}\Sigma^2(t)\,(t-t_0) + (R(t)-D(t))(t-t_0)\right]\cdot\frac{1}{\Sigma(t)\sqrt{t-t_0}}$$

$$:= d_2$$

$$\mathbb{E}\left[\mathbb{1}_{S_t \geq K}\right] = \int_{-\infty}^{+\infty} dg\,\frac{e^{-g^2/2}}{\sqrt{2\pi}}\,\mathbb{1}_{g \leq d_2} = N(d_2)$$

- $\mathbb{E}\left[S_t \mathbb{1}_{S_t \geq K}\right] = ?$

$$\mathbb{E}\left[S_t \mathbb{1}_{S_t \geq K}\right] = S_0\,\mathbb{E}\left[\exp\left[(t-t_0)\left(R(t)-D(t)-\frac{\Sigma^2(t)}{2}\right) - \sqrt{t-t_0}\,\Sigma(t)\,g\right]\mathbb{1}_{g \leq d_2}\right] =$$

$$= S_0 \exp\left[(t-t_0)(R(t)-D(t))\right]\int_{-\infty}^{d_2} \frac{dg}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\Sigma^2(t)(t-t_0) - \Sigma(t)\sqrt{t-t_0}\,g - \frac{g^2}{2}\right] =$$

$$= S_0 \exp\left[(t-t_0)(R(t)-D(t))\right]\int_{-\infty}^{d_2} \frac{dg}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(g+\Sigma(t)\sqrt{t-t_0}\right)^2\right] =$$

$$= \left(g' = g + \Sigma(t)\sqrt{t-t_0}\right) = S_0 \exp\left[(t-t_0)(R(t)-D(t))\right]\int_{-\infty}^{d_2+\Sigma(t)\sqrt{t-t_0}} dg'\,\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{g'^2}{2}\right] =$$

$$= S_0 \exp\left[(R(t)-D(t))(t-t_0)\right] N(d_1)$$

$$C(t_0,t) = B(t_0,t)\left\{ S_0 e^{(R(t)-D(t))(t-t_0)} N(d_1) - K N(d_2) \right\}$$

$$d_{1,2} = \frac{1}{\Sigma(t)\sqrt{t-t_0}}\left[\ln\left(\frac{S_0}{K}\right) + (R(t)-D(t))(t-t_0) \pm \frac{1}{2}\Sigma^2(t)(t-t_0)\right]$$

HULL:

FUTURES OPTIONS vs. SPOT OPTIONS

FUTURES OPTIONS: Exercise of the option gives the holder a position in a futures contract. Right, but not the obligation to enter into a futures contract at a certain futures price by a certain date.

SPOT OPTIONS: provide the holder the right to buy or sell a certain asset by a certain date for a certain price. When the options are exercised, the sale or purchase of the asset at the agreed-on price takes place immediately.

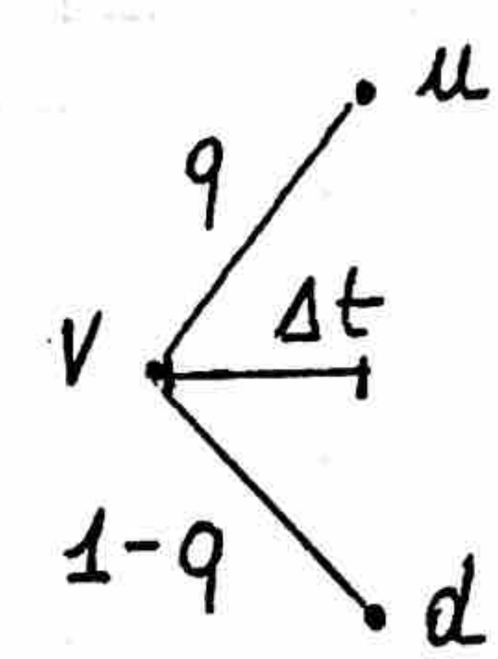
European futures options and European spot options are equivalent when the option contract matures at the same time as the futures contract.

3. CRR TREE AND MONTE CARLO APPROACHES FOR A PLAIN VANILLA.

A useful technique for pricing an option involves constructing a BINOMIAL TREE. This is a diagram representing different possible paths that might be followed by the stock price over the life of an option.
The underlying assumption is that the stock price follows a RANDOM WALK. In each time step, it has a certain probability of moving up or down by a certain percentage amount.

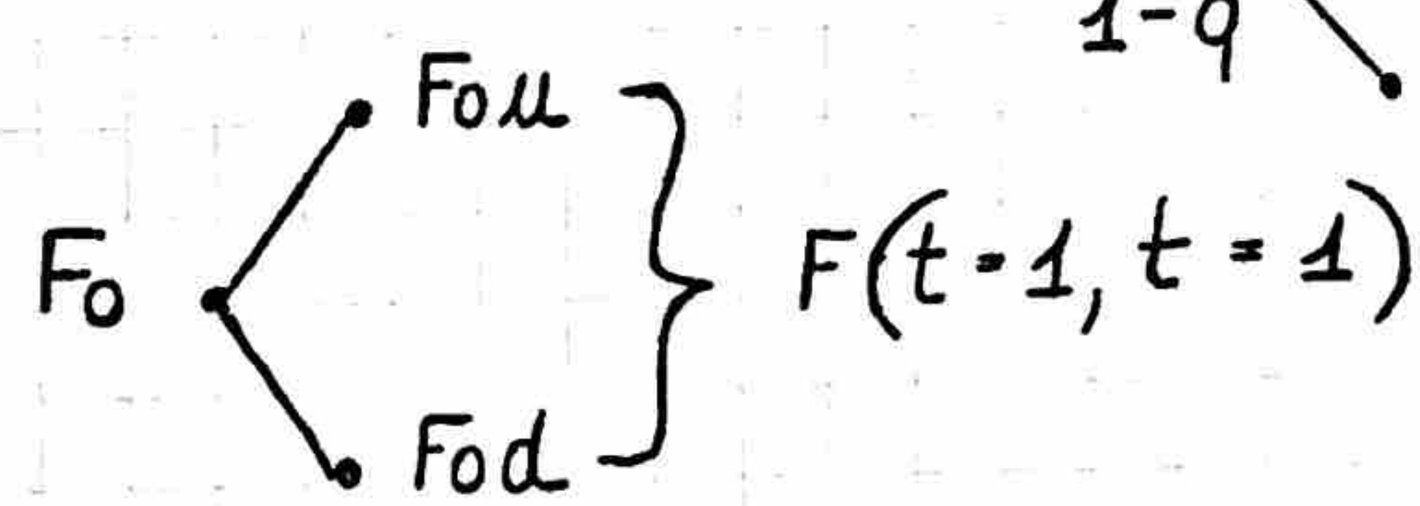
Approach by Cox, Ross and Rubinstein.

1 TIME STEP



Forward described by a random variable $V = u, d$

$$F(t=1, t=1) = F_0 \cdot V$$



Conditions to respect:

1. The forward has a process that should respect the no-arbitrage theory:

$$F_0 = \mathbb{E}_0[F(t=1,t=1)] = F_0 \, \mathbb{E}[V]$$

$$= \mathbb{E}_0[F_0 u q + F_0 d (1-q)] =$$

$$= F_0 \underbrace{\{ q u + (1-q) d \}}_{\mathbb{E}[V]}$$

In order to be a MARTINGALE $\Rightarrow \mathbb{E}[V] = 1$

$$\Longleftrightarrow \quad q u + (1-q) d = 1$$

$$\Longleftrightarrow \quad q = \frac{1-d}{u-d}$$

2 free parameters: we want just one parameter to connect with σ.

2. Trees are very useful tools if they are recombining.
(up-down = down-up)

We need a regular grid to get a recombining tree.

One way to get a recombining tree:

- We consider r.v. $\eta \equiv \log V = \pm \Delta x$ (regular grid)

$\Rightarrow$ $u = e^{\Delta x}$ & $d = 1/u$ ($d = e^{-\Delta x}$)

$\Delta x \to 0$, Taylor 2nd order expansion

$$q = \frac{1-d}{u-d} = \frac{1-e^{-\Delta x}}{e^{\Delta x} - e^{-\Delta x}} = \frac{1-(1-\Delta x + \frac{1}{2}\Delta x^2) + \mathcal{O}(\Delta x^3)}{[1+\Delta x + \frac{1}{2}\Delta x^2] - [1 - \Delta x + \frac{1}{2}\Delta x^2] + \mathcal{O}(\Delta x^3)} =$$

$$= \frac{\Delta x [1 - \frac{1}{2}\Delta x] + \mathcal{O}(\Delta x^3)}{2\Delta x(1+\mathcal{O}(\Delta x^2))} =$$

$$= \frac{1}{2}\left[1 - \frac{\Delta x}{2}\right] + \mathcal{O}(\Delta x^2)$$

We want to relate q, u, d with σ:

$$\begin{cases} \mathbb{E}[\eta] = q\Delta x + (1-q)(-\Delta x) = \Delta x[2q-1] \\ Var[\eta] \overset{*}{=} \mathbb{E}[\eta^2] - \mathbb{E}[\eta]^2 = \Delta x^2\{1-(2q-1)^2\} = \Delta x^2 4q(1-q) \end{cases}$$

$$* \ \mathbb{E}[\eta^2] = q\Delta x^2 + (1-q)\Delta x^2 = \Delta x^2$$

$$q = \frac{1}{2}\left[1 - \frac{\Delta x}{2}\right] + \mathcal{O}(\Delta x^2)$$

$$\Rightarrow \begin{cases} \mathbb{E}[\eta] = \Delta x\left[1 - \frac{\Delta x}{2} - 1\right] + \mathcal{O}(\Delta x^3) \\ Var[\eta] = \Delta x^2 + \mathcal{O}(\Delta x^4) \end{cases}$$

Relation with volatility σ (the only parameter of Black):

- We consider the log variation of the forward price:

$$Var\left[\log \frac{F(t,t)}{F(t_0,t)}\right] \overset{Black}{=} \sigma^2 \Delta t = Var[\log V] \overset{CRR}{=} \Delta x^2$$

log in order to get a normal distribution

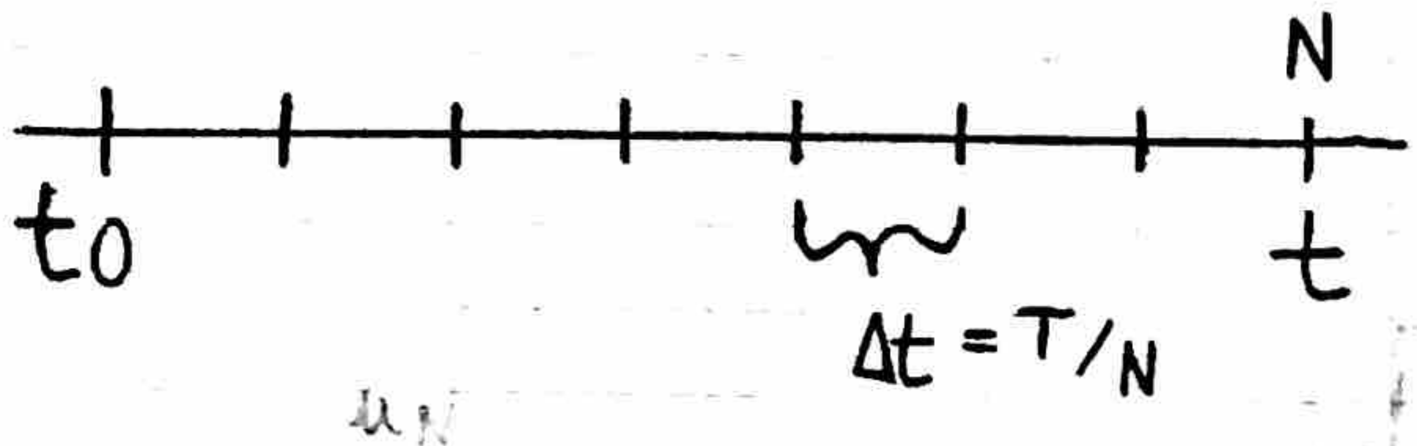
$\Rightarrow$ $\Delta x = \sigma\sqrt{\Delta t}$

$\Rightarrow$ $u = e^{\sigma\sqrt{\Delta t}}$

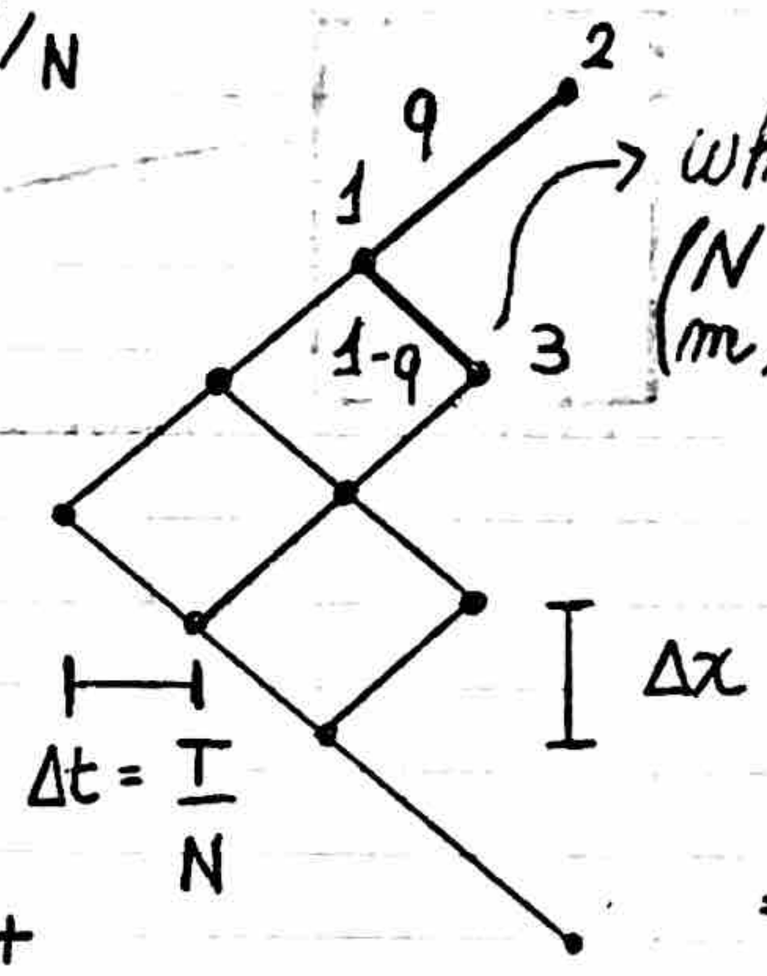
i.e. volatility of a fwd (of a stock) s.t. $\sigma\sqrt{\Delta t}$ is the standard deviation of the fwd-price return in a short period of time Δt.

TREE CONSTRUCTION IN N TIME STEPS

Binomial trees were introduced to value either European or American options. The B&S formula provides analytical valuations for European options but there are no analytic valuations for American ones. Binomial trees are therefore most useful to value these types of options.
We divide T into N intervals of length $\Delta t = T/N$
$T := t - t_0$



$T = N\Delta t$



$\rightarrow$ whose probability $\binom{N}{m} q^m (1-q)^{N-m}$

in the continuous limit becomes:

$$\xrightarrow{CLT} \frac{e^{-\frac{\left(\frac{1}{N}\sum_i \eta_i - \bar{\eta}\right)^2 N}{2\,\mathrm{Var}[\eta]}}}{\sqrt{2\pi\sigma^2 T}} = \frac{e^{-\frac{(x+\sigma^2 T/2)^2}{2\sigma^2 T}}}{\sqrt{2\pi\sigma^2 T}}$$

Relating to the distr. in Black model.

From the payoff at last leafs 2,3, we derive the payoff at leaf 1:

$$q B(t_{N-1}, t_N)[F_0 u^N - K]^+ +$$
$$+(1-q) B(t_{N-1}, t_N)[F_0 u^{N-2} - K]^+$$

(if we have an american option, we have to compare European payoff vs. payoff of early exercise).

up N times $\rightarrow u^N$
up $N-1$ times (down 1 time) $\rightarrow u^{N-2}$

The binomial tree valuation approach involves dividing the life of the option into a large number of small time intervals of length Δt. It assumes that in each time interval the price of the underlying asset moves from its initial value of S to one of the two new values, Su and Sd.

RISK-NEUTRAL VALUATION

The Risk-Neutral valuation principle states that an option (or other derivative) can be valued on the assumption that the world is risk neutral. This means that for valuation purposes, we can use the following procedure:

1. Assume that the expected return from all traded assets is the risk-free interest rate.

2. Value payoffs from the derivative by calculating their expected values and discounting at the risk-free interest rate.

Options are evaluated by starting at the end of the tree (time T) and working backward.
The value of the option is known at time T.
For example, a put option is worth $\max(K - S_T, 0)$ and a call option is worth $\max(S_T - K, 0)$, where S_T is the asset price at time T and K is the strike price.

Because a risk-neutral world is being assumed, the value at each node at time $T-\Delta t$ can be calculated as the expected value at time T discounted at rate r for a time period Δt.

Similarly, the value at each node at time $T-2\Delta t$ can be calculated as the expected value at time $T-\Delta t$ discounted for a time period Δt at rate r, and so on.

If the option is American, it is necessary to check at each node to see whether early exercise is preferable to holding the option for a further time period Δt.

Eventually, by working back through all the nodes, we are able to obtain the value of the option at time zero.

MONTE CARLO SIMULATION when we don't have closed formulas

When used to value an option, Monte Carlo simulation uses the risk-neutral valuation result.

We sample paths to obtain the expected payoff in a risk-neutral world and then discount this payoff at the risk-free rate.

Consider a derivative dependent on a single market variable S that provides a payoff at time T. Assuming that interest rates are constant, we can value the derivative as follows:

1. Sample a random path for S in a risk-neutral world.
2. Calculate the payoff from the derivative.
3. Repeat steps 1 and 2 to get many sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
5. Discount this expected payoff at the risk-free rate to get an estimate of the value of the derivative.

$$F(t,t) \overset{\text{"law"}}{\sim} F_0 \exp\left[-\frac{\sigma^2}{2}(t-t_0) + \sigma\sqrt{t-t_0}\, g\right]$$

We use the equivalence in law and then we simulate directly the r.v.

$$C(t_0,t) = B(t_0,t)\, \mathbb{E}\left[(F(t,t)-K)^+\right]$$

We simulate a vector $\underline{g}$, $\{g_i\}_{i=1..M}$ st.n. r.v., i.i.d.

$$\frac{C(t_0,t)}{B(t_0,t)} = \frac{1}{M}\sum_{i=1}^{M}\left\{F_0 \exp\left[-\frac{\sigma^2}{2}(t-t_0) + \sigma\sqrt{t-t_0}\, g_i\right] - K\right\}^+$$

4. ANTITHETIC VARIABLES IN A MONTE CARLO

↳ Variance reduction procedure that can lead to dramatic savings in computational time.

If the stochastic processes for the variables underlying a derivative are simulated as previously indicated, a very large number of trials is usually necessary to estimate the value of the derivative with reasonable accuracy. This is very expensive in terms of computational time.

In the ANTITHETIC VARIABLES TECHNIQUE, a simulation trial involves calculating two values of the derivative.

The first value f_1 is calculated in the usual way; the second value f_2 is calculated by changing the sign of all the random samples from standard normal distributions. (If g is a sample used to calculate f_1, then $-g$ is the corresponding sample used to calculate f_2).

The sample value of the derivative calculated from a simulation trial is the average of f_1 and f_2.

This works well because one value is above the true value, the other tends to be below, and vice versa.

$$\bar{f} = \frac{f_1 + f_2}{2}$$

The final estimate of the value of the derivative is the average of the $\bar{f}$'s.

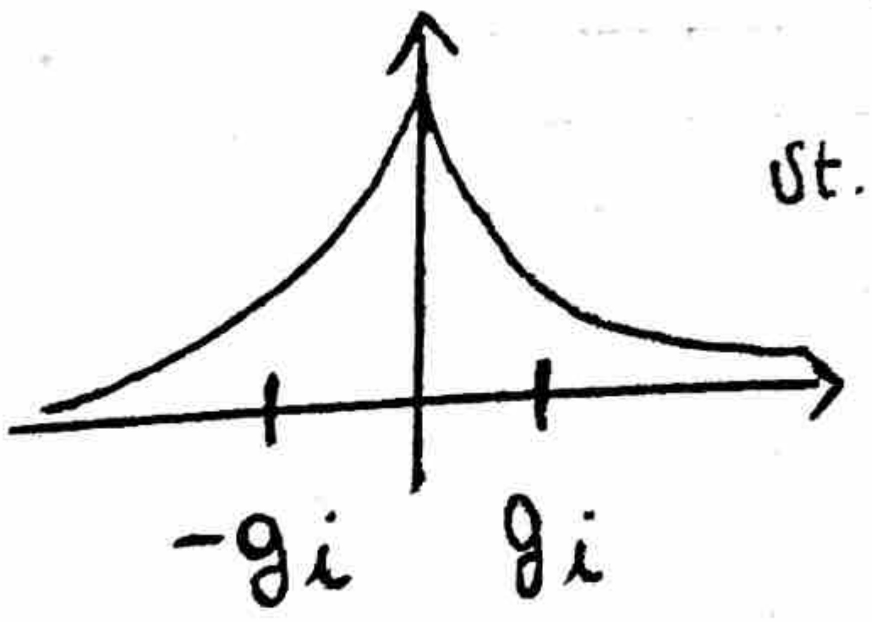
If $\bar{\omega}$ is the standard deviation of the $\bar{f}$'s, and M is the number of simulation trials (i.e., the number of pairs of values calculated), then the standard error of the estimate is:

$$\bar{\omega} / \sqrt{M}$$

This is usually much less than the standard error calculated using $2M$ random trials.

$\{g_i\}_{i=1:M}$ iid gaussian

$$F_t = F_0 \exp\left\{-\frac{1}{2}\sigma^2 (t-t_0) \pm \sigma\sqrt{t-t_0}\, g\right\}$$



st. n. density ⇨ 2M RVs with M drawings!

symmetric drawing

ASS. 1 : As error for the MC method, we can consider the unbiased standard deviation of the MC price. This error rescales as $1/\sqrt{M}$.

We have $\{O_i\}_{i=1:M}$, a set of r.v.s iid (related to the final payoff of our option) with μ and σ^2.

(1)

→ To estimate the average : $\hat{O} = \frac{1}{M} \sum_{i=1}^{M} O_i$,

unbiased : $\mathbb{E}[\hat{O}] = \frac{1}{M} \sum_{i=1}^{M} \mathbb{E}[O_i] = \mu$

This is our price according to MC.

→ We want to estimate $Var[\hat{O}]$, estimator of the variance of the estimator of the mean.

$$Var[\hat{O}] = \mathbb{E}\left[\frac{1}{M^2} \sum_{i,j=1}^{M} (O_i - \mu)(O_j - \mu)\right] =$$

$$= \mathbb{E}\left[\frac{1}{M^2} \sum_{i=1}^{M} (O_i - \mu)^2\right] = \frac{1}{M^2} \sum_{i=1}^{M} \mathbb{E}[O_i - \mu]^2 =$$

$$= \frac{\sigma^2}{M}$$ ⟶ Variance of each iid r.v.

Estimator of σ^2:

i) if μ is known : $\hat{S}_1 = \frac{1}{M} \sum_{i=1}^{M} (O_i - \mu)^2$

$\hat{S}$, estimator of $Var[\hat{O}]$, becomes:

$$\frac{\hat{S}_1}{M} = \frac{1}{M^2} \sum_{i=1}^{M} (O_i - \mu)^2$$

ii) if μ is not known: $\hat{S}_2 = \frac{1}{M-1} \sum_{i=1}^{M} (O_i - \hat{O})^2$

$$\hat{S} = \frac{1}{M(M-1)} \sum_{i=1}^{M} (O_i - \hat{O})^2$$

(2) Replication m times of $\hat{O}_j$, $j = 1, \dots, m$, hence of $\{O_i\}_{i=1:M}$ (much more computations).