POLITECNICO
DI TORINO

e-Lite

Recursion

Tecniche di Programmazione – A.A. 2019/2020

Summary

1. Definition and divide-and-conquer strategies
2. Recursion: design tips
3. Simple recursive algorithms
   1. Fibonacci numbers
   2. Dicothomic search
   3. X-Expansion
   4. Anagrams
   5. Knapsack
4. Recursive vs Iterative strategies
5. More complex examples of recursive algorithms
   1. Knight's Tour
   2. Proposed exercises

Definition and divide-and-conquer strategies

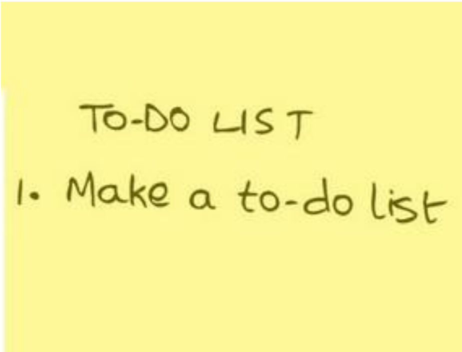
Recursion

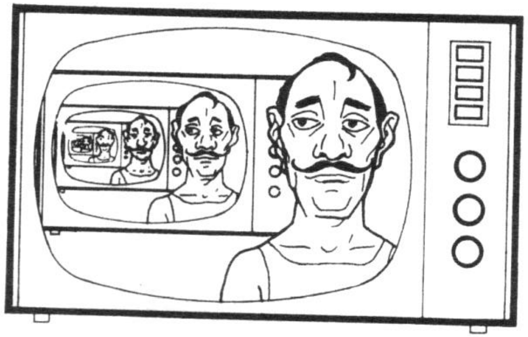
Why recursion?

- Divide et impera
- Systematic exploration/enumeration
- Handling recursive data structures

Definition

- A method (or a procedure or a function) is defined as recursive when:
  - Inside its definition, we have a call to the same method (procedure, function)
  - Or, inside its definition, there is a call to another method that, directly or indirectly, calls the method itself
- An algorithm is said to be recursive when it is based on recursive methods (procedures, functions)






Per capire la ricorsione prima devi capire la ricorsione
@vitadainformatici

Example: Factorial

$$
\begin{cases}
0! \stackrel{\text{def}}{=} 1 \\
\forall N \geq 1: \\
N! \stackrel{\text{def}}{=} N \times (N-1)!
\end{cases}
$$

```
public long recursiveFactorial(long N)
{
  long result = 1 ;

  if ( N == 0 )
    return 1 ;
  else {
    result = recursiveFactorial(N-1) ;
    result = N * result ;
    return result ;
  }
}
```

Motivation

- Many problems lend themselves, naturally, to a recursive description:
  - We define a method to solve sub-problems similar to the initial one, but smaller
  - We define a method to combine the partial solutions into the overall solution of the original problem

Divide et impera

Gaius Julius Caesar

Recursion

- **Divide et Impera**

- Split a problem P into $\{Q_i\}$ where Q_i are still complex, yet *simpler* instances of the same problem.
- Solve $\{Q_i\}$, then merge the solutions
- Merge & split must be "simple"
- A.k.a., *Divide 'n Conquer*

- **Exploration**

- Systematic procedure to enumerate all possible solutions
- Solutions (built stepwise)
  - Paths
  - Permutations
  - Combinations
- Divide et Impera, by "dividing" the possible solutions

Divide et Impera – Divide and Conquer

- Solution = Solve (Problem) ;

- Solve (Problem) {
  - List<SubProblem> subProblems = **Divide** (Problem) ;
  - For (each subP[i] in subProblems) {
    - SubSolution[i] = **Solve** (subP[i]) ;
  - }
  - Solution = **Combine** (SubSolution[1..N]) ;
  - return Solution ;
- }

Divide et Impera – Divide and Conquer

- Solution = Solve (Problem) ;

- Solve (Problem) {
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  - }
  - Solution = **Combine** (SubSolution[1..N]) ;
  - return Solution ;
- }

"a" sub-problems, each "b" times smaller than the initial problem

recursive call

How to stop recursion?

- Recursion **must not** be infinite
  - Any algorithm must always terminate!
- After a sufficient nesting level, sub-problems become so small (and so easy) to be solved:
  - Trivially (ex: sets of just one element)
  - Or, with methods different from recursion

Warnings

- Always remember the “termination condition”
- Ensure that all sub-problems are strictly “smaller” than the initial problem

Divide et Impera – Divide and Conquer

- Solve (Problem) {
  - if(problem is trivial)
    - Solution = **Solve_trivial** (Problem) ;
  - else {
    - List<SubProblem> subProblems = **Divide** (Problem) ;
    - For (each subP[i] in subProblems) {
      - SubSolution[i] = **Solve** (subP[i]) ;
    - }
    - Solution = **Combine** (SubSolution[1..N]) ;
  - }
  - return Solution ;
- }

do recursion

Exploration

- Explore (**S**) {
  - List<Step> steps = **PossibleSteps** (Problem, **S**) ;
  - for (each **p** in steps) {
    - **S**.**Do** (**p**)
    - **Explore** (**S**) ;
    - **S**.**Undo** (**p**) ;
  - }
- }

Exploration

- **Explore** (**S**) {
  - List<Step> steps = **PossibleSteps** (Problem, **S**) ;
  - for (each **p** in steps) {
    - **S**.**Do** (**p**)
    - **Explore** (**S**) ;
    - **S**.**Undo** (**p**) ;
  - }
- }

The "status" of the problem

Local variable

"Try" the step

Recursion

Backtrack!

Design tips

Recursion

Goal

1. Analysis of a problem to be solved with recursive techniques
2. Identification of the main design choices
3. Identification of the main implementation strategies

Analizzare il problema

- Come imposto in generale la ricorsione?
- Che cosa mi rappresenta il "livello"?
- Com'è fatta una soluzione parziale?
- Com'è fatta una soluzione totale?

Generale le possibili soluzioni

- Qual è la regola per generare tutte le soluzioni del livello+1 a partire da una soluzione parziale del livello corrente?
- Come faccio a riconoscere se una soluzione parziale è anche completa? (terminazione con successo)
- Come viene avviata la ricorsione (livello 0)?

Identificare le soluzioni valide

- Data una soluzione **parziale**, come faccio a
  - sapere se è valida (e quindi continuare)?
  - sapere se non è valida (e quindi terminare la ricorsione)?
  - nb. magari non posso…
- Data una soluzione **completa**, come faccio a
  - sapere se è valida?
  - sapere se non è valida?
- Cosa devo fare con le soluzioni complete valide?
  - Fermarmi alla prima?
  - Generarle e memorizzarle tutte?
  - Contarle?

Progettare le strutture dati

- Qual è la struttura dati per memorizzare una soluzione (parziale o completa)?
- Qual è la struttura dati per memorizzare lo stato della ricerca (della ricorsione)?

Scheletro del codice

```
// Struttura di un algoritmo ricorsivo generico

void recursive (... , level) {

  // E -- sequenza di istruzioni che vengono eseguite sempre
  // Da usare solo in casi rari (es. Ruzzle)
  doAlways();

  // A
  if (condizione di terminazione) {
    doSomething;
    return;
  }

  // Potrebbe essere anche un while ()
  for () {

    // B
    generaNuovaSoluzioneParziale;


    if (filtro) { // C
      recursive (..., level + 1);
    }

    // D
    backtracking;
  }

}
```

Riempire lo scheletro (del codice)

Blocco	Frammento di codice
A	
B	
C	
D	
E	

```
// Struttura di un algoritmo ricorsivo

void recursive (... , level) {

  // E -- sequenza di istruzioni che ve
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  doAlways();

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  }

  // Potrebbe essere anche un while ()
  for () {

    // B
    generaNuovaSoluzioneParziale;


    if (filtro) { // C
      recursive (..., level + 1);
    }

    // D
    backtracking;
  }

}
```
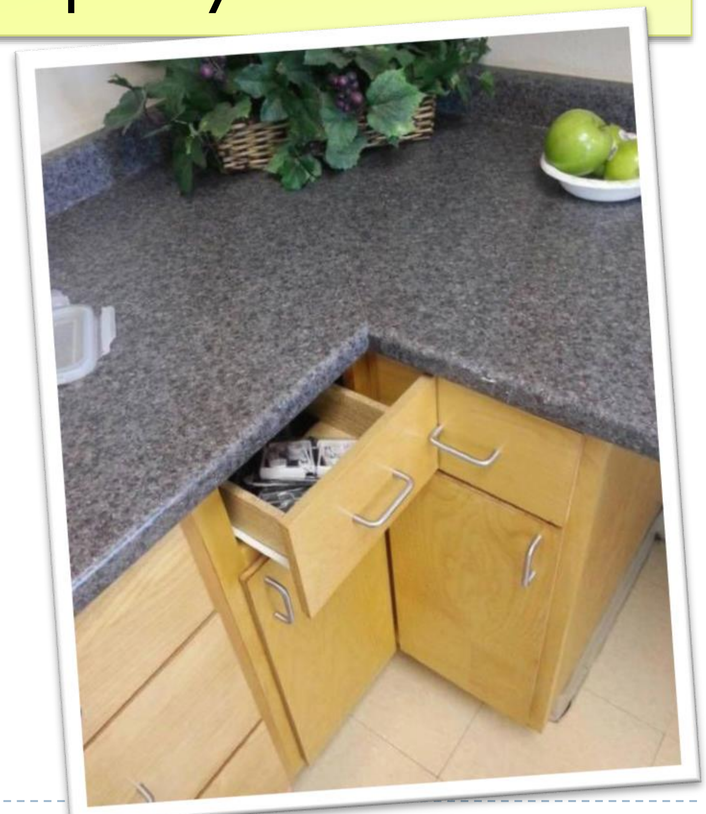

Recursion myths

- Recursive algorithms are O(n log n)
- Recursive algorithms are better than non-recursive ones
- Recursive algorithms can be coded quickly



Complexity

Recursion

What about complexity?

- a = number of sub-problems for a problem
- b = how smaller sub-problems are than the original one
- n = size of the original problem
- T(n) = complexity of **Solve**
  - …our unknown complexity function
- **$\Theta(1)$** = complexity of **Solve_trivial**
  - …otherwise it wouldn't be trivial
- D(n) = complexity of **Divide**
- C(n) = complexity of **Combine**

Divide et Impera – Divide and Conquer

- Solve (Problem) { ← $T(n)$
  - if(problem is trivial)
    - Solution = **Solve_trivial** (Problem) ; ← $\Theta(1)$
  - else {
    - List<SubProblem> subProblems = **Divide** (Problem) ; ← $D(n)$
    - For (each subP[i] in subProblems) { ← a times
      - SubSolution[i] = **Solve** (subP[i]) ; ← $T(n/b)$
    - }
    - Solution = **Combine** (SubSolution[1..a]) ; ← $C(n)$
  - }
  - return Solution ;
- }

Complexity computation

- $T(n) =$
  - $\Theta(1)$ for $n \le c$
  - $D(n) + a\,T(n/b) + C(n)$ for $n > c$

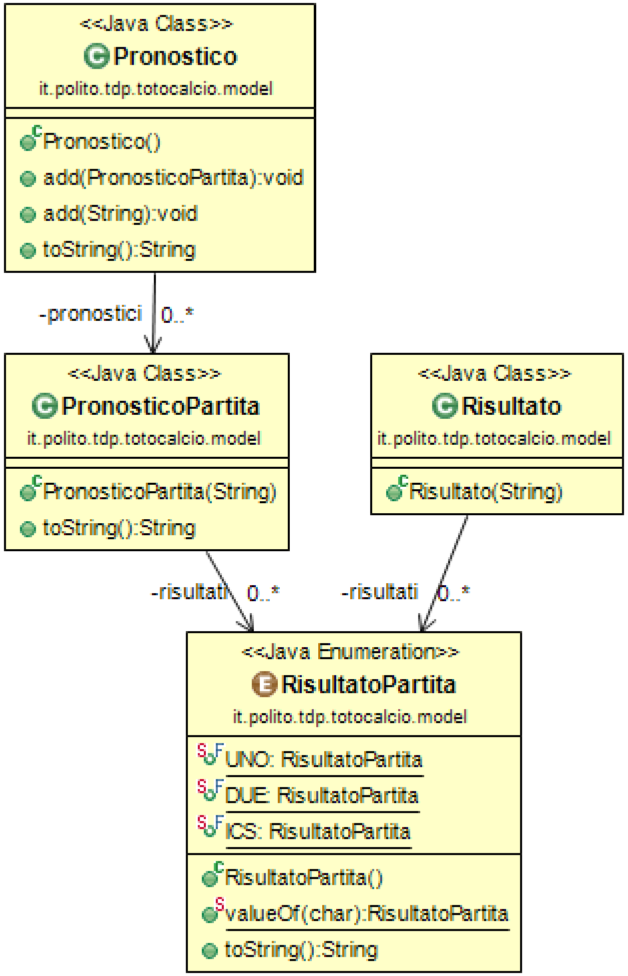
- Recurrence Equation not easy to solve in the general case
- Special case:
  - If $D(n)+C(n)=\Theta(n)$
  - We obtain $\mathbf{T(n) = \Theta(n \log n)}$.

Simple recursive algorithms

Recursion

Schedina Totocalcio

Classi

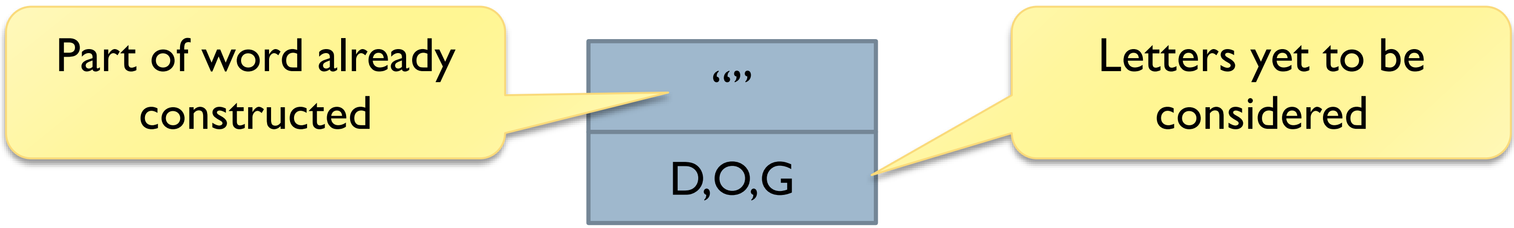


<<Java Class>>
Pronostico
it.polito.tdp.totocalcio.model
Pronostico()
add(PronosticoPartita):void
add(String):void
toString():String
-pronostici
0..*
<<Java Class>>
PronosticoPartita
it.polito.tdp.totocalcio.model
PronosticoPartita(String)
toString():String
<<Java Class>>
Risultato
it.polito.tdp.totocalcio.model
Risultato(String)
-risultati
0..*
-risultati
0..*
<<Java Enumeration>>
RisultatoPartita
it.polito.tdp.totocalcio.model
UNO: RisultatoPartita
DUE: RisultatoPartita
ICS: RisultatoPartita
RisultatoPartita()
valueOf(char):RisultatoPartita
toString():String

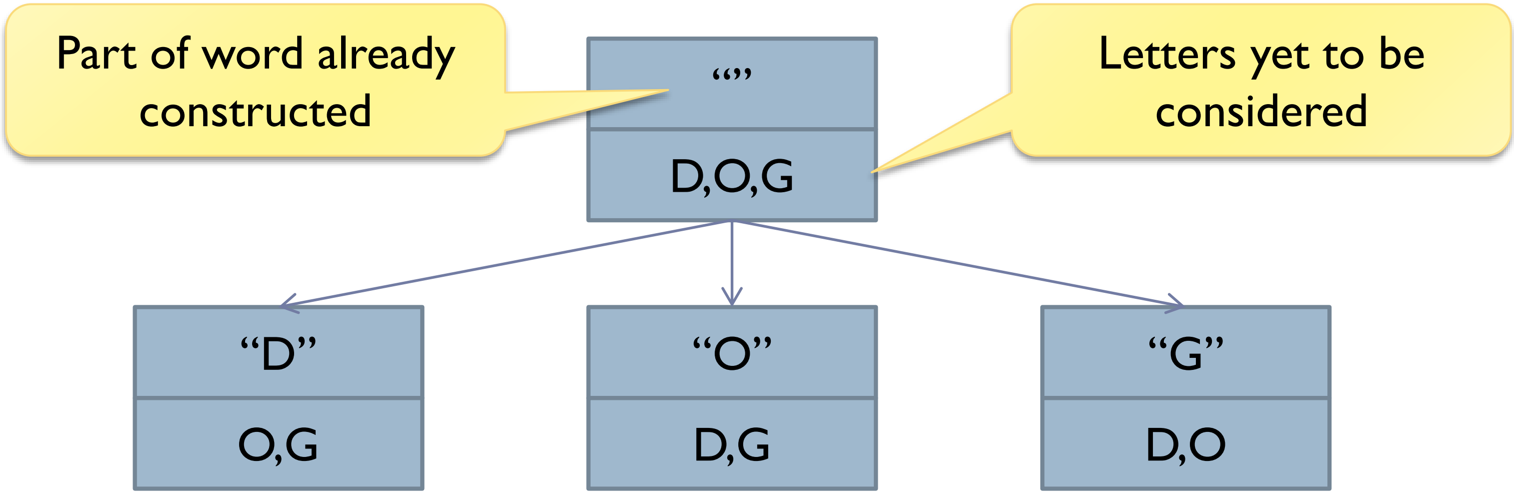
Exercise: Anagram

- Given a word, find all possible anagrams of that word
  - Find all permutations of the elements in a set
  - Permutations are N!
- E.g.: «Dog» → dog, dgo, god, gdo, odg, ogd

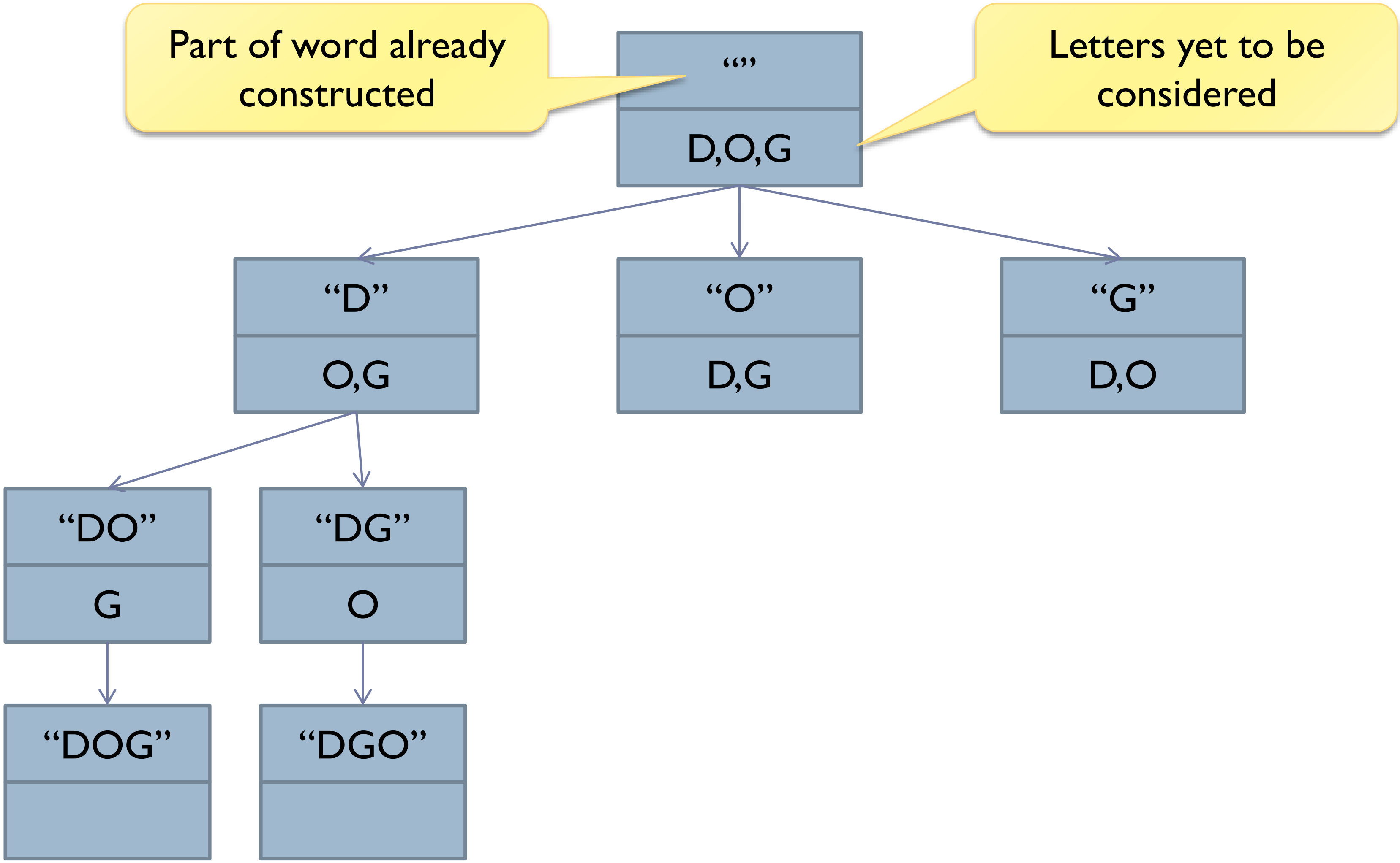
Anagrams: recursion tree



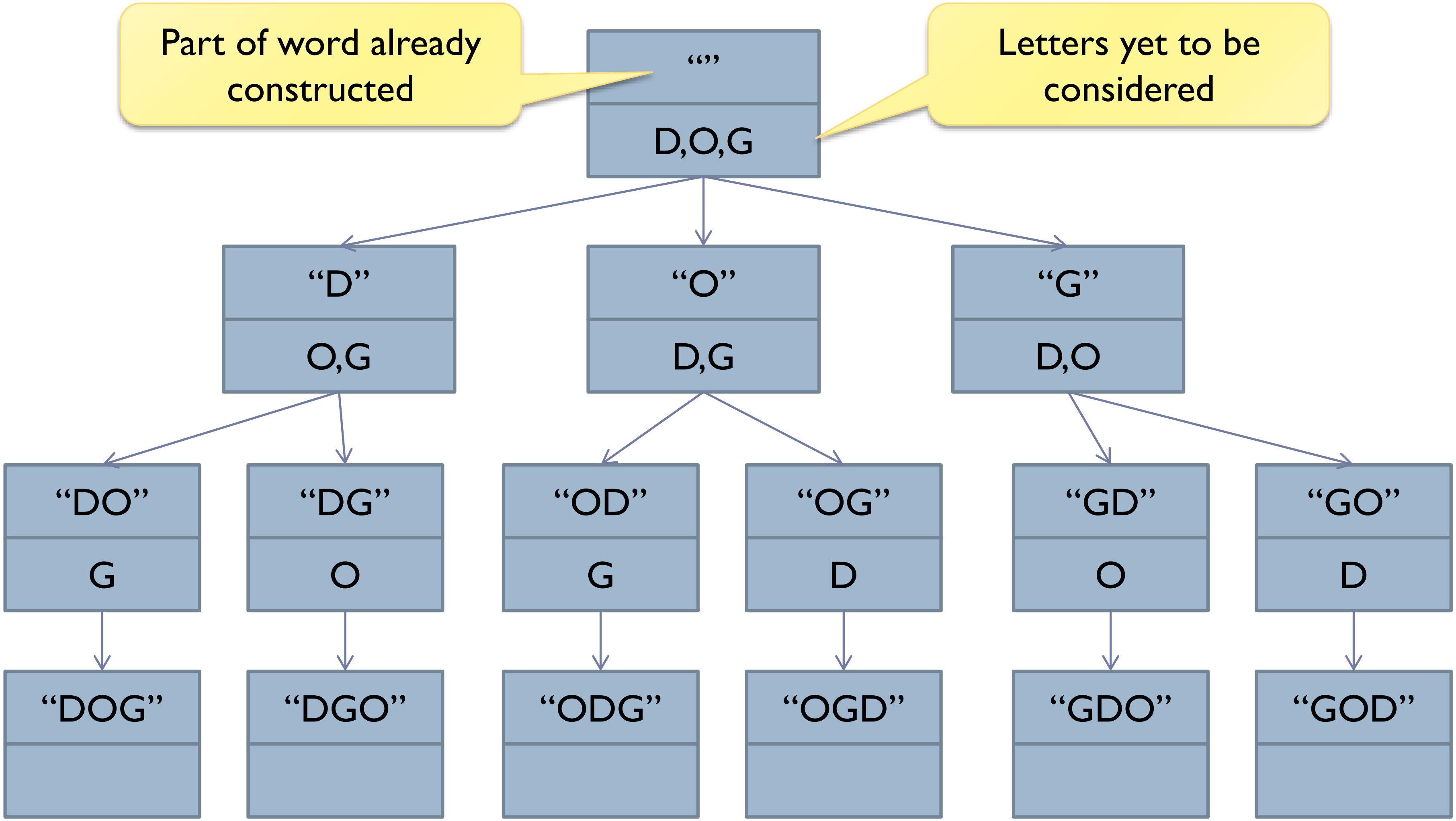
Anagrams: recursion tree



Anagrams: recursion tree



Anagrams: recursion tree

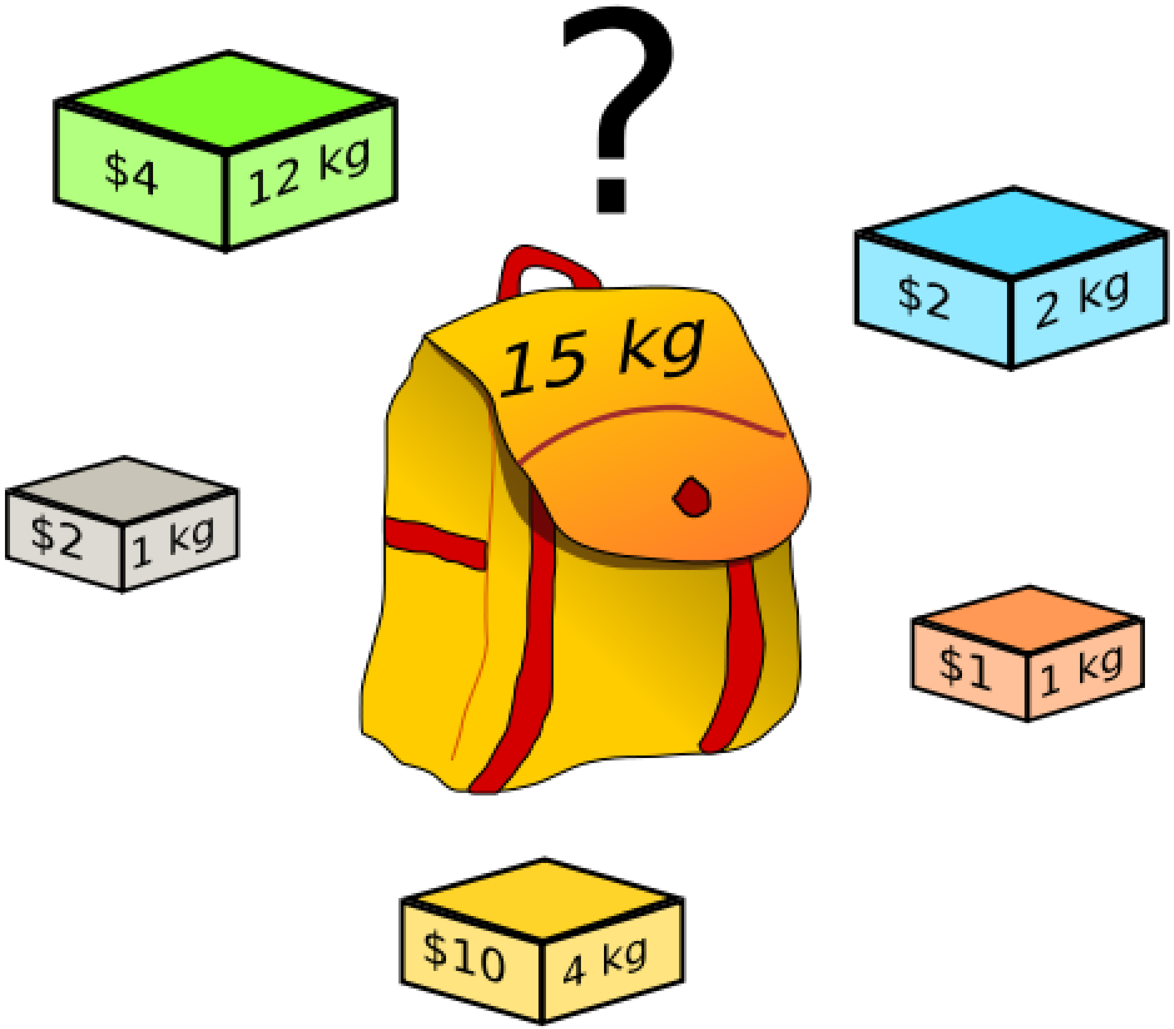


Anagrams: problem variants

- Generate only anagrams that are "valid" words
  - At the end of recursion, check the dictionary
  - During recursion, check whether the current prefix exists in the dictionary
- Handle words with multiple letters: avoid duplicate anagrams
  - E.g., "seas" → seas and seas are the same word
  - Generate all and, at the end or recursion, check if repeated
  - Constrain, during recursion, duplicate letters to always appear in the same order (e.g, s alwaws before s)

http://wordsmith.org/anagram/index.html

The Knapsack Problem



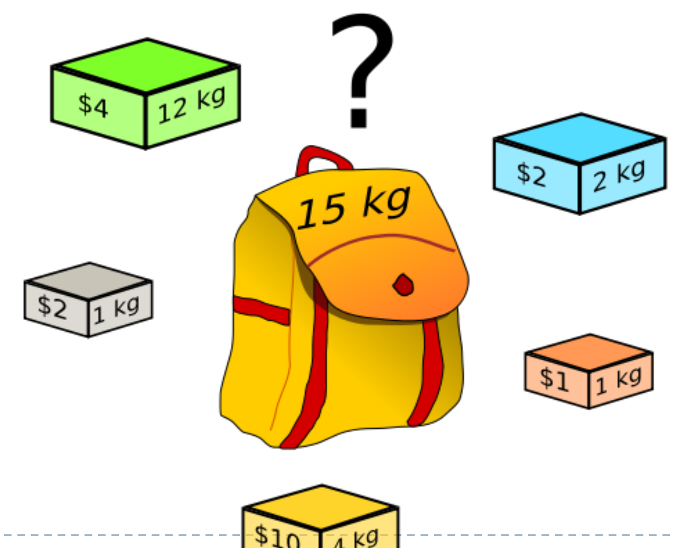
The Knapsack Problem

Input: Weight of N items $\{w_1, w_2, \ldots, w_n\}$
Cost of N items $\{c_1, c_2, \ldots, c_n\}$
Knapsack limit S

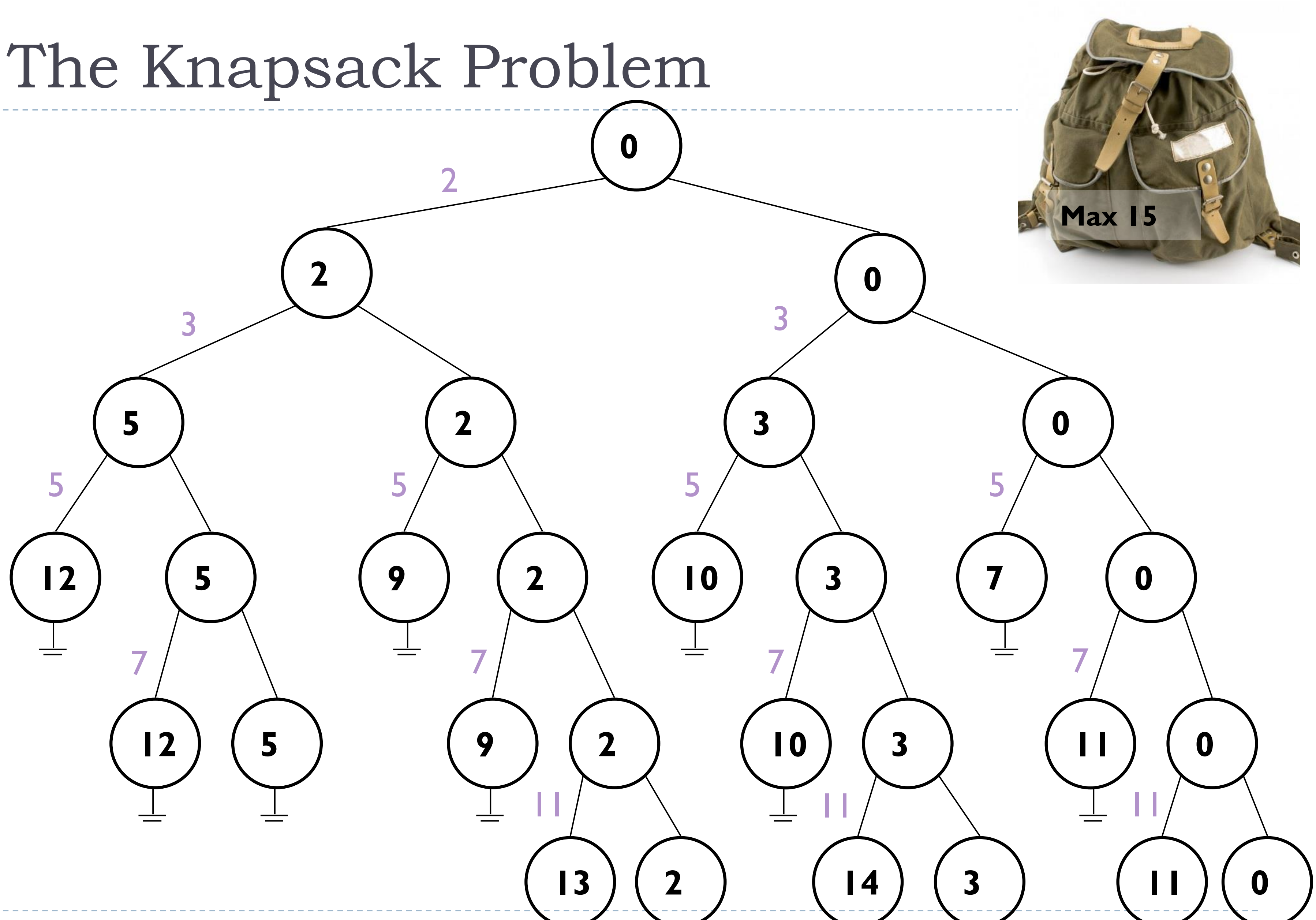
Output: Selection for knapsack: $\{x_1,x_2,\ldots x_n\}$
where $x_i \in \{0,1\}$.

Sample input:

$w_i=\{1,1,2,4,12\}$
$c_i=\{1,2,2,10,4\}$
$S=15$



The Knapsack Problem



Fibonacci Numbers

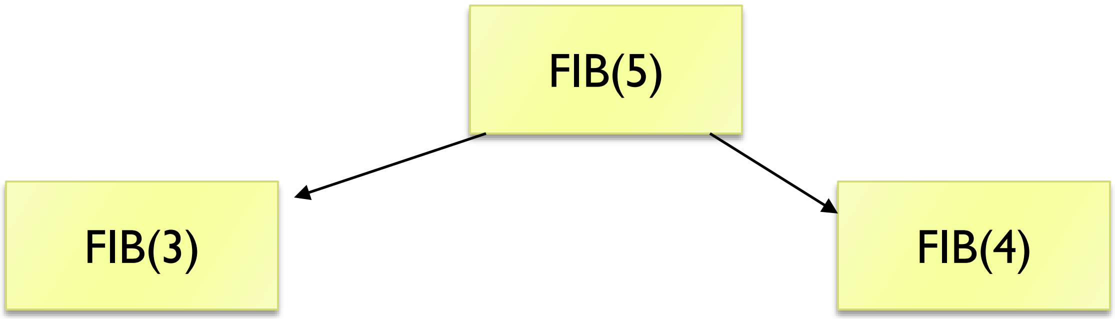
- Problem:
  - Compute the N-th Fibonacci Number
- Definition:
  - $FIB_{N+1} = FIB_N + FIB_{N-1}$ for $N>0$
  - $FIB_1 = 1$
  - $FIB_0 = 0$

Recursive solution

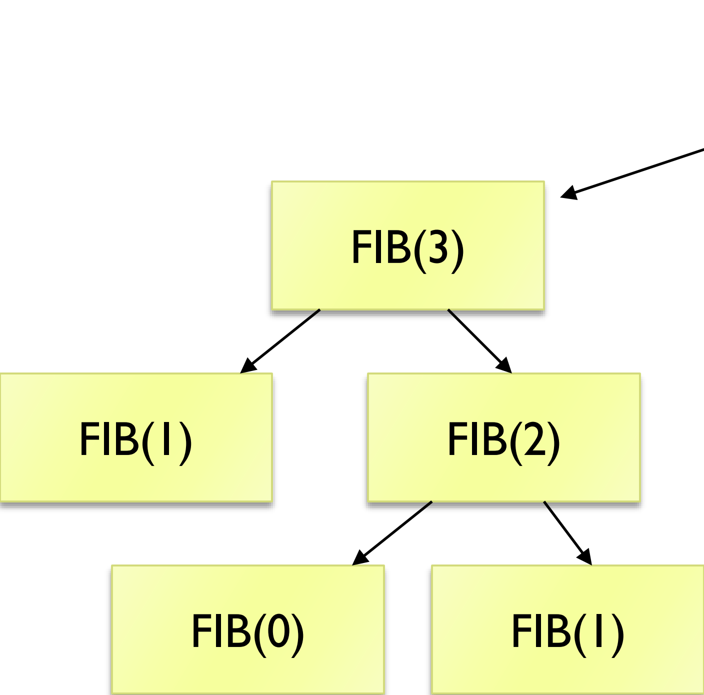
```java
public long recursiveFibonacci(long N) {
    if(N==0)
        return 0 ;
    if(N==1)
        return 1 ;

    long left = recursiveFibonacci(N-1) ;
    long right = recursiveFibonacci(N-2) ;

    return left + right ;
}
```

```
Fib(0)  = 0
Fib(1)  = 1
Fib(2)  = 1
Fib(3)  = 2
Fib(4)  = 3
Fib(5)  = 5
```

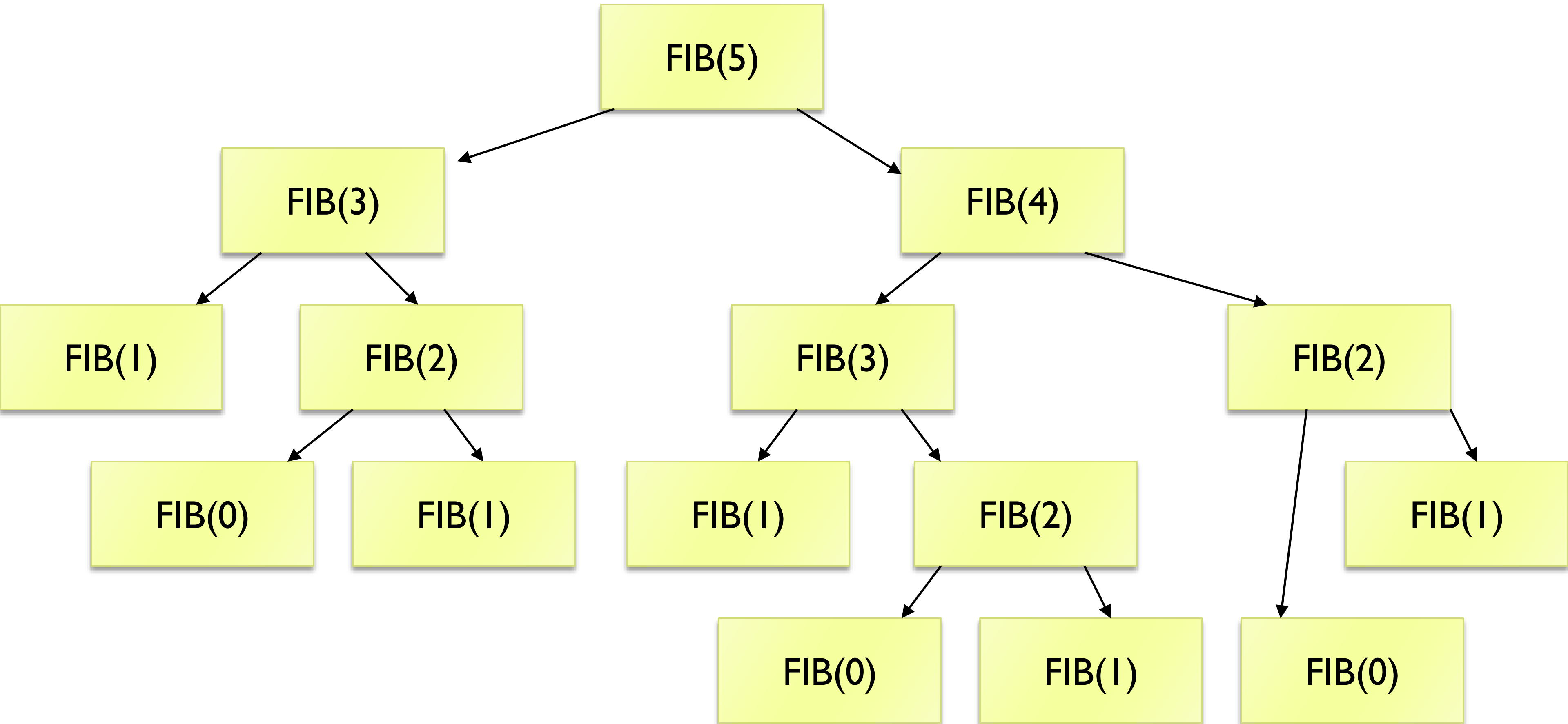
Analysis



Analysis

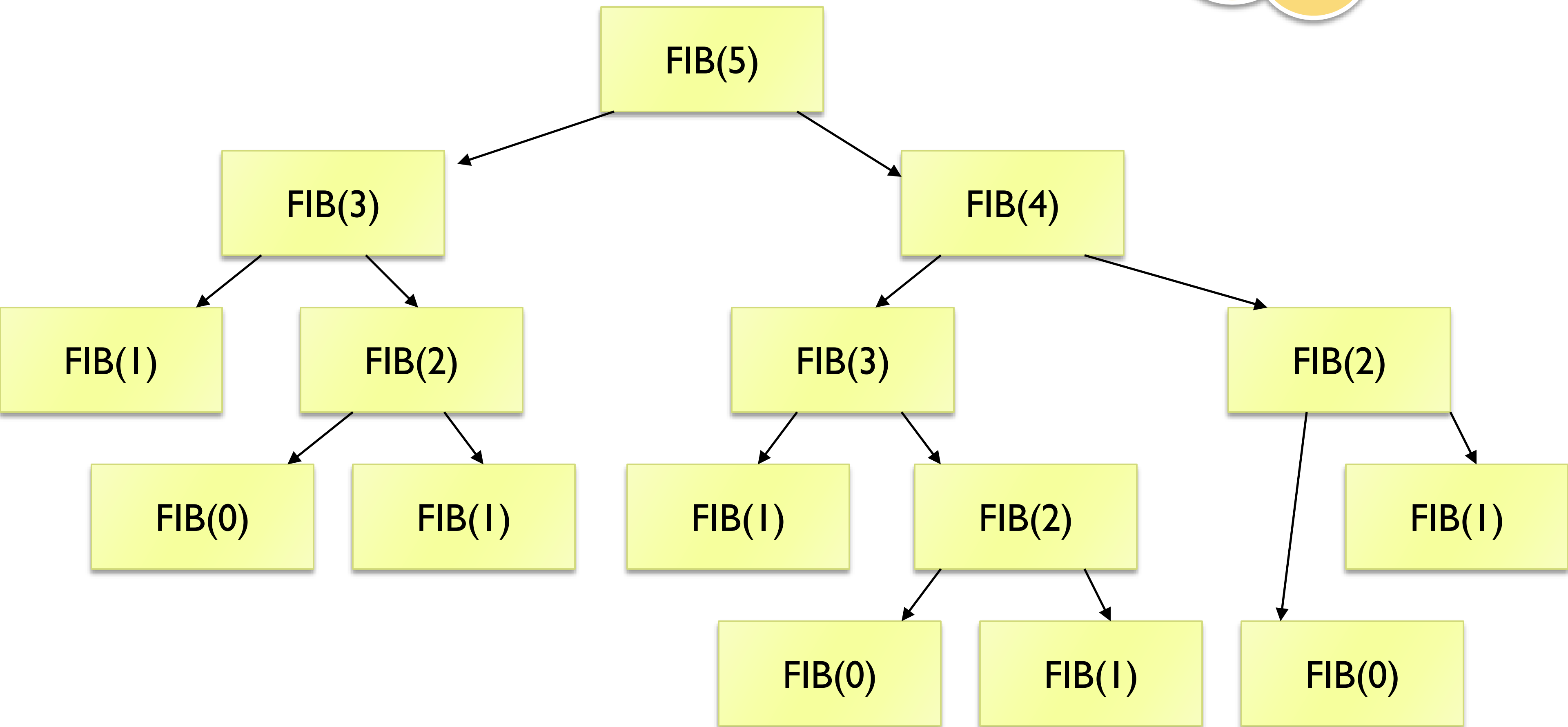


Analysis


FIB(5)
FIB(3)
FIB(4)
FIB(1)
FIB(2)
FIB(3)
FIB(2)
FIB(0)
FIB(1)
FIB(1)
FIB(2)
FIB(1)
FIB(0)
FIB(1)
FIB(0)

Analysis

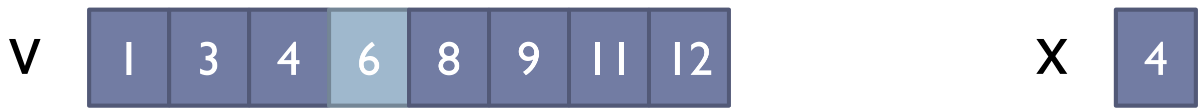
Complexity?



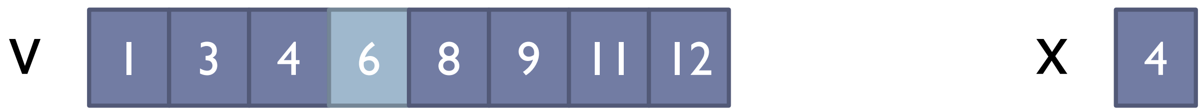
Example: dichotomic search

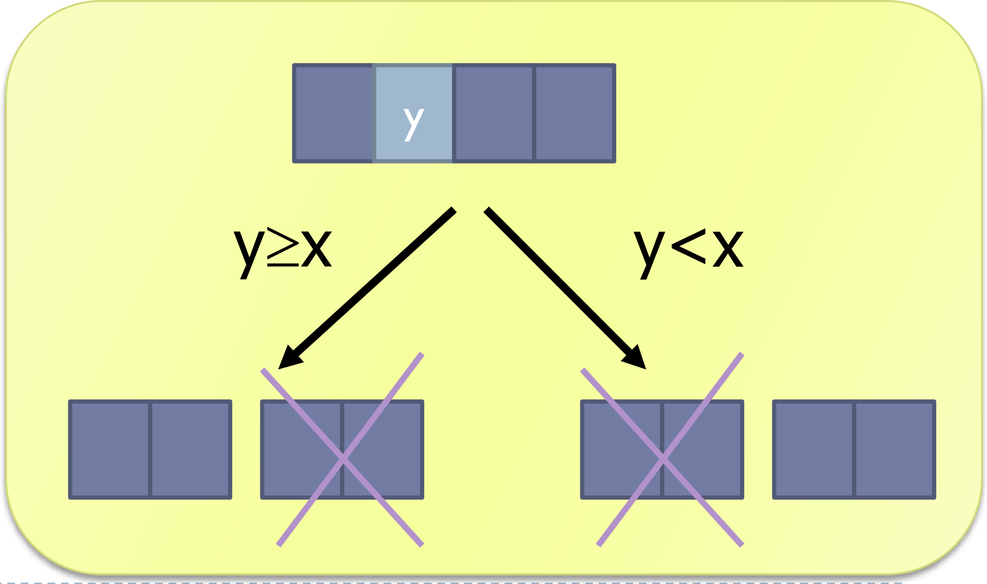
- Problem
  - Determine whether an element x is **present** inside an ordered **vector** v[N]
- Approach
  - Divide the vector in two halves
  - Compare the middle element with x
  - Reapply the problem over one of the two halves (left or right, depending on the comparison result)
  - The other half may be ignored, since the vector is ordered

Example

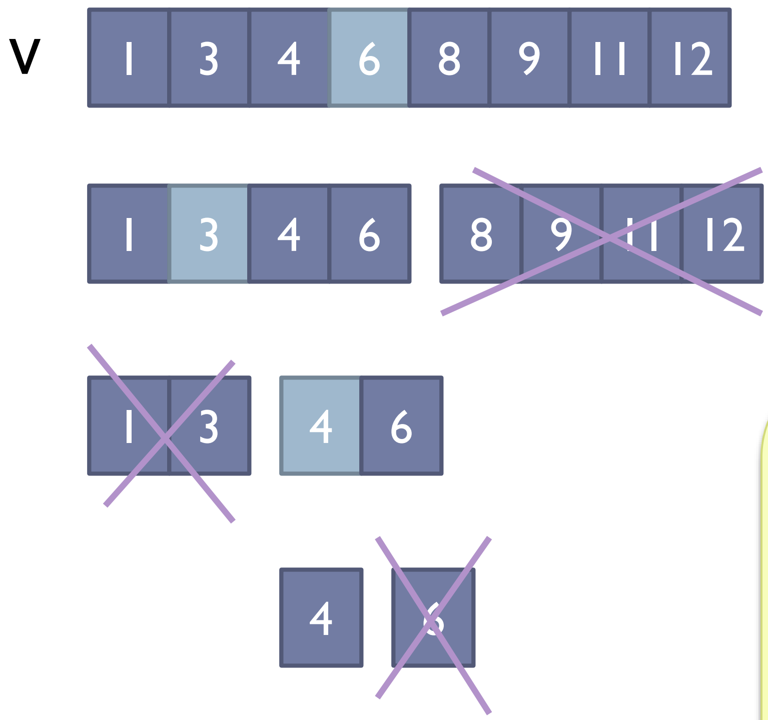


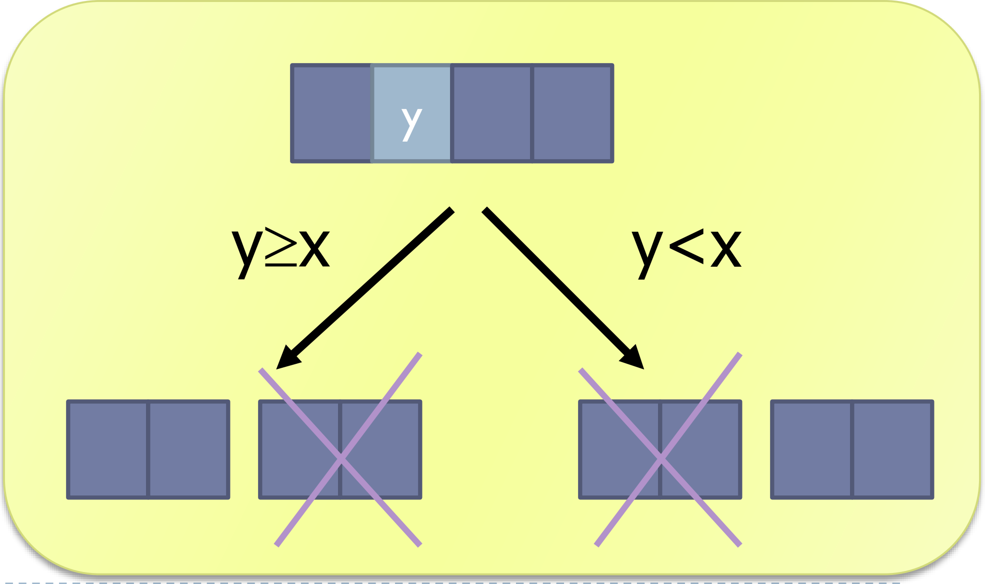
Example





Example





Solution

```
public int find(int[] v, int a, int b, int x)
{
	if(b-a == 0) { // trivial case
		if(v[a]==x) return a ; // found
		else return -1 ;	// not found
	}

	int c = (a+b) / 2 ; // splitting point
	if(v[c] >= x)
		return find(v, a, c, x) ;
	else return find(v, c+1, b, x) ;
}
```

Solution

```
public int fir
{
        if(b-a

        }

        int c = (a+b) / 2            itting point
        if(v[c] >= x)
                return find(v,  a, c, x) ;
        else return find(v, c+1, b, x) ;
}
```

Beware of integer-arithmetic approximations!

Quick reference

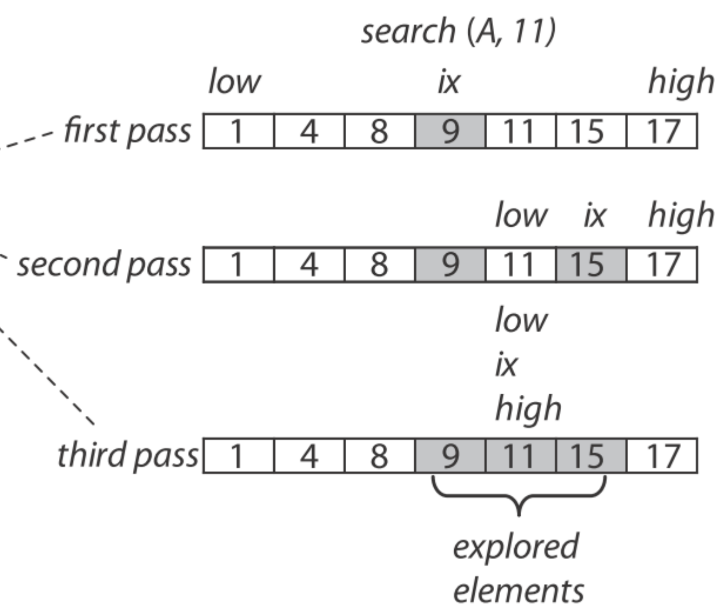
Binary Search

Best	Average	Worst
O (1)	O (log n)	O (log n)

Array

Divide and Conquer

```
search (A, t)
    low = 0
    high = n −1
    while (low ≤ high) do
      ix = (low + high)/2
      if (t = A[ix]) then
          return true
      else if (t < A[ix]) then
          high = ix − 1
      else low = ix + 1
    return false
end
```



Exercise: Value X

- When working with Boolean functions, we often use the symbol X, meaning that a given variable may have indifferently the value `0` or `1`.
- Example: in the OR function, the result is `1` when the inputs are `01`, `10` or `11`. More compactly, if the inputs are X1 or 1X.

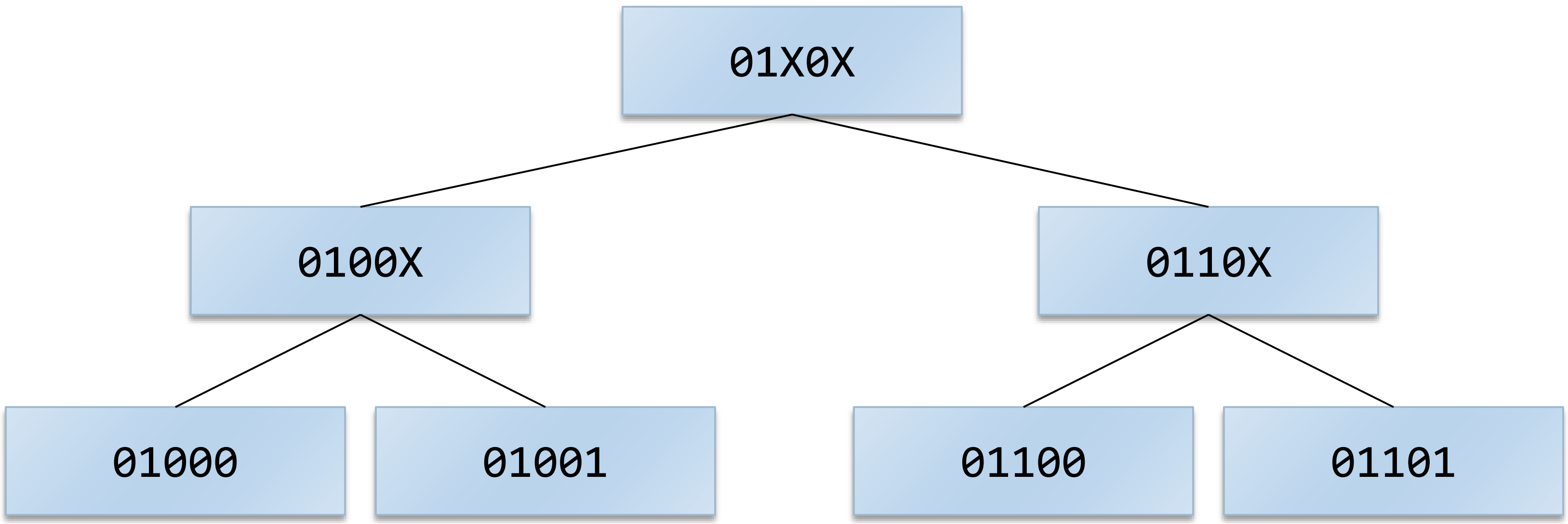
X-Expansion

- We want to devise an algorithm that, given a binary string that includes characters `0`, `1` and `X`, will compute all the possible combinations implied by the given string.
- Example: given the string `01X0X`, algorithm must compute the following combinations
  - `01000`
  - `01001`
  - `01100`
  - `01101`

Solution

- We may devise a recursive algorithm that explores the complete 'tree' of possible compatible combinations:
  - Transforming each X into a `0`, and then into a `1`
  - For each transformation, we recursively seek other X in the string
- The number of final combinations (leaves of the tree) is equal to 2^N, if N is the number of X.
- The tree height is N+1.

Combinations tree



8	2	5	5	6	7	3	9
1	2	4	1	9	2	3	1
2	2	5	2	42	7	9	7
8	2	5	6	6	6	3	9
1	2	4	1	9	2	3	1
2	7	1	1	4	7	8	9
2	3	5	3	1	8	9	9
8	2	3	1	6	7	3	9

4	2	5	5	3	7	3	9
1	2	4	1	9	2	3	1
2	2	5	2	42	7	1	3
8	2	5	6	1	1	1	9
1	2	4	1	9	2	3	1
2	7	1	1	4	7	8	2
2	3	5	3	1	8	9	9
8	2	3	1	6	7	3	9

Exercise: Binomial Coefficient

- Compute the Binomial Coefficient (n m) exploiting the recurrence relations (derived from Tartaglia's triangle):

$$\begin{cases} \binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m} \\ \binom{n}{n} = \binom{n}{0} = 1 \\ 0 \leq n, \quad 0 \leq m \leq n \end{cases}$$

Exercise: Determinant

- Compute the determinant of a square matrix
- Remind that:
  - det(M_{1x1}) = $m_{1,1}$
  - det(M_{NxN}) = sum of the products of all elements of a row (or column), times the determinants of the (N-1)x(N-1) sub-matrices obtained by deleting the row and column containing the multiplying element, with alternating signs $(-1)^{(i+j)}$.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}.$$

Laplace's Formula, at
http://en.wikipedia.org/wiki/Determinant

Recursive vs Iterative strategies

Recursion

Recursion and iteration

- Every **recursive** program can **always** be implemented in an **iterative** manner
- The best solution, in terms of efficiency and code clarity, depends on the problem

Example: Factorial (iterative)

$$\begin{cases} 0! \stackrel{\text{def}}{=} 1 \\ \forall N \geq 1: \\ N! \stackrel{\text{def}}{=} N \times (N-1)! \end{cases}$$

```
public long iterativeFactorial(long N)
{
    long result = 1 ;

    for (long i=2; i<=N; i++)
        result = result * i ;

    return result ;
}
```

Fibonacci (iterative)

```java
public long iterativeFibonacci(long N) {
  if(N==0) return 0 ;
  if(N==1) return 1 ;

  // now we know that N >= 2
  long i = 2 ;
  long fib1 = 1 ; // fib(N-1)
  long fib2 = 0 ; // fib(N-1)

  while( i<=N ) {
    long fib = fib1 + fib2 ;
    fib2 = fib1 ;
    fib1 = fib ;
    i++ ;
  }

  return fib1 ;
}
```

Dichotomic search (iterative)

```java
public int findIterative(int[] v, int x) {
  int a = 0 ;
  int b = v.length-1 ;

  while( a != b ) {
    int c = (a + b) / 2; // middle point
    if (v[c] >= x) {
      // v[c] is too large -> search left
      b = c ;
    } else {
      // v[c] is too small -> search right
      a = c+1 ;
    }
  }
  if (v[a] == x)
    return a;
  else
    return -1;
}
```

Exercises

- Create an iterative version for the computation of the binomial coefficient (n m).

- Analyze a possible iterative version for computing the determinant of a matrix. What are the difficulties?

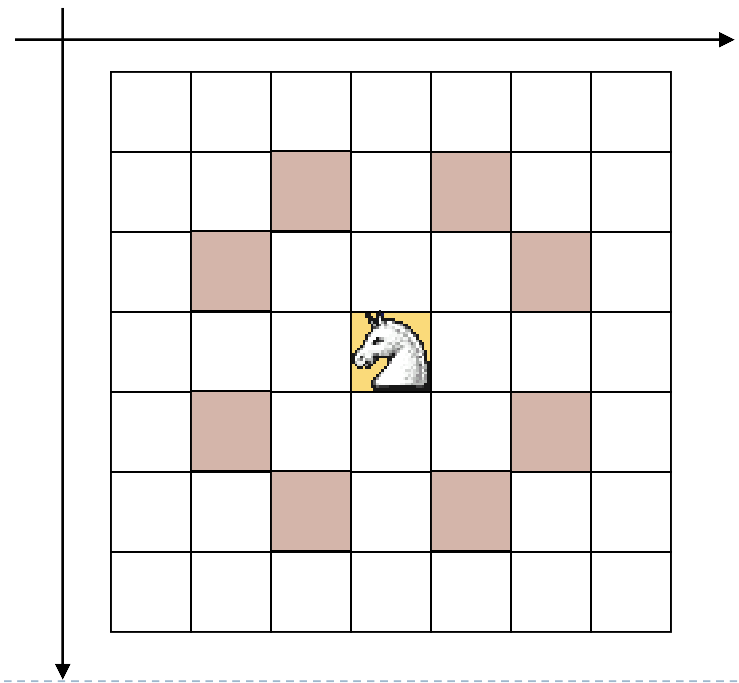
- Can you find a simple iterative solution for the X-Expansion problem? And for the Anagram problem?

More complex examples of recursive algorithms

Recursion

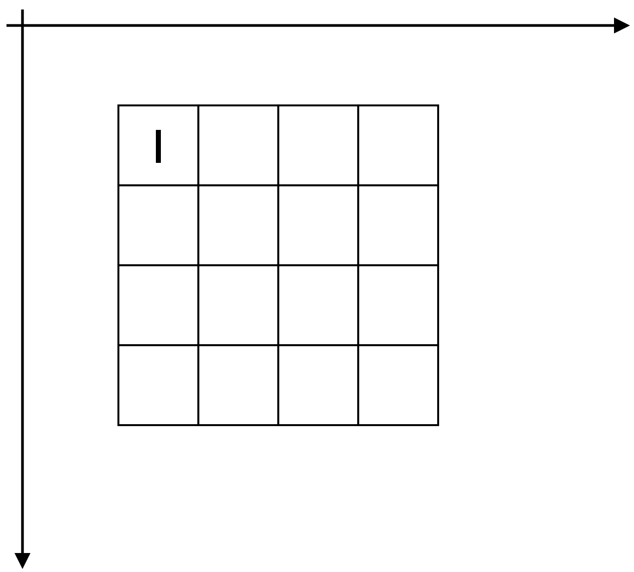
Knight's tour

- Consider a NxN chessboard, with the Knight moving according to Chess rules
  - The Knight may move in 8 different cells
- We want to find a **sequence** of moves for the Knight where
  - **All** cells in the chessboard are visited
  - Each cell is touched exactly **once**
- The starting point is arbitrary

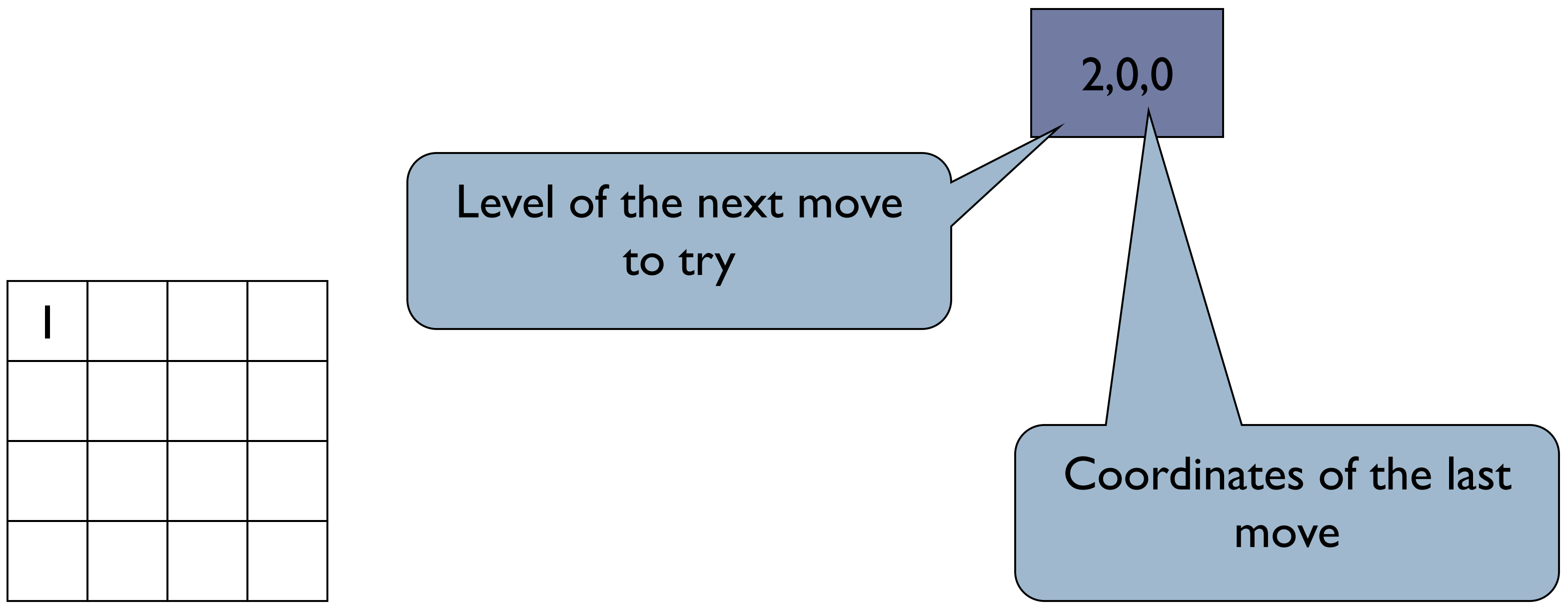


Analysis

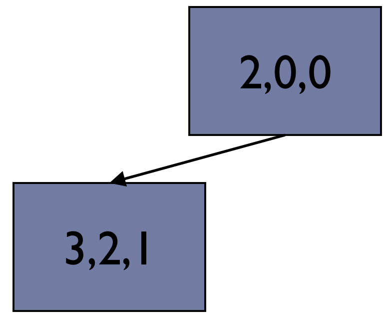
- Assume N=4



Move 1

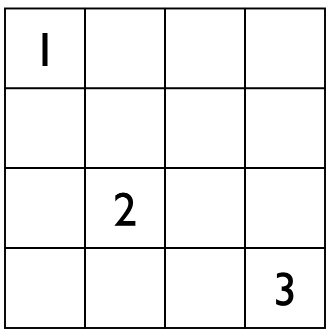

1
2,0,0
Level of the next move to try
Coordinates of the last move

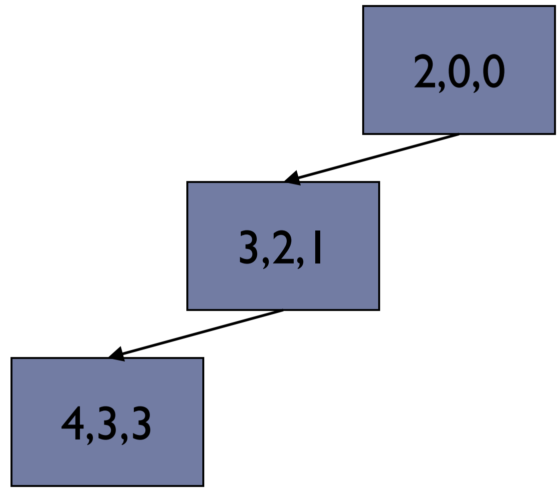
Move 2



1			
	2		

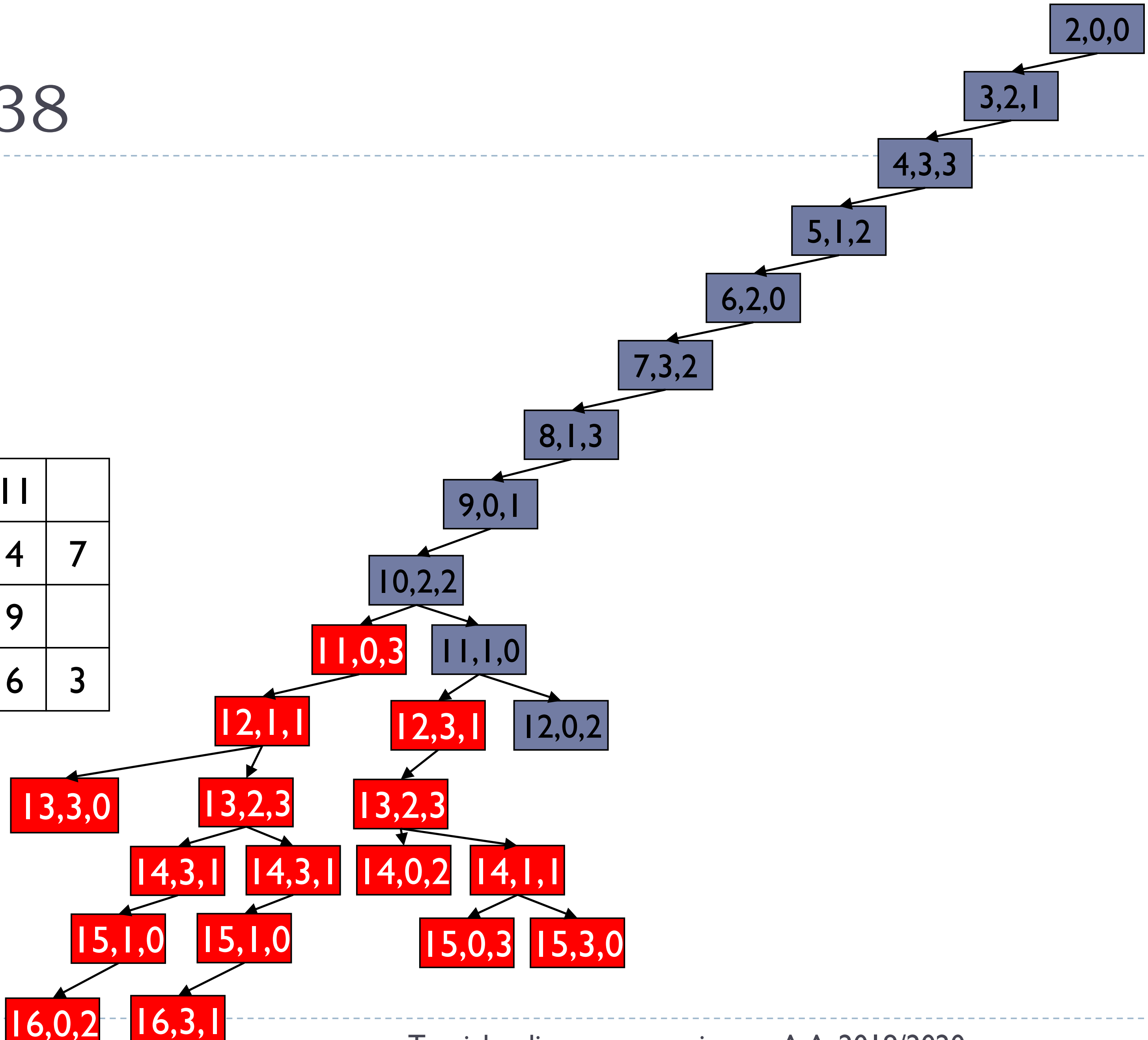
Move 3





Move 38

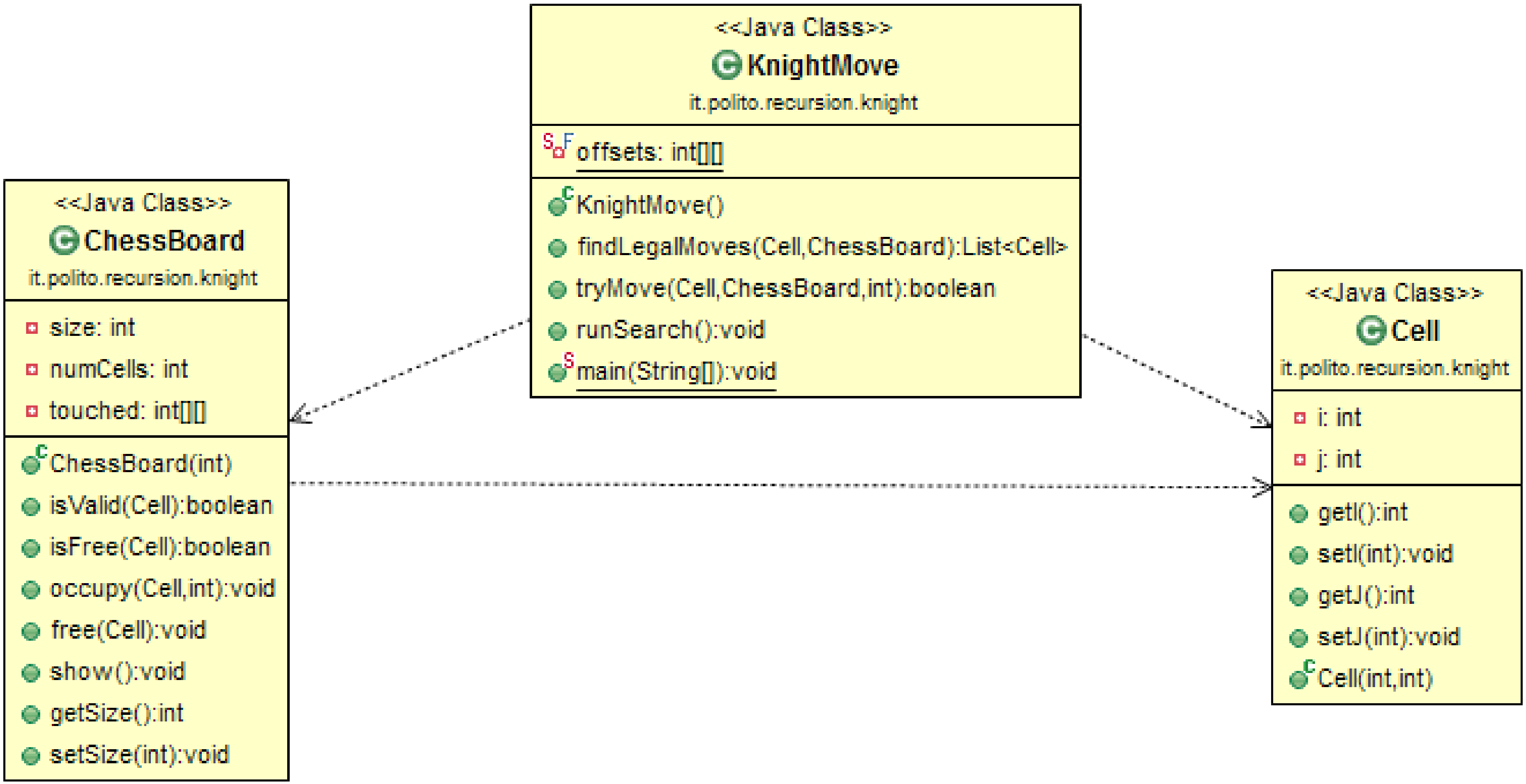
1	8	11	
10		4	7
5	2	9	
		6	3



Complexity

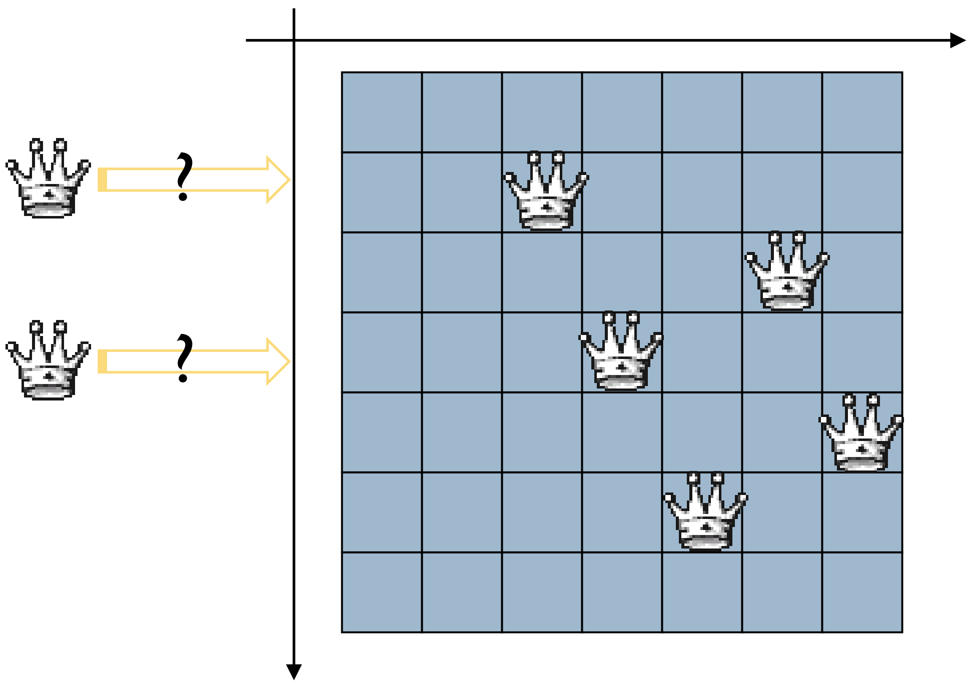
- The number of possible moves, at each step, is at most 8.
- The number of steps is N^2.
- The solution tree has a number of nodes $\leq 8^{N^2}$.
- In the worst case
  - The solution is in the right-most leave of the solution tree
  - The tree is complete
- The number of recursive calls, in the worst case, is therefore $\Theta(8^{N^2})$.

Implementation



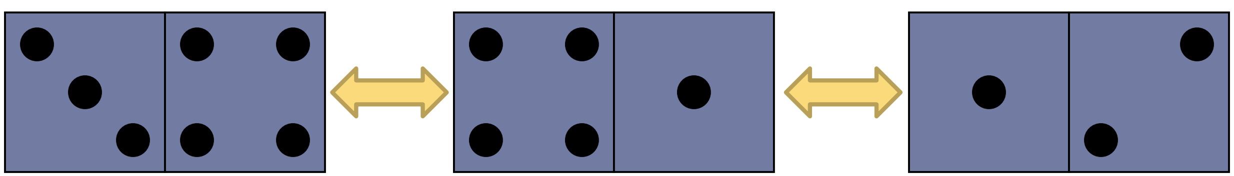
The N Queens

- Consider a NxN chessboard, and N Queens that may act according to the chess rules
- Find a position for the N queens, such that no Queen is able to attack any other Queen



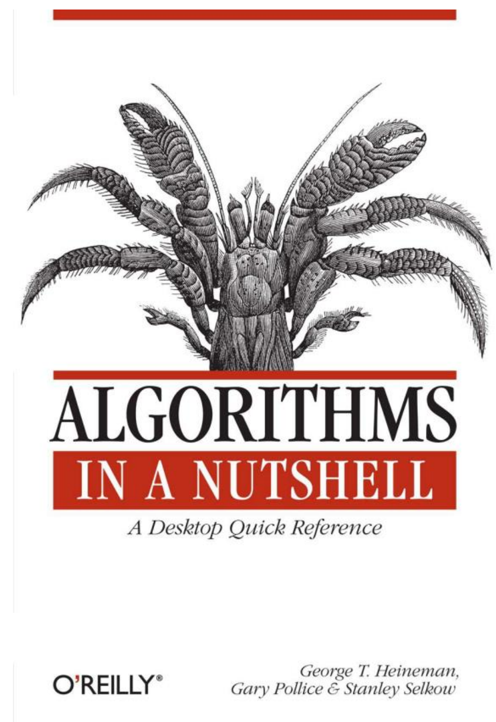
Domino game

- Consider the game of Domino, composed of two-sided pieces: each side is labeled with a number from 0 to 6. All combinations of number pairs are represented exactly once.
- Find the longest possible sequence of pieces, such that consecutive pieces have the same value on the adjacent sides.



Resources

- Algorithms in a Nutshell, By George T. Heineman, Gary Pollice, Stanley Selkow, O'Reilly Media



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