RISK MANAGEMENT ②

1. BASEL & SOLVENCY ACCORDS : A HISTORICAL PERSPECTIVE.

REGULATORY RM FRAMEWORK < BASEL / SOLVENCY ACCORDS

We'll try to understand why risk-managements is so important in the bank, since it has to decide the EQUITY, called the REGULATORY CAPITAL for the bank.

↓ the minimum capital required to run the bank.

FINANCIAL RISK : event or action that may adversely affect an organization's ability to achieve its objectives, generate UL.

The problem is not to have loss in general, there are expected cash flows.
A risk manager looks at the UNEXPECTED cash flows.

Some factors that have contributed to an increased demand for RM:

→ For example, in the 70's, the exchange rates were fixed, because in America it was possible to exchange dollars with the correspondent gold whenever you wanted (Bretton Woods).
But for Vietnam war the USA began to print dollars (to financiate the war) → the exchange rates of dollars began to fluctuate. Abolition of the Bretton-Woods system of fixed exchange rates.

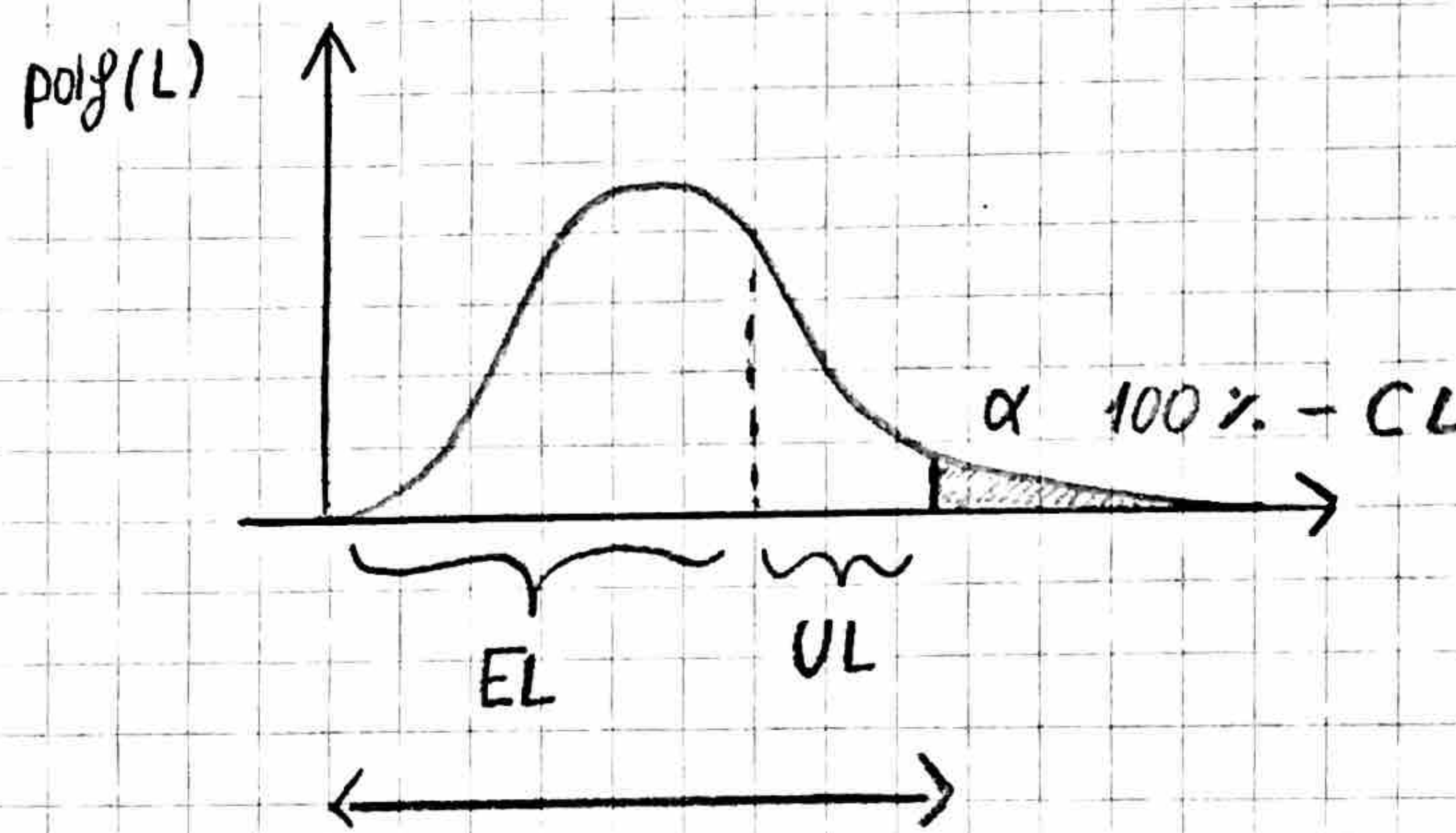
→ Before the 80's, banks were strictly regulated by states, but then there was a world wide deregulation
↳ public banks not owned by the government
(some voices at book value) → at the value at which you bought
↳ otherwise in MtM with real mkt value.

→ Mkt-oriented accounting practice (IASB - Intern. Accounting Standard Boards).

→ The new regulatory RM framework of Basel and Solvency Accords.

?

ROLE OF CAPITAL, RELATION WITH P&L



$VaR_{99\%}$: lowest loss that corresponds to 1% worst scenario

quantile of the distribution of loss.

BANKING REGULATORY RISK MANAGEMENT FRAMEWORK

Basel Accords: made by BIS, where the share holders are the banks (Basel Committee of Banking Supervision at BIS)

BASEL I 1988 → Credit risk
International minimum capital standard

AMENDMENT OF BASEL I 1996 → standardized model for mkt risk (equity, fixed income products, derivatives, exchange rates, interest rate risk).

BASEL II 2004 → PILLAR I: risk-sensitive min. capital requirement

PILLAR II: supervisor's review of bank's capital adequacy

PILLAR III: trasparency of balance sheet

BASEL 2.5 2009 → IRC (Incremental Risk Charge), stressed VaR

BASEL 3 2010 - 2018 → New capital requirement, liquidity.

3types of risk: regulated in Basel I, [illegible] Basel II

Credit risk — 1988, 2004 → BANK CAPITAL

Mkt Risk — 1996, 2004, 2009 introduced with amendment, revised in Basel 2 and 2.5 → BANK CAPITAL

Operational risk — introduced in Basel 2, 2004 → BANK CAPITAL

human error

FAT FINGER
LEGAL LOSSES
(not enough info)

2. BASEL 1 VS. BASEL 2: MAIN DIFFERENCES. STANDARD VS. INTERNAL RATING BASIS (IRB) APPROACHES.

BASEL I (1988)

The main role of the RM is to relate the P&L with the equity (regulative capital).

↳ minimum amount of equity

In Basel I, the formula was based on RISK WEIGHTED ASSETS:

ammontare crediti

$$RC = 8\% \underbrace{\sum_i RW_i A_i}_{RWA}$$

(divided into classes of assets)

i is the index of the obligor in my r.p.

Risk weighted assets divided in four crude categories (goverments, banks, secured loans and others).

$RW_1 = 0\%$ (risk free) → CASH, CENTRAL BANKS, BONDS OF OECD countries (goverment bonds)

$RW_2 = 20\%$ → EIB (European International Bank)
↓
Supranational banks
Public entities of OECD

$RW_3 = 50\%$ → mortgages (because I can take the collateral if they don't pay me)

$RW_4 = 100\%$ → others

LIMITS OF BASEL I:

- Only credit risk, but I can have other types of losses
- No portfolio diversification sensitivity
- Hedging not considered (ex. derivative protection)
- big companies are considered equally risky as small ones (same risk weight to private companies regardless of credit standing)

AMENDMEND (1996): mkt risk

BASEL 2 (2004) RISKS { Credit; Mkt (as in Amendment); Operational: risk of losses resulting from inadequate or failed internal processes

In all countries the regulation is based on the IASB for mkt-oriented accounting.

For RC, bank can choose between:

1) STANDARD MODEL

Mechanism similar to Basel I: for every risky asset of my portfolio there is an ADD-ON (weight) (there's a table, provided by the supervision) (good for small banks)

2) IRB (Internal Rating Basis)

Connect RC with the VaR:

$\alpha = 99\%$, $\Delta = 10$ days

REGULATORY CAPITAL USING IRB FOR MR

Risk-sensitive minimum capital for mkt risk

Compute through last value | mean in past 60 days

$$\rightarrow RC^{t}(MR) = \max\left\{ VaR_{\alpha}^{t,\Delta},\ K \frac{1}{60} \sum_{i=1}^{60} VaR_{\alpha}^{t-i,\Delta} \right\} + C_{SR}$$

in most cases we will have the average VaR

where { $K \in [3,4]$ is a stress factor established by the regulator (supervisor, es BI).

C_{SR} an added capital requested for specific risk, additional capital if you don't trust VaR computation

Only if the new VaR is higher than 3 times the mean, you use that

REGULATORY CAPITAL USING IRB FOR CR

The problems were the weights, so we use the Gordy model.

$$RC_i(CR) \propto LGD_i \times (WCDR_i - PD_i)$$

where { LGD_i is the Loss Given Default for the i^{th} obligor in bank's loan portfolio

PD_i is 1y probability of default for the i^{th} obligor

$WCDR_i$ is the Worst Case Default 1y probability for the i^{th} obligor, obtained via the Asymptotic single risk factor model

(D.4) GORDY

$$WCDR_i = P_i(\bar{y})$$

prob. to default in a very bad situation

More precisely Regulatory Capital in IRB is:

$$RC = \sum_i \underbrace{EAD_i}_{A_i} \times \underbrace{LGD_i \times (WCDR_i^{UL} - PD_i)}_{W_i} \times MA_i$$

where:
- EAD_i is the Exposure at default for the i^{th} obligor in bank's loan portfolio (in case of a single loan, approximately loan's principal amount) value of the assets exposed
- MA_i Maturity adjustment, to correct for loans longer than 1y
- ρ_i^2 correlation is a specific function of PD_i

But Basel II had a problem:

$$\boxed{\text{Two different Capital Requirements (CRs) for Credit Exposures}}$$

Loan
Credit CR based on
- → 1y time horizon
- → a confidence interval of 99.9% — farther tails, much worse scenario ↳ RC higher.

RC(CR)

Bonds
Mkt CR based on
- → 10dd time horizon (60dd average)
- → a confidence interval of 99%.

RC(MR)

⇘ Regulatory Arbitrage ⇙

For example, a loan with ENI can be deem exposed both to credit and mkt risks.

The Credit CR shows always a WORST SITUATION than mkt one, for the same credit exposure.

ROE: I want equity as low as possible (→ transform everything into bonds).

Banks built large exposure to credits via bonds and credit derivatives in trading portfolio, leveraging on the favorable regulatory treatment compared to the loan book.

NOT REAL ARBITRAGE, only a MISBEHAVIOUR.

(to have credit exposure to N < LEND / BUY BOND) in both cases, 2 ways of [illegible], I receive $N(\pi)$.

In order to have a credit exposure vs. ENI (ex.), I have 2 ways: I can lend money or I can buy a bond.

In both cases, if ENI defaults, I will receive $N \cdot \pi$.

But by the regulation I must have a RC for it in both cases.

As a bond, I can compute RC (MR), with VaR method (Δ = 10dd, α = 99%).

As a loan, I can compute RC(CR), again with the VaR (Δ = 1 y, α = 99.9%).

→ for exactly the same exposure, I will have:

$$RC(MR) < RC(CR)$$

I will always prefer the bond "representation", pushing the banks not to keep their loans but to pack them into bonds to be sold.

BASEL 2.5 (2009)
after the collapse of the system.

Not used until 2012, since there was not enough capital.

$$\text{Total Capital Requirement} \begin{cases} \left.\begin{array}{l} sVaR \\ IRC \end{array}\right\} \text{two additional components} \\ RC(MR) \\ RC(CR) \end{cases}$$

→ INCREMENTAL RISK CHARGE : to consider not only the risk connected to default, but also the one connected to MIGRATION (a downgrade in the rating of a company that eventually won't default).

(default & migration for unsecuritized credit products, based on 1y)

→ STRESSED VaR : When we measure VaR at t, we consider the last time window, but it may be not relevant for the risks we are considering, the supervisor asks to compute the VaR not in the last 50dd, but in a great volatility period (ex. Lehman window, 6m after Lehman default).

Related to 1y observation related to significant losses, vs. last 1y

3. BASEL 3 : THE NEW ELEMENTS IN RM

BASEL 3 (2010 - 2018)

The main idea is the same of Basel 2.

1. MORE STRINGENT CAPITAL REQUIREMENTS w.r.t. Basel 2.

   ⇒ different coefficients to put in front of RWA (%).

Consider the liabilities for a bank:

↑ Seniority

EQUITY

HYBRID INSTRUMENTS : some characteristics of bonds and some of equity, subordinated debt.

DEBT

Tier I : is a bond, with coupons and a time horizon (like a debt), but the dividends can be paid or not (like dividends to share holder). Subordinated to the debt.

Tier II (Upper / Lower)

	BASEL II	BASEL III
Common Equity	2.0%	4.5%
Tier I Capital	4.0%	6%

↓

increase of % of risk related to assets.

$$RC = \underbrace{\gamma_1}_{\text{defined by the table}} \; RWA = \sum_i RW_i A_i$$

2. RC ∝ Total Asset,

   but limitation on balance sheet: $RC = \gamma \, TA$

   $$\text{Leverage Ratio} = \frac{\text{Tier I Capital}}{\text{Total Assets}} \quad \text{up to } 3\%$$

3. ADDITIONAL CAPITAL BUFFER

→ mandatory capital conservation buffer of 2.5%

→ additional capital buffer requirement for SIFI banks

→ discretionary counter-cyclical buffer, which allows supervisors to require up to another 2.5% of capital during periods of high credit growth.

4. LIQUIDITY RISK

I have some assets to liquidate, but the mkt is not liquid.
OR: I am a bank, I am managing a fund, but the funding mkt may be not liquid.

2 LIQUIDITY RATIOS (over lags of 30 days or 1 year)

BASEL ACCORDS: don't resolve all problems

→ costs to set up this rules, especially for smaller banks.

→ RM Rearding: similar rules and measures
→ similar behaviours and exit time in crises

→ Procyclical: capital requirements rise in times of recession and fall in time of expansion.

4. GORDY MODEL

We have seen that, in Basel II, the Regulatory Capital using Internal Rating Basis (IRB) for Credit Risk is:

$$RC_i\,(CR) \propto LGD_i\,(WCDR_i - PD_i)$$

Worst Case Default 1y probability for the i-th obligor, obtained via the Asymptotic Single Risk Factor Model.

Conditional probability $p_i(y)$ given a value $\hat{y}$ worst than the 99.9% cases.

GORDY MODEL

Differently from Vasicek, NO HP, and $\rho_i \to \rho_i^2$.

non uniform correlation matrix

$$V_i = \rho_i\, y + \sqrt{1-\rho_i^2}\,\varepsilon_i \qquad V_i, y, \varepsilon_i \text{ std. n. r.v.}$$

$$y, \{\varepsilon_i\}_{i=1:I} \text{ i.i.d. r.v.}$$

$$\mathrm{Corr}(V_i, V_j) = \begin{cases} \rho_i \rho_j & , i \neq j \\ 1 & , i = j \end{cases} \rightarrow \Sigma^2 = \begin{bmatrix} 1 & \rho_i\rho_j \\ & \ddots \\ & & 1 \end{bmatrix}$$

Given y,

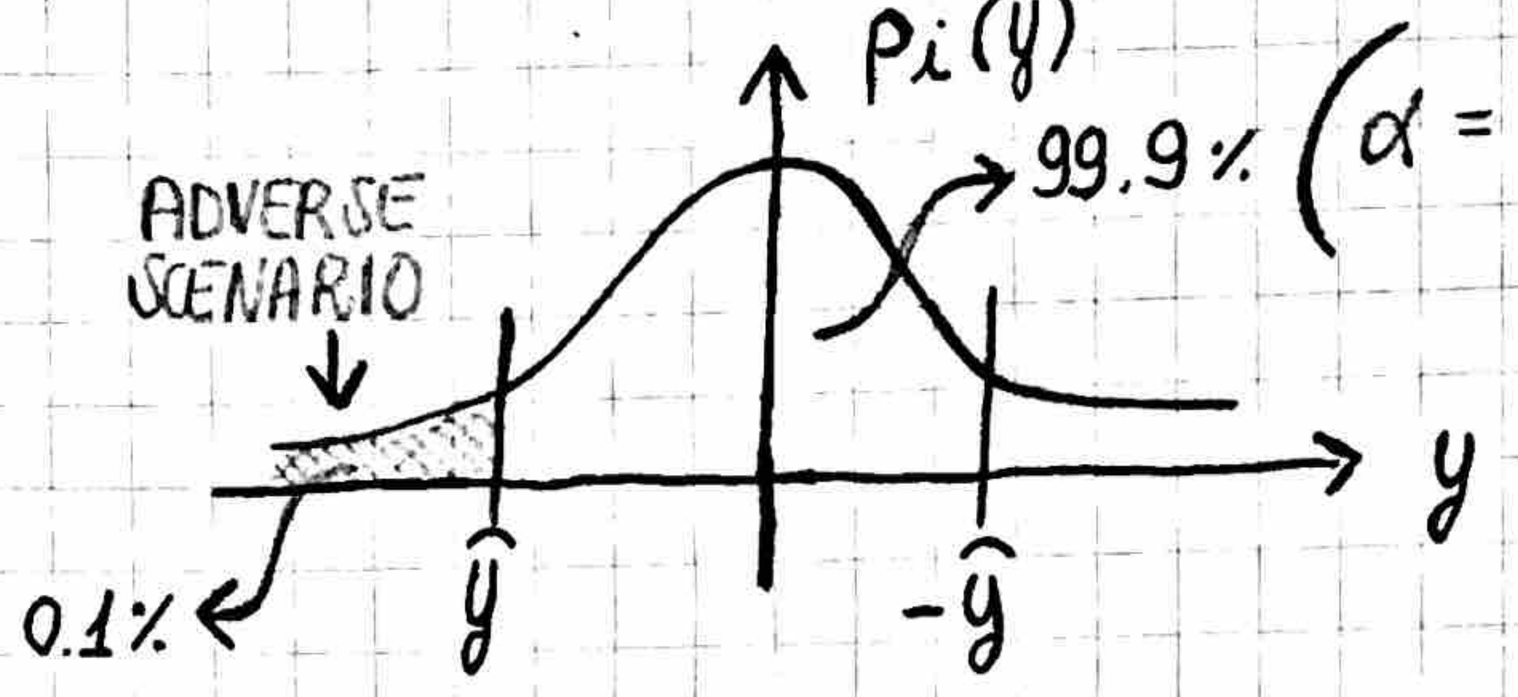
default condition : $V_i \leq K_i$

$$\rho_i y + \sqrt{1-\rho_i^2}\,\varepsilon_i \leq K_i \iff \varepsilon_i \leq \frac{K_i - \rho_i y}{\sqrt{1-\rho_i^2}}$$

$$p_i(y) = N\left(\frac{K_i - \rho_i y}{\sqrt{1-\rho_i^2}}\right) = N\left(\frac{N^{-1}(PD_i) - \rho_i y}{\sqrt{1-\rho_i^2}}\right)$$

$$p_i = PD_i = N(K_i)$$

Which is the adverse scenario?



$$\left(\alpha = N(-\hat{y})\right)$$

$$\Rightarrow WCDR_i = p_i(\hat{y}) = N\left(\frac{N^{-1}(PD_i) - \rho_i \hat{y}}{\sqrt{1-\rho_i^2}}\right)$$

we also know that $\hat{y} = -N^{-1}(\alpha)$ $\quad \alpha = N(-\hat{y})$

$$\alpha = 99.9\%$$

$$\Rightarrow WCDR_i = N\left(\frac{N^{-1}(PD_i) + \rho_i N^{-1}(99.9\%)}{\sqrt{1-\rho_i^2}}\right)$$

ρ_i is usually >0, since most of the economies (businesses) are positively correlated with the mkt.

BASIC RISK MANAGEMENT CONCEPTS

RISK MANAGEMENT (RM) vs. RISK MEASUREMENT (Rm)

Often the first step of RM is risk control, that consists in the measurement of some risks for the (financial) institution of interest.

For example:

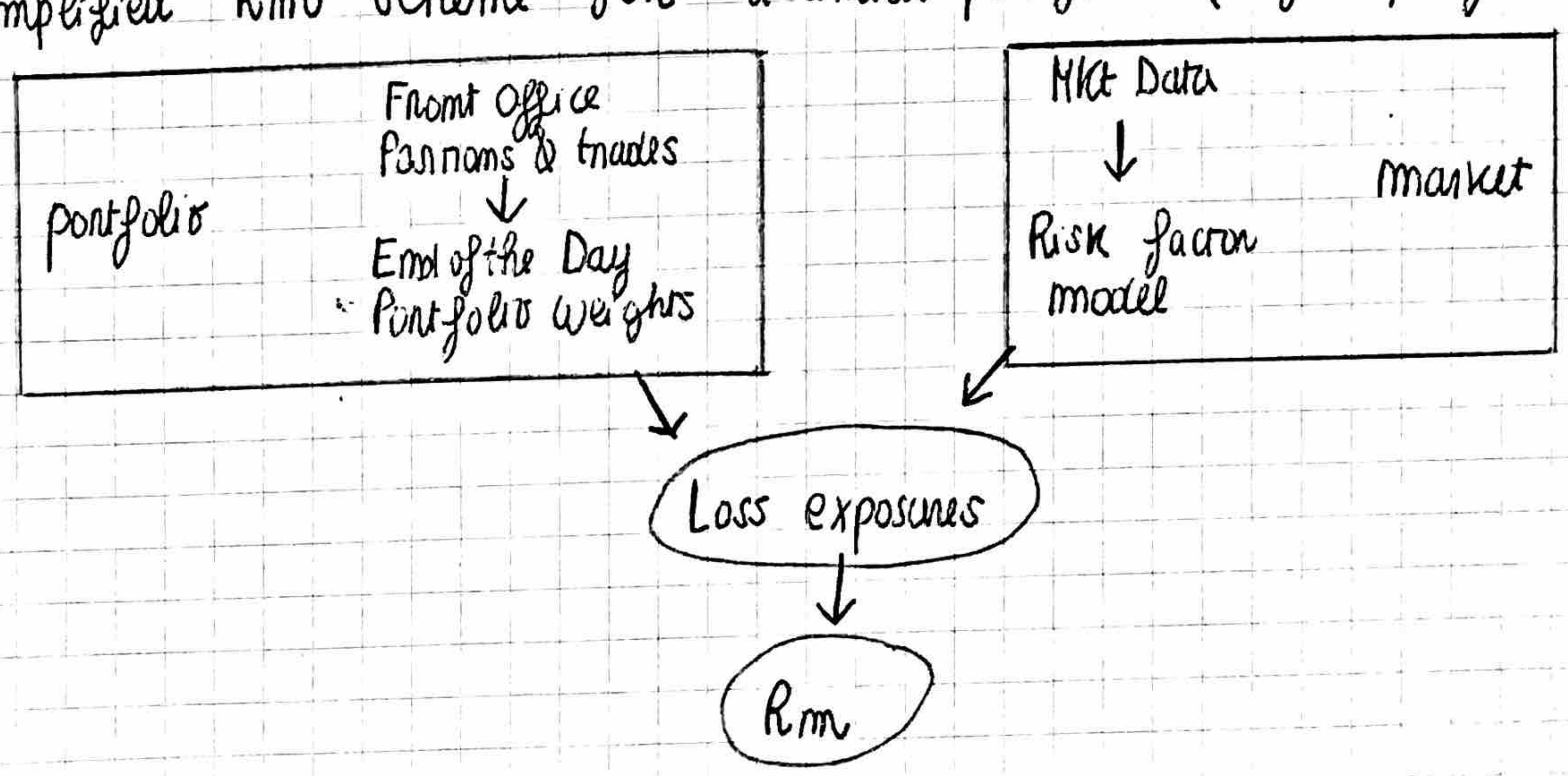
1. The amount of capital a bank needs to hold as a buffer against unexpected future losses on its assets' portfolio.
2. Initial margin requirement from a clearing house

Approaches to Rm:

1. Add-on: the weighted sum of the notionals. The weight (add-on) is related to asset characteristics.
2. Factor-sensitive measures: Greeks.
3. Risk measures based on loss distributions.

$\begin{cases} \text{VaR and ES} \\ \text{Maximum loss scenario (stress test)} \end{cases}$

A simplified Rm scheme for a linear portfolio (e.g. Equity Cash)



The idea behind this approach is a FREEZED PORTFOLIO, as if in the 10dd he is considering for comparisons there weren't changes.

MAIN APPROACHES OF RISK MEASUREMENT
- ↗ PARAMETRIC
  - ↗ statistical estimation of distrib. parameters
  - ↘ PCA
- ↘ NON PARAMETRIC
  - ↗ HS
  - → WHS
  - ↘ statistical Bootstrap

5. VaR & ES : DEFINITIONS. CONTINUOUS CASE. ANALYTIC APPROACH FOR GAUSSIAN AND t-Student CASES.

BASIC DEFINITION FOR P&L

$$\begin{cases} P \equiv V(t+\Delta) - V(t) \\ L \equiv -[V(t+\Delta) - V(t)] \end{cases}$$

$L > 0$ loss
$L < 0$ profit

where $V(t)$ is the value of a given portfolio at time t.

$$\Delta \text{ can be } \begin{cases} 1d \text{ or } 10d \text{ for mkt risk} \\ 1y \text{ for credit risk} \end{cases}$$

and the portfolio is (implicitly) considered unchanged over the lag Δ.

Assumption : The series of risk factor changes $\{X_{t,i}\}_{t \in \mathbb{N}, i=1:m}$ are assumed i.i.d. in time.

Value-at-Risk (VaR) & Expected Shortfall (ES)

- $$\ell_\alpha = VaR_\alpha \equiv \inf_{\ell \in \mathbb{R}} \{ P(L > \ell) \leq 1-\alpha \} =$$
$$= \inf_{\ell \in \mathbb{R}} \{ F(\ell) \geq \alpha \}$$

$\alpha \in (0,1)$
95%, 99%

[illegible] nell'1% [illegible]

where $F(\ell) = P(L \leq \ell)$ is the CDF of the loss

This was an idea of JPM company to report the risk for bank's CEO in a summerized way.

→ VaR is a loss (expressed in €, $)
→ VaR is the quantile of the loss distribution

VaR 99% = lowest loss in the 1% worst scenario.

- $ES_\alpha \equiv \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)\, du$

  → $ES_\alpha \geq VaR_\alpha$

  → ES_α is an average of the <u>tail</u> of the loss distribution

MAIN PROPERTIES IN THE CASE OF CONTINUOUS CDF

$$F(l_\alpha) = \alpha \Rightarrow \begin{cases} VaR_\alpha = \mu + \sigma\, VaR_\alpha^{std} \\ ES_\alpha = \mu + \sigma\, ES_\alpha^{std} \end{cases}$$

How do we calibrate μ, σ? given 24 data, compute returns of shares, to each share 1 risk factor
$\mu = \frac{\Sigma}{n}$, $\sigma = \frac{\Sigma}{n-1}$

PROOF.

1. $F(l_\alpha) = P(L \leq l_\alpha) = P\left(\frac{L-\mu}{\sigma} \leq \underbrace{\frac{l_\alpha - \mu}{\sigma}}_{l_\alpha^{std}}\right)$

   $VaR = \inf_{l \in \mathbb{R}}$ s.t. $F(l) \geq \alpha$ $= \times\times$ cnts

   $F(l_\alpha^{std}) = \alpha$
   $\rightarrow l_\alpha^{std} = F^{-1}(\alpha)$

   $\Rightarrow \overset{VaR}{l_\alpha} = \mu + \sigma\, l_\alpha^{std}$

2. $ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \underbrace{VaR_u}_{\mu + \sigma VaR_u^{std}}\, du =$ linearity of the integral

   $= \mu + \sigma \frac{1}{1-\alpha} \int_\alpha^1 VaR_u^{std}\, du =$

   $= \mu + \sigma\, ES_\alpha^{std}$

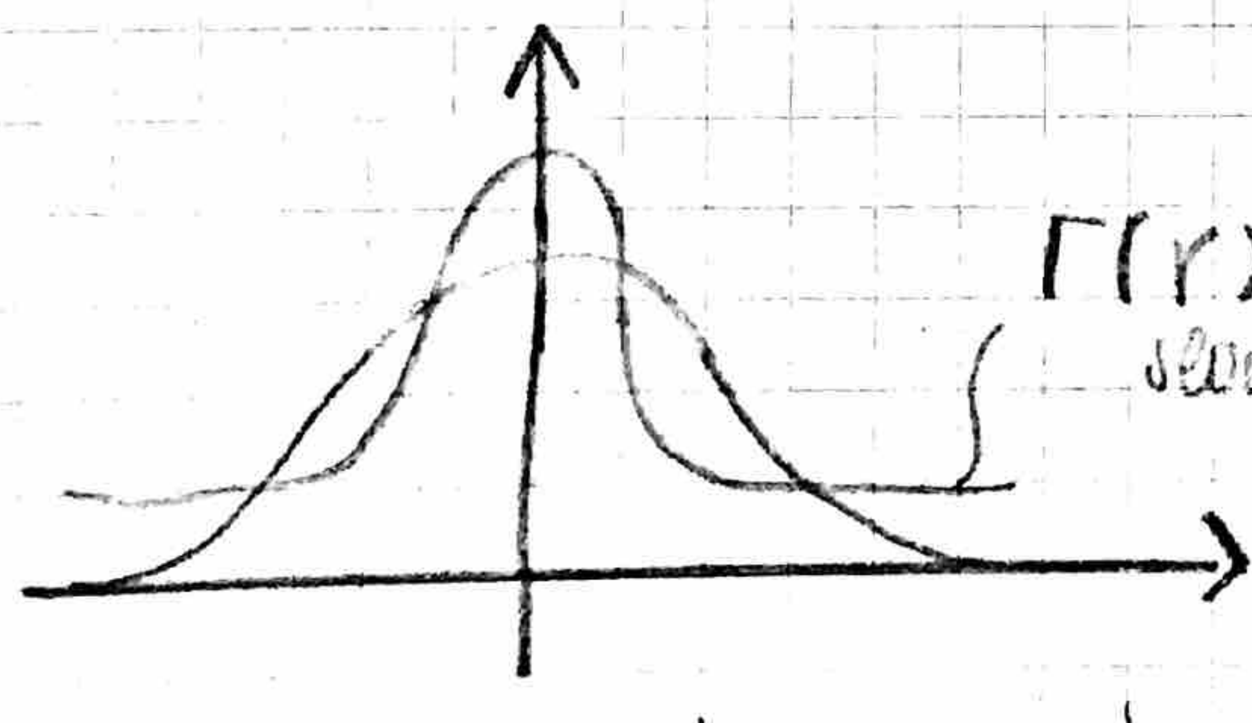
these are general, let's look at examples with a given distribution for the loss!

l_α^{std} OK

ES_α^{std} ?

How can we model the loss distribution?

- standard normal → $e^{-l^2/2}$
- standard t-student → $\Gamma(r) = e^{-(r+1)}$



$\Gamma(r)$ goes to 0 slowly → fat tails.

$t_r(l_\alpha^{std}) = \alpha \rightarrow l_\alpha^{std} = t_r^{-1}(\alpha)$

EXAMPLES ON VaR AND ES: (CONTINUOUS) NORMAL AND t-Student

distribution for the loss!

[illegible]

$F(\ell_\alpha)=\alpha \to \ell_\alpha = \underbrace{F^{-1}(\alpha)}_{VaR}$

NORMAL : $\phi(\ell) = \frac{1}{\sqrt{2\pi}} e^{-\ell^2/2}$

$$VaR_\alpha = \mu + \sigma N^{-1}(\alpha)$$

$$ES_\alpha^{std} = \frac{\phi(N^{-1}(\alpha))}{1-\alpha}$$

da aggiungere nella formula (ℓ−) per l'ES.

t-Student : $\phi_r(\ell) = \underbrace{\frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma(r/2)}}_{c_r} \left(1+\frac{\ell^2}{r}\right)^{-\frac{r+1}{2}}$

$$\Gamma(r) = (r-1)\Gamma(r-1)$$

$$VaR_\alpha = \mu + \sigma t_r^{-1}(\alpha)$$

$$ES_\alpha^{std} = \frac{r + (t_r^{-1}(\alpha))^2}{r-1} \cdot \frac{\phi_r(t_r^{-1}(\alpha))}{1-\alpha}$$

PROOF.

- NORMAL

$$ES_\alpha^{std} = \frac{1}{1-\alpha}\int_\alpha^1 \underbrace{VaR_u^{std}}_{\ell_u^{std}}\, du =$$

$$N(\ell_u^{std}) = u \quad \text{(CONTINUOUS CDF)}$$

$$du = \phi(\ell)\, d\ell$$

$$= \frac{1}{1-\alpha}\int_{\ell_\alpha = N^{-1}(\alpha)}^{+\infty} \ell\,\phi(\ell)\,d\ell = \qquad \frac{d}{d\ell}\phi(\ell) = -\ell\phi(\ell)$$

$$\phi(\ell) = \frac{1}{\sqrt{2\pi}} e^{-\ell^2/2}$$

$$= -\frac{1}{1-\alpha}\int_{\ell_\alpha}^{+\infty} \frac{d}{d\ell}(\phi(\ell))\,d\ell = -\frac{1}{1-\alpha}\phi(-\infty) + \frac{1}{1-\alpha}\phi(\ell_\alpha)$$

$$\Rightarrow ES_\alpha^{std} = \frac{1}{1-\alpha}\phi(N^{-1}(\alpha))$$

- **t-student**

$$ES_\alpha^{std} = \frac{1}{1-\alpha}\int_\alpha^1 \underbrace{VaR_u^{std}}\, du =$$

$$VaR_u^{std} = \ell_u^{std}$$

$$t_r(\ell_u^{std}) = u$$

$$du = \phi_r(\ell)\,d\ell$$

$$= \frac{1}{1-\alpha}\int_{\ell_\alpha = t_r^{-1}(\alpha)}^{+\infty} \underbrace{\ell\,\phi_r(\ell)}\,d\ell$$

$$\ell\cdot c_r\cdot\left(1+\frac{\ell^2}{r}\right)^{-\frac{r+1}{2}} \overset{?}{=} \frac{d}{d\ell}\, c_r\cdot\left(1+\frac{\ell^2}{r}\right)^{-\frac{r}{2}+\frac{1}{2}}$$

$$\boxed{\phi_r(\ell) = c_r\cdot\left(1+\frac{\ell^2}{r}\right)^{-\frac{r+1}{2}}}$$

$$\frac{d}{d\ell}\underbrace{\left(1+\frac{\ell^2}{r}\right)^{-\frac{r+1}{2}+1}\cdot c_r}_{\phi_r(\ell)\cdot\left(1+\frac{\ell^2}{r}\right)} = \left(-\frac{(r-1)}{\cancel{2}}\right)\cdot\left(1+\frac{\ell^2}{r}\right)^{-\frac{r}{2}-\frac{1}{2}}\cdot\frac{\cancel{2}\ell}{r}\cdot c_r =$$

$$= -\frac{(r-1)}{r}\,\ell\left(1+\frac{\ell^2}{r}\right)^{-\frac{r}{2}-\frac{1}{2}}\cdot c_r$$

$$\Rightarrow \frac{r}{r-1}\cdot\frac{d}{d\ell}\left(\left(1+\frac{\ell^2}{r}\right)\cdot\phi_r(\ell)\right) = -\ell\,\phi_r(\ell)$$

$$ES_\alpha^{std} = -\frac{r}{r-1}\,\frac{c_r}{1-\alpha}\int_{\ell_\alpha}^{+\infty}\frac{d}{d\ell}\left(1+\frac{\ell^2}{r}\right)^{-\frac{r}{2}+\frac{1}{2}}d\ell =$$

$$= \frac{r}{r-1}\,\frac{c_r}{1-\alpha}\left(1+\frac{\ell_\alpha^2}{r}\right)^{-\frac{r+1}{2}}\cdot\left(1+\frac{\ell_\alpha^2}{r}\right) =$$

$$= \frac{r+\ell_\alpha^2}{r-1}\,\frac{1}{1-\alpha}\,\phi_r(\ell_\alpha) = \frac{r+(t_r^{-1}(\alpha))^2}{r-1}\,\frac{\phi_r(t_r^{-1}(\alpha))}{1-\alpha} \quad \blacksquare$$

6. MULTINOMIAL GAUSSIAN DISTRIBUTION: DENSITY & CHARACTERISTIC FUNCTION

PARAMETRIC APPROACH: MULTINOMIAL GAUSSIAN DISTRIBUTION

Joint multivariate density is:

$$f(\underline{x}) \equiv \frac{1}{(2\pi)^{d/2}\sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})\cdot \Sigma^{-1}(x-\underline{\mu})\right\}$$

$\underline{x} \in \mathbb{R}^{d}$ # risk factors $\qquad \underline{x} \sim N_d(\underline{\mu}, \Sigma)$

$\underline{\mu} = \mathbb{E}[\underline{x}]$ drift

$\underline{\underline{\Sigma}}$ non singular variance-covariance matrix.
(semi definite positive)

Characteristic function:

$$\phi(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = \exp\left\{i\underline{t}\cdot\underline{\mu} - \frac{\underline{t}\cdot\Sigma\underline{t}}{2}\right\}$$

canonical scalar product in $\mathbb{R}^d$

$$\underline{t}\cdot\underline{x} = \sum_{i=1}^{d} t_i x_i$$

No II variates

CHOLESKY FACTORIZATION: $\exists!$ A lower triangular matrix s.t. $AA' = \Sigma$, $A \equiv \Sigma^{1/2}$

Generate Gaussian r.v.s with mean μ and variance Σ:

- generate a (column) vector $\underline{y}$ of gaussian r.vs iid;
- $\underline{x} = \underline{\mu} + \Sigma^{1/2}\underline{y}$ $\qquad$ every risk factor is normally distributed

- $\mathbb{E}[\underline{x}] = \underline{\mu} + \emptyset$
- $\mathbb{E}[(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})] = \mathbb{E}[(A\underline{y})(A\underline{y})^T] =$

$$= \mathbb{E}[A\underline{y}\underline{y}^T A^T] = \mathbb{E}[\Sigma] = \Sigma \quad \text{COV}$$

PROOF. of characteristic function: (via Cholesky factorization)

$$\phi_c(\underline{t}) = \mathbb{E}[e^{i\underline{t}\cdot\underline{x}}] = \mathbb{E}[e^{i\underline{t}\cdot(\underline{\mu}+\Sigma^{1/2}\underline{y})}] =$$

$$= \mathbb{E}[e^{i\underline{t}\cdot\underline{\mu} + i\underline{t}\cdot A\underline{y}}] = e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}[e^{i\underline{t}\cdot A\underline{y}}] =$$

FORMULA DELL'AGGIUNTA

$$= \left(\underline{t}\cdot A\underline{y} = \underbrace{A'\underline{t}}_{\hat{\underline{t}}}\cdot\underline{y}\right) =$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}[e^{i\hat{\underline{t}}\cdot\underline{y}}] = e^{i\underline{t}\cdot\underline{\mu}}\,\mathbb{E}\Big[\prod_{j=1}^{d} e^{i\hat{t}_j y_j}\Big] =$$

$$= e^{i\underline{t}\cdot\underline{\mu}}\cdot\int_{-\infty}^{+\infty}\prod_{j=1}^{d}\frac{dy_j}{\sqrt{2\pi}}\, e^{-\frac{y_j^2}{2} + i\hat{t}_j y_j}$$

Let's look at 1 integral:

$$\int_{-\infty}^{+\infty}\frac{dy_j}{\sqrt{2\pi}}\, e^{\underbrace{-\left(\frac{y_j^2}{2} - i\hat{t}_j y_j - \frac{1}{2}\hat{t}_j^2\right)}_{\downarrow} - \frac{1}{2}\hat{t}_j^2}$$

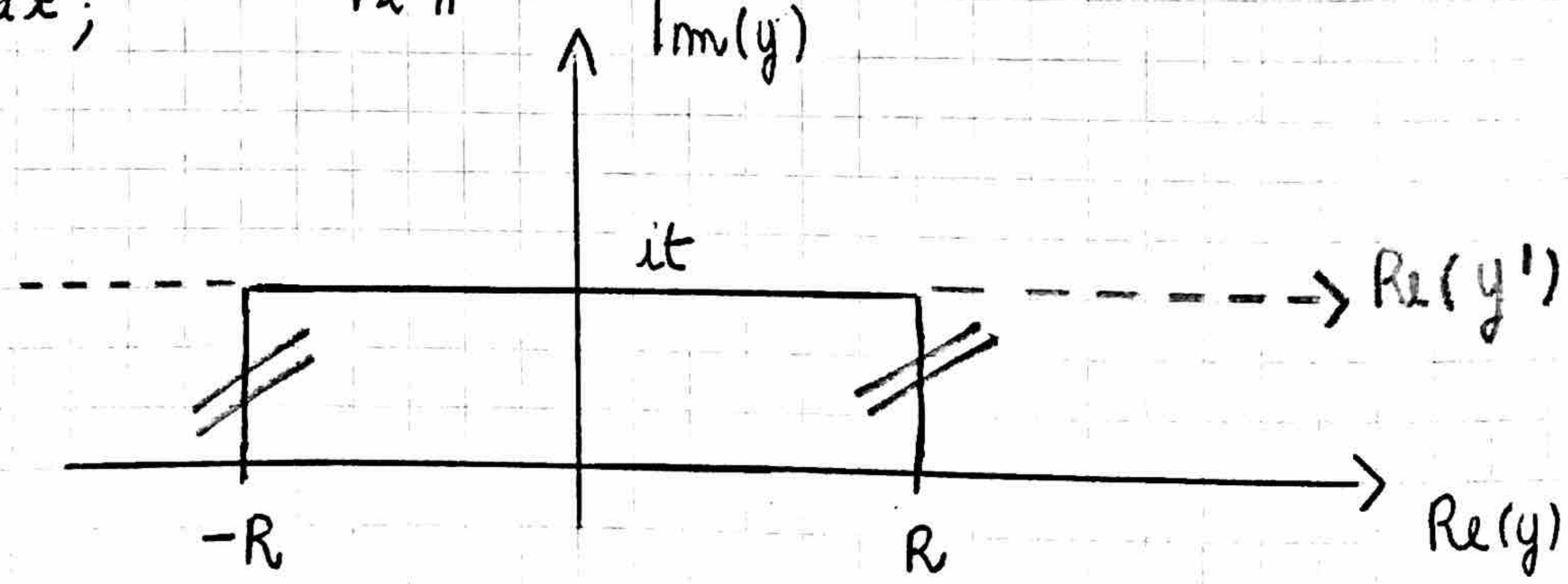
$$-\frac{1}{2}(\underbrace{y_j - i\hat{t}_j}_{y_j'})^2$$

$$\int_{-\infty}^{+\infty}\frac{dy_j}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y_j - i\hat{t}_j)^2} =$$

cambio variabile + teo Cauchy

teo Cauchy

$$= \int_{-\infty+\cancel{i\hat{t}_j}}^{+\infty+\cancel{i\hat{t}_j}}\frac{dy_j'}{\sqrt{2\pi}}\, e^{-\frac{y_j'^2}{2}} = 1$$



$$\Longrightarrow \mathbb{E}\left[e^{i \underline{t} \cdot \underline{x}} \right] = e^{i \underline{t} \cdot \underline{\mu}} \, e^{-\frac{1}{2} \sum_{j=1}^{d} \hat{t}_j^{\,2}} =$$

$$= e^{i \underline{t} \cdot \underline{\mu}} \, e^{-\frac{1}{2} \underline{\hat{t}} \cdot \underline{\hat{t}}} =$$

$$= \left(\underline{\hat{t}} = A^T \underline{t} \;\Longrightarrow (A^T \underline{t}) \cdot A^T \underline{t} = \underline{t} \cdot A A^T \underline{t} \right) =$$

$$= \exp\left\{ i \underline{t} \cdot \underline{\mu} - \frac{1}{2} \underline{t} \cdot \Sigma \underline{t} \right\} \qquad \blacksquare$$

ESTIMATORS OF DRIFT AND COVARIANCE

Given a set of m of i.i.d. observations of a d-dimensional Gaussian risk-factor vector

$$\{ \underline{X}_t \}_{t=1,\dots,m}$$

the unbiased estimators are:

$$\underline{\bar{X}} \equiv \frac{1}{m} \sum_{t=1}^{m} \underline{X}_t$$

$$S \equiv \frac{1}{m-1} \sum_{t=1}^{m} (\underline{X}_t - \underline{\bar{X}})(\underline{X}_t - \underline{\bar{X}})'$$

Natural extension of multivariate Gaussian distributions : Elliptic distributions.

7. SQUARED-ROOT-OF-TIME SCALING RULE: WHAT IS ABOUT?

SCALING: LOSSES OVER TIME LAGS

Generally, we are interested in risk measures over a time lag Δ having daily iid rv.

For example, we want to get the VaR on 10 days:
For 2y data, we have about 500 data, but, if we want to use a 10dd time interval, data are only 50.
How to overcome the problem? ↳ bad quality of estimation

In practice, it is used the so called "squared-root of time scaling rule" for Gaussian i.i.d. returns, i.e.:

$$\mu \to \Delta\mu \;,\; \Sigma \to \Delta\Sigma$$

$$\Downarrow$$

1-d case

Scaling for VaR and ES

$$\begin{cases} VaR_\alpha = \overset{EL}{\Delta\mu} + \overset{UL}{\sqrt{\Delta}\,\sigma\, VaR_\alpha^{std}} \\ ES_\alpha = \Delta\mu + \sqrt{\Delta}\,\sigma\, ES_\alpha^{std} \end{cases}$$

Idea:

$$\underline{X}_t \sim N_d(\underline{\mu}, \Sigma) \Rightarrow \sum_{i=1}^{\Delta} \underline{X}_{t+i} \sim N_d(\Delta\underline{\mu}, \Delta\Sigma)$$

d-dimensional Gaussian risk-factor vector

8. PARAMETRIC METHOD: MEAN VARIANCE/COVARIANCE METHOD

Linearized loss → we can use the linearization since in credit world fluctuations are small.

$$L(\underline{X}_t) = -[V(t+\Delta) - V(t)]$$

(under the assumption of FROZEN PORTFOLIO)

$X_{i,t}$ is the return in our equity portfolio

Equity dynamics (lognormal distribution) $\rightarrow S_{i,t+\Delta} = S_{i,t}\, e^{X_i} \longleftrightarrow X_i = \ln \dfrac{S_{i,t+\Delta}}{S_{i,t}}$

Expression for the value of the portfolio? d different stocks.

$$V(t) = \sum_{i=1}^{d} \underset{\nwarrow \text{ nº shares in the } i\text{-th asset}}{n_{i,t} S_{i,t}} \quad (n_i = \text{nº asset for stock } S_{i,t})$$

$$V(t+\Delta) = \sum_{i=1}^{d} n_{i,t} S_{i,t}\, e^{X_i} = V(t) \sum_{i=1}^{d} \underbrace{\frac{n_{i,t} S_{i,t}}{V(t)}}_{w_{i,t}} e^{X_i} =$$

$w_{i,t}$: weight of the stock in the ptf

linear expansion

$$\overset{\downarrow}{=} V(t) \sum_{i=1}^{d} [1 + x_{i,t}]\, w_{i,t} \qquad V(t) + V(t)\, \underline{W}_t \cdot \underline{X}_t$$

$$\Rightarrow L(\underline{X}_t) = -[V(t+\Delta) - V(t)] =$$

$$= -\Big[V(t) \sum_{i=1}^{d} w_{i,t} [1 + x_{i,t}] - V(t) \Big] = \qquad \sum w_{i,t} = 1$$

$$= -V(t) [\, \underline{W}_t \cdot \underline{x}_t \,]$$

We also have to consider some costs for transactions:

$$L(\underline{X}_t) = -V(t) [\, c_t + \underline{W}_t \cdot \underline{X}_t \,]$$

If returns $\underline{X}_t$ are gaussian, we can compute VaR and ES.

$$\frac{L(\underline{X}_t)}{V_t} \sim N[-(c_t + \underline{W}_t \cdot \underline{\mu}),\ \underline{W}_t \cdot \Sigma\, \underline{W}_t]$$

$\mathrm{VaR}_\alpha = \mathbb{E}[L] + \mathrm{Var}(L)$

CONTRIBUTION TO VaR AND ES (elliptic distributions)

$$\mathrm{CVaR}_i = w_i \beta_i \mathrm{VaR}$$

$$\mathrm{CES}_i = w_i \beta_i \mathrm{ES}$$

$$\beta = \frac{\Sigma \underline{W}(t)}{\underline{W}(t) \cdot \Sigma \underline{W}(t)}, \quad \sum_{i=1}^{d} \beta_i w_i = \underline{W} \cdot \underline{\beta} = 1$$

Sometimes, it may happen that the equity goes below the RC.
I can ask to shareholders to put more money in the bank. In order to get new money, I have to convince the board...

But the best thing to do is to <u>decrease the VaR</u>, since
$RC \propto VaR$

We can associate to each risk a contribution to VaR & ES, or I can select the business to fine (the ones that are absorbing too much capital), instead of the whole bank!

Look at the $RDE_i = \frac{profit_i}{RC_i}$: we can select those activities that don't produce enough w.r.t. investments.